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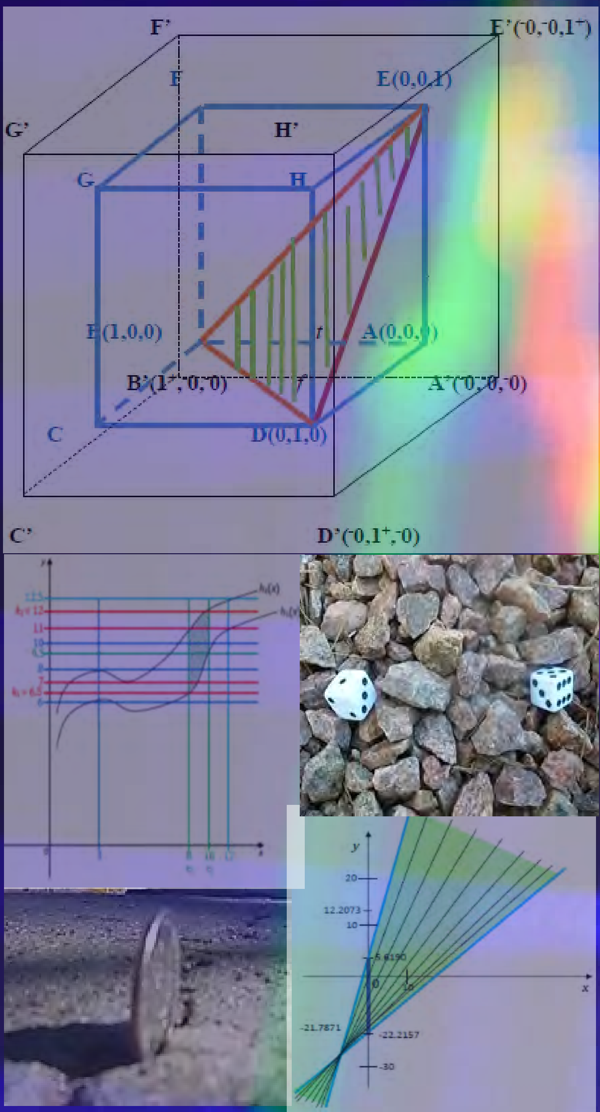

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset
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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
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March 20, 2019

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Univ New Mexico, Gallup Campus

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Sincerely,

Marian Hollingsworth

Marian Hollingsworth
Director, Publisher Relations

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University of New Mexico

Parameter Reduction of Neutrosophic Soft Sets and Their Applications

Abhijit Saha[1] and Said Broumi[2]

[1]Faculty of Mathematics, Techno College of Engineering Agartala, , Tripura, India, Pin-799004; Email: abhijit84.math@gmail.com
[2]Faculty of Science, University of Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco; Email: broumisaid78@gmail.com

* Correspondence: abhijit84.math@gmail.com

Abstract: Parameter reduction can be treated as an effective tool in many fields, including pattern recognition. Many reduction techniques have been reported so far for soft sets, fuzzy soft sets and bipolar fuzzy soft sets to solve decision-making problems. However, there is almost no attention to the parameter reduction of neutrosophic soft sets. In this present paper we focus our discussion on the parameter reduction of neutrosophic soft sets as an extension of parameter reduction of soft sets and fuzzy soft sets. To do that, using the concept of indiscernibility relation, we first define the terms 'dispensable set' and 'indispensible set'. We utilize these definitions to define the terms 'decision partition', 'parameter reduction' and 'degree of importance of a parameter' with a suitable example. Next we present an algorithm based on the concept of degree of importance and parameter reduction of a neutrosophic soft set. An illustrative example is employed to show the feasibility and validity of our proposed algorithm based on parameter reduction of neutrosophic soft sets in real life decision making problem.

Keywords: Neutrosophic set, neutrosophic soft set, parameter reduction, decision making.

1. Introduction

Molodstov **[31]** initiated the concept of soft set theory as a fundamental mathematical tool for modelling uncertainty, vague concepts and not clearly defined objects. Although various traditional tools, including but not limited to rough set theory **[33]**, fuzzy set theory **[41]**, intuitionistic fuzzy set theory **[10]** etc. have been used by many researchers to extract useful information hidden in the uncertain data, but there are inherent complications connected with each of these theories.

Additionally, all these approaches lack in parameterizations of the tools and hence they couldn't be applied effectively in real life problems, especially in areas like environmental, economic and social problems. Soft set theory is standing uniquely in the sense that it is free from the above mentioned impediments and obliges approximate illustration of an object from the beginning, which makes this theory a natural mathematical formalism for approximate reasoning.

The Theory of soft set has excellent potential for application in various directions some of which are reported by Molodtsov **[31]** in his pioneer work. Later on Maji et al. **[27]** introduced some new annotations on soft sets such as subset, complement, union and intersection of soft sets and discussed in detail its applications in decision making problems. Ali et al. **[7]** defined some new operations on soft sets and shown that De Morgan's laws holds in soft set theory with respect to these newly defined operations. Atkas and Cagman **[6]** compared soft sets with fuzzy sets and rough sets to show that every fuzzy set and every rough set may be considered as a soft set. Jun **[24]** connected soft sets to the theory of BCK/BCI-algebra and introduced the concept of soft BCK/BCI-algebras. Feng et al.**[21]** characterized soft semi rings and a few related notions to establish a relation between soft sets and semi rings.

Chen et al. **[15]** introduced the concept of parameter reduction of soft sets in 2005. In 2008, Z. Kong et al **[25]** introduced the definition of normal parameter reduction in soft sets and presented a heuristic algorithm of normal parameter reduction. The soft sets mentioned above are based on complete information. However, incomplete information widely exists in various real life problems. H. Qin et al **[34]** studied the data filling approach of incomplete soft sets. Y. Zou et al **[42]** investigated data analysis approaches of soft sets under incomplete information. In 2001, Maji et al. **[28]** defined the concept of fuzzy soft set by combining of fuzzy sets **[41]** and soft sets **[31]**. Roy and Maji **[35]** proposed a fuzzy soft set based decision making method.

Xiao et al. **[39]** presented a combined forecasting method based on fuzzy soft set. Feng et al. **[22]** discussed the validity of the Roy-Maji method **[35]** and presented an adjustable decision-making method based on fuzzy soft set. Yang et al. **[40]** initiated the idea of interval valued fuzzy soft set (IVFS-set) and analyzed a decision making method using the IVFS-sets. The notion of intuitionistic fuzzy set (IFS) was initiated by Atanassov **[10]** as a significant generalization of fuzzy set **[41]**. Intuitionistic fuzzy sets are very useful in situations when description of a problem by a linguistic variable, given in terms of a membership function only, seems too complicated. Recently intuitionistic fuzzy sets have been applied to many fields such as logic programming, medical diagnosis, decision making problems etc.

Smarandache **[38]** introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Maji **[30]** introduced the concept of neutrosophic soft set and established some operations on these sets. Mukherjee et al **[32]** introduced the concept of interval valued neutrosophic soft sets and studied their basic properties. In 2013, Broumi and Smarandache **[12, 13]** combined the intuitionistic neutrosophic and soft set which lead to a new mathematical model called" intuitionistic neutrosophic soft set". They studied the notions of intuitionistic neutrosophic soft set union, intuitionistic neutrosophic soft set intersection, complement of intuitionistic neutrosophic soft set and several other properties of intuitionistic neutrosophic soft set along with examples and proofs of certain results.

Also, in **[11]** S. Broumi presented the concept of "generalized neutrosophic soft set" by combining the generalized neutrosophic sets **[11]** and soft set models, studied some properties on it, and presented an application of generalized neutrosophic soft set **[11]** in decision making problem. Recently, Deli **[17]** introduced the concept of interval valued neutrosophic soft set as a combination of interval neutrosophic set and soft set. In 2014, S. Broumi et al. **[14]** initiated the concept of relations on interval valued neutrosophic soft sets.I. Deli **[18]** proposed a new notation called expansion and reduction of the neutrosophic classical soft sets that are based on the linguistic modifiers. Saha et al. **[36]** proposed the concept of data filling of neutrosophic soft sets having incomplete/missing data. Few more works on neutrosophic soft sets can be found in **[9, 19, 23, 37]**.

Parameter reduction can be treated an effective tool in many fields, including pattern recognition. Many reduction techniques **[8, 15, 16, 20, 25, 26]** have been reported so far for soft sets, fuzzy soft sets and bipolar fuzzy soft sets to solve decision-making problems. However, there is almost no attention to the parameter reduction of neutrosophic soft sets. In this present paper we focus our discussion on the parameter reduction of neutrosophic soft sets as an extension of parameter reduction of soft sets and fuzzy soft sets.

This present paper is organized as follows:

Section-2 presents some basic definitions related to fuzzy set theory with their generalizations and soft set theory with their generalizations. In section-3, we first present the concept of indiscernibility relations and then based on it, we define the terms 'dispensable set', 'indispensible set', 'decision partition', 'parameter reduction', 'degree of importance of a parameter' with a suitable example in neutrosophic soft environment. In the next section (section-4), we have presented an algorithm based on the concept of degree of importance and parameter reduction supported by an illustrative example to show the feasibility and validity of our algorithm.

2. Preliminaries:

2.1 Definition: [41] Let U be a non empty set. Then afuzzy set τ on U is a set having the form $\tau=\{(\mathrm{x}, \mu_\tau(\mathrm{x})):\mathrm{x}\in \mathrm{U}\}$ where the function $\mu_\tau:\mathrm{U}\to[0, 1]$ is called the membership function and $\mu_\tau(\mathrm{x})$ represents the degree ofmembership of each element $\mathrm{x}\in\mathrm{U}$.

2.2 Definition: [10] Let U be a non empty set. Then an intuitionistic fuzzy set (IFS for short) τ is an object having the form $\tau=\{\langle \mathrm{x}, \mu_\tau(\mathrm{x}), \gamma_\tau(\mathrm{x})\rangle: \mathrm{x}\in\mathrm{U}\}$ where the functions $\mu_\tau:\mathrm{U}\to[0, 1]$ and $\gamma_\tau:\mathrm{U}\to[0, 1]$ are called membership function and non-membership function respectively.

$\mu_\tau(\mathrm{x})$ and $\gamma_\tau(\mathrm{x})$ represent the degree ofmembership and the degree of non-membership respectively of each element $\mathrm{x}\in\mathrm{U}$ and $0\le\mu_\tau(\mathrm{x})+\gamma_\tau(\mathrm{x})\le 1$ for each $\mathrm{x}\in\mathrm{U}$. We denote the class of all intuitionistic fuzzy sets on U by $\mathrm{IFS}^{\mathrm{U}}$.

2.3 Definition: [31] Let U be a universe set and E be a set of parameters. Let $\mathrm{P}(\mathrm{U})$ denotes the power set of U and $\mathrm{A}\subseteq\mathrm{E}$. Then the pair (F, A) is called a soft set over U, where F is a mapping given by $\mathrm{F}: \mathrm{A}\to\mathrm{P}(\mathrm{U})$.
In other words, the soft set is not a kind of set, but a parameterized family of subsets of U. For $\mathrm{e}\in\mathrm{A}$, $F(\mathrm{e})\subseteq\mathrm{U}$ may be considered as the set of e-approximate elements of the soft set (F, A).

2.4 Definition: [28] Let U be a universe set, E be a set of parameters and $\mathrm{A}\subseteq\mathrm{E}$. Then the pair (F, A) is called a fuzzy soft setover U, where F is a mapping given by $\mathrm{F}: \mathrm{A}\to\mathrm{FS}^{\mathrm{U}}$.

2.5 Definition: [29] Let U be a universe set, E be a set of parameters and $\mathrm{A}\subseteq\mathrm{E}$. Then the pair (F, A) is called an intuitionistic fuzzy soft set over U, where F is a mapping given by $\mathrm{F}: \mathrm{A}\to\mathrm{IFS}^{\mathrm{U}}$.
For $\mathrm{e}\in\mathrm{A}$, $F(\mathrm{e})$ is an intuitionistic fuzzy subset of U and is called the intuitionistic fuzzy value set of the parameter ‘e’.
Let us denote $\mu_{F(\mathrm{e})}(\mathrm{x})$ by the membership degree that object ‘x’ holds parameter ‘e’ and $\gamma_{F(\mathrm{e})}(\mathrm{x})$ by the membership degree that object ‘x’ doesn’t hold parameter ‘e’ , where $\mathrm{e}\in\mathrm{A}$ and $\mathrm{x}\in\mathrm{U}$. Then $F(\mathrm{e})$ can be written as an intuitionistic fuzzy set such that $F(\mathrm{e})=\{(\mathrm{x}, \mu_{F(\mathrm{e})}(\mathrm{x}), \gamma_{F(\mathrm{e})}(\mathrm{x})): \mathrm{x}\in\mathrm{U}\}$.

2.6 Definition: [38] A neutrosophicset A on the universe of discourse U is defined as
$A=\{\langle x,\mu_A(x),\gamma_A(x),\delta_A(x)\rangle : x\in U\}$, where $\mu_A,\gamma_A,\delta_A : U\to]^-0,1^+[$ are functions such that the condition: $\forall x\in U, {}^-0\le\mu_A(x)+\gamma_A(x)+\delta_A(x)\le 3^+$ is satisfied.
Here $\mu_A(x),\gamma_A(x),\delta_A(x)$ represent the truth-membership, indeterminacy-membership and falsity-membership (hesitancy membership) respectively of the element $x\in U$.

Smarandache **[25]** applied neutrosophic sets in many directions after giving examples of neutrosophic sets. Then he introduced the neutrosophic set operations namely-complement, union, intersection, difference, Cartesian product etc.

2.7 Definition: [30] Let U be an initial universe, E be a set of parameters and $A \subseteq E$. Let $NP(U)$ denotes the set of all neutrosophic sets of U. Then the pair (f, A) is termed to be the neutrosophicsoftset over U, where f is a mapping given by $f : A \to NP(U)$.

2.8 Example: Let us consider a neutrosophic soft set (f, A) which describes the "attractiveness of the house". Suppose $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the set of six houses under consideration and $E = \{e_1$ (beautiful), e_2 (expensive), e_3 (cheap), e_4 (good location), e_5 (wooden)$\}$ be the set of parameters. Then a neutrosophic soft set (f, A) over U can be given by:

U	e_1	e_2	e_3	e_4	e_5
u_1	(0.8,0.5,0.2)	(0.3,0.4,0.6)	(0.1,0.6,0.4)	(0.7,0.3,0.6)	(0.3,0.4,0.6)
u_2	(0.4,0.1,0.7)	(0.8,0.2,0.4)	(0.4,0.1,0.7)	(0.2,0.4,0.4)	(0.1,0.1,0.3)
u_3	(0.2,0.6,0.4)	(0.5,0.5,0.5)	(0.8,0.1,0.7)	(0.5,0.3,0.5)	(0.5,0.5,0.5)
u_4	(0.3,0.4,0.4)	(0.1,0.3,0.3)	(0.3,0.4,0.4)	(0.6,0.6,0.6)	(0.1,0.1,0.5)
u_5	(0.1,0.1,0.7)	(0.2,0.6,0.7)	(0.4,0.2,0.1)	(0.8,0.6,0.1)	(0.6,0.7,0.7)
u_6	(0.5,0.3,0.9)	(0.3,0.6,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.5)	(0.4,0.4,0.4)

3. Parameter reduction of neutrosophic soft sets:

Suppose $U = \{x_1, x_2, x_3, \ldots, x_n\}$ be the universe set of objects and $E = \{e_1, e_2, e_3, \ldots, e_m\}$ be the set of parameters. Consider a neutrosophic soft set (f, E) given by $f(e) = \{\langle x, m_{f(e)}(x), g_{f(e)}(x), d_{f(e)}(x)\rangle : x \in U\}$ for $e \in E$ $\tilde{f}$ given by: $\tilde{f}_E(x_i) = \sum_j \left(m_{f(e_j)}(x_i) + g_{f(e_j)}(x_i) + d_{f(e_j)}(x_i)\right)$, $x_i \in U$.

We use $\tilde{f}_{e_j}(x_i)$ to denote $m_{f(e_j)}(x_i) + g_{f(e_j)}(x_i) + d_{f(e_j)}(x_i)$.

3.1 Definition: For any subset of parameters $B \in E$, an indiscernibility relation IND_B is defined as: $IND_B = \{(x_i, x_j) \in U \times U : \tilde{f}_B(x_i) = \tilde{f}_B(x_j)\}$.

For the neutrosophic soft set (f, E), we denote $C_E^U = \{\{x_1, x_2, x_3, \ldots, x_i\}_{x_1}, \{x_{i+1}, x_{i+2}, \ldots, x_j\}_{x_2}, \ldots \ldots \{x_k, x_{k+1}, \ldots, x_n\}_{x_s}\}$ as a partition of objects in U which partitions and ranks the objects according to the value of $\tilde{f}_E(x_i)$ based on the indiscernibility relation IND_E. C_E^U is called the decision partition, where the sub classes are: $\{x_1, x_2, x_3, \ldots\ldots, x_i\}, \{x_{i+1}, x_{i+2}, \ldots\ldots, x_j\}, \ldots\ldots, \{x_k, x_{k+1}, \ldots, x_n\}$ where s is the number of sub-classes, and $x_1 \geq x_2 \geq x_3 \geq \ldots \geq x_s$.

For any sub-class $\{x_z, x_{z+1}, ..., x_{z+h}\}_{x_q}$, $\left[\tilde{f}_E(x_z)\right] = \left[\tilde{f}_E(x_{z+1})\right] = ... = \left[\tilde{f}_E(x_{z+h})\right] = x_q$, where $[.]$ denotes the greatest integer function. Thus objects from U with the same value of $\tilde{f}_E(.)$ are included into a same class.

3.2 Example: Let $U = \{x_1, x_2, x_3, ..., x_6\}$ be the set of six houses and $E = \{e_1, e_2, e_3, ..., e_6\}$ be the set of parameters where the parameters $e_1, e_2, e_3, e_4, e_5, e_6$ represents 'beautiful', 'in the main town', 'expensive', 'concrete', 'in green surroundings', 'wooden' respectively. Consider the neutrosophic soft set (f, E) which describes the attractiveness and physical trait of the houses given by the following table (table-1).

Table-1

U	e_1	e_2	e_3	e_4	e_5	e_6	$\tilde{f}_E(.)$
x_1	(0.3,0.7,0.4)	(0.4,0.5,0.1)	(0.2,0.2,0.4)	(0.6,0.3,0.4)	(0.1,0.1,0.3)	(0.2,0.4,0.6)	**6.2**
x_2	(0.4,0.5,0.5)	(0.2,0.2,0.6)	(0.5,0.5,0.1)	(0.2,0.8,0.3)	(0.4,0.3,0.2)	(0.6,0.3,0.4)	**7.0**
x_3	(0.2,0.5,0.7)	(0.3,0.2,0.5)	(0.8,0.2,0.4)	(0.5,0.5,0.3)	(0.2,0.4,0.2)	(0.9,0.6,0.6)	**8.0**
x_4	(0.5,0.3,0.6)	(0.6,0.3,0.1)	(0.2,0.5,0.6)	(0.4,0.4,0.5)	(0.7,0.3,0.2)	(0.5,0.5,0.8)	**8.0**
x_5	(0.3,0.5,0.6)	(0.4,0.4,0.2)	(0.3,0.3,0.5)	(0.6,0.1,0.6)	(0.7,0.8,0.1)	(0.4,0.6,0.6)	**8.0**
x_6	(0.7,0.3,0.4)	(0.3,0.5,0.2)	(0.4,0.8,0.5)	(0.5,0.3,0.5)	(0.1,0.2,0.3)	(0.4,0.4,0.2)	**7.0**

In this case, $C_E^U = \left\{\{x_3, x_4, x_5\}_{x_1}, \{x_2, x_6\}_{x_2}, \{x_1\}_{x_3}\right\}$ as $\tilde{f}_E(x_1) = 6.2, \tilde{f}_E(x_2) = 7.0, \tilde{f}_E(x_3) = 8.0, \tilde{f}_E(x_4) = 8.0,$ $\tilde{f}_E(x_5) = 8.0, \tilde{f}_E(x_6) = 7.0$; where $x_1 = 8, x_2 = 7, x_3 = 6$.

3.3 Definition: For a neutrosophic soft set (f, E) with $E = \{e_1, e_2, e_3, ..., e_m\}$, if there exists a subset $A = \{e_1', e_2', e_3', ..., e_p'\} \subset E$ satisfying $\tilde{f}_A(x_1) = \tilde{f}_A(x_2) = \tilde{f}_A(x_3) = ... = \tilde{f}_A(x_n)$, then we say that A is dispensable, otherwise A is indispensable. Roughly speaking, $A \subset E$ is dispensable means that the difference between among all objects according to the parameters in A doesn't influence the final decision. $A \subset E$ is called a parameter reduction of E if A is indispensible and $\tilde{f}_{E-A}(x_1) = \tilde{f}_{E-A}(x_2)$ $= \tilde{f}_{E-A}(x_3) = ... = \tilde{f}_{E-A}(x_n)$ i.e; *E-A* is the maximal subset of E that keeps the value $\tilde{f}_{E-A}(.)$ constant.

Clearly after the parameter reduction of *E*, we have fewer parameters although the partition of objects have not been changed. In the above definition, $\tilde{f}_A(x_1) = \tilde{f}_A(x_2) = \tilde{f}_A(x_3) = ... = \tilde{f}_A(x_n)$ implies $C_E^U = C_{E-A}^U$.

3.4 Example: Using table-1, we have, $\tilde{f}_{\{e_1,e_2,e_4\}}(x_1)=\tilde{f}_{\{e_1,e_2,e_4\}}(x_2)=\tilde{f}_{\{e_1,e_2,e_4\}}(x_3)=\tilde{f}_{\{e_1,e_2,e_4\}}(x_4)=\tilde{f}_{\{e_1,e_2,e_4\}}(x_5)=\tilde{f}_{\{e_1,e_2,e_4\}}(x_6)=3.7$. Hence the neutrosophic soft set (f,E) given by Table-1 has a parameter reduction $\{e_3,e_5,e_6\}$ and the corresponding neutrosophic soft set (f,A) is displayed in table-2 given below:

Table-2

U	e_3	e_5	e_6	$\tilde{f}_A(.)$
x_1	(0.2,0.2,0.4)	(0.1,0.1,0.3)	(0.2,0.4,0.6)	**2.5**
x_2	(0.5,0.5,0.1)	(0.4,0.3,0.2)	(0.6,0.3,0.4)	**3.3**
x_3	(0.8,0.2,0.4)	(0.2,0.4,0.2)	(0.9,0.6,0.6)	**4.3**
x_4	(0.2,0.5,0.6)	(0.7,0.3,0.2)	(0.5,0.5,0.8)	**4.3**
x_5	(0.3,0.3,0.5)	(0.7,0.8,0.1)	(0.4,0.6,0.6)	**4.3**
x_6	(0.4,0.8,0.5)	(0.1,0.2,0.3)	(0.4,0.4,0.2)	**3.3**

Table-1 shows that $\tilde{f}_E(x_1)=6.2, \tilde{f}_E(x_2)=\tilde{f}_E(x_6)=7, \tilde{f}_E(x_3)=\tilde{f}_E(x_4)=\tilde{f}_E(x_5)=8$ and so x_3 or x_4 or x_5 is the optimal choice, x_2 or x_6 is the sub optional choice and x_1 is the inferior choice. Again according to Table-2, $\tilde{f}_A(x_1)=2.5, \tilde{f}_A(x_2)=\tilde{f}_A(x_6)=3.3, \tilde{f}_A(x_3)=\tilde{f}_A(x_4)=\tilde{f}_A(x_5)=4.3$ and so in this case also x_3 or x_4 or x_5 is the optimal choice, x_2 or x_6 is the sub optional choice and x_1 is the inferior choice. Thus parameter reduction gives the same result as the original one.

We also have $C^U_{E-\{e_1,e_2,e_4\}}=\{\{x_3,x_4,x_5\}_4,\{x_2,x_6\}_3,\{x_1\}_2\}$.

For the neutrosophic soft set (f,E), $E=\{e_1,e_2,e_3,.....,e_m\}$ is the parameter set and $U=\{x_1,x_2,x_3,.....,x_n\}$ is the set of objects, $C^U_E=\{\{x_1,x_2,x_3,.....,x_i\}_{x_1},\{x_{i+1},x_{i+2},.....,x_j\}_{x_2},........\{x_k,x_{k+1},.....,x_n\}_{x_s}\}$ is a decision partition of objects in U. Now deleting the parameter e_i from E, we get a new decision partition deleted e_i denoted by $C^U_{E-\{e_i\}}$, which is given by:

. $C^U_{E-\{e_i\}}=\{\{x_{1'},x_{2'},x_{3'},.....,x_{i'}\}_{x_{1'}},\{x_{i+1'},x_{i+2'},.....,x_{j'}\}_{x_{2'}},........,\{x_{k'},x_{k+1'},.....,x_{n'}\}_{x_{s'}}\}$.

For sake of convenience we denote:

$C^U_E=\{E_{x_1},E_{x_2},........E_{x_s}\}$ and $C^U_{E-\{e_i\}}=\{\widetilde{E-\{e_i\}}_{x_{1'}},\widetilde{E-\{e_i\}}_{x_{2'}},........,\widetilde{E-\{e_i\}}_{x_{s'}}\}$ where

$E_{x_1} = \{x_1, x_2, x_3, \ldots\ldots, x_i\}_{x_1}$,

$E_{x_2} = \{x_{i+1}, x_{i+2}, \ldots\ldots, x_j\}_{x_2}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$,

$E_{x_s} = \{x_k, x_{k+1}, \ldots\ldots, x_n\}_{x_s}$,

$\widetilde{E} - \{e_i\}_{x'_1} = \{x'_1, x'_2, x'_3, \ldots\ldots, x'_i\}_{x'_1}$,

$\widetilde{E} - \{e_i\}_{x'_2} = \{x'_{i+1}, x'_{i+2}, \ldots\ldots, x'_j\}_{x'_2}$,

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$,

$\widetilde{E} - \{e_i\}_{x'_{s'}} = \{x'_k, x'_{k+1}, \ldots\ldots, x'_n\}_{x'_{s'}}$.

3.5 Definition: The degree of importance of e_r for the decision partition is denoted by $\mathrm{Im}(e_r)$ and is defined by $\mathrm{Im}(e_r) = \frac{1}{|U|}\sum_{q=1}^{s} \Omega_{q,e_r}$ where

$$\Omega_{q,e_r} = \begin{cases} \left|E_{x_q} - \widetilde{E} - \{e_r\}_{x'_{y'}}\right|, & \text{if } \exists\ y' \text{ such that } x_q = x_{y'}, 1 \le y' \le s', 1 \le q \le s \\ \left|E_{x_q}\right|, & \text{otherwise} \end{cases}$$

3.6 Definition: For $A = \{e'_1, e'_2, e'_3, \ldots\ldots, e'_p\} \subseteq E$, the decision partition deleted A is denoted by C^U_{E-A} and is given by $C^U_{E-A} = \left\{\widetilde{E} - A_{x'_1}, \widetilde{E} - A_{x'_2}, \ldots\ldots\ldots, \widetilde{E} - A_{x'_{s'}}\right\}$.

The degree of importance of A for the decision partition is defined by:

$\mathrm{Im}(A) = \frac{1}{|U|}\sum_{q=1}^{s} \Omega_{q,\mathrm{A}}$ where

$$\Omega_{q,\mathrm{A}} = \begin{cases} \left|E_{x_q} - \widetilde{E} - A_{x'_{y'}}\right|, & \text{if } \exists\ y' \text{ such that } x_q = x_{y'}, 1 \le y' \le s', 1 \le q \le s \\ \left|E_{x_q}\right|, & \text{otherwise} \end{cases}$$

3.7 Example: Consider the neutrosophic soft set given in example 3.2. Then we have:

$C^U_E = \{\{x_3, x_4, x_5\}_8, \{x_2, x_6\}_7, \{x_1\}_6\}$, s=3 and $C^U_{E-\{e_1\}} = \{\{x_3, x_4, x_5\}_6, \{x_2, x_6\}_5, \{x_1\}_4\}$.

$\therefore\ \Omega_{1,e_1} = \left|\{x_1\} - \{x_1\}\right| = 0, \Omega_{2,e_1} = \left|\{x_2, x_6\}\right| = 2, \Omega_{3,e_1} = \left|\{x_3, x_4, x_5\}\right| = 3.$ So $\mathrm{Im}(e_1) = \frac{1}{6}(0 + 2 + 3) = 0.833.$

3.8 Proposition: For the neutrosophic soft set (f, E) where $E = \{e_1, e_2, \ldots\ldots, e_m\}$, $0 \le \mathrm{Im}(e_r) \le 1, r = 1, 2, \ldots, m$.

Proof:

If $\exists\, y'$ such that $x_q = x_{y'}, 1 \le y' \le s, 1 \le q \le s$, then $\Omega_{q,e_r} = \left|E_{x_q} - \widetilde{E - \{e_r\}}_{x_{y'}}\right| \le \left|E_{x_q}\right|$ and $\Omega_{q,e_r} = \left|E_{x_q}\right|$, otherwise.

$$\therefore \operatorname{Im}(e_r) = \frac{1}{|U|}\sum_{q=1}^{s} \Omega_{q,e_r} \le \frac{1}{|U|}\sum_{q=1}^{s}\left|E_{x_q}\right| = \frac{1}{|U|}\left\{\left|E_{x_1}\right| + \left|E_{x_2}\right| + \ldots\ldots + \left|E_{x_s}\right|\right\} = \frac{1}{|U|} \times |U| = 1.$$

Again it is easy to verify that $\operatorname{Im}(e_r) \ge 0$. Thus we have $0 \le \operatorname{Im}(e_r) \le 1$.

4. Decision making problem solving based on parameter reductionof neutrosophic soft set:

In this section we first develop an algorithm using parameter reduction of neutrosophic soft set and then we illustrate this with a real life application.

- **Algorithm:**

Step-1: Input the neutrosophic soft set (f, E).

Step-2: Choose a parameter reduction A of E.

Step-3: Compute the choice value of the object $x_i \in U$ using the formula given below:

$$c_i = \sum_j \operatorname{Im}(e_j) \times \tilde{f}_{e_j}(x_i) \text{ where } e_j \in A.$$

Step-4: Find k for which $c_k = \max_i c_i$.

Then c_k is the optimal choice object. If k has more than one values, then any one of them can be chosen by the decision maker.

- **An Illustrative example:** Consider the neutrosophic soft set given in example 3.2. Now suppose that Mr. John is interested to buy a house on the basis of his choice parameters $e_1, e_2, e_3, \ldots.., e_6$, which means that out of the available houses in *U*, he will select that house that qualifies with all or maximum number of parameters in *E*.

Step-1: The neutrosophic soft set (f, E) is given below:

U	e_1	e_2	e_3	e_4	e_5	e_6	$\tilde{f}_E(.)$
x_1	(0.3,0.7,0.4)	(0.4,0.5,0.1)	(0.2,0.2,0.4)	(0.6,0.3,0.4)	(0.1,0.1,0.3)	(0.2,0.4,0.6)	**6.2**
x_2	(0.4,0.5,0.5)	(0.2,0.2,0.6)	(0.5,0.5,0.1)	(0.2,0.8,0.3)	(0.4,0.3,0.2)	(0.6,0.3,0.4)	**7.0**
x_3	(0.2,0.5,0.7)	(0.3,0.2,0.5)	(0.8,0.2,0.4)	(0.5,0.5,0.3)	(0.2,0.4,0.2)	(0.9,0.6,0.6)	**8.0**
x_4	(0.5,0.3,0.6)	(0.6,0.3,0.1)	(0.2,0.5,0.6)	(0.4,0.4,0.5)	(0.7,0.3,0.2)	(0.5,0.5,0.8)	**8.0**
x_5	(0.3,0.5,0.6)	(0.4,0.4,0.2)	(0.3,0.3,0.5)	(0.6,0.1,0.6)	(0.7,0.8,0.1)	(0.4,0.6,0.6)	**8.0**
x_6	(0.7,0.3,0.4)	(0.3,0.5,0.2)	(0.4,0.8,0.5)	(0.5,0.3,0.5)	(0.1,0.2,0.3)	(0.4,0.4,0.2)	**7.0**

Step-2: A parameter reduction of E is $A = \{e_3, e_5, e_6\}$. The corresponding neutrosophic soft set is given below:

U	e_3	e_5	e_6	$\tilde{f}_A(.)$
x_1	(0.2,0.2,0.4)	(0.1,0.1,0.3)	(0.2,0.4,0.6)	**2.5**
x_2	(0.5,0.5,0.1)	(0.4,0.3,0.2)	(0.6,0.3,0.4)	**3.3**
x_3	(0.8,0.2,0.4)	(0.2,0.4,0.2)	(0.9,0.6,0.6)	**4.3**
x_4	(0.2,0.5,0.6)	(0.7,0.3,0.2)	(0.5,0.5,0.8)	**4.3**
x_5	(0.3,0.3,0.5)	(0.7,0.8,0.1)	(0.4,0.6,0.6)	**4.3**
x_6	(0.4,0.8,0.5)	(0.1,0.2,0.3)	(0.4,0.4,0.2)	**3.3**

Step-3: $C_A^U = \{\{x_3, x_4, x_5\}_4, \{x_2, x_6\}_3, \{x_1\}_2\}$ and $s = 3$.

$C_{A-\{e_3\}}^U = \{\{x_5\}_3, \{x_4\}_3, \{x_3\}_2, \{x_2\}_2, \{x_1\}_1, \{x_6\}_1\}$,

$C_{A-\{e_5\}}^U = \{\{x_3\}_3, \{x_4\}_3, \{x_5, x_6\}_2, \{x_2\}_2, \{x_1\}_2\}$,

$C_{A-\{e_6\}}^U = \{\{x_5\}_2, \{x_4\}_2, \{x_6\}_2, \{x_3\}_2, \{x_2\}_2, \{x_1\}_1\}$.

$$\therefore \Omega_{1,e_3} = \left|\{x_3, x_4, x_5\}\right| = 3, \Omega_{2,e_3} = \left|\{x_2, x_6\}\right| = 2, \Omega_{3,e_3} = \left|\{x_1\}\right| = 1;$$

$$\Omega_{1,e_5} = \left|\{x_3, x_4, x_5\}\right| = 3, \Omega_{2,e_5} = \left|\{x_2, x_6\}\right| = 2, \Omega_{3,e_5} = \left|\{x_1\} - \{x_1\}\right| = 0;$$

$$\Omega_{1,e_6} = \left|\{x_3, x_4, x_5\}\right| = 3, \Omega_{2,e_6} = \left|\{x_2, x_6\}\right| = 2, \Omega_{3,e_6} = \left|\{x_1\}\right| = 1.$$

Hence $\mathrm{Im}(e_3) = \frac{1}{|U|}\sum_{q=1}^{3} \Omega_{q,e_3} = \frac{1}{6}(3+2+1) = 1, \mathrm{Im}(e_5) = \frac{1}{|U|}\sum_{q=1}^{3} \Omega_{q,e_5} = \frac{1}{6}(3+2+0) = 0.83,$

$$\mathrm{Im}(e_6) = \frac{1}{|U|}\sum_{q=1}^{3} \Omega_{q,e_6} = \frac{1}{6}(3+2+1) = 1.$$

The computation table for obtaining the choice values is given by:

U	e_3	e_5	e_6	c_i
x_1	(0.2,0.2,0.4)	(0.1,0.1,0.3)	(0.2,0.4,0.6)	c_1 =(0.2+0.2+0.4)×1+(0.1+0.1+0.3)×0.83 +(0.2+0.4+0.6)×1=**2.415**
x_2	(0.5,0.5,0.1)	(0.4,0.3,0.2)	(0.6,0.3,0.4)	c_2 =(0.5+0.5+0.5)×1+(0.4+0.3+0.2)×0.83 +(0.6+0.3+0.4)×1=**3.147**
x_3	(0.8,0.2,0.4)	(0.2,0.4,0.2)	(0.9,0.6,0.6)	c_3 =(0.8+0.2+0.4)×1+(0.2+0.4+0.2)×0.83 +(0.9+0.6+0.6)×1=**4.164**
x_4	(0.2,0.5,0.6)	(0.7,0.3,0.2)	(0.5,0.5,0.8)	c_4 =(0.2+0.5+0.6)×1+(0.7+0.3+0.2)×0.83 +(0.5+0.5+0.8)×1=**4.096**
x_5	(0.3,0.3,0.5)	(0.7,0.8,0.1)	(0.4,0.6,0.6)	c_5 =(0.3+0.3+0.5)×1+(0.7+0.8+0.1)×0.83 +(0.4+0.6+0.6)×1=**3.828**
x_6	(0.4,0.8,0.5)	(0.1,0.2,0.3)	(0.4,0.4,0.2)	c_6 =(0.4+0.8+0.5)×1+(0.1+0.2+0.3)×0.83 +(0.4+0.4+0.2)×1=**3.198**

Step-4: Since the choice value c_3 is maximum, so house x_3 is the best option for Mr. John.

Conclusion

In this paper we have proposed the concept of parameter reduction for neutrosophic soft sets and we have used it to solve a decision making problem by developing an algorithm based on degree of importance of parameters. The experimental results prove that our proposed parameter reduction techniques delete the irrelevant parameters while keeping definite decision-making choices unchanged. The parameter reduction presented in this paper may play an important role in some knowledge discovery problem. Using the concept presented in this paper, one can think of parameter reduction of interval valued neutrosophic soft sets, hesitant neutrosophic soft sets and hesitant interval valued neutrosophic soft sets.

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Neutrosophic Geometric Programming (NGP) Problems Subject to $(\vee, .)$ Operator; the Minimum Solution

Huda E. Khalid
Telafer University, Head of the Scientific Affairs and Cultural Relations Department, Telafer, Iraq.
E-mail: hodaesmail@yahoo.com

Abstract. This paper comes as a second step serves the purpose of constructing a neutrosophic optimization model for the relation geometric programming problems subject to (max, product) operator in its constraints. This essay comes simultaneously with my previous paper entitled (Neutrosophic Geometric Programming (NGP) with (max-product) Operator, An Innovative Model) which contains the structure of the maximum solution. The purpose of this article is to set up the minimum solution for the (RNGP) problems, the author faced many difficulties, where the feasible region for this type of problems is already non-convex; furthermore, the negative signs of the exponents with neutrosophic variables $x_j \in [0,1] \cup I$. A new technique to avoid the divided by the indeterminacy component (I) was introduced; Separate the neutrosophic geometric programming into two optimization models, introducing two new matrices named as the distinguishing matrix and the facilitation matrix. All these notions were important for finding the minimum solution of the program. Finally, two numerical examples were presented to enable the reader to understand this work.

Keyword: Relational Neutrosophic Geometric Programming (RNGP); $(\vee, .)$ Operator; Neutrosophic Relation Equations; Distinguishing Matrix; Facilitation Matrix; Minimum Solution; Incompatible Problem.

1. Introduction

As of 1995 so far, dozens of mathematicians and researchers in many fields of sciences trying to study and understand the neutrosophic theory, the first mathematician who set up and put forward the neutrosophic theory was Smarandache F. at 1995 [2,11], he is in the neutrosophic theory as Lotfi A. Zadeh [12] in fuzzy theory and as K. Atanasov [10] in intuitionistic fuzzy theory. The importance of the neutrosophic logic comes from its ability to deal with the indeterminacy component (I), this component makes scholars generalize the fuzzy and intuitionistic fuzzy logics, give them the ability to put the paradoxes in a new framework, and it makes the researchers deal with contradicted information in more relaxation. This paper comes as an establishing article in the relational neutrosophic programming problems (RNGP) with $(\vee, .)$ in its constraints. This kind of problems has many applications in real-world problems, like communication system, civil engineering, mechanical engineering, structural design and optimization, business management …etc. The author published previous articles [1,3,4,6,7,9] to expand the fuzzy theory to be fit with neutrosophic theory, this essay was one of the series of these articles.

This publication includes three original sections, despite the second section goes to the basic concepts, but these pure concepts were originated by the author at the simultaneously published paper, which focused on the form of the maximum solution in the (RNGP) with $(\vee,.)$ operator, the third section was dedicated to many unprecedented mathematical formulas such as pre-distinguishing matrices, pre-facilitation matrices, a new technique to separate the optimization model into two models depending upon the sign of terms powers in the objective function, and a technique to filter all minimum solutions, the forth section was for two numerical examples, they are the same examples that presented in the article [8] which assigned to the maximum solution, the last section includes the conclusion.

2. Basic Concepts

We call

$$\left.\begin{array}{l} \min f(x) = \left(c_1.x_1^{\gamma_1}\right) \vee \left(c_2.x_2^{\gamma_2}\right) \vee \ldots \vee \left(c_n.x_n^{\gamma_n}\right) \\ s.t. \qquad Aox = b \\ x_j \in [0,1] \cup I, \quad 1 \le j \le n \end{array}\right\} \qquad (1)$$

A $(\vee,.)$ (max- product) neutrosophic geometric programming, where $A = (a_{ij})$, $1 \le i \le m$, $1 \le j \le n$, $a_{ij} \in [0,1]$ is $(m \times n)$ dimensional neutrosophic matrix, $x = (x_1, x_2, \ldots, x_n)^T$ an n-dimensional variable vector, $b = (b_1, b_2, \ldots, b_m)^T$ $(b_i \in [0,1] \cup I)$ an m- dimensional constant vector, $c = (c_1, c_2, \ldots, c_n)^T$ $(c_j \ge 0)$ an n- dimensional constant vector, γ_j is an arbitrary real number, and the composition operator $''o''$ is $(\vee,.)$, i.e. $\vee_{j=1}^{n}(a_{ij}.x_j) = b_i$. Note that the program (1) is undefined and has no minimal solution in the case of $\gamma_j < 0$ with all $x_j's$ taking indeterminacy value.

2.1. Definition [8]

$$a_{ij} \bowtie b_i = \begin{cases} \frac{b_i}{a_{ij}}, & if \ a_{ij} > b_i, \ a_{ij} \in [0,1], \ b_i \in [0,1] \\ 1, & if \ a_{ij} \le b_i, \ a_{ij} \in [0,1], \ b_i \in [0,1] \\ 1, & if \ a_{ij} \in [0,1], \ b_i = nI, n \in (0,1] \end{cases} \qquad (2)$$

$$a_{ij}\Theta b_i = \begin{cases} \frac{nI}{a_{ij}}, & if \ a_{ij} > n, \ a_{ij} \in [0,1], b_i = nI, n \in (0,1] \\ 1, & if \ a_{ij} \le n, \ a_{ij} \in [0,1], b_i = nI, n \in (0,1] \\ not\ comp. & if \ a_{ij} = mI, m \in (0,1], b_i \in [0,1] \cup I \\ 1 & if \ a_{ij}, b_{ij} \in [0,1] \end{cases} \qquad (3)$$

Where $\bowtie$ is an operator defined at [0,1], while the operator Θ is defined at $[0,1] \cup I$. Let

$$\hat{x}_j = \wedge_{i=1}^{m}(a_{ij} \bowtie b_i), \qquad (1 \le j \le n) \qquad (4)$$

be the components of the pre maximum solution $\hat{x}_{v1}$.(i.e. $\hat{x}_{v1} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$)

Let $\hat{x}_j = \wedge_{i=1}^{m}(a_{ij}\Theta b_i), \qquad (1 \le j \le n)$, (5)

be the components of the pre maximum solution $\hat{x}_{v2}$. (i.e. $\hat{x}_{v2} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$)

Now the following question will be raised,

Which one $\hat{x}_{v1}$ or $\hat{x}_{v2}$ should be the exact maximum solution?

Neither $\hat{x}_{v1}$ nor $\hat{x}_{v2}$ will be the exact solution! The exact solution is integrated between them.

Before solving $Ao\hat{x} = b$, we first define the matrices A_{v1}, A_{v2}.

Let A_{v1} be a matrix has the same dimension and the same rows elements of A except for those rows of the indexes $i = i_o$ corresponding to those indexes of $b_{i_o} = nI$, those special rows of

A_{v1} will be zeros. Let A_{v2} be a matrix has the same dimension and the same rows elements of A except for those rows of the indexes $i = i_o$ corresponding to those indexes of $b_{i_o} \in [0,1]$, those special rows of A_{v2} will be zeros. Consequently,

$$Ao\hat{x} = b = (A_{v1}o\hat{x}_{v1}) + (A_{v2}o\hat{x}_{v2}) \quad (6)$$

The formula (6) is the greatest solution in $X(A, b)$.

The maximum value of the objective function $f(\hat{x}) = f(\hat{x}_{v1}) \vee f(\hat{x}_{v2})$.

2.2. Theorem [8]

If $\gamma_j < 0$ $(1 \leq j \leq n)$, then the greatest solution to the problem (1) is an optimal solution.

2.3. Definition [5]

If there exists a solution to $x = b$, it's called compatible. Suppose $X(A, b) = \{(x_1, x_2, \dots, x_m)^T \in [0,1]^n \cup I,\ I^n = I, n > 0 | xoA = b, x_i \in [0,1] \cup I\}$ is a solution set of $Aox = b$, we define $x^1 \leq x^2 \Leftrightarrow x_j^1 \leq x_j^2$ $(1 \leq j \leq n), \forall\, x^1, x^2 \in X(A, b)$. Where " $\leq$ " is a partial order relation on $X(A, b)$.

3. The Structure of the Minimum Solution $\breve{x}$.

The feasible region of the solution domain for the neutrosophic geometric programming (NGP) problems subject to (max-product) operator in its constraints is a solution to $Aox = b$, therefore the definition of the solution set $X(A, b)$ and the shape of the maximum and the minimum solutions are very important to optimize the (NGP) model.
The structure of the maximum solution was introduced by Huda E. Khalid in [8].
The definition (2.3) was constructed by Huda E. Khalid at 2016 [5], this definition was dedicated for (RNGP) problems subject to (max-min) operator, this definition is also appropriate for (RNGP) problems with (max, product) operator.

3.1. Definition

If there exists a minimum solution in the solution set $X(A, b)$, then the numbers of the minimum solutions are not lonesome such as the maximum solution. If we denote all minimum elements by $\breve{X}(A, b)$, then another version of $X(A, b)$ can be presented depending upon the minimum and the maximum solutions as follows:

$$X(A, b) = \cup_{\breve{x} \in \breve{X}(A,b)} \ \{x \mid \breve{x} \leq x \leq \hat{x}, x \in X\} \quad (7)$$

The following definitions introduce some important new matrices that were constructed by the author for using them in the filtering rule for finding the minimum solution.

3.2. Definition

Let $S_1 = (s_{ij}{}^1)_{m\times n}$, $S_2 = (s_{ij}{}^2)_{m\times n}$ be two pre - distinguishing matrices of A, where

$$s_{ij}{}^1 = \begin{cases} a_{ij}, & a_{ij}.\hat{x}_j = b_i \\ 0, & a_{ij}.\hat{x}_j \neq b_i \end{cases} \quad (8)$$

In (8), the $\hat{x}_j$'s are the components of the pre - maximum solution $\hat{x}_{v1}$ which supports the fuzzy part of the problem, while the elements a_{ij} are the elements of the matrix A_{v1}.

$$s_{ij}^{\ 2} = \begin{cases} a_{ij}, & a_{ij}.\hat{x}_j = b_i \\ 0, & a_{ij}.\hat{x}_j \neq b_i \end{cases} \qquad (9)$$

In (9), the $\hat{x}_j$'s are the components of the pre - maximum solution $\hat{x}_{v2}$ which supports the neutrosophic part of the problem, while a_{ij} are the elements of the matrix A_{v2}.

Let

$$S = (s_{ij})_{m\times n} = (s_{ij}^{\ 1})_{m\times \mathrm{n}} + (s_{ij}^{\ 2})_{m\times \mathrm{n}} = S_1 + S_2 \qquad (10)$$

The matrix S is called the distinguishing matrix of A. It is obvious that the constraints system $Aox = b$ has a solution if and only if the distinguishing matrix S of A has non zero rows (i.e. S has at least a nonzero element in each row).

3.3. Definition

Let $F_1 = (f_{ij}^{\ 1})_{m\times \mathrm{n}}$, $F_2 = (f_{ij}^{\ 2})_{m\times \mathrm{n}}$ be two pre - facilitation matrices of A, where

$$f_{ij}^{\ 1} = \begin{cases} \hat{x}_{ij}, & a_{ij}.\hat{x}_j = b_i \\ 0, & a_{ij}.\hat{x}_j \neq b_i \end{cases} \qquad (11)$$

In (11), the $\hat{x}_j$'s are the components of the pre- maximum solution $\hat{x}_{v1}$ which supports the fuzzy part of the problem, while the elements a_{ij} are the entries of A_{v1}.

$$f_{ij}^{\ 2} = \begin{cases} \hat{x}_{ij}, & a_{ij}.\hat{x}_j = b_i \\ 0, & a_{ij}.\hat{x}_j \neq b_i \end{cases} \qquad (12)$$

In (12), the $\hat{x}_j$'s are the components of the pre - maximum solution $\hat{x}_{v2}$ which supports the neutrosophic part of the problem,

Let

$$F = (f_{ij})_{m\times n} = (f_{ij}^{\ 1})_{m\times \mathrm{n}} + (f_{ij}^{\ 2})_{m\times \mathrm{n}} = F_1 + F_2 \qquad (13)$$

The matrix F is called the Facilitation matrix of A.

Both matrices S *and* F are first introduced in this paper and they have a key role in finding the set of all quasi-minimum solutions and then the optimal solution for NGP problems.

3.4 The Filtration Method for Finding Minimum Solutions

1. Delete the $i-th$ row of F, for which $b_i = 0$
2. At $b_i > 0$, find an index $z \in \{1,2,\dots,m\}$ such that $z > i$, if for all $j = 1,2,\dots,n$ we find $f_{zj} \neq 0 \Leftrightarrow f_{ij} \neq 0$, then delete the $i-th$ row of F.
3. Denote $\tilde{F}$ for the matrix that gained from the above steps (i.e steps 1&2).
4. To each row of $\tilde{F}$, in each time, the only nonzero value is selected in every row with all entries of the rest seen as zero, perhaps all of the matrices are denoted by $\tilde{F}_1, \tilde{F}_2, \dots., \tilde{F}_p$.
5. To each column of $\tilde{F}_k$ $(1 \leq k \leq p)$, the maximum element is selected, a quasi-minimum solution $\check{x}_j$ can be obtained through such a method

The set composed of all $\check{x}_j$ is called a quasi-minimum solution, and it includes all minimum solutions to $Aox = b$. Delete all repeated solutions, and then all minimum solutions $\check{X}(A, b)$ can be obtained.

As an integrated study for all cases of the exponents (γ_j) of the terms in the objective function $f(x)$, we saw that the theorem (2.2) covered the negative exponents, while the following theorem will cover the positive exponents for the terms of $f(x)$.

3.5 Theorem

If $\gamma_j \geq 0$ $(1 \leq j \leq n)$, then a certain minimum solution $\check{x}$ to $Aox = b$ is an optimal one to the program (1).

Proof

Since $\gamma_j \geq 0$ $(1 \leq j \leq n)$, then $\frac{d(x_j^{\gamma_j})}{dx_j} = \gamma_j x_j^{\gamma_j - 1} \geq 0$.

We have $x_j \in [0,1] \cup I$, so $x_j^{\gamma_j}$ is a monotone increasing function concerning x_j, so is $c_j x_j^{\gamma_j}$ concerning x_j. Hence, $\forall\, x \in X(A, b)$, depending on formula (7), then there exists $\check{x} \in \check{X}(A, b)$, such that $x \geq \check{x}$ (i.e. $x_j \geq \check{x}_j$) $\Longrightarrow c_j . x_j^{\gamma_j} \geq c_j . \check{x}_j^{\gamma_j}$ $(1 \leq j \leq n) \Longrightarrow f(x) \geq f(\check{x})$, this means that the optimal solution to the program (1) must exist in

$\check{X}(A,b).f(\check{x}^*) = \min\{\, f(\check{x}) \mid \check{x} \in \check{X}(A,b)\}$. Then $\forall\, x \in X(A,b)$, there exists $f(x) \geq f(\check{x}^*)$, so $\check{x}^* \in \check{X}(A,b)$ is an optimal solution to the program (1).

3.6 Two Optimization Models Based on the Sign of γ_j

Let $M_1 = \{j \mid \gamma_j < 0, 1 < j < n\}$, $M_2 = \{j \mid \gamma_j > 0, 1 < j < n\}$, then $M_1 \cap M_2 = \emptyset$, $M_1 \cup M_2 = J$, here $J = \{1,2,\ldots,n\}$. It is evident that the terms of the objective function $f(x)$ in the program (1) having negative powers is

$$f_1(x) = \vee_{j \in M_1} \{(c_j . x_j^{\gamma_j})\} \tag{14}$$

While the terms of $f(x)$ that having positive exponents is

$$f_2(x) = \vee_{j \in M_2} \{(c_j . x_j^{\gamma_j})\} \tag{15}$$

Based on (14) and (15), we have the following two optimization models,

$$\begin{array}{c} \min f_1(x) \\ s.t. Aox = b \\ x_j \in [0,1] \cup I \end{array} \tag{16}$$

$$\begin{array}{c} \min f_2(x) \\ s.t. Aox = b \\ x_j \in [0,1] \cup I \end{array} \tag{17}$$

Using theorem (2.2), $\hat{x}$ is an optimal solution for (16). By theorem (3.5), there exists $\check{x}^* \in \check{X}(A,b)$, where $\check{x}^*$ is an optimal solution for (17).

3.7 Important Notes

1. In this type of problems, the first step is to search for the maximum solution which is lonesome for every problem. If the purpose of the program (1) is to optimize it, with the restriction that all powers of the variables x_j are negative, then the greatest solution is the optimal one {i.e. $f(x^*) = f(\hat{x}) = f(\hat{x}_{v1}) \wedge f(\hat{x}_{v2})$}.
2. The second step is to search for the minimum solution which is the set of all minimal solutions $\check{X}(A,b)$. When the purpose of the program (1) is to optimize it, with the restriction that some of the exponents are negative and others are positive, then $f(x^*) = f_1(\hat{x}) \wedge f_2(\check{x})$.

3. It should be noticed that the components of $\hat{x}_{v2}$ containing indeterminate values (I) raised to the negative powers of $f(x)$ must be neglected, otherwise, it will be undefined program.

The upcoming section covering numerical examples, those examples are the same that discussed in [8] for its maximal solution, we could not be remote far away from the paper [8], present paper regarded as the complement of [8] which contained the formula of the maximum solution, while this present paper introduces the set of all minimum solutions.

4 Numerical examples

We now gaze the (max, product) neutrosophic relation geometric programming examples as follows

3.1 Example

Solve

$$\min f(x) = (0.3.\, x_1^2) \vee (1.8I\, .\, x_2^{\frac{1}{3}}) \vee (I\, .\, x_3^{\frac{1}{4}})$$

s. t. $Aox = b$

$x_j \in [0,1] \cup I \quad (1 \le j \le n)$

Where $b = (1, \frac{1}{3}I, \frac{1}{5}I)^T$, $A = \begin{pmatrix} .6 & 1 & .2 \\ .5 & .2 & .1 \\ .3 & .5 & .1 \end{pmatrix}_{3\times 3}$.

Solution:

$\hat{x}_{v1} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)^T = (1,1,1)^T$, $\hat{x}_{v2} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)^T = \left(\frac{2}{3}I, \frac{2}{5}I, 1\right)^T$,

$$A_{v1} = \begin{pmatrix} .6 & 1 & .2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_{v2} = \begin{pmatrix} 0 & 0 & 0 \\ .5 & .2 & .1 \\ .3 & .5 & .1 \end{pmatrix},$$

It is easy to notice that all exponents of $f(x)$ terms are positive. Therefore there will not be a need to separate $f(x)$ into f_1 and f_2.

$f(\hat{x}) = f(\hat{x}_{v1}) \vee f(\hat{x}_{v2}) = 1.8I$ is the maximum solution.

Using theorem (3.5), it is essential to find the set of all minimum solutions for $f(x)$, where the optimal solution occurs at the minimal solution.

$$S_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; S_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0.3 & 0.5 & 0 \end{bmatrix}, S = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0 \\ 0.3 & 0.5 & 0 \end{bmatrix}.$$

$$F_1=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\ F_2=\begin{bmatrix}0&1&0\\\frac{2}{3}I&0&0\\\frac{2}{3}I&\frac{2}{5}I&0\end{bmatrix},\ F=\begin{bmatrix}0&1&0\\\frac{2}{3}I&0&0\\\frac{2}{3}I&\frac{2}{5}I&0\end{bmatrix}.$$

Using the filtration rule stated in section (3.4),

$$\tilde{F}=\begin{bmatrix}\frac{2}{3}I&0&0\\\frac{2}{3}I&\frac{2}{5}I&0\end{bmatrix}\Rightarrow\tilde{F}_1=\begin{bmatrix}\frac{2}{3}I&0&0\\\frac{2}{3}I&0&0\end{bmatrix},\ \tilde{F}_2=\begin{bmatrix}\frac{2}{3}I&0&0\\0&\frac{2}{5}I&0\end{bmatrix},$$

so the minimum solutions that related to $\tilde{F}_1$ and $\tilde{F}_2$ are $\check{x}_1=[\frac{2}{3}I,0,0]$, $\check{x}_2=[\frac{2}{3}I,\frac{2}{5}I,0]$.

$f(\check{x}_1)=f(\check{x}_2)=\frac{2}{15}I$ is the minimum solution.

3.2 Example

Let $\min f(x)=\left(0.2I.x_1^{-\frac{2}{3}}\right)\vee\left(1.3.x_2^{\frac{1}{3}}\right)\vee(I.x_3^{\frac{1}{2}})\vee(0.35.x_4^{-2})$

s. t. $Aox=b$

$x_j\in[0,1]\cup I\quad(1\le j\le n)$

Where $b=(0.3,0.7I,0.5,0.2I)^T$, $A=\begin{pmatrix}.2&.3&.4&.6\\.3&.2&.9&.8\\1&0&.1&1\\0&.5&1&0\end{pmatrix}_{4\times4}$.

Solution

$\hat{x}_{v1}=(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4)^T=\left(0.5,1,\frac{3}{4},0.5\right)^T$, $\hat{x}_{v2}=(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4)^T=\left(\frac{2}{5}I,1,0.2I,0.875I\right)^T$,

The greatest solution for this problem is $f(\hat{x})=f(\hat{x}_{v1})\vee f(\hat{x}_{v2})=1.3$.

The following calculations are for finding the minimum solution.

$$A_{v1}=\begin{pmatrix}.2&.3&.4&.6\\0&0&0&0\\1&0&.1&1\\0&0&0&0\end{pmatrix},\ A_{v2}=\begin{pmatrix}0&0&0&0\\.3&.2&.9&.8\\0&0&0&0\\0&.5&1&0\end{pmatrix}.$$

$$S_1=\begin{pmatrix}0&.3&.4&.6\\0&0&0&0\\1&0&0&1\\0&0&0&0\end{pmatrix},\ S_2=\begin{pmatrix}0&0&0&0\\0&0&0&.8\\0&0&0&0\\0&0&1&0\end{pmatrix},\ \Rightarrow S=\begin{pmatrix}0&.3&.4&.6\\0&0&0&.8\\1&0&0&1\\0&0&1&0\end{pmatrix}.$$

$$F_1=\begin{bmatrix}0&1&\frac{3}{4}&.5\\0&0&0&0\\.5&0&0&.5\\0&0&0&0\end{bmatrix},\ F_2=\begin{bmatrix}0&0&0&0\\0&0&0&.875I\\0&0&0&0\\0&1&.2I&0\end{bmatrix},\Rightarrow F=\begin{bmatrix}0&1&\frac{3}{4}&.5\\0&0&0&.875I\\.5&0&0&.5\\0&1&.2I&0\end{bmatrix}.$$

$$\tilde{F}=\begin{bmatrix}0&0&0&.875I\\.5&0&0&.5\\0&1&.2I&0\end{bmatrix},$$

$$\tilde{F}_1=\begin{bmatrix}0&0&0&.875I\\.5&0&0&0\\0&1&.2I&0\end{bmatrix}\Rightarrow \check{x}_1=(.5,1,.2I,.875I)^T,$$

$$\tilde{F}_2=\begin{bmatrix}0&0&0&.875I\\0&0&0&.5\\0&1&.2I&0\end{bmatrix}\Rightarrow \check{x}_2=(0,1,.2I,.875I)^T,$$

$$\tilde{F}_3=\begin{bmatrix}0&0&0&.875I\\.5&0&0&.5\\0&1&0&0\end{bmatrix}\Rightarrow \check{x}_3=(.5,1,0,.875I)^T,$$

$$\tilde{F}_4=\begin{bmatrix}0&0&0&.875I\\.5&0&0&.5\\0&0&.2I&0\end{bmatrix}\Rightarrow \check{x}_4=(.5,0,.2I,.875I)^T,$$

$$\tilde{F}_5=\begin{bmatrix}0&0&0&.875I\\.5&0&0&0\\0&1&0&0\end{bmatrix}\Rightarrow \check{x}_5=(.5,1,0,.875I)^T,$$

$$\tilde{F}_6=\begin{bmatrix}0&0&0&.875I\\0&0&0&.5\\0&1&0&0\end{bmatrix}\Rightarrow \check{x}_6=(0,1,0,.875I)^T,$$

$$\tilde{F}_7=\begin{bmatrix}0&0&0&.875I\\.5&0&0&0\\0&0&.2I&0\end{bmatrix}\Rightarrow \check{x}_7=(.5,0,.2I,.875I)^T,$$

$$\tilde{F}_8=\begin{bmatrix}0&0&0&.875I\\0&0&0&.5\\0&0&.2I&0\end{bmatrix}\Rightarrow \check{x}_8=(0,0,.2I,.875I)^T.$$

It is clear that there are two repeated solution,

$\check{x}_5=(.5,1,0,.875I)^T=\check{x}_3$, and $\check{x}_7=(.5,0,.2I,.875I)^T=\check{x}_4$, after deleting all repeated solutions, the set of all quasi- minimum solutions $\check{X}(A,b)=\{\check{x}_1,\check{x}_2,\check{x}_3,\check{x}_4,\check{x}_6,\check{x}_8\}$.

Since the powers of some terms in $f(x)$ are positive while others are negative, we separate the objective function $f(x)$ into

$f_1(x)=(0.2I.x_1^{-\frac{2}{3}})\vee(0.35\,.x_4^{-2})$, $f_2(x)=(1.3.x_2^{\frac{1}{3}})\vee(I\,.x_3^{\frac{1}{2}})$,

First, solve for optimizing

$$\begin{aligned}&\min f_1(x)\\&s.t.\,Aox=b\\&x_j\in[0,1]\cup I\end{aligned}$$

By theorem (2.2), we have $\mathrm{f}_1(x^*)=\mathrm{f}_1(\hat{x})=f_1(\hat{x}_{v1})\wedge f_1(\hat{x}_{v2})=1.4$, take care of those terms of $\hat{x}_{v2}$ that holding indeterminate components must be neglected and avoid apply them in the terms of $\mathrm{f}_1(\mathrm{x})$.

Second, solve for optimizing

$\min f_2(x)$

$s.t. Aox = b$

$x_j \in [0,1] \cup I$

$f_2(\check{x}_1) = 1.3, f_2(\check{x}_2) = 1.3, f_2(\check{x}_3) = 1.3, f_2(\check{x}_4) = .447I, f_2(\check{x}_6) = 1.3,$

$f_2(\check{x}_8) = 0.447I,$

$\check{x}_4, \check{x}_8$ are the optimal for $f_2(x)$, (i.e. $f_2(x^*) = 0.447I$).

$\therefore \boldsymbol{f(x^*) = f_1(x^*) \wedge f_2(x^*) = 0.447I}$

5 Conclusion

The importance of this work comes from the unprecedented notions that were firstly introduced in this article which are essential mathematical tools to establish the structure of neutrosophic geometric programming (NGP) problems with $(\vee, .)$ operator. Any optimization problem needs to specify its minimum and maximum solution, in this article the author introduced an effective technique to find the set of all quasi- minimum solution $\check{X}(A, b)$, side by side with the structure of the maximum solution $\hat{x}$. This work contains the theoretical rules with two numerical examples to enable the readers to understand the pure mathematical concepts.

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Ngpr Homeomorphism in Neutrosophic Topological Spaces

K. Ramesh
Department of Mathematics, Government Arts College, Udumalpet - 642126, Tamilnadu, India.E-mail: ramesh251989@gmail.com

* Correspondence: ramesh251989@gmail.com

Abstract: As a generalization of Fuzzy sets introduced by Zadeh [21] in 1965 and Intuitionistic Fuzzy sets introduced by Atanassav [8] in 1983, the Neutrosophic set had been introduced and developed by Smarandache. A Neutrosophic set is characterized by a truth value (membership), an indeterminacy value and a falsity value (non-membership). Salama and Alblowi [17] introduced the new concept of neutrosophic topological space (NTS) in 2012, which had been investigated recently. In 2018, Parimala M et al. introduced and studied the concept of Neutrosophic homeomorphism and Neutrosophic $\alpha\psi$ homeomorphism in Neutrosophic topological spaces. The impact of this article is to introduce and study the concepts of Ngpr homeomorphism and Nigpr homeomorphism in Neutrosophic topological space. Further, the work is extended to Ngpr open mappings, Ngpr closed mappings, Nigpr closed mappings and some of their properties are explored in Neutrosophic topological space.

Keywords: Neutrosophic generalized pre regular closed set, Ngpr open mappings, Ngpr closed mappings, Ngpr homeomorphism and Nigpr homeomorphism.

1. Introduction

Zadeh [21] introduced the concept of fuzzy set in 1965 and Chang C. L. [9] introduced fuzzy topological spaces in 1968. Later, Atanassov [8] proposed the concept of intuitionistic fuzzy sets in 1986, where the degree of membership and degree of non-membership are discussed. Intuitionistic fuzzy topological spaces was introduced by Coker [10] in 1997 using intuitionistic fuzzy sets. As a generalization of Fuzzy sets and Intuitionistic Fuzzy sets, Neutrosophic set have been introduced and developed by Florentin Smarandache [12]. He also defined the Neutrosophic set on three components, namely Truth (membership) (T), Indeterminacy (I) and Falsehood (non-membership) (F).

Neutrosophic concept has wide range of real time applications in the fields of [1 - 6] Information Systems, Computer Science, Artificial Intelligence, Applied Mathematics and Decision Making, Uncertainty assessments of linear time-cost tradeoffs and solving the supply chain problem.

In 2012, Salama A. A and Alblowi [17] introduced the concept of Neutrosophic topological space by using Neutrosophic sets. Salama A. A. [18] introduced Neutrosophic closed set and Neutrosophic continuous function in Neutrosophic topological spaces and their properties are studied by various authors [7 & 11]. Since, Neutrosophic homeomorphism plays an important role in Neutrosophic topology. Parimala M et al. [14] introduced and studied the concept of Neutrosophic homeomorphism and Neutrosophic $\alpha\psi$ homeomorphism in Neutrosophic topological spaces. In this article, introduce and study few properties of Ngpr open mappings, Ngpr closed mappings, Nigpr closed mappings, Ngpr homeomorphism and Nigpr homeomorphism in Neutrosophic topological space. The present study demonstrates some of the related theorems, results and properties.

2. Preliminaries

2.1. Definition: [17] Let X be a non-empty fixed set. A Neutrosophic set (NS for short) A in X is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ where the functions $\mu_A(x)$, $\sigma_A(x)$ and $\nu_A(x)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element $x \in X$ to the set A.

2.2 Remark: [17] A Neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ can be identified to an ordered triple $A = \langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle$ in non-standard unit interval $]^-0, 1^+[$ on X.

2.3 Remark: [17] For the sake of simplicity, we shall use the symbol $A = \langle x, \mu_A, \sigma_A, \nu_A\rangle$ for the neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$.

2.4 Example: [17] Every IFS A is a non-empty set in X is obviously on NS having the form $A = \{\langle x, \mu_A(x), 1 - (\mu_A(x) + \nu_A(x)), \nu_A(x)\rangle : x \in X\}$. Since our main purpose is to construct the tools for developing Neutrosophic set and Neutrosophic topology, we must introduce the NS 0_N and 1_N in X as follows:

0_N may be defined as:

(0_1) $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$

(0_2) $0_N = \{\langle x, 0, 1, 1\rangle : x \in X\}$

(0_3) $0_N = \{\langle x, 0, 1, 0\rangle : x \in X\}$

(0_4) $0_N = \{\langle x, 0, 0, 0\rangle : x \in X\}$

1_N may be defined as:

(1_1) $1_N = \{\langle x, 1, 0, 0\rangle : x \in X\}$

(1_2) $1_N = \{\langle x, 1, 0, 1\rangle : x \in X\}$

(1_3) $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$

(1_4) $1_N = \{\langle x, 1, 1, 1\rangle : x \in X\}$

2.5 Definition: [17] Let $A = \langle \mu_A, \sigma_A, \nu_A\rangle$ be a NS on X, then the complement of the set A [C(A) for short] may be defined as three kind of complements:

(C_1) $C(A) = \{\langle x, 1-\mu_A(x), 1-\sigma_A(x), 1-\nu_A(x)\rangle : x \in X\}$

(C_2) $C(A) = \{\langle x, \nu_A(x), \sigma_A(x), \mu_A(x)\rangle : x \in X\}$

(C_3) $C(A) = \{\langle x, \nu_A(x), 1-\sigma_A(x), \mu_A(x)\rangle : x \in X\}$

2.6 Definition: [17] Let X be a non-empty set and Neutrosophic sets A and B in the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$ and $B = \{\langle x, \mu_B(x), \sigma_B(x), \nu_B(x)\rangle : x \in X\}$. Then we may consider two possible definitions for subsets $(A \subseteq B)$.

(1) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \sigma_A(x) \le \sigma_B(x)$ and $\mu_A(x) \ge \mu_B(x)$ $\forall$ $x \in X$

(2) $A \subseteq B \Leftrightarrow \mu_A(x) \le \mu_B(x), \sigma_A(x) \ge \sigma_B(x)$ and $\mu_A(x) \ge \mu_B(x)$ $\forall$ $x \in X$

2.7 Proposition: [17] For any Neutrosophic set A, the following conditions hold:

$0_N \subseteq A$, $0_N \subseteq 0_N$

$A \subseteq 1_N$, $1_N \subseteq 1_N$

2.8 Definition: [17] Let X be a non-empty set and $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle : x \in X\}$, $B = \{\langle x, \mu_B(x), \sigma_B(x), \nu_B(x)\rangle : x \in X\}$ are NSs. Then $A \cap B$ may be defined as:

(I_1) $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)\rangle$

(I_2) $A \cap B = \langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \vee \nu_B(x)\rangle$

$A \cup B$ may be defined as:

(U_1) $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)\rangle$

(U_2) $A \cup B = \langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x)$ and $\nu_A(x) \wedge \nu_B(x)\rangle$

2.9 Definition: [17] A Neutrosophic topology [NT for short] is a non-empty set X is a family τ of Neutrosophic subsets in X satisfying the following axioms:

(NT_1) 0_N, $1_N \in \tau$,

(NT_2) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$,

(NT_3) $\cup G_i \in \tau$ for every $\{G_i : i \in J\} \subseteq \tau$.

Throughout this paper, the pair (X, τ) is called a Neutrosophic topological space (NTS for short). The elements of τ are called Neutrosophic open sets [NOS for short]. A complement C(A) of a NOS A in NTS (X, τ) is called a Neutrosophic closed set [NCS for short] in X.

2.10 Definition: [17] Let (X, τ) be NTS and A = {⟨x, $\mu_A(x)$, $\sigma_A(x)$, $\nu_A(x)$⟩: x ∈ X} be a NS in X. Then the Neutrosophic closure and Neutrosophic interior of A are defined by

NCl(A) =∩{K : K is a NCS in X and A ⊆ K}

NInt(A) =∪{G : G is a NOS in X and G ⊆ A}

It can be also shown that NCl(A) is NCS and NInt(A) is a NOS in X.

a) A is NOS if and only if A = NInt(A),
b) A is NCS if and only if A = NCl(A).

2.11 Definition: [13] A NS A = {⟨x, $\mu_A(x)$, $\sigma_A(x)$, $\nu_A(x)$⟩: x ∈ X} in a NTS (X, τ) is said to be

(i) Neutrosophic regular closed set (NRCS for short) if A = NCl(NInt(A)),
(ii) Neutrosophic regular open set (NROS for short) if A = NInt(NCl(A)),
(iii) Neutrosophic pre closed set (NPCS for short) if NCl(NInt(A)) ⊆ A,
(iv) Neutrosophic pre open set (NPOS for short) if A ⊆ NInt(NCl(A)),
(v) Neutrosophic α- closed set (NSCS for short) if NCl(NInt(NCl(A))) ⊆ A,
(vi) Neutrosophic α- open set (NSOS for short) if A ⊆ NInt(NCl(NInt(A))).

2.12 Definition: [19] Let (X, τ) be NTS and A = {⟨x, $\mu_A(x)$, $\sigma_A(x)$, $\nu_A(x)$⟩: x ∈ X} be a NS in X. Then the Neutrosophic pre closure and Neutrosophic pre interior of A are defined by

NPCl(A) = ∩{K : K is a NPCS in X and A ⊆ K},

NPInt(A) = ∪{G : G is a NPOS in X and G ⊆ A}.

2.13 Definition: [15] A NS A = {⟨x, $\mu_A(x)$, $\sigma_A(x)$, $\nu_A(x)$⟩: x ∈ X} in a NTS (X, τ) is said to be a Neutrosophic generalized closed set (NGCS for short) if NCl(A) ⊆ U whenever A ⊆ U and U is a NOS in (X, τ). A NS A of a NTS (X, τ) is called a Neutrosophic generalized open set (NGOS for short) if C(A) is a NGCS in (X, τ).

2.14 Definition: [20] A NS A = {⟨x, $\mu_A(x)$, $\sigma_A(x)$, $\nu_A(x)$⟩: x ∈ X} in a NTS (X, τ) is said to be a Neutrosophic generalized pre closed set (NGPCS for short) if NPCl(A) ⊆ U whenever A ⊆ U and U is a NOS in (X, τ). A NS A of a NTS (X, τ) is called a Neutrosophic generalized pre open set (NGPOS for short) if C(A) is a NGPCS in (X, τ).

2.15 Definition: [13] A NS A = {⟨x, $\mu_A(x)$, $\sigma_A(x)$, $\nu_A(x)$⟩: x ∈ X} in a NTS (X, τ) is said to be a Neutrosophic generalized pre regular closed set (NGPRCS for short) if NPCl(A) ⊆ U whenever A ⊆ U and U is a NROS in (X, τ). The family of all NGPRCSs of a NTS(X, τ) is denoted by NGPRC(X). A NS A of a NTS (X, τ) is called a Neutrosophic generalized pre regular open set (NGPROS for short) if C(A) is a NGPRCS in (X, τ).

Every NRCS, NCS, NWCS, NαCS, NGCS, NPCS, NαGCS, NGPCS, NRαGCS, NRGCS is an NGPRCS but the converses are not true in general.

2.16 Definition: [13] A Neutrosophic topological space (X, τ) is called a Neutrosophic pre regular $T_{1/2}$ ($NPRT_{1/2}$ for short) space if every NGPRCS in (X, τ) is NPCS in (X, τ).

2.17 Definition: [13] A Neutrosophic topological space (X, τ) is called a Neutrosophic pre regular $T^*_{1/2}$ ($NPRT^*_{1/2}$ for short) space if every NGPRCS in (X, τ) is NCS in (X, τ).

2.18 Definition: [16] Let (X, τ) and (Y, σ) be two NTSs. A mapping f: (X, τ) → (Y, σ) is called Ngpr continuous (resp. NG continuous, NGP continuous) mapping if $f^{-1}(B)$ is NGPRCS (resp. NGCS, NGPCS) in (X, τ) for every NCS B of (Y, σ).

Every Neutrosophic continuous, NG continuous, NGP continuous is a Ngpr continuous mapping but the converses are not true in general.

2.19 Definition: [16] Let (X, τ) and (Y, σ) be two NTSs. A mapping f: (X, τ) → (Y, σ) is called Ngpr irresolute mapping if $f^{-1}(A)$ is NGPRCS in (X, τ) for every NGPRCS A of (Y, σ).

2.20 Definition: [14] Let (X, τ) and (Y, σ) be two NTSs. A mapping f: (X, τ) → (Y, σ) is called Neutrosophic closed mapping (resp. Neutrosophic open mapping) (NCM (resp. NOM) for short) if the image of every Neutrosophic closed set (resp. Neutrosophic open set) in (X, τ) is a Neutrosophic closed set (resp. Neutrosophic open set) in (Y, σ).

2.21 Definition: [14] Let (X, τ) and (Y, σ) be two NTSs. A bijection f: (X, τ) → (Y, σ) is called a Neutrosophic homeomorphism if f and f^{-1} are Neutrosophic continuous mapping.

3. Ngpr open mappings and Ngpr closed mappings

In this section introduce Ngpr open mapping, Ngpr closed mapping and Nigpr closed mapping in the Neutrosophic topological space and study some of their properties. Also established the relation between the newly introduced mappings and already existing mappings.

3.1 Definition**:** Let (X, τ) and (Y, σ) be two NTSs. A mapping f: (X, τ) → (Y, σ) is called

(i) Neutrosophic generalized open mapping (NGOM for short) if f(A) is NGOS in (Y, σ) for every NOS A of (X, τ).

(ii) Neutrosophic α open mapping (NαOM for short) if f(A) is NαOS in (Y, σ) for every NOS A of (X, τ).

(iii) Neutrosophic pre-open mapping (NPOM for short) if f(A) is NPOS in (Y, σ) for every NOS A of (X, τ).

(iv) Neutrosophic generalized pre-open mapping (NGPOM for short) if f(A) is NGPOS in (Y, σ) for every NOS A of (X, τ).

3.2 Definition**:** Let (X, τ) and (Y, σ) be two NTSs. A mapping f: (X, τ) → (Y, σ) is called Ngpr open mapping (NGPROM for short) if f(A) is NGPROS in (Y, σ) for every NOS A of (X, τ).

3.3 Definition**:** Let (X, τ) and (Y, σ) be two NTSs. A mapping f: (X, τ) → (Y, σ) is called

(i) Neutrosophic generalized closed mapping (NGCM for short) if f(A) is NGCS in (Y, σ) for every NCS A of (X, τ).

(ii) Neutrosophic α closed mapping (NαCM for short) if f(A) is NαCS in (Y, σ) for every NCS A of (X, τ).

(iii) Neutrosophic pre-closed mapping (NPCM for short) if f(A) is NPCS in (Y, σ) for every NCS A of (X, τ).

(iv) Neutrosophic generalized pre-closed mapping (NGPCM for short) if f(A) is NGPCS in (Y, σ) for every NCS A of (X, τ).

3.4 Definition**:** Let (X, τ) and (Y, σ) be two NTSs. A mapping f: (X, τ) → (Y, σ) is called Ngpr closed mapping (NGPRCM for short) if f(A) is NGPRCS in (Y, σ) for every NCS A of (X, τ).

***3.5 Example*:** Let X= {a, b} and Y = {u, v}. Then τ = {0_N, U, 1_N} and σ = {0_N, V_1, V_2, 1_N} are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.4, 0.4, 0.5), (0.6, 0.3, 0.4)⟩ and V_1 = ⟨y, (0.7, 0.5, 0.3), (0.8, 0.4, 0.2)⟩ and V_2 = ⟨y, (0.6, 0.4, 0.4), (0.7, 0.3, 0.3)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here the Neutrosophic set U^c = ⟨x, (0.5, 0.6, 0.4), (0.4, 0.7, 0.6)⟩ is a Neutrosophic closed set in X. Then f(U^c) = ⟨y, (0.5, 0.6, 0.4), (0.4, 0.7, 0.6)⟩ is a NGPRCS in (Y, σ) as f(U^c) ⊆1_N implies Npcl(f(U^c)) = f(U^c) ⊆ 1_N where 1N is a NROS in Y. Therefore f is a Ngpr closed mapping.

***3.6 Proposition*:** Every Neutrosophic closed mapping is Ngpr closed mapping but not conversely in general.

***Proof*:** Let f: (X, τ) → (Y, σ) be a Neutrosophic closed mapping. Let A be a NCS in X. Then f(A) is a NCS in Y. Since every NCS is a NGPRCS in Y, f(A) is a NGPRCS in Y. Hence f is a Ngpr closed mapping.

***3.7 Example*:** Let X= {a, b} and Y = {u, v}. Then τ = {0_N, U, 1_N} and σ = {0_N, V_1, V_2, 1_N} are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.4, 0.4, 0.5), (0.6, 0.3, 0.4)⟩ and V_1 = ⟨y, (0.7, 0.5, 0.3), (0.8, 0.4, 0.2)⟩ and V_2 = ⟨y, (0.6, 0.4, 0.4), (0.7, 0.3, 0.3)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here the Neutrosophic set U^c = ⟨x, (0.5, 0.6, 0.4), (0.4, 0.7, 0.6)⟩ is a NCS in X. Then f(U^c) = ⟨y, (0.5, 0.6, 0.4), (0.4, 0.7, 0.6)⟩ is a NGPRCS in (Y, σ) as f(U^c) ⊆1_N implies Npcl(f(U^c)) = f(U^c) ⊆ 1_N where 1_N is a NROS in Y. Therefore f is a Ngpr closed mapping. But f is not a Neutrosophic closed mapping since U^c is NCS in X but f(U^c) is not a NCS in Y as Ncl(f (U^c)) = 1_N ≠ f(U^c).

***3.8 Proposition*:** Every Neutrosophic generalized closed mapping is Ngpr closed mapping but not conversely in general.

***Proof*:** Let f: (X, τ) → (Y, σ) be a Neutrosophic generalized closed mapping. Let A be a NCS in X. Then f(A) is a NGCS in Y. Since every NGCS is a NGPRCS in Y, f(A) is a NGPRCS in Y. Hence f is a Ngpr closed mapping.

***3.9 Example*:** Let X= {a, b} and Y = {u, v}. Then τ = {0_N, U, 1_N} and σ = {0_N, V, 1_N} are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.3, 0.5, 0.4), (0.2, 0.5, 0.3) ⟩ and V = ⟨y, (0.6, 0.5, 0.2), (0.4, 0.5, 0.2)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here the Neutrosophic set U^c = ⟨x, (0.4, 0.5, 0.3), (0.3, 0.5, 0.2) ⟩ is a NCS in X. Then f(U^c) = ⟨y, (0.4, 0.5, 0.3), (0.3, 0.5, 0.2)⟩ is a NGPRCS in (Y, σ) as f(U^c) ⊆1_N implies Npcl(f(U^c)) = f(U^c) ⊆ 1_N where 1_N is a NROS in Y. Therefore f is a Ngpr closed mapping. But f is not a Neutrosophic generalized closed mapping since U^c is NCS in X but f(U^c) is not a NGCS in Y as f(U^c) ⊆ V implies Ncl(f(U^c)) = 1_N ⊈ V.

***3.10 Proposition*:** Every Neutrosophic α closed mapping is Ngpr closed mapping but not conversely in general.

***Proof*:** Let f: (X, τ) → (Y, σ) be a Neutrosophic α closed mapping. Let A be a NCS in X. Then f(A) is a NαCS in Y. Since every NαCS is a NGPRCS in Y, f(A) is a NGPRCS in Y. Hence f is a Ngpr closed mapping.

***3.11 Example*:** Let X= {a, b} and Y = {u, v}. Then τ = {0_N, U, 1_N} and σ = {0_N, V, 1_N} are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.4, 0.5, 0.4), (0.2, 0.5, 0.3)⟩ and V = ⟨y, (0.7, 0.5, 0.2), (0.3, 0.5, 0.2)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here the Neutrosophic set U^c = ⟨x, (0.4, 0.5, 0.4), (0.3, 0.5, 0.2)⟩ is a NCS in X. Then f(U^c) = ⟨y, (0.4, 0.5, 0.4), (0.3, 0.5, 0.2)⟩ is a NGPRCS in (Y, σ) as f(U^c) ⊆1_N implies Npcl(f(U^c)) = f(U^c) ⊆ 1_N where 1_N is a NROS in Y. Therefore f is a Ngpr closed mapping. But f is not a Neutrosophic α closed mapping since U^c is NCS in X but f(U^c) is not a NαCS in Y as Ncl(Nint(Ncl(f(U^c)))) = 1_N ⊈ f(U^c).

3.12 Proposition: Every Neutrosophic pre-closed mapping is Ngpr closed mapping but not conversely in general.

Proof: Let f: (X, τ) → (Y, σ) be a Neutrosophic pre-closed mapping. Let A be a NCS in X. Then f(A) is a NPCS in Y. Since every NPCS is a NGPRCS in Y, f(A) is a NGPRCS in Y. Hence f is a Ngpr closed mapping.

3.13 Example: Let X= {a, b} and Y = {u, v}. Then $\tau = \{0_N, U, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.4, 0.5, 0.6), (0.2, 0.5, 0.3)⟩ and V = ⟨y, (0.3, 0.5, 0.7), (0.3, 0.5, 0.4)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here the Neutrosophic set U^c = ⟨x, (0.6, 0.5, 0.4), (0.3, 0.5, 0.2)⟩ is a NCS in X. Then $f(U^c)$ = ⟨y, (0.6, 0.5, 0.4), (0.3, 0.5, 0.2)⟩ is a NGPRCS in (Y, σ) as $f(U^c) \subseteq 1_N$ implies $Npcl(f(U^c))$ = ⟨y, (0.7, 0.5, 0.3), (0.4, 0.5, 0.2)⟩ $\subseteq 1_N$ where 1_N is a NROS in Y. Therefore f is a Ngpr closed mapping. But f is not a Neutrosophic pre-closed mapping since U^c is NCS in X but $f(U^c)$ is not a NPCS in Y as $Ncl(Nint(f(U^c))) = V^c \nsubseteq f(U^c)$.

3.14 Proposition: Every Neutrosophic generalized pre-closed mapping is Ngpr closed mapping but not conversely in general.

Proof: Let f: (X, τ) → (Y, σ) be a Neutrosophic generalized pre-closed mapping. Let A be a NCS in X. Then f(A) is a NGPCS in Y. Since every NGPCS is a NGPRCS in Y, f(A) is a NGPRCS in Y. Hence f is a Ngpr closed mapping.

3.15 Example: Let X= {a, b} and Y = {u, v}. Then $\tau = \{0_N, U, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.3, 0.8, 0.5), (0.4, 0.7, 0.6)⟩ and V = ⟨y, (0.5, 0.2, 0.3), (0.6, 0.3, 0.4)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here the Neutrosophic set U^c = ⟨x, (0.5, 0.2, 0.3), (0.6, 0.3, 0.4)⟩ is a NCS in X. Then $f(U^c)$ = ⟨y, (0.5, 0.2, 0.3), (0.6, 0.3, 0.4)⟩ is a NGPRCS in (Y, σ) as $f(U^c) \subseteq 1_N$ implies $Npcl(f(U^c)) = 1_N \subseteq 1_N$ where 1_N is a NROS in Y. Therefore f is a Ngpr closed mapping. But f is not a Neutrosophic generalized pre-closed mapping since U^c is NCS in X but f (U^c) is not a NGPCS in Y as $f(U^c) \subseteq V$ implies $Npcl(f(U^c)) = 1_N \nsubseteq V$ where V is a NOS in Y.

3.16 Definition: Let (X, τ) and (Y, σ) be two NTSs. A mapping f: (X, τ) → (Y, σ) is called Nigpr open mapping (NiGPROM for short) if f(A) is NGPROS in (Y, σ) for every NGPROS A of (X, τ).

3.17 Definition: Let (X, τ) and (Y, σ) be two NTSs. A mapping f: (X, τ) → (Y, σ) is called Nigpr closed mapping (NiGPRCM for short) if f(A) is NGPRCS in (Y, σ) for every NGPRCS A of (X, τ).

3.18 Example: Let X= {a, b} and Y = {u, v}. Then $\tau = \{0_N, U, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.5, 0.4, 0.3), (0.7, 0.8, 0.2)⟩ and V = ⟨y, (0.7, 0.4, 0.5), (0.8, 0.5, 0.5)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Hence f(A) is NGPRCS in (Y, σ) for every NGPRCS A of (X, τ). Therefore f is a Nigpr closed mapping.

3.19 Proposition: Every Nigpr closed mapping is Ngpr closed mapping but not conversely in general.

Proof: Let f: (X, τ) → (Y, σ) be a Nigpr closed mapping. Let A be a NCS in X. Since every NCS is a NGPRCS in X, A is a NGPRCS in X. Then f(A) is a NGPRCS in Y. Hence f is a Ngpr closed mapping.

3.20 Example: Let X= {a, b} and Y = {u, v}. Then $\tau = \{0_N, U, 1_N\}$ and $\sigma = \{0_N, V, 1_N\}$ are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.2, 0.5, 0.7), (0.3, 0.5, 0.6)⟩ and V = ⟨y, (0.3, 0.5, 0.6), (0.4, 0.5, 0.5)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here the Neutrosophic set U^c = ⟨x, (0.7, 0.5, 0.2), (0.6, 0.5, 0.3)⟩ is a NCS in X. Then $f(U^c)$ = ⟨y, (0.7, 0.5, 0.2), (0.6, 0.5, 0.3)⟩ is a NGPRCS in (Y, σ) as $f(U^c) \subseteq 1_N$ implies $Npcl(f(U^c)) = f(U^c) \subseteq 1_N$ where 1_N is a NROS in Y. Therefore f is a Ngpr closed mapping. But f is not a Nigpr closed mapping since W = ⟨x, (0.3, 0.5, 0.6), (0.4, 0.5, 0.5)⟩ is NGPRCS in X but f(W) is not a NGPRCS in Y as f(W) ⊆ V implies Npcl(f(W)) = $V^c \nsubseteq V$ where V is a NROS in Y. Therefore f is not a Nigpr closed mapping.

The relation between various types of Neutrosophic closed mappings is given by

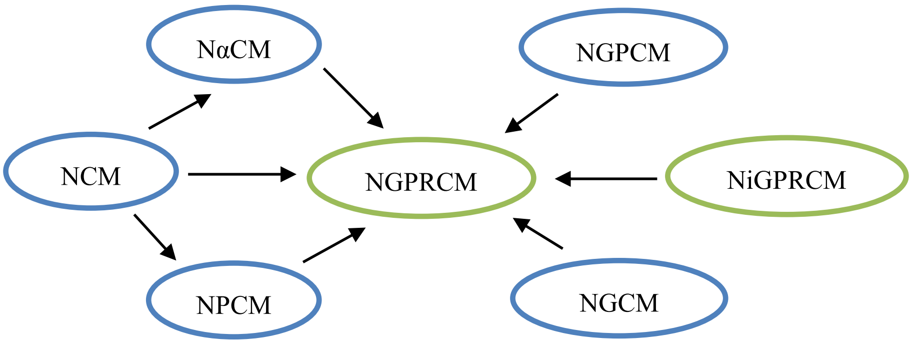

Fig.3.1.1 The reverse implications of Fig.3.1.1 are not true in general in the above diagram.

3.21 Theorem: A mapping f: (X, τ) → (Y, σ) is Ngpr closed mapping if and only if Ngprcl(f(A)) ⊆ f(Ncl(A)).

Proof: Let A ⊆ X and f: (X, τ) → (Y, σ) be a Ngpr closed mapping, then f(Ncl(A)) is NGPRCS in Y which implies Ngprcl(f(Ncl(A))) = f(Ncl(A)). Since f(A) ⊆ f(Ncl(A)), Ngprcl(f(A)) ⊆ Ngprcl(f(Ncl(A))) = f(Ncl(A)) for every NS A of X.
Conversely, let A be any NCS in (X, τ). Then A = Ncl(A) and so f(A) = f(Ncl(A)) ⊇ Ngprcl(f(A)), by hypothesis. Since f(A) ⊆ Ngprcl(f(A)), therefore f(A) = Ngprcl(f(A)). i.e., f(A) is NGPRCS in Y and hence f is Ngpr closed mapping.

3.22 Theorem: If f: (X, τ) → (Y, σ) is Ngpr open mapping iff for every NS A of (X, τ), f(Nint(A)) ⊆ Ngprint(f(A)).

***Proof*: Necessity:** Let A be a NOS in X and f: (X, τ) → (Y, σ) be a Ngpr open mapping then f(Nint(A)) is NGPROS in Y. Since f(Nint(A)) ⊆ f(A) which implies Ngprint(f(Nint(A))) ⊆ Ngprint(f(A)). Since f(Nint(A)) is NGPROS in Y, we have f(Nint(A)) ⊆ Ngprint(f(A)).
Sufficiency: Assume A is a NOS of (X, τ). Then f(A) = f(Nint(A)) ⊆ Ngprint(f(A)). But Ngprint(f(A)) ⊆ f(A). So f(A) = Ngprint(f(A)) which implies f(A) is a NGPROS in (Y, σ) and hence f is a Ngpr open mapping.

3.23 Theorem: If f: (X, τ) → (Y, σ) is a Ngpr open mapping then Nint(f^{-1}(A)) ⊆ f^{-1}(Ngprint(A)) for every NS A of (Y, σ).

Proof: Let A be a NS in (Y, σ). Then Nint(f^{-1}(A)) is a NOS of (X, τ). Since f is Ngpr open mapping which implies f(Nint(f^{-1}(A))) is Neutrosophic gpr open in (Y, σ) and hence f(Nint(f^{-1}(A))) ⊆ Ngprint(f(f^{-1}(A))) ⊆ Ngprint(A). Thus Nint(f^{-1}(A)) ⊆ f^{-1}(Ngprint(A)).

3.24 Theorem: A mapping f: (X, τ) → (Y, σ) is Ngpr open mapping iff for each NS A of (Y, σ) and for each NCS B of (X, τ) containing f^{-1}(A) there is a NGPRCS C of (Y, σ) such that A ⊆ C and f^{-1}(C) ⊆ B.

***Proof*: Necessity:** Assume f: (X, τ) → (Y, σ) is Ngpr open mapping. Let A be the NS of (Y, σ) and B be a NCS of (X, τ) such that f^{-1}(A) ⊆ B. Then C = $(f(B^c))^c$ is NGPRCS of (Y, σ) such that f^{-1}(C) ⊆ B.
Sufficiency: Assume D is a NOS of (X, τ). Then $f^{-1}((f(D))^c \subseteq D^c$ and D^c is NCS in (X, τ). By hypothesis there is a NGPRCS C of (Y, σ) such that $(f(D))^c$ ⊆C and f^{-1}(C) ⊆ D^c. Therefore D ⊆ $(f^{-1}(C))^c$. Hence C^c ⊆

f(D) ⊆ $f((f^{-1}(C))^c)$ ⊆ C^c which implies f(D) = C^c. Since C^c is NGPROS of (Y, σ). Hence f(D) is Neutrosophic gpr open in (Y, σ) and thus f is Ngpr open mapping.

***3.25 Theorem*:** A mapping f: (X, τ) → (Y, σ) is Ngpr open mapping iff f^{-1}(Ngprcl(A)) ⊆ Ncl(f^{-1}(A)) for every NS A of (Y, σ).

***Proof*:** **Necessity:** Assume f is a Ngpr open mapping. For any NS A of (Y, σ), f^{-1}(A) ⊆ Ncl(f^{-1}(A)). Therefore by Theorem 3.24., there exists a NGPRCS C in (Y, σ) such that A ⊆ C and f^{-1}(C) ⊆ Ncl(f^{-1}(A)). Therefore we obtain f^{-1}(Ngprcl(A)) ⊆ f^{-1}(C) ⊆ Ncl(f^{-1}(A)).
Sufficiency: Assume A is a NS of (Y, σ) and B is a NCS of (X, τ) containing f^{-1}(A). Put C = Ncl(A), then A ⊆ C and C is NGPRCS, since f^{-1}(C) ⊆ Ncl(f^{-1}(A)) ⊆ B. Then by Theorem 3.24., f is Ngpr open mapping.

***3.26 Theorem*:** If f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) be two Neutrosophic mappings and g∘f: (X, τ) → (Z, η) Ngpr open mapping. If g is Ngpr irresolute mapping then f is Ngpr open mapping.

***Proof*:** Let A be a NOS of (X, τ). Then g∘f(A) is NGPROS in (Z, η) because g∘f is Ngpr open mapping. Since g is Ngpr irresolute mapping and g∘f(A) is NGPROS of (Z, η) therefore g^{-1}(g∘f(A)) = f(A) is NGPROS in (Y, σ). Hence f is Ngpr open mapping.

***3.27 Theorem*:** If f: (X, τ) → (Y, σ) is Neutrosophic open mapping and g: (Y, σ) → (Z, η) is Ngpr open mapping then g∘f: (X, τ) → (Z, η) is Ngpr open mapping.

***Proof*:** Let A be a NOS of (X, τ). Then f(A) is a NOS in (Y, σ) because f is a Neutrosophic open mapping. Since g is Ngpr open mapping, g(f(A)) = g∘f(A) is NGPROS in (Z, η). Hence g∘f is Ngpr open mapping.

***3.28 Theorem*:** Let f: (X, τ) → (Y, σ) be a bijective mapping then the following statements are equivalent:

(i) f is a Ngpr open mapping.
(ii) f is a Ngpr closed mapping.
(iii) f^{-1} is Neutrosophic continuous mapping.

***Proof*:** **(i) ⇒ (ii):** Let us assume that f is a Ngpr open mapping. By definition, A is a NOS in (X, τ), then f(A) is a NGPROS in (Y, σ). Here A is NCS of (X, τ), then X-A is a NOS of (X, τ). By assumption, f(X-A) is a NGPROS in (Y, σ). Hence, Y-f(X-A) is a NGPRCS in (Y, σ). Therefore, f is a Ngpr closed mapping.
(ii) ⇒ (iii): Let A be a NCS in (X, τ). By (ii), f(A) is a NGPRCS in (Y, σ). Hence, f(A) = $(f^{-1})^{-1}$(A), so f^{-1} is a NGPRCS in (Y, σ). Therefore, f^{-1} is Neutrosophic continuous mapping.
(iii) ⇒ (iv): Let A be a NOS in (X, τ). By (iii), $(f^{-1})^{-1}$(A) = f(A) is a Ngpr open mapping.

3.29 Theorem: Let f: (X, τ) → (Y, σ) be a mapping. Then the following statements are equivalent if Y is a $NPRT_{1/2}$ space:

(i) f is a Ngpr closed mapping.
(ii) Npcl(f(A)) ⊆ f(Ncl(A)) for each NS A of X.

***Proof*:** **(i) ⇒ (ii):** Let A be a NS in X. Then Ncl(A) is a NCS in X. By (i) implies that f(Ncl(A)) is a NGPRCS in Y. Since Y is a $NPRT_{1/2}$ space, f(Ncl(A)) is a NPCS in Y. Therefore Npcl(f(Ncl(A))) = f(Ncl(A)). Now Npcl(f(A)) ⊆ Npcl(f(Ncl(A))) = f(Ncl(A)). Hence Npcl(f(A)) ⊆ f(Ncl(A)) for each NS A of X.

(ii) ⇒ (i): Let A be any NCS in X. Then Ncl(A) = A. By (ii) implies that Npcl(f(A)) ⊆ f(Ncl(A)) = f(A). But f(A) ⊆ Npcl(f(A)). Therefore Npcl(f(A)) = f(A). This implies f(A) is a NPCS in Y. Since every NPCS is NGPRCS in Y, f(A) is NGPRCS in Y. Hence f is a Ngpr closed mapping.

3.30 Theorem: If f: (X, τ) → (Y, σ) is a mapping where X and Y are $NPRT_{1/2}$ space. Then the following statements are equivalent:

(i) f is a Nigpr closed mapping.
(ii) f(A) is a NGPROS in Y for every NGPROS A in X.
(iii) f(Npint(B)) ⊆ Npint(f(B)) for each NS B of X.
(iv) Npcl(f(B)) ⊆ f(Npcl(B)) for each NS B of X.

Proof: **(i) ⇒ (ii):** is obvious by definition of Nigpr closed mapping.
(ii) ⇒ (iii): Let B be any NS in X. Since Npint(B) is a NPOS, it is a NGPROS in X. Then by hypothesis, f(Npint(B)) is a NGPROS in Y. Since Y is $NPRT_{1/2}$ space, f(Npint(B)) is a NPOS in Y. Therefore, f(Npint(B)) = Npint(f(Npint(B))) ⊆ Npint(f(B)).
(iii) ⇒ (iv) is obvious by taking complement in (iii).
(iv) ⇒ (i) Let B be a NGPRCS in X. By Hypothesis, Npcl(f(B)) ⊆ f(Npcl(B)). Since X is a $NPRT_{1/2}$ space, B is a NPCS in X. Therefore, Npcl(f(B)) ⊆ f(Npcl(B)) = f(B) ⊆ Npcl(f(B)) implies f(B) is NPCS in Y and hence f(B) is a NGPRCS in Y. Thus f is Nigpr closed mapping.

4. Ngpr homeomorphism and Nigpr homeomorphism

***4.1 Definition*:** A bijection f: (X, τ) → (Y, σ) is called Ngpr homeomorphism (resp. NG homeomorphism, NGP homeomorphism) if f and f^{-1} are Ngpr continuous (resp. NG continuous, NGP continuous) mapping.

***4.2 Example*:** Let X= {a, b} and Y = {u, v}. Then τ = {0_N, U_1, U_2, 1_N} and σ = {0_N, V, 1_N} are Neutrosophic topologies on X and Y respectively, where U_1 = ⟨x, (0.3, 0.5, 0.6), (0.5, 0.5, 0.5)⟩, U_2 = ⟨x, (0.2, 0.4, 0.7), (0.4, 0.5, 0.6)⟩ and V = ⟨y, (0.2, 0.4, 0.7), (0.4, 0.3, 0.6)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here V^c = ⟨y, (0.7, 0.6, 0.2), (0.6, 0.7, 0.4)⟩ is a Neutrosophic closed set in (Y, σ). Then $f^{-1}(V^c)$ is a NGPRCS in (X, τ). Therefore f is Ngpr continuous mapping. Here U_1^c = ⟨x, (0.6, 0.5, 0.3), (0.5, 0.5, 0.5)⟩ is a Neutrosophic closed set in (X, τ). Then $f(U_1^c)$ is a NGPRCS in (Y, σ). Therefore f^{-1} is a Ngpr continuous mapping. Hence, f and f^{-1} are Ngpr continuous mapping then it is a Ngpr homeomorphism.

***4.3 Theorem*:** Each Neutrosophic homeomorphism is Ngpr homeomorphism but not conversely in general.

Proof: Let a bijection mapping f: (X, τ) → (Y, σ) be Neutrosophic homeomorphism, in which f and f^{-1} are Neutrosophic continuous mapping. Since every Neutrosophic continuous mapping is Ngpr continuous mapping. Hence f and f^{-1} are Ngpr continuous mapping. Therefore, f is Ngpr homeomorphism.

***4.4 Example*:** Let X= {a, b} and Y = {u, v}. Then τ = {0_N, U_1, U_2, 1_N} and σ = {0_N, V, 1_N} are Neutrosophic topologies on X and Y respectively, where U_1 = ⟨x, (0.2, 0.5, 0.7), (0.5, 0.5, 0.5)⟩, U_2 = ⟨x, (0.1, 0.4, 0.7), (0.4, 0.5, 0.6)⟩ and V = ⟨y, (0.4, 0.3, 0.5), (0.3, 0.4, 0.7)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here V^c = ⟨y, (0.5, 0.7, 0.4), (0.7, 0.6, 0.3)⟩ is a NCS in (Y, σ). Then $f^{-1}(V^c)$ is a NGPRCS in (X, τ). Therefore f is Ngpr continuous mapping. Here U_1^c = ⟨x, (0.7, 0.5, 0.2), (0.5, 0.5, 0.5)⟩ is a NCS in (X, τ). Then $f(U_1^c)$ is a NGPRCS in (Y, σ). Therefore f^{-1} is a Ngpr continuous. Hence, f and f^{-1} are Ngpr continuous mapping then it is a Ngpr homeomorphism. However, here V^c is a NCS in (Y, σ) but it is not a NCS in (X, τ). Hence, f is not Neutrosophic continuous mapping. Therefore, f is not a Neutrosophic homeomorphism.

4.5 Theorem: Each NG homeomorphism is Ngpr homeomorphism but not conversely in general.

Proof: Let a bijection mapping f: (X, τ) → (Y, σ) be NG homeomorphism, in which f and f^{-1} are NG continuous mapping. Since every NG continuous mapping is Ngpr continuous mapping. Hence f and f^{-1} are Ngpr continuous mapping. Therefore, f is Ngpr homeomorphism.

4.6 Example: Let X= {a, b} and Y = {u, v}. Then τ = {0_N, U, 1_N} and σ = {0_N, V, 1_N} are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)⟩ and V = ⟨y, (0.8, 0.5, 0.2), (0.7, 0.7, 0.3)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here V^c = ⟨y, (0.2, 0.5, 0.8), (0.3, 0.3, 0.7)⟩ is a NCS in (Y, σ). Then $f^{-1}(V^c)$ is a NGPRCS in (X, τ). Therefore f is Ngpr continuous mapping. Here U^c = ⟨x, (0.6, 0.5, 0.4), (0.5, 0.6, 0.3)⟩ is a NCS in (X, τ). Then $f(U^c)$ is a NGPRCS in (Y, σ). Therefore f^{-1} is a Ngpr continuous mapping. Hence, f and f^{-1} are Ngpr continuous mapping then it is Ngpr homeomorphism. However, here Vc is a NCS in (Y, σ) but it is not a NGCS in (X, τ). Hence, f is not Neutrosophic continuous mapping. Therefore, f is not a NG homeomorphism.

4.7 Theorem: Each NGP homeomorphism is a Ngpr homeomorphism but not conversely in general.

Proof: Let a bijection mapping f: (X, τ) → (Y, σ) be NGP homeomorphism, in which f and f^{-1} are NGP continuous mapping. Since every NGP continuous mapping is Ngpr continuous mapping. Hence f and f^{-1} are Ngpr continuous mapping. Therefore, f is Ngpr homeomorphism.

4.8 Example: Let X= {a, b} and Y = {u, v}. Then τ = {0_N, U_1, U_2, U_3, 1_N} and σ = {0_N, V, 1_N} are Neutrosophic topologies on X and Y respectively, where U_1 = ⟨x, (0.3, 0.5, 0.7), (0.2, 0.5, 0.6)⟩, U_2 = ⟨x, (0.6, 0.5, 0.5), (0.7, 0.5, 0.5)⟩, U_3 = ⟨x, (0.8, 0.5, 0.2), (0.7, 0.5, 0.1)⟩ and V = ⟨y, (0.3, 0.5, 0.7), (0.3, 0.5, 0.7)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here V^c = ⟨y, (0.7, 0.5, 0.3), (0.7, 0.5, 0.3)⟩ is a NCS in (Y, σ). Then $f^{-1}(V^c)$ is a NGPRCS in (X, τ). Therefore f is Ngpr continuous mapping. Here U_1^c = ⟨x, (0.7, 0.5, 0.3), (0.6, 0.5, 0.2)⟩ is a NCS in (X, τ). Then $f(U^c)$ is a NGPRCS in (Y, σ). Therefore f^{-1} is a Ngpr continuous mapping. Hence, f and f^{-1} are Ngpr continuous mapping then it is a Ngpr homeomorphism. However, here V^c is a NCS in (Y, σ) but it is not a NGPCS in (X, τ). Hence, it is not NGP continuous mapping. Therefore, it is not a NGP homeomorphism.
The relation between various types of Neutrosophic homeomorphisms is given by

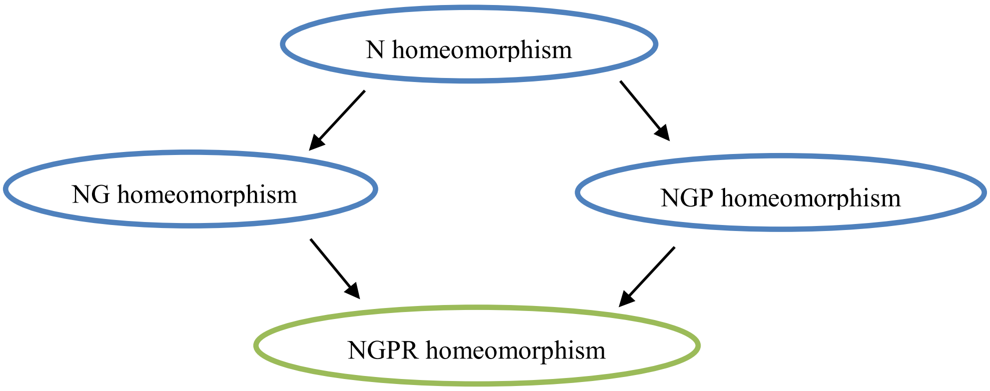

Fig.4.1.1 The reverse implications of Fig.4.1.1 are not true in general in the above diagram.

4.9 Theorem: Let f:(X, τ) → (Y, σ) be a Ngpr homeomorphism, then f is a Neutrosophic homeomorphism if X and Y are $NPRT^*_{1/2}$ space.

Proof: Let A be a NCS in (Y, σ), then $f^{-1}(A)$ is a NGPRCS in (X, τ). Since X is $NPRT^*_{1/2}$ space, $f^{-1}(A)$ is a NCS in (X, τ). Therefore, f is Neutrosophic continuous mapping. By hypothesis, f^{-1} is Ngpr continuous mapping. Let B be a NCS in (X, τ). Then $(f^{-1})^{-1}$ (B) = f(B) is a NGPRCS in Y. Since Y is $NPRT^*_{1/2}$ space, f(B) is NCS in Y. Hence f^{-1} is Neutrosophic continuous mapping. Hence f is a Neutrosophic homeomorphism.

***4.10 Theorem*:** Let f:(X, τ) → (Y, σ) be a bijective mapping. If f is Ngpr continuous mapping then the following statements are equivalent:

(i) f is a Ngpr closed mapping.
(ii) f is a Ngpr open mapping.
(iii) f is a Ngpr homeomorphism.

***Proof:* (i) ⇒ (ii):** Let us assume that f be a bijective mapping and a Ngpr closed mapping. Hence f^{-1} is Ngpr continuous mapping. Since each NOS in (X, τ) is a NGPROS in (Y, σ). Hence, f is a Ngpr open mapping.
(ii) ⇒ (iii): Let f be a bijective mapping and a Ngpr open mapping. Furthermore, f^{-1} is a Ngpr continuous mapping. Hence f and f^{-1} are Ngpr continuous mapping. Therefore, f is a Ngpr homeomorphism.
(iii) ⇒ (i): Let f be a Ngpr homeomorphism. Then f and f^{-1} are Ngpr continuous mapping. Since each NCS in (X, τ) is a NGPRCS in (Y, σ). Hence f is a Ngpr closed mapping.

***4.11 Theorem*:** The composition of two Ngpr homeomorphisms need not be a Ngpr homeomorphism in general.

***4.12 Example*:** Let X = {a, b}, Y = {c, d} and Z = {e, f}. Then τ = {0_N, U, 1_N}, σ = {0_N, V, 1_N} and η = {0_N, W, 1_N} are Neutrosophic topologies on X and Y respectively, where U = ⟨x, (0.2, 0.5, 0.8), (0.3, 0.3, 0.7)⟩, V = ⟨y, (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)⟩, W = ⟨z, (0.8, 0.5, 0.2), (0.7, 0.7, 0.3)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = c and f(b) = d and g: (Y, σ) → (Z, η) by g(c) = e and g(d) = f. Then f and g are Ngpr homeomorphisms but their composition g∘f: (X, τ) → (Z, η) is not a Ngpr homeomorphism. Since W^c is NCS in (Z, η) but it is not NGPRCS in (X, τ).

4.13 Definition: A bijection f: (X, τ) → (Y, σ) is called Nigpr homeomorphism if f and f^{-1} are Ngpr irresolute mappings.

***4.14 Theorem*:** Each Nigpr homeomorphism is a Ngpr homeomorphism but not conversely in general.

Proof: Let a bijection mapping f: (X, τ) → (Y, σ) be Nigpr homeomorphism. Assume that A is a NCS in (Y, σ) implies A is a NGPRCS in (Y, σ). Since f is Ngpr irresolute mapping, $f^{-1}(A)$ is a NGPRCS in (X, τ). Hence f is Ngpr continuous mapping. Therefore, f and f^{-1} are Ngpr continuous mapping. Hence, f is Ngpr homeomorphism.

***4.15 Example*:** Let X= {a, b} and Y = {u, v}. Then τ = {0_N, U_1, U_2, 1_N} and σ = {0_N, V, 1_N} are Neutrosophic topologies on X and Y respectively, where U_1 = ⟨x, (0.2, 0.5, 0.7), (0.4, 0.5, 0.6)⟩, U_2 = ⟨x, (0.2, 0.4, 0.8), (0.3, 0.5, 0.7)⟩ and V = ⟨y, (0.5, 0.4, 0.5), (0.4, 0.5, 0.6)⟩. Define a mapping f: (X, τ) → (Y, σ) by f(a) = u and f(b) = v. Here V^c = ⟨y, (0.5, 0.6, 0.5), (0.6, 0.5, 0.4)⟩ is a NCS in (Y, σ). Then $f^{-1}(V^c)$ is a NGPRCS in (X, τ). Therefore f is Ngpr continuous mapping. Here U_1^c = ⟨x, (0.7, 0.5, 0.2), (0.6, 0.5, 0.4)⟩ is a NCS in (X, τ). Then $f(U_1^c)$ is a NGPRCS in (Y, σ). Therefore f^{-1} is a Ngpr continuous mapping. Hence, f and f^{-1} are Ngpr continuous mapping then it is a Ngpr homeomorphism. However, here A = ⟨y, (0.2, 0.4, 0.7), (0.3, 0.5, 0.6)⟩ is a NGPRCS in (Y, σ) but it is not a NGPRCS in (X, τ). Hence, f is not Neutrosophic irresolute mapping. Therefore, f is not a Nigpr homeomorphism.

***4.16 Theorem*:** If f: (X, τ) → (Y, σ) is a Nigpr homeomorphism then $Ngprcl(f^{-1}(A)) \subseteq f^{-1}(Npcl(A))$ for each NS A in (Y, σ).

Proof: Let A be a NS in (Y, σ). Then Npcl(A) is NPCS in (Y, σ) and since every NPCS is NGPRCS in (Y, σ). Assuming f is Ngpr irresolute mapping, $f^{-1}(Npcl(A))$ is a NGPRCS in (X, τ), then $Ngprcl(f^{-1}(Npcl(A))) = f^{-1}(Npcl(A))$. Here, $Ngprcl(f^{-1}(A)) \subseteq Ngprcl(f^{-1}(Npcl(A))) = f^{-1}(Npcl(A))$. Therefore, $Ngprcl(f^{-1}(A)) \subseteq f^{-1}(Npcl(A))$ for each NS A in (Y, σ).

***4.17 Theorem*:** If f: (X, τ) → (Y, σ) is a Nigpr homeomorphism then Npcl(f^{-1}(A)) = f^{-1}(Npcl(A)) for each NS A in (Y, σ).

Proof: Given f is a Nigpr homeomorphism, then f is a Ngpr irresolute mapping. Let A be a NS in (Y, σ). Clearly, Npcl(A) is a NPCS in (Y, σ). This shows that Npcl(A) is a NGPRCS in (Y, σ). Since f^{-1}(A) ⊆ f^{-1}(Npcl(A)), then Npcl(f^{-1}(A)) ⊆ Npcl(f^{-1}(Npcl(A))) = f^{-1}(Npcl(A)). Therefore, Npcl(f^{-1}(A)) ⊆ f^{-1}(Npcl(A)).

Let f be a Nigpr homeomorphism, f^{-1} is a Ngpr irresolute mapping. Let us consider NS f^{-1}(A) in (X, τ), which bring out that Npcl(f^{-1}(A)) is a NGPRCS in (X, τ). Hence Ngprcl(f^{-1}(A)) is a NGPRCS in (X, τ). This implies that $(f^{-1})^{-1}$(Npcl(f^{-1}(A))) = f(Npcl(f^{-1}(A))) is a NPCS in (Y, σ). This proves A = $(f^{-1})^{-1}$(f^{-1}(A)) ⊆ $(f^{-1})^{-1}$(Npcl(f^{-1}(A))) = f(Npcl(f^{-1}(A))). Therefore, Npcl(A) ⊆ Npcl(f(Npcl(f^{-1}(A)))) = f(Npcl(f^{-1}(A))), since f^{-1} is a Ngpr irresolute mapping. Hence f^{-1}(Npcl(A)) ⊆ f^{-1}(f(Npcl(f^{-1}(A)))) = Npcl(f^{-1}(A)). That is f^{-1}(Npcl(A)) ⊆ Npcl(f^{-1}(A)). Hence, Npcl(f^{-1}(A)) = f^{-1}(Npcl(A)).

4.18 Theorem: If f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, η) are Nigpr homeomorphisms, then the composition g∘f: (X, τ) → (Z, η) is a Nigpr homeomorphism.

Proof: Let f and g be two Nigpr homeomorphisms. Assume C is a NGPRCS in (Z, η). Then g^{-1}(C) is a NGPRCS in (Y, σ). Then by hypothesis, f^{-1}(g^{-1}(C)) is a NGPRCS in (X, τ). Hence g∘f is a Ngpr irresolute mapping. Now, let A be a NGPRCS in (X, τ). By assumption, f(A) is a NGPRCS in (Y, σ). Then by hypothesis, g(f(A)) is a NGPRCS in (Z, η). This implies that g∘f is a Ngpr irresolute mapping. Hence, g∘f is a Nigpr homeomorphism.

5. Conclusion

In this article, the new class of Neutrosophic homeomorphism namely, Ngpr homeomorphism and Nigpr homeomorphism was defined and studied some of their properties in Neutrosophic topological spaces. Furthermore, the work was extended as the Ngpr open mappings, Ngpr closed mappings and Nigpr closed mappings and discussed some of their properties. Many results have been established to show how far topological structures are preserved by this Ngpr homeomorphism.

Also, the relation between Ngpr closed mappings and other existed Neutrosophic closed mappings in Neutrosophic topological spaces were established and derived some of their related attributes. Many examples are given to justify the results.

This concept can be used to drive few more new results of Ngpr connectedness and Ngpr compactness in Neutrosophic topological spaces.

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Generalized Neutrosophic Separation Axioms in Neutrosophic Soft Topological Spaces

A. Mehmood [1,*], F. Nadeem [2], G. Nordo [3], M. Zamir [4], C. Park [5], H. Kalsoom [6], S. Jabeen [7] and M. I. KHAN [8]

[1]Department of Mathematics and Statistics, Riphah International University, Sector I 14, Islamabad, Pakistan; mehdaniyal@gmail.com

[2]Department of Mathematics, University of Science and Technology, Bannu, Khyber Pakhtunkhwa, Pakistan; fawadnadeem2@gmail.com

[3]MIFT – Dipartimento di Scienze Matematiche e Informatiche, scienze Fisiche e scienze della Terra, Messina University, Messina, Italy; giorgio.nordo@unime.it

[4]Department of Mathematics, University of Science and Technology, Bannu, Khyber Pakhtunkhwa, Pakistan; zamirburqi@ustb.edu.pk

[5]Department of Mathematics, Research Institute for Natural Sciences, Hanyang University; baak@hanyang.ac.kr

[6]School of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China; humaira87@zju.edu.cn

[7]School of Mathematics and system Sciences Beihang University Beijing, Beijing, China; shamoona011@buaa.edu.cn

[8]Department of Epidemiology and Biostatistics, Anhui Medical University, China; ddqec@ustb.edu.pk

[*]Correspondence: mehdaniyal@gmail.com

Abstract. The idea of neutrosophic set was floated by Smarandache by considering a truth membership, an indeterminacy membership and a falsehood or falsity membership functions. The engagement between neutrosophic set and soft set was done by Maji. More over it was used effectively to model uncertainty in different areas of application, such as control, reasoning, pattern recognition and computer vision. The first aim of this paper leaks out the notion of neutrosophic soft p-open set,neutrosophic soft p-closed sets and their important characteristics. Also the notion of neutrosophic soft p-neighborhood and neutrosophic soft p-separation axioms in neutrosophic soft topological spaces are developed. Important results are examed marrying to these newly defined notion relative to soft points. The notion of neutrosophic soft p-separation axioms of neutrosophic soft topological spaces is diffused in different results concerning soft points. Furthermore, properties of neutrosophic soft -P^i-space ($i = 0, 1, 2, 3, 4$) and linkage between them is built up.

Keywords: neutrosophic soft set; neutrosophic soft point; neutrosophic soft p-open set; neutrosophic soft p-neighborhood; neutrosophic soft p-separation axioms.)

1. **Introduction**

The outdated fuzzy sets is behaviorized by the membership worth or the grade of membership worth. Some times it may be very difficult to assign the membership worth for a fuzzy sets.This gap was bridged with the introduction of interval valued fuzzy sets. In some real life problems in expert system, belief system and so forth,we must take in account the truth-membership and the falsity-membership simultaneously for appropriate narration of an object in uncertain,ambiguous atmosphere. Fuzzy sets and interval valued fuzzy sets are badly failed to handle this situation. The importance of intuitionistic fuzzy sets is automatically come in play in such a hazardous situation.The intuitionistic fuzzy sets can only handle the imperfect information supposing both the truth-membership or association(or simply membership)and falsity-membership(or non-membership)values. It fails to switch the indeterminate and inconsistent information which exists in belief system. Smarandache [14] bounced up conception of neutrosophic set which is a mathematical technique for facing problems involving imprecise, indeterminacy and inconsistent data.The words neutrosophy and neutrosophic were introduced by Smarandache. Neutrosophy (noun) means knowledge of neutral thought, while neutrosophic (adjective), means having the nature of or having the behavior of neutrosophy. This theory is nothing but just generalization of ordinary sets, fuzzy set theory [15], intuitionistic fuzzy set theory [1] etc. Some work have been supposed on neutrosophic sets by some mathematicians in many area of mathematics [4,12]. Many practical problems in economics, engineering, environment,medical science social science etc.can not be treated by conventional methods, because conventional methods have genetic complexities. These complexities may be taking birth due to the insufficiency of the theories of paramertrization tools. Each of these theories has its transmissible difficulties, as was exposed by Molodtsov [11]. Molodtsov developed an absolutely modern approach to cope with uncertainty and vagueness and applied it more and more in different directions such as smoothness of functions, game theory, operations research, Riemann integration, perron integration, and so forth. Meticulously,theory of soft set is free from the parameterization meagerness condition of fuzzy set theory, rough set theory. probability theory for facing with uncertainty Shabir and Naz [13] floated the conception of soft topological spaces, which are defined over an initial universe of discourse with a fixed set of parameters, and showed that a soft topological space produces a parameterized family of topological spaces. Theoretical studies of soft topological spaces were also done by some authors in [2, 3, 6, 8]. Kattak et al. [9] leaked out the notion of some basic result in soft bi topological spaces with respect to soft points.These results supposed the engagement of soft limit point, soft interior point, soft neighborhood, the relation between soft weak structures and soft weak closures. Moreover the authors also addressed soft sequences uniqueness of limit in soft weak-Hausdorff spaces, the product of soft Hausdorff spaces with respect to soft points

in different soft weak open set and the marriage between soft Hausdorff space and the diagonal. The combination of Neutrosophic set with soft sets was first introduced by Maji [10]. This combination makes entirely a new mathematical model Neutrosophic Soft Set and later this notion was improved by Deli and Broumi [7]. Work was progressively continue,later on mathematician came in action and defined a new mathematical structure known as neutrosophic soft topological spaces. Neutrosophic soft topological spaces were presented by Bera in [5].
M. Abdel-Basset et al. [16] proposed some novel similarity measures for bipolar and interval-valued bipolar neutrosophic set such as the cosine similarity measures and weighted cosine similarity measures. The propositions of these similarity measures are examined, and two multi-attribute decision making techniques are presented based on proposed measures. For verifying the feasibility of proposed measures, two numerical examples are presented in comparison with the related methods for demonstrating the practicality of the proposed method. Finally, the authors applied the proposed measures of similarity for diagnosing bipolar disorder diseases significantly.
M. Abdel-Basset et al. [17] supposed the objective function of scheduling problem to minimize the costs of daily resource fluctuations using the precedence relationships during the project completion time. The authors designed a resource leveling model based on neutrosophic set to overcome the ambiguity caused by the real-world problems. In this model, trapezoidal neutrosophic numbers are used to estimate the activities durations. The crisp model for activities time is obtained by applying score and accuracy functions. The authors produced a numerical example to illustrate the validation of the proposed model in this study.
Arif et al. [18] introduced the notion of most generalized neutrosophic soft open sets in neutrosophic soft topological structures relative to neutrosophic soft points. The authors leaked out the concept of most generalized separation axioms in neutrosophic soft topological spaces with respect to soft points. Gradually the study is extended to deliberate important results related to these newly defined concepts with respect to soft points. Several related properties, structural characteristics have been investigated. The convergence of sequence in neutrosophic soft topological space is defined and its uniqueness in neutrosophic soft most generalized-Hausdorff space relative to soft points is reflected. The authors further studied and switched over neutrosophic monotonous soft function and its characteristics to multifarious results. The authors lastly addressed neutrosophic soft product spaces under most generalized neutrosophic soft open set with respect to crisp points.
The first aim of this article bounces the notion of neutrosophic soft p-open set,neutrosophic soft p-neighborhood and neutrosophic soft p-separation axioms in neutrosophic soft topology which is defined on neutrosophic soft sets. Later on the important results are discussed related to these newly defined concepts with respect to soft points. Finally, the concept of p-separation

axioms of neutrosophic soft topological spaces is diffused in different results with respect to soft points. Furthermore, properties of neutrosophic soft-P^i-space ($i = 0, 1, 2, 3, 4$) and some switch between them are discussed. We hope that these results will best fit for future study on neutrosophic soft topology to carry out a general framework for practical applications.

2. Preliminaries

In this phase we now state certain useful definitions, theorems, and several existing results for neutrosophic soft sets that we require in the next sections.

Definition 2.1. [14] A neutrosophic set A on the universe set X is defined as:
A $= \{\langle x, T^A(x), I^A(x), F^A(x)\rangle : x \in X\}$,
where T, I, $F : X \to]^-0, 1^+[$ and $^-0 \leqq T^A(x) + I^A(x) + F^A(x) \leqq 3^+$.

Definition 2.2. [11] Let X be an initial universe, E be a set of all parameters, and P(x) denote the power set of X. A pair (F, E) is called a soft set over X, where F is a mapping given by $F : E \to P(X)$. In other words, the soft set is a parameterized family of subsets of the set X. For $\lambda \in E$, $F(\lambda)$ may be considered as the set of λ-elements of the soft set (F, E), or as the set of λ-approximate element of the set, i.e.
$(F, E) = \{(\lambda, F(\lambda)) : \lambda \in E, F : E \to P(X)\}$.

After the neutrosophic soft set was defined by Maji [10], this concept was modified by Deli and Broumi [7] as given below:

Definition 2.3. [7] Let X be an initial universe set and E be a set of parameters. Let P(X) denote the set of all neutrosophic sets of X. Then a neutrosophic soft set $(\widetilde{F}, E)$ over X is a set defined by a set valued function $\widetilde{F}$ representing a mapping $\widetilde{F} : E \to P(X)$, where $\widetilde{F}$ is called the approximate function of the neutrosophic soft set $(\widetilde{F}, E)$. In other words, the neutrosophic soft set is a parameterized family of some elements of the set P(X) and therefore it can be written as a set of ordered pairs:
$(\widetilde{F}, E) = \{(\lambda, \langle x, T^{\widetilde{F}(\lambda)}(x), I^{\widetilde{F}(\lambda)}(x), F^{\widetilde{F}(\lambda)}(x)\rangle : x \in X) : \lambda \in E\}$,
where $T^{\widetilde{F}(\lambda)}(x)$, $I^{\widetilde{F}(\lambda)}(x)$, $F^{\widetilde{F}(\lambda)}(x) \in [0, 1]$ are respectively called the truth-membership, indeterminacy-membership, and falsity-membership function of $\widetilde{F}(\lambda)$. Since the supremum of each T, I, F is 1, the inequality $0 \leqq T^{\widetilde{F}(\lambda)}(x) + I^{\widetilde{F}(\lambda)}(x) + F^{\widetilde{F}(\lambda)}(x) \leqq 3$ is obvious.

Definition 2.4. [5] Let $(\widetilde{F}, E)$ be a neutrosophic soft set over the universe set X. The complement of $(\widetilde{F}, E)$ is denoted by $(\widetilde{F}, E)^c$ and is defined by:
$(\widetilde{F}, E)^c = \{(\lambda, \langle x, F^{\widetilde{F}(\lambda)}(x), 1 - I^{\widetilde{F}(\lambda)}(x), T^{\widetilde{F}(\lambda)}(x)\rangle : x \in X) : \lambda \in E\}$.
It is obvious that $((\widetilde{F}, E)^c)^c = (\widetilde{F}, E)$.

Definition 2.5. [10] Let $(\widetilde{F}, E)$ and $(\widetilde{G}, E)$ be two neutrosophic soft sets over the universe set X. $(\widetilde{F}, E)$ is said to be a neutrosophic soft subset of $(\widetilde{G}, E)$ if $T^{\widetilde{F}(\lambda)}(x) \leqq T^{\widetilde{G}(\lambda)}(x)$, $I^{\widetilde{F}(\lambda)}(x)$

$\leqq I^{\widetilde{G}(\lambda)}(x)$, $F^{\widetilde{F}(\lambda)}(x) \geqq F^{\widetilde{G}(\lambda)}(x)$,$\forall\lambda \in E$, $\forall x \in X$. It is denoted by $(\widetilde{F}, E) \subseteqq (\widetilde{G}, E)$. $(\widetilde{F}, E)$ is said to be neutrosophic soft equal to $(\widetilde{G}, E)$ if $(\widetilde{F}, E)$ is a neutrosophic soft subset of $(\widetilde{G}, E)$ and $(\widetilde{G}, E)$ is a neutrosophic soft subset of $(\widetilde{F}, E)$. It is denoted by $(\widetilde{F}, E) = (\widetilde{G}, E)$.

3. Applications of Neutrosophic Soft Point and its Characteristics

Definition 3.1. Let $(\widetilde{F}^1, E)$ and $(\widetilde{F}^2, E)$ be two neutrosophic soft sets over universe set X. Then their union is denoted by $(\widetilde{F}^1, E) \sqcup (\widetilde{F}^2, E) = (\widetilde{F}^3, E)$ and is defined by:

$(\widetilde{F}^3, E) = \{(\lambda, \langle x, T^{\widetilde{F}^3(\lambda)}(x), I^{\widetilde{F}^3(\lambda)}(x), F^{\widetilde{F}^3(\lambda)}(x)\rangle : x \in X) : \lambda \in E\}$,

where $T^{\widetilde{F}^3(\lambda)}(x) = \max\{T^{\widetilde{F}^1(\lambda)}(x), T^{\widetilde{F}^2(\lambda)}(x)\}$,

$I^{\widetilde{F}^3(\lambda)}(x) = \max\{I^{\widetilde{F}^1(\lambda)}(x), I^{\widetilde{F}^2(\lambda)}(x)\}$,

$F^{\widetilde{F}^3(\lambda)}(x) = \max\{F^{\widetilde{F}^1(\lambda)}(x), F^{\widetilde{F}^2(\lambda)}(x)\}$.

Definition 3.2. Let $(\widetilde{F}^1, E)$ and $(\widetilde{F}^2, E)$ be two neutrosophic soft sets over the universe set X. Then their intersection is denoted by $(\widetilde{F}^1, E) \sqcap (\widetilde{F}^2, E) = (\widetilde{F}^3, E)$ and is defined by:

where

$T^(\widetilde{F}^3, E) = \min\{T^{\widetilde{F}^1(\lambda)}(x), T^{\widetilde{F}^2(\lambda)}(x)\}$

$I^{\widetilde{F}^3(\lambda)}(x) = \max\{I^{\widetilde{F}^1(\lambda)}(x), I^{\widetilde{F}^2(\lambda)}(x)\}$,

$F^{\widetilde{F}^3(\lambda)}(x) = \max\{F^{\widetilde{F}^1(\lambda)}(x), F^{\widetilde{F}^2(\lambda)}(x)\}$.

Definition 3.3. A neutrosophic soft set $(\widetilde{F}, E)$ over the universe set X is said to be a null neutrosophic soft set if $T^{\widetilde{F}(\lambda)}(x) = 0$, $I^{\widetilde{F}(\lambda)}(x) = 0$, $F^{\widetilde{F}(\lambda)}(x) = 1$; $\forall\lambda \in E$, $\forall x \in X$. It is denoted by $0^{(X,E)}$.

Definition 3.4. A neutrosophic soft set $(\widetilde{F}, E)$ over the universe set X is said to be an absolute neutrosophic soft set if $T^{\widetilde{F}(\lambda)}(x) = 1$, $I^{\widetilde{F}(\lambda)}(x) = 1$, $F^{\widetilde{F}(\lambda)}(x) = 0$; $\forall\lambda \in E$, $\forall x \in X$. It is denoted by $1^{(X,E)}$.

Clearly, $0^c(X, E) = 1^{(X,E)}$ and $1^c(X, E) = 0^{(X,E)}$.

Definition 3.5. Let NSS(X, E) be the family of all neutrosophic soft sets over the universe set X and $\Im \sqsubset$ NSS(X, E). Then $\Im$ is said to be a neutrosophic soft topology on X if:

1. $0^{(X,E)}$ and $1^{(X,E)}$ belong to $\Im$,
2. the union of any number of neutrosophic soft sets in $\Im$ belongs to $\Im$,
3. the intersection of a finite number of neutrosophic soft sets in $\Im$ belongs to $\Im$.

Then $(X, \Im, E)$ is said to be a neutrosophic soft topological space over X. Each member of $\Im$ is said to be a neutrosophic soft open set.

Definition 3.6. Let $(X, \Im, E)$ be a neutrosophic soft topological space over X and $(\widetilde{F}, E)$ be a subset of neutrosophic soft topological space over X. Then $(\widetilde{F}, E)$ is said to be a neutrosophic soft closed set iff its complement is a neutrosophic soft open set.

Definition 3.7. Let $(X, \Im, E)$ be a neutrosophic soft topological space over X and $(\widetilde{F}, E)$ be a subset of neutrosophic soft topological space over X. Then $(\widetilde{F}, E)$ is said to be a neutrosophic soft p-open (NSPO) set if $(\widetilde{F}, E) \subseteq NSint(NScl((\widetilde{F}, E)))$

Definition 3.8. Let $(X, \Im, E)$ be a neutrosophic soft topological space over X and $(\widetilde{F}, E)$ be a subset of neutrosophic soft topological space over X. Then $(\widetilde{F}, E)$ is said to be a neutrosophic soft p-closed (NSPC) set if $(\widetilde{F}, E) \supseteq NScl(NSint((\widetilde{F}, E)))$

Definition 3.9. Let NS be the family of all neutrosophic sets over the universe set X and x $\in$ X. The neutrosophic set $x^{(\alpha,\beta,\gamma)}$ is called a neutrosophic point, for $0 < \alpha, \beta, \gamma \leqq 1$, and is defined as follow:

$$x^{(\alpha,\beta,\gamma)}(y) = \begin{cases} (\alpha, \beta, \gamma), \ if \ y = x \\ (0, 0, 1), \ if \ y \neq x. \end{cases} \tag{1}$$

It is clear that every neutrosophic set is the union of its neutrosophic points.

Definition 3.10. Suppose that X = $\{x^1, x^2\}$. Then neutrosophic set

A = $\{\langle x^1, 0.1, 0.3, 0.5\rangle, \langle x^2, 0.5, 0.4, 0.7\rangle\}$

is the union of neutrosophic points $x^1(0.1, 0.3, 0.5) and \ x^2(0.5, 0.4, 0.7)$.

Now we define the concept of neutrosophic soft points for neutrosophic soft sets.

Definition 3.11. Let NSS(X, E) be the family of all neutrosophic soft sets over the universe set X. Then neutrosophic soft set $x^{\lambda}(\alpha, \beta, \gamma)$ is called a neutrosophic soft point, for every x $\in$ X, $0 < \alpha, \beta, \gamma \leqq 1$, $\lambda \in$ E, and is defined as follows:

$$x^{\lambda(\alpha,\beta,\gamma)}(\lambda')(y) = \begin{cases} (\alpha, \beta, \gamma) \ if \ \lambda' = \lambda \ and \ y = x \\ (0, 0, 1), \ if \ \lambda' \neq \lambda \ or \ y \neq x. \end{cases} \tag{2}$$

Definition 3.12. Suppose that the universe set X is given by X = $\{x^1, x^2\}$ and the set of parameters by E = $\{\lambda^1, \lambda^2\}$. Let us consider neutrosophic soft sets $(\widetilde{F}, E)$ over the universe X as follows:

$$(\widetilde{F}, E) = \left\{ \begin{array}{c} \lambda^1 = \{\langle x^1, 0.3, 0.7, 0.6\rangle, \langle x^2, 0.4, 0.3, 0.8\rangle\} \\ \lambda^2 = \{\langle x^1, 0.4, 0.6, 0.8\rangle, \langle x^2, 0.3, 0.7, 0.2\rangle\}. \end{array} \right\} \tag{3}$$

It is clear that$(\widetilde{F}, E)$ is union of its neutrosophic soft point $x^1\lambda^{1(0.3,0.7,0.6)}$,$x^1\lambda^{2(0.4,0.6,0.8)}$,$x^2\lambda^1$, and $x^2\lambda^{2(0.3,0.7,0.6)}$. Here

$$x^1\lambda^{1(0.3,0.7,0.6)} = \left\{ \begin{array}{c} \lambda^1 = \{\langle x^1, 0.3, 0.7, 0.6\rangle, \langle x^2, 0, 0, 1\rangle\} \\ \lambda^2 = \{\langle x^1, 0, 0, 1\rangle, \langle x^2, 0, 0, 1\rangle\}. \end{array} \right\} \tag{4}$$

$$x^1\lambda^{2(0.4,0.6,0.8)} = \left\{ \begin{array}{c} \lambda^1 = \{\langle x^1, 0.3, 0.7, 0.6\rangle, \langle x^2, 0, 0, 1\rangle\} \\ \lambda^2 = \{\langle x^1, 0.4, 0.6, 0.8\rangle, \langle x^2, 0, 0, 1\rangle\}. \end{array} \right\}. \tag{5}$$

$$x^2\lambda^{1(0.4,0.3,0.8)} = \left\{ \begin{array}{c} \lambda^1 = \{\langle x^1, 0, 0, 1\rangle, \langle x^2, 0.4, 0.3, 0.8\rangle\} \\ \lambda^2 = \{\langle x^1, 0, 0, 1\rangle, \langle x^2, 0, 0, 1\rangle\}. \end{array} \right\}. \tag{6}$$

$$x^2\lambda^{2(0.3,0.7,0.2)} = \left\{ \begin{array}{c} \lambda^1 = \{\langle x^1, 0, 0, 1\rangle, \langle x^2, 0, 0, 1\rangle\} \\ \lambda^2 = \{\langle x^1, 0, 0, 1\rangle, \langle x^1, 0.3, 0.7, 0.2\rangle\}. \end{array} \right\}. \tag{7}$$

Definition 3.13. Let $(\widetilde{F}, E)$ be a neutrosophic soft set over the universe set X. We say that $x^{\lambda^{(\alpha,\beta,\gamma)}} \in (\widetilde{F}, E)$ read as belonging to the neutrosophic soft set $(\widetilde{F}, E)$ whenever $\alpha \leqq T^{\widetilde{F}(\lambda)}(x)$,$\beta \leqq I^{\widetilde{F}(\lambda)}(x)$ and $\gamma \geqq F^{\widetilde{F}(\lambda)}(x)$.

Definition 3.14. Let $(X, \Im, E)$ be a neutrosophic soft topological space over X. A neutrosophic soft set $(\widetilde{F}, E)$ in $(X, \Im, E)$ is called a neutrosophic soft p-neighborhood of the neutrosophic soft point $x^{\lambda^{(\alpha,\beta,\gamma)}} \in (\widetilde{F}, E)$, if there exists a neutrosophic soft p-open set $(\widetilde{G}, E)$ such that $x^{\lambda^{(\alpha,\beta,\gamma)}} \in (\widetilde{G}, E) \sqsubset (\widetilde{F}, E)$.

Theorem 3.15. *Let $(X, \Im, E)$ be a neutrosophic soft topological space and $(\widetilde{F}, E)$ be a neutrosophic soft set over X. Then $(\widetilde{F}, E)$ is a neutrosophic soft p-open set if and only if $(\widetilde{F}, E)$ is a neutrosophic soft p-neighborhood of its neutrosophic soft points.*

Proof. Let $(\widetilde{F}, E)$ be a neutrosophic soft p-open set and $x^{\lambda^{(\alpha,\beta,\gamma)}} \in (\widetilde{F}, E)$. Then $x^{\lambda^{(\alpha,\beta,\gamma)}} \in (\widetilde{F}, E) \sqsubset (\widetilde{F}, E)$.

Therefore, $(\widetilde{F}, E)$ is a neutrosophic soft p-neighborhood of $x^{\lambda^{(\alpha,\beta,\gamma)}}$.

Conversely, let $(\widetilde{F}, E)$ be a neutrosophic soft p-neighborhood of its neutrosophic soft points. Let $x^{\lambda^{(\alpha,\beta,\gamma)}} \in (\widetilde{F}, E)$. Since $(\widetilde{F}, E)$ is a neutrosophic soft p-neighborhood of the neutrosophic soft point $x^{\lambda^{(\alpha,\beta,\gamma)}}$, there exists $(\widetilde{G}, E) \in \Im$ such that $x^{\lambda^{(\alpha,\beta,\gamma)}} \in (\widetilde{G}, E) \sqsubset (\widetilde{F}, E)$. Since $(\widetilde{F}, E)$ $= \sqcup \{x^{\lambda^{(\alpha,\beta,\gamma)}} : x^{\lambda^{(\alpha,\beta,\gamma)}} \in (\widetilde{F}, E)\}$, it follows that $(\widetilde{F}, E)$ is a union of neutrosophic soft p-open sets and hence $(\widetilde{F}, E)$ is a neutrosophic soft p-open set.

The p-neighborhood system of a neutrosophic soft point $x^{\lambda^{(\alpha,\beta,\gamma)}}$, denoted by U $(x^{\lambda^{(\alpha,\beta,\gamma)}}, E)$, is the family of all its p-neighborhoods. □

Theorem 3.16. *The neighborhood system U $(x^{\lambda^{(\alpha,\beta,\gamma)}}, E)$ at $x^{\lambda^{(\alpha,\beta,\gamma)}}$ in a neutrosophic soft topological space $(X, \Im, E)$ has the following properties.*

1) If $(\widetilde{F}, E) \in U\,(x^{\lambda^{(\alpha,\beta,\gamma)}}, E)$, then $x^{\lambda^{(\alpha,\beta,\gamma)}} \in (\widetilde{F}, E)$;

2) If $(\widetilde{F}, E) \in U\,(x^{\lambda^{(\alpha,\beta,\gamma)}}, E)$ and $(\widetilde{F}, E) \sqsubset (\widetilde{H}, E)$, then $(\widetilde{H}, E)$, then $(\widetilde{H}, E) \in U\,(x^{\lambda^{(\alpha,\beta,\gamma)}}, E)$;

3) If $(\widetilde{F}, E)$, If $(\widetilde{G}, E) \in U\,(x^{\lambda^{(\alpha,\beta,\gamma)}}, E)$, then $(\widetilde{F}, E) \sqcap (\widetilde{G}, E) \in U\,(x^{\lambda^{(\alpha,\beta,\gamma)}}, E)$;

4) If $(\widetilde{F}, E) \in U\,(x^{\lambda^{(\alpha,\beta,\gamma)}}, E)$, then there exists a $(\widetilde{G}, E) \in U\,(x^{\lambda^{(\alpha,\beta,\gamma)}}, E)$ such that $(\widetilde{G}, E) \in U\,(y^{\lambda'^{(\alpha',\beta',\gamma')}}, E)$, for each $y^{\lambda'^{(\alpha',\beta',\gamma')}} \in (\widetilde{G}, E)$.

Proof. The proof of 1), 2), and 3) is obvious from definition 3.12.

4) If $(\widetilde{F}, E) \in$ U $(x^{\lambda(\alpha,\beta,\gamma)}, E)$, then there exists a neutrosophic soft p-open set $(\widetilde{G}, E)$ such that $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}, E) \sqsubset (\widetilde{F}, E)$. From Proposition 3.1, $(\widetilde{G}, E) \in$ U $(x^{\lambda(\alpha,\beta,\gamma)}, E)$, so for each $y^{\lambda'(\alpha',\beta',\gamma')} \in (\widetilde{G}, E)$, $(\widetilde{G}, E) \in$ U $(y^{\lambda'(\alpha',\beta',\gamma')}, E)$ is obtained. □

Definition 3.17. Let $x^{\lambda(\alpha,\beta,\gamma)}$ and $y^{\lambda'(\alpha',\beta',\gamma')}$ be two neutrosophic soft points. For the neutrosophic soft points $x^{\lambda(\alpha,\beta,\gamma)}$ and $y^{\lambda'(\alpha',\beta',\gamma')}$ over a common universe X, we say that neutrosophic soft points are distinct points if $x^{\lambda(\alpha,\beta,\gamma)} \sqcap y^{\lambda'(\alpha',\beta',\gamma')} = 0^{(X,E)}$.

It is clear that $x^{\lambda(\alpha,\beta,\gamma)}$ and $y^{\lambda'(\alpha',\beta',\gamma')}$ are distinct neutrosophic soft points if and only if x $\neq$ y or $\lambda' \neq \lambda$.

4. **Neutrosophic Soft p-Separation Structures**

In this phase, we suppose neutrosophic soft p-separation axioms and neutrosophic soft topological subspace consisting of distinct neutrosophic soft points of neutrosophic soft topological space over X.

Definition 4.1. a) Let $(X, \Im, E)$ be a neutrosophic soft topological space over the crisp set X , and $x^{\lambda(\alpha,\beta,\gamma)} > y^{\lambda'(\alpha',\beta',\gamma')}$ are neutrosophic soft points. If there exist neutrosophic soft p-open sets $(\widetilde{F}, E)$ and $(\widetilde{G}, E)$ such that

$x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{F}, E)$ and $x^{\lambda(\alpha,\beta,\gamma)} \sqcap (\widetilde{G}, E) = 0^{(X,E)}$ or

$y^{\lambda'(\alpha',\beta',\gamma')} \in (\widetilde{G}, E)$ and $y^{\lambda'(\alpha',\beta',\gamma')} \sqcap (\widetilde{F}, E) = 0^{(X,E)}$,

then $(X, \Im, E)$ is called a neutrosophic soft-P^o-space.

b) Let $(X, \Im, E)$ be a neutrosophic soft topological space over the crisp set X and $x^{\lambda(\alpha,\beta,\gamma)} > y^{\lambda'(\alpha',\beta',\gamma')}$ are neutrosophic soft points. If there exist neutrosophic soft p-open sets $(\widetilde{F}, E)$ and $(\widetilde{G}, E)$ such that

$x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{F}, E)$, $x^{\lambda(\alpha,\beta,\gamma)} \sqcap (\widetilde{G}, E) = 0^{(X,E)}$ or

$y^{\lambda'(\alpha',\beta',\gamma')} \in (\widetilde{G}, E)$, $y^{\lambda'(\alpha',\beta',\gamma')} \sqcap (\widetilde{F}, E) = 0^{(X,E)}$,

then $(X, \Im, E)$ is called a neutrosophic soft-P^1-space.

c) Let $(X, \Im, E)$ be a neutrosophic soft topological space over the crisp set X, and $x^{\lambda(\alpha,\beta,\gamma)} > y^{\lambda'(\alpha',\beta',\gamma')}$ are neutrosophic soft points. If there exist neutrosophic soft p-open sets $(\widetilde{F}, E)$ and $(\widetilde{G}, E)$ such that

$x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{F}, E)$, $y^{\lambda'(\alpha',\beta',\gamma')} \in (\widetilde{G}, E)$ and $(\widetilde{F}, E) \sqcap (\widetilde{G}, E) = 0^{(X,E)}$,

then $(X, \Im, E)$ is called a neutrosophic soft-P^2-space.

Example 4.2. Let X = $\{x^1, x^2\}$ be a universe set, E= $\{\lambda^1, \lambda^2\}$ be a parameters set, and $(x^1)\lambda^{1(0.1,0.4,0.7)}$, $(x^1)\lambda^{2(0.2,0.5,0.6)}$, $(x^2)\lambda^{1(0.3,0.3,0.5)}$, and $(x^2)\lambda^{2(0.4,0.4,0.4)}$ be neutrosophic soft points. Then the family $\Im = \{0^{(X,E)}, 1^{(X,E)}, (\widetilde{F}^1, E), (\widetilde{F}^2, E), (\widetilde{F}^3, E), (\widetilde{F}^4, E),$
$(\widetilde{F}^5, E), (\widetilde{F}^6, E), (\widetilde{F}^7, E), (\widetilde{F}^8, E)\}$, where

$(\widetilde{F}^1, E) = \{x^{1^{\lambda^{1}(0.1,0.4,0.7)}}\}$,
$(\widetilde{F}^2, E) = \{(x^1)\lambda^{2(0.2,0.5,0.6)}\}$,
$(\widetilde{F}^3, E) = \{(x^2)\lambda^1(0.3, 0.3, 0.5)\}$,
$(\widetilde{F}^4, E) = (\widetilde{F}^1, E) \sqcup (\widetilde{F}^2, E)$,
$(\widetilde{F}^5, E) = (\widetilde{F}^1, E) \sqcup (\widetilde{F}^3, E)$,
$(\widetilde{F}^6, E) = (\widetilde{F}^2, E) \sqcup (\widetilde{F}^3, E)$,
$(\widetilde{F}^7, E) = (\widetilde{F}^1, E) \sqcup (\widetilde{F}^2, E) \sqcup (\widetilde{F}^3, E)$,
$(\widetilde{F}^8, E) = \{(x^1)\lambda^{1(0.1,0.4,0.7)}, (x^1)\lambda^{2(0.2,0.5,0.6)}, (x^2)\lambda^{2(0.3,0.3,0.5)}, (x^2)\lambda^2(0.4, 0.4, 0.4)\}$,

is a neutrosophic soft topology over X. Hence, $(X, \Im, E)$ is a neutrosophic soft topological space over X. Also, $(X, \Im, E)$ is a neutrosophic soft- P^0-space but not a neutrosophic soft-P^1-space because for neutrosophic soft points $(x^1)\lambda^1(0.1, 0.4, 0.7)$ and $(x^2)\lambda^2(0.4, 0.4, 0.4)$,$(X, \Im, E)$ is not a neutrosophic soft-P^1-space.

Example 4.3. Let X = $\mathbb{N}$ be a natural numbers set and E = $\{\lambda\}$ be a parameters set. Here $n^{\lambda(\alpha n,\beta n,\gamma n)}$ are neutrosophic soft points. Here we can give $(\alpha n, \beta n, \gamma n)$ appropriate values and the neutrosophic soft points $n^{\lambda(\alpha n,\beta n,\gamma n)}$,$m^{\lambda(\alpha n,\beta n,\gamma n)}$ are distinct neutrosophic soft points if and only if n $\neq$ m. It is clear that there is one-to-one compatibitily between the set of natural numbers and the set of neutrosophic soft points $N^\lambda = \{n^{\lambda(\alpha n,\beta n,\gamma n)}\}$.

Then we give cofinite topology on this set. Then neutrosophic soft set $(\widetilde{F}, E)$ is a neutrosophic soft p-open set if and only if the finite neutrosophic soft point is discarded from N^λ. Hence, $(X, \Im, E)$ is a neutrosophic soft-P^1-space but not a neutrosophic soft-P^2-space.

Example 4.4. Let X = $\{x^1, x^2\}$ be a universe set, E = $\{\lambda^1, \lambda^2\}$ be a parameters set, and $x^1\lambda^{1(0.1,0.4,0.7)}$,$x^1\lambda^{2(0.2,0.5,0.6)}$,
$x^2\lambda^{1(0.3,0.3,0.5)}$, and $x^2\lambda^{2(0.4,0.4,0.4)}$, be neutrosophic soft points. Then the family
$\Im = \{0^{(X,E)}, 1^{(X,E)}, (\widetilde{F}^1, E), (\widetilde{F}^2, E), ..., (\widetilde{F}^{15}, E)\}$, where
$(\widetilde{F}^1, E) = \{x^1\lambda^{1(0.1,0.4,0.7)}\}$,
$(\widetilde{F}^2, E) = \{(x^1)\lambda^{2(0.2,0.5,0.6)}\}$,
$(\widetilde{F}^3, E) = \{(x^2)\lambda^{1(0.3,0.3,0.5)}\}$,
$(\widetilde{F}^4, E) = \{(x^2)\lambda^{2(0.4,0.4,0.4)}\}$,
$(\widetilde{F}^5, E) = (\widetilde{F}^1, E) \sqcup (\widetilde{F}^2, E)$,
$(\widetilde{F}^6, E) = (\widetilde{F}^1, E) \sqcup (\widetilde{F}^3, E)$,
$(\widetilde{F}^7, E) = (\widetilde{F}^1, E) \sqcup (\widetilde{F}^4, E)$,
$(\widetilde{F}^8, E) = (\widetilde{F}^2, E) \sqcup (\widetilde{F}^3, E)$,
$(\widetilde{F}^9, E) = (\widetilde{F}^2, E) \sqcup (\widetilde{F}^4, E)$,
$(\widetilde{F}^{10}, E) = (\widetilde{F}^3, E) \sqcup (\widetilde{F}^4, E)$,
$(\widetilde{F}^{11}, E) = (\widetilde{F}^1, E) \sqcup (\widetilde{F}^2, E) \sqcup (\widetilde{F}^3, E)$,

$(\widetilde{F}^{12}, E) = (\widetilde{F}^1, E) \sqcup (\widetilde{F}^2, E) \sqcup (\widetilde{F}^4, E)$,
$(\widetilde{F}^{13}, E) = (\widetilde{F}^2, E) \sqcup (\widetilde{F}^3, E) \sqcup (\widetilde{F}^4, E)$,
$(\widetilde{F}^{14}, E) = (\widetilde{F}^1, E) \sqcup (\widetilde{F}^3, E) \sqcup (\widetilde{F}^4, E)$,
$(\widetilde{F}^{15}, E) = \{(x^1)\lambda^{1(0.1,0.4,0.7)}, (x^1)\lambda^{2(0.2,0.5,0.6)}, (x^2)\lambda^{2(0.3,0.3,0.5)}, (x^2)\lambda^{2(0.4,0.4,0.4)}\}$,

is a neutrosophic soft topology over X. Hence, $(X, \Im, E)$ is a neutrosophic soft topological space over X. Also, $(X, \Im, E)$ is a neutrosophic soft -P^2-space.

Theorem 4.5. *Let $(X, \Im, E)$ be a neutrosophic soft topological space over X. Then $(X, \Im, E)$ is a neutrosophic soft-P^1-space if and only if each neutrosophic soft point is a neutrosophic soft p-closed set.*

Proof. Let$(X, \Im, E)$ be a neutrosophic soft-P^1-space and $x^{\lambda(\alpha,\beta,\gamma)}$ be an arbitrary neutrosophic soft point. We show that $(x^{\lambda(\alpha,\beta,\gamma)})^{\lambda}$ is a neutrosophic soft p-open set. Let $y^{\lambda'(\alpha',\beta',\gamma')} \in (x^{\lambda(\alpha,\beta,\gamma)})^{\lambda}$; then $x^{\lambda(\alpha,\beta,\gamma)}$ and $y^{\lambda'(\alpha',\beta',\gamma')}$ are distinct neutrosophic soft points. Hence, x $\neq$ y or $\lambda' \neq \lambda$.

Since $(X, \Im, E)$ is a neutrosophic soft-P^1-space, there exists a neutrosophic soft p-open set $(\widetilde{G}, E)$ such that

$y^{\lambda'(\alpha',\beta',\gamma')} \in (\widetilde{G}, E)$ and $x^{\lambda(\alpha,\beta,\gamma)} \sqcap (\widetilde{G}, E) = 0^{(X,E)}$.

Then, since $x^{\lambda(\alpha,\beta,\gamma)} \sqcap (\widetilde{G}, E) = 0^{(X,E)}$, we have $y^{\lambda'(\alpha',\beta',\gamma')} \in (\widetilde{G}, E) \sqsubset (x^{\lambda(\alpha,\beta,\gamma)})^{\lambda}$. This implies that $(x^{\lambda(\alpha,\beta,\gamma)})^{\lambda}$ is a neutrosophic soft p-open set, i.e. $x^{\lambda(\alpha,\beta,\gamma)}$ is a neutrosophic soft p-closed set.

Suppose that each neutrosophic soft point $x^{\lambda(\alpha,\beta,\gamma)}$ is a neutrosophic soft p-closed set. Then $(x^{\lambda(\alpha,\beta,\gamma)})^{\lambda}$ is a neutrosophic soft p-open set. Let $x^{\lambda(\alpha,\beta,\gamma)} \sqcap y^{\lambda'(\alpha',\beta',\gamma')} = 0^{(X,E)}$. Thus $y^{\lambda'(\alpha',\beta',\gamma')} \in (x^{\lambda(\alpha,\beta,\gamma)})^{\lambda}$ and $x^{\lambda(\alpha,\beta,\gamma)} \sqcap (x^{\lambda(\alpha,\beta,\gamma)})^{\lambda} = 0^{(X,E)}$. Therefore, $(X, \Im, E)$ is a neutrosophic soft-P^1-space over X. □

Theorem 4.6. *Let $(X, \Im, E)$ be a neutrosophic soft topological space over X. Then $(X, \Im, E)$ is a neutrosophic soft-P^2-space iff for distinct neutrosophic soft points $x^{\lambda(\alpha,\beta,\gamma)}$ and $y^{\lambda'(\alpha',\beta',\gamma')}$, there exists a neutrosophic soft p-open set $(\widetilde{F}, E)$ containing $x^{\lambda}(\alpha, \beta, \gamma)$ but not $y^{\lambda'}(\alpha', \beta', \gamma')$ such that $y^{\lambda'(\alpha',\beta',\gamma')}$ does not belong to $(\overline{\widetilde{F}, E})$.*

Proof. Let $x^{\lambda(\alpha,\beta,\gamma)}$ and $y^{\lambda'(\alpha',\beta',\gamma')}$ be two neutrosophic soft points in neutrosophic soft -P^2-space $(X, \Im, E)$.

Then there exist disjoint neutrosophic soft p-open set $(\widetilde{F}, E)$, $(\widetilde{G}, E)$ such that

$x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{F}, E)$, $y^{\lambda'(\alpha',\beta',\gamma')} \in (\widetilde{G}, E)$.

Since $x^{\lambda(\alpha,\beta,\gamma)} \sqcap y^{\lambda'(\alpha',\beta',\gamma')} = 0^{(X,E)}$ and $(\widetilde{F}, E) \sqcap (\widetilde{G}, E) = 0^{(X,E)}$, $y^{\lambda'(\alpha',\beta',\gamma')}$ does not belong to $(\widetilde{F}, E)$. It implies that $y^{\lambda'(\alpha',\beta',\gamma')}$ does not belong to $(\overline{\widetilde{F}, E})$.

Next suppose that, for distinct neutrosophic soft points $x^{\lambda(\alpha,\beta,\gamma)}$, $y^{\lambda'(\alpha',\beta',\gamma')}$, there exists a neutrosophic soft p-open set $(\widetilde{F}, E)$ containing $x^{\lambda(\alpha,\beta,\gamma)}$ but not $y^{\lambda'(\alpha',\beta',\gamma')}$ such that $y^{\lambda'(\alpha',\beta',\gamma')}$ does not belong to $(\overline{\widetilde{F}, E})$. Then $y^{\lambda'(\alpha',\beta',\gamma')} \in ((\overline{\widetilde{F}, E}))^c$, i.e. $(\widetilde{F}, E)$ and $((\overline{\widetilde{F}, E}))^c$ are disjoint neutrosophic soft p-open sets containing $x^{\lambda(\alpha,\beta,\gamma)}$, $y^{\lambda'(\alpha',\beta',\gamma')}$ respectively. □

Theorem 4.7. *Let $(X, \Im, E)$ be a neutrosophic soft-P^1-space for every neutrosophic soft point $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{F}, E) \in \Im$. If there exists a neutrosophic soft p-open set $(\widetilde{G}, E)$ such that*

$x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}, E) \sqsubset (\overline{\widetilde{G}, E}) \sqsubset (\widetilde{F}, E)$,

then $(X, \Im, E)$ is a neutrosophic soft-P^2-space.

Proof. Suppose that $x^{\lambda(\alpha,\beta,\gamma)} \sqcap y^{\lambda'(\alpha',\beta',\gamma')} = 0^{(X,E)}$. Since $(X, \Im, E)$ is a neutrosophic soft T^1-space, $x^{\lambda(\alpha,\beta,\gamma)}$ and $y^{\lambda'(\alpha',\beta',\gamma')}$ are neutrosophic soft p-closed sets in $\Im$. Thus $x^{\lambda(\alpha,\beta,\gamma)} \in (y^{\lambda'}(\alpha',\beta',\gamma'))^c \in \Im$. Then there exists a neutrosophic soft p-open set $(\widetilde{G}, E)$ in $\Im$ such that

$x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}, E) \sqsubset (\overline{\widetilde{G}, E}) \sqsubset (y^{\lambda'}(\alpha',\beta',\gamma'))^c$.

Hence, we have $y^{\lambda'(\alpha',\beta',\gamma')} \in ((\overline{\widetilde{G}, E}))^c$, $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}, E)$, and $(\widetilde{G}, E) \sqcap ((\overline{\widetilde{G}, E}))^c = 0^{(X,E)}$, i,e. $(X, \Im, E)$ is a neutrosophic soft P^2-space. □

Remark 4.8. Let $(X, \Im, E)$ be a neutrosophic soft-P^1-space for i = 0, 1, 2. For each x ≠ y, neutrosophic points $x^(\alpha,\beta,\gamma)$ and $y^(\alpha',\beta',\gamma')$ have neighborhoods satisfying conditions of-P^i-space in neutrosophic topological space $(X, \Im^\lambda)$ for each $\lambda \in E$ because $x^{\lambda(\alpha,\beta,\gamma)}$ and $y^{\lambda'(\alpha',\beta',\gamma')}$ are distinct neutrosophic soft points.

Definition 4.9. Let $(X, \Im, E)$ be a neutrosophic soft topological space over X, $(\widetilde{F}, E)$ be a neutrosophic soft p-closed set , and $x^{\lambda(\alpha,\beta,\gamma)} \sqcap (\widetilde{F}, E) = 0^{(X,E)}$. If there exist neutrosophic soft p-open open sets $(^1, E)$ and $(\widetilde{G}^2, E)$ such that $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}^1, E)$, $(\widetilde{F}, E) \sqsubset (\widetilde{G}^2, E)$, and $(\widetilde{G}^1, E) \sqcap (\widetilde{G}^2, E) = 0^{(X,E)}$, then $(X, \Im, E)$ is called a neutrosophic soft b-regular space. $(X, \Im, E)$ is said to be a neutrosophic soft-P^3-space if is both a neutrosophic soft p-regular and neutrosophic soft-P^1-space.

Theorem 4.10. *Let $(X, \Im, E)$ be a neutrosophic soft topological space over X, $(X, \Im, E)$ is a neutrosophic soft-P^3-space if and only if for every $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{F}, E) \in \Im$, there exists $(\widetilde{G}, E)$ $\in \Im$ such that $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}, E) \sqsubset (\overline{\widetilde{G}, E}) \sqsubset (\widetilde{F}, E)$.*

Proof. Let $(X, \Im, E)$ be a neutrosophic soft P^3-space and $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{F}, E) \in \Im$. Since $(X, \Im, E)$ is a neutrosophic soft-P^3-space for the neutrosophic soft point $x^{\lambda(\alpha,\beta,\gamma)}$ and neutrosophic soft p-closed set $(\widetilde{F}, E)^c$, there exist $(\widetilde{G}^1, E)$, $(\widetilde{G}^2, E) \in \Im$ such that $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}^1, E)$, $(\widetilde{F}, E)^c \sqsubset (\widetilde{G}^2, E)$, and $(\widetilde{G}^1, E) \sqcap (\widetilde{G}^2, E) = 0^{(X,E)}$. thus, we have $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}^1, E)$ $\sqsubset (\widetilde{G}^2, E)^c \sqsubset (\widetilde{F}, E)$. Since $(\widetilde{G}^2, E)^c$ is a neutrosophic soft p-closed set, $(\overline{\widetilde{G}^1, E}) \sqsubset (\widetilde{G}^2, E)^c$.

Conversely , let $x^{\lambda(\alpha,\beta,\gamma)} \sqcap (\widetilde{H}, E) = 0^{(X,E)}$ and $(\widetilde{H}, E)$ be a neutrosophic soft p-closed set. Thus, $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{H}, E)^c$ and from the condition of the theorem, we have $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}, E) \sqsubset \overline{(\widetilde{G}, E)} \sqsubset (\widetilde{H} \cdot E)^c$.

Then $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{G}, E)$, $(\widetilde{H}, E) \sqsubset (\overline{(\widetilde{G}, E)})^c$, and $(\widetilde{G}, E) \sqcap \overline{(\widetilde{G}, E)}^c = 0^{(X,E)}$ are satisfied, i.e. $(X, \Im, E)$ is a neutrosophic soft-P^3-space. □

Definition 4.11. A neutrosophic soft topological space $(X, \Im, E)$ over X is called a neutrosophic soft p-normal space if for every pair of disjoint neutrosophic soft b-closed set $(\widetilde{F}^1, E)$, $(\widetilde{F}^2, E)$, there exists disjoint neutrosophic soft p-open sets $(\widetilde{G}^1, E)$, $(\widetilde{G}^2, E)$ such that $(\widetilde{F}^1, E) \sqsubset (\widetilde{G}^1, E)$ and $(\widetilde{F}^2, E) \sqsubset (\widetilde{G}^2, E)$.

$(X, \Im, E)$ is said to be a neutrosophic soft b-T^4-space if it is both a neutrosophic soft p-normal and neutrosophic soft-P^1-space.

Theorem 4.12. *Let $(X, \Im, E)$ be a neutrosophic soft topological space over X . Then $(X, \Im, E)$ is a neutrosophic soft-P^4-space if and only if, for each neutrosophic soft p-closed set $(\widetilde{F}, E)$ and neutrosophic soft p-open set $(\widetilde{G}, E)$ with $(\widetilde{F}, E) \sqsubset (\widetilde{G}, E)$, there exists a neutrosophic soft p-open set $(\widetilde{D}, E)$ such that*

$(\widetilde{F}, E) \sqsubset (\widetilde{D}, E) \sqsubset \overline{(\widetilde{D}, E)} \sqsubset (\widetilde{G}, E).$

Proof. Let $(X, \Im, E)$ be a neutrosophic soft-P^4-space, $(\widetilde{F}, E)$ be a neutrosophic soft p-closed set and $(\widetilde{F}, E) \sqsubset (\widetilde{G}, E) \in \Im$. Then $(\widetilde{G}, E)^c$ is a neutrosophic soft p-closed set and $(\widetilde{F}, E) \sqcap (\widetilde{G}, E)^c = 0^{(X,E)}$. Since $(X, \Im, E)$ is a neutrosophic soft-P^4-space, there exist neutrosophic soft p-open sets $(\widetilde{D}^1, E)$ and $(\widetilde{D}^2, E)$ such that $(\widetilde{F}, E) \sqsubset (\widetilde{D}^1, E)$, $(\widetilde{G}, E)^c \sqsubset (\widetilde{D}^2, E)$, and $(\widetilde{D}^1, E) \sqcap (\widetilde{D}^2, E) = 0^{(X,E)}$. This implies that

$(\widetilde{F}, E) \sqsubset (\widetilde{D}^1, E) \sqsubset (\widetilde{D}^2, E)^c \sqsubset (\widetilde{G}, E).$

$(\widetilde{D}^2, E)^c$ is a neutrosophic soft p-closed set and $\overline{(\widetilde{D}^1, E)} \sqsubset (\widetilde{D}^2, E)^c$ is satisfied. Thus,

$(\widetilde{F}, E) \sqsubset (\widetilde{D}^1, E) \sqsubset \overline{(\widetilde{D}^1, E)} \sqsubset (\widetilde{G}, E)$

is obtained.

Conversely, let $(\widetilde{F}^1, E)$, $(\widetilde{F}^2, E)$ be two disjoint neutrosophic soft p-closed sets. Then $(\widetilde{F}^1, E) \sqsubset (\widetilde{F}^2, E)^c$. From the condition of theorem, there exists a neutrosophic soft p-open set $(\widetilde{D}, E)$ such that

$(\widetilde{F}^1, E) \sqsubset (\widetilde{D}, E) \sqsubset \overline{(\widetilde{D}^1, E)} \sqsubset (\widetilde{F}^2, E)^c$.

Thus, $(\widetilde{D}, E)$, $(\overline{(\widetilde{D}, E)})^c$ are neutrosophic soft p-open sets and $(\widetilde{F}^1, E) \sqsubset (\widetilde{D}, E)$, $(\widetilde{F}^2, E) \sqsubset (\overline{(\widetilde{D}, E)})^c$, and $(\widetilde{D}, E) \sqcap (\overline{(\widetilde{D}, E)})^c = 0^{(X,E)}$ are obtained. Hence, $(X, \Im, E)$ is a neutrosophic soft-P^4-space. □

Definition 4.13. Let $(X, \Im, E)$ be a neutrosophic soft topological space over X and $(\widetilde{F}, E)$ be an arbitrary neutrosophic soft set. Then $\Im^{(\widetilde{F},E)} = \{(\widetilde{F}, E) \sqcap (\widetilde{H}, E) : (\widetilde{H}, E) \in \Im\}$ is said to be neutrosophic soft topology on $(\widetilde{F}, E)$ and $((\widetilde{F}, E), \Im^{(\widetilde{F},E)}, E)$ is called a neutrosophic soft topological subspace of $(X, \Im, E)$.

Theorem 4.14. *Let $(X, \Im, E)$ be a neutrosophic soft topological space over X. If $(X, \Im, E)$ is a neutrosophic soft-P^i-space, then the neutrosophic soft topological subspace $((\widetilde{F}, E), \Im^{(\widetilde{F},E)}, E)$ is a neutrosophic soft-P^i-space for i = 0, 1, 2, 3.*

Proof. Let $x^{\lambda(\alpha,\beta,\gamma)}$, $y^{\lambda'(\alpha',\beta',\gamma')} \in ((\widetilde{F}, E), \Im^{(\widetilde{F},E)}, E)$ such that $x^{\lambda(\alpha,\beta,\gamma)} \sqcap y^{\lambda'(\alpha',\beta',\gamma')} = 0^{(X,E)}$. Thus , there exist neutrosophic soft p-open set $(\widetilde{F}^1, E)$ and $(\widetilde{F}^2, E)$ satisfying the conditions of neutrosophic soft -P^i-space such that $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{F}^1, E)$, $y^{\lambda'(\alpha',\beta',\gamma')} \in (\widetilde{F}^2, E)$. Then $x^{\lambda(\alpha,\beta,\gamma)} \in (\widetilde{F}^1, E) \sqcap (\widetilde{F}, E)$ and $y^{\lambda'(\alpha',\beta',\gamma')} \in (\widetilde{F}^2, E) \sqcap (\widetilde{F}, E)$. Also, the neutrosophic soft p-open set $(\widetilde{F}^1, E) \sqcap (\widetilde{F}, E)$, $(\widetilde{F}^2, E) \sqcap (\widetilde{F}, E)$ in $\Im^{(\widetilde{F},E)}$ satisfy the conditions of neutrosophic soft-P^i-space for i = 0, 1, 2, 3. □

Theorem 4.15. *Let $(X, \Im, E)$ be a neutrosophic soft topological space over X. If $(X, \Im, E)$ is a neutrosophic soft-P^4-space and $(\widetilde{F}, E)$ is a neutrosophic soft p-closed set in $(X, \Im, E)$, then $((\widetilde{F}, E), \Im^{(\widetilde{F},E)}, E)$ is a neutrosophic soft -P^4-space.*

Proof. Let $(X, \Im, E)$ be a neutrosophic soft P^4-space and $(\widetilde{F}, E)$ be a neutrosophic soft p-closed set in $(X, \Im, E)$. Let $(\widetilde{F}^1, E)$ and $(\widetilde{F}^2, E)$ be two neutrosophic soft p-closed sets in $((\widetilde{F}, E), \Im^{(\widetilde{F},E)}, E)$ such that $(\widetilde{F}^1, E) \sqcap (\widetilde{F}^2, E) = 0^{(X,E)}$. When $(\widetilde{F}, E)$ is a neutrosophic soft p-closed set in $(X, \Im, E)$, $(\widetilde{F}^1, E)$ and $(\widetilde{F}^2, E)$ are neutrosophic soft p-closed sets in $(X, \Im, E)$. Since $(X, \Im, E)$ is a neutrosophic soft-P^4-space, there exist neutrosophic soft p-open sets $(\widetilde{G}^1, E)$ and $(\widetilde{G}^2, E)$ such that $(\widetilde{F}^1, E) \sqsubset (\widetilde{G}^1, E)$, $(\widetilde{F}^2, E) \sqsubset (\widetilde{G}^2, E)$ and $(\widetilde{G}^1, E)$ $\sqcap$ $(\widetilde{G}^2, E) = 0^{(X,E)}$. Then $(\widetilde{F}^1, E) = (\widetilde{G}^1, E) \sqcap (\widetilde{F}, E)$, $(\widetilde{F}^2, E) = (\widetilde{G}^2, E) \sqcap (\widetilde{F}, E)$ and $((\widetilde{G}^1, E) \sqcap (\widetilde{F}, E)) \sqcap ((\widetilde{G}^2, E) \sqcap (\widetilde{F}, E)) = 0^{(X,E)}$. This implies that $((\widetilde{F}, E), \Im^{(\widetilde{F},E)}, E)$ is a neutrosophic soft-P^4-space. □

5. **Conclusion**

Neutrosophic soft p-separation structures are the most imperative and fascinating notions in neutrosophic soft topology.We have introduced neutrosophic soft p-separation axioms in neutrosophic soft topological structures with respect to soft points, which are defined over an initial universe of discourse with a fixed set of parameters(data,decision variables). We further investigated and scrutinized some essential features of the initiated neutrosophic soft p-separation structures. It is supposed that these results will be very very useful for future studies on neutrosophic soft topology to carry out a general framework for practical applications. Applications of neutrosophic soft p-separation structures in neutrosophic soft topological spaces can be traced out in decision making problems.

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Interval Valued Neutrosophic Topological Spaces

T. Nanthini [1,*] and A. Pushpalatha [2]

[1] Research Scholar, Department of Mathematics, Government Arts College, Udumalpet-642 126, Tamil Nadu, India, Email: tnanthinimaths@gmail.com
[2] Assistant Professor, Department of Mathematics, Government Arts College, Udumalpet-642 126, Tamil Nadu, India, Email:velu_pushpa@yahoo.co.in

* Correspondence: tnanthinimaths@gmail.com

Abstract: Within this paper, we present and research the definition of interval valued neutrosophic topological space along with interval valued neutrosophic finer and interval valued neutrosophic coarser topologies. We also describe interval valued neutrosophic interior and closer of an interval valued neutrosophic set. Interval valued neutrosophic subspace topology also studied. Some examples and theorems are presented concerning this concept.

Keywords: Interval valued neutrosophic topology, Interval valued neutrosophic subspace topology

1. Introduction

The notion of fuzzy set has invaded almost all branches of mathematics since its introduction by Zadeh[20]. Fuzzy sets and fuzzy logic has been applied in many real applications to handle uncertaintely fuzzy set theory is very successful in handling uncertainties arising from vagueness or partial belongingness of an element in a set, it cannot model all type of uncertainties pre – veiling in different real physical problems such as problems involving incomplete information. Turksen [18] introducted the idea of interval valued fuzzy sets.

Later, Atanassov[10] introduced the concept generalization of fuzzy set, which is known as intuitionistic fuzzy sets. Intuitionistic fuzzy sets take into account both the degree of membership and non – membership. Further, intuitionistic fuzzy sets were extended to the interval valued intuitionistic fuzzy sets[11]. The interval valued intuitionistic fuzzy set uses a pair of interval $[t^-, t^+], 0 \leq t^- \leq t^+ \leq 1$ and $[f^-, f^+]$, $0 \leq f^- \leq f^+ \leq 1$ with $t^+ + f^+ \leq 1$, to describe the degree of true belief and false belief. Because of the restriction that $t^+ + f^+ \leq 1$, intuitionistic fuzzy sets and interval valued intuitionistic fuzzy sets can only handle incomplete information not the indeterminate information and inconsistent information which exists commonly in belief systems.

As a generalization of fuzzy set and intuitionistic fuzzy set, neutrosophic set have been introduced and developed by F. Smaramdache[15,16 & 17]. It is a logic in which each proposition is calculated to have degree of truth(T), a degree of indeterminacy(I) and a degree of falsity(F). Smarandache's neutrosophic concept have wide range of real applications for many fields of

[1,2,3,4,5,6,7 & 8] information system, computer science, artificial intelligence, applied mathematics, decision making, mechanics, electrical and electronics, medicine and management science etc.

Salama, Albloe[14] proposed the concept of neutrosophic topological space. Later, Wang, Smarandache, Zhang and Sunderraman introduced the notion of interval valued neutrosophic set[19]. An interval valued neutrosophic set A defined on X, $x = x(T, I, F) \in A$ with T, I and F being the subinterval of [0,1]. Lupianez discusses the relation between interval value neutrosophic sets and topology [12]

The purpose of this article is to propose the idea of interval valued neutrosophic topological space and discuss the some of the basic properties.

2. Preliminaries

Definition 2.1[19] Let X be a space of points (objects), with a generic element in X denoted by x. An interval valued neutrosophic set(INS) A in X is characterized by truth – membership function T_A, indeterminacy – membership function I_A and falsity – membership function F_A. For each point x in X, $T_A(x), I_A(x), F_A(x) \subseteq [0,1]$.

Example 2.2[19] Suppose, $X = \{x_1, x_2, x_3\}$. The strength is x_1, the trust is x_2 and the price is x_3. The x_1, x_2 and x_3 values are given in $[0,1]$. They're obtained from some domain experts ' questionnaire, their choice could be degree of goodness, degree of indeterminacy, and degree of poorness. A and B are the interval neutrosophic sets of X define by $A =< \frac{[0.2,0.4],[0.3,0.5],[0.3,0.5]}{x_1}, \frac{[0.5,0.7],[0,0.2],[0.2,0.3]}{x_2}, \frac{[0.6,0.8],[0.2,0.3],[0.2,0.3]}{x_3} >$ $B =< \frac{[0.5,0.7],[0.1,0.3],[0.1,0.3]}{x_1}, \frac{[0.2,0.3],[0.2,0.4],[0.5,0.8]}{x_2}, \frac{[0.4,0.6],[0,0.1],[0.3,0.4]}{x_3} >$

Definition 2.3[19] An interval neutrosophic set A is empty $if\ and\ only\ if$ its $\inf T_A(x) = \sup T_A(x) = 0$, $\inf I_A(x) = \sup I_A(x) = 1$ and $\inf F_A(x) = \sup F_A(x) = 0$, for all x in X.

Definition 2.4(Containment) [19] An interval neutrosophic set A is contained in the other interval neutrosophic set B, $A \subseteq B$, $if\ and\ only\ if$

$$\inf T_A(x) \leq \inf T_B(x), \sup T_A(x) \leq \sup T_B(x)$$
$$\inf I_A(x) \geq \inf I_B(x), \sup I_A(x) \geq \sup T_B(x)$$
$$\inf F_A(x) \geq \inf F_B(x), \sup F_A(x) \geq \sup F_B(x)$$

for all x in X.

Definition 2.5[19] Two interval neutrosophic sets A and B are equal, written as $A = B, if\ and\ only\ if\ A \subseteq B$ and $B \subseteq A$. Let $0_N =< 0,1,1 >$ and $1_N =< 1,0,0 >$.

Definition 2.6[19] The complement of an interval neutrosophic set A is denoted by $\bar{A}$ and is defined by $T_{\bar{A}}(x) = F_A(x)$; $\inf I_{\bar{A}}(x) = 1 - \sup I_A(x)$;$\sup I_{\bar{A}}(x) = 1 - \inf I_A(x)$; $F_{\bar{A}}(x) = T_A(x)$ for all x in X.

Example 2.7[19] Let A be the interval neutrosophic set defined in Example 2.3, then $\bar{A} =< \frac{[0.3,0.5],[0.5,0.7],[0.3,0.4]}{x_1}, \frac{[0.2,0.3],[0.8,0],[0.5,0.7]}{x_2}, \frac{[0.2,0.3],[0.7,0.8],[0.6,0.8]}{x_3} >$

Definition 2.8 (Intersection) [19] The intersection of two interval neutrosophic sets A and B is an interval neutrosophic set $C = A \cap B$, whose truth-membership, indeterminacy – membership and false – membership are related to those of A and B by

$$\inf T_C(x) = \min(\inf T_A(x), \inf T_B(x)), \quad \sup T_C(x) = \min(\sup T_A(x), \sup T_B(x))$$
$$\inf I_C(x) = \max(\inf I_A(x), \inf I_B(x)), \quad \sup T_C(x) = \max(\sup I_A(x), \sup I_B(x))$$
$$\inf F_C(x) = \max(\inf F_A(x), \inf F_B(x)), \quad \sup T_C(x) = \max(\sup F_A(x), \sup F_B(x))$$

for all x in X.

Example 2.9[19] Let A and B be the interval neutrosophic sets defined in Example 2.3, then $A \cap B = < \frac{[0.2,0.4],[0.3,0.5],[0.3,0.5]}{x_1}, \frac{[0.2,0.3],[0.2,0.4],[0.5,0.8]}{x_2}, \frac{[0.4,0.6],[0.2,0.3],[0.3,0.4]}{x_3} >$.

Theorem 2.10[19] $A \cap B$ is the largest interval neutrosophic set contained in both A and B.

Definition 2.11(Union) [19] The union of two interval neutrosophic sets A and B is an interval neutrosophic set C, written as $C = A \cup B$, whose truth – membership, indeterminacy – membership and false membership are related to those of A and B by

$$\inf T_C(x) = \max(\inf T_A(x), \inf T_B(x)), \quad \sup T_C(x) = \max(\sup T_A(x), \sup T_B(x))$$
$$\inf I_C(x) = \min(\inf I_A(x), \inf I_B(x)), \quad \sup T_C(x) = \min(\sup I_A(x), \sup I_B(x))$$
$$\inf F_C(x) = \min(\inf F_A(x), \inf F_B(x)), \quad \sup T_C(x) = \min(\sup F_A(x), \sup F_B(x))$$

for all x in X.

Example 2.12[19] Let A and B be the interval neutrosophic sets defined in Example 2.3, then $A \cup B =< \frac{[0.5,0.7],[0.1,0.3],[0.1,0.3]}{x_1}, \frac{[0.5,0.7],[0,0.2],[0.2,0.3]}{x_2}, \frac{[0.6,0.8],[0,0.1],[0.2,0.3]}{x_3} >$.

Theorem 2.13[19] $A \cup B$ is the smallest interval neutrosophic set containing both A and B.

3. Interval Valued Neutrosophic Topological Spaces

With some examples and results, we give the concept of interval valued neutrosophic topologi cal spaces.

Definition 3.1 An interval valued neutrosophic topological space of interval valued neutrosophic set (In short IVN topological space) is a pair (X, τ_N) where X is a nonempty set and τ_N is a family of IVN sets on X satisfying the following axioms:

1. $0_N, 1_N \in \tau_N$
2. $A, B \in \tau_N \Rightarrow A \cap B \in \tau_N$
3. $A_i \in \tau_N, i \in I \Rightarrow \cup_{i \in I} A_i \in \tau_N$

τ_N is called an interval valued neutrosophic topology on X. τ_N members are called interval valued neutrosophic open sets (In Short IVN open sets).

Example 3.2 Assume that $X = \{a, b\}$. Here a is denoted by quality of Computers, b is denoted by Price of Computers. The value of a and b are in [0,1]. These are collected from some domain expects questionnaire; their choices could be degree of excellence, degree of indeterminacy, degree of poorness. The IVN set are

$0_N = \langle [0,0],[1,1],[1,1] \rangle$, $1_N = \langle [1,1],[0,0],[0,0] \rangle$, $A = \langle \frac{([0.1,0.4],[0.2,0.7],[0.4,0.6])}{a}, \frac{([0.6,0.8],[0.2,0.3],[0.2,0.3])}{b} \rangle$, $B = \langle \frac{([0.1,0.3],[0.3,0.8],[0.5,0.8])}{a}, \frac{([0.2,0.7],[0.4,0.8],[0.3,0.7])}{b} \rangle$,. $\tau_N = \{0_N, 1_N, A, B\}$ is called an IVN topology on X. (X, τ_N) is called an $IVNTS$.

Example3.3 Let $X = \{a, b\}$ and the IVN sets are

$C = \langle \frac{([0.4,0.7],[0.5,0.7],[0.4,0.9])}{a}, \frac{([0.2,0.3],[0.4,0.5],[0.7,0.9])}{b} \rangle$, $D = \langle \frac{([0.5,0.8],[0.3,0.5],[0.2,0.7])}{a}, \frac{([0.5,0.7],[0.1,0.5],[0.3,0.7])}{b} \rangle$.

$\tau_N = \{0_N, 1_N, C, D\}$ is called an IVN topology on X. (X, τ_N) is called an IVN topological space.

Theorem 3.4 Let $\{\tau_{N_i} : i \in I\}$ be a family of IVN topologies of IVN sets on X. Then $\cap_i \{\tau_{N_i} : i \in I\}$ is also an IVN topology of IVN sets on X.

Proof: (i) $0_N, 1_N \in \tau_{N_i}$ for each $i \in I$, Hence $0_N, 1_N \in \bigcap_{i \in I} \tau_{N_i}$.(ii) Let $\{A_i : i \in I\}$ be a arbitrary family of IVN sets where $A_i \in \bigcap_{i \in I} \tau_{N_i}$ for each $i \in I$.Then for each $i \in I$. $A_i \in \tau_{N_i}$ for $i \in I$ and since for each $i \in I$, τ_{N_i} is a IVN topology, Therefore $\bigcup_{i \in I} A_i \in \tau_{N_i}$ for each $i \in I$. Hence $\bigcup_{i \in I} A_i \in \bigcap_{i \in I} \tau_{N_i}$

But union of IVN topologies as seen in the following example need not be an IVN topology.

Example: 3.5 In example 3.2 and 3.3 the families $\tau_{N_1} = \{0_N, 1_N, A, B\}$ and $\tau_{N_2} = \{0_N, 1_N, C, D\}$ are IVN topologies in X. For X, however their union $\tau_{N_1} \cup \tau_{N_2} = \{0_N, 1_N, A, B, C, D\}$ is not a IVN topology.

Definition 3.6 Let (X, τ_N) be an IVN topological space. An IVN set A of X is called an interval valued neutrosophic closed set (in short IVN -closed set) if its complement A^c is an IVN open set in τ_N.

Example 3.7 Let us consider the Example 3.2, the IVN closed sets in (X, τ_N) are $A^c = \langle \frac{([0.4,0.6],[0.3,0.8],[0.1,0.4])}{a}, \frac{([0.2,0.3],[0.7,0.8],[0.6,0.8])}{b} \rangle$, $B^c = \langle \frac{([0.5,0.8],[0.2,0.7],[0.1,0.3])}{a}, \frac{([0.3,0.7],[0.2,0.6],[0.2,0.7])}{b} \rangle$, $0_N^c = 1_N$ and $1_N^c = 0_N$ are the IVN – closed sets in (X, τ_N).

Theorem 3.8 Let (X, τ_N) be an IVN topological space. Then (i) $0_N, 1_N$ are IVN – closed sets. (ii) Arbitrary intersection of IVN – closed sets is IVN – closed set. (iii) Finite union of IVN – closed sets is IVN – closed set.

Proof: (i) since $0_N, 1_N \in \tau_N$, $0_N^c = 1_N$ and $1_N^c = 0_N$, therefore 0_N^c and 1_N^c are IVN – closed sets. (ii) Let $\{A_i : i \in I\}$be an arbitrary family of IVN – closed sets in (X, τ_N) and let $A = \bigcap_{i \in I} A_i$ Now $A^c = \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} (A_i)^c$ and $A^c \in \tau_N$ for each $i \in I$, hence $\bigcup_{i \in I} (A_i)^c \in \tau_N$, therefore $A^c \in \tau_N$. Thus A is an IVN– closed set. (iii) Let $\{A_k : k = 1,2, \ldots \ldots \ldots nn\}$ be a family of IVN – closed set in

(X,τ_N) and let $G=\bigcup_{k=1}^{n} A_k$. Now $(G)^c=\left(\bigcup_{k=1}^{n} A_k\right)^c=\bigcap_{k=1}^{n} A_k^c$ and $(A_k)^c\in\tau_N$ for $k=1,2,.........n$

, so $\bigcap_{k=1}^{n} A_k^c\in\tau_N$. Hence $G^c\in\tau_N$, thus G is IVN – closed set.

Definition 3.9 Let both (X,τ_{N_1}) and (X,τ_{N_2}) be two $IVNTS$. If each $A\in\tau_{N_2}$ implies $A\in\tau_{N_1}$, then $\tau_{N_1}$is called interval valued neutrosophic finer topology than τ_{N_2} and τ_{N_2} is called interval valued neutrosophic coarser topology than τ_{N_1}

Example 3.10 Let $X=\{a,b\}$ and IVN sets are $A=\langle\frac{([0.5,0.7],[0.3,0.6],[0.2,0.8])}{a},\frac{([0.4,0.6],[0.3,0.5],[0.4,0.7])}{b}\rangle$, $B=\langle\frac{([0.3,0.7],[0.4,0.6],[0.3,0.8])}{a},\frac{([0.1,0.7],[0.3,0.8],[0.2,0.6])}{b}\rangle$, $C=\langle\frac{([0.5,0.7],[0.3,0.6],[0.2,0.8])}{a},\frac{([0.4,0.7],[0.3,0.5],[0.2,0.6])}{b}\rangle$, $D=\langle\frac{([0.3,0.7],[0.4,0.6],[0.3,0.8])}{a},\frac{([0.1,0.7],[0.3,0.8],[0.4,0.7])}{b}\rangle$. Let $\tau_{N_1}=\{0_N,1_N,A,B,C,D\}$ and $\tau_{N_2}=\{0_N,1_N,A,C\}$ be an IVN topologies on X and let (X,τ_{N_1}) and (X,τ_{N_2})be a IVN topological spaces. If τ_{N_1} is IVN finer topology than τ_{N_2} andτ_{N_2} is IVN coarser topology than τ_{N_1}

Definition 3.11 Let (X,τ_N) be a IVN topological space. A subcollection $\mathfrak{B}$ of τ_N is said to be base of τ_N if every element of τ_N can be expressed as the arbitray IVN union of some elements of $\mathfrak{B}$, then $\mathfrak{B}$ is called an IVN basis for the IVN topology τ_N.

Example 3.12 In Example 3.10, for the IVN topology $\tau_{N_1}=\{0_N,1_N,A,B,C,D\}$. The sub collection $\mathfrak{B}=\{0_N,1_N,A,B,C\}$ of $P(X)$ is a IVN basis for the IVN topology τ_{N_1}.

Definition 3.13 Let (X,τ_N) be a IVN topological space and $A\in IVNs(X)$, the interior and closure of A is denoted by $IVN\ Int(A)$ and $IVN\ Cl(A)$ are defined as
$IVN\ Int(A)=\cup\{G\in\tau_N: G\subseteq A\}$, $IVN\ Cl(A)=\cap\{K\in\tau_N^c: A\subseteq K\}$

Example 3.14 Let us take an Example 3.3 and consider an IVN set
$E=\langle\frac{([0.4,0.6],[0.4,0.7],[0.2,0.7])}{a},\frac{([0.3,0.5],[0.3,0.6],[0.3,0.5])}{b}\rangle$. Now $IVN\ Int(E)=0_N$ and $IVN\ Cl(E)=1_N$.

Theorem 3.15 Let (X,τ_N) be a IVN topological space and $A,B\in IVNs(X)$ then the following properties holds:

(i) $IVN\ Int(A)\subseteq A$

(ii) $A\subseteq B\Rightarrow IVN\ Int(A)\subseteq IVN\ Int(B)$

(iii) $IVN\ Int(A)\in\tau_N$

(iv) $A\in\tau_N\ iff\ IVN\ Int(A)=A$

(v) $IVN\ Int(IVN\ Int(A))=IVN\ Int(A)$

(vi) $IVN\ Int(0_N)=0_N,\ IVN\ Int(1_N)=1_N$

Proof:

(i) Straight forward.

(ii) $A \subseteq B \Rightarrow$ All of the IVN open sets in A that are also in B. Both IVN open sets included in A also included in B. $ie., \{K \in \tau_N : K \subseteq A\} \subseteq \{G \in \tau_N : G \subseteq B\}$. $ie., \cup \{K \in \tau_N : K \subseteq A\} \subseteq \cup \{G \in \tau_N : G \subseteq B\}$. $ie., IVN\ Int(A) \subseteq IVNInt(B)$.

(iii) $IVN\ Int(A) = \cup \{K \in \tau_N : K \subseteq A\}$. It is clear that $\cup \{K \in \tau_N : K \subseteq A\} \in \tau_N$. So, $IVN\ Int(A) \in \tau_N$.

(iv) Let $A \in \tau_N$, then by(i), $IVN\ Int(A) \subseteq A$. Now since $A \in \tau_N$ and $IVN\ Int(A) \subseteq A$. Therefore $A \subseteq \cup \{G \in \tau_N : G \subseteq A\} = IVN\ Int(A)$, $A \subseteq INV\ Int(A)$. Thus $IVN\ Int(A) = A$. Conversely, let $IVN\ Int(A) = A$. Since by (iii), $IVN\ Int(A) \in \tau_N$. Therefore $A \in \tau_N$.

(v) By (iii), $IVN\ Int(A) \in \tau_N$. Therefore by (iv), $IVN\ Int(IVN\ Int(A)) = IVN\ Int(A)$.

(vi) We know that $0_N, 1_N \in \tau_N$, by (iv), $IVN\ Int(0_N) = 0_N$, $IVN\ Int(1_N) = 1_N$.

Theorem 3.16 Let (X, τ_N) be a $IVNTS$ and $A, B \in IVNs(X)$ then possess the following properties:

(i) $A \subseteq IVN\ Cl(A)$

(ii) $A \subseteq B \Rightarrow IVN\ Cl(A) \subseteq IVN\ Cl(B)$

(iii) $(IVN\ Cl(A))^c \in \tau_N$

(iv) $A^c \in \tau_N\ iff\ IVN\ Cl(A) = A$

(v) $IVN\ Cl(IVN\ Cl(A)) = IVN\ Cl(A)$

(vi) $IVN\ Cl(0_N) = 0_N$, $IVN\ Cl(1_N) = 1_N$

Proof:

Straight forward.

Theorem 3.17 Let (X, τ_N) be a IVN topological space and $A, B \in IVNs(X)$then hold the following properties:

(i) $IVN\ Int(A \cap B) = IVN\ Int(A) \cap IVN\ Int(B)$

(ii) $IVN\ Int(A \cup B) \supseteq IVNInt(A) \cup IVNInt(B)$

(iii) $IVN\ Cl(A \cup B) = IVN\ Cl(A) \cup IVNInt(B)$

(iv) $IVN\ Cl(A \cap B) \subseteq IVN\ Cl(A) \cap IVN\ Int(B)$

(v) $(IVN\ Int(A))^c = IVN\ Cl\ (A^c)$

(vi) $(IVN\ Cl(A))^c = IVN\ Int(A^c)$

Proof:

(i) By Theorem 3.15(i), $IVN\ Int\ (A) \subseteq A$ and $IVN\ Int(B) \subseteq B$. Thus $IVN\ Int(A) \cap IVN\ Int(B) \subseteq A \cap B$. Hence $IVN\ Int(A) \cap IVN\ Int(B) \subseteq IVNInt(A \cap B)$ -----------(1)

Again since $A \cap B \subseteq A$, by Theorem 3.15(ii). $IVN\ Int(A \cap B) \subseteq IVNInt(A)$. Similarly $IVN\ Int(A \cap B) \subseteq IVNInt(B)$.

Hence $IVN\ Int(A \cap B) \subseteq IVNInt(A) \cap IVNInt(B)$ --------(2) from (1) and (2) we get, $IVN\ Int(A \cap B) = IVNInt(A) \cap IVN\ Int(B)$.

(ii) Since $A \subseteq A \cup B$. $IVN\ Int(A) \subseteq IVNInt(A \cup B)$ by Theorem 3.15(ii). Similarly $IVN\ Int(B) \subseteq IVNInt(A \cup B)$. Hence $IVN\ Int(A) \cup IVN\ Int(B) \subseteq IVNInt(A \cup B)$.

(iii) By Theorem 3.16(i), $A \subseteq IVN\ Cl(A)$ and $B \subseteq IVN\ Cl(B)$. Thus $A \cup B \subseteq IVN\ Cl(A) \cup IVNCl(B)$, $IVN\ Cl(A \cup B) \subseteq IVN\ Cl(A) \cup IVNCl(B)$-----------(1)

Again since $A \subseteq A \cup B$, by Theorem 3.16(ii). $IVN\ Cl(A) \subseteq IVNCl(A \cup B)$. Similarly $IVN\ Cl(B) \subseteq IVNCl(A \cup B)$. Hence $IVN\ Cl(A) \cup IVN\ Cl(B) \subseteq IVNCl(A \cup B)$------(2) from (1) and (2) we get $IVN\ Cl(A) \cup IVN\ Cl(B) = IVNCl(A \cup B)$.

(iv) Since $A \cap B \subseteq A$, $IVN\ Cl(A \cap B) \subseteq IVN\ Cl(A)$ by Theorem 3.16(ii), Similarly, $IVN\ Cl(A \cap B) \subseteq IVN\ Cl(B)$. Hence $IVN\ Cl(A \cap B) \subseteq IVN\ Cl(A) \cap IVNCl(B)$.

(v) $\{IVN\ Int(A)\}^c = [\cup \{G \in \tau_N : G \subseteq A\}]^c = \cap \{G \in \tau_N^c : A^c \subseteq G\}$,
$\{IVN\ Int(A)\}^c = IVN\ Cl(A)^c$.

(vi) $\{IVN\ Cl(A)\}^c = [\cap \{G \in \tau_N^c : A^c \subseteq G\}]^c = \cup \{G \in \tau_N : G \subseteq A\}$,
$\{IVN\ Cl(A)\}^c = IVN\ Int(A)^c$.

In theorem 3.17((ii) and (iv)), the equality does not hold. Let us display this by an example below

Example 3.18 Let $X = \{a, b\}$ and the IVN sets are $0_N = < \frac{[0,0],[0,0],[1,1]}{a}, \frac{[0,0],[0,0],[1,1]}{b} >$;

$1_N = < \frac{[1,1],[0,0],[0,0]}{a}, \frac{[1,1],[0,0],[0,0]}{b} >$; $A = < \frac{[0.1,0.4],[0.2,0.7],[0.4,0.6]}{a}, \frac{[0.6,0.8],[0.2,0.3],[0.2,0.3]}{b} >$;

$B = < \frac{[0.1,0.3],[0.3,0.8],[0.5,0.8]}{a}, \frac{[0.2,0.7],[0.4,0.8],[0.3,0.7]}{b} >$, $\tau_N = \{0_N, 1_N, A, B\}$ is an IVN topology on X. Let us consider two IVN sets $C = < \frac{[0.1,0.4],[0.3,0.7],[0.5,0.6]}{a}, \frac{[0.4,0.8],[0.2,0.3],[0.2,0.3]}{b} >$ and $D = <$ $\frac{[0,0.3],[0.2,0.8],[0.4,0.9]}{a}, \frac{[0.6,0.7],[0.3,0.6],[0.2,0.5]}{b} >$; Now $C \cup D = < \frac{[0.1,0.4],[0.2,0.7],[0.4,0.6]}{a}, \frac{[0.6,0.8],[0.2,0.3],[0.2,0.3]}{b} > =$ A; $IVN\ Int(C \cup D) = IVN\ Int(A) = A$; $IVN\ Int(C) = 0_N$, $IVN\ Int(D) = 0_N$, $IVN\ Int(C) \cup IVN\ Int(D) = 0_N$;

Therefore $IVN\ Int(C \cup D) \neq IVN\ Int(C) \cup IVN\ Int(D)$.

By Theorem 3.17(v), $IVN\ Cl(C)^c = (IVN\ Int(C))^c = (0_N)^c = 1_N$, $IVN\ Cl(D)^c = (IVN\ Int(D))^c = (0_N)^c = 1_N$, $IVN\ Int(C) \cap IVN\ Int(D) = 1_N$; $IVN\ Cl(C^c \cap D^c) = IVN\ Cl((C \cup D)^c) = (IVN\ Int(C \cup D))^c = (IVN\ Int(A))^c = A^c$; $IVN\ Cl(C^c \cap D^c) \neq IVN\ Cl(C^c) \cup IVN\ Cl(D^c)$.

4. Interval Valued Neutrosophic Subspace Topology

In this section we present, along with some examples and findings, the definition of interval valued neutrosophic subspace topology.

Theorem 4.1 Let (X, τ_N) be a IVN topological space on X and $Y \in P(X)$. Then the collection $\tau_{NY} = \{Y \cap G : G \in \tau_N\}$ is a IVN topology on X.

Proof:

(i) Since $0_N, 1_N \in \tau_N$, therefore $Y \cap 0_N = 0_N \in \tau_{NY}$ and $Y \cap 1_N = Y \in \tau_{NY}$.

(ii) Let $Y_k \in \tau_{NY}, \forall\, k \in I$, then $Y_k = Y \cap G_k$ where $G_k \in \tau_N$ for each $k \in I$. Now

$$\bigcup_{k \in I} Y_k = \bigcup_{k \in I} (Y \cap G_k) = Y \cap \left(\bigcup_{k \in I} G_k \right) \in \tau_{NY}.$$ Since $\bigcup_{k \in I} G_k \in \tau_N$ as each $G_k \in \tau_N$.

(iii) Let $Y_1, Y_2 \in \tau_{NY}$, $Y_1 = Y \cap G_1$ and $Y_2 = Y \cap G_2$ where $G_1, G_2 \in \tau_N$ Now $Y_1 \cap Y_2 = (Y \cap G_1) \cap (Y \cap G_2) = Y \cap (G_1 \cap G_2) \in \tau_{NY}$, since $G_1 \cap G_2 \in \tau_N$ as $G_1, G_2 \in \tau_N$.

Definition 4.2 Let (X, τ_N) be an IVN topological space on X and Y is a interval values neutrosophic subset (In short IVN subset) of X, the collection $\tau_{NY} = \{Y \cap G: G \in \tau_N\}$ is called interval valued neutrosophic subspace (In short IVN subspace) of Y. Y is called IVN subspace of X.

Example 4.3 Let us consider the IVN topology $\tau_{N_1} = \{0_N, 1_N, A, B, C, D\}$ as in Example 3.10 and an IVN set $Y =< \frac{[0.4,0.6],[0.3,0.7],[0.1,0.5]}{a}, \frac{[0.5,0.9],[0.4,1],[0.2,0.6]}{b} >$, $0_N = Y \cap 0_N = 0_N$;

$G_1 = Y \cap A$, $G_1 =< \frac{[0.4,0.6],[0.3,0.7],[0.2,0.8]}{a}, \frac{[0.4,0.6],[0.4,1],[0.4,0.7]}{b} >$;

$G_2 = Y \cap B$, $G_2 =< \frac{[0.3,0.6],[0.4,0.7],[0.3,0.8]}{a}, \frac{[0.1,0.7],[0.4,1],[0.2,0.6]}{b} >$;

$G_3 = Y \cap C$, $G_3 =< \frac{[0.4,0.6],[0.3,0.7],[0.2,0.8]}{a}, \frac{[0.4,0.7],[0.4,1],[0.2,0.6]}{b} >$;

$G_4 = Y \cap D$, $G_4 =< \frac{[0.3,0.6],[0.4,0.7],[0.3,0.8]}{a}, \frac{[0.1,0.7],[0.4,1],[0.4,0.7]}{b} >$; Then $\tau_{NY} = \{0_N, 1_N, G_1, G_2, G_3\}$ is an IVN subspace topology for τ_{N_1} and τ_{NY} is called IVN subspace of (X, τ_{N_1}).

Theorem 4.4 Let (X, τ_N) be an IVN topological space, $\mathfrak{B}$ be an IVN basis for τ_N and Y is an IVN subset of X. Then the family $\mathfrak{B}_Y = \{Y \cap G: G \in \mathfrak{B}\}$ is an IVN basis for IVN subspace topology τ_{NY}.

Proof:

Let $U \in \tau_{NY}$ be arbitrary, then there exists an IVN set $G \in \tau_N$ such that $U = Y \cap G$. Since $\mathfrak{B}$ is an IVN basis for τ_N, therefore there exists a sub collection $\{\chi_i: i \in I\}$ of $\mathfrak{B}$ such that $G = \bigcup_{i \in I} \chi_i$. Now, $U = Y \cap G = \bigcup_{i \in I} \chi_i = \bigcup_{i \in I} (Y \cap \chi_i)$. Since $Y \cap \chi_i \in B_Y$, therefore B_Y is a IVN basis for an IVN subspace topology τ_{NY}.

5. Conclusion

The concept of interval valued neutrosophic topological space, interval valued neutrosophic interior and interval valued neutrosophic closure of an interval valued neutrosophic sets were introduced. An interval valued neutrosophic subspace topology of interval valued neutrosophic sets are also introduced. The newly introduced' Interval Valued Neutrophic Topological Spaces' is a stronger version of 'Neutrosophic Topological Spaces'.

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Arithmetic and Geometric Operators of Pentagonal Neutrosophic Number and its Application in Mobile Communication Service Based MCGDM Problem

Avishek Chakraborty [1,2*],Baisakhi Banik [3],Sankar Prasad Mondal[4] and Shariful Alam[2]

[1]Department of Basic Science, Narula Institute of Technology, Agarpara, Kolkata-700109, India.
Email- avishek.chakraborty@nit.ac.in
[2]Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah-711103, India.
Email- salam50in@yahoo.co.in
[3]Department of Basic Science, Techno Engineering College (Banipur), Habra, W.Bengal-743233, India.
Email-baisakhibanik14@gmail.com.
[4]Department of Applied Science, MaulanaAbulKalam Azad University of Technology, West Bengal, Haringhata-721249, Nadia,WestBengal, India. Email-sankar.res07@gmail.com
*Correspondence: avishek.chakraborty@nit.ac.in.

Abstract: In this paper, the theory of pentagonal neutrosophic number has been studied in a disjunctive frame of reference. Moreover, the dependency and independency of the membership functions for the pentagonal neutrosophic number are also classified here. Additionally, the development of a new score function and its computation have been formulated in distinct rational perspectives. Further, weighted arithmetic averaging operator and weighted geometric averaging operator in the pentagonal neutrosophic environment are introduced here using an influx of different logical & innovative thought. Also, a multi-criteria group decision-making problem (MCGDM) in a mobile communication system is formulated in this paper as an application in the pentagonal neutrosophic arena. Lastly, the sensitivity analysis portion reflects the variation of this noble work.

Keywords: Pentagonal neutrosophic number, Weighted arithmetic and geometric averaging operator, Score functions, MCGDM.

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1. Introduction

1.1 Neutrosophic Sets

Handling the notion of vagueness and uncertainty concepts, fuzzy set theory is a dominant field, was first presented by Zadeh [1] in his paper (1965).Vagueness theory has a salient feature for solving engineering and statistical problem very lucidly. It has a great impact on social-science, networking, decision making and numerous kinds of realistic problems. On the basis of ideas of Zadeh's research paper, Atanassov [2] invented the prodigious concept of intuitionistic fuzzy set where he precisely interpreted the idea of membership as well as non membership function very aptly. Further, researchers developed the formulation of triangular [3], trapezoidal [4], pentagonal [5] fuzzy numbers in uncertainty arena. Also, Liu & Yuan [6] established the concept of the triangular intuitionistic fuzzy set;Ye [7] put forth the basic idea of trapezoidal intuitionistic fuzzy set in the research field. Naturally, the question arises, how can we evolve the idea of uncertainty concepts in mathematical modelling? Researchers have invented disjunctive kinds of methodologies to define elaborately the concepts and have suggested some new kinds of ambivalent parameters. To deal with those kinds of problems, the decision-makers' choice varies in different areas. F. Smarandache [8] in 1998 generated the concept of a neutrosophic set having three different integrants namely, (i) truthiness, (ii) indeterminacies, and (iii) falseness. Each and every characteristic of the neutrosophic set are very pertinent factors in our real-life models. Later, Wang et al. [9] proceeded with the idea of a single typed neutrosophic set, which is very productive to sort out the solution of any complicated kind of problem. Recently, Chakraborty et al. [10, 11] conceptualized the dynamic idea of triangular and trapezoidal neutrosophic numbers in the research domain and applied it in different real-life problem. Also, Maity et al. [12] built the perception of ranking and defuzzification in a completely different type of attributes. To handle human decision making procedure on the basis of positive and negative sides, Bosc and Pivert [13] cultivated the notion of bipolarity. With that continuation, Lee [14] elucidated the perception of bipolar fuzzy set in their research article. Further, Kang and Kang [15] broadened this concept into semi-groups and group structures field. As research proceeded, Deli et al. [16] germinated the idea of a bipolar neutrosophic set and used it as an implication to a decision-making related problem. Broumi et al. [17] produced the idea of bipolar neutrosophic graph theory and, subsequently, Ali and Smarandache [18] put forth the concept of the uncertain complex neutrosophic set. Chakraborty [19] introduced the triangular bipolar number in different aspects. In succession; Wang et al. [20] also introduced the idea of operators in a bipolar neutrosophic set and applied it in a decision-making problem. The multi-criteria decision making (MCDM) problem is a supreme interest to the researchers who deal with the decision scientific analysis. Presently, it is more acceptable in such issues where a group of criteria is utilized. Such cases of problems relating to multi-criteria group decision making (MCGDM) have shown its fervent influence. Also MCDM has broad applications in disjunctive fields under various uncertainty contexts.We can find many applications and development of neutrosophic theory in multi-criteria decision making problem in the literature surveys presented in [21–25], graph theory [26-30], optimization techniques [31-33] etc. In this current era, Basset [34-40] presented some worthy articles related to neutrosophic sphere and applied it in many different well-known fields.Also, K.Mondal [41,42] successfully applied the notion of neutrosophic number

in faculty recruitment MCDM problem in education purpose. Recently, the viewpoint of plithogenic set is being constructed by Abdel [43] and it has an immense influential motivation in impreciseness field in various sphere of research field. Also, Chakraborty [44] developed the conception of cylindrical neutrosophic number is minimal tree problem.

Neutrosophic concept is very fruitful & vibrant in a realistic approach in the recent research field. R. Helen [45] first germinated the idea of the pentagonal fuzzy number then Christi [46] utilized the conception of pentagonal fuzzy number into pentagonal intuitionistic number and skillfully applied it to solve a transportation problem. Additionally, Chakraborty [47, 48] put forward the notion of pentagonal neutrosophicnumber and its different and disjunctive representation in transportation problem and graph-theoretical research arenas. Subsequently, Karaaslan [51-56] put forth some innovative idea on multi-attribute decision making in neutrosophic domain. Also, Karaaslan [57-61] presented the notion of soft set theory with the appropriate justification of neutrosophic fuzzy number. Recently, Broumi et.al [62-66] manifested the conception of the graph-theoretical shortest path problem under neutrosophic environment. Further, Broumi [67] implemented the concept of neutrosophic membership functions using MATLAB programming. A few works [68-71] are also established recently, based on impreciseness domain.

In this article, we mainly focus on the different representation of pentagonal neutrosophic number and its dependency, independency portions. We generate a new logical score function for crispification of pentagonal neutrosophic number. Additionally, we introduce two different logical operators namely i) pentagonal neutrosophic weighted arithmetic averaging operator (PNWAA), ii) pentagonal neutrosophic weighted geometric averaging operator (PNWGA) and established its theoretical developments along with its different properties. Also, we discussed the utility of these operators in real-life problems. Later, we consider a mobile communication based MCGDM problem in neutrosophic domain and solve it using the established two operators & score function.Sensitivity analysis of this problem is also addressed here which will show distinct results in different aspects. Finally, comparison analysis is performed here with the established methods which give an important impact in the research arena. This noble thought will help us to solve a plethora of daily life problems in uncertainty arena.

1.2 Motivation for the study

With the advent of vagueness theory the arena of numerous realistic mathematical modeling, engineering structural issues, multi-criteria problem have immensely achieved a productive and impulsive effect.Naturally it is very intriguing to the researchers that if someone sheds light on the pentagonal neutrosophic number then what will be it in the form of linearity and its classification? Based on this perception we impose three components on a pentagonal neutrosophic number i.e. truthiness, indeterminacy and falsity. Proceeding with the PNNWAA and PNNWGA operators and based on the score function of pentagonal neutrosophic numbers, an MCGDM method is built up and some interesting and worthy conclusions are tried to extract from this research article.

1.3 Novelties of the work

Recently, researchers are utmost persevere to develop theories connecting neutrosophic field and constantly try to generate its distinct application in various sphere of neutrosophic arena. However, justifying all the perspectives related to pentagonal neutrosophic fuzzy set theory; numerous theories and problems are yet to be solved. In this research article our ultimate objective is to shed light some unfocussed points in the pentagonal domain.

(1) Classification of Pentagonal Neutrosophic Number.
(2) Illustrative demonstration of aggregation operations and geometric operations on Pentagonal Neutrosophic Number's.
(3) Proposed new score function and its utility.
(4) Execute the idea of Pentagonal Neutrosophic Number's in MCGDM problem.

2. Preliminaries

Definition 2.1: Fuzzy Set: [1] Let $\tilde{A}$ be a set such that $\tilde{A} = \{(\beta, \alpha_{\tilde{A}}(\beta)) : \beta \epsilon A, \alpha_{\tilde{A}}(\beta) \epsilon [0,1]\}$ which is normally denoted by this ordered pair$(\beta, \alpha_{\tilde{A}}(\beta))$, here β is a member of the set$A$and $0 \leq \alpha_{\tilde{A}}(\beta) \leq 1$, then set $\tilde{A}$ is called a fuzzy set.

Definition 2.2: Neutrosophic Set:[8] A set $\tilde{A}_{Neu}$ in the domain of discourse A, most commonly stated as $\in$ is called a neutrosophic set if $\tilde{A}_{Neu} = \{\langle \in; [\varphi_{\widetilde{A_{Neu}}}(\in), \gamma_{\widetilde{A_{Neu}}}(\in), \delta_{\widetilde{A_{Neu}}}(\in)] \rangle : \in \epsilon A\}$,where $\varphi_{\widetilde{A_{Neu}}}(\in) : A \to]-0,1+[$ symbolizes the index of confidence, $\gamma_{\widetilde{A_{Neu}}}(\in) : A \to]-0,1+[$symbolizes the index of uncertainty and $\delta_{\widetilde{A_{Neu}}}(\in) : A \to]-0,1+[$symbolizes the degree of falseness in the decision making procedure. Where,$[\varphi_{\widetilde{A_{Neu}}}(\in), \gamma_{\widetilde{A_{Neu}}}(\in), \delta_{\widetilde{A_{Neu}}}(\in)]$ satisfies the in the equation

$$-0 \leq \varphi_{\widetilde{A_{Neu}}}(\in) + \gamma_{\widetilde{A_{Neu}}}(\in) + \delta_{\widetilde{A_{Neu}}}(\in) \leq 3+.$$

Definition 2.3: Single Typed Neutrosophic Number: [8]Single Typed Neutrosophic Number $(\tilde{n})$ is denoted as $\tilde{n} = \langle [(u^1, v^1, w^1, x^1); \alpha], [(u^2, v^2, w^2, x^2); \beta], [(u^3, v^3, w^3, x^3); \gamma] \rangle$where$\alpha, \beta, \gamma \in [0,1]$, where$(\varphi_{\tilde{n}}) : \mathbb{R} \to [0, \alpha]$, $(\gamma_{\tilde{n}}) : \mathbb{R} \to [\beta, 1]$ and $(\delta_{\tilde{n}}) : \mathbb{R} \to [\gamma, 1]$ is given as:

$$\varphi_{\tilde{n}}(\in) = \begin{cases} €_{\widetilde{nl}}(\in) \ when \ u^1 \leq \in \leq v^1 \\ \alpha \quad when \ v^1 \leq \in \leq w^1 \\ €_{\widetilde{nu}}(\in) \ when \ w^1 \leq \in \leq x^{1\prime} \\ 0 \qquad otherwise \end{cases}$$

$$£_{\tilde{n}}(\in) = \begin{cases} \gamma_{\widetilde{nl}}(\in) \ when \ u^2 \leq \in \leq v^2 \\ \beta \quad when \ v^2 \leq \in \leq w^2 \\ \gamma_{\widetilde{nu}}(\in) \ when \ w^2 \leq \in \leq x^2 \\ 1 \qquad otherwise \end{cases}$$

and

$$\delta_{\tilde{n}}(\in) = \begin{cases} \mu_{\widetilde{nl}}(\in) \ when \ u^3 \leq \in \leq v^3 \\ \gamma \quad when \ v^3 \leq \in \leq w^3 \\ \mu_{\widetilde{nu}}(\in) \ when \ w^3 \leq \in \leq x^3 \\ 1 \qquad otherwise \end{cases}$$

Definition 2.4: Single-Valued Neutrosophic Set:[9] A Neutrosophic set in the definition 2.2 is $\tilde{A}_{Neu}$said to be a Single-Valued Neutrosophic Set $(\tilde{A}_{Neu})$ if $\in$ is a single-valued independent variable. $\tilde{A}_{Neu} = \{\langle \in; [\alpha_{\tilde{A}_{Neu}}(\in), \beta_{\tilde{A}_{Neu}}(\in), \gamma_{\tilde{A}_{Neu}}(\in)] \rangle : \in \epsilon A\}$, where $\alpha_{\tilde{A}_{Neu}}(\in), \beta_{\tilde{A}_{Neu}}(\in) \& \gamma_{\tilde{A}_{Neu}}(\in)$ denote the idea of accuracy, ambiguity and falsity membership functions respectively.$\widetilde{SnS}$is named as neut-convex, which implies that $\widetilde{SnS}$ is a subset of R by satisfying the following criterion:

i. $\alpha_{\tilde{A}_{Neu}} \langle \delta a_1 + (1-\delta) a_2 \rangle \geq min \langle \alpha_{\tilde{A}_{Neu}}(a_1), \alpha_{\tilde{A}_{Neu}}(a_2) \rangle$
ii. $\beta_{\tilde{A}_{Neu}} \langle \delta a_1 + (1-\delta) a_2 \rangle \leq max \langle \beta_{\tilde{A}_{Neu}}(a_1), \beta_{\tilde{A}_{Neu}}(a_2) \rangle$
iii. $\gamma_{\tilde{A}_{Neu}} \langle \delta a_1 + (1-\delta) a_2 \rangle \leq max \langle \gamma_{\tilde{A}_{Neu}}(a_1), \gamma_{\tilde{A}_{Neu}}(a_2) \rangle$

where$a_1 \& a_2 \epsilon \mathbb{R} and \delta \epsilon [0,1]$

3. Single Type Linear Pentagonal Neutrosophic Number:

In this section we introduce different type single type linear pentagonal neutrosophic number. For the help of the researchers we pictorially draw the following block diagram as follows:

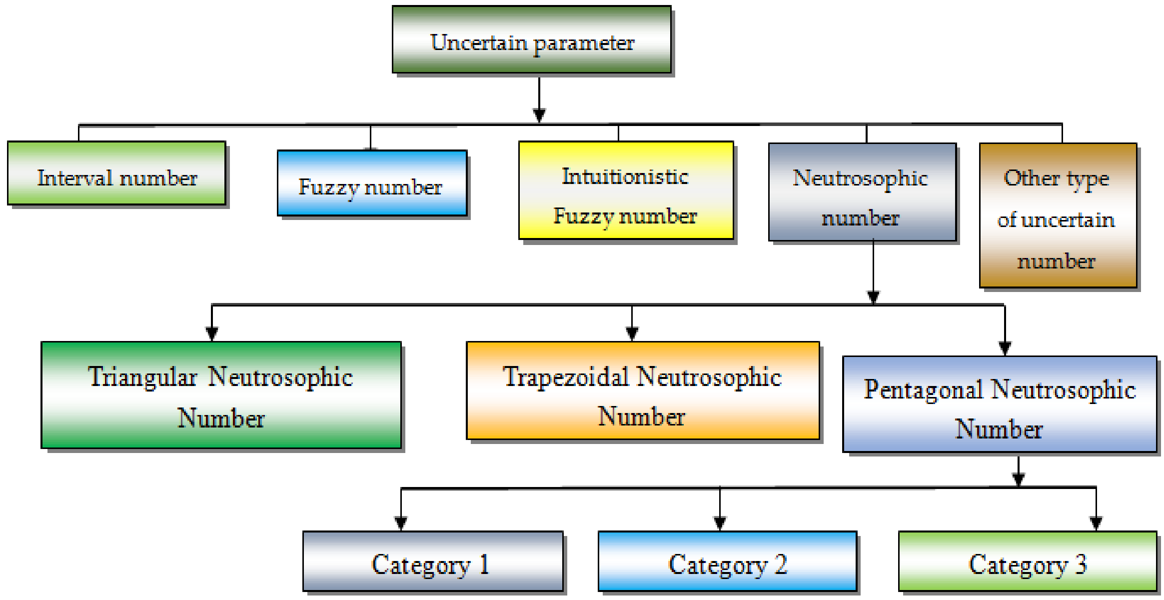

Figure 3.1: Block diagram for a different type of uncertain numbers and their categories

Definition 3.1: Single-Valued Pentagonal Neutrosophic Number: [47]A Single-Valued Pentagonal Neutrosophic Number $(\widetilde{N_{Pen}})$ is defined as$\widetilde{N_{Pen}} =$ $\langle[(h_1,h_2,h_3,h_4,h_5);\pi],[(h_1,h_2,h_3,h_4,h_5);\mu],[(h_1,h_2,h_3,h_4,h_5);\sigma]\rangle$, where$\pi,\mu,\sigma \in [0,1]$. The accuracy membership function$(\tau_{\tilde{S}}): \mathbb{R} \to [0,\pi]$, the ambiguity membership function $(\vartheta_{\tilde{S}}): \mathbb{R} \to [\rho,1]$ and the falsity membership function $(\varepsilon_{\tilde{S}}): \mathbb{R} \to [\sigma,1]$ are defined by:

$$\tau_{\tilde{S}}(x) = \begin{cases} \frac{\pi(x-h_1)}{(h_2-h_1)} & when\ h_1 \le x \le h_2 \\ \frac{\pi(x-h_2)}{(h_3-h_2)} & when\ h_2 \le x < h_3 \\ \pi & when\ x = h_3 \\ \frac{\pi(h_4-x)}{(h_4-h_3)} & when\ h_3 < x \le h_4 \\ \frac{\pi(h_4-x)}{(h_5-h_4)} & when\ h_4 \le x \le h_5 \\ 0 & otherwise \end{cases},$$

$$\vartheta_{\tilde{S}}(x) = \begin{cases} \frac{h_2 - x + \mu(x-h_1)}{(h_2-h_1)} & when\ h_1 \le x \le h_2 \\ \frac{h_3 - x + \mu(x-h_2)}{(h_3-h_2)} & when\ h_2 \le x < h_3 \\ \mu & when\ x = h_3 \\ \frac{x - h_3 + \mu(h_4-x)}{(h_4-h_3)} & when\ h_3 < x \le h_4 \\ \frac{x - h_4 + \mu(h_5-x)}{(h_5-h_4)} & when\ h_4 \le x \le h_5 \\ 1 & otherwise \end{cases}$$

and

$$\varepsilon_{\tilde{S}}(x)=\begin{cases}\frac{h_2-x+\sigma(x-h_1)}{(h_2-h_1)} & when\ h_1\le x\le h_2\\ \frac{h_3-x+\sigma(x-h_2)}{(h_3-h_2)} & when\ h_2\le x<h_3\\ \sigma & when\ x=h_3\\ \frac{x-h_3+\sigma(h_4-x)}{(h_4-h_3)} & when\ h_3<x\le h_4\\ \frac{x-h_4+\sigma(h_5-x)}{(h_5-h_4)} & when\ h_4\le x\le h_5\\ 1 & otherwise\end{cases}$$

4. Proposed Score Function:

Score function of a pentagonal neutrosophic number entirely depends on the value of truth membership indicator degree, falsity membership indicator degree and uncertainty membership indicator degree. The necessity of score function is to draw a comparison or transfer a pentagonal neutrosophic fuzzy number into a crisp number. In this section, we will generate a score function as follows.For any Pentagonal Single typed Neutrosophic Number (PSNN)

$$\tilde{A}_{Pt}=(s_1,s_2,s_3,s_4,s_5;\pi,\mu,\sigma)$$

We define the score function as

$$S_{Pt}=\frac{1}{15}(s_1+s_2+s_3+s_4+s_5)\times(2+\pi-\sigma-\mu)$$

Here, S_{Pt} belongs to [0,1].

4.1Relationship between any two pentagonal neutrosophic fuzzy numbers:

Let us consider any two pentagonal neutrosophic fuzzy number defined as follows

$\tilde{A}_{Pt1}=(s_{Pt11},s_{Pt12},s_{Pt13},s_{Pt14},s_{Pt15};\pi_{Pt1},\mu_{Pt1},\sigma_{Pt1})and\tilde{A}_{Pt2}=$ $(s_{Pt21},s_{Pt22},s_{Pt23},s_{Pt24},s_{Pt25};\pi_{Pt2},\mu_{Pt2},\sigma_{Pt2})$

The score function for the are

$$S_{Pt1}=\frac{1}{15}(s_{Pt11}+s_{Pt12}+s_{Pt13}+s_{Pt14}+s_{Pt15})\times(2+\pi_{Pt1}-\sigma_{Pt1}-\mu_{Pt1})$$

and

$$S_{Pt2}=\frac{1}{15}(s_{Pt21}+s_{Pt22}+s_{Pt23}+s_{Pt24}+s_{Pt25})\times(2+\pi_{Pt2}-\sigma_{Pt2}-\mu_{Pt2})$$

Then we can say the following

1) $\tilde{A}_{Pt1}>\tilde{A}_{Pt2}$ if $S_{Pt1}>S_{Pt2}$
2) $\tilde{A}_{Pt1}<\tilde{A}_{Pt2}$if $S_{Pt1}<S_{Pt2}$
3) $\tilde{A}_{Pt1}=A_{Pt2}$ if $S_{Pt1}=S_{Pt2}$

Table 4.1: Numerical Examples

Pentagonal Neutrosophic Number ($\tilde{A}_{Pt}$)	Score Value (S_{Pt})	Ordering
$\tilde{A}_{Pt1}=<(0.2,0.3,0.4,0.5,0.6;0.4,0.5,0.6)>$	0.17333	$A_{Pt4}>A_{Pt2}>A_{Pt1}>A_{Pt3}$
$\tilde{A}_{Pt2}=<(0.35,0.4,0.45,0.5,0.55;0.6,0.3,0.4)>$	0.28500	
$\tilde{A}_{Pt3}=<(0.15,0.2,0.25,0.3,0.35;0.6,0.4,0.5)>$	0.14167	
$\tilde{A}_{Pt4}=<(0.7,0.75,0.8,0.85,0.9;0.3,0.2,0.6)>$	0.40000	

4.1 Basic Operations for pentagonal neutrosophic fuzzy number:

Let $\widetilde{p_1} = \langle (c_1, c_2, c_3, c_4, c_5); \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} \rangle$ and $\widetilde{p_2} = \langle (d_1, d_2, d_3, d_4, d_5); \pi_{\widetilde{p_2}}, \mu_{\widetilde{p_2}}, \sigma_{\widetilde{p_2}} \rangle$ be two IPFNs and $\alpha \geq 0$. Then the following operational relations hold:

4.1.1 Addition:

$$\widetilde{p_1} + \widetilde{p_2} = \langle (c_1 + d_1, c_2 + d_2, c_3 + d_3, c_4 + d_4, c_5 + d_5); \pi_{\widetilde{p_1}} + \pi_{\widetilde{p_2}} - \pi_{\widetilde{p_1}}\pi_{\widetilde{p_2}}, \mu_{\widetilde{p_1}}\mu_{\widetilde{p_2}}, \sigma_{\widetilde{p_1}}\sigma_{\widetilde{p_2}} \rangle$$

4.1.2Multipliction:

$$\widetilde{p_1}\widetilde{p_2} = \langle (c_1 d_1, c_2 d_2, c_3 d_3, c_4 d_4, c_5 d_5);\ \pi_{\widetilde{p_1}}\pi_{\widetilde{p_2}}, \mu_{\widetilde{p_1}} +, \mu_{\widetilde{p_2}} - \mu_{\widetilde{p_1}}\mu_{\widetilde{p_2}}, \sigma_{\widetilde{p_1}} + \sigma_{\widetilde{p_2}} - \sigma_{\widetilde{p_1}}\sigma_{\widetilde{p_2}} \rangle$$

4.1.3 Multiplication by scalar:

$$\alpha\widetilde{p_1} = \langle (\alpha c_1, \alpha c_2, \alpha c_3, \alpha c_4, \alpha c_5); 1 - (1 - \pi_{\widetilde{p_1}})^{\alpha}, \mu_{\widetilde{p_1}}{}^{\alpha}, \sigma_{\widetilde{p_1}}{}^{\alpha}) \rangle$$

4.1.4 Power:

$$\widetilde{p_1}^{\alpha} = \langle (c_1{}^{\alpha}, c_2{}^{\alpha}, c_3{}^{\alpha}, c_4{}^{\alpha}, c_5{}^{\alpha}); \pi_{\widetilde{p_1}}{}^{\alpha}, (1 - \mu_{\widetilde{p_1}})^{\alpha}, (1 - \sigma_{\widetilde{p_1}})^{\alpha} \rangle$$

5. Arithmetic and Geometric Operators:

5.1 Two weighted aggregation operators of Pentagonal Neutrosophic Numbers

Aggregation operators are such pertinent tool for aggregating information to tactfully handle the decision making procedure, this section generates a brief understanding between two weighted aggregation operators to aggregate PNNs as a generalization of the weighted aggregation operators for PNNs, which are broadly and aptly used in decision making.

5.1.1 Pentagonal neutrosophic weighted arithmetic averaging operator

Let $\tilde{p}_j = \langle (c_{j1}, c_{j2}, c_{j3}, c_{j4}, c_{j5});\ \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} \rangle (j = 1,2,3, \dots, n)$ be a set of PNNs, then a PNWAA operator is defined as follows:

$$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \textstyle\sum_{j=1}^{n} \omega_j\, \tilde{p}_j \qquad (5.1)$$

whereω_j is the weight of $\tilde{p}_j (j = 1,2,3, \dots, n)$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.

In accordance with the result ofSection 4.1 and equation (5.1) we can introduce the following theorems:

Theorem 5.1. Let $\tilde{p}_j = \langle (c_{j1}, c_{j2}, c_{j3}, c_{j4}, c_{j5});\ \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} \rangle (j = 1,2,3, \dots, n)$ be a set of PNNs, then according to Section 4.1 and equation (5.1) we can give the following PNWAA operator

$$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \sum_{j=1}^{n} \omega_j\, \tilde{p}_j$$

$$=\langle (\textstyle\sum_{j=1}^{n} \omega_j\, c_{j1}, \sum_{j=1}^{n} \omega_j\, c_{j2}, \sum_{j=1}^{n} \omega_j\, c_{j3}, \sum_{j=1}^{n} \omega_j\, c_{j4}, \sum_{j=1}^{n} \omega_j\, c_{j5}\ ); 1 - \prod_{j=1}^{n}(1 - \pi_{\widetilde{p_j}})^{\omega_j}\ , \prod_{j=1}^{n} \mu_{\widetilde{p_j}}{}^{\omega_j}\ , \prod_{j=1}^{n} \sigma_{\widetilde{p_j}}{}^{\omega_j} \rangle$$

Where ω_j is the weight of $\tilde{p}_j (j = 1,2,3, \dots, n)$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 5.1 can be proved with the help of mathematical induction.

Proof: When $n = 2$ then,

$$\omega_1\tilde{p}_1 = \langle (\omega_1 c_{11}, \omega_1 c_{12}, \omega_1 c_{13}, \omega_1 c_{14}, \omega_1 c_{15}\ );\ 1 - \left(1 - \pi_{\widetilde{p_1}}\right)^{\omega_1} \sigma_{\widetilde{p_2}}{}^{\omega_2}, \mu_{\widetilde{p_1}}{}^{\omega_1}, \sigma_{\widetilde{p_1}}{}^{\omega_1} \rangle$$

and $\omega_2\tilde{p}_2 = \langle (\omega_2 c_{21}, \omega_2 c_{22}, \omega_2 c_{23}, \omega_2 c_{24}, \omega_2 c_{25}\ );\ 1 - \left(1 - \pi_{\widetilde{p_2}}\right)^{\omega_2}, \mu_{\widetilde{p_2}}{}^{\omega_2}, \sigma_{\widetilde{p_2}}{}^{\omega_2} \rangle$

Thus, $PNWAA\ (\widetilde{p_1},\widetilde{p_2}) = \omega_1\tilde{p}_1 + \omega_2\tilde{p}_2$

$=\langle(\omega_1c_{11} + \omega_2c_{21} + \omega_1c_{12} + \omega_2c_{22} + \omega_1c_{13} + \omega_2c_{23} + \omega_1c_{14} + \omega_2c_{24} + \omega_1c_{15} + \omega_2c_{25}); 1 - \left(1-\pi_{\widetilde{p_1}}\right)^{\omega_1} + 1 - \left(1-\pi_{\widetilde{p_2}}\right)^{\omega_2}(1-\left(1-\pi_{\widetilde{p_1}}\right)^{\omega_1})(1-\left(1-\pi_{\widetilde{p_2}}\right)^{\omega_2}), \mu_{\widetilde{p_1}}{}^{\omega_1}\mu_{\widetilde{p_2}}{}^{\omega_2}, \sigma_{\widetilde{p_1}}{}^{\omega_1}\sigma_{\widetilde{p_2}}{}^{\omega_2}\rangle$

When applying $n = k$, by applying equation (5.1) , we get

$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_k) = \sum_{j=1}^{k} \omega_j\, \tilde{p}_j$ (5.2)

$= \langle(\sum_{j=1}^{k}\omega_j\, c_{j1}, \sum_{j=1}^{k}\omega_j\, c_{j2}, \sum_{j=1}^{k}\omega_j\, c_{j3}, \sum_{j=1}^{k}\omega_j\, c_{j4}, \sum_{j=1}^{k}\omega_j\, c_{j5}\ );\ 1 - \prod_{j=1}^{k}(1 - \pi_{\widetilde{p_j}})^{\omega_j}\ , \prod_{j=1}^{k}\mu_{\widetilde{p_j}}{}^{\omega_j}, \prod_{j=1}^{k}\sigma_{\widetilde{p_j}}{}^{\omega_j}\rangle$

When $n = k + 1$, by applying equations (5.1) and (5.2) we can yield

$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_{k+1}) = \sum_{j=1}^{k+1} \omega_j\, \tilde{p}_j$ (5.3)

$= \langle(\sum_{j=1}^{k+1}\omega_j\, c_{j1}, \sum_{j=1}^{k+1}\omega_j\, c_{j2}, \sum_{j=1}^{k+1}\omega_j\, c_{j3}, \sum_{j=1}^{k+1}\omega_j\, c_{j4}, \sum_{j=1}^{k+1}\omega_j\, c_{j5}\ );\ 1 - \prod_{j=1}^{k}\left(1-\pi_{\widetilde{p_j}}\right)^{\omega_j} + 1 - \left(1-\pi_{\widetilde{p_{k+1}}}\right)^{\omega_{k+1}}, \prod_{j=1}^{k+1}\mu_{\widetilde{p_j}}{}^{\omega_j}, \prod_{j=1}^{k+1}\sigma_{\widetilde{p_j}}{}^{\omega_j}\rangle$

$= \langle(\sum_{j=1}^{k+1}\omega_j\, c_{j1}, \sum_{j=1}^{k+1}\omega_j\, c_{j2}, \sum_{j=1}^{k+1}\omega_j\, c_{j3}, \sum_{j=1}^{k+1}\omega_j\, c_{j4}, \sum_{j=1}^{k+1}\omega_j\, c_{j5}\ ); 1 - \prod_{j=1}^{k+1}(1 - \pi_{\widetilde{p_j}})^{\omega_j}\ , \prod_{j=1}^{k+1}\mu_{\widetilde{p_j}}{}^{\omega_j}, \prod_{j=1}^{k+1}\sigma_{\widetilde{p_j}}{}^{\omega_j}\rangle$

This completes the proof.

Obviously, the $PNWAA$ operator satisfies the following properties:

i) Idempotency: Let $\tilde{p}_j (j = 1,2,3, \dots, n)$ be a set of PNNs. If $\tilde{p}_j (j = 1,2,3, \dots, n)$ is equal , i.e. $\tilde{p}_j = \tilde{p}$ for j=1,2,3,….,n then $PNWAA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \tilde{p}$.

Proof: Since $\tilde{p}_j = \tilde{p}$ for $j = 1,2,3, \dots, n$ we have,

$$PNWAA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \sum_{j=1}^{n} \omega_j\, \tilde{p}_j$$

$=\langle(\sum_{j=1}^{n}\omega_j\, c_{j1}, \sum_{j=1}^{n}\omega_j\, c_{j2}, \sum_{j=1}^{n}\omega_j\, c_{j3}, \sum_{j=1}^{n}\omega_j\, c_{j4}, \sum_{j=1}^{n}\omega_j\, c_{j5}\ );\ 1 - \prod_{j=1}^{n}(1 - \pi_{\widetilde{p_j}})^{\omega_j}\ , \prod_{j=1}^{n}\mu_{\widetilde{p_j}}{}^{\omega_j}, \prod_{j=1}^{n}\sigma_{\widetilde{p_j}}{}^{\omega_j}\rangle$

$= \langle\left(c_1\sum_{j=1}^{n}\omega_j\, , c_2\sum_{j=1}^{n}\omega_j\, , c_3\sum_{j=1}^{n}\omega_j\, , c_4\sum_{j=1}^{n}\omega_j\, , c_5\sum_{j=1}^{n}\omega_j\right); (1 - \left(1 - \pi_{\widetilde{p_j}}\right))^{\sum_{j=1}^{n}\omega_j}, \mu_{\widetilde{p_j}}{}^{\sum_{j=1}^{n}\omega_j}\ , \sigma_{\widetilde{p_j}}{}^{\sum_{j=1}^{n}\omega_j}\rangle$

$=\langle(\ c_1, c_2, c_3, c_4, c_5); 1 - (1 - \pi_{\widetilde{p_1}}), \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}}\rangle\ = \tilde{p}$

ii) Boundedness: Let $\tilde{p}_j$(j=1,2,3,….,n) be a set of PNNs and let

$\tilde{p}^- = \langle\left(min_j\left(c_{j1}\ \right), min_j\left(c_{j2}\ \right), min_j\left(c_{j3}\ \right), min_j\left(c_{j4}\ \right), min_j\left(c_{j5}\ \right)\right); min_j\left(\pi_{\widetilde{p_j}}\right), max_j\left(\mu_{\widetilde{p_j}}\right), max_j\left(\sigma_{\widetilde{p_j}}\right)\rangle$

and $\tilde{p}^+ = \langle\left(max_j\left(c_{j1}\ \right), max_j\left(c_{j2}\ \right), max_j\left(c_{j3}\ \right), max_j\left(c_{j4}\ \right), max_j\left(c_{j5}\ \right)\right); max_j\left(\pi_{\widetilde{p_j}}\right), min_j\left(\mu_{\widetilde{p_j}}\right), min_j\left(\sigma_{\widetilde{p_j}}\right)\rangle$

Then $\tilde{p}^- \leq PNWAA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \leq\ \tilde{p}^+$.

Proof: Since the minimum PNN is $\tilde{p}^-$ and the maximum is $\tilde{p}^+$there is $\tilde{p}^- \leq \tilde{p}_j \leq \tilde{p}^+$. Thus there is $\sum_{j=1}^{n} \omega_j \tilde{p}^- \leq \sum_{j=1}^{n} \omega_j \tilde{p}_j \leq \sum_{j=1}^{n} \omega_j \tilde{p}^+$.According to the above property (i) there is $\tilde{p}^- \leq \sum_{j=1}^{n} \omega_j \tilde{p}_j \leq \tilde{p}^+$,
i.e.,$\tilde{p}^- \leq$ PNWAA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \leq \tilde{p}^+$.

iii) Monotonicity: Let $\tilde{p}_j (j = 1,2,3, \dots, n)$ be a set of PNNs. If $\tilde{p}_j \leq \tilde{p}_j{}^*$ for j= j=1,2,3,….,n, then

PNWAA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \leq$ PNWAA$(\tilde{p}_{1,}^* \tilde{p}_{2,}^* \tilde{p}_{3,}^* \tilde{p}_{4,}^* \tilde{p}_{5,}^*)$

Proof: Since $\tilde{p}_j \leq \tilde{p}_j{}^*$ for $j = j = 1,2,3, \dots, n$ there is $\sum_{j=1}^{n} \omega_j \tilde{p}_j \leq \sum_{j=1}^{n} \omega_j \tilde{p}_j{}^*$ i.e. PNWAA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \leq$ PNWAA$(\tilde{p}_{1,}^* \tilde{p}_{2,}^* \tilde{p}_{3,}^* \tilde{p}_{4,}^* \tilde{p}_5^*)$.Thus we complete the proofs of all the properties.

5.2 Pentagonal neutrosophic weighted geometric averaging operator

Let $\tilde{p}_j = \langle (c_{j1}, c_{j2}, c_{j3}, c_{j4}, c_{j5}); \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} \rangle (j = 1,2,3, \dots, n)$ be a set of PNNs, then a PNWGAA operator is defined as follows:
$PNWGA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \prod_{j=1}^{n} \widetilde{p_j{}^{\omega_j}}$(5.4)
whereω_j is the weight of $\tilde{p}_j (j = 1,2,3, \dots, n)$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.

Theorem 5.2. Let $\tilde{p}_j =< (c_{j1}, c_{j2}, c_{j3}, c_{j4}, c_{j5}); \pi_{\widetilde{p_1}}, \mu_{\widetilde{p_1}}, \sigma_{\widetilde{p_1}} > (j = 1,2,3, \dots, n)$ be a set of PNNs, then according to Section 4.1 and equation (5.4) we can give the following PNWGA operator
$PNWGA\ (\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \prod_{j=1}^{n} \widetilde{p_j{}^{\omega_j}}$ (5.5)
= $\langle (\prod_{j=1}^{n} c_{j1}{}^{\omega_j}, \prod_{j=1}^{n} c_{j2}{}^{\omega_j}, \prod_{j=1}^{n} c_{j3}{}^{\omega_j}, \prod_{j=1}^{n} c_{j4}{}^{\omega_j}, \prod_{j=1}^{n} c_{j5}{}^{\omega_j}; \prod_{j=1}^{n} \pi_{\widetilde{p_j}}{}^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - \mu_{\widetilde{p_j}})^{\omega_j}, 1 - \prod_{j=1}^{n} (1 - \sigma_{\widetilde{p_j}})^{\omega_j} \rangle$
whereω_j is the weight of $\tilde{p}_j (j = 1,2,3, \dots, n)$ such that $\omega_j > 0$ and $\sum_{j=1}^{n} \omega_j = 1$.
By the similar proof manner of Theorem 5.1 we can prove the Theorem 5.2 which is not repeated here.
Obviously, the PNWGA operator satisfies the following properties:

i) Idempotency: Let $\tilde{p}_j (j = 1,2,3, \dots, n)$ be a set of PNNs.
If $\tilde{p}_j (j = 1,2,3, \dots, n)$ is equal , i.e. $\tilde{p}_j = \tilde{p}$ for $j = 1,2,3, \dots, n$ then PNWGA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) = \tilde{p}$.

ii) Boundedness: Let $\tilde{p}_j (j = 1,2,3, \dots, n)$ be a set of PNNs and let

$\tilde{p}^- = \langle (min_j(c_{j1}), min_j(c_{j2}), min_j(c_{j3}), min_j(c_{j4}), min_j(c_{j5}); min_j \left(\pi_{\widetilde{p_j}}\right), max_j \left(\mu_{\widetilde{p_j}}\right), max_j \left(\sigma_{\widetilde{p_j}}\right) \rangle$

and

$\tilde{p}^+ = \langle (max_j(c_{j1}), max_j(c_{j2}), max_j(c_{j3}), max_j(c_{j4}), max_j(c_{j5}); max_j \left(\pi_{\widetilde{p_j}}\right), min_j \left(\mu_{\widetilde{p_j}}\right), min_j \left(\sigma_{\widetilde{p_j}}\right) \rangle$

Then $\tilde{p}^- \leq$ PNWGA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \leq \tilde{p}^+$.

iii) Monotonicity: Let $\tilde{p}_j (j = 1,2,3, \dots, n)$ be a set of PNNs. If $\tilde{p}_j \leq \tilde{p}_j{}^*$for $j = j = 1,2,3, \dots, n$, then

PNWGA $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_n) \leq$ PNWGA$(\tilde{p}_{1,}^* \tilde{p}_{2,}^* \tilde{p}_{3,}^* \tilde{p}_{4,}^* \tilde{p}_5^*)$

As the proofs of these properties are similar to the proofs of the above properties, so we don't repeat them.

6. Multi-Criteria Group Decision Making Problem in Pentagonal Neutrosophic Environment

Multi-criteria group decision-making problem is one of the reliable, logistical and mostly used topics in this current era. The main goal of this process is to find out the best alternatives among a finite number of distinct alternatives based on finite different attribute values. Such decision-making program may be raised powerfully by the methods of multi-criteria group decision analysis (MCGDA) which is extremely beneficial to produce decision counselling and offers procedure benefits in terms of upgraded decision attributes,delivers improvised communication techniques and enrichesresolutions of decision-makers.The execution process is not so much easy to evaluate in the pentagonal neutrosophic environment. Using some mathematical operators, score function technique, we developed an algorithm to tackle this MCGDM problem.

In this section, we consider a multi-criteria group decision-making problem based on mobile communication provider services in which we need to select the best service according to different opinions from people. The developed algorithm is described briefly as follows:

6.1 Illustration of the MCGDM problem

We consider the problem as follows:

Suppose $G = \{ G_1, G_2, G_3 \dots\dots\dots.. G_m\}$ is a distinctive alternative set and $H = \{ H_1, H_2, H_3 \dots\dots\dots.. H_n\}$ is the distinctive attribute set respectively. Let $\omega = \{ \omega_1, \omega_2, \omega_3 \dots\dots\dots.. \omega_n\}$ be the corresponding weight set attributes where each $\omega \geq 0$ and also satisfies the relation$\sum_{i=1}^{n} \omega_i = 1$. Thus we consider the set of decision-maker $\lambda = \{ \lambda_1, \lambda_2, \lambda_3 \dots\dots\dots.. \lambda_K\}$ associated with alternatives whose weight vector is stated as $\Omega = \{\Omega_1, \Omega_2, \Omega_3 \dots\dots\dots.. \Omega_k\}$ where each $\Omega_i \geq 0$ and also satisfies the relation$\sum_{i=1}^{k} \Omega_i = 1$, this weight vector will be chosen in accordance with the decision-makers capability of judgment, experience, innovative thinking power etc.

6.2 Normalisation Algorithm of MCGDM Problem:

Step 1: Composition of Decision Matrices

Here, we construct all decision matrices proposed by the decision maker's choice connected with finite alternatives and finite attribute functions. The interesting fact is that the member's s_{ij} for each matrix are of pentagonal neutrosophic numbers. Thus, we finalize the matrix and is given as follows:

$$X^K = \begin{pmatrix} . & H_1 & H_2 & H_3 & . & . & . & H_n \\ G_1 & s_{11}^k & s_{12}^k & s_{13}^k & . & .. & . & s_{1n}^k \\ G_2 & s_{21}^k & s_{22}^k & s_{23}^k & . & . & . & s_{2n}^k \\ G_3 & . & . & . & . & . & . & . \\ . & \vdots & . & . & . & . & . & . \\ G_m & s_{m1}^k & s_{m2}^k & s_{m3}^k & . & . & . & s_{mn}^k \end{pmatrix} \quad (6.1)$$

Step 2: Composition of Single decision matrix

For generating a single group decision matrix X we have promoted the logical pentagonal neutrosophic weighted arithmetic averaging operator (PNWAA) as, $s'_{ij} = \sum_{j=1}^{n} \omega_j \, s_{ij}^{k}$, for individual decision matrix X^k, where$k = 1,2,3 \ldots . n$. hence, we finalize the matrix and defined as follows:

$$X = \begin{pmatrix} . & H_1 & H_2 & H_3 & . & . & . & H_n \\ G_1 & s'_{11} & s'_{12} & s'_{13} & . & .. & . & s'_{1n} \\ G_2 & s'_{21} & s'_{22} & s'_{23} & . & . & . & s'_{2n} \\ G_3 & \cdot & \cdot & \cdot & . & . & . & . \\ . & \vdots & . & . & . & . & . & . \\ G_m & s'_{m1} & s'_{m2} & s'_{m3} & . & . & . & s'_{mn} \end{pmatrix} \quad (6.2)$$

Step 3: Composition of leading matrix

To illustrate the single decision matrix we have promoted the logical pentagonal neutrosophic weighted geometric averaging operator (PNWGA) as, $s''_{ij} = \prod_{j=1}^{n} \widetilde{s_{ij}}^{\omega_j}$ for each individual column and finally, we construct the decision matrix as below,

$$X = \begin{pmatrix} . & H_1 \\ G_1 & s''_{11} \\ G_2 & s''_{21} \\ \vdots & \vdots \\ G_m & s''_{m1} \end{pmatrix} \quad (6.3)$$

Step 4: Ranking

Now, considering the score value and transforming the matrix (6.3) into crisp form, we can evaluate the best substitute corresponding to the best attributes. We align the values as increasing order according to their score values and then detect the best fit result. The best result will be the highest magnitude and the worst ones will be the least one.

6.3.1 Flowchart:

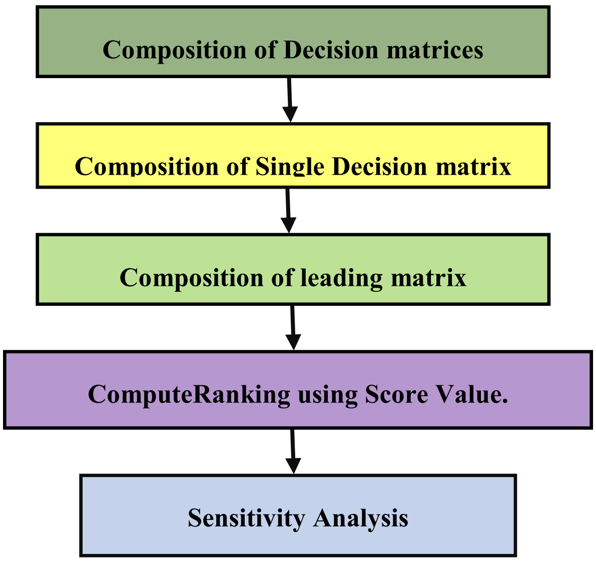

Figure 6.3.1: Flowchart for the problem

6.3 Illustrative Example:

Here, we consider a mobile communication service provider based problem in which there are three different companies are accessible. Among those companies, our problem is to find out the best mobile communication service provider in a logical and meaningful way. Normally, mobile communication service providers mostly depend on attributes such as Service & Reliability, Price & Availability, and Quality & Features of the system. Here, we also consider three different categories of people i) youth age ii) adult age iii) old age people as a decision-maker. According to their opinions we formulate the different decision matrices in the pentagonal neutrosophic environment described below:

$$G_1 = Mobilecommunicationserviceprovider\ 1,$$
$$G_2 = Mobilecommunicationserviceprovider\ 2,$$
$$G_3 = Mobilecommunicationserviceprovider\ 3$$

are the alternatives.

Also

$$H_1 = \text{Service \& Reliability},$$
$$H_2 = \text{Price \& Availability},$$
$$H_3 = \text{Quality \& Features}$$

arethe attributes.

Let, $D_1 = Youthagepeople$, $D_2 = Adultagepeople, D_3 = Senioragepeople$ having weight allocation $D = \{ 0.31, 0.35, 0.34 \}$ and the weight allocation in different attribute function is $\Delta = \{0.3,0.4,0.3\}$.A verbal matrix is built up by the decision maker's to assist the classification of the decision matrix. Attribute vs. Verbal Phrase matrix is given below in Table 6.3.1. The total MCGDM problem is graphically described as below:

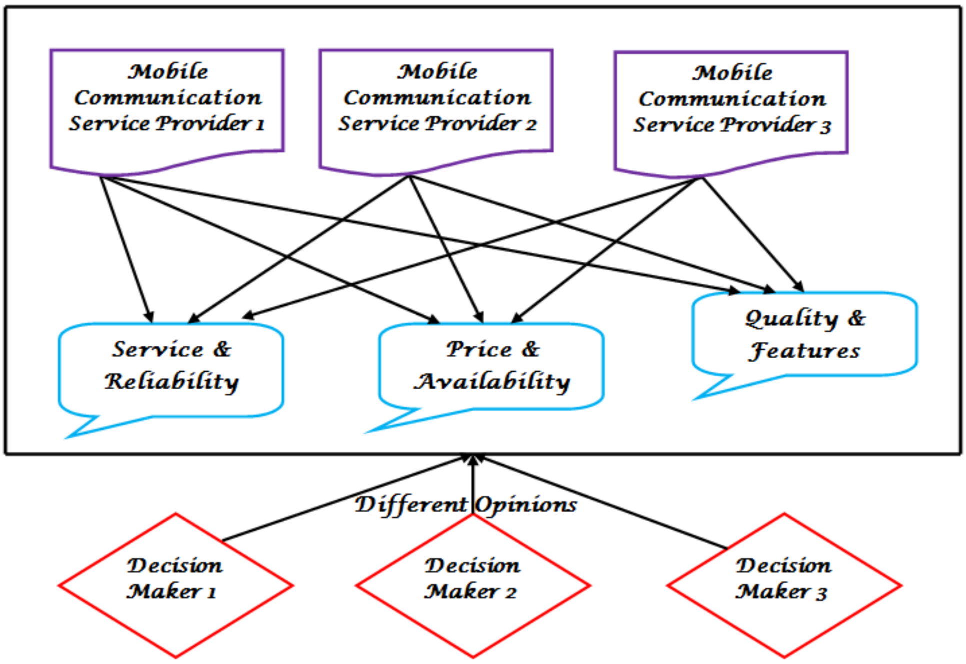

Table 6.3.1: List of Verbal Phrase

Sl no.	Attribute	Verbal phrase
	Quantitative Attributes	
1	Service & Reliability	Very High (VH), High (L), Intermediate (I), Small (S), Very small (VS)
2	Price & Availability	Very high (VH), High (H), Mid (M), Low (L), Very low (VL)
3	Quality & Features	Very high (VH), High (H), Standard (SD), Low (L), Very low (VL)

Table 6.3.2: Relationship between Verbal Phrase and PNN

Verbal Phase	Linguistic Pentagonal Neutrosophic Number (PNN)
Very Low (VL)	$< (0.1,0.1,0.1,0.1,0.1;0.4,0.4,0.4) >$
Low (L)	$< (0.2,0.3,0.4,0.5,0.6;0.5,0.3,0.3) >$
Moderate (M)	$< (0.4,0.5,0.6,0.7,0.8;0.7,0.2,0.2) >$
Little High (LH)	$< (0.5,0.6,0.7,0.8,0.9;0.75,0.18,0.18) >$
High (H)	$< (0.6,0.7,0.8,0.9,1.0;0.8,0.15,0.15) >$
Very High (VH)	$< (1.0,1.0,1.0,1.0,1.0;0.95,0.05,0.05) >$

Step 1

In accordance with finite alternatives and finite attribute functions the decision matrices are constructed by the proposal of decision maker's choice. The noteworthy fact is that the entity s_{ij} for each matrix are of pentagonal neutrosophic numbers. Finally, the matrices are presented as follows:

$$D^1 = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & < 0.2,0.3,0.4,0.5,0.6;0.4,0.6,0.5 > & < 0.1,0.2,0.3,0.4,0.5;0.5,0.6,0.7 > & < 0.3,0.4,0.5,0.6,0.7;0.6,0.3,0.3 > \\ G_2 & < 0.15,0.25,0.35,0.45,0.5;0.5,0.6,0.5 > & < 0.3,0.4,0.5,0.6,0.7;0.7,0.3,0.5 > & < 0.4,0.5,0.55,0.6,0.7;0.8,0.7,0.3 > \\ G_3 & < 0.4,0.5,0.6,0.7,0.8;0.6,0.4,0.3 > & < 0.25,0.3,0.35,0.4,0.45;0.4,0.6,0.5 > & < 0.35,0.4,0.45,0.5,0.55;0.6,0.3,0.4 > \end{pmatrix}$$

$$Youth's\ opinion$$

$$D^2 = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & < 0.15,0.2,0.25,0.3,0.35;0.6,0.4,0.5 > & < 0.1,0.15,0.3,0.35,0.4;0.7,0.5,0.3 > & < 0.7,0.75,0.8,0.85,0.9;0.3,0.2,0.6 > \\ G_2 & < 0.2,0.25,0.3,0.35,0.4;0.7,0.5,0.4 > & < 0.2,0.25,0.3,0.4,0.45;0.6,0.3,0.3 > & < 0.4,0.5,0.55,0.6,0.7;0.8,0.7,0.4 > \\ G_3 & < 0.3,0.35,0.4,0.45,0.5;0.7,0.5,0.3 > & < 0.5,0.55,0.6,0.7,0.8;0.5,0.6,0.7 > & < 0.6,0.7,0.75,0.8,0.9;0.6,0.5,0.6 > \end{pmatrix}$$

$$Adult's\ Opinion$$

$$D^3 = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & < 0.2,0.25,0.3,0.4,0.45;0.6,0.3,0.3 > & < 0.2,0.3,0.4,0.5,0.6;0.4,0.6,0.5 > & < 0.7,0.75,0.8,0.85,0.9;0.3,0.2,0.6 > \\ G_2 & < 0.3,0.4,0.5,0.6,0.7;0.7,0.3,0.5 > & < 0.6,0.7,0.75,0.8,0.9;0.6,0.5,0.6 > & < 0.7,0.75,0.8,0.85,0.9;0.3,0.2,0.6 > \\ G_3 & < 0.3,0.35,0.4,0.45,0.5;0.7,0.5,0.3 > & < 0.4,0.5,0.55,0.6,0.7;0.8,0.7,0.3 > & < 0.15,0.2,0.25,0.3,0.35;0.6,0.4,0.5 > \end{pmatrix}$$

$$Senior\ 's\ Opinion$$

Step 2: Composition of Single decision matrix

In this step we generate a single group decision matrix M and have incorporated the idea of logical pentagonal neutrosophic weighted arithmetic averaging operator (PNWAA) as, $s'_{ij} = \sum_{j=1}^{n} \omega_j \, s_{ij}^{k}$, for individual decision matrix D^k, where$k = 1,2,3 \ldots . n$. Thus we finalize the matrix which is presented as follows:

$$M = \begin{pmatrix} . & H_1 & H_2 & H_3 \\ G_1 & < 0.18,0.25,0.31,0.4,0.46; 1.00,0.41,0.42 > & < 0.13,0.22,0.33,0.42,0.50; 0.99,0.56,0.46 > & < 0.58,0.64,0.70,0.77,0.84; 0.98,0.23,0.43 > \\ G_2 & < 0.22,0.30,0.38,0.47,0.53; 1.00,0.44,0.46 > & < 0.38,0.45,0.52,0.60,0.68; 1.00,0.36,0.44 > & < 0.42,0.48,0.53,0.58,0.65; 1.00,0.40,0.39 > \\ G_3 & < 0.33,0.40,0.46,0.53,0.59; 1.00,0.47,0.30 > & < 0.39,0.46,0.51,0.57,0.66; 1.00,0.41,0.47 > & < 0.37,0.48,0.49,0.58,0.60; 1.00,0.40,0.50 > \end{pmatrix}$$

Step 3: Composition of leading matrix

To define the single decision matrix we have employed the concept of the logical pentagonal neutrosophic weighted geometric averaging operator (PNWGA) as, $s''_{ij} = \prod_{j=1}^{n} \widetilde{s_{ij}}^{\omega_j}$ for each individual column and finally, we present the decision matrix as below

$$M = \begin{pmatrix} \langle 0.26, 0.35, 0.44, 0.56, 0.60; 0.99,0.98,0.99 \rangle \\ \langle 0.33, 0.41, 0.48,0.55,0.62; 1.00,0.98,0.99 \rangle \\ \langle 0.36,0.43,0.48,0.54,0.62; 1.00,0.99,0.99 \rangle \end{pmatrix}$$

Step 4: Ranking

Now, we examine the proposed score value for crispification of the PNN into a real number, thus we get the ultimate decision matrix as

$$M = \begin{pmatrix} < 0.1503 > \\ < 0.1641 > \\ < 0.1652 > \end{pmatrix}$$

Here, ordering is $0.1503 < 0.1641 < 0.1652$. Hence, the ranking of the mobile communication service provider is $G_3 > G_2 > G_1$.

6.4 Results and Sensitivity Analysis

To understand how the attribute weights of each criterion affect the relative matrix and their ranking a sensitivity analysis is done. The basic idea of sensitivity analysis is to exchange weights of the attribute values keeping the rest of the terms are fixed. The below table is the evaluation table which shows the sensitivity results.

Attribute Weight	Final Decision Matrix	Ordering
<$(\mathbf{0.3, 0.3, 0.4})$>	$\begin{pmatrix} < 0.1367 > \\ < 0.1617 > \\ < 0.1650 > \end{pmatrix}$	$G_3 > G_2 > G_1$
<$(\mathbf{0.33, 0.35, 0.32})$>	$\begin{pmatrix} < 0.1387 > \\ < 0.1641 > \\ < 0.1666 > \end{pmatrix}$	$G_3 > G_2 > G_1$
<$(\mathbf{0.3, 0.37, 0.33})$>	$\begin{pmatrix} < 0.1394 > \\ < 0.1621 > \\ < 0.1692 > \end{pmatrix}$	$G_3 > G_2 > G_1$
<$(\mathbf{0.45, 0.25, 0.3})$>	$\begin{pmatrix} < 0.1415 > \\ < 0.1641 > \\ < 0.1699 > \end{pmatrix}$	$G_3 > G_2 > G_1$
<$(\mathbf{0.25, 0.45, 0.3})$>	$\begin{pmatrix} < 0.1799 > \\ < 0.1559 > \\ < 0.1623 > \end{pmatrix}$	$G_1 > G_3 > G_2$

$<(\mathbf{0.25, 0.3, 0.45})>$	$\begin{pmatrix} <0.1367> \\ <0.1669> \\ <0.1680> \end{pmatrix}$	$G_3 > G_2 > G_1$
$<(\mathbf{0.4, 0.3, 0.3})>$	$\begin{pmatrix} <0.1544> \\ <0.1675> \\ <0.1666> \end{pmatrix}$	$G_2 > G_3 > G_1$
$<(\mathbf{0.3, 0.4, 0.3})>$	$\begin{pmatrix} <0.1503> \\ <0.1641> \\ <0.1652> \end{pmatrix}$	$G_3 > G_2 > G_1$

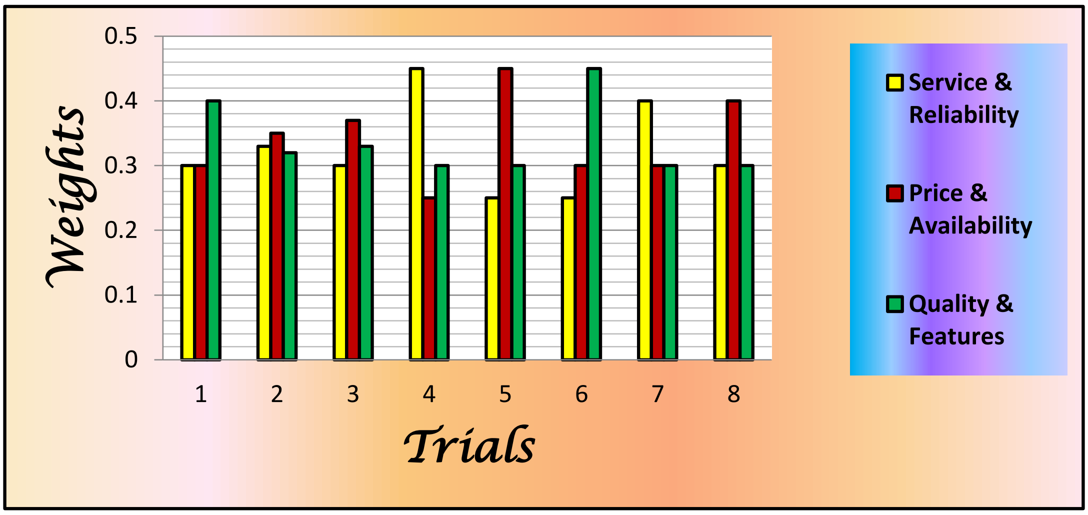

Figure 6.4.1: Sensitivity analysis on attribute functions

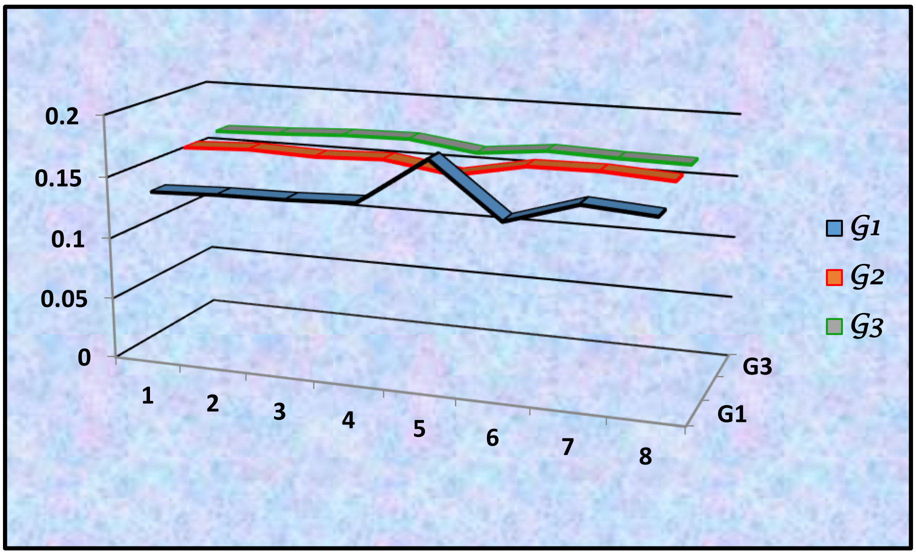

Figure 6.4.2: Best Alternative Mobile Communication Service

6.5 Comparison Table

This section actually contains a comparative study among the established work and proposed work.Comparing with [49,50] , we find that the best service provider among those three and it is noticed that in each case G_3becomes the best mobile communication service provider. The comparison table is given as follows:

Approach	Ranking
Deli[49]	$G_3 > G_2 > G_1$
Garg[50]	$G_3 > G_1 > G_2$
Proposed method	$G_3 > G_2 > G_1$

7. Conclusion and future research scope

The idea of pentagonal neutrosophic number is intriguing, competent and has ample scope of utilization in various research domains. In this research article, we vigorously erect the perception of pentagonal neutrosophic number from different aspects. We also resort to the perception of truthiness, falsity and ambiguity functions in case of pentagonal neutrosophic number when the membership functions are interconnected to each other and a new score function is formulated here. Also, two logical operators have been developed here theoretically as well as applied it in MCGDM problem. Finally we perform a sensitivity analysis and also demonstrate a comparative study with the other results derived from other research articles to enumerate our proposed work and conclude that our result is pretty satisfactory as we consider the pentagonal neutrosophic value in the problem of multi-criteria decision making.

Further, researchers can immensely apply this idea of neutrosophic number in numerous flourishing research fields like an engineering problem, mobile computing problems, diagnoses problem, realistic mathematical modelling, cloud computing issues, pattern recognition problems, an architecture based structural modelling, image processing, linear programming, big data analysis, neural network etc. Apart from these there is an immense scope of application basis works in various fields which can be constructed by taking the help of pentagonal neutrosophic numbers.

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On Q-Neutrosophic Soft Fields

Majdoleen Abu Qamar[1,*], Abd Ghafur Ahmad[2], Nasruddin Hassan[3]

[1,2,3]School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor, Malaysia.

E-mail: [1]p90675@siswa.ukm.edu.my, mjabuqamar@gmail.com, [2]ghafur@ukm.edu.my, [3]nas@ukm.edu.my

*Correspondence: mjabuqamar@gmail.com

Abstract: As an extension of neutrosophic soft sets, Q-neutrosophic soft sets were established to deal with two-dimensional indeterminate data. Different hybrid models of fuzzy sets were utilized to different algebraic structures, for example groups, rings, fields and lie-algebras. A field is an essential algebraic structure, which is widely used in algebra and several domains of mathematics. The motivation of the current work is to extend the thought of Q-neutrosophic soft sets to fields. In this paper, we define the notion of Q-neutrosophic soft fields. Structural characteristics of it are investigated. Moreover, the concepts of homomorphic image and pre-image of Q-neutrosophic soft fields are discussed. Finally, the Cartesian product of Q-neutrosophic soft fields is defined and some related properties are discussed.

Keywords: Neutrosophic soft field, Neutrosophic soft set, Q-neutrosophic soft field, Q-neutrosophic soft set.

1 Introduction

Fuzzy sets were established by Zadeh [1] as a tool to deal with uncertain data. Since then, fuzzy logic has been utilized in several real-world problems in uncertain environments. Consequently, numerous analysts discussed many results using distinct directions of fuzzy-set theory, for instance, interval valued fuzzy set [2] and intuitionistic fuzzy set [3]. These extensions can deal with uncertain real-world problems but it does not cope with indeterminate data. Thus, Smarandache [4] initiated the neutrosophic idea to overcome this problem. A neutrosophic set (NS) [5] is a mathematical notion serving issues containing inconsistent, indeterminate, and imprecise data. Molodtsov [6] introduced the concept of soft sets as another way to handle uncertainty. Since its initiation, a plenty of hybrid models of soft set have been produced, for example, fuzzy soft sets [7], neutrosophic soft sets (NSSs) [8]. Accordingly, NSSs became an important notion for more deep discussions [9–17]. NSSs were extended to Q-neutrosophic soft sets (Q-NSSs) [18] a new model that deals with two-dimensional uncertain data. Q-NSSs were further investigated and their basic operations and relations were discussed in [18, 19].

Different hybrid models of fuzzy sets and soft sets were utilized in different branches of mathematics, including algebra. This was started by Rosenfeld in 1971 [20] when he established the idea of fuzzy subgroup. Since then, the theories and approaches of fuzzy soft sets on different algebraic structures developed rapidly. Mukherjee and Bhattacharya [21] studied fuzzy groups, Sharma [22] discussed intuitionistic fuzzy groups. Recently, many researchers have applied different hybrid models of fuzzy sets and soft sets to several algebraic structures such as groups, semigroups, rings, fields and BCK/BCI-algebras [23–32]. NSs and NSSs have received more attention in studying the algebraic structure of set theories dealing with uncertainty. Cetkin and Aygun [33] established the concept of neutrosophic subgroup. Bera and Mahapatra introduced the notion of neutrosophic soft group [34], neutrosophic soft fields [35]. Moreover, two-dimensional hybrid models of fuzzy sets and soft sets were also applied to different algebraic structures. Solairaju and Nagarajan [36] introduced the notion of Q-fuzzy groups. Thiruveni and Solairaju defined the concept of neutrosophic Q-fuzzy subgroup [37], while Rasuli [38] established the notion of Q-fuzzy subring and anti Q-fuzzy subring. The concept of Q-NSSs was also implemented in the theories of groups and rings [39, 40].

Inspired by the above works and to utilize Q-NSSs to different algebraic structures, in the current paper, we continue the work presented in [41] about Q-neutrosophic soft fields (Q-NSFs) and investigate some of its structural characteristics; we give some theorems that simplifies the main definition, also we discuss the intersection and union of two Q-NSFs . The concepts of homomorphic image and pre-image of Q-NSFs are investigated. Also, we discuss the Cartesian product of Q-NSFs and discuss some related properties.

2 Preliminaries

In this section, we recall the basic definitions related to this work.

Definition 2.1 ([18]). Let X be a universal set, Q be a nonempty set and $A \subseteq E$ be a set of parameters. Let $\mu^l QNS(X)$ be the set of all multi Q-NSs on X with dimension $l = 1$. A pair (Γ_Q, A) is called a Q-NSS over X, where $\Gamma_Q : A \to \mu^l QNS(X)$ is a mapping, such that $\Gamma_Q(e) = \phi$ if $e \notin A$.

Definition 2.2 ([19]). The union of two Q-NSSs (Γ_Q, A) and (Ψ_Q, B) is the Q-NSS (Λ_Q, C) written as $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cup B$ and for all $c \in C, (x, q) \in X \times Q$, the truth-membership, indeterminacy-membership and falsity-membership of (Λ_Q, C) are as follows:

$$T_{\Lambda_{Q(c)}}(x,q) = \begin{cases} T_{\Gamma_{Q(c)}}(x,q) & \text{if } c \in A - B, \\ T_{\Psi_{Q(c)}}(x,q) & \text{if } c \in B - A, \\ \max\{T_{\Gamma_{Q(c)}}(x,q), T_{\Psi_{Q(c)}}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$I_{\Lambda_{Q(c)}}(x,q) = \begin{cases} I_{\Gamma_{Q(c)}}(x,q) & \text{if } c \in A - B, \\ I_{\Psi_{Q(c)}}(x,q) & \text{if } c \in B - A, \\ \min\{I_{\Gamma_{Q(c)}}(x,q), I_{\Psi_{Q(c)}}(x,q)\} & \text{if } c \in A \cap B, \end{cases}$$

$$F_{\Lambda_{Q(c)}}(x,q) = \begin{cases} F_{\Gamma_{Q(c)}}(x,q) & \text{if } c \in A - B, \\ F_{\Psi_{Q(c)}}(x,q) & \text{if } c \in B - A, \\ \min\{F_{\Gamma_{Q(c)}}(x,q), F_{\Psi_{Q(c)}}(x,q)\} & \text{if } c \in A \cap B. \end{cases}$$

Definition 2.3 ([19]). The intersection of two Q-NSSs (Γ_Q, A) and (Ψ_Q, B) is the Q-NSS (Λ_Q, C) written as $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, C)$, where $C = A \cap B$ and for all $c \in C$ and $(x, q) \in X \times Q$ the truth-membership, indeterminacy-membership and falsity-membership of (Λ_Q, C) are as follows:

$$T_{\Lambda_{Q(c)}}(x,q) = \min\{T_{\Gamma_{Q(c)}}(x,q), T_{\Psi_{Q(c)}}(x,q)\},$$
$$I_{\Lambda_{Q(c)}}(x,q) = \max\{I_{\Gamma_{Q(c)}}(x,q), I_{\Psi_{Q(c)}}(x,q)\},$$
$$F_{\Lambda_{Q(c)}}(x,q) = \max\{F_{\Gamma_{Q(c)}}(x,q), F_{\Psi_{Q(c)}}(x,q)\}.$$

3 Q-Neutrosophic Soft Fields

In this section, we define the notion of Q-NSF and discuss several related properties.

Definition 3.1. Let (Γ_Q, A) be a Q-NSS over a field $(F,+,.)$. Then, (Γ_Q, A) is said to be a Q-NSF over $(F,+,.)$ if for all $e \in A$, $\Gamma_Q(e)$ is a Q-neutrosophic subfield of $(F,+,.)$, where $\Gamma_Q(e)$ is a mapping given by $\Gamma_Q(e) : F \times Q \to [0,1]^3$.

Definition 3.2. Let $(F,+,.)$ be a field and (Γ_Q, A) be a Q-NSS over $(F,+,.)$. Then, (Γ_Q, A) is called a Q-NSF over $(F,+,.)$ if for all $x, y \in F, q \in Q$ and $e \in A$ it satisfies:

1. $T_{\Gamma_{Q(e)}}(x+y,q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\right\}, I_{\Gamma_{Q(e)}}(x+y,q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\right\}$ and $F_{\Gamma_{Q(e)}}(x+y,q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\right\}$.
2. $T_{\Gamma_{Q(e)}}(-x,q) \geq T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(-x,q) \leq I_{\Gamma_{Q(e)}}(x,q)$ and $F_{\Gamma_{Q(e)}}(-x,q) \leq F_{\Gamma_{Q(e)}}(x,q)$.
3. $T_{\Gamma_{Q(e)}}(x.y,q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\right\}, I_{\Gamma_{Q(e)}}(x.y,q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\right\}$ and $F_{\Gamma_{Q(e)}}(x.y,q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\right\}$.
4. $T_{\Gamma_{Q(e)}}(x^{-1},q) \geq T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(x^{-1},q) \leq I_{\Gamma_{Q(e)}}(x,q)$ and $F_{\Gamma_{Q(e)}}(x^{-1},q) \leq F_{\Gamma_{Q(e)}}(x,q)$.

Example 3.3. Let $F = (\mathbb{R},+,.)$ be the field of real numbers and $A = \mathbb{N}$ the set of natural numbers be the parametric set. Define a Q-NSS (Γ_Q, A) as follows for $q \in Q, x \in \mathbb{R}$ and $m \in \mathbb{N}$

$$T_{\Gamma_{Q(m)}}(x,q) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ \frac{1}{9m} & \text{if } x \text{ is irrational} \end{cases},$$
$$I_{\Gamma_{Q(m)}}(x,q) = \begin{cases} 1-\frac{1}{3m} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases},$$
$$F_{\Gamma_{Q(m)}}(x,q) = \begin{cases} 1+\frac{3}{m} & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

It is clear that $(\Gamma_Q, \mathbb{N})$ is a Q-NSF over F.

Proposition 3.4. *Let (Γ_Q, A) be a Q-NSF over $(F,+,.)$. Then, for the additive identity 0_F and the multiplicative identity 1_F, for all $x \in F, q \in Q$ and $e \in A$ the following hold*

1. $T_{\Gamma_{Q(e)}}(0_F,q) \geq T_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(0_F,q) \leq I_{\Gamma_{Q(e)}}(x,q)$ *and* $F_{\Gamma_{Q(e)}}(0_F,q) \leq F_{\Gamma_{Q(e)}}(x,q)$.

2. $T_{\Gamma_{Q(e)}}(1_F, q) \geq T_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(1_F, q) \leq I_{\Gamma_{Q(e)}}(x, q)$ *and* $F_{\Gamma_{Q(e)}}(1_F, q) \leq F_{\Gamma_{Q(e)}}(x, q)$, *for* $x \neq 0_F$.

3. $T_{\Gamma_{Q(e)}}(0_F, q) \geq T_{\Gamma_{Q(e)}}(1_F, q), I_{\Gamma_{Q(e)}}(0_F, q) \leq I_{\Gamma_{Q(e)}}(1_F, q)$ *and* $F_{\Gamma_{Q(e)}}(0_F, q) \leq F_{\Gamma_{Q(e)}}(1_F, q)$.

Proof. $\forall x \in F, q \in Q$ and $e \in A$

1. $T_{\Gamma_{Q(e)}}(0_F, q) = T_{\Gamma_{Q(e)}}(x - x, q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(x, q)\right\} = T_{\Gamma_{Q(e)}}(x, q)$,
$I_{\Gamma_{Q(e)}}(0_F, q) = I_{\Gamma_{Q(e)}}(x - x, q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(x, q)\right\} = I_{\Gamma_{Q(e)}}(x, q)$,
$F_{\Gamma_{Q(e)}}(0_F, q) = F_{\Gamma_{Q(e)}}(x - x, q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(x, q)\right\} = F_{\Gamma_{Q(e)}}(x, q)$.

2. $T_{\Gamma_{Q(e)}}(1_F, q) = T_{\Gamma_{Q(e)}}(x.x^{-1}, q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(x, q)\right\} = T_{\Gamma_{Q(e)}}(x, q)$,
$I_{\Gamma_{Q(e)}}(1_F, q) = I_{\Gamma_{Q(e)}}(x.x^{-1}, q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(x, q)\right\} = I_{\Gamma_{Q(e)}}(x, q)$,
$F_{\Gamma_{Q(e)}}(1_F, q) = F_{\Gamma_{Q(e)}}(x.x^{-1}, q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(x, q)\right\} = F_{\Gamma_{Q(e)}}(x, q)$.

3. Follows directly by applying 1. □

Theorem 3.5. *A Q-NSS* (Γ_Q, A) *over the field* $(F, +, .)$ *is a Q-NSF if and only if for all* $x, y \in F, q \in Q$ *and* $e \in A$

1. $T_{\Gamma_{Q(e)}}(x - y, q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(y, q)\right\}, I_{\Gamma_{Q(e)}}(x - y, q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(y, q)\right\}, F_{\Gamma_{Q(e)}}(x - y, q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(y, q)\right\}$.

2. $T_{\Gamma_{Q(e)}}(x.y^{-1}, q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(y, q)\right\}, I_{\Gamma_{Q(e)}}(x.y^{-1}, q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(y, q)\right\}, F_{\Gamma_{Q(e)}}(x.y^{-1}, q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(y, q)\right\}$.

Proof. Suppose that (Γ_Q, A) is a Q-NSF over $(F, +, .)$. Then,

$$\begin{aligned} T_{\Gamma_{Q(e)}}(x - y, q) &\geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(-y, q)\right\} \geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(y, q)\right\}, \\ I_{\Gamma_{Q(e)}}(x - y, q) &\leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(-y, q)\right\} \leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(y, q)\right\}, \\ F_{\Gamma_{Q(e)}}(x - y, q) &\leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(-y, q)\right\} \leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(y, q)\right\}. \end{aligned}$$

Also,

$$\begin{aligned} T_{\Gamma_{Q(e)}}(x.y^{-1}, q) &\geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(y^{-1}, q)\right\} \geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(y, q)\right\}, \\ I_{\Gamma_{Q(e)}}(x.y^{-1}, q) &\leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(y^{-1}, q)\right\} \leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(y, q)\right\}, \\ F_{\Gamma_{Q(e)}}(x.y^{-1}, q) &\leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(y^{-1}, q)\right\} \leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(y, q)\right\}. \end{aligned}$$

Conversely, Suppose that conditions 1 and 2 are satisfied. We show that for each $e \in A$, (Γ_Q, A) is a Q-neutrosophic subfield

$$\begin{aligned} T_{\Gamma_{Q(e)}}(-x, q) = T_{\Gamma_{Q(e)}}(0_F - x, q) &\geq \min\left\{T_{\Gamma_{Q(e)}}(0_F, q), T_{\Gamma_{Q(e)}}(x, q)\right\} \\ &\geq \min\left\{T_{\Gamma_{Q(e)}}(x, q), T_{\Gamma_{Q(e)}}(x, q)\right\} = T_{\Gamma_{Q(e)}}(x, q), \\ I_{\Gamma_{Q(e)}}(-x, q) = I_{\Gamma_{Q(e)}}(0_F - x, q) &\leq \max\left\{I_{\Gamma_{Q(e)}}(0_F, q), I_{\Gamma_{Q(e)}}(x, q)\right\} \\ &\leq \max\left\{I_{\Gamma_{Q(e)}}(x, q), I_{\Gamma_{Q(e)}}(x, q)\right\} = I_{\Gamma_{Q(e)}}(x, q), \\ F_{\Gamma_{Q(e)}}(-x, q) = F_{\Gamma_{Q(e)}}(0_F - x, q) &\leq \max\left\{F_{\Gamma_{Q(e)}}(0_F, q), F_{\Gamma_{Q(e)}}(x, q)\right\} \\ &\leq \max\left\{F_{\Gamma_{Q(e)}}(x, q), F_{\Gamma_{Q(e)}}(x, q)\right\} = F_{\Gamma_{Q(e)}}(x, q)\} \end{aligned}$$

also,

$$T_{\Gamma_{Q(e)}}(x+y,q) = T_{\Gamma_{Q(e)}}(x-(-y),q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\right\},$$
$$I_{\Gamma_{Q(e)}}(x+y,q) = I_{\Gamma_{Q(e)}}(x-(-y),q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\right\},$$
$$F_{\Gamma_{Q(e)}}(x+y,q) = F_{\Gamma_{Q(e)}}(x-(-y),q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\right\}.$$

Next,

$$\begin{aligned}
T_{\Gamma_{Q(e)}}(x^{-1},q) = T_{\Gamma_{Q(e)}}(1_F.x^{-1},q) &\geq \min\left\{T_{\Gamma_{Q(e)}}(1_F,q), T_{\Gamma_{Q(e)}}(x,q)\right\} \\
&\geq \min\left\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(x,q)\right\} = T_{\Gamma_{Q(e)}}(x,q), \\
I_{\Gamma_{Q(e)}}(x^{-1},q) = I_{\Gamma_{Q(e)}}(1_F.x^{-1},q) &\leq \max\left\{I_{\Gamma_{Q(e)}}(1_F,q), I_{\Gamma_{Q(e)}}(x,q)\right\} \\
&\leq \max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(x,q)\right\} = I_{\Gamma_{Q(e)}}(x,q), \\
F_{\Gamma_{Q(e)}}(x^{-1},q) = F_{\Gamma_{Q(e)}}(1_F.x^{-1},q) &\leq \max\left\{F_{\Gamma_{Q(e)}}(1_F,q), F_{\Gamma_{Q(e)}}(x,q)\right\} \\
&\leq \max\left\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(x,q)\right\} = F_{\Gamma_{Q(e)}}(x,q)\}
\end{aligned}$$

and

$$T_{\Gamma_{Q(e)}}(x.y,q) = T_{\Gamma_{Q(e)}}(x(y^{-1})^{-1},q) \geq \min\left\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\right\},$$
$$I_{\Gamma_{Q(e)}}(x.y,q) = I_{\Gamma_{Q(e)}}(x(y^{-1})^{-1},q) \leq \max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\right\},$$
$$F_{\Gamma_{Q(e)}}(x.y,q) = F_{\Gamma_{Q(e)}}(x(y^{-1})^{-1},q) \leq \max\left\{F_{\Gamma_{Q(e)}}(x,q), F_{\Gamma_{Q(e)}}(y,q)\right\}.$$

This completes the proof. □

Theorem 3.6. *Let* (Γ_Q, A) *and* (Ψ_Q, B) *be two Q-NSFs over* $(F, +, .)$. *Then,* $(\Gamma_Q, A) \cap (\Psi_Q, B)$ *is also Q-NSF over* $(F, +, .)$.

Proof. Let $(\Gamma_Q, A) \cap (\Psi_Q, B) = (\Lambda_Q, A \cap B)$. Now, $\forall x, y \in F, q \in Q$ and $e \in A \cap B$,

$$\begin{aligned}
T_{\Lambda_{Q(e)}}(x-y,q) &= \min\Big\{T_{\Gamma_{Q(e)}}(x-y,q), T_{\Psi_{Q(e)}}(x-y,q)\Big\} \\
&\geq \min\Big\{\min\left\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\right\}, \min\left\{T_{\Psi_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(y,q)\right\}\Big\} \\
&= \min\Big\{\min\left\{T_{\Gamma_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(x,q)\right\}, \min\left\{T_{\Gamma_{Q(e)}}(y,q), T_{\Psi_{Q(e)}}(y,q)\right\}\Big\} \\
&= \min\Big\{T_{\Lambda_{Q(e)}}(x,q), T_{\Lambda_{Q(e)}}(y,q)\Big\},
\end{aligned}$$

also,

$$\begin{aligned}
I_{\Lambda_{Q(e)}}(x-y,q) &= \max\Big\{I_{\Gamma_{Q(e)}}(x-y,q), I_{\Psi_{Q(e)}}(x-y,q)\Big\} \\
&\leq \max\Big\{\max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\right\}, \max\left\{I_{\Psi_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(y,q)\right\}\Big\} \\
&= \max\Big\{\max\left\{I_{\Gamma_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(x,q)\right\}, \max\left\{I_{\Gamma_{Q(e)}}(y,q), I_{\Psi_{Q(e)}}(y,q)\right\}\Big\} \\
&= \max\Big\{I_{\Lambda_{Q(e)}}(x,q), I_{\Lambda_{Q(e)}}(y,q)\Big\},
\end{aligned}$$

similarly, $F_{\Lambda_{Q(e)}}(x-y,q) \leq \max\Big\{F_{\Lambda_{Q(e)}}(x,q), F_{\Lambda_{Q(e)}}(y,q)\Big\}$. Next,

$$\begin{aligned}
T_{\Lambda_{Q(e)}}(x.y^{-1},q) &= \min\Big\{T_{\Gamma_{Q(e)}}(x.y^{-1},q), T_{\Psi_{Q(e)}}(x.y^{-1},q)\Big\} \\
&\geq \min\Big\{\min\big\{T_{\Gamma_{Q(e)}}(x,q), T_{\Gamma_{Q(e)}}(y,q)\big\}, \min\big\{T_{\Psi_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(y,q)\big\}\Big\} \\
&= \min\Big\{\min\big\{T_{\Gamma_{Q(e)}}(x,q), T_{\Psi_{Q(e)}}(x,q)\big\}, \min\big\{T_{\Gamma_{Q(e)}}(y,q), T_{\Psi_{Q(e)}}(y,q)\big\}\Big\} \\
&= \min\Big\{T_{\Lambda_{Q(e)}}(x,q), T_{\Lambda_{Q(e)}}(y,q)\Big\},
\end{aligned}$$

also,

$$\begin{aligned}
I_{\Lambda_{Q(e)}}(x.y^{-1},q) &= \max\Big\{I_{\Gamma_{Q(e)}}(x.y^{-1},q), I_{\Psi_{Q(e)}}(x.y^{-1},q)\Big\} \\
&\leq \max\Big\{\max\big\{I_{\Gamma_{Q(e)}}(x,q), I_{\Gamma_{Q(e)}}(y,q)\big\}, \max\big\{I_{\Psi_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(y,q)\big\}\Big\} \\
&= \max\Big\{\max\big\{I_{\Gamma_{Q(e)}}(x,q), I_{\Psi_{Q(e)}}(x,q)\big\}, \max\big\{I_{\Gamma_{Q(e)}}(y,q), I_{\Psi_{Q(e)}}(y,q)\big\}\Big\} \\
&= \max\Big\{I_{\Lambda_{Q(e)}}(x,q), I_{\Lambda_{Q(e)}}(y,q)\Big\}
\end{aligned}$$

similarly, we can show $F_{\Lambda_{Q(e)}}(x.y^{-1},q) \leq \max\Big\{F_{\Lambda_{Q(e)}}(x,q), F_{\Lambda_{Q(e)}}(y,q)\Big\}$. This completes the proof. □

Remark 3.7. For two Q-NSFs (Γ_Q, A) and (Ψ_Q, B) over $(F,+,.)$, $(\Gamma_Q, A) \cup (\Psi_Q, B)$ is not generally a Q-NSF.

For example, let $F = (\mathbb{Q},+,.)$, $E = 2\mathbb{Z}$. Consider two Q-NSFs (Γ_Q, E) and (Ψ_Q, E) over F as follows: for $x \in \mathbb{Q}, q \in Q$ and $m \in \mathbb{Z}$

$$T_{\Gamma_{Q(4m)}}(x,q) = \begin{cases} 0.50 & \text{if } x = 4tm, \exists t \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases}$$

$$I_{\Gamma_{Q(4m)}}(x,q) = \begin{cases} 0 & \text{if } x = 4tm, \exists t \in \mathbb{Z}, \\ 0.25 & \text{otherwise}, \end{cases}$$

$$F_{\Gamma_{Q(4m)}}(x,q) = \begin{cases} 0.40 & \text{if } x = 4tm, \exists t \in \mathbb{Z}, \\ 0.10 & \text{otherwise}, \end{cases}$$

and

$$T_{\Psi_{Q(4m)}}(x,q) = \begin{cases} 0.70 & \text{if } x = 6tm, \exists t \in \mathbb{Z}, \\ 0 & \text{otherwise}, \end{cases}$$

$$I_{\Psi_{Q(4m)}}(x,q) = \begin{cases} 0 & \text{if } x = 6tm, \exists t \in \mathbb{Z}, \\ 0.50 & \text{otherwise}, \end{cases}$$

$$F_{\Psi_{Q(4m)}}(x,q) = \begin{cases} 0.20 & \text{if } x = 6tm, \exists t \in \mathbb{Z}, \\ 0.40 & \text{otherwise}. \end{cases}$$

Let $(\Gamma_Q, A) \cup (\Psi_Q, B) = (\Lambda_Q, E)$. For $m = 2, x = 8, y = 12$ we have

$$T_{\Lambda_Q(8)}(8-12, q) = T_{\Lambda_Q(8)}(-4, q) = \max\left\{T_{\Gamma_Q(8)}(-4, q), T_{\Psi_Q(8)}(-4, q)\right\} = \max\{0, 0\} = 0$$

and

$$\begin{aligned}\min\left\{T_{\Lambda_Q(8)}(8, q), T_{\Lambda_Q(8)}(12, q)\right\} \\ &= \min\left\{\max\left\{T_{\Gamma_Q(8)}(8, q), T_{\Psi_Q(8)}(8, q)\right\}, \max\left\{T_{\Gamma_Q(8)}(12, q), T_{\Psi_Q(8)}(12, q)\right\}\right\} \\ &= \min\left\{\max\left\{0.50, 0\right\}, \max\left\{0, 0.7\right\}\right\} \\ &= \min\left\{0.50, 0.70\right\} = 0.50.\end{aligned}$$

Hence, $T_{\Lambda_Q(8)}(8-12, q) < \min\left\{T_{\Lambda_Q(8)}(8, q), T_{\Lambda_Q(8)}(12, q)\right\}$. Thus, the union is not a Q-NSF.

4 Q-Neutrosophic Soft Homomorphism

In this section, we define the Q-neutrosophic soft function, then define the image and pre-image of a Q-NSS under a Q-neutrosophic soft function. In continuation, we introduce the notion of Q-neutrosophic soft homomorphism along with some of it's properties.

Definition 4.1. Let $g : X \times Q \to Y \times Q$ and $h : A \to B$ be two functions where A and B are parameter sets. Then, the pair (g, h) is called a Q-neutrosophic soft function from $X \times Q$ to $Y \times Q$.

Definition 4.2. Let (Γ_Q, A) and (Ψ_Q, B) be two Q-NSSs defined over $X \times Q$ and $Y \times Q$, respectively, and (g, h) be a Q-neutrosophic soft function from $X \times Q$ to $Y \times Q$. Then,

1. The image of (Γ_Q, A) under (g, h), denoted by $(g, h)(\Gamma_Q, A)$, is a Q-NSS over $Y \times Q$ and is defined by:

$$(g, h)(\Gamma_Q, A) = \Big(g(\Gamma_Q), h(A)\Big) = \left\{\left\langle b, g(\Gamma_Q)(b) : b \in h(A)\right\rangle\right\},$$

where for all $b \in h(A), y \in Y$ and $q \in Q$,

$$T_{g(\Gamma_Q)(b)}(y, q) = \begin{cases}\max_{g(x,q)=(y,q)} \max_{h(a)=b}[T_{\Gamma_Q(a)}(x, q)] & \text{if } (x, q) \in g^{-1}(y, q), \\ 0 & \text{otherwise,}\end{cases}$$

$$I_{g(\Gamma_Q)(b)}(y, q) = \begin{cases}\min_{g(x,q)=(y,q)} \min_{h(a)=b}[I_{\Gamma_Q(a)}(x, q)] & \text{if } (x, q) \in g^{-1}(y, q), \\ 1 & \text{otherwise,}\end{cases}$$

$$F_{g(\Gamma_Q)(b)}(y, q) = \begin{cases}\min_{g(x,q)=(y,q)} \min_{h(a)=b}[F_{\Gamma_Q(a)}(x, q)] & \text{if } (x, q) \in g^{-1}(y, q), \\ 1 & \text{otherwise,}\end{cases}$$

2. The preimage of (Ψ_Q, B) under (g, h), denoted by $(g, h)^{-1}(\Psi_Q, B)$, is a Q-NSS over X and is defined

by:

$$(g,h)^{-1}(\Psi_Q,B)=\Big(g^{-1}(\Psi_Q),h^{-1}(B)\Big)=\Big\{\Big\langle a,g^{-1}(\Psi_Q)(a):a\in h^{-1}(B)\Big\rangle\Big\},$$

where for all $a\in h^{-1}(B), x\in X$ and $q\in Q$,

$$\begin{aligned}T_{g^{-1}(\Psi_Q)(a)}(x,q)&=T_{\Psi_Q[h(a)]}(g(x,q)),\\ I_{g^{-1}(\Psi_Q)(a)}(x,q)&=I_{\Psi_Q[h(a)]}(g(x,q)),\\ F_{g^{-1}(\Psi_Q)(a)}(x,q)&=F_{\Psi_Q[h(a)]}(g(x,q)).\end{aligned}$$

If g and h are injective (surjective), then (g,h) is injective (surjective).

Definition 4.3. Let (g,h) be a Q-neutrosophic soft function from $X\times Q$ to $Y\times Q$. If g is a homomorphism from $X\times Q$ to $Y\times Q$, then (g,h) is said to be a Q-neutrosophic soft homomorphism. If g is an isomorphism from $X\times Q$ to $Y\times Q$ and h is a one-to-one mapping from A to B, then (g,h) is said to be a Q-neutrosophic soft isomorphism.

Example 4.4. Let $A=\mathbb{N}$ (the set of natural numbers) be the parametric set and $F=(\mathbb{Z}_5,+,.)$ be a field. Define a Q-NSS (Γ_Q,A) as follows, for any $a\in A, q\in Q$ and $x\in\mathbb{Z}_5$,

$$T_{\Gamma_Q(a)}(x,q)=\begin{cases}0 & \text{if } x\in\{\bar{1},\bar{3}\}\\ \frac{1}{3a} & \text{if } x\in\{\bar{0},\bar{2},\bar{4}\}\end{cases},$$

$$I_{\Gamma_Q(a)}(x,q)=\begin{cases}1-\frac{1}{a} & \text{if } x\in\{\bar{1},\bar{3}\}\\ 0 & \text{if } x\in\{\bar{0},\bar{2},\bar{4}\}\end{cases},$$

$$F_{\Gamma_Q(a)}(x,q)=\begin{cases}\frac{3}{a+1} & \text{if } x\in\{\bar{1},\bar{3}\}\\ 0 & \text{if } x\in\{\bar{0},\bar{2},\bar{4}\}\end{cases}.$$

Now, let $g:\mathbb{Z}_5\times Q\to\mathbb{Z}_5\times Q$ and $h:\mathbb{N}\to\mathbb{N}$ be given by $g(x,q)=3x+1$ and $h(a)=a^2$. Then for $b\in\mathbb{N}^2, y\in 3\mathbb{Z}_5+1$, the image of (Γ_Q,A) under (g,h) as follows :

$$T_{g(\Gamma_Q)(b)}(y,q)=\begin{cases}0 & \text{if } y\in\{\bar{0},\bar{2},\bar{4}\}\\ \frac{1}{3\sqrt{b}} & \text{if } y\in\{\bar{1},\bar{3}\}\end{cases},$$

$$I_{g(\Gamma_Q)(b)}(y,q)=\begin{cases}1-\frac{1}{\sqrt{b}} & \text{if } y\in\{\bar{0},\bar{2},\bar{4}\}\\ 0 & \text{if } y\in\{\bar{1},\bar{3}\}\end{cases},$$

$$F_{g(\Gamma_Q)(b)}(y,q)=\begin{cases}\frac{1}{1+\sqrt{b}} & \text{if } y\in\{\bar{0},\bar{2},\bar{4}\}\\ 0 & \text{if } y\in\{\bar{1},\bar{3}\}\end{cases}.$$

Theorem 4.5. *Let (Γ_Q,A) be a Q-NSF over F_1 and $(g,h):F_1\times Q\to F_2\times Q$ be a Q-neutrosophic soft homomorphism. Then, $(g,h)(\Gamma_Q,A)$ is a Q-NSF over F_2.*

Proof. Let $b\in h(A)$ and $y_1,y_2\in F_2$. For $g^{-1}(y_1,q)=\phi$ or $g^{-1}(y_2,q)=\phi$, the proof is straight forward.

So, assume there exists $x_1, x_2 \in F_1$ such that $g(x_1, q) = (y_1, q)$ and $g(x_2, q) = (y_2, q)$. Then,

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(y_1 - y_2, q) &= \max_{g(x,q)=(y_1-y_2,q)} \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x, q) \Big] \\ &\geq \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_1 - x_2, q) \Big] \\ &\geq \max_{h(a)=b} \Big[\min \Big\{ T_{\Gamma_Q(a)}(x_1, q), T_{\Gamma_Q(a)}(-x_2, q) \Big\} \Big] \\ &\geq \max_{h(a)=b} \Big[\min \Big\{ T_{\Gamma_Q(a)}(x_1, q), T_{\Gamma_Q(a)}(x_2, q) \Big\} \Big] \\ &= \min \Big\{ \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_1, q) \Big], \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_2, q) \Big] \Big\} \end{aligned}$$

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(y_1.y_2^{-1}, q) &= \max_{g(x,q)=(y_1.y_2^{-1},q)} \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x, q) \Big] \\ &\geq \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_1.x_2^{-1}, q) \Big] \\ &\geq \max_{h(a)=b} \Big[\min \Big\{ T_{\Gamma_Q(a)}(x_1, q), T_{\Gamma_Q(a)}(x_2^{-1}, q) \Big\} \Big] \\ &\geq \max_{h(a)=b} \Big[\min \Big\{ T_{\Gamma_Q(a)}(x_1, q), T_{\Gamma_Q(a)}(x_2, q) \Big\} \Big] \\ &= \min \Big\{ \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_1, q) \Big], \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_2, q) \Big] \Big\} \end{aligned}$$

Since, the inequality is satisfied for each $x_1, x_2 \in F_1$, satisfying $g(x_1, q) = (y_1, q)$ and $g(x_2, q) = (y_2, q)$. Then,

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(y_1 - y_2, q) &\geq \min \Big\{ \max_{g(x_1,q)=(y_1,q)} \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_1, q) \Big], \max_{g(x_2,q)=(y_1,q)} \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_2, q) \Big] \Big\} \\ &= \min \Big\{ T_{g(\Gamma_Q)(b)}(y_1, q), T_{g(\Gamma_Q)(b)}(y_2, q) \Big\}. \end{aligned}$$

$$\begin{aligned} T_{g(\Gamma_Q)(b)}(y_1.y_2^{-1}, q) &\geq \min \Big\{ \max_{g(x_1,q)=(y_1,q)} \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_1, q) \Big], \max_{g(x_2,q)=(y_1,q)} \max_{h(a)=b} \Big[T_{\Gamma_Q(a)}(x_2, q) \Big] \Big\} \\ &= \min \Big\{ T_{g(\Gamma_Q)(b)}(y_1, q), T_{g(\Gamma_Q)(b)}(y_2, q) \Big\}. \end{aligned}$$

Similarly, we show that

$I_{g(\Gamma_Q)(b)}(y_1 - y_2, q) \leq \max \Big\{ I_{g(\Gamma_Q)(b)}(y_1, q), I_{g(\Gamma_Q)(b)}(y_2, q) \Big\}$,

$I_{g(\Gamma_Q)(b)}(y_1.y_2^{-1}, q) \leq \max \Big\{ I_{g(\Gamma_Q)(b)}(y_1, q), I_{g(\Gamma_Q)(b)}(y_2, q) \Big\}$,

$F_{g(\Gamma_Q)(b)}(y_1 - y_2, q) \leq \max \Big\{ F_{g(\Gamma_Q)(b)}(y_1, q), F_{g(\Gamma_Q)(b)}(y_2, q) \Big\}$,

$F_{g(\Gamma_Q)(b)}(y_1.y_2^{-1}, q) \leq \max \Big\{ F_{g(\Gamma_Q)(b)}(y_1, q), F_{g(\Gamma_Q)(b)}(y_2, q) \Big\}$.

□

Theorem 4.6. *Let (Ψ_Q, B) be a Q-NSF over F_2 and (g,h) be a Q-neutrosophic soft homomorphism from $F_1 \times Q$ to $F_2 \times Q$. Then, $(g,h)^{-1}(\Psi_Q, B)$ is a Q-NSF over over F_1.*

Proof. For $a \in h^{-1}(B)$ and $x_1, x_2 \in F_1$, we have

$$\begin{aligned} T_{g^{-1}(\Psi_Q)(a)}(x_1 - x_2, q) &= T_{\Psi_Q[h(a)]}(g(x_1 - x_2, q)) \\ &= T_{\Psi_Q[h(a)]}(g(x_1, q) - g(x_2, q)) \\ &\geq \min\left\{T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(-g(x_2, q))\right\} \\ &\geq \min\left\{T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(g(x_2, q))\right\} \\ &= \min\left\{T_{g^{-1}(\Psi_Q)(a)}(x_1, q), T_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\} \end{aligned}$$

and

$$\begin{aligned} T_{g^{-1}(\Psi_Q)(a)}(x_1.x_2^{-1}, q) &= T_{\Psi_Q[h(a)]}(g(x_1.x_2^{-1}, q)) \\ &= T_{\Psi_Q[h(a)]}(g(x_1, q).g(x_2^{-1}, q)) \\ &\geq \min\left\{T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(g(x_2, q)^{-1})\right\} \\ &\geq \min\left\{T_{\Psi_Q[h(a)]}(g(x_1, q)), T_{\Psi_Q[h(a)]}(g(x_2, q))\right\} \\ &= \min\left\{T_{g^{-1}(\Psi_Q)(a)}(x_1, q), T_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\} \end{aligned}$$

Similarly, we can obtain

$$\begin{aligned} I_{g^{-1}(\Psi_Q)(a)}(x_1 - x_2, q) &\leq \max\left\{I_{g^{-1}(\Psi_Q)(a)}(x_1, q), I_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}, \\ I_{g^{-1}(\Psi_Q)(a)}(x_1.x_2^{-1}, q) &\leq \max\left\{I_{g^{-1}(\Psi_Q)(a)}(x_1, q), I_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}, \\ F_{g^{-1}(\Psi_Q)(a)}(x_1 - x_2, q) &\leq \max\left\{F_{g^{-1}(\Psi_Q)(a)}(x_1, q), F_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}, \\ F_{g^{-1}(\Psi_Q)(a)}(x_1.x_2^{-1}, q) &\leq \max\left\{F_{g^{-1}(\Psi_Q)(a)}(x_1, q), F_{g^{-1}(\Psi_Q)(a)}(x_2, q)\right\}. \end{aligned}$$

Thus, the theorem is proved. □

5 Cartesian Product of Q-Neutrosophic Soft Fields

In this section, we define the Cartesian product of Q-NSFs and prove that it is also a Q-NSF.

Definition 5.1. Let (Γ_Q, A) and (Ψ_Q, B) be two Q-NSFs over $(F_1, +, .)$ and $(F_2, +, .)$, respectively. Then, their Cartesian product $(\Lambda_Q, A \times B) = (\Gamma_Q, A) \times (\Psi_Q, B)$, where $\Lambda_Q(a,b) = \Gamma_Q(a) \times \Psi_Q(b)$ for $(a,b) \in A \times B$. Analytically, for $x \in F_1, y \in F_2$ and $q \in Q$

$$\Lambda_Q(a,b) = \left\{\left\langle \big((x,y),q\big), T_{\Lambda_Q(a,b)}\big((x,y),q\big), I_{\Lambda_Q(a,b)}\big((x,y),q\big), F_{\Lambda_Q(a,b)}\big((x,y),q\big)\right\rangle\right\}, \text{ where}$$

$$\begin{aligned} T_{\Lambda_Q(a,b)}\big((x,y),q\big) &= \min\big\{T_{\Gamma_Q(a)}\big(x,q\big), T_{\Psi_Q(b)}\big(y,q\big)\big\}, \\ I_{\Lambda_Q(a,b)}\big((x,y),q\big) &= \max\big\{I_{\Gamma_Q(a)}\big(x,q\big), I_{\Psi_Q(b)}\big(y,q\big)\big\}, \\ F_{\Lambda_Q(a,b)}\big((x,y),q\big) &= \max\big\{F_{\Gamma_Q(a)}\big(x,q\big), F_{\Psi_Q(b)}\big(y,q\big)\big\}. \end{aligned}$$

Theorem 5.2. *Let (Γ_Q, A) and (Ψ_Q, B) be two Q-NSFs over $(F_1, +, .)$ and $(F_2, +, .)$, respectively. Then, their Cartesian product $(\Gamma_Q, A) \times (\Psi_Q, B)$ is a Q-NSF over $(F_1 \times F_2)$.*

Proof. Let $(\Lambda_Q, A \times B) = (\Gamma_Q, A) \times (\Psi_Q, B)$, where $\Lambda_Q(a,b) = \Gamma_Q(a) \times \Psi_Q(b)$ for $(a,b) \in A \times B$. Then, for $\big((x_1,y_1),q\big), \big((x_2,y_2),q\big) \in (F_1 \times F_2) \times Q$ we have,

$$\begin{aligned} &T_{\Lambda_Q(a,b)}\Big(\big((x_1,y_1)-(x_2,y_2),q\big)\Big) \\ &= T_{\Lambda_Q(a,b)}\big((x_1-x_2, y_1-y_2),q\big) \\ &= \min\big\{T_{\Gamma_Q(a)}\big((x_1-x_2),q\big), T_{\Psi_Q(b)}\big((y_1-y_2),q\big)\big\} \\ &\geq \min\Big\{\min\big\{T_{\Gamma_Q(a)}\big(x_1,q\big), T_{\Gamma_Q(a)}\big(-x_2,q\big)\big\}, \min\big\{T_{\Psi_Q(b)}\big(y_1,q\big), T_{\Psi_Q(b)}\big(-y_2,q\big)\big\}\Big\} \\ &\geq \min\Big\{\min\big\{T_{\Gamma_Q(a)}\big(x_1,q\big), T_{\Gamma_Q(a)}\big(x_2,q\big)\big\}, \min\big\{T_{\Psi_Q(b)}\big(y_1,q\big), T_{\Psi_Q(b)}\big(y_2,q\big)\big\}\Big\} \\ &= \min\Big\{\min\big\{T_{\Gamma_Q(a)}\big(x_1,q\big), T_{\Psi_Q(b)}\big(y_1,q\big)\big\}, \min\big\{T_{\Gamma_Q(a)}\big(x_2,q\big), T_{\Psi_Q(b)}\big(y_2,q\big)\big\}\Big\} \\ &= \min\Big\{T_{\Lambda_Q(a,b)}\big((x_1,y_1),q\big), T_{\Lambda_Q(a,b)}\big((x_2,y_2),q\big)\Big\} \end{aligned}$$

also,

$$\begin{aligned} &I_{\Lambda_Q(a,b)}\Big(\big((x_1,y_1)-(x_2,y_2),q\big)\Big) \\ &= I_{\Lambda_Q(a,b)}\big((x_1-x_2, y_1-y_2),q\big) \\ &= \max\big\{I_{\Gamma_Q(a)}\big((x_1-x_2),q\big), I_{\Psi_Q(b)}\big((y_1-y_2),q\big)\big\} \\ &\leq \max\Big\{\max\big\{I_{\Gamma_Q(a)}\big(x_1,q\big), I_{\Gamma_Q(a)}\big(-x_2,q\big)\big\}, \max\big\{I_{\Psi_Q(b)}\big(y_1,q\big), I_{\Psi_Q(b)}\big(-y_2,q\big)\big\}\Big\} \\ &\leq \max\Big\{\max\big\{I_{\Gamma_Q(a)}\big(x_1,q\big), I_{\Gamma_Q(a)}\big(x_2,q\big)\big\}, \max\big\{I_{\Psi_Q(b)}\big(y_1,q\big), I_{\Psi_Q(b)}\big(y_2,q\big)\big\}\Big\} \\ &= \max\Big\{\max\big\{I_{\Gamma_Q(a)}\big(x_1,q\big), I_{\Psi_Q(b)}\big(y_1,q\big)\big\}, \max\big\{I_{\Gamma_Q(a)}\big(x_2,q\big), I_{\Psi_Q(b)}\big(y_2,q\big)\big\}\Big\} \\ &= \max\Big\{I_{\Lambda_Q(a,b)}\big((x_1,y_1),q\big), I_{\Lambda_Q(a,b)}\big((x_2,y_2),q\big)\Big\}, \end{aligned}$$

similarly, $F_{\Lambda_Q(a,b)}\Big(\big((x_1,y_1)-(x_2,y_2),q\big)\Big) \leq \max\Big\{F_{\Lambda_Q(a,b)}\big((x_1,y_1),q\big), F_{\Lambda_Q(a,b)}\big((x_2,y_2),q\big)\Big\}$. Next,

$$\begin{aligned} &T_{\Lambda_Q(a,b)}\Big(\big((x_1,y_1).(x_2,y_2)^{-1},q\big)\Big) \\ &= T_{\Lambda_Q(a,b)}\big((x_1.x_2^{-1}, y_1.y_2^{-1}),q\big) \\ &= \min\big\{T_{\Gamma_Q(a)}\big((x_1.x_2^{-1}),q\big), T_{\Psi_Q(b)}\big((y_1.y_2^{-1}),q\big)\big\} \\ &\geq \min\Big\{\min\big\{T_{\Gamma_Q(a)}\big(x_1,q\big), T_{\Gamma_Q(a)}\big(x_2^{-1},q\big)\big\}, \min\big\{T_{\Psi_Q(b)}\big(y_1,q\big), T_{\Psi_Q(b)}\big(y_2^{-1},q\big)\big\}\Big\} \end{aligned}$$

$$\geq \min\Big\{\min\big\{T_{\Gamma_Q(a)}\big(x_1,q\big),T_{\Gamma_Q(a)}\big(x_2,q\big)\big\},\min\big\{T_{\Psi_Q(b)}\big(y_1,q\big),T_{\Psi_Q(b)}\big(y_2,q\big)\big\}\Big\}$$
$$=\min\Big\{\min\big\{T_{\Gamma_Q(a)}\big(x_1,q\big),T_{\Psi_Q(b)}\big(y_1,q\big)\big\},\min\big\{T_{\Gamma_Q(a)}\big(x_2,q\big),T_{\Psi_Q(b)}\big(y_2,q\big)\big\}\Big\}$$
$$=\min\Big\{T_{\Lambda_Q(a,b)}\big((x_1,y_1),q\big),T_{\Lambda_Q(a,b)}\big((x_2,y_2),q\big)\Big\},$$

$$I_{\Lambda_Q(a,b)}\Big(\big((x_1,y_1).(x_2,y_2)^{-1},q\big)\Big)$$
$$=I_{\Lambda_Q(a,b)}\big((x_1.x_2^{-1},y_1.y_2^{-1}),q\big)$$
$$=\max\big\{I_{\Gamma_Q(a)}\big((x_1.x_2^{-1}),q\big),I_{\Psi_Q(b)}\big((y_1.y_2^{-1}),q\big)\big\}$$
$$\leq \max\Big\{\max\big\{I_{\Gamma_Q(a)}\big(x_1,q\big),I_{\Gamma_Q(a)}\big(x_2^{-1},q\big)\big\},\max\big\{I_{\Psi_Q(b)}\big(y_1,q\big),I_{\Psi_Q(b)}\big(y_2^{-1},q\big)\big\}\Big\}$$
$$\leq \max\Big\{\max\big\{I_{\Gamma_Q(a)}\big(x_1,q\big),I_{\Gamma_Q(a)}\big(x_2,q\big)\big\},\max\big\{I_{\Psi_Q(b)}\big(y_1,q\big),I_{\Psi_Q(b)}\big(y_2,q\big)\big\}\Big\}$$
$$=\max\Big\{\max\big\{I_{\Gamma_Q(a)}\big(x_1,q\big),I_{\Psi_Q(b)}\big(y_1,q\big)\big\},\max\big\{I_{\Gamma_Q(a)}\big(x_2,q\big),I_{\Psi_Q(b)}\big(y_2,q\big)\big\}\Big\}$$
$$=\max\Big\{I_{\Lambda_Q(a,b)}\big((x_1,y_1),q\big),I_{\Lambda_Q(a,b)}\big((x_2,y_2),q\big)\Big\},$$

similarly, $F_{\Lambda_Q(a,b)}\Big(\big((x_1,y_1),q\big).\big((x_2,y_2)^{-1},q\big)\Big) \leq \max\Big\{F_{\Lambda_Q(a,b)}\big((x_1,y_1),q\big),F_{\Lambda_Q(a,b)}\big((x_2,y_2),q\big)\Big\}$. This completes the proof. □

6 Conclusions

In this study, we have introduced the concept of Q-neutrosophic soft fields. We have investigated some of its structural characteristics. Also, we have discussed the concepts of homomorphic image and pre-image of Q-neutrosophic soft fields. Moreover, we have defined the Cartesian product of Q-neutrosophic soft fields and discussed some related properties. The proposed notion enriches knowledge on neutrosophic sets in the branch of algebra. Also, it illuminates the way for more further deep discussion in algebra under neutrosophic and Q-neutrosophic soft environment for example, by establishing the notions of n-valued neutrosophic soft fields Q-neutrosophic soft modules and more.

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Neutrosophic projective G-submodules

Binu R.[1,*] and P. Isaac[2]

[1]Department of Mathematics,Rajagiri School of Engineering and Technology, India; 1984binur@gmail.com

[2]Department of Mathematics,Bharata Mata College, Thrikkakara, India; pibmct@gmail.com

[*]Correspondence: 1984binur@gmail.com; Tel.: (91-9946167116)

Abstract. A significant area of module theory is the concept of free modules, projective modules and injective modules. The goal of this study is to characterize the projective G-modules under a single-valued neutrosophic set. So we define neutrosophic G-submodule as a generic version of projective G-submodule. It also describes and derives fundamental algebraic properties including quotient space and direct sum of neutrosophic projective G-submodules

Keywords: Neutrosophic set; Neutrosophic G-module;Direct sum; Projective G-module; Neutrosophic projective G-module

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1. Introduction

The projective G-module in the abstract algebra plays a pivotal role to analyze the algebraic structure G-module and its characteristics. Cartan and Eilemberge [16] introduced the concept of projective modules that offer significant ideas through the theoretical approach to module theory. The algebraic structure G-module widely used to study the representation of finite groups developed by Frobenius G and Burnside [11] in the 19th century. Several researchers have studied the algebraic structure in pure mathematics associated with uncertainty. Since Zadeh [35] introduced fuzzy sets, fuzzification of algebraic structures was an important milestone in classical algebraic studies. The notion of a fuzzy submodule was introduced by Negoita and Ralescu [25] and further developed by Mashinchi and Zahedi [24]. This basic notion has been generalized in several ways after Zadeh's implementation of fuzzy sets [4, 5]. In 1986 Atanassov [6] put forward intuitionistic fuzzy set theory in which each element coincides with membership grades and non-membership grades. Biswas [9] applied the idea of the intuitionistic fuzzy set to the algebraic structure group and K. Hur et.al. [21] additionally studied it. In

2011 P. Isaac, P.P.John [22] studied about algebraic nature of intuitionistic fuzzy submodule of a classical module.

The theory of neutrosophy first appeared in philosophy [30] and then evolved neutrosophic set as a mathematical tool. In 1995, Smarandache [31] outlined the neutrosophicset as a combination of tri valued logic with non-standard analysis in which three different types of membership values represent each element of a set.The main objective of the neutrosophic set is to narrow the gap between the vague, ambiguous and imprecise real-world situations. Neutrosophic set theory gives a thorough scientific and mathematical model knowledge in which speculative and uncertain hypothetical phenomena can be managed by hierarchal membership of the components " truth / indeterminacy / falsehood " [2,3,32]. Neutrosophic set generalizes a classical set, fuzzy set, interval-valued fuzzy set and intuitionistic fuzzy set that can be used to make a mathematical model for the real problems of science and engineering. From a scientific and engineering perspective, Wang et.al. [20] specified the definition of a neutrosophic set, which is called a single-valued neutrosophic set. Several scientists dealt with the neutrosophic set notion as a new evolving instrument for uncertain information processing and a general framework for uncertainty analysis in data set [1, 7, 17, 28].

The consolidation of the neutrosophic set hypothesis with algebraic structures is a growing trend in mathematical research. Among the various branches of applied and pure mathematics, abstract algebra was one of the first few topics where the research was carried out using the neutrosophic set concept. W. B. Vasantha Kandasamy and Florentin Smarandache [23] initially presented basic algebraic neutrosophic structures and their application to advanced neutrosophic models. Vidan Cetkin [12, 13] consolidated the neutrosophic set theory and algebraic structures, creating neutrosophic subgroups and neutrosophic submodules. F. Sherry [18, 19] introduced the concept of fuzzy G-modules in which the concept of fuzzy sets was combined with G-module and the theory of group representation. One of the key developments in the neutrosophic set theory is the hybridization of the neutrosophic set with the algebraic structure G-module. The above fact leads to inspiration for conducting an exploratory study in the field of abstract algebra, especially in the theory of G-modules in conjunction with neutrosophic set. In this paper we described neutrosophic projective G-submodule as the general case of projective G-module and derived its algebraic properties.

The reminder of this work is structured as follows. Section 2 briefs about necessary preliminary definitions and results which are basic for a better and clear cognizance of next sections. Section 3 defines neutrosophic projective G-modules, algebraic extension of projective G-submodules and derive the theorems related to quotient space and direct sum of neutrosophic G-submodules. A comprehensive overview, relevance and future study of this work is defined at the end of the paper in Section 4.

2. Preliminaries

In this section, we recall some of the preliminary definitions and results which are essential for a better and clear comprehension of the upcoming sections.

Definition 2.1. [14] Let $(G, *)$ be a group. A vector space M over the field K is called a G-*module*, denoted as G_M, if for every $g \in G$ and $m \in M$; $\exists$ a product (called the action of G on M), $g \cdot m \in M$ satisfies the following axioms

(1) $1_G \cdot m = m; \ \ \forall m \in M$ (1_G being the identity element of G)
(2) $(g * h) \cdot m = g \cdot (h \cdot m); \ \ \forall m \in M$ and $g, h \in G$
(3) $g \cdot (k_1 m_1 + k_2 m_2) = k_1(g \cdot m_1) + k_2(g \cdot m_2); \forall\ k_1, k_2 \in K; m_1, m_2 \in M$ ".

Example 2.1. [18] Let $G = \{1, -1, i, -i\}$ and $M = \mathbb{C}^n$; $(n \geq 1)$. Then M is a vector space over $\mathbb{C}$ and under the usual addition and multiplication of complex numbers we can show that M is a G-module.

Definition 2.2. [15] Let M be a G-module. A vector subspace N of M is a G-submodule if N is also a G-module under the same action of G.

Definition 2.3. [15] Let M and M^* be G-modules. A mapping $f : M \to M^*$ is called a G module homomorphism ($Hom_G(M, M^*)$) if $\forall\ k_1, k_2 \in K, m_1, m_2 \in M, g \in G$ satisfies the following conditions

(1) $f(k_1 m_1 + k_2 m_2) = k_1 f(m_1) + k_2 f(m_2)$
(2) $f(gm) = gf(m)$

Definition 2.4. [10, 29] A G-module M is projective if for any G-module M^* and any G-submodule N^* of M^*, every homomorphism $\varphi : M \to M^*/N^*$ can be lifted to a homomorphism $\psi : M \to M^*$ or $\pi \circ \psi = \varphi$ where $\pi : M^* \to M^*/N^*$.

Remark 2.1. A G-module M is projective if and only if M is M^* projective for every G-module M^*

Theorem 2.2. [29] Let M and M^* be G-modules such that M is M^* projective. Let N^* be any G-submodule of M^*. Then M is N^* projective and M is M^*/N^* projective.

Proposition 2.1. [29] Let M *and* M_i be G-modules.Then M is $\oplus_{i=1}^n M_i$-projective if and only if M is M_i-projective $\forall\ i$

Definition 2.5. [32, 34] A neutrosophic set P of the universal set X is defined as $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\}$ where $t_P, i_P, f_P : X \to (^-0, 1^+)$. The three components t_P, i_P *and* f_P represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval $(^{-}0, 1^{+})$ [27].

Remark 2.3. [20, 32]

(1) If $t_P, i_P, f_P : X \to [0,1]$, then P is known as single valued neutrosophic set(SVNS).
(2) In this paper, we discuss about the algebraic structure R-module with underlying set as SVNS. For simplicity SVNS will be called neutrosophic set.
(3) U^X denotes the set of all neutrosophic subset of X or neutrosophic power set of X.

Definition 2.6. [26, 32] Let $P, Q \in U^X$. Then P is contained in Q, denoted as $P \subseteq Q$ if and only if $P(\eta) \leqslant Q(\eta)\ \forall \eta \in X$, this means that $t_P(\eta) \leq t_Q(\eta), i_P(\eta) \leq i_Q(\eta), f_P(\eta) \geq f_Q(\eta),\ \forall\ \eta \in X$.

Definition 2.7. [26, 33]For any neutrosophic subset $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\}$, the support P^* of the neutrosophic set P can be defined as $P^* = \{\eta \in X, t_P(\eta) > 0, i_P(\eta) > 0, f_P(\eta) < 1\}$.

Definition 2.8. [8] Let $(G, *)$ be a group and M be a G module over a field K. A neutrosphic G-submodule is a neutrosophic set $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in M\}$ in G_M such that the following conditions are satisfied;

(1) $t_P(\varrho\eta + \tau\theta) \geq t_P(\eta) \wedge t_P(\theta)$
$i_P(\varrho\eta + \tau\theta) \geq i_P(\eta) \wedge i_P(\theta)$
$f_P(\varrho\eta + \tau\theta) \leq f_P(\eta) \vee f_P(\theta)$,
$\forall\ \eta, \theta \in M, \varrho, \tau \in K$
(2) $t_P(\xi\eta) \geq t_P(\eta)$
$i_P(\xi\eta) \geq i_P(\eta)$
$f_P(\xi\eta) \leq f_P(\eta)\ \forall\ \xi \in G, \eta \in M$

Remark 2.4. We denote neutrosophic G-submodules using single valued neutrosophic set by $U(G_M)$.

Example 2.2. Consider the example 2.1 for G-module M. Define a neutrosophic set

$$P = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta) : \eta \in M\}$$

of M where

$$t_P(\eta) = \begin{cases} 1 & if\ \eta = 0 \\ 0.5 & if\ \eta \neq 0 \end{cases},\ i_P(\eta) = \begin{cases} 1 & if\ \eta = 0 \\ 0.5 & if\ \eta \neq 0 \end{cases},\ f_P(\eta) = \begin{cases} 0 & if\ \eta = 0 \\ 0.25 & if\ \eta \neq 0 \end{cases}$$

Then P is a neutrosophic G-submodule of M.

Definition 2.9. [8] Let $P = \{(x, t_P(x), i_P(x), f_P(x)) : x \in X\} \in U(^{(G_M)})$. The support P^* of the neutrosophic G-submodule P can be defined as $P^* = \{x \in X, t_P(x) > 0, i_P(x) > 0, f_P(x) < 1, \forall x \in G_M\}$.

Proposition 2.2. If $P \in U(G_M)$, then the support $P^* \in G_M$.

Definition 2.10. [8] Let $P \in U(G_M)$ and N be a G-submodule of M. Then the restriction of P to N is denoted by $P|_N$ and it is a neutrosophic set of N defined as follows $P|_N(\eta) = (\eta, t_{P|_N}(\eta), i_{P|_N}(\eta), f_{P|_N}(\eta))$ where

$$t_{P|_N}(\eta) = t_P(\eta),\ i_{P|_N}(\eta) = i_P(\eta),\ f_{P|_N}(\eta) = f_P(\eta),\ \forall \eta \in N.$$

Proposition 2.3. [8] Let $P \in U(G_M)$ and $N \subseteq M$ then $P|_N \in U(G_N)$.

Definition 2.11. [8] Let $M \in G_M$ and N be a G-submodule of M. Then the neutrosophic set P_N of M/Ndefined as $P_N(\eta + N) = \{\eta + N, t_{P_N}(\eta + N), i_{P_N}(\eta + N), f_{P_N}(\eta + N)\}$,where

$$t_{P_N}(\eta + N) = \vee t_P(\eta + n) : n \in N$$

$$i_{P_N}(\eta + N) = \vee i_P(\eta + n) : n \in N$$

$$f_{P_N}(\eta + N) = \wedge f_P(\eta + n) : n \in N, \forall \eta \in M$$

Proposition 2.4. [8] Let $M \in G_M$. Let N be a G-submodule of M. Then $P_N \in U(G_{M/N})$.

Proposition 2.5. [8] Let $P \in U(G_M)$ and $Q \in U(G_{M^*})$ where M and M^* are G-modules over the field K. Let $r \in [0, 1]$, the neutrosophic set $Q_r = \{\eta, t_{Q_r}(\eta), i_{Q_r}(\eta), f_{Q_r}(\eta) : \eta \in M^*\}$ defined by $t_{Q_r}(\eta) = t_Q(\eta) \wedge r,\ i_{Q_r}(\eta) = i_Q(\eta) \wedge r,\ f_{Q_r}(\eta) = f_Q(\eta) \vee (1 - r)\ \forall\ \eta \in M^\star$ be a neutrosophic G-submodule.

Definition 2.12. [8] Let M and M^* be G-modules over K and a mapping $\Upsilon : M \to M^*$ is a G-module homomorphism. Also $P \in U(G_M)$ *and* $Q \in U(G_{M^*})$. A homomorphism Υ of M on to M^* is called weak neutrosophic G-submodule homomorphism of P into Q if $\Upsilon(P) \subseteq Q$. If Υ is a **weak neutrosophic G-module homomorphism** of P into Q, then P is weakly homomorphic to Q and we write $P \sim Q$.

A homomorphism Υ of M on to M^* is called a **neutrosophic G-module homomorphism** of P onto Q if $\Upsilon(P) = Q$ and we represent it as $P \approx Q$.

3. **Neutrosophic Projective G module**

In this section we discuss the generalized notion of projective G-modules, called neutrosophic projective G-modules, and study several characteristics of projective G-modules in the neutrosophic domain.

Definition 3.1. Let M and M^* be G-modules. Let $P = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta) : \eta \in M\}$ be neutrosophic G submodule of M and $Q = \{\eta, t_Q(\eta), i_Q(\eta), f_Q(\eta) : \eta \in M^*\}$ be neutrosophic G-submodule of M^*. Then P is said to be Q projective, if the following conditions are satisfied;

(1) M is M^* projective
(2) $t_P(\eta) \leq t_Q(\psi(\eta))$
(3) $i_P(\eta) \leq i_Q(\psi(\eta))$
(4) $f_P(\eta) \geq f_Q(\psi(\eta)),\ \forall\ \psi \in Hom(M, M^*),\ \eta \in M$

Theorem 3.1. Let P and Q be neutrosophic G-submodules of finite dimensional G-modules of M and M^* respectively and M is M^* projective. Let $\{\beta_1, \beta_2, ..., \beta_n\}$ be a basis for M^*. If

(1) $t_P(\eta) \leq min\{t_Q(\beta_j); j = 1, 2, ..., n\}$
(2) $i_P(\eta) \leq min\{i_Q(\beta_j); j = 1, 2, ..., n\}$
(3) $f_P(\eta) \geq max\{f_Q(\beta_j); j = 1, 2, ..., n\}, \forall\ \eta \in M$

Then P is Q-projective.

Proof. Let $Q = \{\eta, t_B(\eta), i_B(\eta), f_B(\eta) : \eta \in M^*\}$ be a neutrosophic G submodule of M^*. Then $\forall\ \eta_1, \eta_2 \in M^*; \varrho, \tau \in K$;

(1) $t_Q(\varrho\eta_1 + \tau\eta_2) \geq t_Q(\eta_1) \wedge t_Q(\eta_2)$
(2) $i_Q(\varrho\eta_1 + \tau\eta_2) \geq i_Q(\eta_1) \wedge i_Q(\eta_2)$
(3) $f_Q(\varrho\eta_1 + \tau\eta_2) \leq f_Q(\eta_1) \vee f_Q(\eta_2)$
(4) $t_Q(\xi\eta) \geq t_P(\eta),\ i_Q(\xi\eta) \geq i_Q(\eta),\ f_Q(\xi\eta) \leq f_Q(\eta)\ \forall\ \eta \in M^*,\ \xi \in G$

Also P is a neutrosophic G-submodule of M and M is M^* projective G-module and $\psi \in Hom(M, M^*)$ be any G-module homomorphism. For any $\eta \in M,\ \psi(\eta) \in M^*$.

$\therefore \psi(\eta) = \alpha_1\beta_1 + \alpha_2\beta_2 + ... + \alpha_n\beta_n,\ \alpha_i \in K, \beta_i \in M^*,\ i = 1, 2, ..., n$

$$\begin{aligned} t_Q(\psi(\eta)) &= t_Q(\alpha_1\beta_1 + \alpha_2\beta_2 + ... + \alpha_n\beta_n) \\ &\geq t_Q(\beta_1) \wedge t_Q(\beta_2) \wedge ... \\ &\quad \wedge t_Q(\beta_n) \\ &= mini\{t_Q(\beta_1),\ t_Q(\beta_2),\ ..., \\ &\quad t_Q(\beta_n)\} \\ &\geq t_P(\eta) \end{aligned}$$

Similarly $i_Q(\psi(\eta)) \geq i_P(\eta)$

$$\begin{aligned} f_Q(\psi(\eta)) &= f_Q(\alpha_1\beta_1 + \alpha_2\beta_2 + ... + \alpha_n\beta_n) \\ &\leq f_Q(\beta_1) \wedge t_Q(\beta_2) \wedge ... \wedge t_Q(\beta_n) \\ &= max\{f_Q(\beta_1),\ f_Q(\beta_2), \\ &\quad ...\ ,\ f_Q(\beta_n)\} \\ &\leq f_P(\eta) \end{aligned}$$

$\therefore$ P is Q projective. □

Theorem 3.2. Let $P \in U(G_M),\ Q \in U(G_{M^*})$ and P is Q projective. If N^* is a G-submodule of M^* and $C \in U(G_{N^*})$, then P is C-Projective if $Q|_{N^*} \subseteq C$

Proof. Given P is Q projective, then

(1) M is M^* projective

(2) $t_P(\eta) \leq t_Q(\psi(\eta))$,
$i_P(\eta) \leq i_Q(\psi(\eta))$
$f_P(\eta) \geq f_Q(\psi(\eta))$

$\forall\ \psi \in Hom_G(M, M^*),\ \eta \in M$. Since N^* is a G-submodule of M^*, by a theorem 2.2, M is N^* projective. Let $\varphi \in Hom_G(M, N^*)$ and $\theta : N^* \to M^*$ be the inclusion homomorphism. Then $\theta \circ \varphi = \psi$

$\therefore$ from the condition 2

$t_P(\eta) \leq t_Q(\psi(\eta)) = t_Q(\theta \circ \varphi)(\eta)$

$= t_Q(\theta(\varphi(\eta))) = t_Q(\varphi(\eta))$.

Similarly $i_P(\eta) \leq i_Q(\varphi(\eta))$ *and* $f_P(\eta) \geq f_Q(\varphi(\eta)\ \forall\ \eta \in M,\ \varphi \in Hom_G(M, N^*)$.

Given $C \in U(G_{N^*})$, $\varphi(\eta) \in N^*$ and $Q|_{N^*} \subseteq C$

$$t_{Q|_{N^*}}(\varphi(\eta)) = t_Q(\varphi(\eta)) \leq t_C(\varphi(\eta)$$

$\Rightarrow$ $t_P(\eta) \leq t_C(\varphi(\eta))$. Similarly, $i_P(\eta) \leq i_C(\varphi(\eta))$ and $f_P(\eta) \geq f_C(\varphi(\eta))$. Hence P is C-Projective. □

Theorem 3.3. Let M and M^* be G-modules where P and Q are neutrosophic G-submodules of M and M^* respectively. Let $r \in [0, 1]$, the neutrosophic set $Q_r = \{\eta, t_{Q_r}(\eta), i_{Q_r}(\eta), f_{Q_r}(\eta) : \eta \in M^*\}$ defined by $t_{Q_r}(\eta) = t_Q(\eta) \wedge r,\ i_{Q_r}(\eta) = i_Q(\eta) \wedge r,\ f_{Q_r}(\eta) = f_Q(\eta) \vee (1 - r)\ \forall\ \eta \in M^\star$ be a neutrosophic G- submodule. If P is Q_r projective, then P is Q projective.

Proof. Consider P as Q_r projective where $r \in [0, 1]$. Then

(1) M is M^* projective

(2) $t_P(\eta) \leq t_{Q_r}(\psi(\eta))$,
$i_P(\eta) \leq i_{Q_r}(\psi(\eta))$,
$f_P(\eta) \geq f_{Q_r}(\psi(\eta))$,
$\psi \in Hom_G(M, M^*)$ *and* $\eta \in M$

Since $Q_r \subseteq Q$, $\Rightarrow t_{Q_r}(\psi(\eta)) \leq t_Q(\psi(\eta)$,
$i_{Q_r}(\psi(\eta)) \leq i_Q(\psi(\eta))$ *and*
$f_{Q_r}(\psi(\eta)) \geq f_Q(\psi(\eta)), \forall\ \psi(\eta) \in M^*$.
$\Rightarrow t_P(\eta) \leq t_Q(\psi(\eta))$,
$i_P(\eta) \leq i_Q(\psi(\eta))$ *and*
$f_P(\eta) \geq f_Q(\psi(\eta))\ \forall\ \eta \in M$.
$\therefore P$ is Q projective. □

Proposition 3.1. Let $M = \oplus_{i=1}^{n} M_i$ be a G-module where $M_i^{\prime}s$ are G-submodules of M. If $P_i \in U(G_{M_i})$ $(1 \leq i \leq n)$, then the neutrosophic set P of M defined by $t_P(\eta) = \wedge\{t_{p_i}(\eta_i) : i = 1, 2, ..., n\}$,$i_P(\eta) = \wedge\{i_{p_i}(\eta_i) : i = 1, 2, ..., n\}$ and $f_P(\eta) = \vee\{f_{P_i}(\eta_i) : i = 1, 2, ..., n\}$ where $\eta = \sum_{i=1}^{i=n}(\eta_i),\ \eta_i \in M_i$, is a neutrosophic G-submodule of M.

Proof. Let $\eta, \nu \in M$ where $\eta = \sum_{i=1}^{i=n} \eta_i$ and $\nu = \sum_{i=1}^{i=n} \nu_i$. Each η_i, $\nu_i \in M_i$ and $\varrho, b \in K$. Then by definition, $\varrho\eta + \tau\nu = \sum_{i=1}^{i=n}[\varrho\eta_i + \tau\nu_i]$ where $\varrho\eta_i + \tau\nu_i \in M_i$ $(1 \leq i \leq n)$. Now

$$\begin{aligned} t_P(\varrho\eta + \tau\nu) &= \wedge\ t_{P_i}(\varrho\eta_i + \tau\nu_i) \\ &\geq \wedge\ \{t_{P_i}(\eta_i), t_{P_i}(\nu_i)\} \\ &= \{\wedge\ t_{P_i}(\eta_i)\} \wedge \{\wedge\ t_{P_i}(\nu_i)\} \\ &= t_P(\eta) \wedge t_P(\nu) \end{aligned}$$

Similarly $i_P(\varrho\eta + \tau\nu) \geq i_P(\eta) \wedge i_P(\nu)$
Now consider

$$\begin{aligned} f_P(\varrho\eta + \tau\nu) &= \vee\ f_{P_i}(\varrho\eta_i + \tau\nu_i) \\ &\leq \vee\ \{f_{P_i}(\eta_i), f_{P_i}(\nu_i)\} \\ &= \{\vee\ f_{A_i}(\eta_i)\} \vee \{\vee\ f_{P_i}(\nu_i)\} \\ &= f_P(\eta) \vee f_P(\nu) \end{aligned}$$

Now, for $g \in G, \eta \in M$

$$\begin{aligned} t_P(g\eta) &= \wedge\ t_{P_i}(g\eta_i) \\ &\geq \wedge\ \{t_{P_i}(\eta_i)\} \\ &= t_P(\eta) \end{aligned}$$

Similarly $i_P(g\eta) \geq i_P(\eta)$, $f_P(g\eta) \leq f_P(\eta) \therefore P \in U(G_M)$. □

Definition 3.2. Let $M = \oplus_{i=1}^{n} M_i$ be a G module where $M_i's$ are G-submodules of M. If $P_i \in U(G_{M_i})$ $(1 \leq i \leq n)$ and $P \in U(G_{M=\oplus_{i=1}^{n} M_i})$ with $t_P(0) = t_{P_i}(0), i_P(0) = i_{P_i}(0)$ *and* $f_P(0) = f_{P_i}(0)$ $\forall$ i Then P is called the direct sum of P_i and it is denoted as $P = \oplus_{i=1}^{n} P_i$.

Theorem 3.4. Let $M = \oplus_{i=1}^{n} M_i$ be G module where $M_i's$ are G submodules of M. Let $P \in U(G_M)$ and $Q_i \in U(G_{M_i})$ such that $Q = \oplus_{i=1}^{n} Q_i$. Then P is Q projective if and only if P is Q_i projective $\forall i$.

Proof. Assume that P is Q-projective, then

(1) M is M projective

(2) $t_P(\eta) \leq t_Q(\psi(\eta))$,
$i_P(\eta) \leq i_Q(\psi(\eta)$
$f_P(\eta) \geq f_Q(\psi(\eta)$
$\psi \in Hom_G(M, M); \eta \in M$

To prove that P is Q_i projective where $i = 1, 2, ..., n$, it is enough to prove the following conditions.

(1) M is M_i -projective

(2) $t_P(\eta) \leq t_{Q_i}(\varphi(\eta))$,
$i_P(\eta) \leq i_{Q_i}(\varphi(\eta))$
$f_P(\eta) \geq f_{Q_i}(\varphi(\eta))$
where $\forall$ $\varphi \in Hom_G(M, M_i), \eta \in M$.

Here M is $M = \oplus_{i=1}^{n} M_i$-projective and by the the proposition 2.2, M is M_i projective $\forall$ $i = 1, 2, ..., n$. Let $\varphi \in Hom_G(M, M_i)$ and $\theta : M_i \rightarrow M \in Hom_G(M_i, M)$ (inclusion) such that $\psi = \theta \circ \varphi$. Then $\forall$ $\varphi \in Hom_G(M, M_i)$

$$\begin{aligned} t_P(\eta) &\leq t_Q(\psi(\eta) \\ &= t_Q((\theta \circ \varphi)(\eta)) \\ &= t_Q(\theta(\varphi(\eta))) \\ &= t_Q(\varphi(\eta)) \end{aligned}$$

Similarly $i_P(\eta) \leq i_Q(\varphi(\eta))$ and

$$\begin{aligned} f_P(\eta) &\geq f_Q(\psi(\eta)) \\ &= f_Q((\theta \circ \varphi)(\eta)) \\ &= f_Q(\theta(\varphi(\eta))) \\ &= f_Q(\varphi(\eta)) \end{aligned}$$

Now $\varphi(\eta) \in M_i \subseteq M$ and $\eta \in M$ and consider

$\varphi(\eta) = 0 + 0 + ... + \varphi(\eta) + ... + 0$

Then

$$\begin{aligned} t_Q(\varphi(\eta)) &= t_Q(0 + 0 + ... + \varphi(\eta) + ... + 0) \\ &= t_{Q_1}(0) \wedge t_{Q_2}(0) \wedge ... \wedge t_{Q_i}\varphi(\eta) \wedge ... \wedge t_{Q_n}(0) \\ &= t_{Q_i}(\varphi(\eta)) \end{aligned}$$

Similarly $i_Q(\varphi(\eta)) = i_{Q_i}(\varphi(\eta))$ and

$f_Q(\varphi(\eta)) = f_{Q_i}(\varphi(\eta)) \ \forall \ i$

$\Rightarrow \ t_P(\eta) \leq t_Q(\varphi(\eta)) = t_{Q_i}(\varphi(\eta))$.

Also $i_P(\eta) \leq i_Q(\varphi(\eta)) = i_{Q_i}(\varphi(\eta)$ and

$f_P(\eta) \geq f_Q(\varphi(\eta)) = f_{Q_i}(\varphi(\eta), \forall \ \eta \in M, \varphi \in Hom_G(M, M_i)$.

Then P is Q_i projective.

Conversely Assume that P is Q_i projective where $i = 1, 2, ..., n$.Then

(1) M is M_i-projective

(2) $t_P(m) \leq t_{Q_i}(\varphi_i(m))$,
$i_P(m) \leq i_{Q_i}(\varphi_i(m) \ and$
$f_P(m) \geq f_{Q_i}(\varphi_i(m)$,
$\varphi_i \in Hom_G(M, M_i); m \in M$

To prove P is Q projective, it is enough to prove the following conditions

(1) M is M projective

(2) $t_P(\eta) \leq t_Q(\psi(\eta))$,
$i_P(\eta) \leq i_Q(\psi(\eta))$
$f_P(\eta) \geq f_Q(\psi(\eta)), \ \psi \in Hom_G(M, M); \eta \in M$

1. :- Since P is Q_i projective and proposition 2.1, M is M-Projective where $M = \oplus_{i=1}^{n} M_i$.

2. :- Let $\psi \in Hom_G(M, M) \ where \ M = \oplus_{i=1}^{n} M_i$ such that $\forall \ \eta \in M$,
$\psi(\eta) \in M, \ i.e. \ \psi(\eta) = \eta_1 + \eta_2 + ... + \eta_n, \forall \ \eta_i \in M_i, 1 \leq i \leq n$ and $\pi_i : M \rightarrow M_i$ be the

projection map where $i = 1, 2, ..., n$ such that $\pi_i(\psi(\eta)) = \eta_i,\ \forall\ i.$, then

$$\begin{aligned}
\psi(\eta) &= \eta_1 + \eta_2 + ... + \eta_n, \\
&\quad \forall\ \eta_i \in M_i, 1 \leq i \leq n \\
&= \pi_1(\psi(\eta)) + \pi_2(\psi(\eta)) + ... \\
&\quad ... + \pi_n(\psi(\eta)) \\
&= (\pi_1 \circ \psi)(\eta) + (\pi_2 \circ \psi)(\eta) + ... + \\
&\quad (\pi_n \circ \psi)(\eta) \\
&= \varphi_1(\eta) + \varphi_2(\eta) + ... + \varphi_n(\eta)
\end{aligned}$$

Also

$$\begin{aligned}
t_Q(\psi(\eta)) &= t_Q(\varphi_1(\eta)) + t_Q(\varphi_2(\eta)) + ... + \\
&\quad t_Q(\varphi_n(\eta)) \\
&= \wedge\{t_{Q_i}(\varphi_i(\eta)) : 0 \leq i \leq n\} \\
&\quad [\textit{by the proposition } 3.1] \\
&\geq t_P(\eta)
\end{aligned}$$

Similarly $i_Q(\psi(\eta)) \geq i_P(\eta)$ and

$$\begin{aligned}
f_Q(\psi(\eta)) &= f_Q(\varphi_1(\eta)) + f_Q(\varphi_2(\eta)) + \\
&\quad ... + f_Q(\varphi_n(\eta)) \\
&\leq \vee\{f_{Q_i}(\varphi_i(\eta)) : 0 \leq i \leq n\} \\
&\leq f_P(m)
\end{aligned}$$

$\therefore$ A is Q projective. □

4. **Conclusion**

The study of G-module in a neutrosophic set domain using a single-valued neutrosophic set provides a new step in the algebra sector and helps to analyze group action in application level on a vector space. Projective G-modules expand the free G-modules class by maintaining a portion of the free module's primary properties. Neutrosophic projective G-module is one of the most generalizations of classical projective G-module. This paper has developed, the notion of projectivity of neutrosophic G-modules and its quotient and direct sum properties of M projectivity. This analysis leads to the extension of the quasi projective module, neutrosophic injective & projective modules and its features in neutrosophic domain.

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Quadripartitioned Single valued Neutrosophic Dombi Weighted Aggregation Operators for Multiple Attribute Decision Making

M.Mohanasundari[1], K.Mohana[2]

[1]Department of Mathematics, Bannari Amman Institute of Technology; mohanadhianesh@gmail.com

[2]Department of mathematics, Nirmala college for women; riyaraju1116@gmail.com

*Correspondence: mohanadhianesh@gmail.com

Abstract. In this paper we have introduced the concept of score and accuracy function of the Quadripartitioned Single valued Neutrosophic Numbers (QSVNN) and also defined ranking methods between two QSVNNs which is based on its score function. Dombi operators are used in solving many Multicriteria Attribute Group decision making (MAGDM) problems because of its very good flexibility with a general parameter.Here Dombi T-norm and T-conorm operations of two QSVNNs are defined. Based on this Dombi operations, we introduced two Dombi weighted aggregation operators QSVNDWAA and QSVNDWGA under Quadripartitioned Single valued Neutrosophic environment and also studied its properties. Finally, we discussed about Multicriteria Attribute Decision making method (MADM) using QSVNDWAA or QSVNDWGA operator and also an illustrative example is given for the proposed method which gives a detailed results to select the best alternative based upon the ranking orders.

Keywords: Quadripartitioned single valued neutrosophic sets, Score and Accuracy functions, Dombi Weighted Aggregation Operators .

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1. **Introduction**

Fuzzy sets which allows the elements to have a degrees of membership in the set and it was introduced by Zadeh [31] in 1965. The degrees of membership lies in the real unit interval $[0, 1]$. Intuitionstic fuzzy set (IFS) allows both membership and non membership to the elements and this was introduced by Atnassov [1] in 1983. By introducing one more component in IFS set neutrosophic set was introduced by Smarandache [19] in 1998. Neutrosophic set has three components truth membership function, indeterminacy membership function and falsity membership function respectively. This neutrosophic set helps to handle the indeterminate and inconsistent information effectively. Later Wang [21] (2010) introduced the concept of Single valued Neutrosophic set (SVNS) which is a generalization of classic set, fuzzy set, interval valued fuzzy set and intuitionstic fuzzy set.

In 1982 Pawlak [17] defined the standard version of rough set theory which is given in terms of a pair of sets that is lower and upper approximation sets. It provided a new approach to vagueness which is defined by a boundary region of a set. Later Yang(2017) [23] defined a new hybrid model of single valued neutrosophic rough set model and it has many applications in medical diagnosis, decision making problems, image processing etc., Neutrosophic set helps to solve many real life world problems [2–6] because of its uncertainty analysis in data sets. K.Mohana, M.Mohanasundari [15] studied On Some Similarity Measures of Single Valued Neutrosophic Rough Sets and applied the concept in Medical Diagnosis problem. When indeterminacy component in neutrosophic set is divided into two parts namely 'Contradiction' (both true and false) 'Unknown' (neither true nor false) we get four components that is T,C,U,F which define a new set called 'Quadripartitioned Single valued neutrosophic set' (QSVNS)introduced by Rajashi Chatterjee., et al. [18] And this is completely based on Belnap's four valued logic and Smarandache's 'Four Numerical valued neutrosophic logic'. By combining the concept of rough set and QSVNS a new hybrid model of 'Quadripartitioned Single valued neutrosophic Rough set' (QSVNRS) was introduced by K.Mohana and M.Mohanasundari. [16]

Many mathematical operations like average, aggregate, sum, count, max, min are performed with the help of aggregation operations.Multicriteria Attribute decision making (MADM) is an approach which is used to select a best one when several alternatives are included under consideration of many attributes. So many researchers [8, 11, 24–27, 29] pay attention to solve the Multicriteria Attribute decision making problems using the concept of various correlation coefficients of the different sets like fuzzy set, IFS, SVNS, QSVNS. And also many researchers [12–14, 20, 22, 28, 30, 33] used aggregation operators as one of the tool to solve a Multicriteria decision making problem and also studied its properties. Dombi Bonferroni mean operators were introduced by Dombi [10] in 1982 which is used in many Multicriteria Attribute Group decision making (MAGDM) problems because of its very good flexibility with a general parameter. J.Chen and J.Ye [7] studied Some Single-valued Neutrosophic Dombi Weighted Aggregation Operators for Multiple Attribute Decision-Making problem.

In this paper Section 2 deals about the basic definitions of Quadripartitioned Single valued neutrosophic sets, Score and accuracy function of single valued neutrosophic number, Dombi T norm and T conorm operations of two single valued neutrosophic numbers(SVNN) and its properties. We have defined Score and accuracy function of quadripartitioned single valued neutrosophic number, Dombi T norm and T conorm operations of two quadripartitioned single valued neutrosophic numbers(QSVNN) in Section 3. Based on the operations of Dombi T norm and T conorm on two QSVNNs we have defined two aggregation operators QSVNDWAA and QSVNDWGA and also studied its properties. Section 4 deals about Multicriteria Attribute Decision making (MADM) method using the above proposed operators QSVNDWAA and QSVNDWGA. Finally an illustrative example is given in the method which we have discussed in Section 4.

2. **Preliminaries**

2.1 Quadripartitioned single valued neutrosophic sets

Definition 2.1. [19]
Neutrosophic set is defined over the non-standard unit interval $]^{-}0,\ 1^{+}[$ whereas single valued neutrosophic set is defined over standard unit interval $[0,1]$.It means a single valued neutrosophic set A is defined by

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$$

where $T_A(x), I_A(x), F_A(x) : X \to [0,1]$ such that $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$

Definition 2.2. [18]
Let X be a non-empty set. A quadripartitioned single valued neutrosophic set (QSVNS) A over X characterizes each element in X by a truth-membership function $T_A(x)$, a contradiction membership function $C_A(x)$, an ignorance membership function $U_A(x)$ and a falsity membership function $F_A(x)$ such that for each $x \in X$, $T_A, C_A, U_A, F_A \in [0,1]$ and $0 \leq T_A(x) + C_A(x) + U_A(x) + F_A(x) \leq 4$ when X is discrete, A is represented as $A = \sum_{i=1}^{n} \langle T_A(x_i), C_A(x_i), U_A(x_i), F_A(x_i)\rangle / x_i, x_i \in X$.

Definition 2.3. [18]
The complement of a QSVNS A is denoted by A^C and is defined as,
$A^C = \sum_{i=1}^{n} \langle F_A(x_i), U_A(x_i), C_A(x_i), T_A(x_i)\rangle / x_i, x_i \in X$ i.e., $T_{A^C}(x_i) = F_A(x_i)$, $C_{A^C}(x_i) = U_A(x_i), U_{A^C}(x_i) = C_A(x_i), F_{A^C}(x_i) = T_A(x_i), x_i \in X$

Definition 2.4. [18]
Consider two QSVNS A and B, over X. A is said to be contained in B, denoted by $A \subseteq B$ iff $T_A(x) \leq T_B(x), C_A(x) \leq C_B(x), U_A(x) \geq U_B(x)$, and $F_A(x) \geq F_B(x)$

Definition 2.5. [18]
The union of two QSVNS A and B is denoted by $A \cup B$ and is defined as,

$A \cup B = \sum_{i=1}^{n} \langle T_A(x_i) \vee T_B(x_i), C_A(x_i) \vee C_B(x_i), U_A(x_i) \wedge U_B(x_i), F_A(x_i) \wedge F_B(x_i) \rangle /$
$x_i, x_i \in X$

Definition 2.6. [18]

The intersection of two QSVNS A and B is denoted by $A \cap B$ and is defined as,

$A \cap B = \sum_{i=1}^{n} \langle T_A(x_i) \wedge T_B(x_i), C_A(x_i) \wedge C_B(x_i), U_A(x_i) \vee U_B(x_i), F_A(x_i) \vee F_B(x_i) \rangle /$
$x_i, x_i \in X$

Definition 2.7. [21]

Let X be a universal set. A $SVNS$ N in X is described by a truth-membership function $t_N(x)$, an indeterminacy-membership function $u_N(x)$, and a falsity-membership function $v_N(x)$. Then a $SVNS$ N can be denoted as the following form:

$$N = \{\langle x, t_N(x), u_N(x), v_N(x) \rangle | x \in X\}$$

where the functions $t_N(x), u_N(x), v_N(x) \in [0, 1]$ satisfy the condition $0 \leq t_N(x) + u_N(x) + v_N(x) \leq 3$ for $x \in X$. For convenient expression, a basic element $\langle x, t_N(x), u_N(x), v_N(x) \rangle$ in N is denoted by $s = \langle t, u, v, \rangle$ which is called a SVNN. For any SVNN $s = \langle t, u, v, \rangle$, its score and accuracy functions can be introduced, respectively as follows:

$$E(s) = (2 + t - u - v)/3, \quad E(s) \in [0, 1],$$
$$H(s) = t - v, \quad H(s) \in [-1, 1]$$

According to the two functions $E(s)$ and $H(s)$, the comparison and ranking of two SVNNs are introduced by the following definition.

Definition 2.8. [32] Let $s_1 = \langle t_1, u_1, v_1 \rangle$ and $s_2 = \langle t_2, u_2, v_2 \rangle$ be two SVNNs. Then the ranking method for s_1 and s_2 is defined as follows:

(1) If $E(s_1) > E(s_2)$ then $s_1 \succ s_2$,

(2) If $E(s_1) = E(s_2)$ and $H(s_1) > H(s_2)$ then $s_1 \succ s_2$,

(3) If $E(s_1) = E(s_2)$ and $H(s_1) = H(s_2)$ then $s_1 = s_2$.

Definition 2.9. [10] Let p and q be any two real numbers. Then, the Dombi T-norm and T-conorm between p and q are defined as follows:

$$O_D(p, q) = \frac{1}{1+\left\{\left(\frac{1-p}{p}\right)^{\rho}+\left(\frac{1-q}{q}\right)^{\rho}\right\}^{1/\rho}},$$
$$O_D^c(p, q) = 1 - \frac{1}{1+\left\{\left(\frac{p}{1-p}\right)^{\rho}+\left(\frac{q}{1-q}\right)^{\rho}\right\}^{1/\rho}},$$

where $\rho \geq 1$ and $(p,q) \in [0,1] \times [0,1]$.
According to the Dombi T-norm and T-conorm, we define the Dombi operations of SVNNs.

Definition 2.10. [7] Let $s_1 = \langle t_1, u_1, v_1 \rangle$ and $s_2 = \langle t_2, u_2, v_2 \rangle$ be two SVNNs, $\rho \geq 1$, and $\lambda > 0$. Then, the Dombi T-norm and T-conorm operations of SVNNs are defined below:

(1) $s_1 \oplus s_2 = \left\langle 1 - \frac{1}{1+\left\{\left(\frac{t_1}{1-t_1}\right)^{\rho}+\left(\frac{t_2}{1-t_2}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\left(\frac{1-u_1}{u_1}\right)^{\rho}+\left(\frac{1-u_2}{u_2}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\left(\frac{1-v_1}{v_1}\right)^{\rho}+\left(\frac{1-v_2}{v_2}\right)^{\rho}\right\}^{1/\rho}} \right\rangle$

(2) $s_1 \otimes s_2 = \left\langle \frac{1}{1+\left\{\left(\frac{1-t_1}{t_1}\right)^{\rho}+\left(\frac{1-t_2}{t_2}\right)^{\rho}\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\left(\frac{u_1}{1-u_1}\right)^{\rho}+\left(\frac{u_2}{1-u_2}\right)^{\rho}\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\left(\frac{v_1}{1-v_1}\right)^{\rho}+\left(\frac{v_2}{1-v_2}\right)^{\rho}\right\}^{1/\rho}} \right\rangle$

(3) $\lambda s_1 = \left\langle 1 - \frac{1}{1+\left\{\lambda\left(\frac{t_1}{1-t_1}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\lambda\left(\frac{1-u_1}{u_1}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\lambda\left(\frac{1-v_1}{v_1}\right)^{\rho}\right\}^{1/\rho}} \right\rangle$

(4) $s_1^{\lambda} = \left\langle \frac{1}{1+\left\{\lambda\left(\frac{1-t_1}{t_1}\right)^{\rho}\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\lambda\left(\frac{u_1}{1-u_1}\right)^{\rho}\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\lambda\left(\frac{v_1}{1-v_1}\right)^{\rho}\right\}^{1/\rho}} \right\rangle$

Definition 2.11. [7] Let $s_j = \langle t_j, u_j, v_j \rangle \, (j = 1, 2, ..., n)$ be a collection of SVNNs and $w = (w_1, w_2, ..., w_n)$ be the weight vector for s_j with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$. Then, the SVNDWAA and SVNDWGA operators are defined respectively as follows:

$$\text{SVNDWAA}\ (s_1, s_2, ..., s_n) = \bigoplus_{j=1}^{n} w_j s_j$$

$$\text{SVNDWGA}\ (s_1, s_2, ..., s_n) = \bigotimes_{j=1}^{n} s_j^{w_j}$$

3. Quadripartitioned single valued Neutrosophhic Dombi Operations

Definition 3.1. For an QSVNNs $q = \langle t, c, u, f \rangle$ its score and accuracy functions are defined by,

$$E(q) = (3 + t - c - u - f)/4, \quad E(q) \in [0,1], \tag{1}$$

$$H(q) = t - f, \quad H(q) \in [-1,1] \tag{2}$$

The following definition defined the comparison and ranking of any two QSVNNs based on the two functions E(s) and H(s).

Definition 3.2. Let $q_1 = \langle t_1, c_1, u_1, f_1 \rangle$ and $q_2 = \langle t_2, c_2, u_2, f_2 \rangle$ be two QSVNNs. Then the ranking method for q_1 and q_2 is defined as follows:

(1) If $E(q_1) > E(q_2)$ then $q_1 \succ q_2$,

(2) If $E(q_1) = E(q_2)$ and $H(q_1) > H(q_2)$ then $q_1 \succ q_2$,

(3) If $E(q_1) = E(q_2)$ and $H(q_1) = H(q_2)$ then $q_1 = q_2$.

Definition 3.3. The Dombi T-norm and T-conorm operations of any two QSVNNs $q_1 = \langle t_1, c_1, u_1, f_1\rangle$ and $q_2 = \langle t_2, c_2, u_2, f_2\rangle$ are defined as follows:

(1) $$q_1 \oplus q_2 = \left\langle 1 - \frac{1}{1+\left\{\left(\frac{t_1}{1-t_1}\right)^\rho+\left(\frac{t_2}{1-t_2}\right)^\rho\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\left(\frac{c_1}{1-c_1}\right)^\rho+\left(\frac{c_2}{1-c_2}\right)^\rho\right\}^{1/\rho}}, \frac{1}{1+\left\{\left(\frac{1-u_1}{u_1}\right)^\rho+\left(\frac{1-u_2}{u_2}\right)^\rho\right\}^{1/\rho}}, \frac{1}{1+\left\{\left(\frac{1-f_1}{f_1}\right)^\rho+\left(\frac{1-f_2}{f_2}\right)^\rho\right\}^{1/\rho}} \right\rangle$$

(2) $$q_1 \otimes q_2 = \left\langle \frac{1}{1+\left\{\left(\frac{1-t_1}{t_1}\right)^\rho+\left(\frac{1-t_2}{t_2}\right)^\rho\right\}^{1/\rho}}, \frac{1}{1+\left\{\left(\frac{1-c_1}{c_1}\right)^\rho+\left(\frac{1-c_2}{c_2}\right)^\rho\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\left(\frac{u_1}{1-u_1}\right)^\rho+\left(\frac{u_2}{1-u_2}\right)^\rho\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\left(\frac{f_1}{1-f_1}\right)^\rho+\left(\frac{f_2}{1-f_2}\right)^\rho\right\}^{1/\rho}} \right\rangle$$

(3) $$\lambda q_1 = \left\langle 1 - \frac{1}{1+\left\{\lambda\left(\frac{t_1}{1-t_1}\right)^\rho\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\lambda\left(\frac{c_1}{1-c_1}\right)^\rho\right\}^{1/\rho}}, \frac{1}{1+\left\{\lambda\left(\frac{1-u_1}{u_1}\right)^\rho\right\}^{1/\rho}}, \frac{1}{1+\left\{\lambda\left(\frac{1-f_1}{f_1}\right)^\rho\right\}^{1/\rho}} \right\rangle$$

(4) $$q_1^\lambda = \left\langle \frac{1}{1+\left\{\lambda\left(\frac{1-t_1}{t_1}\right)^\rho\right\}^{1/\rho}}, \frac{1}{1+\left\{\lambda\left(\frac{1-c_1}{c_1}\right)^\rho\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\lambda\left(\frac{u_1}{1-u_1}\right)^\rho\right\}^{1/\rho}}, 1 - \frac{1}{1+\left\{\lambda\left(\frac{f_1}{1-f_1}\right)^\rho\right\}^{1/\rho}} \right\rangle$$

4. **Dombi Weighted Aggregation Operators of QSVNNs**

In this section we introduce two Dombi weighted aggregation operators QSVNDWAA and QSVNDWGA which is based on the Dombi operations of QSVNNs in Definition 3.3 and also studied its properties.

Definition 4.1. A collection of QSVNNs is denoted by $q_j = \langle t_j, c_j, u_j, f_j\rangle\,(j = 1, 2, ..., n)$ and $w = (w_1, w_2, ..., w_n)$ be the weight vector for q_j with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. Then the QSVNDWAA and QSVNDWGA operators are defined as follows.

$$\text{QSVNDWAA}\ (q_1, q_2, ..., q_n) = \bigoplus_{j=1}^{n} w_j q_j$$

$$\text{QSVNDWGA}\ (q_1, q_2, ..., q_n) = \bigotimes_{j=1}^{n} q_j^{w_j}$$

Theorem 3.1 A collection of QSVNNs is denoted by $q_j = \langle t_j, c_j, u_j, f_j\rangle\,(j = 1, 2, ..., n)$ and $w = (w_1, w_2, ..., w_n)$ be the weight vector for q_j with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$. Then the aggregated value of the QSVNDWAA operator is still a QSVNN and is calculated by the following formula,

$$QSVNDWAA(q_1, q_2, ..., q_n) = \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{n} w_j \left(\frac{t_j}{1-t_j} \right)^{\rho} \right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{n} w_j \left(\frac{c_j}{1-c_j} \right)^{\rho} \right\}^{1/\rho}}, \right.$$
$$\left. \frac{1}{1 + \left\{ \sum_{j=1}^{n} w_j \left(\frac{1-u_j}{u_j} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ \sum_{j=1}^{n} w_j \left(\frac{1-f_j}{f_j} \right)^{\rho} \right\}^{1/\rho}} \right\rangle \quad (3)$$

We can prove this theorem using mathematical induction.
Proof: When $n = 2$ by using the Dombi operations of QSVNNs in Definition (3.3) we can have the following result

$$QSVNDWAA(q_1, q_2) = q_1 \oplus q_2$$
$$= \left\langle 1 - \frac{1}{1 + \left\{ w_1 \left(\frac{t_1}{1-t_1} \right)^{\rho} + w_2 \left(\frac{t_2}{1-t_2} \right)^{\rho} \right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{ w_1 \left(\frac{c_1}{1-c_1} \right)^{\rho} + w_2 \left(\frac{c_2}{1-c_2} \right)^{\rho} \right\}^{1/\rho}}, \right.$$
$$\left. \frac{1}{1 + \left\{ w_1 \left(\frac{1-u_1}{u_1} \right)^{\rho} + w_2 \left(\frac{1-u_2}{u_2} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ w_1 \left(\frac{1-f_1}{f_1} \right)^{\rho} + w_2 \left(\frac{1-f_2}{f_2} \right)^{\rho} \right\}^{1/\rho}} \right\rangle$$
$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} w_j \left(\frac{t_j}{1-t_j} \right)^{\rho} \right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{2} w_j \left(\frac{c_j}{1-c_j} \right)^{\rho} \right\}^{1/\rho}}, \right.$$
$$\left. \frac{1}{1 + \left\{ \sum_{j=1}^{2} w_j \left(\frac{1-u_j}{u_j} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ \sum_{j=1}^{2} w_j \left(\frac{1-f_j}{f_j} \right)^{\rho} \right\}^{1/\rho}} \right\rangle$$

when $n = k$, Equation (1) becomes,

$$QSVNDWAA(q_1, q_2, ..., q_k) = \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k} w_j \left(\frac{t_j}{1-t_j} \right)^{\rho} \right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k} w_j \left(\frac{c_j}{1-c_j} \right)^{\rho} \right\}^{1/\rho}}, \right.$$
$$\left. \frac{1}{1 + \left\{ \sum_{j=1}^{k} w_j \left(\frac{1-u_j}{u_j} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k} w_j \left(\frac{1-f_j}{f_j} \right)^{\rho} \right\}^{1/\rho}} \right\rangle$$

When $n = k + 1$ we have the following result

$$QSVNDWAA(q_1, q_2, ..., q_k, q_{k+1}) = \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k} w_j \left(\frac{t_j}{1-t_j} \right)^{\rho} \right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k} w_j \left(\frac{c_j}{1-c_j} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k} w_j \left(\frac{1-u_j}{u_j} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k} w_j \left(\frac{1-f_j}{f_j} \right)^{\rho} \right\}^{1/\rho}} \right\rangle \oplus w_{k+1} q_{k+1}$$

$$= \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k+1} w_j \left(\frac{t_j}{1-t_j} \right)^{\rho} \right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{k+1} w_j \left(\frac{c_j}{1-c_j} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k+1} w_j \left(\frac{1-u_j}{u_j} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ \sum_{j=1}^{k+1} w_j \left(\frac{1-f_j}{f_j} \right)^{\rho} \right\}^{1/\rho}} \right\rangle$$

Hence we proved that Theorem 3.1 is true for $n = k + 1$. Thus Equation (1) is true for all n. The operator QSVNDWAA satisfies the following properties.

(1) Reducibility : If $w = (1/n, 1/n, ..., 1/n)$, then it is obvious that there exits,

$$QSVNDWAA(q_1, q_2, ..., q_n) = \left\langle 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{n} \frac{1}{n} \left(\frac{t_j}{1-t_j} \right)^{\rho} \right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{ \sum_{j=1}^{n} \frac{1}{n} \left(\frac{c_j}{1-c_j} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ \sum_{j=1}^{n} \frac{1}{n} \left(\frac{1-u_j}{u_j} \right)^{\rho} \right\}^{1/\rho}}, \frac{1}{1 + \left\{ \sum_{j=1}^{n} \frac{1}{n} \left(\frac{1-f_j}{f_j} \right)^{\rho} \right\}^{1/\rho}} \right\rangle$$

(2) Idempotency : Let all the QSVNNs be denoted by $q_j = \langle t_j, c_j, u_j, f_j \rangle = q(j = 1, 2, ..., n)$. Then QSVNDWAA $(q_1, q_2, ..., q_n) = q$.

(3) Commutativity: Let any QSVNS $(q_1^{'}, q_2^{'}, ..., q_n^{'})$ be any permutation of $(q_1, q_2, ..., q_n)$. Then there is QSVNDWAA $(q_1^{'}, q_2^{'}, ..., q_n^{'}) =$ QSVNDWAA $(q_1, q_2, ..., q_n)$.

(4) Boundedness: Let $q_{min} = min(s_1, s_2, ..., s_n)$ and $q_{max} = max(s_1, s_2, ..., s_n)$. Then $q_{min} \leq QSVNDWAA(q_1, q_2, ..., q_n) \leq q_{max}$

Proof: (1) Given $q_j = \langle t_j, c_j, u_j, f_j \rangle = q(j = 1, 2, ..., n)$ Property (1) is trivially true based on equation (3)

(2) The following result is derived from the equation (3) and we get,

$$QSVNDWAA(q_1,q_2,...,q_n)=\left\langle 1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{t_j}{1-t_j}\right)^{\rho}\right\}^{1/\rho}},1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{c_j}{1-c_j}\right)^{\rho}\right\}^{1/\rho}},\right.$$
$$\left.\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{1-u_j}{u_j}\right)^{\rho}\right\}^{1/\rho}},\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{1-f_j}{f_j}\right)^{\rho}\right\}^{1/\rho}}\right\rangle$$
$$=\left\langle 1-\frac{1}{1+\left\{\left(\frac{t}{1-t}\right)^{\rho}\right\}^{1/\rho}},1-\frac{1}{1+\left\{\left(\frac{c}{1-c}\right)^{\rho}\right\}^{1/\rho}},\frac{1}{1+\left\{\left(\frac{1-u}{u}\right)^{\rho}\right\}^{1/\rho}},\frac{1}{1+\left\{\left(\frac{1-f}{f}\right)^{\rho}\right\}^{1/\rho}}\right\rangle$$
$$=\left\langle 1-\frac{1}{1+\frac{t}{1-t}},1-\frac{1}{1+\frac{c}{1-c}},1-\frac{1}{1+\frac{1-u}{u}},1-\frac{1}{1+\frac{1-f}{f}}\right\rangle=\langle t,c,u,f\rangle=q$$

QSVNDWAA $(q_1,q_2,...,q_n)=q$ holds.

(3) This property is obvious.

(4) Consider $q_{min}=min(q_1,q_2,...,q_n)=\langle t^-,c^-,u^-,f^-\rangle$ and $q_{max}=max(q_1,q_2,...,q_n)=\langle t^+,c^+,u^+,f^+\rangle$ Then,

$t^-=\min_j(t_j), c^-=\min_j(c_j), u^-=\max_j(u_j), f^-=\max_j(f_j)$

$t^+=\max_j(t_j), c^+=\max_j(c_j), u^+=\min_j(u_j), f^+=\min_j(f_j)$

Therefore we get the following inequalities.

$$1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{t^-}{1-t^-}\right)^{\rho}\right\}^{1/\rho}}\le 1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{t_j}{1-t_j}\right)^{\rho}\right\}^{1/\rho}}\le 1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{t^+}{1-t^+}\right)^{\rho}\right\}^{1/\rho}}$$

$$1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{c^-}{1-c^-}\right)^{\rho}\right\}^{1/\rho}}\le 1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{c_j}{1-c_j}\right)^{\rho}\right\}^{1/\rho}}\le 1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{c^+}{1-c^+}\right)^{\rho}\right\}^{1/\rho}}$$

$$\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{1-u^+}{u^+}\right)^{\rho}\right\}^{1/\rho}}\le\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{1-u_j}{u_j}\right)^{\rho}\right\}^{1/\rho}}\le\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{1-u^-}{u^-}\right)^{\rho}\right\}^{1/\rho}}$$

$$\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{1-f^+}{f^+}\right)^{\rho}\right\}^{1/\rho}}\le\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{1-f_j}{f_j}\right)^{\rho}\right\}^{1/\rho}}\le\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(\frac{1-f^-}{f^-}\right)^{\rho}\right\}^{1/\rho}}$$

Hence $q_{min}\le QSVNDWAA(q_1,q_2,...,q_n)\le q_{max}$ holds.

Theorem 3.2 A collection of QSVNNs is denoted by $q_j=\langle t_j,c_j,u_j,f_j\rangle\,(j=1,2,...,n)$ and $w=(w_1,w_2,...,w_n)$ be the weight vector for q_j with $w_j\in[0,1]$ and $\sum_{j=1}^{n} w_j=1$. Then the aggregated value of the QSVNDWGA operator is still a QSVNN and is calculated by the following formula:

$$QSVNDWGA(q_1, q_2, ..., q_n) = \left\langle \frac{1}{1+\left\{\sum_{j=1}^{n} w_j \left(\frac{1-t_j}{t_j}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\sum_{j=1}^{n} w_j \left(\frac{1-c_j}{c_j}\right)^{\rho}\right\}^{1/\rho}}, \right.$$
$$\left. 1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j \left(\frac{u_j}{1-u_j}\right)^{\rho}\right\}^{1/\rho}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j \left(\frac{f_j}{1-f_j}\right)^{\rho}\right\}^{1/\rho}} \right\rangle \tag{4}$$

The proof is similar to the proof of Theorem (3.1).

This QSVNDWGA operator also satisfies the following properties.

(1) Reducibility : If $w = (1/n, 1/n, ..., 1/n)$, then it is obvious that there exits,

$$QSVNDWGA(q_1, q_2, ..., q_n) = \left\langle \frac{1}{1+\left\{\sum_{j=1}^{n} \frac{1}{n} \left(\frac{1-t_j}{t_j}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1+\left\{\sum_{j=1}^{n} \frac{1}{n} \left(\frac{1-c_j}{c_j}\right)^{\rho}\right\}^{1/\rho}}, \right.$$
$$\left. 1-\frac{1}{1+\left\{\sum_{j=1}^{n} \frac{1}{n} \left(\frac{u_j}{1-u_j}\right)^{\rho}\right\}^{1/\rho}}, 1-\frac{1}{1+\left\{\sum_{j=1}^{n} \frac{1}{n} \left(\frac{f_j}{1-f_j}\right)^{\rho}\right\}^{1/\rho}} \right\rangle$$

(2) Idempotency : Let all the QSVNNs be denoted by $q_j = \langle t_j, c_j, u_j, f_j \rangle = q(j = 1, 2, ..., n)$. Then QSVNDWGA $(q_1, q_2, ..., q_n) = q$.
(3) Commutativity: Let any QSVNS $(q_1^{'}, q_2^{'}, ..., q_n^{'})$ be any permutation of $(q_1, q_2, ..., q_n)$. Then there is QSVNDWGA $(q_1^{'}, q_2^{'}, ..., q_n^{'})$ = QSVNDWGA $(q_1, q_2, ..., q_n)$.
(4) Boundedness: Let $q_{min} = min(q_1, q_2, ..., q_n)$ and $q_{max} = max(q_1, q_2, ..., q_n)$. Then $q_{min} \leq QSVNDWGA(q_1, q_2, ..., q_n) \leq q_{max}$

To prove the above properties it is similar to the operator properties of QSVNDWAA. Hence it is not repeated here.

5. **MADM method using QSVNDWAA operator or QSVNDWGA operator**

This section deals about the MADM method to handle the MADM problems effectively with QSVNN information by using the QSVNDWAA operator or QSVNDWGA operator.
Let $A = \{A_1, A_2, ..., A_m\}$ and $C = \{C_1, C_2, ..., C_n\}$ be a discrete set of alternatives and attributes respectively. The weight vector of the above attributes is given by $w = \{w_1, w_2, ..., w_n\}$ such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

To make a better decision to choose the alternative $A_i(i = 1, 2, ..., m)$, a decision maker needs to analyse the attributes $C_j(j = 1, 2, ..., n)$ by the QSVNN $q_{ij} = \langle t_{ij}, c_{ij}, u_{ij}, f_{ij}\rangle\, (i = 1, 2, ..., m; j = 1, 2, ..., n)$ then we get a QSVNN decision matrix $D = (d_{ij})_{m\times n}$

The following decision steps are needed to handle the MADM problems under QSVNN information by using the operator QSVNDWAA or QSVNDWGA.

Step 1 : Collect the QSVNN $q_i(i = 1, 2, ..., m)$ for the given alternative $A_i(i = 1, 2, ..., m)$ by using the operator QSVNDWAA

$$q_i = QSVNDWAA(q_{i1}, q_{i2}, ..., q_{in})$$
$$= \left\langle 1 - \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j \left(\frac{t_{ij}}{1-t_{ij}}\right)^{\rho}\right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j \left(\frac{c_{ij}}{1-c_{ij}}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j \left(\frac{1-u_{ij}}{u_{ij}}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j \left(\frac{1-f_{ij}}{f_{ij}}\right)^{\rho}\right\}^{1/\rho}} \right\rangle$$

or by using QSVNDWGA operator

$$q_i = QSVNDWGA(q_{i1}, q_{i2}, ..., q_{in})$$
$$= \left\langle \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j \left(\frac{1-t_{ij}}{t_{ij}}\right)^{\rho}\right\}^{1/\rho}}, \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j \left(\frac{1-c_{ij}}{c_{ij}}\right)^{\rho}\right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j \left(\frac{u_{ij}}{1-u_{ij}}\right)^{\rho}\right\}^{1/\rho}}, 1 - \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j \left(\frac{f_{ij}}{1-f_{ij}}\right)^{\rho}\right\}^{1/\rho}} \right\rangle$$

where $w = (w_1, w_2, ..., w_n)$ is the weight vector such that $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$

Step 2: Score values $E(q_i)$ can be calculated by using Equation (1) with the collective QSVNN $q_i(i = 1, 2, ..., m)$

Step 3: Select the best one according to rank given to the alternatives.

6. **Illustrative Example**

This section illustrates an example for a MADM problem about investment alternatives under a QSVNN environment. An investment company chooses three possible alternatives for investing their money by considering the four attributes. Let A_1, A_2, A_3 be three alternatives which represent food, car and computer company respectively. Let C_1, C_2, C_3, C_4 be the four attributes which denotes i) Knowledge (or) Expertise ii) Start up costs iii) Market or Demand iv) Competition respectively. Here the alternatives under given attributes are expressed by the form of QSVNNs. When three alternatives under four attributes are evaluated we get a quadripartitioned single valued neutrosophic decision matrix $D = (q_{ij})_{m\times n}$ where $q_{ij} = \langle t_{ij}, c_{ij}, u_{ij}, f_{ij}\rangle\,(i = 1, 2, 3; j = 1, 2, 3, 4)$ which is given below.

$$D = \begin{bmatrix} \langle 0.5, 0.6, 0.2, 0.1\rangle & \langle 0.4, 0.2, 0.3, 0.1\rangle & \langle 0.4, 0.2, 0.3, 0.1\rangle & \langle 0.6, 0.7, 0.1, 0.5\rangle \\ \langle 0.5, 0.1, 0.8, 0.7\rangle & \langle 0.2, 0.1, 0.8, 0.7\rangle & \langle 0.5, 0.4, 0.7, 0.3\rangle & \langle 0.5, 0.4, 0.7, 0.3\rangle \\ \langle 0.1, 0.2, 0.5, 0.7\rangle & \langle 0.1, 0.5, 0.3, 0.4\rangle & \langle 0.3, 0.2, 0.7, 0.8\rangle & \langle 0.9, 0.8, 0.4, 0.1\rangle \end{bmatrix}$$

The weight vector for the above four attributes is given as $w = (0.35, 0.25, 0.25, 0.15)$. Hence the proposed operator of QSVNDWAA (or) QSVNDWGA are used here to solve MADM problem under QSVNN information.

The following steps are needed to solve MADM problem when we use the operator QSVND-WAA. Step 1 : By using Equation(1) for $\rho = 1$ derive the collective QSVNNs of q_i for the alternative $A_i (i = 1, 2, 3)$ which is given below.

$q_1 = \langle 0.4760, 0.6667, 0.2034, 0.1136\rangle$,

$q_2 = \langle 0.4483, 0.25, 0.7568, 0.4565\rangle$,

$q_3 = \langle 0.6038, 0.5, 0.4414, 0.3404\rangle$

Step 2 : Score values $E(q_i)$ can be calculated by using Equation (1) of the collective QSVNN $q_i (i = 1, 2, 3)$ for the alternatives $A_i (i = 1, 2, 3)$ gives the following results.

$E(q_1) = 0.6231, E(q_2) = 0.4962, E(q_3) = 0.5805$

Step 3: The ranking order is given according to the obtained score values

$q_1 > q_3 > q_2$ and the best one is q_1

The same MADM problem can also be solved by using the another proposed operator that is QSVNDWGA. The following steps are needed to solve the MADM problem.

Step 1 : By using Equation (4) for $\rho = 1$ derive the collective QSVNNs of q_i for the alternative $A_i (i = 1, 2, 3)$ which is given below.

$q_1 = \langle 0.4545, 0.3033, 0.2416, 0.1965\rangle$,

$q_2 = \langle 0.3636, 0.1429, 0.7692, 0.6111\rangle$,

$q_3 = \langle 0.1429, 0.2712, 0.5328, 0.6667\rangle$

Step 2 : Score values $E(q_i)$ can be calculated by using Equation (1) of the collective QSVNN $q_i(i = 1, 2, 3)$ for the alternatives $A_i(i = 1, 2, 3)$ gives the following results.

$E(q_1) = 0.6783, E(q_2) = 0.4601, E(q_3) = 0.4181$

Step 3: The ranking order is given according to the obtained score values
$q_1 > q_2 > q_3$ and the best one is q_1

The following Table 1 and 2 shows the ranking results for the parameters of $\rho \in [1, 10]$ of the quadripartitioned single valued neutrosophic Dombi weighted arithmetic average (QSVND-WAA) operator and quadripartitioned single valued neutrosophic Dombi weighted geometric average (QSVNDWGA) operator respectively.

We can observe the following results from Tables 1 and 2.

1) Different aggregation operators that is QSVNDWAA and QSVNDWGA shows different ranking orders. But the ranking orders due to different operational parameters are same according to the one operator. This results that the operational parameter ρ is not sensitive in this decision making problem since we get the same ranking orders corresponding to the QSVNDWAA and QSVNDWGA operator.

Table 1. Ranking results of the operator QSVNDWAA for different operational parameters.

ρ	$E(q_1), E(q_2), E(q_3)$	**Ranking Order**
1	0.6231,0.4962,0.5805	$q_1 > q_3 > q_2$
2	0.6596,0.5044,0.6323	$q_1 > q_3 > q_2$
3	0.6601,0.5089,0.6468	$q_1 > q_3 > q_2$
4	0.6619,0.5118,0.6535	$q_1 > q_3 > q_2$
5	0.6637,0.5139,0.6577	$q_1 > q_3 > q_2$
6	0.6652,0.5154,0.6605	$q_1 > q_3 > q_2$
7	0.6665,0.5166,0.6626	$q_1 > q_3 > q_2$
8	0.6675,0.5181,0.6641	$q_1 > q_3 > q_2$
9	0.6683,0.5184,0.6654	$q_1 > q_3 > q_2$
10	0.6689,0.5191,0.6664	$q_1 > q_3 > q_2$

Table 2. Ranking results of the operator QSVNDWGA for different operational parameters.

ρ	$E(q_1), E(q_2), E(q_3)$	**Ranking Order**
1	0.6783,0.4601,0.4181	$q_1 > q_2 > q_3$
2	0.6619,0.4424,0.3994	$q_1 > q_2 > q_3$
3	0.6475,0.4308,0.3877	$q_1 > q_2 > q_3$
4	0.6380,0.4236,0.3799	$q_1 > q_2 > q_3$
5	0.6315,0.4196,0.3745	$q_1 > q_2 > q_3$
6	0.6269,0.4158,0.3706	$q_1 > q_2 > q_3$
7	0.6236,0.4136,0.3678	$q_1 > q_2 > q_3$
8	0.6204,0.4119,0.3656	$q_1 > q_2 > q_3$
9	0.6185,0.4105,0.3638	$q_1 > q_2 > q_3$
10	0.6167,0.4094,0.3624	$q_1 > q_2 > q_3$

1) The ranking orders according to the operators QSVNDWAA and QSVNDWGA are different
2) Ranking orders are not affected by different operational parameters of $\rho \in [0, 1]$ in both the operators which shows that ρ is not sensitive in this decision making problem.
3) These aggregation methods of the operators QSVNDWAA and QSVNDWGA provides new method to solve MADM problems under an QSVNN environment.

7. Conclusion

In this paper we have studied the Dombi operations of QSVNN based on the Dombi T-norm and T-conorm operations and also we have proposed the two weighted aggregation operators QSVNDWAA , QSVNDWGA and investigate their properties. Multiple Attribute Decision making is one of the effective approach which helps us to the problems involving a selection from a finite number of alternatives are included under finite number of attributes. To solve these type of MADM problems ranking orders are used to select the best one among the given alternatives. This paper also deals about MADM method by using the proposed QSVNDWAA and QSVNDWGA operator under a QSVNN environment. Using these aggregation operators we calculate the score function of the alternatives with respect to the given attributes and this score function helps us to rank the alternatives and choose the best one. Finally we illustrated an example of a MADM problem for the proposed aggregation operators.

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Polarity of generalized neutrosophic subalgebras in BCK/BCI-algebras

Rajab Ali Borzooei[1], Florentin Smarandache[2,*] and Young Bae Jun[1,3]

[1]First Department of Mathematics, Shahid Beheshti University, Tehran 1983963113, Iran

E-mail: borzooei@sbu.ac.ir

[2]Mathematics & Science Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA.

E-mail: fsmarandache@gmail.com

[3]Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea.

E-mail: skywine@gmail.com

*Correspondence: F. Smarandache (fsmarandache@gmail.com)

Abstract: k-polar generalized neutrosophic set is introduced, and it is applied to BCK/BCI-algebras. The notions of k-polar generalized subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra are defined, and several properties are investigated. Characterizations of k-polar generalized neutrosophic subalgebra and k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra are discussed, and the necessity and possibility operator of k-polar generalized neutrosophic subalgebra are are considered. We show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets subalgebras by using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra. A k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra is established by using the generaliged neutrosophic $\in\vee q$-sets, conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic subalgebra and a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra are provided.

Keywords: k-polar generalized neutrosophic subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

1 Introduction

In the fuzzy set which is introduced by Zadeh [35], the membership degree is expressed by only one function so called the truth function. As a generalization of fuzzy set, intuitionistic fuzzy set is introduced by Atanassove by using membership function and nonmembership function. The membership (resp. nonmembership) function represents truth (resp. false) part. Smarandache introduced a new notion so called neutrosophic set by using three functions, i.e., membership function (t), nonmembership function (f) and neutalitic/indeterministic membership function (i) which are independent components. Neutrosophic set is applied to BCK/BCI-algebras which are discussed in the papers [13, 19, 20, 21, 22, 26, 27, 30]. Indeterministic membership function is leaning to one side, membership function or nonmembership function, in the application of neutrosophic set to algebraic structures. In order to divide the role of the indeterministic membership function, Song et al.

[31] introduced the generalized neutralrosophic set, and discussed its application in BCK/BCI-algebras. Borzooei et al. [8] introduced the notion of a commutative generalized neutrosophic ideal in a BCK-algebra, and investigated related properties. They considered characterizations of a commutative generalized neutrosophic ideal. Using a collection of commutative ideals in BCK-algebras, they established a commutative generalized neutrosophic ideal. They also introduced the notion of equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras, and investigated related properties. Zhang [36] introduced the notion of bipolar fuzzy sets as an extension of fuzzy sets, and it is applied in several (algebraic) structures such as (ordered) semigroups (see [12, 7, 10, 28]), (hyper) BCK/BCI-algebras (see [6, 14, 15, 23, 16, 17]) and finite state machines (see [18, 32, 33, 34]). The bipolar fuzzy set is an extension of fuzzy sets whose membership degree range is $[-1, 1]$. So, it is possible for a bipolar fuzzy set to deal with positive information and negative information at the same time. Chen et al. [9] raised a question: "How to generalize bipolar fuzzy sets to multipolar fuzzy sets and how to generalize results on bipolar fuzzy sets to the case of multipolar fuzzy sets?" To solve their question, they tried to fold the negative part into positive part, that is, they used positive part instead of negative part in bipolar fuzzy set. And then they introduced introduced an m-polar fuzzy set which is an extension of bipolar fuzzy sets. It is applied to BCK/BCI-algebra, graph theory and decision-making problems etc. (see [4, 2, 1, 3, 29, 5, 25]).

In this paper, we introduce k-polar generalized neutrosophic set and apply it to BCK/BCI-algebras to study. We define k-polar generalized neutrosophic subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra and study various properties. We discuss characterization of k-polar generalized neutrosophic subalgebra and k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We show that the necessity and possibility operator of k-polar generalized neutrosophic subalgebra are also a k-polar generalized neutrosophic subalgebra. Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets subalgebras. Using the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets are subalgebras. Using the generaliged neutrosophic $\in\vee q$-sets, we establish a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We provide conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic subalgebra and a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

2 Preliminaries

If a set X has a special element 0 and a binary operation $*$ satisfying the conditions:

(I) $(\forall u, v, w \in X)\ (((u * v) * (u * w)) * (w * v) = 0)$,

(II) $(\forall u, v \in X)\ ((u * (u * v)) * v = 0)$,

(III) $(\forall u \in X)\ (u * u = 0)$,

(IV) $(\forall u, v \in X)\ (u * v = 0,\ v * u = 0\ \Rightarrow\ u = v)$,

then we say that X is a *BCI-algebra*. If a BCI-algebra X satisfies the following identity:

(V) $(\forall u \in X)\ (0 * u = 0)$,

then X is called a BCK*-algebra.*

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall u \in X)\,(u * 0 = u)\,, \tag{2.1}$$
$$(\forall u, v, w \in X)\,(u \leq v \;\Rightarrow\; u * w \leq v * w,\; w * v \leq w * u)\,, \tag{2.2}$$
$$(\forall u, v, w \in X)\,((u * v) * w = (u * w) * v) \tag{2.3}$$

where $u \leq v$ if and only if $u * v = 0$. A subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $u * v \in S$ for all $u, v \in S$.

See the books [11] and [24] for more information on BCK/BCI-algeebras.

A fuzzy set μ in a BCK/BCI-algebra X is called a *fuzzy subalgebra* of X if $\mu(u * v) \geq \min\{\mu(u), \mu(v)\}$ for all $u, v \in X$.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1, 2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

3 k-polar generalized neutrosophic subalgebras

A *k-polar generalized neutrosophic set* over a universe X is a structure of the form:

$$\widehat{\mathcal{L}} := \left\{ \frac{z}{\left(\widehat{\ell}_T(z), \widehat{\ell}_{IT}(z), \widehat{\ell}_{IF}(z), \widehat{\ell}_F(z)\right)} \mid z \in X,\; \widehat{\ell}_{IT}(z) + \widehat{\ell}_{IF}(z) \leq \hat{1} \right\} \tag{3.1}$$

where $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are mappings from X into $[0,1]^k$. The membership values of every element $z \in X$ in $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are denoted by

$$\begin{aligned} \widehat{\ell}_T(z) &= \left((\pi_1 \circ \widehat{\ell}_T)(z), (\pi_2 \circ \widehat{\ell}_T)(z), \cdots, (\pi_k \circ \widehat{\ell}_T)(z)\right), \\ \widehat{\ell}_{IT}(z) &= \left((\pi_1 \circ \widehat{\ell}_{IT})(z), (\pi_2 \circ \widehat{\ell}_{IT})(z), \cdots, (\pi_k \circ \widehat{\ell}_{IT})(z)\right), \\ \widehat{\ell}_{IF}(z) &= \left((\pi_1 \circ \widehat{\ell}_{IF})(z), (\pi_2 \circ \widehat{\ell}_{IF})(z), \cdots, (\pi_k \circ \widehat{\ell}_{IF})(z)\right), \\ \widehat{\ell}_F(z) &= \left((\pi_1 \circ \widehat{\ell}_F)(z), (\pi_2 \circ \widehat{\ell}_F)(z), \cdots, (\pi_k \circ \widehat{\ell}_F)(z)\right), \end{aligned} \tag{3.2}$$

respectively, and satisfies the following condition

$$(\pi_i \circ \widehat{\ell}_{IT})(z) + (\pi_i \circ \widehat{\ell}_{IF})(z) \leq 1$$

for all $i = 1, 2, \cdots, k$.

We shall use the ordered quadruple $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ for the k-polar generalized neutrosophic set in (3.1).

Note that for every k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over X, we have

$$(\forall z \in X)\left(\hat{0} \leq \widehat{\ell}_T(z) + \widehat{\ell}_{IT}(z) + \widehat{\ell}_{IF}(z) + \widehat{\ell}_F(z) \leq \hat{3}\right),$$

that is, $0 \leq (\pi_i \circ \widehat{\ell}_T)(z) + (\pi_i \circ \widehat{\ell}_{IT})(z) + (\pi_i \circ \widehat{\ell}_{IF})(z) + (\pi_i \circ \widehat{\ell}_F)(z) \leq 3$ for all $z \in X$ and $i = 1, 2, \cdots, k$.

Unless otherwise stated in this section, X will represent a BCK/BCI-algebra.

Definition 3.1. A k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over X is called a *k-polar generalized neutrosophic subalgebra* of X if it satisfies:

$$(\forall z, y \in X)\left(\begin{array}{l} \widehat{\ell}_T(z*y) \geq \widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y) \\ \widehat{\ell}_{IT}(z*y) \geq \widehat{\ell}_{IT}(z) \wedge \widehat{\ell}_{IT}(y) \\ \widehat{\ell}_{IF}(z*y) \leq \widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(y) \\ \widehat{\ell}_F(z*y) \leq \widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) \end{array}\right), \tag{3.3}$$

that is,

$$\left\{\begin{array}{l} (\pi_i \circ \widehat{\ell}_T)(z*y) \geq (\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y) \\ (\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) \\ (\pi_i \circ \widehat{\ell}_{IF})(z*y) \leq (\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y) \\ (\pi_i \circ \widehat{\ell}_F)(z*y) \leq (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y) \end{array}\right. \tag{3.4}$$

for $i = 1, 2, \cdots, k$.

Example 3.2. Consider a BCK-algebra $X = \{0, \alpha, \beta, \gamma\}$ with the binary operation "$*$" which is given below.

$*$	0	α	β	γ
0	0	0	0	0
α	α	0	α	α
β	β	β	0	β
γ	γ	γ	γ	0

Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a 4-polar neutrosophic set over X in which $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are defined as follows:

$$\widehat{\ell}_T : X \to [0,1]^4,\ z \mapsto \left\{\begin{array}{ll} (0.6, 0.7, 0.8, 0.9) & \text{if } z = 0, \\ (0.4, 0.4, 0.8, 0.5) & \text{if } z = \alpha, \\ (0.5, 0.6, 0.7, 0.3) & \text{if } z = \beta, \\ (0.3, 0.5, 0.4, 0.7) & \text{if } z = \gamma, \end{array}\right.$$

$$\widehat{\ell}_{IT} : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.7, 0.6, 0.8, 0.9) & \text{if } z = 0, \\ (0.6, 0.4, 0.7, 0.5) & \text{if } z = \alpha, \\ (0.5, 0.5, 0.4, 0.8) & \text{if } z = \beta, \\ (0.2, 0.6, 0.5, 0.7) & \text{if } z = \gamma, \end{cases}$$

$$\widehat{\ell}_{IF} : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.2, 0.3, 0.4, 0.5) & \text{if } z = 0, \\ (0.4, 0.7, 0.5, 0.8) & \text{if } z = \alpha, \\ (0.5, 0.5, 0.8, 0.6) & \text{if } z = \beta, \\ (0.7, 0.3, 0.6, 0.7) & \text{if } z = \gamma, \end{cases}$$

$$\widehat{\ell}_{F} : X \to [0,1]^4,\ z \mapsto \begin{cases} (0.4, 0.4, 0.3, 0.2) & \text{if } z = 0, \\ (0.8, 0.7, 0.5, 0.3) & \text{if } z = \alpha, \\ (0.6, 0.5, 0.6, 0.6) & \text{if } z = \beta, \\ (0.4, 0.6, 0.8, 0.4) & \text{if } z = \gamma, \end{cases}$$

It is routine to verify that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a 4-polar generalized neutrosophic subalgebra of X.

If we take $z = y$ in (3.3) and use (III), then we have the following lemma.

Lemma 3.3. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic subalgebra of a BCK/BCI-algebr X. Then*

$$(\forall z, y \in X) \quad \left(\begin{array}{l} \widehat{\ell}_T(0) \geq \widehat{\ell}_T(z),\ \widehat{\ell}_{IT}(0) \geq \widehat{\ell}_{IT}(z) \\ \widehat{\ell}_{IF}(0) \leq \widehat{\ell}_{IF}(z),\ \widehat{\ell}_F(0) \leq \widehat{\ell}_F(z) \end{array} \right). \tag{3.5}$$

Proposition 3.4. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. If there exists a sequence $\{z_n\}$ in X such that $\lim_{n\to\infty} \widehat{\ell}_T(z_n) = \hat{1} = \lim_{n\to\infty} \widehat{\ell}_{IT}(z_n)$ and $\lim_{n\to\infty} \widehat{\ell}_{IF}(z_n) = \hat{0} = \lim_{n\to\infty} \widehat{\ell}_F(z_n)$, then $\widehat{\ell}_T(0) = \hat{1} = \widehat{\ell}_{IT}(0)$ and $\widehat{\ell}_{IF}(0) = \hat{0} = \widehat{\ell}_F(0)$.*

Proof. Using Lemma 3.3, we have

$$\begin{aligned} \hat{1} &= \lim_{n\to\infty} \widehat{\ell}_T(z_n) \leq \widehat{\ell}_T(0) \leq \hat{1} = \lim_{n\to\infty} \widehat{\ell}_{IT}(z_n) \leq \widehat{\ell}_{IT}(0) \leq \hat{1}, \\ \hat{0} &= \lim_{n\to\infty} \widehat{\ell}_{IF}(z_n) \geq \widehat{\ell}_{IF}(0) \geq \hat{0} = \lim_{n\to\infty} \widehat{\ell}_F(z_n) \geq \widehat{\ell}_F(0) \geq \hat{0}. \end{aligned}$$

This completes the proof. □

Proposition 3.5. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic subalgebra of X such that*

$$(\forall z, y \in X) \quad \left(\begin{array}{l} \widehat{\ell}_T(z * y) \geq \widehat{\ell}_T(y),\ \widehat{\ell}_{IT}(z * y) \geq \widehat{\ell}_{IT}(y) \\ \widehat{\ell}_{IF}(z * y) \leq \widehat{\ell}_{IF}(y),\ \widehat{\ell}_F(z * y) \leq \widehat{\ell}_F(y) \end{array} \right). \tag{3.6}$$

Then $\widehat{\mathcal{L}}$ is constant on X, that is, $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are constants on X.

Proof. Since $z * 0 = z$ for all $z \in X$, it follows from the condition (3.6) that

$$\widehat{\ell}_T(z) = \widehat{\ell}_T(z * 0) \geq \widehat{\ell}_T(0),\ \widehat{\ell}_{IT}(z) = \widehat{\ell}_{IT}(z * 0) \geq \widehat{\ell}_{IT}(0), \tag{3.7}$$

$$\widehat{\ell}_{IF}(z) = \widehat{\ell}_{IF}(z * 0) \leq \widehat{\ell}_{IF}(0),\ \widehat{\ell}_F(z) = \widehat{\ell}_F(z * 0) \leq \widehat{\ell}_F(0) \tag{3.8}$$

for all $z \in X$. Combining (3.5) and (3.7) induces $\widehat{\ell}_T(z) = \widehat{\ell}_T(0)$, $\widehat{\ell}_{IT}(z) = \widehat{\ell}_{IT}(0)$, $\widehat{\ell}_{IF}(z) = \widehat{\ell}_{IF}(0)$ and $\widehat{\ell}_F(z) = \widehat{\ell}_F(0)$ for all $z \in X$. Therefore $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are constants on X, that is, $\widehat{\mathcal{L}}$ is constant on X. □

Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over a universe X, consider the following cut sets.

$$U(\widehat{\ell}_T, \hat{n}_T) := \{z \in X \mid \widehat{\ell}_T(z) \geq \hat{n}_T\},$$
$$U(\widehat{\ell}_{IT}, \hat{n}_{IT}) := \{z \in X \mid \widehat{\ell}_{IT}(z) \geq \hat{n}_{IT}\},$$
$$L(\widehat{\ell}_{IF}, \hat{n}_{IF}) := \{z \in X \mid \widehat{\ell}_{IF}(z) \leq \hat{n}_{IF}\},$$
$$L(\widehat{\ell}_F, \hat{n}_F) := \{z \in X \mid \widehat{\ell}_F(z) \leq \hat{n}_F\}$$

for $\hat{n}_T, \hat{n}_{IT}, \hat{n}_{IF}, \hat{n}_F \in [0, 1]^k$, that is,

$$U(\widehat{\ell}_T, \hat{n}_T) := \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) \geq \hat{n}_T^i \text{ for all } i = 1, 2, \cdots, k\},$$
$$U(\widehat{\ell}_{IT}, \hat{n}_{IT}) := \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) \geq \hat{n}_{IT}^i \text{ for all } i = 1, 2, \cdots, k\},$$
$$L(\widehat{\ell}_{IF}, \hat{n}_{IF}) := \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) \leq \hat{n}_{IF}^i \text{ for all } i = 1, 2, \cdots, k\},$$
$$L(\widehat{\ell}_F, \hat{n}_F) := \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) \leq \hat{n}_F^i \text{ for all } i = 1, 2, \cdots, k\}$$

where $\hat{n}_T = (n_T^1, n_T^2, \cdots, n_T^k)$, $\hat{n}_{IT} = (n_{IT}^1, n_{IT}^2, \cdots, n_{IT}^k)$, $\hat{n}_{IF} = (n_{IF}^1, n_{IF}^2, \cdots, n_{IF}^k)$ and $\hat{n}_F = (n_F^1, n_F^2, \cdots, n_F^k)$. It is clear that $U(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^k U(\widehat{\ell}_T, \hat{n}_T)^i$, $U(\widehat{\ell}_{IT}, \hat{n}_{IT}) = \bigcap_{i=1}^k U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i$, $L(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$ and $L(\widehat{\ell}_F, \hat{n}_F) = \bigcap_{i=1}^k L(\widehat{\ell}_F, \hat{n}_F)^i$, where

$$U(\widehat{\ell}_T, \hat{n}_T)^i := \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) \geq \hat{n}_T^i\},$$
$$U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i := \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) \geq \hat{n}_{IT}^i\},$$
$$L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i := \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) \leq \hat{n}_{IF}^i\},$$
$$L(\widehat{\ell}_F, \hat{n}_F)^i := \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) \leq \hat{n}_F^i\}$$

for $i = 1, 2, \cdots, k$.

We handle the characterization of k-polar generalized neutrosophic subalgebra.

Theorem 3.6. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X if and only if the cut sets $U(\widehat{\ell}_T, \hat{n}_T)$, $U(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $L(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT}, \hat{n}_{IF}, \hat{n}_F \in [0, 1]^k$.*

Proof. Assume that $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X. Let $z, y \in X$. If $z, y \in U(\widehat{\ell}_T, \hat{n}_T)$ for all $\hat{n}_T \in [0, 1]^k$, then $(\pi_i \circ \widehat{\ell}_T)(z) \geq n_T^i$ and $(\pi_i \circ \widehat{\ell}_T)(y) \geq n_T^i$ for $i = 1, 2, \cdots, k$. It follows that

$$(\pi_i \circ \widehat{\ell}_T)(z * y) \geq (\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y) \geq n_T^i$$

$i = 1, 2, \cdots, k$. Hence $z * y \in U(\widehat{\ell}_T, \hat{n}_T)$, and so $U(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X. If $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$ for all $\hat{n}_F \in [0,1]^k$, then $(\pi_i \circ \widehat{\ell}_F)(z) \leq n_F^i$ and $(\pi_i \circ \widehat{\ell}_F)(y) \leq n_F^i$ for $i = 1, 2, \cdots, k$. Hence

$$(\pi_i \circ \widehat{\ell}_F)(z * y) \leq (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y) \leq n_F^i$$

$i = 1, 2, \cdots, k$, and so $z * y \in L(\widehat{\ell}_F, \hat{n}_F)$. Therefore $L(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X. Similarly, we can verify that $U(\widehat{\ell}_{IT}, \hat{n}_{IT})$ and $L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ are subalgebras of X.

Conversely, suppose that the cut sets $U(\widehat{\ell}_T, \hat{n}_T)$, $U(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $L(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT}, \hat{n}_{IF}, \hat{n}_F \in [0,1]^k$. If there exists $\alpha, \beta \in X$ such that $\widehat{\ell}_{IT}(\alpha * \beta) < \widehat{\ell}_{IT}(\alpha) \wedge \widehat{\ell}_{IT}(\beta)$, that is,

$$(\pi_i \circ \widehat{\ell}_{IT})(\alpha * \beta) < (\pi_i \circ \widehat{\ell}_{IT})(\alpha) \wedge (\pi_i \circ \widehat{\ell}_{IT})(\beta)$$

for $i = 1, 2, \cdots, k$, then $\alpha, \beta \in U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $\alpha * \beta \notin U(\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ where $\hat{n}_{IT}^i = (\pi_i \circ \widehat{\ell}_{IT})(\alpha) \wedge (\pi_i \circ \widehat{\ell}_{IT})(\beta)$ for for $i = 1, 2, \cdots, k$. This is a contradiction, and so

$$\widehat{\ell}_{IT}(z * y) \geq \widehat{\ell}_{IT}(z) \wedge \widehat{\ell}_{IT}(y)$$

for all $z, y \in X$. By the similarly way, we know that $\widehat{\ell}_T(z * y) \geq \widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y)$ for all $z, y \in X$. Now, suppose that $\widehat{\ell}_F(\alpha * \beta) > \widehat{\ell}_F(\alpha) \vee \widehat{\ell}_F(\beta)$ for some $\alpha, \beta \in X$. Then

$$(\pi_i \circ \widehat{\ell}_F)(\alpha * \beta) > (\pi_i \circ \widehat{\ell}_F)(\alpha) \vee (\pi_i \circ \widehat{\ell}_F)(\beta)$$

for $i = 1, 2, \cdots, k$. If we take $n_F^i = (\pi_i \circ \widehat{\ell}_F)(\alpha) \vee (\pi_i \circ \widehat{\ell}_F)(\beta)$ for $i = 1, 2, \cdots, k$, then $\alpha, \beta \in L(\widehat{\ell}_F, \hat{n}_F)^i$ but $\alpha * \beta \notin L(\widehat{\ell}_F, \hat{n}_F)^i$, a contradiction. Hence

$$\widehat{\ell}_F(z * y) \leq \widehat{\ell}_F(z) \vee \widehat{\ell}_F(y)$$

for all $z, y \in X$. Similarly, we can check that $\widehat{\ell}_{IF}(z * y) \leq \widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(y)$ for all $z, y \in X$. Therefore $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X. □

Theorem 3.7. *Let* $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ *be a* k*-polar generalized neutrosophic set over* X*. Then* $\widehat{\mathcal{L}}$ *is a* k*-polar generalized neutrosophic subalgebra of* X *if and only if the fuzzy sets* $\pi_i \circ \widehat{\ell}_T$, $\pi_i \circ \widehat{\ell}_{IT}$, $\pi_i \circ \widehat{\ell}_F^c$ *and* $\pi_i \circ \widehat{\ell}_{IF}^c$ *are fuzzy subalgebras of* X *where* $(\pi_i \circ \widehat{\ell}_F^c)(z) = 1 - (\pi_i \circ \widehat{\ell}_F)(z)$ *and* $(\pi_i \circ \widehat{\ell}_{IF}^c)(z) = 1 - (\pi_i \circ \widehat{\ell}_{IF})(z)$ *for all* $z \in X$ *and* $i = 1, 2, \cdots, k$*.*

Proof. Suppose that $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X. For any $i = 1, 2, \cdots, k$, it is clear that $\pi_i \circ \widehat{\ell}_T$ and $\pi_i \circ \widehat{\ell}_{IT}$ are fuzzy subalgebras of X. For any $z, y \in X$, we get

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_F^c)(z * y) &= 1 - (\pi_i \circ \widehat{\ell}_F)(z * y) = 1 - (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y) \\ &= (1 - (\pi_i \circ \widehat{\ell}_F)(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_F)(y)) \\ &= (\pi_i \circ \widehat{\ell}_F^c)(z) \wedge (\pi_i \circ \widehat{\ell}_F^c)(y)\end{aligned}$$

and

$$(\pi_i \circ \widehat{\ell}^c_{IF})(z * y) = 1 - (\pi_i \circ \widehat{\ell}_{IF})(z * y) = 1 - (\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)$$
$$= (1 - (\pi_i \circ \widehat{\ell}_{IF})(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_{IF})(y))$$
$$= (\pi_i \circ \widehat{\ell}^c_{IF})(z) \wedge (\pi_i \circ \widehat{\ell}^c_{IF})(y).$$

Hence $\pi_i \circ \widehat{\ell}^c_F$ and $\pi_i \circ \widehat{\ell}^c_{IF}$ are fuzzy subalgebras of X .

Conversely, suppose that the fuzzy sets $\pi_i \circ \widehat{\ell}_T$, $\pi_i \circ \widehat{\ell}_{IT}$, $\pi_i \circ \widehat{\ell}^c_F$ and $\pi_i \circ \widehat{\ell}^c_{IF}$ are fuzzy subalgebras of X for $i = 1, 2, \cdots, k$ and let $z, y \in X$. Then

$$(\pi_i \circ \widehat{\ell}_T)(z * y) \geq (\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y),$$
$$(\pi_i \circ \widehat{\ell}_{IT})(z * y) \geq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y)$$

for all $i = 1, 2, \cdots, k$. Also we have

$$1 - (\pi_i \circ \widehat{\ell}_F)(z * y) = (\pi_i \circ \widehat{\ell}^c_F)(z * y) \geq (\pi_i \circ \widehat{\ell}^c_F)(z) \wedge (\pi_i \circ \widehat{\ell}^c_F)(y)$$
$$= (1 - (\pi_i \circ \widehat{\ell}_F)(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_F)(y))$$
$$= 1 - ((\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y))$$

and

$$1 - (\pi_i \circ \widehat{\ell}_{IF})(z * y) = (\pi_i \circ \widehat{\ell}^c_{IF})(z * y) \geq (\pi_i \circ \widehat{\ell}^c_{IF})(z) \wedge (\pi_i \circ \widehat{\ell}^c_{IF})(y)$$
$$= (1 - (\pi_i \circ \widehat{\ell}_{IF})(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_{IF})(y))$$
$$= 1 - ((\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y))$$

which imply that $(\pi_i \circ \widehat{\ell}_F)(z * y) \leq (\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y)$ and

$$(\pi_i \circ \widehat{\ell}_{IF})(z * y) \leq (\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)$$

for all $i = 1, 2, \cdots, k$. Hence $\widehat{\mathcal{L}}$ is a k-polar generalized neutrosophic subalgebra of X . □

Theorem 3.8. *If* $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ *is a k-polar generalized neutrosophic subalgebra of X, then so are* $\Box\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}^c_{IT}, \widehat{\ell}^c_T\right)$ *and* $\Diamond\widehat{\mathcal{L}} := \left(\widehat{\ell}^c_{IF}, \widehat{\ell}^c_F, \widehat{\ell}_F, \widehat{\ell}_{IF}\right)$.

Proof. Note that $(\pi_i \circ \widehat{\ell}_{IT})(z) + (\pi_i \circ \widehat{\ell}^c_{IT})(z) = (\pi_i \circ \widehat{\ell}_{IT})(z) + 1 - (\pi_i \circ \widehat{\ell}_{IT})(z) = 1$ and $(\pi_i \circ \widehat{\ell}_F)(z) + (\pi_i \circ \widehat{\ell}^c_F)(z) = (\pi_i \circ \widehat{\ell}_F)(z) + 1 - (\pi_i \circ \widehat{\ell}_F)(z) = 1$, that is, $\widehat{\ell}_{IT}(z) + \widehat{\ell}^c_{IT}(z) = \hat{1}$ and $\widehat{\ell}_F(z) + \widehat{\ell}^c_F(z) = \hat{1}$ for all $z \in X$. Hence $\Box\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}^c_{IT}, \widehat{\ell}^c_T\right)$ and $\Diamond\widehat{\mathcal{L}} := \left(\widehat{\ell}^c_{IF}, \widehat{\ell}^c_F, \widehat{\ell}_F, \widehat{\ell}_{IF}\right)$ are k-polar generalized neutrosophic sets over X. For any $z, y \in X$, we get

$$(\pi_i \circ \widehat{\ell}^c_{IT})(z * y) = 1 - (\pi_i \circ \widehat{\ell}_{IT})(z * y) \leq 1 - ((\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y))$$
$$= (1 - (\pi_i \circ \widehat{\ell}_{IT})(z)) \vee (1 - (\pi_i \circ \widehat{\ell}_{IT})(y))$$
$$= (\pi_i \circ \widehat{\ell}^c_{IT})(z) \vee (\pi_i \circ \widehat{\ell}^c_{IT})(y),$$

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_T)(z * y) &= 1 - (\pi_i \circ \widehat{\ell}_T)(z * y) \leq 1 - ((\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y)) \\ &= (1 - (\pi_i \circ \widehat{\ell}_T)(z)) \vee (1 - (\pi_i \circ \widehat{\ell}_T)(y)) \\ &= (\pi_i \circ \widehat{\ell}^c_T)(z) \vee (\pi_i \circ \widehat{\ell}^c_T)(y),\end{aligned}$$

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_{IF})(z * y) &= 1 - (\pi_i \circ \widehat{\ell}_{IF})(z * y) \geq 1 - ((\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)) \\ &= (1 - (\pi_i \circ \widehat{\ell}_{IF})(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_{IF})(y)) \\ &= (\pi_i \circ \widehat{\ell}^c_{IF})(z) \wedge (\pi_i \circ \widehat{\ell}^c_{IF})(y),\end{aligned}$$

and

$$\begin{aligned}(\pi_i \circ \widehat{\ell}^c_F)(z * y) &= 1 - (\pi_i \circ \widehat{\ell}_F)(z * y) \geq 1 - ((\pi_i \circ \widehat{\ell}_F)(z) \vee (\pi_i \circ \widehat{\ell}_F)(y)) \\ &= (1 - (\pi_i \circ \widehat{\ell}_F)(z)) \wedge (1 - (\pi_i \circ \widehat{\ell}_F)(y)) \\ &= (\pi_i \circ \widehat{\ell}^c_F)(z) \wedge (\pi_i \circ \widehat{\ell}^c_F)(y).\end{aligned}$$

Therefore $\Box\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}^c_{IT}, \widehat{\ell}^c_T\right)$ and $\Diamond\widehat{\mathcal{L}} := \left(\widehat{\ell}^c_{IF}, \widehat{\ell}^c_F, \widehat{\ell}_F, \widehat{\ell}_{IF}\right)$ are $k$polar generalized neutrosophic subalgebras of X. $\square$

Theorem 3.9. *Let $\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_k \subseteq [0,1]^k$, that is, $\Lambda_i \subseteq [0,1]$ for $i = 1, 2, \cdots, k$. Let $\mathcal{S}_i := \{S_{t_i} \mid t_i \in \Lambda_i\}$ be a family of subalgebras of X for $i = 1, 2, \cdots, k$ such that*

$$X = \bigcup_{t_i \in \Lambda_i} S_i, \tag{3.9}$$

$$(\forall s_i, t_i \in \Lambda_i)\,(s_i > t_i \ \Rightarrow\ S_{s_i} \subset S_{t_i}) \tag{3.10}$$

for $i = 1, 2, \cdots, k$. Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X defined by

$$(\forall z \in X) \quad \left.\begin{array}{l}(\pi_i \circ \widehat{\ell}_T)(z) = \bigvee\{q_i \in \Lambda_i \mid z \in S_{q_i}\} = (\pi_i \circ \widehat{\ell}_{IT})(z), \\ (\pi_i \circ \widehat{\ell}_{IF})(z) = \bigwedge\{r_i \in \Lambda_i \mid z \in S_{r_i}\} = (\pi_i \circ \widehat{\ell}_F)(z)\end{array}\right) \tag{3.11}$$

for $i = 1, 2, \cdots, k$. Then $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X.

Proof. For any $i = 1, 2, \cdots, k$, we consider the following two cases.

$$t_i = \bigvee\{q_i \in \Lambda_i \mid q_i < t_i\} \text{ and } t_i \neq \bigvee\{q_i \in \Lambda_i \mid q_i < t_i\}.$$

The first case implies that

$$z \in U(\widehat{\ell}_T, t_i) \Leftrightarrow (\forall q_i < t_i)(z \in S_{q_i}) \Leftrightarrow z \in \bigcap_{q_i < t_i} S_{q_i},$$

$$z \in U(\widehat{\ell}_{IT}, t_i) \Leftrightarrow (\forall q_i < t_i)(z \in S_{q_i}) \Leftrightarrow z \in \bigcap_{q_i < t_i} S_{q_i}.$$

Hence $U(\widehat{\ell}_T, t_i) = \bigcap_{q_i<t_i} S_{q_i} = U(\widehat{\ell}_{IT}, t_i)$, and so $U(\widehat{\ell}_T, t_i)$ and $U(\widehat{\ell}_{IT}, t_i)$ are subalgebras of X for all $i = 1, 2, \ldots, k$. Hence $U(\widehat{\ell}_T, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_T, t_i)$ and $U(\widehat{\ell}_{IT}, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_{IT}, t_i)$ are subalgebras of X. For the second case, we will show that $U(\widehat{\ell}_T, t_i) = \bigcup_{q_i\geq t_i} S_{q_i} = U(\widehat{\ell}_{IT}, t_i)$ for all $i = 1, 2, \ldots, k$. If $z \in \bigcup_{q_i\geq t_i} S_{q_i}$, then $z \in S_{q_i}$ for some $q_i \geq t_i$. Hence $(\pi_i \circ \widehat{\ell}_{IT})(z) = (\pi_i \circ \widehat{\ell}_T)(z) \geq q_i \geq t_i$, and so $z \in U(\widehat{\ell}_T, t_i)$ and $z \in U(\widehat{\ell}_{IT}, t_i)$. If $z \notin \bigcup_{q_i\geq t_i} S_{q_i}$, then $z \notin S_{q_i}$ for all $q_i \geq t_i$. The condition $t_i \neq \bigvee\{q_i \in \Lambda_i \mid q_i < t_i\}$ induces $(t_i - \varepsilon_i, t_i) \cap \Lambda_i = \emptyset$ for some $\varepsilon_i > 0$. Hence $z \notin S_{q_i}$ for all $q_i > t_i - \varepsilon_i$, which means that if $z \in S_{q_i}$ then $q_i \leq t_i - \varepsilon_i$. Hence $(\pi_i \circ \widehat{\ell}_{IT})(z) = (\pi_i \circ \widehat{\ell}_T)(z) \leq t_i - \varepsilon_i < t_i$ and so $z \notin U(\widehat{\ell}_{IT}, t_i) = U(\widehat{\ell}_T, t_i)$. Therefore $U(\widehat{\ell}_T, t_i) = U(\widehat{\ell}_{IT}, t_i) \subseteq \bigcup_{q_i\geq t_i} S_{q_i}$. Consequently, $U(\widehat{\ell}_T, t_i) = U(\widehat{\ell}_{IT}, t_i) = \bigcup_{q_i\geq t_i} S_{q_i}$ which is a subalgebra of X, and therefore $U(\widehat{\ell}_T, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_T, t_i)$ and $U(\widehat{\ell}_{IT}, \hat{t}) = \bigcap_{i=1,2,\ldots,k} U(\widehat{\ell}_{IT}, t_i)$ are subalgebras of X. Now, we consider the following two cases.

$$s_i = \bigwedge\{r_i \in \Lambda_i \mid r_i > s_i\} \text{ and } s_i \neq \bigwedge\{r_i \in \Lambda_i \mid r_i > s_i\}.$$

For the first case, we get

$$z \in L(\widehat{\ell}_{IF}, s_i) \Leftrightarrow (\forall s_i < r_i)(z \in S_{r_i}) \Leftrightarrow z \in \bigcap_{r_i>s_i} S_{r_i},$$
$$z \in L(\widehat{\ell}_F, s_i) \Leftrightarrow (\forall s_i < r_i)(z \in S_{r_i}) \Leftrightarrow z \in \bigcap_{r_i>s_i} S_{r_i}.$$

It follows that $L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i) = \bigcap_{r_i>s_i} S_{r_i}$, which is a subalgebra of X. The second case induces $(s_i, s_i + \varepsilon_i) \cap \Lambda_i = \emptyset$ for some $\varepsilon_i > 0$. If $z \in \bigcup_{r_i\leq s_i} S_{r_i}$, then $z \in S_{r_i}$ for some $r_i \leq s_i$, and thus $(\pi_i \circ \widehat{\ell}_{IF})(z) = (\pi_i \circ \widehat{\ell}_F)(z) \leq r_i \leq s_i$, i.e., $z \in L(\widehat{\ell}_{IF}, s_i)$ and $z \in L(\widehat{\ell}_F, s_i)$. Hene $\bigcup_{r_i\leq s_i} S_{r_i} \subseteq L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i)$. If $z \notin \bigcup_{r_i\leq s_i} S_{r_i}$, then $z \notin S_{r_i}$ for all $r_i \leq s_i$ which implies that $z \notin S_{r_i}$ for all $r_i \leq s_i + \varepsilon_i$, that is, if $z \in S_{r_i}$ then $r_i \geq s_i + \varepsilon_i$. Thus $(\pi_i \circ \widehat{\ell}_{IF})(z) = (\pi_i \circ \widehat{\ell}_F)(z) \geq s_i + \varepsilon_i \geq s_i$ and so $z \notin L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i)$. This shows that $L(\widehat{\ell}_{IF}, s_i) = L(\widehat{\ell}_F, s_i) = \bigcup_{r_i\leq s_i} S_{r_i}$, which is a subalgebra of X. Therefore $L(\widehat{\ell}_F, \hat{s}) = \bigcap_{i=1,2,\ldots,k} L(\widehat{\ell}_F, s_i)$ and $U(\widehat{\ell}_{IF}, \hat{s}) = \bigcap_{i=1,2,\ldots,k} L(\widehat{\ell}_{IF}, s_i)$ are subalgebras of X. Using Theorem 3.6, we know that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X. □

4 k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebras

Let $\hat{n}_T = (n_T^1, n_T^2, \cdots, n_T^k)$, $\hat{n}_{IT} = (n_{IT}^1, n_{IT}^2, \cdots, n_{IT}^k)$, $\hat{n}_{IF} = (n_{IF}^1, n_{IF}^2, \cdots, n_{IF}^k)$ and $\hat{n}_F = (n_F^1, n_F^2, \cdots, n_F^k)$ in $[0, 1]^k$. Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over a universe X, we consider the following sets.

$$T_q(\widehat{\ell}_T, \hat{n}_T) := \{z \in X \mid \widehat{\ell}_T(z) + \hat{n}_T > \hat{1}\},$$
$$IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}) := \{z \in X \mid \widehat{\ell}_{IT}(z) + \hat{n}_{IT} > \hat{1}\},$$
$$IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) := \{z \in X \mid \widehat{\ell}_{IF}(z) + \hat{n}_{IF} < \hat{1}\},$$
$$F_q(\widehat{\ell}_F, \hat{n}_F) := \{z \in X \mid \widehat{\ell}_F(z) + \hat{n}_F < \hat{1}\},$$

which are called *generaliged neutrosophic q-sets*, and

$$T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) := \{z \in X \mid \widehat{\ell}_T(z) \geq \hat{n}_T \text{ or } \widehat{\ell}_T(z) + \hat{n}_T > \hat{1}\},$$
$$IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) := \{z \in X \mid \widehat{\ell}_{IT}(z) \geq \hat{n}_{IT} \text{ or } \widehat{\ell}_{IT}(z) + \hat{n}_{IT} > \hat{1}\},$$
$$IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) := \{z \in X \mid \widehat{\ell}_{IF}(z) \leq \hat{n}_{IF} \text{ or } \widehat{\ell}_{IF}(z) + \hat{n}_{IF} < \hat{1}\},$$
$$F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) := \{z \in X \mid \widehat{\ell}_F(z) \leq \hat{n}_F \text{ or } \widehat{\ell}_F(z) + \hat{n}_F < \hat{1}\}$$

which are called *generaliged neutrosophic $\in\vee q$-sets*. Then

$$T_q(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^{k} T_q(\widehat{\ell}_T, \hat{n}_T)^i, \; IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}) = \bigcap_{i=1}^{k} IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})^i,$$
$$IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^{k} IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i, \; F_q(\widehat{\ell}_F, \hat{n}_F) = \bigcap_{i=1}^{k} F_q(\widehat{\ell}_F, \hat{n}_F)^i$$

and

$$T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^{k} T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)^i, \; IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = \bigcap_{i=1}^{k} IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})^i,$$
$$IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^{k} IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})^i, \; F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = \bigcap_{i=1}^{k} F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)^i$$

where

$$T_q(\widehat{\ell}_T, \hat{n}_T)^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1\},$$
$$IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) + n_{IT}^i > 1\},$$
$$IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1\},$$
$$F_q(\widehat{\ell}_F, \hat{n}_F)^i = \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1\}$$

and

$$
\begin{aligned}
T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_T)(z) \geq n_T^i \text{ or } (\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1\},\\
IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IT})(z) \geq n_{IT}^i \text{ or } (\pi_i \circ \widehat{\ell}_{IT})(z) + n_{IT}^i > 1\},\\
IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_{IF})(z) \leq n_{IF}^i \text{ or } (\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1\},\\
F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)^i &= \{z \in X \mid (\pi_i \circ \widehat{\ell}_F)(z) \leq n_F^i \text{ or } (\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1\}.
\end{aligned}
$$

It is clear that $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = U(\widehat{\ell}_T, \hat{n}_T) \cup T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = U(\widehat{\ell}_{IT}, \hat{n}_{IT}) \cup IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = L(\widehat{\ell}_{IF}, \hat{n}_{IF}) \cup IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$, and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = L(\widehat{\ell}_F, \hat{n}_F) \cup F_q(\widehat{\ell}_F, \hat{n}_F)$.

By routine calculations, we have the following properties.

Proposition 4.1. *Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over a universe X, we have*

1. *If $\hat{n}_T, \hat{n}_{IT} \in [0, 0.5]^k$, then $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = U(\widehat{\ell}_T, \hat{n}_T)$ and $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = U(\widehat{\ell}_{IT}, \hat{n}_{IT})$.*
2. *If $\hat{n}_F, \hat{n}_{IF} \in [0.5, 1]^k$, then $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = L(\widehat{\ell}_F, \hat{n}_F)$.*
3. *If $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$, then $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = T_q(\widehat{\ell}_T, \hat{n}_T)$ and $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}) = IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$.*
4. *If $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$, then $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}) = IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F) = F_q(\widehat{\ell}_F, \hat{n}_F)$.*

Unless otherwise stated in this section, X will represent a BCK/BCI-algebra.

Definition 4.2. Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is called a *k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra* of X if it satisfies:

$$
\begin{aligned}
&z \in U(\widehat{\ell}_T, \hat{n}_T),\ y \in U(\widehat{\ell}_T, \hat{n}_T) \ \Rightarrow\ z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T),\\
&z \in U(\widehat{\ell}_{IT}, \hat{n}_{IT}),\ y \in U(\widehat{\ell}_{IT}, \hat{n}_{IT}) \ \Rightarrow\ z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}),\\
&z \in L(\widehat{\ell}_{IF}, \hat{n}_{IF}),\ y \in L(\widehat{\ell}_{IF}, \hat{n}_{IF}) \ \Rightarrow\ z * y \in IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}),\\
&z \in L(\widehat{\ell}_F, \hat{n}_F),\ y \in L(\widehat{\ell}_F, \hat{n}_F) \ \Rightarrow\ z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)
\end{aligned}
\tag{4.1}
$$

for all $z, y \in X$, $\hat{n}_T, \hat{n}_{IT} \in (0, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 1)^k$.

Example 4.3. Consider a BCI-algebra $X = \{0, 1, 2, \alpha, \beta\}$ with the binary operation "$*$" which is given below.

$*$	0	1	2	α	β
0	0	0	0	α	α
1	1	0	1	β	α
2	2	2	0	α	α
α	α	α	α	0	0
β	β	α	β	1	0

Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a 3-polar neutrosophic set over X in which $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are defined as follows:

$$\widehat{\ell}_T : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.5, 0.5) & \text{if } z = 0, \\ (0.7, 0.7, 0.2) & \text{if } z = 1, \\ (0.7, 0.8, 0.5) & \text{if } z = 2, \\ (0.3, 0.4, 0.5) & \text{if } z = \alpha, \\ (0.3, 0.4, 0.2) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IT} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.5, 0.6) & \text{if } z = 0, \\ (0.4, 0.3, 0.7) & \text{if } z = 1, \\ (0.6, 0.8, 0.4) & \text{if } z = 2, \\ (0.7, 0.4, 0.1) & \text{if } z = \alpha, \\ (0.4, 0.3, 0.1) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IF} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.3, 0.1, 0.5) & \text{if } z = 0, \\ (0.8, 0.3, 0.7) & \text{if } z = 1, \\ (0.3, 0.8, 0.5) & \text{if } z = 2, \\ (0.7, 0.9, 0.6) & \text{if } z = \alpha, \\ (0.8, 0.9, 0.7) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_F : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.2, 0.2, 0.5) & \text{if } z = 0, \\ (0.3, 0.9, 0.8) & \text{if } z = 1, \\ (0.5, 0.2, 0.4) & \text{if } z = 2, \\ (0.6, 0.4, 0.6) & \text{if } z = \alpha, \\ (0.6, 0.9, 0.8) & \text{if } z = \beta, \end{cases}$$

It is routine to verify that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is 3-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 4.4. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized neutrosophic subalgebra of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0,1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0,1)^k$.*

Proof. Let $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Then $\widehat{\ell}_T(z) + \hat{n}_T > \hat{1}$ and $\widehat{\ell}_T(y) + \hat{n}_T > \hat{1}$, that is, $(\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1$ and $(\pi_i \circ \widehat{\ell}_T)(y) + n_T^i > 1$ for $i = 1, 2, \cdots, k$. It follows that

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_T)(z * y) + n_T^i &\geq ((\pi_i \circ \widehat{\ell}_T)(z) \wedge (\pi_i \circ \widehat{\ell}_T)(y)) + n_T^i \\ &= ((\pi_i \circ \widehat{\ell}_T)(z) + n_T)^i \wedge ((\pi_i \circ \widehat{\ell}_T)(y) + n_T)^i > 1\end{aligned}$$

for $i = 1, 2, \cdots, k$. Hence $\widehat{\ell}_T(z * y) + \hat{n}_T > \hat{1}$, that is, $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Therefore $T_q(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X. Let $z, y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Then $(\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1$ and $(\pi_i \circ \widehat{\ell}_{IF})(y) + n_{IF}^i < 1$ for $i = 1, 2, \cdots, k$.

Hence

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_{IF})(z*y) + n_{IF}^i &\leq ((\pi_i \circ \widehat{\ell}_{IF})(z) \vee (\pi_i \circ \widehat{\ell}_{IF})(y)) + n_{IF}^i \\ &= ((\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF})^i \vee ((\pi_i \circ \widehat{\ell}_{IF})(y) + n_{IF})^i < 1\end{aligned}$$

for $i = 1, 2, \cdots, k$ and so $\widehat{\ell}_{IF}(z*y) + \hat{n}_{IF} < \hat{1}$. Thus $z*y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X. By the similar way, we can verify that $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X. □

We handle characterizations of a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 4.5. *Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall z, y \in X) \begin{pmatrix} \widehat{\ell}_T(z*y) \geq \bigwedge\{\widehat{\ell}_T(z), \widehat{\ell}_T(y), \widehat{0.5}\} \\ \widehat{\ell}_{IT}(z*y) \geq \bigwedge\{\widehat{\ell}_{IT}(z), \widehat{\ell}_{IT}(y), \widehat{0.5}\} \\ \widehat{\ell}_{IF}(z*y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\} \\ \widehat{\ell}_F(z*y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \end{pmatrix}, \tag{4.2}$$

that is,

$$\begin{cases} (\pi_i \circ \widehat{\ell}_T)(z*y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(z), (\pi_i \circ \widehat{\ell}_T)(y), 0.5\}, \\ (\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}, \\ (\pi_i \circ \widehat{\ell}_{IF})(z*y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(z), (\pi_i \circ \widehat{\ell}_{IF})(y), 0.5\}, \\ (\pi_i \circ \widehat{\ell}_F)(z*y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z), (\pi_i \circ \widehat{\ell}_F)(y), 0.5\} \end{cases} \tag{4.3}$$

for all $z, y \in X$ and $i = 1, 2, \cdots, k$.

Proof. Suppose that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X and let $z, y \in X$. For any $i = 1, 2, \ldots, k$, assume that $(\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) < 0.5$. Then

$$(\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y)$$

because if $(\pi_i \circ \widehat{\ell}_{IT})(z*y) < (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y)$, then there exists $n_{IT}^i \in (0, 0.5)$ such that

$$(\pi_i \circ \widehat{\ell}_{IT})(z*y) < n_{IT}^i \leq (\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y).$$

It follows that $z \in U(\widehat{\ell}_{IT}, n_{IT})^i$ and $y \in U(\widehat{\ell}_{IT}, n_{IT})^i$ but $z*y \notin U(\widehat{\ell}_{IT}, n_{IT})^i$. Also $(\pi_i \circ \widehat{\ell}_{IT})(z*y) + n_{IT}^i < 1$, i.e., $z*y \notin IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Hence $z*y \notin IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ which is a contradiction. Therefore

$$(\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}$$

for all $z, y \in X$ with $(\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) < 0.5$. Now suppose that $(\pi_i \circ \widehat{\ell}_{IT})(z) \wedge (\pi_i \circ \widehat{\ell}_{IT})(y) \geq 0.5$. Then $z \in U(\widehat{\ell}_{IT}, 0.5)^i$ and $y \in U(\widehat{\ell}_{IT}, 0.5)^i$, and so $z*y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, 0.5)^i = U(\widehat{\ell}_{IT}, 0.5)^i \cup IT_q(\widehat{\ell}_{IT}, 0.5)^i$.

Hence $z * y \in U(\widehat{\ell}_{IT}, 0.5)^i$. Otherwise, $(\pi_i \circ \widehat{\ell}_{IT})(z * y) + 0.5 < 0.5 + 0.5 = 1$, a contradiction. Consequently,

$$(\pi_i \circ \widehat{\ell}_{IT})(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}$$

for all $z, y \in X$. Similarly, we know that

$$(\pi_i \circ \widehat{\ell}_{T})(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{T})(z), (\pi_i \circ \widehat{\ell}_{T})(y), 0.5\}$$

for all $z, y \in X$. Suppose that $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) > \widehat{0.5}$. If $\widehat{\ell}_F(z*y) > \widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) := \hat{n}_F$, then $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$, $z * y \notin L(\widehat{\ell}_F, \hat{n}_F)$ and $\widehat{\ell}_F(z * y) + \hat{n}_F > 2\hat{n}_F > 1$, i.e., $z * y \notin F_q(\widehat{\ell}_F, \hat{n}_F)$. This is a contradiction, and so $\widehat{\ell}_F(z*y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\}$ whenever $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) > \widehat{0.5}$. Now assume that $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) \leq \widehat{0.5}$. Then $z, y \in L(\widehat{\ell}_F, \widehat{0.5})$ and thus $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \widehat{0.5}) = L(\widehat{\ell}_F, \widehat{0.5}) \cup F_q(\widehat{\ell}_F, \widehat{0.5})$. If $z * y \notin L(\widehat{\ell}_F, \widehat{0.5})$, that is, $\widehat{\ell}_F(z * y) > \widehat{0.5}$, then $\widehat{\ell}_F(z * y) + \widehat{0.5} > \widehat{0.5} + \widehat{0.5} = \hat{1}$, i.e., $z * y \notin F_q(\widehat{\ell}_F, \widehat{0.5})$. This is a contradiction. Hence $\widehat{\ell}_F(z*y) \leq \widehat{0.5}$ and so $\widehat{\ell}_F(z*y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\}$ whenever $\widehat{\ell}_F(z) \vee \widehat{\ell}_F(y) \leq \widehat{0.5}$. Therefore $\widehat{\ell}_F(z*y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\}$ for all $z, y \in X$. By the similar way, we have $\widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\}$ for all $z, y \in X$.

Conversely, let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X which satisfies the condition (4.2). Let $z, y \in X$ and $\hat{n}_T = (n_T^1, n_T^2, \cdots, n_T^k) \in [0, 1]^k$. If $z, y \in U(\widehat{\ell}_T, \hat{n}_T)$, then $\widehat{\ell}_T(z) \geq \hat{n}_T$ and $\widehat{\ell}_T(y) \geq \hat{n}_T$. If $\widehat{\ell}_T(z * y) < \hat{n}_T$, then $\widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y) \geq \widehat{0.5}$. Otherwise, we get

$$\widehat{\ell}_T(z * y) \geq \bigwedge\{\widehat{\ell}_T(z), \widehat{\ell}_T(y), \widehat{0.5}\} = \widehat{\ell}_T(z) \wedge \widehat{\ell}_T(y) \geq \hat{n}_T,$$

which is a contradiction. Hence

$$\widehat{\ell}_T(z * y) + \hat{n}_T > 2\widehat{\ell}_T(z * y) \geq 2\bigwedge\{\widehat{\ell}_T(z), \widehat{\ell}_T(y), \widehat{0.5}\} = \hat{1}$$

and so $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T) \subseteq T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. Similarly, if $z, y \in U(\widehat{\ell}_{IT}, \hat{n}_{IT})$, then $z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ for $\hat{n}_{IT} = (n_{IT}^1, n_{IT}^2, \cdots, n_{IT}^k) \in [0, 1]^k$. Now, let $z, y \in L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ for $\hat{n}_{IF} = (n_{IF}^1, n_{IF}^2, \cdots, n_{IF}^k) \in [0, 1]^k$. Then $\widehat{\ell}_{IF}(z) \leq \hat{n}_{IF}$ and $\widehat{\ell}_{IF}(y) \leq \hat{n}_{IF}$. If $\widehat{\ell}_{IF}(z * y) > \hat{n}_{IF}$, then $\widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(z) \leq \widehat{0.5}$ because if not, then $\widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\} \leq \widehat{\ell}_{IF}(z) \vee \widehat{\ell}_{IF}(y) \leq \hat{n}_{IF}$, which is a contradiction. Thus

$$\widehat{\ell}_{IF}(z * y) + \hat{n}_{IF} < 2\widehat{\ell}_{IF}(z * y) \leq 2\bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\} = \hat{1}$$

and so $z * y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) \subseteq IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Similarly, we know that if $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$, then $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ for $\hat{n}_F = (n_F^1, n_F^2, \cdots, n_F^k) \in [0, 1]^k$. Therefore $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets subalgebras.

Theorem 4.6. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

Proof. Suppose that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X. Let $z, y \in X$. If $z, y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ for $\hat{n}_{IT} \in (0.5, 1]^k$, then $\widehat{\ell}_{IT}(z) + \hat{n}_{IT} > \hat{1}$ and $\widehat{\ell}_{IT}(y) + \hat{n}_{IT} > \hat{1}$. It follows from Theorem 4.5 that

$$\begin{aligned}\widehat{\ell}_{IT}(z*y) + \hat{n}_{IT} &\geq \bigwedge\{\widehat{\ell}_{IT}(z), \widehat{\ell}_{IT}(y), \widehat{0.5}\} + \hat{n}_{IT} \\ &= \bigwedge\{\widehat{\ell}_{IT}(z) + \hat{n}_{IT}, \widehat{\ell}_{IT}(y) + \hat{n}_{IT}, \widehat{0.5} + \hat{n}_{IT}\} \\ &> \hat{1},\end{aligned}$$

i.e., $z*y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Thus $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X. Suppose that $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$ for $\hat{n}_F \in [0, 0.5)^k$. Then $(\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1$ and $(\pi_i \circ \widehat{\ell}_F)(z) + n_F^i < 1$. Using Theorem 4.5, we have

$$\begin{aligned}(\pi_i \circ \widehat{\ell}_F)(z*y) + n_F^i &\leq \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z), (\pi_i \circ \widehat{\ell}_F)(y), 0.5\} + n_F^i \\ &= \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z) + n_F^i, (\pi_i \circ \widehat{\ell}_F)(y) + n_F^i, 0.5 + n_F^i\} \\ &< 1\end{aligned}$$

and thus $z*y \in F_q(\widehat{\ell}_F, \hat{n}_F)^i$ for all $i = 1, 2, \cdots, k$. Hence $z*y \in \bigcap_{i=1}^k F_q(\widehat{\ell}_F, \hat{n}_F)^i = F_q(\widehat{\ell}_F, \hat{n}_F)$, and therefore $F_q(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X. Similarly, we can induce that $T_q(\widehat{\ell}_T, \hat{n}_T)$ and $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ are subalgebras of X for $\hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F \in [0, 0.5)^k$. □

Using the generaliged neutrosophic $\in\vee q$-sets, we establish a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 4.7. *Given a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ over X, if the generaliged neutrosophic $\in\vee q$-sets $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 1)^k$, then $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X.*

Proof. Assume that there exist $\alpha, \beta \in X$ such that

$$(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) < \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(\alpha), (\pi_i \circ \widehat{\ell}_T)(\beta), 0.5\}$$

for $i = 1, 2, \cdots, k$. Then there exists $n_T^i \in (0, 0.5]$ such that

$$(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) < n_T^i \leq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(\alpha), (\pi_i \circ \widehat{\ell}_T)(\beta), 0.5\}.$$

Hence $\alpha, \beta \in U(\widehat{\ell}_T, \hat{n}_T)^i$, and so $\alpha, \beta \in \bigcap_{i=1}^k U(\widehat{\ell}_T, \hat{n}_T)^i = U(\widehat{\ell}_T, \hat{n}_T) \subseteq T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. Since $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X, it follows that $\alpha * \beta \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^k T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)^i$. Thus $(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) \geq n_T^i$ or $(\pi_i \circ \widehat{\ell}_T)(\alpha * \beta) + n_T^i > 1$ for $i = 1, 2, \cdots, k$. This is a contradiction, and thus $(\pi_i \circ \widehat{\ell}_T)(z * y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_T)(z), (\pi_i \circ \widehat{\ell}_T)(y), 0.5\}$ for all $z, y \in X$ and $i = 1, 2, \cdots, k$. Now, if there exist $\alpha, \beta \in X$ such that

$$(\pi_i \circ \widehat{\ell}_{IF})(\alpha * \beta) > \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(\alpha), (\pi_i \circ \widehat{\ell}_{IF})(\beta), 0.5\}$$

for $i = 1, 2, \cdots, k$, then

$$(\pi_i \circ \widehat{\ell}_{IF})(\alpha * \beta) > n^i_{IF} \geq \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(\alpha), (\pi_i \circ \widehat{\ell}_{IF})(\beta), 0.5\} \tag{4.4}$$

for some $n^i_{IF} \in [0.5, 1)$. Hence $\alpha, \beta \in L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$, and so $\alpha, \beta \in \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = L(\widehat{\ell}_{IF}, \hat{n}_{IF}) \subseteq IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$. This implies that $\alpha * \beta \in IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$, and (4.4) induces $\alpha * \beta \notin L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$ and $(\pi_i \circ \widehat{\ell}_{IF})(\alpha * \beta) + n^i_{IF} > 2n^i_{IF} > 1$ for $i = 1, 2, \cdots, k$. Thus $\alpha * \beta \notin \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = L(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $\alpha * \beta \notin \bigcap_{i=1}^k IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Hence $\alpha * \beta \notin IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ which is a contradiction. Therefore

$$(\pi_i \circ \widehat{\ell}_{IF})(z * y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_{IF})(z), (\pi_i \circ \widehat{\ell}_{IF})(y), 0.5\}$$

for for all $z, y \in X$ and $i = 1, 2, \cdots, k$, i.e., $\widehat{\ell}_{IF}(z * y) \leq \bigvee\{\widehat{\ell}_{IF}(z), \widehat{\ell}_{IF}(y), \widehat{0.5}\}$ for all $z, y \in X$. Similarly, we show that $(\pi_i \circ \widehat{\ell}_{IT})(z*y) \geq \bigwedge\{(\pi_i \circ \widehat{\ell}_{IT})(z), (\pi_i \circ \widehat{\ell}_{IT})(y), 0.5\}$ and $(\pi_i \circ \widehat{\ell}_F)(z*y) \leq \bigvee\{(\pi_i \circ \widehat{\ell}_F)(z), (\pi_i \circ \widehat{\ell}_F)(y), 0.5\}$ for all $z, y \in X$ and $i = 1, 2, \cdots, k$. Using Theorem 4.5, we conclude that $\widehat{\mathcal{L}}$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X. □

Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic $\in\vee q$-sets subalgebras.

Theorem 4.8. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic $\in\vee q$-sets $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0, 0.5]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0.5, 1)^k$.*

Proof. Let $z, y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Then

$$z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i \text{ or } z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$$

and

$$y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i \text{ or } y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$$

for $i = 1, 2, \cdots, k$. Thus we get the following four cases:

(i) $z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$,

(ii) $z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$,

(iii) $z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$,

(iv) $z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ and $y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$.

For the first case, we have $z * y \in IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ for $i = 1, 2, \cdots, k$ and so

$$z * y \in \bigcap_{i=1}^k IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i = IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}).$$

In the the case (ii) (resp., (iii)), $y \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ (resp., $z \in IT_q((\widehat{\ell}_{IT}, \hat{n}_{IT})^i)$ induce $\widehat{\ell}_{IT}(y) > 1 - n^i_{IT} \geq n^i_{IT}$ (resp., $\widehat{\ell}_{IT}(z) > 1 - n^i_{IT} \geq n^i_{IT}$), that is, $y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ (resp., $z \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i)$. Thus $z * y \in IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ for $i = 1, 2, \cdots, k$ which implies that

$$z * y \in \bigcap_{i=1}^{k} IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i = IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}).$$

The last case induces $\widehat{\ell}_{IT}(z) > 1 - n^i_{IT} \geq n^i_{IT}$ and $\widehat{\ell}_{IT}(y) > 1 - n^i_{IT} \geq n^i_{IT}$, i.e., $z, y \in U((\widehat{\ell}_{IT}, \hat{n}_{IT})^i$ for $i = 1, 2, \cdots, k$. It follows that

$$z * y \in \bigcap_{i=1}^{k} IT_{\in\vee q}((\widehat{\ell}_{IT}, \hat{n}_{IT})^i = IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}).$$

Therefore $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X for all $\hat{n}_{IT} \in (0, 0.5]^k$. Similarly, we can show that the set $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X for all $\hat{n}_T \in (0, 0.5]^k$. Let $z, y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Then

$$\widehat{\ell}_F(z) \leq \hat{n}_F \text{ or } \widehat{\ell}_F(z) + \hat{n}_F < \hat{1}$$

and

$$\widehat{\ell}_F(y) \leq \hat{n}_F \text{ or } \widehat{\ell}_F(y) + \hat{n}_F < \hat{1}.$$

If $\widehat{\ell}_F(z) \leq \hat{n}_F$ and $\widehat{\ell}_F(y) \leq \hat{n}_F$, then

$$\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \leq \hat{n}_F \vee \widehat{0.5} = \hat{n}_F$$

by Theorem 4.5, and so $z * y \in L(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $\widehat{\ell}_F(z) \leq \hat{n}_F$ or $\widehat{\ell}_F(y) + \hat{n}_F < \hat{1}$, then

$$\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \leq \bigvee\{\hat{n}_F, \hat{1} - \hat{n}_F, \widehat{0.5}\} = \hat{n}_F$$

by Theorem 4.5. Hence $z * y \in L(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Similarly, if $\widehat{\ell}_F(z) + \hat{n}_F < \hat{1}$ and $\widehat{\ell}_F(y) \leq \hat{n}_F$, then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $\widehat{\ell}_F(z) + \hat{n}_F < \hat{1}$ and $\widehat{\ell}_F(y) + \hat{n}_F < \hat{1}$, then

$$\widehat{\ell}_F(z * y) \leq \bigvee\{\widehat{\ell}_F(z), \widehat{\ell}_F(y), \widehat{0.5}\} \leq (\hat{1} - \hat{n}_F) \vee \widehat{0.5} = \widehat{0.5} < \hat{n}_F$$

by Theorem 4.5. Thus $z * y \in L(\widehat{\ell}_F, \hat{n}_F) \subseteq F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Consequencly, $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X for all $\hat{n}_F \in [0.5, 1)^k$. By the similar way, we can verify that $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X for all $\hat{n}_{IF} \in [0.5, 1)^k$. □

5 k-polar generalized $(q,\ \in\vee q)$-neutrosophic subalgebras

Definition 5.1. Let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X. Then $\widehat{\mathcal{L}}$ is called a *k-polar generalized $(q,\ \in\vee q)$-neutrosophic subalgebra* of X if it satisfies:

$$\begin{aligned}
&z \in T_q(\widehat{\ell}_T, \hat{n}_T),\ y \in T_q(\widehat{\ell}_T, \hat{n}_T) \ \Rightarrow\ z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T),\\
&z \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}),\ y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT}) \ \Rightarrow\ z * y \in IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT}),\\
&z \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}),\ y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) \ \Rightarrow\ z * y \in IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF}),\\
&z \in F_q(\widehat{\ell}_F, \hat{n}_F),\ y \in F_q(\widehat{\ell}_F, \hat{n}_F) \ \Rightarrow\ z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)
\end{aligned} \tag{5.1}$$

for all $z, y \in X$, $\hat{n}_T, \hat{n}_{IT} \in (0,1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0,1)^k$.

Example 5.2. Let $X = \{0, 1, 2, \alpha, \beta\}$ be the BCI-algebra which is given in Example 4.3. Let $\widehat{\mathcal{L}} := \big(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\big)$ be a 3-polar generalized neutrosophic set over X in which $\widehat{\ell}_T$, $\widehat{\ell}_{IT}$, $\widehat{\ell}_{IF}$ and $\widehat{\ell}_F$ are defined as follows:

$$\widehat{\ell}_T : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.7, 0.8) & \text{if } z = 0,\\ (0.7, 0.0, 0.0) & \text{if } z = 1,\\ (0.0, 0.0, 0.9) & \text{if } z = 2,\\ (0.0, 0.0, 0.0) & \text{if } z = \alpha,\\ (0.0, 0.0, 0.0) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IT} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.6, 0.7, 0.8) & \text{if } z = 0,\\ (0.7, 0.0, 0.0) & \text{if } z = 1,\\ (0.5, 0.8, 0.9) & \text{if } z = 2,\\ (0.0, 0.0, 0.7) & \text{if } z = \alpha,\\ (0.0, 0.0, 0.0) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_{IF} : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.2, 0.3, 0.1) & \text{if } z = 0,\\ (1.0, 1.0, 0.2) & \text{if } z = 1,\\ (0.3, 0.4, 1.0) & \text{if } z = 2,\\ (0.4, 1.0, 1.0) & \text{if } z = \alpha,\\ (1.0, 1.0, 1.0) & \text{if } z = \beta, \end{cases}$$

$$\widehat{\ell}_F : X \to [0,1]^3,\ z \mapsto \begin{cases} (0.2, 0.4, 0.4) & \text{if } z = 0,\\ (0.4, 1.0, 1.0) & \text{if } z = 1,\\ (1.0, 0.2, 0.1) & \text{if } z = 2,\\ (1.0, 0.3, 1.0) & \text{if } z = \alpha,\\ (1.0, 1.0, 1.0) & \text{if } z = \beta, \end{cases}$$

It is routine to verify that $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a 3-polar generalized $(q,\ \in\vee q)$-neutrosophic subalgebra of X.

Using the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra, we show that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets are subalgebras.

Theorem 5.3. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

Proof. Let $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Then $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, and so $z * y \in U(\widehat{\ell}_T, \hat{n}_T)$ or $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. If $z * y \in U(\widehat{\ell}_T, \hat{n}_T)$, then $(\pi_i \circ \widehat{\ell}_T)(z * y) \geq n_T^i > 1 - n_T^i$ since $n_T^i > 0.5$ for all $i = 1, 2, \cdots, k$. Hence $z*y \in T_q(\widehat{\ell}_T, \hat{n}_T)$, and so $T_q(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X. By the similar way, we can verify that $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X. Let $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$. Then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$, and so $z * y \in L(\widehat{\ell}_F, \hat{n}_F)$ of $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F)$. If $z * y \in L(\widehat{\ell}_F, \hat{n}_F)$, then $(\pi_i \circ \widehat{\ell}_F)(z * y) \leq n_F^i < 1 - n_F^i$ since $n_F^i < 0.5$ for all $i = 1, 2, \cdots, k$. Thus $z * y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, and hence $F_q(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X. Similarly, the set $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X. □

Theorem 5.4. *If $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X, then the generaliged neutrosophic $\in\vee q$-sets $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$, $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

Proof. Let $z, y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ for $\hat{n}_T \in (0.5, 1]^k$. If $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$, then obviously $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. If $z \in U(\widehat{\ell}_T, \hat{n}_T)$ and $y \in T_q(\widehat{\ell}_T, \hat{n}_T)$, then $\widehat{\ell}_T(z) + \hat{n}_T \geq 2\hat{n}_T > \hat{1}$, i.e., $z \in T_q(\widehat{\ell}_T, \hat{n}_T)$. It follows that $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. We can prove $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ whenever $y \in U(\widehat{\ell}_T, \hat{n}_T)$ and $z \in T_q(\widehat{\ell}_T, \hat{n}_T)$ in the same way. If $z, y \in U(\widehat{\ell}_T, \hat{n}_T)$, then $\widehat{\ell}_T(z) + \hat{n}_T \geq 2\hat{n}_T > \hat{1}$ and $\widehat{\ell}_T(y) + \hat{n}_T \geq 2\hat{n}_T > \hat{1}$ and so $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T)$. Thus $z * y \in T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$. Therefore $T_{\in\vee q}(\widehat{\ell}_T, \hat{n}_T)$ is a subalgebra of X for $\hat{n}_T \in (0.5, 1]^k$. Now, let $z, y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ for $\hat{n}_F \in [0, 0.5)^k$. If $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then obviously $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $z \in L(\widehat{\ell}_F, \hat{n}_F)$ and $y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then $\widehat{\ell}_F(z) + \hat{n}_F \leq 2\hat{n}_F < \hat{1}$, i.e., $z \in F_q(\widehat{\ell}_F, \hat{n}_F)$. Hence $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Similarly, we can prove that if $y \in L(\widehat{\ell}_F, \hat{n}_F)$ and $z \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. If $z, y \in L(\widehat{\ell}_F, \hat{n}_F)$, then $\widehat{\ell}_F(z) + \hat{n}_F \leq 2\hat{n}_F < \hat{1}$ and $\widehat{\ell}_F(y) + \hat{n}_F \leq 2\hat{n}_F < \hat{1}$, that is, $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$. Hence $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Therefore $F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$ is a subalgebra of X for all $\hat{n}_F \in [0, 0.5)^k$. In the same way, we can show that $IT_{\in\vee q}(\widehat{\ell}_{IT}, \hat{n}_{IT})$ is a subalgebra of X for $\hat{n}_{IT} \in (0.5, 1]^k$ and $IF_{\in\vee q}(\widehat{\ell}_{IF}, \hat{n}_{IF})$ is a subalgebra of X for all $\hat{n}_{IF} \in [0, 0.5)^k$. □

We provide conditions for a k-polar generalized neutrosophic set to be a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

Theorem 5.5. *For a subalgebra S of X, let $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ be a k-polar generalized neutrosophic set over X such that*

$$(\forall z \in S)(\widehat{\ell}_T(z) \geq \widehat{0.5},\ \widehat{\ell}_{IT}(z) \geq \widehat{0.5},\ \widehat{\ell}_{IF}(z) \leq \widehat{0.5},\ \widehat{\ell}_F(z) \leq \widehat{0.5}), \tag{5.2}$$

$$(\forall z \in X \setminus S)(\widehat{\ell}_T(z) = \widehat{0} = \widehat{\ell}_{IT}(z),\ \widehat{\ell}_{IF}(z) = \widehat{1} = \widehat{\ell}_F(z)). \tag{5.3}$$

Then $\widehat{\mathcal{L}}$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X.

Proof. Let $z, y \in T_q(\widehat{\ell}_T, \hat{n}_T) = \bigcap_{i=1}^k T_q(\widehat{\ell}_T, \hat{n}_T)^i$. Then $(\pi_i \circ \widehat{\ell}_T)(z) + n_T^i > 1$ and $(\pi_i \circ \widehat{\ell}_T)(y) + n_T^i > 1$ for all $i = 1, 2, \cdots, k$. If $z * y \notin S$, then $z \in X \setminus S$ or $y \in X \setminus S$ since S is a subalgebra of X. Hence $(\pi_i \circ \widehat{\ell}_T)(z) = 0$ or $(\pi_i \circ \widehat{\ell}_T)(y) = 0$, which imply that $n_T^i > 1$, a contradiction. Thus $z * y \in S$ and so $(\pi_i \circ \widehat{\ell}_T)(z * y) \geq 0.5$ by (5.2). If $n_T^i > 0.5$, then $(\pi_i \circ \widehat{\ell}_T)(z * y) + n_T^i > 1$, ie., $z * y \in T_q(\widehat{\ell}_T, \hat{n}_T)^i$ for all $i = 1, 2, \cdots, k$. Hence $z * y \in \bigcap_{i=1}^k T_q(\widehat{\ell}_T, \hat{n}_T)^i = T_q(\widehat{\ell}_T, \hat{n}_T)$. Similarly, if $z, y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, then $z * y \in IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$. Let $z, y \in IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF}) = \bigcap_{i=1}^k IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i$. Then $(\pi_i \circ \widehat{\ell}_{IF})(z) + n_{IF}^i < 1$ and $(\pi_i \circ \widehat{\ell}_{IF})(y) + n_{IF}^i < 1$ for all $i = 1, 2, \cdots, k$, which implies that $z * y \in S$. If $n_{IF}^i \geq 0.5$, then $(\pi_i \circ \widehat{\ell}_{IF})(z * y) \leq 0.5 \leq n_{IF}^i$ for all $i = 1, 2, \cdots, k$ which shows that $z * y \in \bigcap_{i=1}^k L(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = L(\widehat{\ell}_{IF}, \hat{n}_{IF})$. If $n_{IF}^i < 0.5$, then $(\pi_i \circ \widehat{\ell}_{IF})(z * y) + n_{IF}^i < 1$ for all $i = 1, 2, \cdots, k$ and so $z * y \in \bigcap_{i=1}^k IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})^i = IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$. Similarly way is to show that if $z, y \in F_q(\widehat{\ell}_F, \hat{n}_F)$, then $z * y \in F_{\in\vee q}(\widehat{\ell}_F, \hat{n}_F)$. Therefore $\widehat{\mathcal{L}}$ is a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra of X. □

Combining Theorems 5.3 and 5.5, we have the following corollary.

Corollary 5.6. *If a k-polar generalized neutrosophic set $\widehat{\mathcal{L}} := \left(\widehat{\ell}_T, \widehat{\ell}_{IT}, \widehat{\ell}_{IF}, \widehat{\ell}_F\right)$ satisfies two conditions (5.2) and (5.3) for a subalgebra S of X, then the generaliged neutrosophic q-sets $T_q(\widehat{\ell}_T, \hat{n}_T)$, $IT_q(\widehat{\ell}_{IT}, \hat{n}_{IT})$, $IF_q(\widehat{\ell}_{IF}, \hat{n}_{IF})$ and $F_q(\widehat{\ell}_F, \hat{n}_F)$ are subalgebras of X for all $\hat{n}_T, \hat{n}_{IT} \in (0.5, 1]^k$ and $\hat{n}_F, \hat{n}_{IF} \in [0, 0.5)^k$.*

6 Conclusions

We have introduced k-polar generalized neutrosophic set and have applied it to BCK/BCI-algebras. We have defined k-polar generalized neutrosophic subalgebra, k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra and k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra and have studid various properties. We have discussed characterization of k-polar generalized neutrosophic subalgebra and k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We have shown that the necessity and possibility operator of k-polar generalized neutrosophic subalgebra are also a k-polar generalized neutrosophic subalgebra. Using the k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra, we have shown that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets subalgebras. Using the k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra, we have shown that the generaliged neutrosophic q-sets and the generaliged neutrosophic $\in\vee q$-sets are subalgebras. Using the generaliged neutrosophic $\in\vee q$-sets, we have established a k-polar generalized $(\in, \in\vee q)$-neutrosophic subalgebra. We have provided conditions for a k-polar generalized neutrosophic set to be a k-polar generalized neutrosophic subalgebra and a k-polar generalized $(q, \in\vee q)$-neutrosophic subalgebra.

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Neutrosophic N-Soft Sets with TOPSIS method for Multiple Attribute Decision Making

Muhammad Riaz [1], Khalid Naeem[2], Iqra Zareef [3], Deeba Afzal[4]

[1] Department of Mathematics, University of the Punjab Lahore, Pakistan. E-mail: mriaz.math@pu.edu.pk
[2] Department of Mathematics & Statistics, The University of Lahore, Pakistan. E-mail: khalidnaeem333@gmail.com
[3] Department of Mathematics, University of the Punjab Lahore, Pakistan. E-mail: iqramaan90@yahoo.com
[4] Department of Mathematics & Statistics, The University of Lahore, Pakistan. Pakistan. E-mail: deebafzal@gmail.com

Abstract: The objective of this article is to introduce a new hybrid model of neutrosophic N-soft set which is combination of neutrosophic set and N-soft set. We introduce some basic operations on neutrosophic N-soft sets along with their fundamental properties. For multi-attribute decision-making (MADM) problems with neutrosophic N-soft sets, we propose an extended TOPSIS (technique based on order preference by similarity to ideal solution) method. In this method, we first propose a weighted decision matrix based comparison method to identify the positive and the negative ideal solutions. Afterwards, we define a separation measurement of these solutions. Finally, we calculate relative closeness to identify the optimal alternative. At length, a numerical example is rendered to illustrate the developed scheme in medical diagnosis via hypothetical case study.

Keywords: Neutosophic N-soft set, operations on neutosophic N-soft sets, MADM, TOPSIS, medical diagnosis.

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1. Introduction

In contemporary decision-making science, multi-attribute decision-making (MADM) phenomenon plays a significant role in solving many real world problems. To deal with uncertainties, researchers have introduced different theories including, Fuzzy set (FS) [54] that comprises a mapping communicating the degree of association and intuitionistic fuzzy set (IFS) [10, 11] that comprises a pair of mappings communicating the degree of association and the degree of non-association of members of the universe to the unit closed interval with the restriction that sum of degree of association and degree of non-association should not exceed one. Smarandache [46, 47] introduced neutrosophic sets as an extension of IFSs. A neutrosophic object comprises three degrees, namely, degree of association, indeterminacy, and the degree of non-association to each alternative.

Smarandache's Neutrosophic Set [50] is a generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and n-Hyper-Spherical Fuzzy Set; while Neutrosophication is a generalization of Regret Theory, Grey System Theory, and Three-Ways Decision. In 1999, Molodtsov [32] presented the notion of soft set as an important mathematical tool to deal with uncertainties. In 2007, Aktas and Cagman [6] extended the idea of soft sets to soft groups. In 2010, Feng *et al*. [18, 19] presented several results on soft sets, fuzzy soft sets and rough sets. In 2009 and 2011, Ali *et al*. [7, 8] introduced various properties of soft sets, fuzzy soft sets and rough sets. In 2011, Cagman *et al*. [12], and Shabir and Naz [51] independently presented soft topological spaces. Arockiarani *et al*. [9], in 2013, introduced the notion of fuzzy neutrosophic soft toplogical spaces. In 2016, Davvaz and Sadrabadi [16] presented an interesting application of IFSs in medicine. Nabeeh *et al*. [33, 34] worked on neutrosophic multi-criteria decision making approach for IoT-based enterprises and for personnel selection used the neutrosophic-TOPSIS approach in 2019. Chang *et al*. [35] worked towards a reuse strategic decision pattern framework-from theories to practices. Garg and Arora [20]-[23] introduced generalized intuitionistic fuzzy soft power aggregation operator, Dual hesitant fuzzy soft aggregation operators, a novel scaled prioritized intuitionistic fuzzy soft interaction averaging aggregation operators and their application to multi criteria decision-making. Peng and Dai [36] presented some approaches to single-valued neutrosophic MADM based on MABAC, TOPSIS and new similarity measure with score function. Hashmi *et al*. [24] introduced m-polar neutrosophic topology with applications to multi-criteria decision-making in medical diagnosis and clustering analysis. In 2019, Naeem *et al*. [29] presented pythagorean fuzzy soft MCGDM methods based on TOPSIS, VIKOR and aggregation operators. In 2019, Naeem *et al*. [30] established pythagorean m-polar fuzzy sets and TOPSIS method for the selection of advertisement mode. In 2019, Riaz *et al*. [37] introduced N-soft topology and its applications to multi-criteria group decision making (MCGDM). Riaz and Hashmi [38] introduced the concept of cubic m-polar fuzzy set and presented multi-attribute group decision making (MAGDM) method for agribusiness in the environment of various cubic m-polar fuzzy averaging aggregation operators. Riaz and Hashmi [39] introduced the notion of linear Diophantine fuzzy Set (LDFS) and its applications towards multi-attribute decision making problems. Riaz and Hashmi [40] introduced soft rough Pythagorean m-polar fuzzy sets and Pythagorean m-polar fuzzy soft rough sets with application to decision-making. Riaz and Tehrim [41, 42, 43] substantiated the idea of bipolar fuzzy soft topology, cubic bipolar fuzzy set and cubic bipolar fuzzy ordered weighted geometric aggregation operators and their application using internal and external cubic bipolar fuzzy data. Riaz and Tahrim [44] introduced the concept of bipolar fuzzy soft mappings with application to bipolar disorders.

Smarandache [48] introduced a unifying field in logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics. Smarandache [49] introduced Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics.

Soft sets provide binary evaluation of the objects and other mathematical models like fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets associate values in the interval $[0,1]$. These models fail to deal with the situation when modeling on real world problems associate non-binary evaluations. Non-binary evaluations are also expected in rating or ranking positions. The ranking can be expressed in multinary values in the form of number of stars, dots, grades or any generalized notation. Motivated by these concerns, in 2017, Fatimah *et al*. [17] floated the idea of N-soft set as an extended model of soft set, in order to describe the importance of grades in real life. In 2018 and 2019, Akram *et al*. [1]-[3] introduced group decision-making methods based on hesitant N-soft sets and intuitionistic fuzzy N-soft rough set.

The technique for the order of preference by similarity to ideal solution (TOPSIS) was initially developed by Hwang and Yoon [26] in 1981. The core idea in the TOPSIS method is that selected alternative should have least geometric distance from positive ideal solution and maximum geometric distance from negative ideal solution. Positive ideal solution represents the condition for best solution whereas negative ideal solution represents the condition for the worst. In 2000, Chen [13] extended the TOPSIS method to fuzzy environment and solved a decision making problem based on fuzzy information. Later, in 2008, Chen and Tsao [14] developed interval-valued fuzzy TOPSIS method. TOPSIS method in intuitionistic fuzzy framework was proposed by Li and Nan [31] in 2011. Joshi and Kumar [28] discussed TOPSIS method based on intuitionistic fuzzy entropy and distance measure for multi-criteria decision making. Recently, in 2016 Dey *et al*. [15] employed TOPSIS method for solving decision making problem under bipolar neutrosophic environment. In 2013, Xu and Zhang [53] developed a novel approach based on maximizing deviation and TOPSIS method for the explanation of multi-attribute decision making problems. In 2014, Zhang and Xu [55] presented an extension of TOPSIS in multiple criteria decision making with the help of Pythagorean fuzzy sets. Chen and Tsao [14] proposed interval-valued fuzzy TOPSIS method and its experimental analysis in 2016. In 2018, Akram and Arshad [4] presented a novel trapezoidal bipolar fuzzy TOPSIS method for group decision-making. In 2019, Akram and Adeel [5] presented TOPSIS approach for MAGDM based on interval-valued hesitant fuzzy N-soft environment. In 2019, Tehrim and Riaz [45] presented a novel extension of TOPSIS method with bipolar neutrosophic soft topology and its applications to multi-criteria group decision making (MCGDM). Riaz *et al*. [56]-[57] introduced novel concepts of soft rough topology with applications to MAGDM.

The goal of this paper is to present a new hybrid model "neutrosophic N-soft set" and their applications to the decision making (DM). Neutrosophic N-soft set is the generalization of N-soft set, fuzzy N-soft set and intuitionistic fuzzy N-soft.

The comparison analysis of the proposed model with some existing models is given in Table 1.

Sets	Parametrization	Non Binary Evaluation	Truth Membership	Falsity Membership	Indeterminacy
Fuzzy set [54]	×	×		×	×
Intutionistic fuzzy set [10]	×	×		✓	×
Neutrosophic set [46]	×	×		✓	✓
Soft Set [12]	✓	×	×	×	×
N-soft Set [17]	✓		×	×	×
Fuzzy N-soft Set[1]	✓		✓	×	×
Intutionistic N-soft Set [3]	✓		✓	✓	×
Neutrosophic N-soft Set (Proposed)	✓		✓	✓	✓

Table 1: Comparison with other existing theories

The rest of paper is organized as follows. In Section 2, we recall some fundamental concepts of N-soft set, fuzzy neutrosophic set and fuzzy neutrosophic soft set. In Section 3, we propose our new hybrid model fuzzy neutrosophic N-soft set along with their examples. We also present some basic operations on fuzzy neutrosophic N-soft set with illustrations. We also investigate fundamental properties of the proposed model by using defined operations. In Section 4, we construct relations by using fuzzy neutrosophic N-soft set and define composition of fuzzy neutrosophic N-soft sets using relations. We also define some new choice functions and score functions in connection with fuzzy neutrosophic N-soft sets. In Section 5, we proposed DM method for medical diagnosis by the model. In Section 6, we give a numerical example of this diagnosis method via conjectural case study. In Section7, we conclude with some future directions and give suggestions for future work.

2. Preliminaries

In this segment, we review some essential definitions and a few aftereffects of N-soft and neutrosophic sets that would be accommodating in the following segments.

Definition 2.1 [54] A *fuzzy set* ϑ in $\mathbb{X}$ is assessed up by a mapping with $\mathbb{X}$ as domain and membership degree in $[0,1]$. The accumulation of all *fuzzy sets* (FSs) in the universal set $\mathbb{X}$ is signified by $\vartheta(\mathbb{X})$.

Definition 2.2 [46, 47] A *neutrosophic set* (NS) $\mathbb{P}$ over the universe of discourse $\mathbb{X}$ is defined as

$$\mathbb{P} = \{\langle\varphi, (\mathbb{T}_{\mathbb{P}}(\varphi), \mathbb{I}_{\mathbb{P}}(\varphi), \mathbb{F}_{\mathbb{P}}(\varphi))\rangle : \varphi \in \mathbb{X}\}$$

where $\mathbb{T}_{\mathbb{P}}, \mathbb{I}_{\mathbb{P}}, \mathbb{F}_{\mathbb{P}} : \mathbb{X} \to]^{-}0, 1^{+}[$ and $^{-}0 \leq \mathbb{T}_{\mathbb{P}}(\varphi) + \mathbb{I}_{\mathbb{P}}(\varphi) + \mathbb{F}_{\mathbb{P}}(\varphi) \leq 3^{+}$.

The mapping $\mathbb{T}_{\mathbb{P}}$ stands for degree of membership, $\mathbb{I}_{\mathbb{P}}$ is the degree of indeterminacy and $\mathbb{F}_{\mathbb{P}}$ is the degree of falsity of points of the given set. From philosophical perspective, the neutrosophic set takes the entries from some subset of $]^{-}0, 1^{+}[$. But it many actual applications, it is inconvenient to utilize neutrosophic set with entries from such subsets. Therefore, we consider the neutrosophic set which takes the entries from some subset of $[0,1]$.

Definition 2.3 [9] Let $\mathbb{X}$ be a space of objects (points). A *fuzzy neutrosophic set* (FNS) $\mathbb{P}$ in $\mathbb{X}$ is dispirit by a truth-membership function $\mathbb{T}_P$, an indeterminacy membership-function $\mathbb{I}_P$ and a falsity-membership function $\mathbb{F}_P$. In mathematical form, this collection is expressed as

$$\mathbb{P} = \{\langle\varphi, (\mathbb{T}_{\mathbb{P}}(\varphi), \mathbb{I}_{\mathbb{P}}(\varphi), \mathbb{F}_{\mathbb{P}}(\varphi))\rangle : \varphi \in \mathbb{X}, \mathbb{T}_{\mathbb{P}}, \mathbb{I}_{\mathbb{P}}, \mathbb{F}_{\mathbb{P}} \in [0,1]\}$$

with the constraint that sum of $\mathbb{T}_{\mathbb{P}}(\varphi)$, $\mathbb{I}_{\mathbb{P}}(\varphi)$ and $\mathbb{F}_{\mathbb{P}}(\varphi)$ should fall in $[0,3]$ i.e.

$$0 \leq \mathbb{T}_{\mathbb{P}}(\varphi) + \mathbb{I}_{\mathbb{P}}(\varphi) + \mathbb{F}_{\mathbb{P}}(\varphi) \leq 3$$

Definition 2.4 [32] Let $\mathbb{X}$ be the set of points and E be the set of attributes with $\mathcal{L}$ in E. Assume that $\mathrm{P}(\mathbb{X})$ denotes collection of subsets of $\mathbb{X}$. The pair $(\zeta, \mathcal{L})$ is said to be a *soft set* (SS) over $\mathbb{X}$, where ζ is a function given by $\zeta : \mathcal{L} \to \mathrm{P}(\mathbb{X})$

Thus, an SS is expressed in mathematical form as

$$(\zeta, \mathcal{L}) = \{(\xi, \zeta(\xi)) : \xi \in \mathcal{L}\}.$$

Definition 2.5 [9] Let $\mathbb{X}$ be the initial universal set and E be the set of parameters. We consider the non-empty set $\mathcal{L} \subseteq E$. Let $\widehat{\mathrm{P}}(\mathbb{X})$ signifies the set of all NSs of $\mathbb{X}$. The accretion $\Omega_{\mathcal{L}}$ is called the *neutrosophic soft set* (NSS) over $\mathbb{X}$, where $\Omega_{\mathcal{L}}$ is a function given by $\Omega_{\mathcal{L}} : \mathcal{L} \to \widehat{\mathrm{P}}(\mathbb{X})$. We can write it as

$$\Omega_{\mathcal{L}} = \{(\xi, \{\langle\varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi)\rangle : \varphi \in \mathbb{X}\}) : \xi \in E\}$$

Notice that if $\Omega_{\mathcal{L}}(\xi) = \{\langle\varphi, 0,1,1\rangle : \varphi \in \mathbb{X}\}$, then NS-element $(\xi, \Omega_{\mathcal{L}}(\xi))$ does not seem to appear in the NSS $\Omega_{\mathcal{L}}$. The set of all NSSs over $\mathbb{X}$ is symbolized by $\mathrm{NS}(\mathbb{X}_E)$.

Definition 2.6 [17] Let $\mathbb{X}$ be a set of points and E be a set of attributes with $\mathcal{L}$ in E. Let $\mathcal{G} = \{0,1,2, \cdots, N-1\}$ be the set of ordered grades where $N \in \{2,3, \cdots\}$. The *N-soft set* (NSS) on $\mathbb{X}$ is denoted by $(\zeta, \mathcal{L}, N)$ where $\zeta : \mathcal{L} \to 2^{\mathbb{X} \times \mathcal{G}}$ is a map characterized by

$$\zeta(\xi) = (\varphi, r_{\mathcal{L}(\xi)})$$

$\forall \varphi \in \mathbb{X}, \xi \in \mathcal{L}, r_{\mathcal{L}(\xi)} \in \mathcal{G}$.

Definition 2.7 [17] A *weak complement of N-soft set* $(\zeta, \mathcal{L}, N)$ is another N-soft set $(\zeta^{C}, \mathcal{L}, N)$ gratifying $\zeta(\xi)^{C} \sqcap \zeta(\xi) = \phi, \ \forall \xi \in \mathbb{X}$.

Definition 2.8 [17] A *top weak complement of N-soft set* $(\zeta, \mathcal{L}, N)$ is an N-soft set $(\zeta^{\star}, \mathcal{L}, N)$, where

$$(\zeta^{\star}, \mathcal{L}, N) = \begin{cases} \zeta(\xi) = (\varphi, N-1), & if\, r_{\mathcal{L}(\xi)}(\varphi) < N-1 \\ \zeta(\xi) = (\varphi, 0), & if\, r_{\mathcal{L}(\xi)}(\varphi) = N-1 \end{cases}$$

Definition 2.9 [17] A *bottom weak complement of N-soft set* $(\zeta, \mathcal{L}, N)$ is one more N-soft set $(\zeta_{\star}, \mathcal{L}, N)$, where

$$(\zeta_{\star}, \mathcal{L}, N) = \begin{cases} \zeta(\xi) = (\varphi, 0), & if\, r_{\mathcal{L}(\xi)}(\varphi) > 0, \\ \zeta(\xi) = (\varphi, N-1), & if\, r_{\mathcal{L}(\xi)}(\varphi) = 0. \end{cases}$$

3 Neutrosophic N-soft Set

In this section, we propose a novel structure *neutrosophic N-soft set* (NNSS), which is blend of NS and NSS. We present some definitions and operations on NNSS too. Some properties of NNSS associated with these operations also have been set up.

Definition 3.1 Let $\mathbb{X}$ be the initial universe set, E the set of attributes and $\mathcal{G}$ the aggregate of ordered grades. We consider non-empty subset $\mathcal{L}$ of E. Let $\widehat{\mathrm{P}}(\mathbb{X} \times \mathcal{G})$ be the collection of all NSSs of $\mathbb{X} \times \mathcal{G}$. A *neutrosophic N-soft set* (NNSS) is signified by (λ, Ω, N), where $\Omega = (\zeta, \mathcal{L}, N)$ is an NSS. If there is no ambiguity, we can abbreviate it as $\lambda_{\mathcal{L}}$ represented by the mapping

$$\lambda_{\mathcal{L}}: \mathcal{L} \to \widehat{\mathrm{P}}(\mathbb{X} \times \mathcal{G})$$

Mathematically,

$$\lambda_{\mathcal{L}} = \{(\xi, \Gamma_{\mathcal{L}}(\xi)): \Gamma_{\mathcal{L}}(\xi) = \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi)), r_{\mathcal{L}} \in \mathcal{G},$$
$$\varphi \in \mathbb{X}, \mathbb{T}_{\mathcal{L}}, \mathbb{I}_{\mathcal{L}}, \mathbb{F}_{\mathcal{L}} \in [0,1]\}, \xi \in E\}$$

In short form, we may write

$$\lambda_{\mathcal{L}} = \{(\xi, \Gamma_{\mathcal{L}}(\xi)): \xi \in E\}$$

where

$$\Gamma_{\mathcal{L}}(\xi) = \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi)): r_{\mathcal{L}} \in \mathcal{G}, \varphi \in \mathbb{X}, \mathbb{T}_{\mathcal{L}}, \mathbb{I}_{\mathcal{L}}, \mathbb{F}_{\mathcal{L}} \in [0,1]\}$$

The accretion of all NNSSs is denoted by NNS($\mathbb{X}$).

Our proposed structure is more generalized then other existing models. The existing models are special cases of our proposed model, as shown in Table 2

Neutrosophic N-soft Set (Proposed)	$(\xi, (\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi))$
Intutionistic N-soft Set [3]	$(\xi, (\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), 0, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi))$
Fuzzy N-soft Set [1]	$(\xi, (\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi), 0, 0\rangle, r_{\mathcal{L}(\xi)}(\varphi))$
N-soft Set [17]	$(\xi, r_{\mathcal{L}(\xi)}(\varphi))$

Table 2: Comparison with N-soft set and it's other existing generalization

Example 3.2 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define N8SS as $\lambda_{\mathcal{L}} = \{(\xi_i, \Gamma_{\mathcal{L}}(\xi_i)): \xi_i \in \mathcal{L}, i = 1,2\}$, where 8SS is given in Table 3 below:

$(\zeta, \mathcal{L}, 8)$	ξ_1	ξ_2
φ_1	6	3
φ_2	4	5

Table 3: Tabular representation of 8SS

Now, we define N8SS as

$$\Gamma_{\mathcal{L}}(\xi_1) = \{(\langle\varphi_1, 0.8,0.5,0.1\rangle,6), (\langle\varphi_2, 0.6,0.2,0.9\rangle,4)\}$$
$$\Gamma_{\mathcal{L}}(\xi_2) = \{(\langle\varphi_1, 0.5,0.7,0.3\rangle,3), (\langle\varphi_2, 0.7,0.4,0.8\rangle,5)\}$$

The tabular representation of N8SS is given in Table 4.

$\lambda_{\mathcal{L}}$	ξ_1	ξ_2
φ_1	$(\langle 0.8,0.5,0.1\rangle,6)$	$(\langle 0.5,0.7,0.3\rangle,3)$
φ_2	$(\langle 0.6,0.2,0.9\rangle,4)$	$(\langle 0.7,0.4,0.8\rangle,5)$

Table 4: Tabular representation of N8SS

Remarks:

1. Every N2SS $(\lambda, \Omega, 2)$ is generally equal to NSS.
2. Any arbitrary NNSS over the universe $\mathbb{X}$ can also be thought of as N$(N+1)$-soft set. For example an N8SS can also be treated as an N9SS for the grade 8 is never used as can be seen in Table 4. This observation may be extended on the parallel track.

Now, we head towards presenting some arithmetical notions related to NNSS.

Definition 3.3 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$NNS($\mathbb{X}$). $\lambda_{\mathcal{L}}$ is said to be NNS- *subset* of $\lambda_{\mathcal{M}}$, if

$$\mathcal{L} \sqsubseteq \mathcal{M},$$
$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \leq \mathbb{T}_{\mathcal{M}(\xi)}(\varphi),$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \geq \mathbb{I}_{\mathcal{M}(\xi)}(\varphi),$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \geq \mathbb{F}_{\mathcal{M}(\xi)}(\varphi),$$
$$r_{\mathcal{L}(\xi)}(\varphi) \leq r_{\mathcal{M}(\xi)}(\varphi)$$

$\forall \xi \in E, \varphi \in \mathcal{X}, r_{\mathcal{L}} \in \mathcal{G}$. We demonstrate it by $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{M}}$. $\lambda_{\mathcal{M}}$ is said to be NNS- *superset* of $\lambda_{\mathcal{L}}$.

Example 3.4 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Consider N8SS $\lambda_{\mathcal{L}}$ as given in Example 3.2. Let $\mathcal{M} = E$. Define N8SS $\lambda_{\mathcal{M}}$ as

$$\lambda_{\mathcal{M}} = \{(\xi_i, \Gamma_{\mathcal{M}}(\xi_i)) : \xi_i \in \mathcal{M}, i = 1,2,3\}$$

where 8SS is given in Table 5 below.

$(\zeta, \mathcal{M}, 8)$	ξ_1	ξ_2	ξ_3
φ_1	7	4	6
φ_2	5	7	3

Table 5: Tabular representation of 8SS

Now, we define N8SS

$$\Gamma_{\mathcal{M}}(\xi_1) = \{(\langle\varphi_1, 0.9,0.4,0.0\rangle,7), (\langle\varphi_2, 0.7,0.1,0.8\rangle,5)\}$$
$$\Gamma_{\mathcal{M}}(\xi_2) = \{(\langle\varphi_1, 0.6,0.5,0.2\rangle,4), (\langle\varphi_2, 0.9,0.3,0.8\rangle,7)\}$$
$$\Gamma_{\mathcal{M}}(\xi_3) = \{(\langle\varphi_1, 0.8,0.5,0.1\rangle,6), (\langle\varphi_3, 0.5,0.7,0.3\rangle,3)\}$$

having tabular form

$\lambda_{\mathcal{M}}$	ξ_1	ξ_2	ξ_3
φ_1	(⟨0.9,0.4,0.0⟩,7)	(⟨0.6,0.5,0.2⟩,4)	(⟨0.8,0.5,0.1⟩,6)
φ_2	(⟨0.7,0.1,0.8⟩,5)	(⟨0.9,0.3,0.8⟩,7)	(⟨0.5,0.7,0.3⟩,3)

Table 6: Tabular representation of N8SS

It can be seen from Table 4 and Table 6 that $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{M}}$.

Definition 3.5 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$NNS($\mathbb{X}$). Then $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ are said to be NNS- *equal*, if

$$\mathcal{L} = \mathcal{M},$$
$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{T}_{\mathcal{M}(\xi)}(\varphi),$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{I}_{\mathcal{M}(\xi)}(\varphi),$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{F}_{\mathcal{M}(\xi)}(\varphi),$$
$$r_{\mathcal{L}(\xi)}(\varphi) = r_{\mathcal{M}(\xi)}(\varphi)$$

$\forall \xi \in E, \varphi \in \mathcal{X}, r_{\mathcal{L}} \in \mathcal{G}$. We demonstrate it by $\lambda_{\mathcal{L}} = \lambda_{\mathcal{M}}$.

Definition 3.6 Let $\lambda_{\mathcal{L}} \in$NNS($\mathbb{X}$). If $\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = 0, \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = 1, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = 1$ and $r_{\mathcal{L}(\xi)}(\varphi) = 0$, $\forall \xi \in E$, $\varphi \in \mathcal{X}$, $r_{\mathcal{L}} \in \mathcal{G}$; then $\lambda_{\mathcal{L}}$ is called *null* NNSS and symbolized by $\lambda_{\mathcal{L}_\phi}$.

Example 3.7 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define null N8SS as $\lambda_{\mathcal{L}_\phi} = \{(\xi_i, \Gamma_{\mathcal{L}_\phi}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1,2\}$ where

$$\Gamma_{\mathcal{L}_\phi}(\xi_1) = \{(\langle\varphi_1, 0,1,1\rangle, 0), (\langle\varphi_2, 0,1,1\rangle, 0)\}$$
$$\Gamma_{\mathcal{L}_\phi}(\xi_2) = \{(\langle\varphi_1, 0,1,1\rangle, 0), (\langle\varphi_2, 0,1,1\rangle, 0)\}$$

The tabular form given in Table 7

$\lambda_{\mathcal{L}_\phi}$	ξ_1	ξ_2
φ_1	(⟨0,1,1⟩,0)	(⟨0,1,1⟩,0)
φ_2	(⟨0,1,1⟩,0)	(⟨0,1,1⟩,0)

Table 7: Tabular representation of null N8SS

Definition 3.8 Let $\lambda_{\mathcal{L}} \in$NNS($\mathbb{X}$). If $\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = 1, \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = 0, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = 0$ and $r_{\mathcal{L}(\xi)}(\varphi) = N - 1$, $\forall \xi \in E, \varphi \in \mathbb{X}, r_{\mathcal{L}} \in \mathcal{G}$, then $\lambda_{\mathcal{L}}$ is called *absolute* NNSS and symbolized by $\lambda_{\hat{\mathcal{L}}}$.

Example 3.9 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define absolute N8SS as $\lambda_{\hat{\mathcal{L}}} = \{(\xi_i, \Gamma_{\hat{\mathcal{L}}}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1,2\}$ where

$$\Gamma_{\hat{\mathcal{L}}}(\xi_1) = \{(\langle\varphi_1, 1,0,0\rangle, 7), (\langle\varphi_2, 1,0,0\rangle, 7)\}$$
$$\Gamma_{\hat{\mathcal{L}}}(\xi_2) = \{(\langle\varphi_1, 1,0,0\rangle, 7), (\langle\varphi_2, 1,0,0\rangle, 7)\}$$

having tabular representation that is given in Table 8:

$\lambda_{\hat{\mathcal{L}}}$	ξ_1	ξ_2
φ_1	(⟨1,0,0⟩,7)	(⟨1,0,0⟩,7)
φ_2	(⟨1,0,0⟩,7)	(⟨1,0,0⟩,7)

Table 8: Tabular representation of absolute N8SS

Proposition 3.10 *Let* $\lambda_{\mathcal{K}}, \lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$*NNS*($\mathbb{X}$). *Then,*

1. $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\hat{\mathcal{L}}}$.
2. $\lambda_{\mathcal{L}_\phi} \sqsubseteq \lambda_{\mathcal{L}}$.
3. $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{L}}$.
4. $\lambda_{\mathcal{K}} \sqsubseteq \lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{M}} \Rightarrow \lambda_{\mathcal{K}} \sqsubseteq \lambda_{\mathcal{M}}$.

Proof. The proof follows directly from definitions of related terms.

Proposition 3.11 *Let* $\lambda_{\mathcal{K}}, \lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in NNS(\mathbb{X})$. *Then,*

1. $\lambda_{\mathcal{K}} = \lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{L}} = \lambda_{\mathcal{M}} \Rightarrow \lambda_{\mathcal{K}} = \lambda_{\mathcal{M}}$.
2. $\lambda_{\mathcal{L}} \sqsubseteq \lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{M}} \sqsubseteq \lambda_{\mathcal{L}} \Rightarrow \lambda_{\mathcal{L}} = \lambda_{\mathcal{M}}$.

Proof. Straight forward.

Definition 3.12 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then weak complement of NNSS* $\lambda_{\mathcal{L}}$ *is symbolized by* $\lambda_{\mathcal{L}}^{\mathsf{C}}$ *and defined as*

$$\lambda_{\mathcal{L}}^{\mathsf{C}} = \{(\xi, \Gamma_{\mathcal{L}}^{\mathsf{C}}) : \xi \in E\}$$

where

$$\Gamma_{\mathcal{L}}^{\mathsf{C}} = \{(\langle \varphi, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi), 1 - \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{T}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}^{\mathsf{C}}(\varphi)) : \varphi \in X\}$$

Here $r_{\mathcal{L}(\xi)}^{\mathsf{C}}(\varphi)$ denotes weak complement defined in Definition 2.7.

Example 3.13 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define complement of N8SS $\lambda_{\mathcal{L}}$ given in Example 3.2 as $\lambda_{\mathcal{L}}^{\mathsf{C}} = \{(\xi_i, \Gamma_{\mathcal{L}}^{\mathsf{C}}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1,2\}$ i.e.

$$\Gamma_{\mathcal{L}}^{\mathsf{C}}(\xi_1) = \{(\langle \varphi_1, 0.1,0.5,0.8\rangle,5), (\langle \varphi_2, 0.9,0.8,0.6\rangle,7)\}$$

$$\Gamma_{\mathcal{L}}^{\mathsf{C}}(\xi_2) = \{(\langle \varphi_1, 0.3,0.3,0.5\rangle,4), (\langle \varphi_2, 0.8,0.6,0.7\rangle,2)\}$$

The tabular form is given in Table 9.

$\lambda_{\mathcal{L}}^{\mathsf{C}}$	ξ_1	ξ_2
φ_1	(⟨0.8,0.5,0.1⟩,5)	(⟨0.5,0.7,0.3⟩,4)
φ_2	(⟨0.6,0.2,0.9⟩,7)	(⟨0.7,0.4,0.8⟩,2)

Table 9: Tabular representation of weak complement of N8SS

Proposition 3.14 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$, *then*

1. $(\lambda_{\mathcal{L}}^{\mathsf{C}})^{\mathsf{C}} \neq \lambda_{\mathcal{L}}$.
2. $\lambda_{\mathcal{L}_\phi}^{\mathsf{C}} \neq \lambda_{\hat{\mathcal{L}}}$.
3. $\lambda_{\hat{\mathcal{L}}}^{\mathsf{C}} \neq \lambda_{\mathcal{L}_\phi}$.

Proof. Straight forward.

Definition 3.15 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then top weak complement of NNSS* $\lambda_{\mathcal{L}}$ *is symbolized by* $\lambda_{\mathcal{L}}^{\star}$ *and defined as*

$$\lambda_{\mathcal{L}}^{\star} = \{(\xi, \Gamma_{\mathcal{L}}^{\star}) : \xi \in E\}$$

Where,

$$\Gamma_{\mathcal{L}}^{\star} = \{(\langle \varphi, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi), 1 - \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{T}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}^{\star}(\varphi)) : \varphi \in X\}$$

where, $r_{\mathcal{L}(\xi)}^{\star}(\varphi)$ denotes top weak complement defined in Definition 2.8.

Example 3.16 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Define complement of N8SS $\lambda_{\mathcal{L}}$ given in Example 3.2 as $\lambda_{\mathcal{L}}^{\star} = \{(\xi_i, \Gamma_{\mathcal{L}}^{\star}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1,2\}$ i.e.

$$\Gamma_{\mathcal{L}}^{\star}(\xi_1) = \{(\langle \varphi_1, 0.1,0.5,0.8\rangle,7), (\langle \varphi_2, 0.9,0.8,0.6\rangle,7)\}$$

$$\Gamma^{\star}_{\mathcal{L}}(\xi_2) = \{(\langle\varphi_1, 0.3,0.3,0.5\rangle,7), (\langle\varphi_2, 0.8,0.6,0.7\rangle,7)\}$$

In tabular form given in Table 10.

$\lambda^{\star}_{\mathcal{L}}$	ξ_1	ξ_2
φ_1	$(\langle 0.8,0.5,0.1\rangle,7)$	$(\langle 0.5,0.7,0.3\rangle,7)$
φ_2	$(\langle 0.6,0.2,0.9\rangle,7)$	$(\langle 0.7,0.4,0.8\rangle,7)$

Table 10: Tabular representation of top weak complement of N8SS

Proposition 3.17 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then,*

1. $(\lambda^{\star}_{\mathcal{L}})^{\star} \neq \lambda_{\mathcal{L}}$.
2. $\lambda^{\star}_{\mathcal{L}_\phi} = \lambda_{\hat{\mathcal{L}}}$.
3. $\lambda^{\star}_{\hat{\mathcal{L}}} = \lambda_{\mathcal{L}_\phi}$.

Proof. The proof follows quickly from definitions of relevant terms.

Definition 3.18 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then bottom weak complement of NNSS* $\lambda_{\mathcal{L}}$ *is symbolized by* $\lambda_{\mathcal{L}_\star}$ *and defined as follows*

$$\lambda_{\mathcal{L}_\star} = \{(\xi, \Gamma_{\mathcal{L}_\star}) : \xi \in E\}$$

where

$$\Gamma_{\mathcal{L}_\star} = \{(\langle\varphi, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi), 1 - \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), \mathbb{T}_{\mathcal{L}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)_\star}(\varphi)) : \varphi \in X\}$$

Here $r_{\mathcal{L}(\xi)_\star}(\varphi)$ denotes top weak complement defined in Definition 2.9.

Example 3.19 Let $\mathbb{X} = \{\varphi_1, \varphi_2\}$ and $E = \{\xi_1, \xi_2, \xi_3\}$. Consider $E \supseteq \mathcal{L} = \{\xi_1, \xi_2\}$. Bottom weak complement of N8SS $\lambda_{\mathcal{L}}$ defined in Example 3.2 as $\lambda_{\mathcal{L}_\star} = \{(\xi_i, \Gamma_{\mathcal{L}_\star}(\xi_i)) : \xi_i \in \mathcal{L}, i = 1,2\}$ where

$$\Gamma_{\mathcal{L}_\star}(\xi_1) = \{(\langle\varphi_1, 0.1,0.5,0.8\rangle,7), (\langle\varphi_2, 0.9,0.8,0.6\rangle,7)\}$$

$$\Gamma_{\mathcal{L}_\star}(\xi_2) = \{(\langle\varphi_1, 0.3,0.3,0.5\rangle,7), (\langle\varphi_2, 0.8,0.6,0.7\rangle,7)\}$$

In tabular form the bottom weak complement of N8SS is given in Table 11.

$\lambda_{\mathcal{L}_\star}$	ξ_1	ξ_2
φ_1	$(\langle 0.8,0.5,0.1\rangle,0)$	$(\langle 0.5,0.7,0.3\rangle,0)$
φ_2	$(\langle 0.6,0.2,0.9\rangle,0)$	$(\langle 0.7,0.4,0.8\rangle,0)$

Table 11: Tabular representation of bottom weak complement of N8SS

Proposition 3.20 *Let* $\lambda_{\mathcal{L}} \in NNS(\mathbb{X})$. *Then,*

1. $(\lambda_{\mathcal{L}_\star})_\star \neq \lambda_{\mathcal{L}}$.
2. $(\lambda_{\mathcal{L}_\phi})_\star = \lambda_{\hat{\mathcal{L}}}$.
3. $\lambda_{\hat{\mathcal{L}}_\star} = \lambda_{\mathcal{L}_\phi}$.

Proof. Straight forward.

Definition 3.21 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$NNS($\mathbb{X}$). Then *difference* of $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ is symbolized by $\lambda_{\mathcal{L}} \backslash \lambda_{\mathcal{M}}$ and is defined as

$$\lambda_{\mathcal{L}} \backslash \lambda_{\mathcal{M}} = \{(\xi, \{(\langle\varphi, \mathbb{T}_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi)) :$$
$$\varphi \in \mathbb{X}\}) : \xi \in E\}$$

where $\mathbb{T}_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi)$ and $\mathbb{F}_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi)$ are defined as

$$\mathbb{T}_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi) = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\}$$
$$\mathbb{I}_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi) = \max\{\mathbb{I}_{\mathcal{L}(\xi)}(\varphi), 1 - \mathbb{I}_{\mathcal{M}(\xi)}(\varphi)\}$$
$$\mathbb{F}_{\mathcal{L}(\xi)\backslash\mathcal{M}(\xi)}(\varphi) = \max\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi), \mathbb{T}_{\mathcal{M}(\xi)}(\varphi)\}$$

$$r_{\mathcal{L}(\xi)\setminus\mathcal{M}(\xi)}(\varphi) = \begin{cases} r_{\mathcal{L}(\xi)}(\varphi) - r_{\mathcal{M}(\xi)}(\varphi), & if\, r_{\mathcal{L}(\xi)}(\varphi) > r_{\mathcal{M}(\xi)}(\varphi), \\ 0, & otherwise \end{cases}$$

Definition 3.22 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$NNS$(\mathbb{X})$. Then *addition* of $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ is symbolized by $\lambda_{\mathcal{L}} \oplus \lambda_{\mathcal{M}}$ and is defined as

$$\lambda_{\mathcal{L}} \oplus \lambda_{\mathcal{M}} = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi)) : \varphi \in \mathbb{X}\}) : \xi \in E\}$$

where $\mathbb{T}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi)$ and $\mathbb{F}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi)$ are given as

$$\mathbb{T}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi) = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), 1\}$$

$$\mathbb{I}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi) = \min\{\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), 1\}$$

$$\mathbb{F}_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi) = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{F}_{\mathcal{M}(\xi)}(\varphi), 1\}$$

$$r_{\mathcal{L}(\xi)\oplus\mathcal{M}(\xi)}(\varphi) = \begin{cases} r_{\mathcal{L}(\xi)}(\varphi) + r_{\mathcal{M}(\xi)}(\varphi), & if\, 0 \leq r_{\mathcal{L}(\xi)}(\varphi) + r_{\mathcal{M}(\xi)}(\varphi) < N-1, \\ N-1, & if\, r_{\mathcal{L}(\xi)}(\varphi) + r_{\mathcal{M}(\xi)}(\varphi) \geq N-1 \end{cases}$$

Definition 3.23 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$ NNS $(\mathbb{X})$ be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then their *restricted union* is symbolized by $(\lambda_1, \Omega_1, N_2) \sqcup_{\mathfrak{R}} (\lambda_2, \Omega_2, N_1)$ and defined as $(w, \Omega_1 \sqcup_{\mathfrak{R}} \Omega_2, \max(N_1, N_2))$ where $\Omega_1 \sqcup_{\mathfrak{R}} \Omega_2 = (W, \mathcal{L} \sqcap \mathcal{M}, \max(N_1, N_2))$ i.e.

$$(\lambda_1, \Omega_1, N_2) \sqcup_{\mathfrak{R}} (\lambda_2, \Omega_2, N_1)$$
$$= \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi) \vee r_{\mathcal{M}(\xi)}(\varphi)) : \varphi \in \mathbb{X}\}) : \xi \in \mathcal{L} \sqcap \mathcal{M}\}$$

Example 3.24 Consider again $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}}$ as given in Examples 3.2 and 3.4 respectively. The restricted union $\lambda_{\mathcal{L}} \sqcup_{\mathfrak{R}} \lambda_{\mathcal{M}}$ is given in Table 12.

$\lambda_{\mathcal{L}} \sqcup_{\mathfrak{R}} \lambda_{\mathcal{M}}$	ξ_1	ξ_2
φ_1	(⟨0.9,0.4,0.0⟩,7)	(⟨0.6,0.5,0.2⟩,4)
φ_2	(⟨0.7,0.1,0.8⟩,5)	(⟨0.9,0.3,0.8⟩,7)

Table 12: Tabular representation of restricted union of two N8SSs

Definition 3.25 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$ NNS $(\mathbb{X})$ be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then their *extended union* is symbolized by $(\lambda_1, \Omega_1, N_2) \sqcup_{\mathcal{E}} (\lambda_2, \Omega_2, N_1)$ and defined as $(w, \Omega_1 \sqcup_{\mathcal{E}} \Omega_2, \max(N_1, N_2))$ where $\Omega_1 \sqcup_{\mathcal{E}} \Omega_2 = (W, \mathcal{L} \sqcup \mathcal{M}, \max(N_1, N_2))$ i.e.

$$(\lambda_1, \Omega_1, N_2) \sqcup_{\mathcal{E}} (\lambda_2, \Omega_2, N_1) = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\rangle, r_{\mathcal{L}(\xi)}(\varphi) \vee r_{\mathcal{M}(\xi)}(\varphi)) : \varphi \in \mathbb{X}\}) : \xi \in \mathcal{L} \sqcup \mathcal{M}\}$$

Example 3.26 Consider again $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}}$ as given in Examples 3.2 and 3.4 respectively. The extended union $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}}$ is given in Table 13.

$\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}}$	ξ_1	ξ_2	ξ_3
φ_1	(⟨0.9,0.4,0.0⟩,7)	(⟨0.6,0.5,0.2⟩,4)	(⟨0.8,0.5,0.1⟩,6)
φ_2	(⟨0.7,0.1,0.8⟩,5)	(⟨0.9,0.3,0.8⟩,7)	(⟨0.5,0.7,0.3⟩,3)

Table 13: Tabular representation of extended union of two N8SSs

Theorem 3.27 *Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in NNS(\mathbb{X})$. Then their extended-union $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}}$ is the smallest NNSS containing both $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$.*

Proof. Straight forward.

Definition 3.28 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$ NNS $(\mathbb{X})$ be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then their *restricted intersection* is symbolized by $(\lambda_1, \Omega_1, N_2) \sqcap_{\Re} (\lambda_2, \Omega_2, N_1)$ and is defined as $(y, \Omega_1 \sqcup_{\Re} \Omega_2, \min(N_1, N_2))$ where $\Omega_1 \sqcap_{\Re} \Omega_2 = (Y, \mathcal{L} \sqcap \mathcal{M}, \min(N_1, N_2))$ i.e.

$$(\lambda_1, \Omega_1, N_2) \sqcap_{\Re} (\lambda_2, \Omega_2, N_1) = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{F}_{\mathcal{M}(\xi)}(\varphi) \rangle, r_{\mathcal{L}(\xi)}(\varphi) \wedge r_{\mathcal{M}(\xi)}(\varphi)) : \varphi \in \mathbb{X}\}) : \xi \in \mathcal{L} \sqcap \mathcal{M}\}$$

Example 3.29 Consider again $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}}$ as given in Examples 3.2, 3.4 respectively. The restricted intersection $\lambda_{\mathcal{L}} \sqcap_{\Re} \lambda_{\mathcal{M}}$ is given in Table 14.

$\lambda_{\mathcal{L}} \sqcap_{\Re} \lambda_{\mathcal{M}}$	ξ_1	ξ_2
φ_1	(⟨0.8,0.5,0.1⟩,6)	(⟨0.5,0.7,0.3⟩,3)
φ_2	(⟨0.6,0.2,0.9⟩,4)	(⟨0.7,0.4,0.8⟩,5)

Table 14: Tabular representation of restricted intersection of two N8SSs

Theorem 3.30 *Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in NNS(\mathbb{X})$. Then their restricted-intersection $\lambda_{\mathcal{L}} \sqcap_{\Re} \lambda_{\mathcal{M}}$ is the largest NNSS contained in both $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$.*

Proof. Straight forward.

Definition 3.31 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$ NNS $(\mathbb{X})$ be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then their *restricted intersection* is symbolized by $(\lambda_1, \Omega_1, N_2) \sqcap_{\mathcal{E}} (\lambda_2, \Omega_2, N_1)$ and defined as $(y, \Omega_1 \sqcap_{\mathcal{E}} \Omega_2, \min(N_1, N_2))$, where $\Omega_1 \sqcap_{\mathcal{E}} \Omega_2 = (Y, \mathcal{L} \sqcup \mathcal{M}, \min(N_1, N_2))$ i.e.

$$(\lambda_1, \Omega_1, N_2) \sqcap_{\mathcal{E}} (\lambda_2, \Omega_2, N_1) = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \wedge \mathbb{T}_{\mathcal{M}(\xi)}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{I}_{\mathcal{M}(\xi)}(\varphi), \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \vee \mathbb{F}_{\mathcal{M}(\xi)}(\varphi) \rangle, r_{\mathcal{L}(\xi)}(\varphi) \wedge r_{\mathcal{M}(\xi)}(\varphi)) : \varphi \in \mathbb{X}\}) : \xi \in \mathcal{L} \sqcup \mathcal{M}\}$$

Example 3.32 Consider again $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}}$ as given in Examples 3.2, 3.4 respectively. The extended intersection $\lambda_{\mathcal{L}} \sqcap_{\mathcal{E}} \lambda_{\mathcal{M}}$ is given in Table 15.

$\lambda_{\mathcal{L}} \sqcap_{\mathcal{E}} \lambda_{\mathcal{M}}$	ξ_1	ξ_2	ξ_3
φ_1	(⟨0.8,0.5,0.1⟩,6)	(⟨0.5,0.7,0.3⟩,3)	(⟨0.8,0.5,0.1⟩,6)
φ_2	(⟨0.6,0.2,0.9⟩,4)	(⟨0.7,0.4,0.8⟩,5)	(⟨0.5,0.7,0.3⟩,3)

Table 15: Tabular representation of extended intersection of two N8SSs

For any two NNSS $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ over same set of points $\mathbb{X}$ and using the operations defined above, we conclude the following proposition:

Proposition 3.33 Let $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ be two NNSS

(1) $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{L}} = \lambda_{\mathcal{L}}$

(2) $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}} = \lambda_{\mathcal{M}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{L}}$

(3) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{L}} = \lambda_{\mathcal{L}}$

(4) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{M}} = \lambda_{\mathcal{M}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{L}}$

(5) $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{L}_{\phi}} = \lambda_{\mathcal{L}}$

(6) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{L}_{\phi}} = \lambda_{\mathcal{L}_{\phi}}$

For any three NNSS $\lambda_{\mathcal{L}}$, $\lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{N}}$ over same set of points $\mathbb{X}$ and using the operations defined above, we conclude the following proposition:

Proposition 3.34 Let $\lambda_{\mathcal{L}}$, $\lambda_{\mathcal{M}}$ and $\lambda_{\mathcal{N}}$ be three NNSS

(1) $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} (\lambda_{\mathcal{M}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{N}}) = (\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}}) \sqcup_{\mathcal{E}} \lambda_{\mathcal{N}}$

(2) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} (\lambda_{\mathcal{M}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{N}}) = (\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{M}}) \sqcap_{\mathcal{R}} \lambda_{\mathcal{N}}$

(3) $\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} (\lambda_{\mathcal{M}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{N}}) = (\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{M}}) \sqcap_{\mathcal{R}} (\lambda_{\mathcal{L}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{N}})$

(4) $\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} (\lambda_{\mathcal{M}} \sqcup_{\mathcal{E}} \lambda_{\mathcal{N}}) = (\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{M}}) \sqcup_{\mathcal{E}} (\lambda_{\mathcal{L}} \sqcap_{\mathcal{R}} \lambda_{\mathcal{N}})$

Definition 3.35 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$ NNS ($\mathbb{X}$) be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then *AND Operation* symbolized by $(\lambda_1, \Omega_1, N_2) \wedge (\lambda_2, \Omega_2, N_1)$ or shortly $\lambda_{\mathcal{L}} \wedge \lambda_{\mathcal{M}}$ and is defined as $(\lambda_1, \Omega_1, N_2) \wedge (\lambda_2, \Omega_2, N_1) = (\lambda_{\mathcal{K}}, \mathcal{L} \times \mathcal{M}, \min(N_1, N_2))$, where degree of membership , indeterminacy and non-membership are given as follows:

$$\mathbb{T}_{\mathcal{K}(\xi_i,\xi_j)}(\varphi) = \min\{\mathbb{T}_{\mathcal{L}(\xi_i)}(\varphi), \mathbb{T}_{\mathcal{M}(\xi_j)}(\varphi)\},$$

$$\mathbb{I}_{\mathcal{K}(\xi_i,\xi_j)}(\varphi) = \frac{\{\mathbb{I}_{\mathcal{L}(\xi_i)}(\varphi) + \mathbb{I}_{\mathcal{M}(\xi_j)}(\varphi)\}}{2},$$

$$\mathbb{F}_{\mathcal{K}(\xi_i,\xi_j)}(\varphi) = \max\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\},$$

$$r_{\mathcal{K}(\xi_i,\xi_j)}(\varphi) = \max\{r_{\mathcal{L}(\xi_i)}(\varphi), r_{\mathcal{M}(\xi_j)}(\varphi)\}, \forall \xi_i \in \mathcal{L}, \xi_j \in \mathcal{M}$$

for all $\varphi \in \mathbb{X}$.

Definition 3.36 Let $\lambda_{\mathcal{L}}, \lambda_{\mathcal{M}} \in$NNS($\mathbb{X}$) be two NNS be expressed as $\lambda_{\mathcal{L}} = (\lambda_1, \Omega_1, N)$ and $\lambda_{\mathcal{M}} = (\lambda_2, \Omega_2, N_1)$ where $\Omega_1 = (\zeta_1, \mathcal{L}, N_2)$ and $\Omega_2 = (\zeta_2, \mathcal{M}, N_1)$ are NSSs. Then *OR operation* is symbolized by $(\lambda_1, \Omega_1, N_2) \vee (\lambda_2, \Omega_2, N_1)$ or shortly $\lambda_{\mathcal{L}} \vee \lambda_{\mathcal{M}}$ and is defined as $(\lambda_1, \Omega_1, N_2) \vee (\lambda_2, \Omega_2, N_1) = (\lambda_{\mathcal{K}}, \mathcal{L} \times \mathcal{M}, \min(N_1, N_2))$, where degree of membership ,indeterminacy and non-membership are given as follows:

$$\mathbb{T}_{\mathcal{H}(\xi_i,\xi_j)}(\varphi) = \max\{\mathbb{T}_{\mathcal{L}(\xi_i)}(\varphi), \mathbb{T}_{\mathcal{M}(\xi_j)}(\varphi)\},$$

$$\mathbb{I}_{\mathcal{H}(\xi_i,\xi_j)}(\varphi) = \frac{\{\mathbb{I}_{\mathcal{L}(\xi_i)}(\varphi) + \mathbb{I}_{\mathcal{M}(\xi_j)}(\varphi)\}}{2},$$

$$\mathbb{F}_{\mathcal{H}(\xi_i,\xi_j)}(\varphi) = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi), \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)\},$$

$$r_{\mathcal{H}(\xi_i,\xi_j)}(\varphi) = \min\{r_{\mathcal{L}(\xi_i)}(\varphi), r_{\mathcal{M}(\xi_j)}(\varphi)\}, \forall \xi_i \in \mathcal{L}, \xi_j \in \mathcal{M}$$

for all $\varphi \in \mathbb{X}$.

Definition 3.37 The *Truth-favorite* of an NNSS $\lambda_{\mathcal{L}}$ is denoted by $\lambda_{\mathcal{M}} = \widehat{\triangle}\, \lambda_{\mathcal{L}}$ and is defined by

$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), 1\}$$

$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = 0$$

$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{F}_{\mathcal{M}(\xi)}(\varphi)$$

$$r_{\mathcal{L}(\xi)}(\varphi) = r_{\mathcal{M}(\xi)}(\varphi)$$

for all $\varphi \in \mathbb{X}, \xi \in \mathcal{L}$.

Definition 3.38 *The Falsity-favorite of an NNSS* $\lambda_{\mathcal{L}}$ *is denoted by* $\lambda_{\mathcal{M}} = \widehat{\nabla}\, \lambda_{\mathcal{L}}$ *and is defined by*

$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) = \mathbb{T}_{\mathcal{M}(\xi)}(\varphi)$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) = 0$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) + \mathbb{I}_{\mathcal{L}(\xi)}(\varphi), 1\}$$
$$r_{\mathcal{L}(\xi)}(\varphi) = r_{\mathcal{M}(\xi)}(\varphi)$$

for all $\varphi \in \mathbb{X}, \xi \in \mathcal{L}$.

Proposition 3.39 *Let* $\lambda_{\mathcal{L}}$ *be an NNSS, then*

1. $\widehat{\triangle}\widehat{\triangle}\, \lambda_{\mathcal{L}} = \widehat{\triangle}\, \lambda_{\mathcal{L}}$.
2. $\widehat{\nabla}\widehat{\nabla}\, \lambda_{\mathcal{L}} = \widehat{\nabla}\, \lambda_{\mathcal{L}}$.

Proof. Follows immediately from definitions.

Definition 3.40 Let $\lambda_{\mathcal{L}} \in$NNS($\mathbb{X}$). Then *scalar multiplication* of $\lambda_{\mathcal{L}}$ with α is symbolized by $\lambda_{\mathcal{L}} \otimes \alpha$ and is defined as

$$\lambda_{\mathcal{L}} \otimes \alpha = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha, \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha \rangle, r_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha) : \varphi \in \mathbb{X}\}) : \xi \in E\}$$

where $\mathbb{T}_{\mathcal{L}(\xi)\otimes\alpha}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)\otimes\alpha}(\varphi) \mathbb{F}_{\mathcal{L}(\xi)\otimes\alpha}(\varphi)$ and $r_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha$ are defined by

$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \times \alpha, 1\}$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha = \min\{\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \times \alpha, 1\}$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \times \alpha, 1\}$$
$$r_{\mathcal{L}(\xi)}(\varphi) \otimes \alpha = \begin{pmatrix} r_{\mathcal{L}(\xi)}(\varphi) \times \alpha, & if 0 \leq r_{\mathcal{L}(\xi)}(\varphi) \times \alpha < N-1, \\ N-1, & otherwise \end{pmatrix}$$

Definition 3.41 Let $\lambda_{\mathcal{L}} \in$NNS($\mathbb{X}$). Then *scalar division* of $\lambda_{\mathcal{L}}$ by α is symbolized by $\lambda_{\mathcal{L}} \tilde{/} \alpha$ and is defined as

$$\lambda_{\mathcal{L}} \tilde{/} \alpha = \{(\xi, \{(\langle \varphi, \mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha, \mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha, \mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha \rangle, r_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha) : \varphi \in \mathbb{X}\}) : \xi \in E\}$$

where $\mathbb{T}_{\mathcal{L}(\xi)\tilde{/}\alpha}(\varphi), \mathbb{I}_{\mathcal{L}(\xi)\tilde{/}\alpha}(\varphi) \mathbb{F}_{\mathcal{L}(\xi)\tilde{/}\alpha}(\varphi)$ and $r_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha$ are defined by

$$\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha = \min\{\mathbb{T}_{\mathcal{L}(\xi)}(\varphi) / \alpha, 1\}$$
$$\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha = \min\{\mathbb{I}_{\mathcal{L}(\xi)}(\varphi) / \alpha, 1\}$$
$$\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha = \min\{\mathbb{F}_{\mathcal{L}(\xi)}(\varphi) / \alpha, 1\}$$
$$r_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha = \begin{pmatrix} r_{\mathcal{L}(\xi)}(\varphi) / \alpha, & if 0 \leq r_{\mathcal{L}(\xi)}(\varphi) / \alpha < N-1, \\ N-1, & otherwise \end{pmatrix}$$

$$r_{\mathcal{L}(\xi)}(\varphi) \tilde{/} \alpha = \begin{cases} r_{\mathcal{L}(\xi)}(\varphi) / \alpha, & if 0 \leq r_{\mathcal{L}(\xi)}(\varphi) / \alpha < N-1, \\ N-1, & otherwise \end{cases}$$

4 Relations On Neutrosophic N-Soft Sets

Definition 4.1 Let $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ be two NNSSs defined over the universe $(\mathbb{X}, \mathcal{L})$ and $(\mathbb{X}, \mathcal{M})$ respectively. *Neutrosophic N-soft relation* $\breve{\mathfrak{R}}$ is defined as $\breve{\mathfrak{R}}(\xi_i, \xi_j) = \lambda_{\mathcal{L}}(\xi_i) \sqcap_{\mathcal{R}} \lambda_{\mathcal{M}}(\xi_j)$, $\forall \xi_i \in \mathcal{L}$ and $\forall \xi_j \in \mathcal{M}$, where

$$\breve{\mathfrak{R}}: \mathcal{N} \to \mathrm{P}(\mathbb{X})$$

is an NNSS over $(\mathbb{X}, \mathcal{N})$, where $\mathcal{N} \sqsubseteq \mathcal{L} \times \mathcal{M}$.

Definition 4.2 The *composition* $\diamond$ of two neutrosophic N-soft relations $\breve{\mathfrak{R}}_1$ and $\breve{\mathfrak{R}}_2$ is defined by

$$(\breve{\mathfrak{R}}_1 \diamond \breve{\mathfrak{R}}_2)(l, n) = \breve{\mathfrak{R}}_1(l, m) \sqcap \breve{\mathfrak{R}}_2(m, n)$$

where $\breve{\mathfrak{R}}_1$ is neutrosophic N-soft relations from $\lambda_{\mathcal{L}}$ to $\lambda_{\mathcal{M}}$ over the universe $(\mathbb{X}, \mathcal{L})$ and $(\mathbb{X}, \mathcal{M})$ respectively and $\breve{\mathfrak{R}}_2$ is neutrosophic N-soft relations from $\lambda_{\mathcal{M}}$ to $\lambda_{\mathcal{N}}$ over the universe $(\mathbb{X}, \mathcal{M})$ and $(\mathbb{X}, \mathcal{N})$ respectively.

Definition 4.3 Let $\breve{\mathfrak{R}}_1$ is neutrosophic N-soft relation over the universe $(\mathbb{X}, \mathcal{L})$ and $\breve{\mathfrak{R}}_2$ is neutrosophic N-soft relation over the universe $(\mathbb{X}, \mathcal{M})$. The *union* and *intersection* of $\breve{\mathfrak{R}}_1$ and $\breve{\mathfrak{R}}_2$ defined as below

$$(\breve{\mathfrak{R}}_1 \sqcup \breve{\mathfrak{R}}_2)(l, m) = \max\{\breve{\mathfrak{R}}_1(l, m), \breve{\mathfrak{R}}_2(l, m)\}$$

$$(\breve{\mathfrak{R}}_1 \sqcap \breve{\mathfrak{R}}_2)(l, m) = \min\{\breve{\mathfrak{R}}_1(l, m), \breve{\mathfrak{R}}_2(l, m)\}$$

where $\breve{\mathfrak{R}}_1: \mathcal{L} \times \mathcal{M} \to \mathbb{P}(\mathbb{X})$ and $\breve{\mathfrak{R}}_2: \mathcal{L} \times \mathcal{M} \to \mathbb{P}(\mathbb{X})$.

Definition 4.4 Let $\lambda_{\mathcal{L}}$ in $(\mathbb{X}, \mathcal{L})$ be a neutrosophic N-soft set. Let $\breve{\mathfrak{R}}$ for $\lambda_{\mathcal{L}}$ to $\lambda_{\mathcal{M}}$. Then *max-min-max composition* of neutrosophic N-soft set with $\lambda_{\mathcal{L}}$ is another neutrosophic N-soft set $\lambda_{\mathcal{M}}$ of $(\mathbb{X}, \mathcal{M})$ which is denoted by $\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}$. The membership function, indeterminate function, non-membership function and grading function of $\lambda_{\mathcal{M}}$ are defined, respectively, as

$$\mathbb{T}_{\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}}(m) = \max_l\{\min(\mathbb{T}_{\mathcal{L}}(l), \mathbb{T}_{\mathcal{L}}(l, m))\},$$

$$\mathbb{I}_{\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}}(m) = \min_l\{\max(\mathbb{I}_{\mathcal{L}}(l), \mathbb{I}_{\mathcal{L}}(l, m))\},$$

$$\mathbb{F}_{\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}}(m) = \min_l\{\max(\mathbb{F}_{\mathcal{L}}(l), \mathbb{F}_{\mathcal{L}}(l, m))\},$$

$$r_{\breve{\mathfrak{R}} \diamond \lambda_{\mathcal{L}}}(m) = \max_l\{\min(r_{\mathcal{L}}(l), r_{\mathcal{L}}(l, m))\},$$

$$\forall l \in \mathcal{L}, m \in \mathcal{M}, r_{\mathcal{L}} \in \mathcal{G}.$$

Definition 4.5 Let $\lambda_{\mathcal{L}}$ be a neutrosophic N-soft set. Then the *choice function* of $\lambda_{\mathcal{L}}$ is defined as

$$C(\lambda_{\mathcal{L}}) = r_{\mathcal{L}} + \mathbb{T}_{\mathcal{L}} - \mathbb{I}_{\mathcal{L}} - \mathbb{F}_{\mathcal{L}}$$

Definition 4.6 Let $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ be two neutrosophic N-soft sets. Then the *score function* of $\lambda_{\mathcal{L}}$ and $\lambda_{\mathcal{M}}$ is defined as

$$\mathcal{S}_{LM} = C(\lambda_{\mathcal{L}}) - C(\lambda_{\mathcal{M}})$$

Definition 4.7 Let $\lambda_{\mathcal{L}}$ be a neutrosophic N-soft set. We define score function for $\lambda_{\mathcal{L}}$ as

$$\mathcal{S}_L = r_i + \mathbb{T}_i - \mathbb{I}_i \mathbb{F}_i$$

5 Application of Neutrosophic N-Soft Set to Medical Diagnosis

In this Section, we discuss the execution of N-soft set and neutrosophic set in medical diagnosis . In some previous studies of the neutrosophic set and neutrosophic soft set, there are many examples of medical diagnosis but all of them have lack of parameterized evaluation characterization. First we propose Algorithm 1 as given below.

Algorithm 1

Step 1: Input a set $\mathfrak{P}$ of patients, a set $\mathcal{S}$ of symptoms as parameter set and a set $\mathfrak{D}$ of diseases .

Step 2: Construct a relation $\mathfrak{L}(\mathfrak{P} \hookrightarrow \mathcal{S})$ between the patients and symptoms.

Step 3: Construct a relation a relation $\mathfrak{M}(\mathcal{S} \hookrightarrow \mathfrak{D})$ between the symptoms and the diseases.

Step 4: Compute the composition relation $\mathfrak{N}(\mathfrak{P} \hookrightarrow \mathfrak{D})$ the relation of patients and diseases by using Definition 4.4.

Step 5: Obtain the choice function of $\mathfrak{N}$ by using Definition 4.5.

Step 6: Choose the highest choice value of patient corresponding to disease gives the higher possibility of the patient affected with the respective disease.

Flow chart portrayal of Algorithm 1 is given in Figure 1:

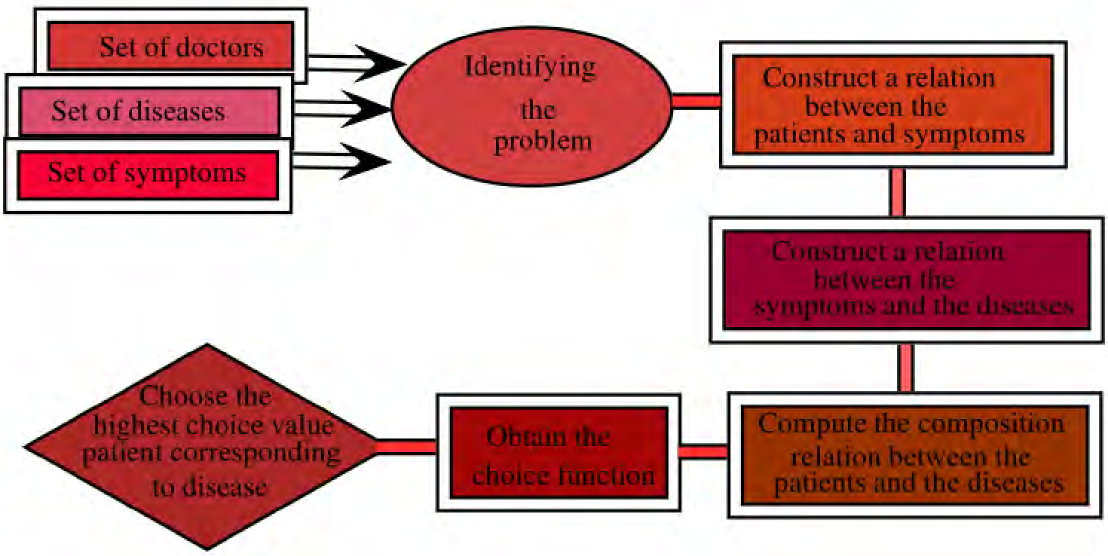

Figure 1: Flow chart representation of Algorithm 1

Now we demonstrate how neutrosophic N-soft set (NNSS) can be efficiently employed in multi-criteria group decision making (MCGDM). First of all, we propose an extension of TOPSIS to NNSS. In this study, we choose TOPSIS because our goal is to solve a medical diagnosis decision making problem. Since medical diagnosis involves similarities (in symptoms) and TOPSIS method is most appropriate method for handling such problems. A detailed study of TOPSIS may be found in [26]. The procedural steps of Neutrosophic N-soft set TOPSIS Method to examine critical situation of each patient is given in Algorithm 2.

Algorithm 2 (Neutrosophic N-soft set TOPSIS Method)

Step 1: Constructing weighed parameter matrix $\mathcal{H}$ by using ranking values obtained in Step 4 of Algorithm 1 composition relation $\mathfrak{N}(\mathfrak{P} \hookrightarrow \mathfrak{D})$ and relates it with linguistic ratings from Table 26.

$$\mathcal{H} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{12} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{i1} & r_{i2} & \cdots & r_{in} \\ \vdots & \vdots & & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{bmatrix} = [r_{ij}]_{m\times n}$$

Step 2: Creating normalized decision matrix $\mathcal{B}$. Throughout from now, we shall use

$L_n = \{1,2,3,\cdots,n\}\ \forall n \in N$

$$b_{ij} = \frac{r_{ij}}{\sqrt{\sum_{k=1}^{m} r_{kj}^2}} \tag{1}$$

$$\mathcal{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{12} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix} = [b_{ij}]_{m\times n}$$

Step 3: Creating weighted vector $\mathbf{W} = \{\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \cdots, \mathbf{W}_n\}$ by using the expression

$$\mathbf{W}_j = \frac{\mathrm{w}_j}{\sum_{k=1}^{m} \mathrm{w}_k}, \mathbf{w}_k = \frac{1}{m}\sum_{i=1}^{m} b_{ij} \tag{2}$$

Step 4: Constructing weighted decision matrix μ.

$$\mu = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1n} \\ \mu_{21} & \mu_{12} & \cdots & \mu_{2n} \\ \vdots & \vdots & & \vdots \\ \mu_{i1} & \mu_{i2} & \cdots & \mu_{in} \\ \vdots & \vdots & & \vdots \\ \mu_{m1} & \mu_{m2} & \cdots & \mu_{mn} \end{bmatrix} = [\mu_{ij}]_{m\times n}$$

where $\mu_{ij} = \mathbf{W}_j b_{ij}$ (3)

Step 5: Finding positive ideal solution (PIS) and negative ideal solution (NIS) by using the Equations

$$PIS = \{\mu_1^+, \mu_2^+, \mu_3^+, \cdots, \mu_j^+ \cdots, \mu_n^+\} = \{\max(\mu_{ij}) : i \in L_n\} \tag{4}$$

$$NIS = \{\mu_1^-, \mu_2^-, \mu_3^-, \cdots, \mu_j^- \cdots, \mu_n^-\} = \{\min(\mu_{ij}) : i \in L_n\} \tag{5}$$

Step 6: Calculate separation measurements of PIS ($\mathcal{S}_i^+$) and NIS ($\mathcal{S}_i^-$) for each parameter by using the equations

$$\mathcal{S}_i^+ = \sqrt{\sum_{j=1}^{n} (\mu_{ij} - \mu_j^+)^2}, \quad \forall i \in L_m \tag{6}$$

and

$$\mathcal{S}_i^- = \sqrt{\sum_{j=1}^{n} (\mu_{ij} - \mu_j^-)^2}, \quad \forall i \in L_m \tag{7}$$

Step 7: Calculating of relative closeness of alternative to the ideal solution by using the equation

$$\mathcal{C}_i^+ = \frac{\mathcal{S}_i^-}{\mathcal{S}_i^- + \mathcal{S}_i^+}, \quad 0 \le \mathcal{C}_i^+ \le 1, \ \forall i \in L_m \tag{8}$$

Step 8: Ranking the preference order.

Flow chart portrayal of neutrosophic N-soft set TOPSIS method is shown in Figure 2.

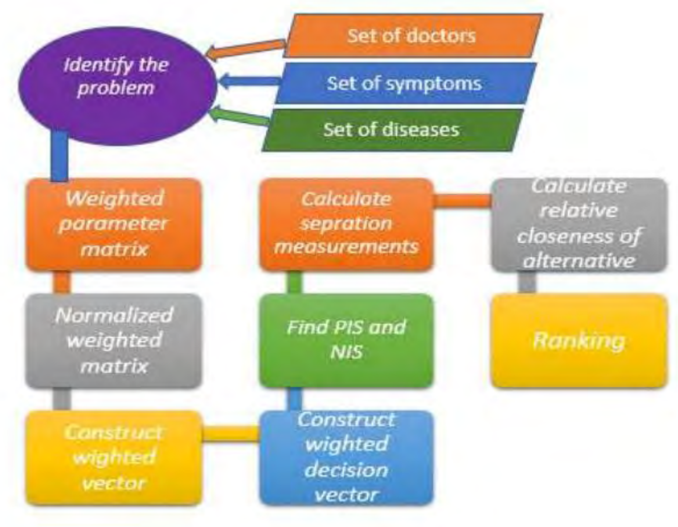

Figure 2: Flow chart of neutrosophic N-soft set TOPSIS method

5.1 Numerical Example

Now we employ the above **Algorithm 1** to find the decision factor about the following top four deadliest diseases in the world. Due to the following risk factors, these diseases progress slowly. Here is some detail about these diseases:

D_1: Coronary artery disease (CAD)

CAD occurs when the vessels that transfer blood towards heart become narrowed. CAD leads to heart failure, arrhythmias and chest pain. Risk factors for CAD are

High blood pressure	High cholesterol	Smoking
Family history of CAD	Diabetes	Obesity

Table 16: Risk factors for CAD

D_2: Stroke

This fatal disease occurs when some artery is in brain blocked or leaks. The risk factors for Stroke are:

High blood pressure	Being female	Smoking
Family history of stroke	Being American	Being African

Table 17: Risk factors for Stroke

D_3: Lower respiratory infections (LRI)

This disease occurs due to tuberculosis, pneumonia, influenza, flu, or bronchitis. Risk factors for LRI contain

Poor air quality	Asthma	Smoking
Weak immune system	HIV	Crowded child-care settings

Table 18: Risk factors for LRI

D_4: Chronic obstructive pulmonary disease (COPD)

This disease is a long-term, progressive lung disease that makes breathing difficult. Risk factors for COPD are

Family history	Lungs irritation
History of respiratory infections	Smoking

Table 19: Risk factors for COPD

D_5: Trachea, bronchus and lungs cancers

Respiratory cancers incorporate diseases of the bronchus, larynx, lungs and trachea. The risk factors for Trachea, bronchus and lungs cancers involve

Use of coal for cooking	Tobacco usage	Poor air quality
Family history of disease	Smoking	Diesel fumes

Table 20: Risk factors for Trachea, bronchus and lungs cancers

Core in certain sense is the most basic part occurring in the considered knowledge. Core can be translated as the arrangement of most trademark some portion of knowledge, which cannot be abstained from when decreasing the data. The core risk factor of all diseases discussed above is "smoking". For computational purpose, let's decide the grading values depending upon the degree of membership function as in Table 21:

Degree of membership function	Grading values
$\mathbb{T} = 0$	0
$0 < \mathbb{T} \leq 0.2$	1
$0.2 < \mathbb{T} \leq 0.4$	2
$0.4 < \mathbb{T} \leq 0.6$	3
$0.6 < \mathbb{T} \leq 0.8$	4
$0.8 < \mathbb{T} \leq 1.0$	5

Table 21: Ranking scale

Table 22 yields relation between symptoms and patients:

$\mathfrak{L}$	Headache$(\mathfrak{s}_1)$	Shortness of breath$(\mathfrak{s}_2)$	Angina$(\mathfrak{s}_3)$
$\mathfrak{p}_1$	$(\langle 0.7,0.2,0.5\rangle,4)$	$(\langle 0.6,0.3,0.4\rangle,3)$	$(\langle 0.4,0.6,0.5\rangle,2)$
$\mathfrak{p}_2$	$(\langle 0.9,0.3,0.1\rangle,5)$	$(\langle 0.7,0.4,0.3\rangle,4)$	$(\langle 0.8,0.5,0.2\rangle,4)$
$\mathfrak{p}_3$	$(\langle 0.6,0.6,0.4\rangle,3)$	$(\langle 0.5,0.5,0.8\rangle,3)$	$(\langle 0.2,0.4,0.8\rangle,1)$
$\mathfrak{p}_4$	$(\langle 0.2,0.5,0.8\rangle,1)$	$(\langle 0.3,0.1,0.7\rangle,2)$	$(\langle 0.7,0.1,0.3\rangle,4)$

Table 22: Relation between symptoms and patients

The relation between the symptoms and the diseases is given in Table 23:

$\mathfrak{M}$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{D}_4$
Headache$(\mathfrak{s}_1)$	$(\langle 0.8,0.4,0.2\rangle,4)$	$(\langle 0.9,0.2,0.1\rangle,5)$	$(\langle 0.6,0.3,0.4\rangle,3)$	$(\langle 0.7,0.5,0.3\rangle,4)$
Shortness of breath$(\mathfrak{s}_2)$	$(\langle 0.1,0.8,0.9\rangle,1)$	$(\langle 0.2,0.9,0.8\rangle,1)$	$(\langle 0.5,0.7,0.5\rangle,3)$	$(\langle 0.3,0.7,0.6\rangle,2)$
Angina$(\mathfrak{s}_3)$	$(\langle 0.5,0.7,0.5\rangle,3)$	$(\langle 0.4,0.6,0.6\rangle,2)$	$(\langle 0.3,0.5,0.7\rangle,2)$	$(\langle 0.9,0.1,0.1\rangle,5)$

Table 23: Relation between the symptoms and the diseases

The composition relation of patients and diseases in Table 24:

$\mathfrak{N}$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{D}_4$
$\mathfrak{p}_1$	$(\langle 0.7,0.4,0.5\rangle,4)$	$(\langle 0.7,0.2,0.5\rangle,4)$	$(\langle 0.6,0.3,0.5\rangle,3)$	$(\langle 0.7,0.5,0.5\rangle,4)$
$\mathfrak{p}_2$	$(\langle 0.8,0.4,0.2\rangle,4)$	$(\langle 0.9,0.3,0.1\rangle,5)$	$(\langle 0.6,0.3,0.4\rangle,3)$	$(\langle 0.7,0.5,0.2\rangle,4)$
$\mathfrak{p}_3$	$(\langle 0.6,0.6,0.4\rangle,3)$	$(\langle 0.6,0.6,0.4\rangle,3)$	$(\langle 0.6,0.6,0.4\rangle,3)$	$(\langle 0.60.4,0.4\rangle,3)$
$\mathfrak{p}_4$	$(\langle 0.5,0.5,0.5\rangle,3)$	$(\langle 0.4,0.5,0.6\rangle,2)$	$(\langle 0.3,0.5,0.7\rangle,2)$	$(\langle 0.7,0.5,0.3\rangle,4)$

Table 24: Composition relation of patients and diseases

Table 25 gives choice values of the relation $\mathfrak{N}$:

$\mathfrak{N}$	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{D}_4$
$\mathfrak{p}_1$	3.8	4	2.8	3.7
$\mathfrak{p}_2$	4.2	5.5	2.9	4
$\mathfrak{p}_3$	2.6	2.6	2.6	2.8
$\mathfrak{p}_4$	2.5	1.3	1.1	3.9

Table 25: Choice values of relation $\mathfrak{N}$

From Table 25, we conclude that the patients $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are likely to be suffering from $\mathfrak{D}_2$ whereas $\mathfrak{p}_3$ and $\mathfrak{p}_4$ are suffering from $\mathfrak{D}_4$.

In order to examine the intensity level of the disease of the patients, we use **neutrosophic N-soft TOPSIS** method which is demonstrated in **Algorithm 2**. First, we decide the grading values as a function of linguistic terms as Table 26:

Linguistic Terms	Grading Values
Undetermined (U)	0
Very Stable (VS)	1
Stable (S)	2
Grave (G)	3
Critical (C)	4
Very Critical (VC)	5

Table 26: Linguistic terms for evaluation of parameters

Now we construct weighted parameter matrix by using Step 9 and Table 26 as

$$\mathcal{H} = \begin{bmatrix} 4 & 4 & 3 \\ 4 & 5 & 3 \\ 3 & 3 & 3 \\ 3 & 2 & 2 \end{bmatrix} = \begin{bmatrix} C & C & G \\ C & VC & G \\ G & G & G \\ G & S & S \end{bmatrix}$$

Creating normalized decision matrix $\mathcal{B}$ by using Equation 1

$$\mathcal{B} = \begin{bmatrix} 0.57 & 0.54 & 0.54 & 0.53 \\ 0.57 & 0.68 & 0.54 & 0.53 \\ 0.43 & 0.41 & 0.54 & 0.40 \\ 0.43 & 0.27 & 0.36 & 0.53 \end{bmatrix}$$

Now by using Equation 2 construct weight vector

$$\mathbf{W} = \{\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4\} = \{0.58, 0.14, 0.14, 0.14\}$$

By using Equation 3 the weighted decision matrix μ is

$$\mu = \begin{bmatrix} 0.33 & 0.07 & 0.07 & 0.07 \\ 0.33 & 0.09 & 0.07 & 0.07 \\ 0.25 & 0.06 & 0.07 & 0.06 \\ 0.25 & 0.04 & 0.05 & 0.07 \end{bmatrix}$$

The positive ideal solution (PIS) and negative ideal solution (NIS) by using the Equations 4 and 5 as

$$PIS = \{0.33, 0.09, 0.07, 0.07\}$$

$$NIS = \{0.25, 0.04, 0.05, 0.06\}$$

The separation measurements of PIS and NIS for each parameter by using the Equations 6 and 7 are

$$\mathcal{S}_1^+ = 0.11$$

$$\mathcal{S}_2^+ = 0.06$$

$$\mathcal{S}_3^+ = 0.02$$

$$\mathcal{S}_4^+ = 0.01$$

$$\mathcal{S}_1^- = 0.11$$

$$\mathcal{S}_2^- = 0.06$$

$$\mathcal{S}_3^- = 0.03$$

$$\mathcal{S}_4^- = 0.02$$

The relative closeness of alternatives to the ideal solution by using Equation 8 are

$$\mathcal{C}_1^+ = 0.5$$

$$\mathcal{C}_2^+ = 0.5$$

$$\mathcal{C}_3^+ = 0.6$$

$$\mathcal{C}_4^+ = 0.7$$

Ranking the preference order is

$$\mathcal{C}_4^+ \geq \mathcal{C}_3^+ \geq \mathcal{C}_2^+ \geq \mathcal{C}_1^+$$

which indicates that condition of patient $\mathfrak{p}_4$ is most critical. The pictorial representation of the rankings of the patients is demonstrated with the assistance of a chart as given in Figure 3.

Figure 3: Ranking of patients w.r.t. intensity level of disease

5. Conclusion

The purpose of this work is to lay the foundation of theory of neutrosophic N-soft set as a hybrid model of neutrosophic sets and N-soft sets. We established some basic operations on neutrosophic N-soft sets along with their fundamental properties. We introduced the notions of NNS-subset, null-NNS, absolute-NNS, complements of NNS, truth-favorite, falsity-favorite, relations on NNS, composition of NNSS and score function of NNS. We explained these concepts with the help of illustrations. We presented a novel application of multi-attribute decision-making (MADM) based on neutrosophic N-soft set by using Algorithm 1. We proposed neutrosophic N-soft sets TOPSIS method as demonstrated in Algorithm 2 for MADM in medical diagnosis. We defined separation measurements of positive ideal solution and negative ideal solution to compute a relative closeness to identify the optimal alternative. Lastly, a numerical example is given to illustrate the developed method for medical diagnosis.

This may be the starting point for neutrosophic N-soft set mathematical concepts and information structures that are based on neutrosophic set and N-soft set theoretic operations. We have studied a few concepts only, it will be necessary to carry out more theoretical research to recognize a general framework for the practical applications. The proposed model of neutrosophic N-soft set can be elaborated with new research topics such as image processing, expert systems, soft computing techniques, fusion rules, cognitive maps, graph theory and decision-making of real world problems.

We hope that this study will prove a ground-breaking and will open new doors for the vibrant researchers in this field.

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Conflicts of Interest

The authors declare that they have no conflict of interest.

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A Novel of neutrosophic τ-Structure Ring $ExtB$ and $ExtV$ Spaces

R. Narmada Devi

Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R & D Institute of Science and Technology, Chennai, India.

E-mail: narmadadevi23@gmail.com

Abstract: In this paper, the concepts of a neutrosophic τ-structure ring spaces, neutrosophic τ-structure ring $G_\delta T_{1/2}$ spaces and neutrosophic τ-structure ring exterior B spaces and neutrosophic τ-structure ring exterior V spaces are introduced. Some interesting functions that preserve neutrosophic τ-structure ring exterior B spaces and neutrosophic τ-structure ring exterior V spaces in the context of image and preimage are derived with the necessary examples.

Keywords: neutrosophic τ-structure ring space, neutrosophic τ-structure ring $G_\delta T_{1/2}$ space, neutrosophic τ-structure ring $ExtB$ space and neutrosophic τ-structure ring $ExtV$ space.

1 Introduction

The concept of fuzzy sets was introduced by Zadeh [16]. consequent to the introduction of fuzzy sets, fuzzy logic has been applied in many real life situations to handle uncertainty. Chang [7] introduced the concept of fuzzy topological spaces. There are several kinds of fuzzy set extensions such as intuitionistic fuzzy set, interval-valued fuzzy sets, etc. After the introduction of intuitionistic fuzzy sets and its topological spaces by Atanassov [6] and Coker [8], the concept of imprecise data called neutrosophic sets was introduced by Smarandache [9]. The concept of neutrosophic topological space was introduced by Salama [15]. Later R.Narmada Devi [10,11,12,13,14] introduced the concepts of intuitionistic fuzzy G_δ sets, intuitionistic fuzzy exterior spaces and neutrosophic complex topological spaces. Moreover, the neutrosophic theory plays a viral role in all fields of branches like medial diagnosis [1,2,5], multiple criteria group decision making [3,4], etc. In this paper, the concepts of neutrosophic τ-structure ring spaces, neutrosophic G_δ rings, neutrosophic first category rings, neutrosophic τ-structure ring $G_\delta T_{1/2}$ spaces and neutrosophic τ-structure ring exterior B spaces and neutrosophic τ-structure ring exterior V spaces are introduced. Further, neutrosophic τ-structure ring continuous (resp. open, hardly open) functions and somewhat neutrosophic τ-structure ring continuous functions are presented. Some interesting properties among of functions along with the spaces are discussed and necessary examples are provided.

2 Preliminiaries

We need the following basic definitions for our study.

Definition 2.1. [9] Let X be a nonempty set. A neutrosophic set A in X is defined as an object of the form $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$ such that $T_A, I_A, F_A : X \to [0,1]$. and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2.2. [9] Let $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$ and $B = \langle x, T_B(x), I_B(x), F_B(x)\rangle$ be any two neutrosophic sets in X. Then

(i) $A \cup B = \langle x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)\rangle$ where $T_{A\cup B}(x) = T_A(x) \vee T_B(x), I_{A\cup B}(x) = I_A(x) \vee I_B(x)$ and $F_{A\cup B}(x) = F_A(x) \wedge F_B(x)$.

(ii) $A \cap B = \langle x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)\rangle$ where $T_{A\cap B}(x) = T_A(x) \wedge T_B(x), I_{A\cap B}(x) = I_A(x) \wedge I_B(x)$ and $F_{A\cap B}(x) = F_A(x) \vee F_B(x)$.

(iii) $A \subseteq B$ if $T_A(x) \leq T_B(x), I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x),$ for all $x \in X$.

(iv) the complement of A is defined as $C(A) = \langle x, T_{C(A)}(x), I_{C(A)}(x), F_{C(A)}(x)\rangle$ where $T_{C(A)}(x) = 1 - T_A(x), I_{C(A)}(x) = 1 - I_A(x)$ and$F_{C(A)}(x) = 1 - F_A(x)$.

(v) $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$ and $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$

Definition 2.3. [10,11] Let (X, T) be an intuitionistic fuzzy topological space. Let $A = \langle x, \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy set on an intuitionistic fuzzy topological space (X, T). Then A is said be an intuitionistic fuzzy G_δ set if $A = \bigcap_{i=1}^{\infty} A_i$, where $A_i = \langle x, \mu_{A_i}, \gamma_{A_i}\rangle$ is an intuitionistic fuzzy open set in an intuitionistic fuzzy topological space (X, T). The complement of an intuitionistic fuzzy G_δ set is said to be an intuitionistic fuzzy F_σ set.

Definition 2.4. [12,13] Let $A = \langle \mu_A, \gamma_A\rangle$ be an intuitionistic fuzzy set on an intuitionistic fuzzy topological space (X, τ). An intuitionistic fuzzy exterior of A is defines as follows: if $IFExt(A) = IFint(\overline{A})$

Definition 2.5. [12,13] Let R be a ring. An intuitionistic fuzzy set $A = \langle x, \mu_A, \gamma_A\rangle$ in R is called an intuitionistic fuzzy ring on R if it satisfies the following conditions on the membership and nonmembership values:

(i) $\mu_A(x + y) \geq \mu_A(x) \wedge \mu_A(y)$,

(ii) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$,

(iii) $\gamma_A(x + y) \leq \gamma_A(x) \vee \gamma_A(y)$,

(iv) $\gamma_A(xy) \leq \mu_A(x) \vee \gamma_A(y)$,

for all $x, y \in R$.

3 Properties of neutrosophic τ-Structure Ring Exterior B Spaces

Definition 3.1. Let R be a ring. A neutrosophic set $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$ in R is called a neutrosophic ring on R if it satisfies the following conditions:

(i) $T_A(x + y) \geq T_A(x) \wedge T_A(y)$ and $T_A(xy) \geq T_A(x) \wedge T_A(y)$

(ii) $I_A(x + y) \geq I_A(x) \wedge I_A(y)$ and $I_A(xy) \geq I_A(x) \wedge I_A(y)$

(iii) $F_A(x+y) \leq F_A(x) \vee F_A(y)$ and $F_A(xy) \leq F_A(x) \vee F_A(y)$, for all $x, y \in R$.

Definition 3.2. Let R be a ring. A family $\mathscr{S}$ of a neutrosophic rings in R is said to be neutrosophic τ-structure ring on R if it satisfies the following conditions:

(i) $0_N, 1_N \in \mathscr{S}$.

(ii) $G_1 \cap G_2 \in \mathscr{S}$ for any $G_1, G_2 \in \mathscr{S}$.

(iii) $\cup G_i \in \mathscr{S}$ for arbitrary family $\{G_i \mid i \in I\} \subseteq \mathscr{S}$.

The ordered pair $(R, \mathscr{S})$ is called a neutrosophic τ-structure ring space. Every member of $\mathscr{S}$ is called a neutrosophic τ-open ring in $(R, \mathscr{S})$. The complement $C(A)$ of a neutrosophic τ-open ring A is a neutrosophic τ-closed ring in $(R, \mathscr{S})$.

Example 3.1. Let $R = \{0, 1\}$ be a set of integers module 2 with two binary operations '+' and '.' are specified by the following tables:

+	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings B and D on R as follows: $T_B(0) = 0.5, T_B(1) = 0.7, I_B(0) = 0.5, I_B(1) = 0.7, F_B(0) = 0.3, F_B(1) = 0.2, T_D(0) = 0.3, T_D(1) = 0.4, I_D(0) = 0.3, I_D(1) = 0.4, F_D(0) = 0.5, F_D(1) = 0.6$. Then $\mathscr{S} = \{0_N, B, D, 1_N\}$ is a neutrosophic τ-structure ring on R. Thus the pair $(R, \mathscr{S})$ is a neutrosophic τ- structure ring space.

Notation 3.1. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Then $NO(R)$ (resp. $NC(R)$) denotes the family of all neutrosophic τ-open(resp. closed) rings of $(R, \mathscr{S})$.

Definition 3.3. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A be a neutrosophic ring in R. Then the neutrosophic ring interior and neutrosophic ring closure A are defined and denoted as $NF_Rint(A) = \cup\{B \mid B \in NO(R)$ and $B \subseteq A\}$ and $NF_Rcl(A) = \cap\{B \mid B \in NC(R)$ and $A \subseteq B$ respectively.

Remark 3.1. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A be any neutrosophic ring in R. Then the following statements hold:

(i) $NF_Rcl(A) = A$ if and only if A is a neutrosophic τ-closed ring.

(ii) $NF_Rint(A) = A$ if and only if A is a neutrosophic τ-open ring.

(iii) $NF_Rint(A) \subseteq A \subseteq NF_Rcl(A)$.

(iv) $NF_Rint(1_N) = 1_N$ and $NF_Rint(0_N) = 0_N$.

(v) $NF_Rcl(1_N) = 1_N$ and $NF_Rcl(0_N) = 0_N$.

(vi) $NF_Rcl(C(A)) = C(NF_Rint(A))$ and $NF_Rint(C(A)) = C(NF_Rcl(A))$.

(vii) $\cup_{i=1}^{\infty} NF_Rcl(A_i) \subseteq NF_Rcl(\cup_{i=1}^{\infty} A_i)$.

(viii) $\cap_{i=1}^{n} NF_Rcl(A_i) = NF_Rcl(\cup_{i=1}^{n} A_i)$.

(ix) $\cap_{i=1}^{\infty} NF_R cl(A_i) \subseteq NF_R cl(\cup_{i=1}^{\infty} A_i)$.

(x) $\cup_{i=1}^{\infty} NF_R int(A_i) \subseteq NF_R int(\cup_{i=1}^{\infty} A_i)$.

Definition 3.4. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A be a neutrosophic ring in R. Then $NF_R int(C(A))$ is called a neutrosophic ring exterior of A and is denoted by $NF_R Ext(A)$.

Proposition 3.1. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A and B be any two neutrosophic rings in R. Then the following statements hold:

(i) $NF_R Ext(A) \subseteq C(A)$.

(ii) $NF_R Ext(A) = C(NF_R cl(A))$.

(iii) $NF_R Ext(NF_R Ext(A)) = NF_R int(NF_R cl(A))$.

(iv) If $A \subseteq B$ then $NF_R Ext(A) \supseteq NF_R Ext(B)$.

(v) $NF_R Ext(1_N) = 0_N$ and $NF_R Ext(0_N) = 1_N$.

(vi) $NF_R Ext(A \cup B) = NF_R Ext(A) \cap NF_R Ext(B)$.

Definition 3.5. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Let A be any neutrosophic ring in R. Then A is said be to a neutrosophic G_δ ring in $(R, \mathscr{S})$ if $A = \bigcap_{i=1}^{\infty} A_i$, where $A_i = \langle x, T_{A_i}, I_{A_i}, F_{A_i} \rangle$ is a neutrosophic τ-open ring in $(R, \mathscr{S})$. The complement of a neutrosophic G_δ ring is a neutrosophic F_σ ring in $(R, \mathscr{S})$.

Definition 3.6. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Let A be any neutrosophic ring in R. Then A is said be to a

(i) neutrosophic dense ring if there exists no neutrosophic τ-closed ring B in $(R, \mathscr{S})$ such that $A \subset B \subset 1_N$.

(ii) neutrosophic nowhere dense ring if there exists no neutrosophic τ-open ring B in $(R, \mathscr{S})$ such that $B \subset NF_R cl(A)$. That is, $NF_R int(NF_R cl(A)) = 0_N$.

Definition 3.7. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Let A be any neutrosophic fuzzy ring in R. Then A is said be to a neutrosophic first category ring in $(R, \mathscr{S})$ if $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. The complement of a neutrosophic first category ring is a neutrosophic residual ring in $(R, \mathscr{S})$.

Proposition 3.2. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. If A is a neutrosophic G_δ ring and the neutrosophic ring exterior of $C(A)$ is a neutrosophic dense ring in $(R, \mathscr{S})$, then $C(A)$ is a neutrosophic first category ring in $(R, \mathscr{S})$.

Proof:

A being a neutrosophic G_δ ring in $(R, \mathscr{S})$, $A = \cap_{i=1}^{\infty} A_i$ where A_i's are neutrosophic τ-open rings. Since the neutrosophic ring exterior of $C(A)$ is a neutrosophic dense ring in $(R, \mathscr{S})$, $NF_R cl(NF_R Ext(C(A))) = 1_N$. Because $NF_R Ext(C(A)) \subseteq A \subseteq NF_R cl(A)$, one has $NF_R Ext(C(A)) \subseteq NF_R cl(A)$.
This implies that $NF_R cl(NF_R Ext(C(A))) \subseteq NF_R cl(A)$, that is, $1_N \subseteq NF_R cl(A)$. Therefore, $NF_R cl(A) = 1_N$. That is, $NF_R cl(A) = NF_R cl(\cap_{i=1}^{\infty} A_i) = 1_N$. However, $IF_R cl(\cap_{i=1}^{\infty} A_i) \subseteq \cap_{i=1}^{\infty} NF_R cl(A_i)$. Hence,

$1_N \subseteq \cap_{i=1}^{\infty} NF_Rcl(A_i)$. That is, $\cap_{i=1}^{\infty} NF_Rcl(A_i) = 1_N$. This implies that $NF_Rcl(A_i) = 1_N$, for each $A_i \in \mathscr{S}$. Hence $NF_Rcl(NF_Rint(A_i)) = 1_N$. Now, $NF_Rint(NF_Rcl(C(A_i))) = NF_Rint(C(NF_Rint(A_i))) = C(NF_Rcl(NF_Rint(A_i))) = 0_N$. Therefore, $C(A_i)$ is a neutrosophic nowhere dense ring in $(R, \mathscr{S})$. Now, $C(A) = C(\cap_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} C(A_i)$. Hence, $C(A) = \cup_{i=1}^{\infty} C(A_i)$ where $C(A_i)$'s are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Consequently, $C(A)$ is a neutrosophic first category ring in $(R, \mathscr{S})$.

Proposition 3.3. If A is a neutrosophic first category ring in a neutrosophic τ-structure ring space $(R, \mathscr{S})$ such that $B \subseteq C(A)$ where B is non-zero neutrosophic G_δ ring and the neutrosophic ring exterior of $C(B)$ is a neutrosophic dense ring in $(R, \mathscr{S})$, then A is a neutrosophic nowhere dense ring in $(R, \mathscr{S})$.

Proof:

Let A be a neutrosophic first category ring in $(R, \mathscr{S})$. Then $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Now $C(NF_Rcl(A_i))$ is a neutrosophic τ-open ring in $(R, \mathscr{S})$. Let $B = \cap_{i=1}^{\infty} C(NF_Rcl(A_i))$. Then B is non-zero neutrosophic G_δ ring in $(R, \mathscr{S})$. Now, $B = \cap_{i=1}^{\infty} C(NF_Rcl(A_i)) = C(\cup_{i=1}^{\infty} NF_Rcl(A_i)) \subseteq C(\cup_{i=1}^{\infty} A_i) = C(A)$. Hence $B \subseteq C(A)$. Then $A \subseteq C(B)$. Now,

$$\begin{aligned} NF_Rint(NF_Rcl((A)) &\subseteq NF_Rint(NF_Rcl((C(B))) \\ &= NF_Rint(C(NF_Rint(B))) \\ &= C(NF_Rcl(NF_Rint(B))) \\ &= C(NF_Rcl(NF_RExt(C(B))) \end{aligned}$$

Since $NF_RExt(C(B))$ is a neutrosophic dense ring in $(R, \mathscr{S})$, $NF_Rcl(Ext(C(B)))$ $= 1_N$. Therefore, $NF_Rint(NF_Rcl(A)) \subseteq 0_N$. Then, $NF_Rint(NF_Rcl(A)) = 0_N$. Hence A is a neutrosophic nowhere dense ring in $(R, \mathscr{S})$.

Definition 3.8. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Let A be any neutrosophic ring in R. Then A is said to be a neutrosophic τ-regular closed ring in $(R, \mathscr{S})$ if $NF_Rcl(NF_Rint(A)) = A$. The complement of a neutrosophic τ-regular closed ring in $(R.\mathscr{S})$ is a neutrosophic τ-regular open ring in $(R.\mathscr{S})$.

Remark 3.2. Every neutrosophic τ-regular closed ring is a neutrosophic τ-closed ring.

Definition 3.9. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Then $(R, \mathscr{S})$ is called a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space if every non-zero neutrosophic G_δ ring in $(R, \mathscr{S})$ is a neutrosophic τ-open ring in $(R, \mathscr{S})$.

Proposition 3.4. If the neutrosophic τ-structure ring space $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space and if A is a neutrosophic first category ring in $(R, \mathscr{S})$, then A is not a neutrosophic dense ring in $(R, \mathscr{S})$.

Proof:

Assume the contrary. Suppose that A is a neutrosophic first category ring in $(R, \mathscr{S})$ such that A is a neutrosophic dense ring in $(R, \mathscr{S})$, that is, $NF_Rcl(A) = 1_N$. Then, $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Now, $C(NF_Rcl(A_i))$ is a neutrosophic τ-open ring in $(R, \mathscr{S})$. Let $B = \cap_{i=1}^{\infty} C(NF_Rcl(A_i))$. Then, B is non-zero neutrosophic G_δ ring in $(R, \mathscr{S})$. Now, $B = \cap_{i=1}^{\infty} C(NF_Rcl(A_i)) = C(\cup_{i=1}^{\infty} NF_Rcl(A_i)) \subseteq C(\cup_{i=1}^{\infty} A_i) = C(A)$. Hence $B \subseteq C(A)$. Then, $NF_Rint(B) \subseteq NF_Rint(C(A)) \subseteq C(NF_Rcl(A)) = 0_N$. That is, $NF_Rint(B) = 0_N$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space, $B = NF_Rint(B)$, which implies that $B = 0_N$. This is a contradiction. Hence A is not a neutrosophic dense ring in $(R, \mathscr{S})$.

Proposition 3.5. If $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space, then $NF_R Ext(\cup_{i=1}^{\infty} C(A_i)) = \cap_{i=1}^{\infty} A_i$.

Proof:

Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring $G_\delta T_{1/2}$ space. Assume that A_i's are neutrosophic τ-regular closed rings in $(R, \mathscr{S})$. Then, the A_i's are neutrosophic τ-closed rings in $(R, \mathscr{S})$, which implies that $C(A_i)$'s are neutrosophic τ-open rings in $(R, \mathscr{S})$. Let $B = \cap_{i=1}^{\infty} A_i$. Then B is a non-zero neutrosophic G_δ ring in $(R, \mathscr{S})$. Since $(R, \mathscr{S})$ is a neutrosophic τ-ring $G_\delta T_{1/2}$ space, $B = NF_R int(B)$ is a neutrosophic τ-open ring, which implies that $NF_R int(\cap_{i=1}^{\infty} A_i) = \cap_{i=1}^{\infty} A_i$. Now, $NF_R Ext(\cup_{i=1}^{\infty} C(A_i)) = NF_R int(C(\cup_{i=1}^{\infty} C(A_i))) = NF_R int(\cap_{i=1}^{\infty} A_i) = \cap_{i=1}^{\infty} A_i$. Hence the proof.

Definition 3.10. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Then $(R, \mathscr{S})$ is called a neutrosophic τ-structure ring exterior B (in short, $ExtB$) space if $NF_R Ext(\cap_{i=1}^{\infty} C(A_i)) = 0_N$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$.

Example 3.2. Let $R = \{0, 1\}$ be a set of integers of module 2 with two binary operations provided by the following tables:

+	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings A, B, M, D, E, F and G on R as follows: $T_A(0) = 0.5, T_A(1) = 0.7, I_A(0) = 0.5, I_A(1) = 0.7, F_A(0) = 0.3, F_A(1) = 0.3, T_B(0) = 0.5, T_B(1) = 0.7, I_B(0) = 0.5, I_B(1) = 0.7, F_B(0) = 0.3, F_B(1) = 0.2, T_M(0) = 0.3, T_M(1) = 0.4, I_M(0) = 0.3, I_M(1) = 0.4, F_M(0) = 0.5, F_M(1) = 0.6, T_D(0) = 0.4, T_D(1) = 0.5, I_D(0) = 0.4, I_D(1) = 0.5, F_D(0) = 0.3, F_D(1) = 0.5, T_E(0) = 0.3, T_E(1) = 0.2, I_E(0) = 0.3, I_E(1) = 0.2, F_E(0) = 0.5, F_E(1) = 0.7, T_F(0) = 0.3, T_F(1) = 0.2, I_F(0) = 0.3, I_F(1) = 0.2, F_F(0) = 0.5, F_F(1) = 0.8, T_G(0) = 0.3, T_G(1) = 0.2, I_G(0) = 0.3, I_G(1) = 0.2, F_G(0) = 0.6, F_G(1) = 0.7, T_H(0) = 0.3, T_H(1) = 0.2, I_H(0) = 0.3, I_H(1) = 0.2, F_H(0) = 0.6, F_H(1) = 0.8$. Then $\mathscr{S} = \{0_N, A, B, M, D, 1_N\}$ is a neutrosophic τ-structure ring on R. Thus the pair $(R, \mathscr{S})$ is a neutrosophic τ-structure ring space. Let $\{E, F, G, H\}$ be neutrosophic nowhere dense rings in $(R, \mathscr{S})$.

Then $NF_R Ext(\cap\{C(E), C(F), C(G), C(H)\}) = NF_R Ext(C(E)) = NF_R int(E) = 0_N$. Therefore, $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Proposition 3.6. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. Then the following statements are equivalent:

(i) $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

(ii) $NF_R int(A) = 0_N$, for every neutrosophic first category ring A in $(R, \mathscr{S})$.

(iii) $NF_R cl(A) = 1_N$, for every neutrosophic residual ring A in $(R, \mathscr{S})$.

Proof:

(i)⇒(ii)

Let A be any neutrosophic first category ring in $(R, \mathscr{S})$. Then $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Now, $NF_R int(A) = NF_R int(\cup_{i=1}^{\infty} A_i) = NF_R int(C(\cap_{i=1}^{\infty} C(A_i))) = NF_R Ext(\cap_{i=1}^{\infty} C(A_i))$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space, $NF_R Ext(\cap_{i=1}^{\infty} C(A_i)) = 0_N$. Therefore, $NF_R int(A) = 0_N$. Hence (i) ⇒ (ii).

(ii)⇒(iii)

Let A be any neutrosophic residual ring in $(R, \mathscr{S})$. Then $C(A)$ is a neutrosophic first category ring in $(R, \mathscr{S})$. By (ii), $NF_Rint(C(A)) = 0_N$. That is, $NF_Rint(C(A)) = 0_N = C(NF_Rcl(A))$. Therefore, $NF_Rcl(A) = 1_N$. Hence (ii) $\Rightarrow$ (iii).

(iii)⇒(i)

Let A be any neutrosophic first category ring in $(R, \mathscr{S})$. Then $A = \cup_{i=1}^{\infty} A_i$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Since A is a neutrosophic first category ring, $C(A)$ is a neutrosophic residual ring in $(R, \mathscr{S})$. Then by (iii), $NF_Rcl(C(A)) = 1_N$. Now, $NF_RExt(\cap_{i=1}^{\infty} C(A_i)) = NF_Rint(C(\cap_{i=1}^{\infty} C(A_i))) = NF_Rint(\cup_{i=1}^{\infty} A_i) = NF_Rint(A) = C(NF_Rcl(C(A))) = 0_N$. Hence, $NF_RExt(\cap_{i=1}^{\infty} C(A_i)) = 0_N$ where A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Therefore, $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Proposition 3.7. If A is a neutrosophic first category ring in a neutrosophic τ-structure ring space $(R, \mathscr{S})$ such that $B \subseteq C(A)$ where B is non-zero neutrosophic G_δ ring and the neutrosophic ring exterior of $C(B)$ is a neutrosophic dense ring in $(R, \mathscr{S})$, then $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Proof:

Let A be any neutrosophic first category ring in $(R, \mathscr{S})$ such that $B \subseteq C(A)$ where B is non-zero neutrosophic G_δ ring and the neutrosophic ring exterior of $C(B)$ is aneutrosophic dense ring in $(R, \mathscr{S})$. Then by Proposition 3.3., A is a neutrosophic nowhere dense ring $(R, \mathscr{S})$, that is, $NF_Rint(NF_Rcl(A)) = 0_N$. Then, $NF_Rint(A) \subseteq NF_Rint(NF_Rcl(A))$ implies that $NF_Rint(A) = 0_N$. By Proposition 3.6., $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Proposition 3.8. If $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space and if $\cup_{i=1}^{\infty} A_i = 1_N$ where A_i's are neutrosophic τ-regular closed rings in $(R, \mathscr{S})$, then $NF_Rcl(\cup_{i=1}^{\infty} NF_RExt(C(A_i))) = 1_N$.

Proof:

Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring $ExtB$ space. Assume that A_i's are neutrosophic τ-regular closed rings in $(R, \mathscr{S})$. Suppose that $NF_Rint(A_i) = 0_N$, for each $i \in J$. Since A_i is a neutrosophic τ- regular closed ring in $(R, \mathscr{S})$, A_i is a neutrosophic τ-closed ring in $(R, \mathscr{S})$. Also, $NF_Rint(A_i) = 0_N$ implies that $NF_Rint(NF_Rcl(A_i)) = 0_N$. Therefore, A_i's are neutrosophic nowhere dense rings in $(R, \mathscr{S})$. Since $\cup_{i=1}^{\infty} A_i = 1_N$, $NF_RExt(\cap_{i=1}^{\infty} C(A_i)) = NF_RExt(C(\cup_{i=1}^{\infty} A_i)) = NF_Rint(\cup_{i=1}^{\infty} A_i) = NF_Rint(1_N) = 1_N$. Hence, $NF_RExt(\cap_{i=1}^{\infty} C(A_i)) = 1_N$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space, $NF_RExt(\cap_{i=1}^{\infty} C(A_i)) = 0_N$, which is a contradiction. Hence $NF_Rint(A_i) \neq 0_N$, for atleast one $i \in J$. Therefore, $\cup_{i=1}^{\infty} NF_Rint(A_i) \neq 0_N$. Since A_i is a neutrosophic τ-regular closed rings in $(R, \mathscr{S})$ and $\cup_{i=1}^{\infty} NF_Rcl(A_i) \subseteq NF_Rcl(\cup_{i=1}^{\infty} A_i)$,

$$\begin{aligned}
&\Rightarrow \cup_{i=1}^{\infty} NF_Rcl(NF_Rint(A_i)) \subseteq NF_Rcl(\cup_{i=1}^{\infty} NF_Rint(A_i)) \\
&\Rightarrow \cup_{i=1}^{\infty} A_i \subseteq NF_Rcl(\cup_{i=1}^{\infty} NF_Rint(A_i)) \\
&\Rightarrow \cup_{i=1}^{\infty} A_i \subseteq NF_Rcl(\cup_{i=1}^{\infty} NF_RExt(C(A_i))) \\
&\Rightarrow 1_N \subseteq NF_Rcl(\cup_{i=1}^{\infty} NF_RExt(C(A_i))).
\end{aligned}$$

But $1_N \supseteq NF_Rcl(\cup_{i=1}^{\infty} NF_RExt(C(A_i)))$. Hence, $NF_Rcl(\cup_{i=1}^{\infty} NF_RExt(C(A_i))) = 1_N$.

4 On neutrosophic τ-Structure Ring Exterior V Spaces

Definition 4.1. Let $(R, \mathscr{S})$ be any neutrosophic τ-structure ring space. Then $(R, \mathscr{S})$ is called a neutrosophic τ-structure ring exterior V (in short, $ExtV$)space if $NF_Rcl(\cap_{i=1}^{n} A_i) = 1_N$ where A_i's are neutrosophic G_δ

rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$.

Example 4.1. Let $R = \{0, 1, 2\}$ be a set of integers of module 3 together with two binary operations as follows:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings A, B and D on R as follows: $T_A(0) = 1, T_A(1) = 0.2, T_A(2) = 0.9, I_A(0) = 1, I_A(1) = 0.2, I_A(2) = 0.9, F_A(0) = 0, F_A(1) = 0.8, F_A(2) = 0.1, T_B(0) = 0.3, T_B(1) = 1, T_B(2) = 0.2, I_B(0) = 0.3, I_B(1) = 1, I_B(2) = 0.2, F_B(0) = 0.7, F_B(1) = 0, F_B(2) = 0.8, T_D(0) = 0.7, T_D(1) = 0.4, T_D(2) = 1, I_D(0) = 0.7, I_D(1) = 0.4, I_D(2) = 1, F_D(0) = 0.3, F_D(1) = 0.6, F_D(2) = 0$.
Then $\mathscr{S} = \{0_N, A, B, D, A \cap B, A \cup B, A \cap D, A \cup D, B \cap D, B \cup D, D \cap (A \cup B), A \cup (B \cap D), B \cup (A \cap D), 1_N\}$ is a neutrosophic τ-structure ring on R. Thus the pair $(R, \mathscr{S})$ is a neutrosophic τ-structure ring space.

Now, $A \cap D = \cap\{B \cup (A \cap D), D \cap (A \cup B), D, A\}$ and $D \cap (A \cup B) = \cap\{A \cup B, D \cap (A \cup B), A \cup D\}$ are neutrosophic G_δ rings in $(R, \mathscr{S})$. Also, the neutrosophic ring exterior of $C(A \cap D)$ and $C(D \cap (A \cup B))$ are neutrosophic dense rings in $(R, \mathscr{S})$. Now, $NF_Rcl(\cap\{A \cap D, D \cap (A \cup B)\}) = NF_Rcl(A \cap D) = 1_N$.Therefore, $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space.

Proposition 4.1. Let $(R, \mathscr{S})$ be a neutrosophic structure ring space. Then $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space iff $NF_Rint(\cup_{i=1}^n C(A_i)) = 0_N$ where A_i's are neutrosophic G_δ rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$.
Proof:

Let $(R, \mathscr{S})$ be a neutrosophic ring $ExtV$ space. Assume that A_i's are neutrosophic G_δ rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space, $NF_Rcl(\cap_{i=1}^n A_i) = 1_N$. Now, $NF_Rint(\cup_{i=1}^n C(A_i)) = NF_Rint(C(\cap_{i=1}^n A_i)) = C(NF_Rcl(\cap_{i=1}^n A_i)) = 0_N$. Therefore, $NF_Rint(\cup_{i=1}^n C(A_i)) = 0_N$ where A_i's are neutrosophic G_δ rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$.

Conversely, let $NF_Rint(\cup_{i=1}^n C(A_i)) = 0_N$ where A_i's are neutrosophic G_δ rings and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$. Now, $NF_Rcl(\cap_{i=1}^n A_i) = NF_Rcl(C(\cup_{i=1}^n A_i)) = C(NF_Rint(\cup_{i=1}^n A_i)) = 1_N$. Therefore, $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space.

Proposition 4.2. Let $(R, \mathscr{S})$ be a neutrosophic τ-structure ring space. If every neutrosophic first category ring in $(R, \mathscr{S})$ is formed from the neutrosophic G_δ rings and the neutrosophic ring exterior of its complements are neutrosophic dense rings in a neutrosophic τ-structure ring $ExtV$ space $(R, \mathscr{S})$, then $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.
Proof:

Assume that A_i's are neutrosophic G_δ rings in $(R, \mathscr{S})$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$, for $i = 1, ..., n$. Since $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtV$ space and by Proposition 4.1., $NF_Rint(\cup_{i=1}^n C(A_i)) = 0_N$. But $\cup_{i=1}^n NF_Rint(C(A_i)) \subseteq NF_Rint(\cup_{i=1}^n C(A_i))$, which implies that $\cup_{i=1}^n NF_Rint(C(A_i)) = 0_N$. Then $NF_Rint(C(A_i)) = 0_\sim$. Since A_i's are neutrosophic G_δ rings in $(R, \mathscr{S})$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R, \mathscr{S})$, for $i = 1, ..., n$. By Proposition 3.2., $C(A_i)$'s are neutrosophic first category rings in $(R, \mathscr{S})$, for $i = 1, ..., n$. Therefore, $NF_Rint(C(A_i)) = 0_N$, for every $C(A_i)$ is a neutrosophic first category rings in $(R, \mathscr{S})$. By Proposition 3.6., $(R, \mathscr{S})$ is a neutrosophic τ-structure ring $ExtB$ space.

Definition 4.2. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. Let $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ be any function. Then f is said to be a

(i) neutrosophic τ-structure ring continuous function if $f^{-1}(A)$ is a neutrosophic τ-open ring in $(R_1, \mathscr{S}_1)$, for every neutrosophic τ-open ring A in $(R_2, \mathscr{S}_2)$.

(ii) somewhat neutrosophic τ-structure ring continuous function if $A \in \mathscr{S}_2$ and $f^{-1}(A) \neq 0_\sim$ implies that there exists a neutrosophic τ-open ring B in $(R_1, \mathscr{S}_1)$ such that $B \neq 0_N$ and $B \subseteq f^{-1}(A)$.

(iii) neutrosophic τ-structure ring hardly open function if for each neutrosophic dense ring A in $(R_2, \mathscr{S}_2)$ such that $A \subseteq B \subset 1_N$ for some neutrosophic τ-open ring B in $(R_2, \mathscr{S}_2)$, $f^{-1}(A)$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$.

(iv) neutrosophic τ-structure ring open function if $f(A)$ is a neutrosophic τ-open ring in $(R_2, \mathscr{S}_2)$, for every neutrosophic τ-open ring A in $(R_1, \mathscr{S}_1)$.

Proposition 4.3. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. Let $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ be any function. Then the following statements are equivalent:

(i) f is a neutrosophic τ-structure ring continuous function.

(ii) $f^{-1}(B)$ is a neutrosophic τ-closed ring in $(R_1, \mathscr{S}_1)$, for every neutrosophic τ-closed ring B in $(R_2, \mathscr{S}_2)$.

(iii) $NF_Rcl(f^{-1}(A)) \subseteq f^{-1}(NF_Rcl(A))$, for each neutrosophic ring A in $(R_2, \mathscr{S}_2)$.

(iv) $f^{-1}(NF_Rint(A)) \subseteq NF_Rint(f^{-1}(A))$, for each neutrosophic ring A in $(R_2, \mathscr{S}_2)$.

Remark 4.1. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. If $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring continuous function, then $f^{-1}(NF_RExt(C(A)) \subseteq NF_RExt(C(f^{-1}(A)))$, for each neutrosophic ring A in $(R_2, \mathscr{S}_2)$.
Proof: The proof follows from the Definition 3.4 and Proposition 4.3..

Proposition 4.4. If a function $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ from a neutrosophic τ-structure ring space $(R_1, \mathscr{S}_1)$ into another neutrosophic τ-structure ring space $(R_2, \mathscr{S}_2)$ is neutrosophic τ-structure ring continuous, 1-1 and if A is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$, then $f(A)$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$.
Proof:

Suppose that $f(A)$ is not a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Then there exists a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$ such that $f(A) \subset D \subset 1_N$. Then, $f^{-1}(f(A)) \subset f^{-1}(D) \subset f^{-1}(1_N)$. Since f is 1-1, $f^{-1}(f(A)) = A$. Hence $A \subset f^{-1}(D) \subset 1_N$. Since f is a neutrosophic τ-structure ring continuous function and D is a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$, $f^{-1}(D)$ is a neutrosophic τ-closed ring in $(R_1, \mathscr{S}_1)$. Then $NF_Rcl(A) \neq 1_N$, which is a contradiction. Therefore $f(A)$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$.

Remark 4.2. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. Then

(i) the neutrosophic τ-structure ring continuous image of a neutrosophic τ-structure ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be a neutrosophic τ-structure ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

(ii) the neutrosophic τ-structure ring open image of a neutrosophic τ-structure ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be a neutrosophic τ-structure ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

Proof: It is clear from the following Examples.

Example 4.2. Let $R = \{0, 1, 2\}$ be a set of integers of module 3 together with two binary operations as follows:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

$\cdot$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings $A, B, V, D, E,$ and F on R as follows: $T_A(0) = 1, T_A(1) = 0.2, T_A(2) = 0.9, I_A(0) = 1, I_A(1) = 0.2, I_A(2) = 0.9, F_A(0) = 0, F_A(1) = 0.8, F_A(2) = 0.1, T_B(0) = 0.3, T_B(1) = 1, T_B(2) = 0.2, I_B(0) = 0.3, I_B(1) = 1, I_B(2) = 0.2, F_B(0) = 0.7, F_B(1) = 0, F_B(2) = 0.8, T_V(0) = 0.7, T_V(1) = 0.4, T_V(2) = 1, I_V(0) = 0.7, I_V(1) = 0.4, I_V(2) = 1 F_V(0) = 0.3, F_V(1) = 0.6, F_V(2) = 0, T_D(0) = 0.9, T_D(1) = 1, T_D(2) = 0.2, I_D(0) = 0.9, I_D(1) = 1, I_D(2) = 0.2, F_D(0) = 0.1, F_D(1) = 0, F_D(2) = 0.8, T_E(0) = 0.2, T_E(1) = 0.2, T_E(2) = 1, I_E(0) = 0.2, I_E(1) = 0.2, I_E(2) = 1, F_E(0) = 0.8, F_E(1) = 0.8, F_E(2) = 0, T_F(0) = 1, T_F(1) = 0.7, T_F(2) = 0.4, I_F(0) = 1, I_F(1) = 0.7, I_F(2) = 0.4, F_F(0) = 0, F_F(1) = 0.3, F_F(2) = 0.6.$

Then $\mathscr{S}_1 = \{0_N, A, B, V, A \cap B, A \cup B, A \cap V, A \cup V, B \cap V, B \cup V, V \cap (A \cup B), A \cup (B \cap V), B \cup (A \cap V), 1_N\}$ and $\mathscr{S}_2 = \{0_N, D, E, F, D \cap E, D \cup E, D \cap F, D \cup F, E \cap F, E \cup F, F \cap (D \cup E), D \cup (E \cap F), E \cup (D \cap F), 1_N\}$ are two neutrosophic τ-structure rings on R. Thus the pair $(R, \mathscr{S}_1)$ and $(R, \mathscr{S}_2)$ are neutrosophic τ-structure ring spaces. Now, $A \cap V = \cap\{B \cup (A \cap V), V \cap (A \cup B), V, A\}$ and $V \cap (A \cup B) = \cap\{A \cup B, V \cap (A \cup B), A \cup V\}$ are neutrosophic G_δ rings in $(R, \mathscr{S}_1)$. Also, the neutrosophic ring exterior of $C(A \cap V)$ and $C(V \cap (A \cup B))$ are neutrosophic dense rings in $(R, \mathscr{S}_1)$. Now, $NF_Rcl(\cap\{A \cap V, V \cap (A \cup B)\}) = NF_Rcl(A \cap V) = 1_N$. Therefore, $(R, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space. Define a function $f : (R, \mathscr{S}_1) \to (R, \mathscr{S}_2)$ by $f(0) = 1, f(1) = 2$ and $f(2) = 0$. Clearly, f is a neutrosophic τ-structure ring continuous function. Also, $f(A) = D, f(B) = E$ and $f(V) = F$. Now, $D = \cap\{D, D \cup E, D \cup (E \cap F)\}$, $D \cap F = \cap\{F, D \cup F, D \cap F, F \cap (D \cup E)\}$ and $E = \cap\{E, E \cup F, E \cup (D \cap F)\}$ are neutrosophic G_δ rings in $(R, \mathscr{S}_2)$. Also, the neutrosophic ring exterior of $C(D)$, $C(F)$ and $C(D \cap F)$ are neutrosophic G_δ rings in $(R, \mathscr{S}_2)$. But, $NF_Rcl(\cap\{D, E, D \cap F\}) = C(E \cap F) \neq 1_N$. Therefore, $(R, \mathscr{S}_2)$ is not a neutrosophic τ-structure ring $ExtV$ space. Therefore the neutrosophic τ-structure ring continuous image of a neutrosophic τ-structure ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be a neutrosophic τ-structure ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

Example 4.3. Let $R = \{0, 1, 2\}$ be a set of integers of module 3 together with two binary operations as follows:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

and

.	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Then $(R, +, \cdot)$ is a ring. Define neutrosophic rings A, B, V and D on R as follows: $T_A(0) = 1, T_A(1) = 0.2, T_A(2) = 0.9, I_A(0) = 1, I_A(1) = 0.2, I_A(2) = 0.9, F_A(0) = 0, F_A(1) = 0.8, F_A(2) = 0.1, T_B(0) = 0.3, T_B(1) = 1, T_B(2) = 0.2, I_B(0) = 0.3, I_B(1) = 1, I_B(2) = 0.2, F_B(0) = 0.7, F_B(1) = 0, F_B(2) = 0.8, T_V(0) = 0.7, T_V(1) = 0.4, T_V(2) = 1, I_V(0) = 0.7, I_V(1) = 0.4, I_V(2) = 1, F_V(0) = 0.3, F_V(1) = 0.6, F_V(2) = 0, T_D(0) = 0.5, T_D(1) = 0.6, T_D(2) = 0.4, I_D(0) = 0.5, I_D(1) = 0.6, I_D(2) = 0.4, F_D(0) = 0.5, F_D(1) = 0.4, F_D(2) = 0.6.$

Then $\mathscr{S}_1 = \{0_N, A, B, V, A \cap B, A \cup B, A \cap V, A \cup V, B \cap V, B \cup V, V \cap (A \cup B), A \cup (B \cap V), B \cup (A \cap V), 1_N\}$ and $\mathscr{S}_2 = \{0_N, A, B, V, D, A \cup B, A \cup V, A \cup D, B \cup V, B \cup D, V \cup D, A \cap B, A \cap V, A \cap D, B \cap V, B \cap$

$D, V \cap D, D \cup (A \cap V), V \cap (A \cup B), A \cup (B \cap V), B \cup (A \cap V), 1_N\}$ are two neutrosophic τ-structure rings on R. Thus the pair $(R, \mathscr{S}_1)$ and $(R, \mathscr{S}_2)$ are neutrosophic τ-structure ring spaces. Now, $A \cap V = \cap\{B \cup (A \cap V), V \cap (A \cup B), V, A\}$ and $V \cap (A \cup B) = \cap\{A \cup B, V \cap (A \cup B), A \cup V\}$ are neutrosophic G_δ rings in $(R, \mathscr{S}_1)$. Also, the neutrosophic ring exterior of $C(A \cap V)$ and $C(V \cap (A \cup B))$ are neutrosophic dense rings in $(R, \mathscr{S}_1)$. Now, $NF_Rcl(\cap\{A \cap V, V \cap (A \cup B)\}) = NF_Rcl(A \cap V) = 1_V$. Therefore, $(R, \mathscr{S}_1)$ is a neutrosophic ring $ExtV$ space. Define a function $f : (R, \mathscr{S}_1) \to (R, \mathscr{S}_2)$ by $f(0) = 0, f(1) = 1$ and $f(2) = 2$. Clearly, f is a neutrosophic τ-structure ring open function. Also, $f(A) = A, f(B) = B, f(V) = V$ and $f(D) = D$. Now, $A = \cap\{A, A \cup B, A \cup V, A \cup (B \cap V)\}$, $D \cup (A \cap V) = \cap\{V, V \cup D, A \cap V, D \cup (A \cap V), V \cap (A \cup B)\}$ and $B = \cap\{B, B \cup V, B \cup D, B \cup (A \cap V)\}$ are neutrosophic G_δ rings in $(R, \mathscr{S}_2)$. Also, the neutrosophic ring exterior of $C(A), C(B)$ and $C(D \cup (A \cap V))$ are neutrosophic G_δ rings in $(R, \mathscr{S}_2)$. But, $NF_Rcl(\cap\{A, B, D \cup (A \cap V)\}) = C(B \cap V) \neq 1_N$. Therefore, $(R, \mathscr{S}_2)$ is not a neutrosophic τ-ring $ExtV$ space. Therefore the neutrosophic τ-structure ring open image of a neutrosophic τ-structure ring $ExtV$ space $(R_1, \mathscr{S}_1)$ may fail to be a neutrosophic τ-structure ring $ExtV$ space $(R_2, \mathscr{S}_2)$.

Proposition 4.5. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. If $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ is onto function, then the following statements are equivalent:

(i) f is a neutrosophic τ-structure ring hardly open function.

(ii) $NF_Rint(f(A)) \neq 0_N$, for all neutrosophic ring A in $(R_1, \mathscr{S}_1)$ with $NF_Rint(A) \neq 0_N$ and there exists a neutrosophic τ-closed ring $B \neq 0_N$ in $(R_2, \mathscr{S}_2)$ such that $B \subseteq f(A)$.

(iii) $NF_Rint(f(A)) \neq 0_N$, for all neutrosophic ring A in $(R_1, \mathscr{S}_1)$ with $NF_Rint(A) \neq 0_N$ and there exists a neutrosophic τ-closed ring $B \neq 0_N$ in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(B) \subseteq A$.

Proof:
(i)⇒(ii)
Assume that (i) is true. Let A be any neutrosophic ring A in $(R_1, \mathscr{S}_1)$ with $NF_Rint(A) \neq 0_N$ and $B \neq 0_N$ be a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$ such that $B \subseteq f(A)$. Suppose that $NF_Rint(A) = 0_N$. This implies that $NF_Rcl(C(f(A))) = 1_N$. Thus, $C(f(A))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$ and $C(f(A)) \subseteq C(B)$. By assumption, $f^{-1}(C(f(A)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. That is, $NF_Rcl(f^{-1}(C(f(A)))) = 1_N$. Now, $NF_Rint(A) = NF_Rint(f^{-1}(f(A))) = C(NF_Rcl(C(f^{-1}(f(A))))) = C(NF_Rcl(f^{-1}(C(f(A))))) = 0_N$. This is a contradiction. Hence (i)⇒(ii).
(ii)⇒(iii)
Assume that (ii) is true. Since f is onto function and by assumption, $B \subseteq f(A)$. This implies that $f^{-1}(B) \subseteq f^{-1}(f(A))$, that is, $f^{-1}(B) \subseteq A$. Hence (ii)⇒(iii).
(iii)⇒(i)
Let $V \subseteq C(D)$ where C is a neutrosophic dense ring and D is non-zero neutrosophic τ-open ring in $(R_2, \mathscr{S}_2)$. Let $A = f^{-1}(C(V))$ and $B = C(D)$. Now, $f^{-1}(B) = f^{-1}(C(D)) \subseteq f^{-1}(C(V)) = A$.

Consider, $NF_Rint(f(A)) = NF_Rint(f(f^{-1}(C(V))) = NF_Rint(C(V)) = C(NF_Rint(V)) = 0_N$. Therefore, $NF_Rint(A) = 0_N$, which implies that $NF_Rint(f^{-1}(C(V))) = NF_Rint(C(f^{-1}(V))) = 0_N$. Therefore, $C(NF_Rcl(f^{-1}(V))) = 0_N$. Thus, $NF_Rcl(f^{-1}(V)) = 1_N$. Therefore, $f^{-1}(V)$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. This implies that f is a neutrosophic τ-structure ring hardly open function. Hence (iii)⇒(i). This completes the proof.

Proposition 4.6. If a function $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ from a neutrosophic τ-structure ring space $(R_1, \mathscr{S}_1)$ onto another neutrosophic τ-structure ring space $(R_2, \mathscr{S}_2)$ is neutrosophic τ-structure ring continuous, 1-1 and neutrosophic τ-structure ring hardly open function and if $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space, then $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space.

Proof:

Let $(R_1, \mathscr{S}_1)$ be a neutrosophic τ-structure ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_2, \mathscr{S}_2)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Then $NF_Rcl(NF_RExt(C(A_i))) = 1_N$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$. Hence

$$f^{-1}(A_i) = f^{-1}(\cap_{j=1}^{\infty} B_{ij}) = \cap_{j=1}^{\infty} f^{-1}(B_{ij}) \tag{4.1}$$

Since f is a neutrosophic τ-structure ring continuous function and B_{ij}'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$, $f^{-1}(B_{ij})$'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$. Hence $f^{-1}(A_i) = \cap_{j=1}^{\infty} f^{-1}(B_{ij})$ is an neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$. Since f is a neutrosophic τ-structure ring hardly open function and $NF_RExt(C(A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$, $f^{-1}(NF_RExt(C(A_i)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Now,

$$\begin{aligned} f^{-1}(NF_RExt(C(A_i))) &= f^{-1}(NF_Rint(A_i)) \\ &\subseteq NF_Rint(f^{-1}(A_i)) \\ &= NF_RExt(C(f^{-1}(A_i))). \end{aligned}$$

Therefore $1_N = NF_Rcl(f^{-1}(NF_RExt(C(A_i)))) \subseteq NF_Rcl(NF_RExt(C(f^{-1}(A_i))))$, which implies that $1_N = NF_Rcl(NF_RExt(C(f^{-1}(A_i))))$. Hence $NF_RExt(C(f^{-1}(A_i)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Since $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-strucuture ring $ExtV$ space, $NF_Rcl(\cap_{i=1}^{n} f^{-1}(A_i)) = 1_N$ where $f^{-1}(A_i)$'s are neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$ and the neutrosophic ring exterior of $C(f^{-1}(A_i))$'s are neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Thus, $NF_Rcl(\cap_{i=1}^{n} f^{-1}(A_i)) = 1_N = NF_Rcl(f^{-1}(\cap_{i=1}^{n} A_i))$. Therefore, $f^{-1}(\cap_{i=1}^{n} A_i)$ is a neutrosophic dense rings in $(R_1, \mathscr{S}_1)$. Since f is a neutrosophic τ-structure ring continuous, 1-1 and by Proposition 3.4., $f(f^{-1}(\cap_{i=1}^{n} A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Hence $NF_Rcl(f(f^{-1}(\cap_{i=1}^{n} A_i))) = 1_N$. Since f is 1-1, $f(f^{-1}(\cap_{i=1}^{n} A_i)) = \cap_{i=1}^{n} A_i$. Then, $NF_Rcl(\cap_{i=1}^{n} A_i) = 1_N$. Therefore, $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space.

Conversely, let $(R_2, \mathscr{S}_2)$ be a neutrosophic τ-structure ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_2, \mathscr{S}_2)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R_2, \mathscr{S}_2)$.

Then $NF_Rcl(NF_RExt(C(A_i))) = 1_N$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$. Hence

$$f^{-1}(A_i) = f^{-1}(\cap_{j=1}^{\infty} B_{ij}) = \cap_{j=1}^{\infty} f^{-1}(B_{ij}) \tag{4.2}$$

Since f is a neutrosophic τ-structure ring continuous function and B_{ij}'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$, $f^{-1}(B_{ij})$'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$. Hence $f^{-1}(A_i) = \cap_{j=1}^{\infty} f^{-1}(B_{ij})$ is a neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$. Since f is a neutrosophic τ-structure ring hardly open function and $NF_RExt(C(A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$, $f^{-1}(NF_RExt(C(A_i)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. By Remark 4.2., $f^{-1}(NF_RExt(C(A_i))) \subseteq NF_RExt(C(f^{-1}(A_i)))$.
Thus, $NF_Rcl(f^{-1}(NF_RExt(C(A_i)))) = 1_N \subseteq NF_Rcl(NF_RExt(C(f^{-1}(A_i))))$. Hence, $NF_RExt(C(f^{-1}(A_i)))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Suppose that $NF_Rcl(\cap_{i=1}^{n} f^{-1}(A_i)) \neq 1_N$. This implies that

$$\begin{gathered} \overline{NF_Rcl(\cap_{i=1}^{n} f^{-1}(A_i))} \neq 0_N \\ \Rightarrow NF_Rint(\cup_{i=1}^{n} C(f^{-1}(A_i))) \neq 0_N \\ \Rightarrow NF_Rint(\cup_{i=1}^{n} f^{-1}(C(A_i))) \neq 0_N. \end{gathered}$$

Then, there is a non-zero neutrosophic τ-open ring E_i in $(R_1, \mathscr{S}_1)$ such that $E_i \subseteq \cup_{i=1}^{n} f^{-1}(C(A_i))$. Now,

$$\begin{aligned} f(E_i) &\subseteq f(\cup_{i=1}^{n} f^{-1}(C(A_i))) \\ &\subseteq \cup_{i=1}^{n} f(f^{-1}(C(A_i))) \\ &\subseteq \cup_{i=1}^{n} C(A_i) \\ &= C(\cap_{i=1}^{n} A_i). \end{aligned}$$

$$\text{Then, } NF_R int(f(E_i)) \subseteq NF_R int(C(\cap_{i=1}^{n} A_i)) = C(NF_R cl(\cap_{i=1}^{n} A_i)). \tag{4.3}$$

Since $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space, $NF_R cl(\cap_{i=1}^{n} A_i) = 1_N$. Hence from (4.3), $NF_R int(f(E_i)) \subseteq 0_N$. This implies that $NF_R int(f(E_i)) = 0_N$, which is a contradiction. Hence $NF_R cl(\cap_{i=1}^{n} f^{-1}(A_i)) = 1_N$. Therefore, $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space.

Proposition 4.7. Let $(R_1, \mathscr{S}_1)$ and $(R_2, \mathscr{S}_2)$ be any two neutrosophic τ-structure ring spaces. Let $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ be any bijective function. Then the following statements are equivalent:

(i) f is somewhat neutrosophic τ-structure ring continuous function.

(ii) If A is a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(A) \neq 1_N$, then there exists a neutrosophic τ-closed ring $0_N \neq E \neq 1_N$ in $(R_1, \mathscr{S}_1)$ such that $f^{-1}(A) \subset E$.

(iii) If A is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$, then $f(A)$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$.

Proof:
(i)⇒(ii)
Assume that (i) is true. Let A be a neutrosophic τ-closed ring in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(A) \neq 1_N$. Then $C(A)$ is a neutrosophic τ-open ring in $(R_2, \mathscr{S}_2)$ such that $C(f^{-1}(A)) = f^{-1}(C(A)) \neq 0_N$. Since f is somewhat neutrosophic τ-structure ring continuous, there exists a neutrosophic τ-open ring E in $(R_1, \mathscr{S}_1)$ such that $E \subseteq f^{-1}(C(A))$. Then there exists a neutrosophic τ-closed ring $C(E) \neq 0_N$ in $(R_1, \mathscr{S}_1)$ such that $C(E) \subset f^{-1}(A)$. Hence (i)⇒(ii).
(ii)⇒(iii)
Assume that (ii) is true. Let A be a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$ such that $f(A)$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Then, there exists a neutrosophic τ-closed ring C in $(R_2, \mathscr{S}_2)$ such that

$$f(A) \subset E \subset 1_N.$$

This implies that $f^{-1}(E) \neq 1_N$. Then by (ii), there exists a neutrosophic τ-closed ring $0_N \neq D \neq 1_N$ such that $A \subset f^{-1}(E) \subset D \subset 1_N$. This is a contradiction. Hence (ii)⇒(iii).
(iii)⇒(ii)
Assume that (iii) is true. Suppose (ii) is not true. Then there exists a neutrosophic τ-closed ring A in $(R_2, \mathscr{S}_2)$ such that $f^{-1}(A) \neq 1_N$. But there is no neutrosophic τ-closed ring $0_N \neq E \neq 1_N$ in $(R_1, \mathscr{S}_1)$ such that $f^{-1}(A) \subseteq E$. This implies that $f^{-1}(A)$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. But from hypothesis $f(f^{-1}(A)) = A$ must be neutrosophic dense ring in $(R_2, \mathscr{S}_2)$, which is a contradiction. Hence (iii)⇒(ii).
(ii)⇒(i)
Let A be a neutrosophic τ-open ring in $(R_2, \mathscr{S}_2)$ and $f^{-1}(A) \neq 0_N$. Then, $f^{-1}(C(A)) = C(f^{-1}(A)) = 0_N$. Then by (ii), there exists a neutrosophic τ-closed ring $0_N \neq B \neq 1_N$ such that $f^{-1}(C(A)) \subset B$. This implies that $C(B) \subset f^{-1}(A)$ and $C(B) \neq 0_N$ is a neutrosophic τ-open ring in $(R_1, \mathscr{S}_1)$. Hence (ii)⇒(i). Hence the proof.

Proposition 4.8. If a function $f : (R_1, \mathscr{S}_1) \to (R_2, \mathscr{S}_2)$ from a neutrosophic τ-structure ring space $(R_1, \mathscr{S}_1)$ onto another neutrosophic τ-structure ring space $(R_2, \mathscr{S}_2)$ is somewhat neutrosophic τ-structure ring continuous, 1-1 and neutrosophic τ-structure ring open function and if $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space, then $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space.
Proof:

Let $(R_1, \mathscr{S}_1)$ be a neutrosophic τ-structure ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R_1, \mathscr{S}_1)$. Then, $NF_Rcl(NF_RExt(C(A_i))) = 1_N$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$. Since f is a neutrosophic τ-structure ring open function, $f(B_{ij})$'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$. Now, $\cap_{j=1}^{\infty} f(B_{ij})$ is a neutrosophic G_δ rings in $(R_2, \mathscr{S}_2)$. Since f is 1-1,

$$f^{-1}(\cap_{j=1}^{\infty} f(B_{ij})) = \cap_{j=1}^{\infty} f^{-}(f(B_{ij})) = \cap_{j=1}^{\infty} B_{ij} = A_i \tag{4.4}$$

$$\text{Since } f \text{ is onto, } f(A_i) = f(f^{-1}(\cap_{j=1}^{\infty} f(B_{ij}))) = \cap_{j=1}^{\infty} f(B_{ij}) \tag{4.5}$$

Therefore, $f(A_i)$ is a neutrosophic G_δ rings in $(R_2, \mathscr{S}_2)$. Since f is somewhat neutrosophic τ-structure ring continuous function, $NF_RExt(C(A_i`))$ is a neutrosophic dense ring in $(R_1, \mathscr{S}_1)$ and by Proposition 4.7., $f(NF_RExt(C(A_i)))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$, which implies that $NF_RExt((f(A_i)))$. Now we claim that $NF_Rcl(\cap_{i=1}^{\infty} f(A_i)) = 1_N$. Suppose that $NF_Rcl(\cap_{i=1}^{n} f(A_i)) \neq 1_N$. This implies that

$$\begin{aligned} & C(NF_Rcl(\cap_{i=1}^{n} f(A_i))) \neq 0_N \\ & \Rightarrow NF_Rint(\cup_{i=1}^{n} C(f(A_i))) \neq 0_N \\ & \Rightarrow NF_Rint(\cup_{i=1}^{n} f(C(A_i))) \neq 0_N. \end{aligned}$$

Therefore there is an non-zero neutrosophic τ-open ring E_i in $(R_2, \mathscr{S}_2)$ such that $E_i \subseteq \cup_{i=1}^{n} f(C(A_i))$. Then $f^{-1}(E_i) \subseteq f^{-1}(\cup_{i=1}^{n} f(C(A_i)))$. Since f is somewhat neutrosophic τ-structure ring continuous function and $E_i \in \mathscr{S}_2$, $NF_Rint(f^{-1}(E_i)) \neq 0_N$ implies that $NF_Rint(f^{-1}(\cup_{i=1}^{n} f(C(A_i)))) \neq 0_N$.
Then $NF_Rint(\cup_{i=1}^{n} f^{-1}(f(C(A_i)))) \neq 0_N$. Since f is a bijective function, $NF_Rint(\cap_{i=1}^{n} C(A_i)) \neq 0_N$, which implies that $C(NF_Rcl(\cap_{i=1}^{n} A_i)) \neq 0_N$. That is, $NF_Rcl(\cap_{i=1}^{n} A_i) \neq 1_N$. This is a contradiction. Hence $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space.

Conversely, let $(R_2, \mathscr{S}_2)$ be a neutrosophic τ-structure ring $ExtV$ space. Assume that A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense ring in $(R_1, \mathscr{S}_1)$. Then $NF_Rcl(NF_RExt(C(A_i))) = 1_N$ and $A_i = \cap_{j=1}^{\infty} B_{ij}$ where B_{ij}'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$. Since f is somewhat neutrosophic τ-structure ring continuous function, $NF_RExt(C(A_i))$'s are neutrosophic dense rings in $(R_1, \mathscr{S}_1)$ and By Proposition 4.7., $f(NF_RExt(C(A_i)))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. That is, $NF_Rcl(NF_RExt(C(A_i))) = 1_N$. Since f is a neutrosophic τ-structure ring open function and B_{ij}'s are neutrosophic τ-open rings in $(R_1, \mathscr{S}_1)$, $f(B_{ij})$'s are neutrosophic τ-open rings in $(R_2, \mathscr{S}_2)$. Hence $\cap_{j=1}^{\infty} f(B_{ij})$ is a neutrosophic G_δ ring in $(R_2, \mathscr{S}_2)$. Since f is 1-1,

$$f^{-1}(\cap_{i=1}^{n} f(B_{ij})) = \cap_{i=1}^{n} (f^{-1}(f(B_{ij})) = \cap_{i=1}^{n} B_{ij}. \tag{4.6}$$

Since f is onto,

$$f(A_i) = f(f^{-1}(\cap_{j=1}^{\infty} f(B_{ij}))) = \cap_{j=1}^{\infty} f(B_{ij}). \tag{4.7}$$

Hence $f(A_i)$ is a neutrosophic G_δ ring in $(R_2, \mathscr{S}_2)$. Now,

$$\begin{aligned} NF_Rcl(NF_RExt(C(f(A_i))) &= NF_Rcl(NF_RExt(f(C(A_i))) \\ &= NF_Rcl(NF_Rint(f(A_i)) \\ &\supseteq NF_Rcl(f(NF_Rint(A_i)) \\ &\supseteq f(NF_Rcl(NF_Rint(A_i))) \\ &= f(1_N) = 1_N. \end{aligned}$$

This implies that $NF_RExt(C(f(A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Hence the neutrosophic ring exterior of $C(f(A_i))$ is a neutrosophic dense ring in $(R_2, \mathscr{S}_2)$. Since $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space, $NF_Rcl(\cap_{i=1}^n f(A_i)) = 1_N$. Now we claim that $NF_Rcl(\cap_{i=1}^n f(A_i)) = 1_N$ where A_i's $(i = 1, ..., n)$ are neutrosophic G_δ rings in $(R_1, \mathscr{S}_1)$ and the neutrosophic ring exterior of $C(A_i)$'s are neutrosophic dense rings in $(R_1, \mathscr{S}_1)$. Suppose that $NF_Rcl(\cap_{i=1}^n A_i) \neq 1_N$. This implies that

$$\begin{aligned} &C(NF_Rcl(\cap_{i=1}^n A_i)) \neq 0_N \\ &\Rightarrow NF_Rint(C(\cap_{i=1}^n A_i)) \neq 0_N \\ &\Rightarrow NF_Rint(\cup_{i=1}^n C(A_i)) \neq 0_N. \end{aligned}$$

Then there is a non-zero neutrosophic τ-open ring E_i in $(R_1, \mathscr{S}_1)$ such that $E_i \subseteq \cup_{i=1}^n C(A_i)$. Now,

$$\begin{aligned} f(E_i) &\subseteq f(\cup_{i=1}^n C(A_i)) \\ &\subseteq \cup_{i=1}^n f(C(A_i)) \\ &\subseteq \cup_{i=1}^n C(f(A_i)) \\ &= C(\cap_{i=1}^n f(A_i)). \end{aligned}$$

$$\text{Then, } NF_Rint(f(E_i)) \subseteq NF_Rint(C(\cap_{i=1}^n f(A_i))) \subseteq C(NF_Rcl(\cap_{i=1}^n f(A_i))) \qquad (4.8)$$

Since $(R_2, \mathscr{S}_2)$ is a neutrosophic τ-structure ring $ExtV$ space, $NF_Rcl(\cap_{i=1}^n f(A_i)) = 1_N$. Hence from (4.8), $NF_Rint(f(E_i)) \subseteq 0_N$, which implies that $NF_Rint(f(E_i)) = 0_N$, which is a contradiction. Hence $NF_Rcl(\cap_{i=1}^n A_i) = 1_N$. Therefore $(R_1, \mathscr{S}_1)$ is a neutrosophic τ-structure ring $ExtV$ space.

5 Conclusion

A neutrosophic set model provides a mechanism for solving the modeling problems which involve indeterminacy, and inconsistent information in which human knowledge is necessary and human evaluation is needed. It deals more flexibility and compatibility to the system as compared to the classical theory, fuzzy theory and intuitionistic fuzzy models. In this paper, a new idea of a neutrosophic τ-structure ring spaces, neutrosophic τ-structure ring $G_\delta T_{1/2}$ spaces and neutrosophic τ-structure ring exterior B spaces and neutrosophic τ-structure ring exterior V spaces have been introduced. Further, neutrosophic τ-structure ring continuous (resp. open,hardly open)functions, somewhat neutrosophic τ-structure ring continuous functions are studied. Their characterization are derived and illustrated with examples.

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An MCDM Method under Neutrosophic Cubic Fuzzy Sets with Geometric Bonferroni Mean Operator

D. Ajay [1,*], Said Broumi [2], J. Aldring [3]

[1] Assistant Professor, Department of Mathematics, Sacred Heart College, Tamilnadu, India ; dajaypravin@gmail.com

[2] Laboratory of Information Processing, Faculty of Science Ben MSik, University Hassan II, Casablanca, Morocco ; broumisaid78@gmail.com

[3] Research Scholar, Department of Mathematics, Sacred Heart College, Tamilnadu, India ; jaldring24@gmail.com

*Correspondence: dajaypravin@gmail.com

Abstract. Neutrosophic cubic fuzzy sets ($NCFSs$) involve interval valued and single valued neutrosophic sets, and are used to describe uncertainty or fuzziness in a more efficient way. Aggregation of neutrosopic cubic fuzzy information is crucial and necessary in a decision making theory. In order to get a better solution to decision making problems under neutrosophic cubic fuzzy environment, this paper introduces an aggregating operator to neutrosophic cubic fuzzy sets with the help of Bonferroni mean and geometric mean, and proposes neutrosophic cubic fuzzy geometric Bonferroni mean operator ($NCFGBM^{u,v}$) with its properties. Then, an efficient decision making technique is introduced based on weighted operator $WNCFGBM^{u,v}_w$. An application of the established method is also examined for a real life problem.

Keywords: Neutrosophic Sets; Cubic Fuzzy Sets; Bonferroni Geometric Mean; Aggregation Operators; MCDM

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1. **Introduction**

Fuzzy set [1] deals with fuzziness in terms of degree of truthness or membership within the range of interval $[0, 1]$. The traditional fuzzy sets are not efficient when the decision makers face more complex problems and it is difficult to quantify their truth values. Y.B.Jun et al. [2] introduced the notion of cubic sets which represents the degree of belongingness or certainty by interval valued fuzzy sets and single valued fuzzy sets simultaneously. Therefore, cubic sets are made up of two parts, where the first one is the interval valued fuzzy sets which represents belongingness in a particular range of interval, and the second one is exact belongingness or fuzzy sets.

Smarandache [3] introduced the philosophical idea of neutrosophic sets (NS) which is formulated from the general concept of fuzzy sets and many real life applications are avaliable under NS. Ajay, D., et al. used neutrosophic theory in fuzzy SAW method [4] and Abdel-Basset.M et al. utilized neutrosophic sets to asseses the uncertainty of linear time-cost tradeoffs [5] and also they applied to resource levelling problem in construction projects [6]. Further, biploar neutrosophic sets have been used in medical diagnosis [7] and decision making suituations [8]. Moreover, Y.B.Jun et al. [9] and M.Ali et al. [10] effectively utilized cubic fuzzy sets to the neutrosophic sets and introduced the concept of neutrosophic cubic fuzzy sets (NCFSs) with some basic operations. Therefore the hybrid form of neutrosophic cubic fuzzy set may be more adequate to address problems of more complexity using interval valued and exact valued neutrosophic information and it has been broadly used in the fields of MCDM [12–19]. Neutrosophic cubic fuzzy sets contain more information than general form of NS and therefore NCFSs provide better and efficient solution in MCDM.

Aggregating the fuzzy information plays an important role in decision theory and in particular decision making in real life problems. Variety of aggregating operators exist, but very few aggregating operators are available under neutrosophic cubic fuzzy numbers such as Heronian mean operators [21], Einstein Hybrid Geometric Aggregation Operators [22, 23], Dombi Aggregation Operators [24], weighted arithmetic averaging (NCNWAA) operator and weighted geometric averaging (NCNWGA) operator [25]. Still the Bonferroni geometric mean aggregating operator has not been studied in NCF environment. So the main purposes of this study are: (1) to establish a neutrosophic cubic fuzzy Bonferroni weighted geometric mean operator $WNCFBWGM_w^{u,v}$.(2) to develop an MCDM method using $WNCFBWGM_w^{u,v}$ operator to rank the alternatives under NCFS environment.

The content of the paper is organized as follows. Section 2 and 3 briefly introduce the basic concepts and operations of neutrosophic cubic fuzzy sets. The concepts of Bonferroni mean and geometric Bonferroni mean are explained in section 4. The neutrosophic cubic fuzzy geometric Bonferroni mean $NCFGBM^{u,v}$ and weighted neutrosophic cubic fuzzy geometric Bonferroni mean $WNCFGBM_w^{u,v}$ operators are established and examined with their properties in section 5. An MCDM method based on $WNCFGBM_w^{u,v}$ is presented in section 6. Finally conclusions and scope for future research are given in section 7.

2. **Neutrosophic Cubic Fuzzy Set**

Definition 2.1. [9] Let X be a non empty universal set or universe of discourse. A neutrosophic cubic fuzzy set $\tilde{S}$ in X is constructed in the following form:

$$\tilde{S} = \{x, \langle T(x), I(x), F(x)\rangle\, ; \langle T_\lambda(x), I_\lambda(x), F_\lambda(x)\rangle\, |x \in X\}$$

where $T(x), I(x), F(x)$ are interval valued neutrosophic sets; $T(x) = [T^-(x), T^+(x)] \subseteq [0,1]$ is the degree of truth interval values; $I(x) = [I^-(x), I^+(x)] \subseteq [0,1]$ is the degree of indeterminacy interval values; $F(x) = [F^-(x), F^+(x)] \subseteq [0,1]$ is the degree of falsity interval values; and $\langle T_\lambda(x), I_\lambda(x), F_\lambda(x)\rangle \in [0,1]$ are truth, indeterminacy, and falsity degrees of membership values respectively. For convenience, a neutrosophic cubic fuzzy element in a neutrosophic cubic fuzzy set (NCFSs) $\tilde{S}$ is simply denoted by $\tilde{S} = \{\langle T, I, F\rangle ; \langle T_\lambda, I_\lambda, F_\lambda\rangle\}$, where $\langle T, I, F\rangle \subseteq [0,1]$ and $\langle T_\lambda, I_\lambda, F_\lambda\rangle \in [0,1]$, satisfying the conditions that $0 \leq \langle T^+, I^+, F^+\rangle \leq 3$ and $0 \leq \langle T_\lambda, I_\lambda, F_\lambda\rangle \leq 3$.

Definition 2.2. [10] Let $\tilde{S}$ be a neutrosophic cubic fuzzy set in X given by

$$\tilde{S} = \left\{ \left[T^-(x), T^+(x)\right], \left[I^-(x), I^+(x)\right], \left[F^-(x), F^+(x)\right]; \langle T_\lambda(x), I_\lambda(x), F_\lambda(x)\rangle \,|x \in X \right\}$$

$\tilde{S}$ is said to be internal NCFSs if $T^-(x) \leq T_\lambda(x) \leq T^+(x), I^-(x) \leq I_\lambda(x) \leq I^+(x), F^-(x) \leq F_\lambda(x) \leq F^+(x) \forall x$; $\tilde{S}$ is said to be external NCFSs if $T_\lambda(x) \notin [T^-(x), T^+(x)], I_\lambda(x) \notin [I^-(x), I^+(x)], F_\lambda(x) \notin [F^-(x), F^+(x)] \forall x$.

Definition 2.3. Let $\tilde{S}$ be a neutrosophic cubic fuzzy set in X. Then the support of neutrosophic cubic fuzzy set $\tilde{S}^*$ is defined by

$$\begin{aligned}\tilde{S}^* = \{ &\left[T^-(x), T^+(x)\right] \supset [0,0], \left[I^-(x), I^+(x)\right] \supset [0,0], \left[F^-(x), F^+(x)\right] \subset [1,1];\\ &\langle T_\lambda(x) > 0, I_\lambda(x) > 0, F_\lambda(x) < 1\rangle \,|x \in X\}\end{aligned}$$

Definition 2.4. [25] Let $\tilde{S}$ be a non empty neutrosophic cubic fuzzy number given by

$$\begin{aligned}\tilde{S} &= \{x, \langle T(x), I(x), F(x)\rangle ; \langle T_\lambda(x), I_\lambda(x), F_\lambda(x)\rangle \,|x \in X\}\\ &= \left\{ \left[T^-(x), T^+(x)\right], \left[I^-(x), I^+(x)\right], \left[F^-(x), F^+(x)\right]; \langle T_\lambda(x), I_\lambda(x), F_\lambda(x)\rangle \,|x \in X \right\},\end{aligned}$$

then its score, accuracy and certainty functions can be defined respectively, as follows:

$$s(\tilde{S}) = \frac{\frac{\left[4+T^-(x)-I^-(x)-F^-(x)+T^+(x)-I^+(x)-F^+(x)\right]}{6} + \frac{\left[2+T_\lambda(x)-I_\lambda(x)-F_\lambda(x)\right]}{3}}{2}, \tag{1}$$

$$a(\tilde{S}) = \frac{\left[\left(T^-(x) - F^-(x) + T^+(x) - F^+(x)\right)/2 + T_\lambda(x) - F_\lambda(x)\right]}{2}, \tag{2}$$

$$c(\tilde{S}) = \frac{\left[\left(T^-(x) + T^+(x)\right)/2 + T_\lambda(x)\right]}{2}; \qquad s(\tilde{S}), a(\tilde{S}), c(\tilde{S}) \in [0,1] \tag{3}$$

3. Operations on NCFNs

Let $A_i(x) = \left\{ \left[T_i^-, T_i^+\right], \left[I_i^-, I_i^+\right], \left[F_i^-, F_i^+\right]; \langle T_{\lambda i}, I_{\lambda i}, F_{\lambda i}\rangle \,|x \in X \right\} (i = 1,2,3,\cdots n)$ and $A_j(y) = \left\{ \left[T_j^-, T_j^+\right], \left[I_j^-, I_j^+\right], \left[F_j^-, F_j^+\right]; \langle T_{\lambda j}, I_{\lambda j}, F_{\lambda j}\rangle \,|y \in Y \right\} (j = 1,2,3,\cdots n)$ be two collections of NCFNs. Then the following operations are defined [25]:

(1) **Union**

$$A_i(x) \cup A_j(y) = \Big\{ \Big[min(T_i^-, T_j^-), max(T_i^+, T_j^+) \Big], \Big[max(I_i^-, I_j^-), min(I_i^+, I_j^+) \Big], \Big[max(F_i^-, F_j^-), min(F_i^+, F_j^+) \Big]; \langle max(T_{\lambda i}, T_{\lambda j}), min(I_{\lambda i}, I_{\lambda j}), min(F_{\lambda i}, F_{\lambda j}) \rangle \Big\}$$

(2) **Intersection**

$$A_i(x) \cap A_j(y) = \Big\{ \Big[max(T_i^-, T_j^-), min(T_i^+, T_j^+) \Big], \Big[min(I_i^-, I_j^-), max(I_i^+, I_j^+) \Big], \Big[min(F_i^-, F_j^-), max(F_i^+, F_j^+) \Big]; \langle min(T_{\lambda i}, T_{\lambda j}), max(I_{\lambda i}, I_{\lambda j}), max(F_{\lambda i}, F_{\lambda j}) \rangle \Big\}$$

(3) **Complement**

$$A_i^c(x) = \left\{ \left[F_i^-, F_i^+ \right], \left[1 - I_i^-, 1 - I_i^+ \right], \left[T_i^-, T_i^+ \right]; \langle F_{\lambda i}, 1 - I_{\lambda i}, T_{\lambda i} \rangle \, | x \in X \right\}$$

(4) $A_i(x) \subseteq A_j(y)$ if and only if $\left[T_i^-, T_i^+ \right] \subseteq \Big[T_j^-, T_j^+ \Big], \left[I_i^-, I_i^+ \right] \supseteq \Big[I_j^-, I_j^+ \Big], \left[F_i^-, F_i^+ \right] \supseteq \Big[F_j^-, F_j^+ \Big]$ and $T_{\lambda i} \leq T_{\lambda i}, I_{\lambda i} \geq I_{\lambda j}, F_{\lambda i} \geq F_{\lambda j} \forall x \in X, y \in Y$.

(5) $A_i(x) = A_j(y)$ if and only if $A_i(x) \subseteq A_j(y)$ and $A_i(x) \supseteq A_j(y)$ i.e. $\left[T_i^-, T_i^+ \right] = \Big[T_j^-, T_j^+ \Big], \left[I_i^-, I_i^+ \right] = \Big[I_j^-, I_j^+ \Big], \left[F_i^-, F_i^+ \right] = \Big[F_j^-, F_j^+ \Big]; \langle T_{\lambda i}, I_{\lambda i}, F_{\lambda i} \rangle = \langle T_{\lambda j}, I_{\lambda j}, F_{\lambda j} \rangle$

(6) For $\omega > 0$

$$\omega A_i = \left\{ \left[1 - \left(1 - T_i^- \right)^\omega, 1 - \left(1 - T_i^+ \right)^\omega \right], \left[\left(I_i^- \right)^\omega, \left(I_i^+ \right)^\omega \right], \left[\left(F_i^- \right)^\omega, \left(F_i^+ \right)^\omega \right]; \langle 1 - (1 - T_{\lambda i})^\omega, (I_{\lambda i})^\omega, (F_{\lambda i})^\omega \rangle \right\}$$

(7) For $\omega > 0$

$$(A_i)^\omega = \left\{ \left[\left(T_i^- \right)^\omega, \left(T_i^+ \right)^\omega \right], \left[1 - \left(1 - I_i^- \right)^\omega, 1 - \left(1 - I_i^+ \right)^\omega \right], \left[1 - \left(1 - F_i^- \right)^\omega, 1 - \left(1 - F_i^+ \right)^\omega \right]; \langle (T_{\lambda i})^\omega, 1 - (1 - I_{\lambda i})^\omega, 1 - (1 - F_{\lambda i})^\omega \rangle \right\}$$

(8) **Algebraic Sum**

$$A_i(x) \oplus A_j(y) = \Big\{ \Big[T_i^- + T_j^- - T_i^- T_j^-, T_i^+ + T_j^+ - T_i^+ T_j^+ \Big], \Big[I_i^- I_j^-, I_i^+ I_j^+ \Big], \Big[F_i^- F_j^-, F_i^+ F_j^+ \Big]; \langle T_{\lambda i} + T_{\lambda j} - T_{\lambda i} T_{\lambda j}, I_{\lambda i} I_{\lambda j}, F_{\lambda i} F_{\lambda j} \rangle \Big\}$$

(9) **Algebraic Product**

$$A_i(x) \otimes A_j(y) = \Big\{ \Big[T_i^- T_j^-, T_i^+ T_j^+ \Big], \Big[I_i^- + I_j^- - I_i^- I_j^-, I_i^+ + I_j^+ - I_i^+ I_j^+ \Big], \Big[F_i^- + F_j^- - F_i^- F_j^-, F_i^+ + F_j^+ - F_i^+ F_j^+ \Big]; \langle T_{\lambda i} T_{\lambda j}, I_{\lambda i} + I_{\lambda j} - I_{\lambda i} I_{\lambda j}, F_{\lambda i} + F_{\lambda j} - F_{\lambda i} F_{\lambda j} \rangle \Big\}$$

4. Geometric Bonferroni Mean

Bonferroni proposed the concept of Bonferroni mean (BM) which is defined as follows:

Definition 4.1. [11] Let $s_i(i = 1, 2, \ldots, n)$ be n number of positive crisp data. For any $u, v \geq 0$,

$$B^{u,v}(s_1, s_2, \ldots, s_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1,\\ i\neq j}}^{n} \left(s_i^u s_j^v \right) \right)^{\frac{1}{u+v}} \tag{4}$$

We call Eq.(4) as the Bonferroni mean (BM) operator. Especially, if v=0, Eq.(4) reduces to the generalized mean operator given by

$$\begin{aligned} B^{u,0}(s_1, s_2, \ldots, s_n) &= \left(\frac{1}{n} \sum_{i=1}^{n} s_i^u \left(\frac{1}{(n-1)} \sum_{\substack{j=1,\\ i\neq j}}^{n} \left(s_j^0 \right) \right) \right)^{\frac{1}{u+0}} \\ &= \left. \frac{1}{n} \sum_{i=1}^{n} s_i^u \right)^{\frac{1}{u}} \end{aligned} \tag{5}$$

If $u = 1$ and $v = 0$, the above equation produces the very known arithmetic mean (AM):

$$B^{1,0}(s_1, s_2, \ldots, s_n) = \frac{1}{n} \sum_{i=1}^{n} s_i^u \tag{6}$$

With the usual notion of geometric mean and the BM, the geometric Bonferroni mean operator is formulated.

Definition 4.2. Let $u, v > 0$, and $s_i(i = 1, 2, \ldots, n)$ be a collection of non negative crisp numbers. If

$$GB^{u,v}(s_1, s_2, \ldots, s_n) = \frac{1}{(u+v)} \prod_{\substack{i,j=1,\\ i\neq j}}^{n} (us_i + vs_j)^{\frac{1}{n(n-1)}} \tag{7}$$

then $GB^{u,v}$ is called the geometric Bonferroni mean (GBM).

Obviously, the GBM statisfies the following properties:

(1) $GB^{u,v}(0, 0, \ldots, 0) = 0$
(2) $GB^{u,v}(s_1, s_2, \ldots, s_n) = s$ if $s_i = s$, for all $i = 1, 2, \ldots, n$.
(3) $GB^{u,v}(s_1, s_2, \ldots, s_n) \geq GB^{u,v}(t_1, t_2, \ldots, t_n)$ if $s_i \geq t_i$ $\forall i$ that is, $GB^{u,v}$ is monotonic.
(4) $\text{Min}(s_i) \leq GB^{u,v} \leq \text{Max}(s_i)$.

Furthermore, if $v = 0$, $Eq.$(7) generates the geometric mean:

$$GB^{u,0}(s_1, s_2, \ldots, s_n) = \frac{1}{u} \prod_{\substack{i,j=1,\\ i\neq j}}^{n} (us_i)^{\frac{1}{n(n-1)}} = \prod_{i=1}^{n} (s_i)^{\frac{1}{n}} \tag{8}$$

5. Neutrosophic Cubic Fuzzy Geometric Bonferroni Mean

Definition 5.1. Let $\tilde{S}_i = \left\{ \left[T_i^-, T_i^+\right], \left[I_i^-, I_i^+\right], \left[F_i^-, F_i^+\right]; \langle T_{\lambda i}, I_{\lambda i}, F_{\lambda i}\rangle \right\}$ be a collection of neutrosophic cubic fuzzy numbers $(NCFN)$. For any $u, v > 0$,

$$NCFGBM^{u,v}\left(\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n\right) = \frac{1}{u+v}\left(\bigotimes_{\substack{i,j=1,\\ i\neq j}}^{n} \left((u\tilde{S}_i \oplus v\tilde{S}_j)^{\frac{1}{n(n-1)}} \right) \right)$$

is called the neutrosophic cubic fuzzy geometric bonferroni mean operator.

Theorem 5.2. *Let $u, v > 0$ and $\tilde{S}_i = \left\{ \left[T_i^-, T_i^+\right], \left[I_i^-, I_i^+\right], \left[F_i^-, F_i^+\right]; \langle T_{\lambda i}, I_{\lambda i}, F_{\lambda i}\rangle \right\}$ be a collection of neutrosophic cubic fuzzy numbers $(NCFN)$. Then the aggregated value is calculated using the operator $NCFGBM^{u,v}$*

$$NCFGBM^{u,v}\left(\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n\right) = \frac{1}{u+v}\left(\bigotimes_{\substack{i,j=1,\\ i\neq j}}^{n} \left((u\tilde{S}_i \oplus v\tilde{S}_j)^{\frac{1}{n(n-1)}} \right) \right)$$

$$= \left\{ \left[1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (1-T_i^-)^u (1-T_j^-)^v \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{u+v}}, \right.\right.$$

$$\left. 1 - \left(1 - \prod_{\substack{i,j=1,\\ i\neq j}}^{n} \left(1 - (1-T_i^+)^u (1-T_j^+)^v \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{u+v}} \right],$$

$$\left[\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (I_i^-)^u (I_j^-)^v \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{u+v}}, \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (I_i^+)^u (I_j^+)^v \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{u+v}} \right], \tag{9}$$

$$\left[\left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (F_i^-)^u (F_j^-)^v \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{u+v}}, \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} \left(1 - (F_i^+)^u (F_j^+)^v \right)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{u+v}} \right];$$

$$\left\langle 1 - \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} (1 - (1-T_{\lambda i})^u (1-T_{\lambda j})^v)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{u+v}}, \right.$$

$$\left.\left. \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} (1 - (I_{\lambda i})^u (I_{\lambda j})^v)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{u+v}}, \left(1 - \prod_{\substack{i,j=1\\ i\neq j}}^{n} (1 - (F_{\lambda i})^u (F_{\lambda j})^v)^{\frac{1}{n(n-1)}} \right)^{\frac{1}{u+v}} \right\rangle \right\}.$$

Proof. . Using the operational laws on $NCFN$ described in section (3), we have

$$u\tilde{S}_i = \{[1-(1-T_i^-)^u, 1-(1-T_i^+)^u], [(I_i^-)^u, (I_i^+)^u], [(F_i^-)^u, (F_i^+)^u]; \langle 1-(1-T_{\lambda i})^u, (I_{\lambda i})^u, (F_{\lambda i})^u \rangle\}$$

$$v\tilde{S}_j = \left\{\left[1-\left(1-T_j^-\right)^v, 1-\left(1-T_j^+\right)^v\right], \left[\left(I_j^-\right)^v, \left(I_j^+\right)^v\right], \left[\left(F_j^-\right)^v, \left(F_j^+\right)^v\right]; \langle 1-(1-T_{\lambda j})^v, (I_{\lambda j})^v, (F_{\lambda j})^v \rangle\right\}$$

$$u\tilde{S}_i \oplus v\tilde{S}_j = \left\{\left[1-(1-T_i^-)^u(1-T_j^-)^v, 1-(1-T_i^+)^u(1-T_j^+)^v\right], \left[(I_i^-)^u(I_j^-)^v, (I_i^+)^u(I_j^+)^v\right], \left[(F_i^-)^u(F_j^-)^v, (F_i^+)^u(F_j^+)^v\right]; \langle 1-(1-T_{\lambda i})^u(1-T_{\lambda j})^v, (I_{\lambda i})^u(I_{\lambda j})^v, (F_{\lambda i})^u(F_{\lambda j})^v \rangle\right\}.$$

Next, we have the following equation which has been derived by Xu and Yager [28].

$$\bigotimes_{\substack{i,j=1,\\ i\neq j}}^{n} \left(u\tilde{S}_i \oplus v\tilde{S}_j\right)^{\frac{1}{n(n-1)}} = \left\{\left[\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-(1-T_i^-)^u(1-T_j^-)^v\right)^{\frac{1}{n(n-1)}}, \prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-(1-T_j^+)^u(1-T_j^+)^v\right)^{\frac{1}{n(n-1)}}\right],\right.$$
$$\left[1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-(I_i^-)^u(I_j^-)^v\right)^{\frac{1}{n(n-1)}}, 1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-(I_i^+)^u(I_j^+)^v\right)^{\frac{1}{n(n-1)}}\right],$$
$$\left[1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-(F_i^-)^u(F_j^-)^v\right)^{\frac{1}{n(n-1)}}, 1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-(F_i^+)^u(F_j^+)^v\right)^{\frac{1}{n(n-1)}}\right]; \tag{10}$$
$$\left\langle \prod_{\substack{i,j=1,\\ i\neq j}}^{n}(1-(1-T_{\lambda i})^u(1-T_{\lambda j})^v)^{\frac{1}{n(n-1)}}, 1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-(I_{\lambda i}^u)\,(I_{\lambda j}^v)\right)^{\frac{1}{n(n-1)}},\right.$$
$$\left.\left. 1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-(F_{\lambda i}^u)\,(F_{\lambda j}^v)\right)^{\frac{1}{n(n-1)}}\right\rangle\right\}.$$

Using NCF operational laws, Eq.(10) yields neutrosophic cubic fuzzy geometric bonferroni mean operator $NCFGBM^{u,v}(\tilde{S}_1, \tilde{S}_2, \cdots, \tilde{S}_n)$ given by $Eq.(9)$. In addition, it satisfies the following conditions

$$\left[1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(1-T_i^-)^u(1-T_j^-)^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}},\right.$$

$$\left.1-\left(1-\prod_{\substack{i,j=1,\\i\neq j}}^{n}\left(1-(1-T_i^+)^u(1-T_j^+)^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\right]\subseteq[0,,1],$$

$$\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(I_i^-)^u(I_j^-)^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(I_i^+)^u(I_j^+)^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\right]\subseteq[0,1],$$

$$\left[\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(F_i^-)^u(F_j^-)^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}},\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}\left(1-(F_i^+)^u(F_j^+)^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\right]\subseteq[0,1];$$

$$0\leq 1-\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}(1-(1-T_{\lambda i})^u(1-T_{\lambda j})^v)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\leq 1,$$

$$0\leq\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}(1-(I_{\lambda i})^u(I_{\lambda j})^v)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\leq 1,$$

$$0\leq\left(1-\prod_{\substack{i,j=1\\i\neq j}}^{n}(1-(F_{\lambda i})^u(F_{\lambda j})^v)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\leq 1$$

which completes the proof of the theorem. □

We discuss some of the important properties of the $NCFGBM^{u,v}$:

(1) **Idempotency**: Suppose the colletive data of neutrosophic cubic fuzzy numbers $\tilde{S}_i = \{[T_i^-, T_i^+], [I_i^-, I_i^+], [F_i^-, F_i^+]; \langle T_{\lambda i}, I_{\lambda i}, F_{\lambda i}\rangle\}\,(i = 1, 2, 3, \cdots n)$ are equal, for

any $u, v > 0$, the aggregate operator be

$$\begin{aligned} NCFGBM^{u,v}(\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n) &= NCFGBM^{u,v}(\tilde{S}, \tilde{S}, ..., \tilde{S}) \\ &= \frac{1}{u+v}\left(\bigotimes_{\substack{i,j=1, \\ i\neq j}}^{n}\left(\left(u\tilde{S}\oplus v\tilde{S}\right)^{\frac{1}{n(n-1)}}\right)\right) \\ &= \frac{1}{u+v}\left(\bigotimes_{\substack{i,j=1, \\ i\neq j}}^{n}\left((u+v)\,\tilde{S}\right)^{\frac{1}{n(n-1)}}\right) \\ &= \frac{1}{u+v}\left((u+v)\,\tilde{S}\right)^{\frac{n(n-1)}{n(n-1)}} = \tilde{S} \end{aligned} \tag{11}$$

(2) **Commuatativity**: Let $\tilde{S}_i (i = 1, 2, 3, \cdots n)$ be a collection of neutrosophic cubic numbers. For any $u, v > 0$,

$$NCFGBM^{u,v}\left(\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n\right) = NCFGBM^{u,v}\left(\dot{\tilde{S}}_1, \dot{\tilde{S}}_2, \ldots, \dot{\tilde{S}}_n\right) \tag{12}$$

Let $\left(\dot{\tilde{S}}_1, \dot{\tilde{S}}_2, \ldots, \dot{\tilde{S}}_n\right)$ be any permuation of $\left(\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n\right)$. Then

$$\begin{aligned} NCFGBM^{u,v}\left(\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n\right) &= \frac{1}{u+v}\left(\bigotimes_{\substack{i,j=1, \\ i\neq j}}^{n}\left((u\tilde{S}_i\oplus v\tilde{S}_j)^{\frac{1}{n(n-1)}}\right)\right) \\ &= \frac{1}{u+v}\left(\bigotimes_{\substack{i,j=1, \\ i\neq j}}^{n}\left((u\dot{\tilde{S}}_i\oplus v\dot{\tilde{S}}_j)^{\frac{1}{n(n-1)}}\right)\right) \\ &= NCFGBM^{u,v}\left(\dot{\tilde{S}}_1, \dot{\tilde{S}}_2, \ldots, \dot{\tilde{S}}_n\right) \end{aligned}$$

(3) **Monotonicity**: Let $\tilde{S}_i$ $(i = 1, 2, 3, \cdots n)$ and $\tilde{S}_j$ $(j = 1, 2, 3, \cdots n)$ be two collections of neutrosophic cubic numbers. For any $u, v > 0$, if $[T_i^-, T_i^+], \subseteq [T_j^-, T_j^+], [I_i^-, I_i^+] \supseteq [I_j^-, I_j^+], [F_i^-, F_i^+] \supseteq [F_j^-, F_j^+]; T_{\lambda i} \leq T_{\lambda j}, I_{\lambda i} \geq I_{\lambda j}, F_{\lambda i} \geq F_{\lambda j} \quad (\forall i, j = 1, 2, 3, \ldots n)$, Then

$$NCFGBM^{u,v}\left(\tilde{S}_i\right) \leq NCFGBM^{u,v}\left(\tilde{S}_j\right) \tag{13}$$

(4) **Boundedness**: Let $\tilde{S}_i = \left\{\left[T_i^-, T_i^+\right], \left[I_i^-, I_i^+\right], \left[F_i^-, F_i^+\right]; \langle T_{\lambda i}, I_{\lambda i}, F_{\lambda i}\rangle\right\} (i = 1, 2, 3, \cdots n)$ be a collection of neutrosophic cubic fuzzy numbers, and let

$$\tilde{S}_i^- = \left\{inf\left(\left[T_i^-, T_i^+\right]\right), sup\left(\left[I_i^-, I_i^+\right]\right), sup\left(\left[F_i^-, F_i^+\right]\right); min\left(T_{\lambda i}\right), max\left(I_{\lambda i}\right), max\left(F_{\lambda i}\right)\right\},$$

$$\tilde{S}_i^+ = \left\{sup\left(\left[T_i^-, T_i^+\right]\right), inf\left(\left[I_i^-, I_i^+\right]\right), inf\left(\left[F_i^-, F_i^+\right]\right); max\left(T_{\lambda i}\right), min\left(I_{\lambda i}\right), min\left(F_{\lambda i}\right)\right\}.$$

For any $u, v > 0$,

$$\tilde{S}_i^- \leq NCFGBM^{u,v}\left(\tilde{S}_i\right)(i = 1, 2, 3, \ldots n) \leq \tilde{S}_i^+ \tag{14}$$

Thus the boundedness is easily obtained.

If parameters u and v are modified in $NCFGBM^{u,v}$, then a special case can be obtained as follows:

If $v \to 0$, then by equation (9), we have

$$NCFGBM^{u,v}\left(\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n\right) = \frac{1}{u+v}\left(\bigotimes_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\left(u\tilde{S}_i \oplus v\tilde{S}_j\right)^{\frac{1}{n(n-1)}}\right)\right) = \frac{1}{u}\left(\bigotimes_{i=1}^{n}\left((u\tilde{S}_i)^{\frac{1}{n}}\right)\right)$$

$$= \Bigg\{\Bigg[1-\left(1-\prod_{i=1}^{n}\left(1-(1-T_i^-)^u\right)^{\frac{1}{n}}\right)^{\frac{1}{u}}, 1-\quad 1-\prod_{i=1}^{n}\left(1-(1-T_i^+)^u\right)^{\frac{1}{n}}\Bigg)^{\frac{1}{u}}\Bigg],$$

$$\left[\;1-\prod_{i=1}^{n}\left(1-(I_i^-)^u\right)^{\frac{1}{n}}\Bigg)^{\frac{1}{u}},\;\; 1-\prod_{i=1}^{n}\left(1-(I_i^+)^u\right)^{\frac{1}{n}}\Bigg)^{\frac{1}{u}}\right],$$

$$\left[\;1-\prod_{i=1}^{n}\left(1-(F_i^-)^u\right)^{\frac{1}{n}}\Bigg)^{\frac{1}{u}},\;\; 1-\prod_{i=1}^{n}\left(1-(F_i^+)^u\right)^{\frac{1}{n}}\Bigg)^{\frac{1}{u}}\right];$$

$$\left\langle 1-\quad 1-\prod_{i=1}^{n}\left(1-(1-T_{\lambda i})^u\right)^{\frac{1}{n}}\Bigg)^{\frac{1}{u}},\;\; 1-\prod_{i=1}^{n}\left(1-(I_{\lambda i})^u\right)^{\frac{1}{n}}\Bigg)^{\frac{1}{u}},\;\; 1-\prod_{i=1}^{n}\left(1-(F_{\lambda i})^u\right)^{\frac{1}{n}}\Bigg)^{\frac{1}{u}}\right\rangle\Bigg\}.$$

which we call the generalized neutrosophic cubic fuzzy geometric mean ($NCFBGM^{u,v}$).

5.1. *Weighted Neutrosophic Cubic Fuzzy Bonferroni Geometric Mean*

Generally weighted aggregating operator plays a significant role in decision-making processes to aggregate information. Therefore we propose a weighted aggregate operator based on neutrosophic cubic fuzzy bonferroni geometric mean ($WNCFGBM_w^{u,v}$).

Definition 5.3. Let $\tilde{S}_i = \left\{\left[T_i^-, T_i^+\right], \left[I_i^-, I_i^+\right], \left[F_i^-, F_i^+\right]; \langle T_{\lambda i}, I_{\lambda i}, F_{\lambda i}\rangle\right\}$ be a collection of neutrosophic cubic numbers (NCN), and $w = (W_1, W_2, \ldots, W_n)^T$ the wieght vector of $\tilde{S}_i = \tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_n$, where w_i indicates the importance degree of $\tilde{S}_i$ such that $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$ $(i = 1, 2, 3, \ldots, n)$. For any $u, v > 0$,

$$WNCFGBM_w^{u,v}\left(\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n\right) = \frac{1}{u+v}\left(\bigotimes_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\left(u(\tilde{S}_i)^{w_i} \oplus v(\tilde{S}_j)^{w_j}\right)^{\frac{1}{n(n-1)}}\right)\right) \tag{15}$$

is called the weighted neutrosophic cubic fuzzy geometric bonferroni mean operator.

Theorem 5.4. *Let $u, v > 0$ and $\tilde{S}_i$ $(i = 1, 2, 3, \ldots, n)$ be a collection of neutrosophic cubic fuzzy numbers (NCFN), whose weight vector is $w_i = (W_1, W_2, \ldots, W_n)^T$, which satisfies that*

$w_i > 0$, *and* $\sum_{i=1}^{n} w_i = 1$ $(i = 1, 2, 3, \ldots, n)$. *Then the aggregated value using the operator is*

$$WNCFGBM_w^{u,v}(\tilde{S}_1, \tilde{S}_2, ..., \tilde{S}_n) = \frac{1}{u+v}\left(\bigotimes_{\substack{i,j=1,\\ i\neq j}}^{n}\left(\left(u(\tilde{S}_i)^{w_i} \oplus v(\tilde{S}_j)^{w_j}\right)^{\frac{1}{n(n-1)}}\right)\right)$$

$$= \left\{\left[1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(T_i^-)^{w_i})^u(1-(T_j^-)^{w_j})^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}},\right.\right.$$

$$\left.1-\left(1-\prod_{\substack{i,j=1,\\ i\neq j}}^{n}\left(1-(1-(T_i^+)^{w_i})^u(1-(T_j^+)^{w_j})^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\right],$$

$$\left[\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(1-I_i^-)^{w_i})^u(1-(1-I_j^-)^{w_j})^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}},\right.$$

$$\left.\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(1-I_i^+)^{w_i})^u(1-(1-I_j^+)^{w_j})^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\right], \tag{16}$$

$$\left[\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(1-F_i^-)^{w_i})^u(1-(1-F_j^-)^{w_j})^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}},\right.$$

$$\left.\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}\left(1-(1-(1-F_i^+)^{w_i})^u(1-(1-F_j^+)^{w_j})^v\right)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\right];$$

$$\left\langle 1-\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-(T_{\lambda i})^{w_i})^u(1-(T_{\lambda j})^{w_j})^v)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}},\right.$$

$$\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-(1-I_{\lambda i})^{w_i})^u(1-(1-I_{\lambda j})^{w_j})^v)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}},$$

$$\left.\left.\left(1-\prod_{\substack{i,j=1\\ i\neq j}}^{n}(1-(1-(1-F_{\lambda i})^{w_i})^u(1-(1-F_{\lambda j})^{w_j})^v)^{\frac{1}{n(n-1)}}\right)^{\frac{1}{u+v}}\right\rangle\right\}.$$

Proof. The proof is identical with the proof of theorem (5.2) and therefore is omitted. □

6. An application of weighted neutrosophic cubic fuzzy geometric bonferroni mean operator to MCDM problems

In this section, we propose an algorithm for MCDM method based on neutrosophic cubic fuzzy geometric Bonferroni mean operators and illustrate it with a numerical example.

Algorithm. Let $\tilde{A}_i = \{\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots, \tilde{\gamma}_n\}$ and $\tilde{C}_j = \{\tilde{\eta}_1, \tilde{\eta}_2, \ldots, \tilde{\eta}_m\}$ be collections of n alternatives and m attributes respectively. According to the appropriate weight of attributes $(\widehat{\omega}_j)^T = \{\tilde{\omega}_1, \tilde{\omega}_2, \ldots \tilde{\omega}_m\}$ is determined, which satisfies the condition that $\tilde{\omega}_j > 0$ and $\sum \widehat{\omega}_j = 1$. Then the following steps are used in process of MCDM method.

Step 1. Construct neutrosophic cubic fuzzy decision matrix $D = [N_{ij}]_{n \times m}$.

Step 2. The decision matrix is aggregated using $NCFGBM^{u,v}$ or $WNCFGBM_w^{u,v}$ to m attributes.

Step 3. Utilize the score formula (Eq.1) to calculate the values of $s(\tilde{A}_i)$

Step 4. The n alternatives are ranked according to their score values

6.1. *Numerical Example and Investigation*

An illustrative example on the selection problem of investment alternatives is adapted (Ref. [25, 26]) to validate the proposed MCDM method with NCF data. A company wants a sum of money to be invested in an industry. Then the committee suggests the following four feasible alternatives: (a) $\tilde{\gamma}_1$ is a textile company; (b) $\tilde{\gamma}_2$ is an automobile company; (c) $\tilde{\gamma}_3$ is a computer company; (d) $\tilde{\gamma}_4$ is a software company. Suppose that three attributes namely, (1) $\tilde{\eta}_1$ is the risk; (2) $\tilde{\eta}_2$ is the growth; (3) $\tilde{\eta}_3$ is the environmental impact; are taken into the evaluation requirements of the alternatives. The weight vectors of the three attributes $\tilde{\eta}_j (j = 1, 2, 3)$ are $(\widehat{\omega}_j)^T = (0.32, 0.38, 0.3)$ respectively. Then the experts or decision makers are asked to evaluate each alternative on attributes by the form of NCFNs. Thus, the assessment data can be represented by neutrosophic cubic decision matrix $D = [S_{ij}]_{m \times n}$.

step 1. Neutrosophic cubic fuzzy decision matrix $D = [S_{ij}]_{4 \times 3}$

$$D = \begin{bmatrix} \begin{pmatrix} [0.5,0.6], & [0.1,0.3], \\ [0.2,0.4]; & \langle 0.6,0.2,0.3 \rangle \end{pmatrix}, & \begin{pmatrix} [0.5,0.6], & [0.1,0.3], \\ [0.2,0.4]; & \langle 0.6,0.2,0.3 \rangle \end{pmatrix}, & \begin{pmatrix} [0.2,0.4], & [0.7,0.8], \\ [0.8,0.9]; & \langle 0.3,0.8,0.9 \rangle \end{pmatrix} \\ \begin{pmatrix} [0.6,0.8], & [0.1,0.2], \\ [0.2,0.3]; & \langle 0.7,0.1,0.2 \rangle \end{pmatrix}, & \begin{pmatrix} [0.6,0.7], & [0.1,0.2], \\ [0.2,0.3]; & \langle 0.6,0.1,0.2 \rangle \end{pmatrix}, & \begin{pmatrix} [0.3,0.4], & [0.6,0.7], \\ [0.8,0.9]; & \langle 0.3,0.6,0.9 \rangle \end{pmatrix} \\ \begin{pmatrix} [0.4,0.6], & [0.2,0.3], \\ [0.1,0.3]; & \langle 0.6,0.2,0.2 \rangle \end{pmatrix}, & \begin{pmatrix} [0.5,0.6], & [0.2,0.3], \\ [0.3,0.4]; & \langle 0.6,0.3,0.4 \rangle \end{pmatrix}, & \begin{pmatrix} [0.3,0.5], & [0.7,0.8], \\ [0.6,0.7]; & \langle 0.4,0.8,0.7 \rangle \end{pmatrix} \\ \begin{pmatrix} [0.7,0.8], & [0.1,0.2], \\ [0.1,0.2]; & \langle 0.8,0.1,0.2 \rangle \end{pmatrix}, & \begin{pmatrix} [0.6,0.7], & [0.1,0.2], \\ [0.1,0.3]; & \langle 0.7,0.1,0.2 \rangle \end{pmatrix}, & \begin{pmatrix} [0.3,0.4], & [0.6,0.7], \\ [0.7,0.8]; & \langle 0.3,0.7,0.8 \rangle \end{pmatrix} \end{bmatrix}$$

step 2. The decision matrix is aggregated by $WNCFGBM_w^{u,v}(\tilde{S}_{i1}, \tilde{S}_{i2}, \tilde{S}_{i3})(i = 1, 2, \ldots n)$ operators (Using Eq.16) to the three $(\tilde{\eta}_j, j = 1, 2, 3)$ attributes.

If we take the parameter values $u = v = 1$, then using $\tilde{A}_i = WNCFGBM_w^{(1,1)}$, we get the following values

$$\tilde{A}_1 = \{[0.7345, 0.8126], [0.0881, 0.1861], [0.1453, 0.2523]; \langle 0.7951, 0.1453, 0.2093\rangle\},$$
$$\tilde{A}_2 = \{[0.7951, 0.8635], [0.0790, 0.1307], [0.1453, 0.2093]; \langle 0.8124, 0.0790, 0.1642\rangle\},$$
$$\tilde{A}_3 = \{[0.7378, 0.8287], [0.1307, 0.1861], [0.1195, 0.1876]; \langle 0.8126, 0.1674, 0.1703\rangle\},$$
$$\tilde{A}_4 = \{[0.8124, 0.8635], [0.0790, 0.1307], [0.0881, 0.1674]; \langle 0.8491, 0.0881, 0.1453\rangle\}.$$

step 3. Utilizing Eq.(1), the score values $s(\tilde{A}_i)$ are found

$$s(\tilde{A}_1) = 0.8130, s(\tilde{A}_2) = 0.8527, s(\tilde{A}_3) = 0.8244, s(\tilde{A}_4) = 0.8702.$$

step 4. Since the values $s(\tilde{A}_4) > s(\tilde{A}_2) > s(\tilde{A}_3) > s(\tilde{A}_1)$, the rank of alternatives are in the order of $\tilde{\gamma}_4 > \tilde{\gamma}_2 > \tilde{\gamma}_3 > \tilde{\gamma}_1$.

From the results, we could see that the ranking order and the best choice of alternatives are the same as the results in [25, 26].

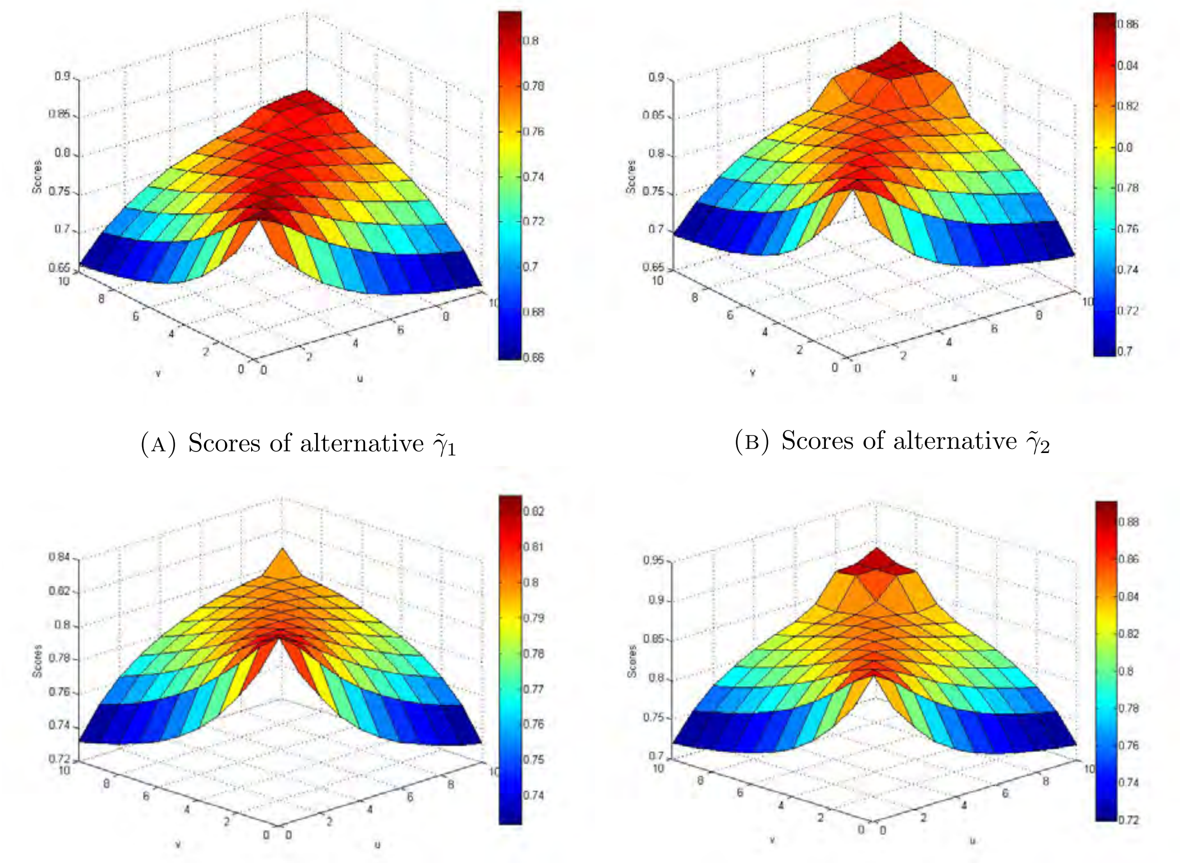
If the parameters $u = v = 2$, then using $\tilde{A}_i = WNCFGBM_w^{(2,2)}$, we get the following aggregate values

$$\tilde{A}_1 = \{[0.7306, 0.8111], [0.0950, 0.1940], [0.1542, 0.2619]; \langle 0.7916, 0.1542, 0.2204\rangle\},$$
$$\tilde{A}_2 = \{[0.7916, 0.8563], [0.0847, 0.1376], [0.1542, 0.2204]; \langle 0.8055, 0.0847, 0.1757\rangle\},$$
$$\tilde{A}_3 = \{[0.7371, 0.8283], [0.1376, 0.1940], [0.1354, 0.1945]; \langle 0.8111, 0.1797, 0.1841\rangle\},$$
$$\tilde{A}_4 = \{[0.8055, 0.8563], [0.0847, 0.1376], [0.0950, 0.1797]; \langle 0.8395, 0.0950, 0.1542\rangle\}.$$

Then we calculate the score of the alternatives $s(\tilde{A}_1) = 0.8059$, $s(\tilde{A}_2) = 0.8451$, $s(\tilde{A}_3) = 0.8165$, $s(\tilde{A}_4) = 0.8621$.

Since $s(\tilde{A}_4) > s(\tilde{A}_2) > s(\tilde{A}_3) > s(\tilde{A}_1)$, the order of the rank is $\tilde{\gamma}_4 > \tilde{\gamma}_2 > \tilde{\gamma}_3 > \tilde{\gamma}_1$.

As the values of parameters u and v change according to the subjective preference of the decision maker, we can find that the ranking order of the alternatives are the same, which indicates that the proposed method can obtain the most optimistic results than the existing MCDM methods based on GBM [29]. For a detailed comparision, we represent the scores of each alternatives in Fig.1 by changing the values of parameters u, v between 0 and 10.



(A) Scores of alternative $\tilde{\gamma}_1$

(B) Scores of alternative $\tilde{\gamma}_2$

(C) Scores of alternative $\tilde{\gamma}_3$

(D) Scores of alternative $\tilde{\gamma}_4$

FIGURE 1. Scores of alternative $\tilde{\gamma}_i$ obtained by $WNCFGBM_w^{u,v}$

7. Conclusions

In this paper, we have applied geometric Bonferroni mean to neutrosophic cubic fuzzy sets. A new aggregating operator $NCFGBM^{u,v}$ has been established and its properties are discussed. The MCDM method is developed based on the weighted operator $WNCFGBM_w^{u,v}$ and is verified with a numerical example where four alternatives are ranked under three criteria. The graphical representation of the results depicted above shows that the ranking of the alternatives remains unaffected when the parameters are changed due to subjective preferences. This proves that the method is objective and moreover the result obtained, when compared with the results of existing techniques, shows that the proposed method is more effective in dealing with neutrosophic fuzzy information. In future, $NCFGBM^{u,v}$ operator could be applied to various other MCDM methods.

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University of New Mexico

Single valued neutrosophic mappings defined by single valued neutrosophic relations with applications

Abdelkrim Latreche[1], Omar Barkat[2], Soheyb Milles[2,*] and Farhan Ismail[3]

[1]Department of Technology, Faculty of Technology, University of Skikda, Algeria; a.latreche@univ-skikda.dz

[2]Laboratory of Pure and Applied Mathematics, University of Msila, Algeria; omar.bark@gmail.com

[3]Faculty of Technology, Sakarya University, Turkey; farhanismail@subu.edu.tr

*Correspondence: soheyb.milles@univ-msila.dz; Tel.: +213664081002

Abstract: In this paper, we introduce the notion of single valued neutrosophic mapping defined by single valued neutrosophic relation which is considered as a generalization of fuzzy mapping defined by fuzzy relation and several properties related to this notion are studied. Moreover, we generalize the notion of fuzzy topology on fuzzy sets introduced by Kandil et al. to the setting of single valued neutrosophic sets. As applications, we establish the property of continuity in single valued neutrosophic topological space and investigate relationships among various types of single valued neutrosophic continuous mapping.

Keywords: Single valued neutrosophic set; Binary relation; Mapping; Topology; Continuous mapping.

1 Introduction

It is a well-known fact by now that mappings in crisp set theory are among the oldest acquaintances of modern mathematics and, play an important role in many mathematical branches (both pure and applied), as well as in topology and its analysis approaches. The uses of mappings appear also in formal logic [13], category theory [35], graph theory [11], group theory [6] and in computer science [31]. In general, it was and still more common.

In fuzzy setting, the concept of fuzzy mapping has received far attention. It has appeared in many papers, for instance, S. Heilpern [12] introduced this concept and proved a fixed point theorem for fuzzy contraction mappings. In [17], S. Lou and L. Cheng proved that fuzzy controllers can be regarded as a fuzzy mapping from the set of linguistic variables describing the observed object to that of linguistic variables describing the controlled objects. Thereafter, Lim et al. [18] investigated the equivalence relations and mappings for fuzzy sets and relationship between them. Ismail and Massa'deh [9] defined L-fuzzy mappings and studied their operations, also they developed many properties of classical mappings into L-fuzzy case. For the study of fuzzy continuous mappings in fuzzy topological space, an extended approaches are proposed, R.N. Bhaumik and M.N Mukherjee [5] investigated some properties of fuzzy completely continuous mapping. Mukherjee and B. Ghosh [27] pay attention to the introduction and studying of the concepts of certain classes of mappings between fuzzy topological spaces. Each of these mappings presents a stronger form of the fuzzy continuous mappings. In this regard, we find that other authors also contributed a lot to this field, like M. K. Single and A. R. Single [36], B. Ahmed [1] and M. K. Mishra et al. [26].

In [3], Attanassov introduced the concept of intuitionistic fuzzy set which is an extension of fuzzy set, characterized by a membership (truth-membership) function and a non-membership (falsity-membership) function for the elements of a universe X. Moreover, there is a restriction that the sum of both values is less and equal to one. Recently, F. Smarandache [32] generalized the Atanassov's intuitionistic fuzzy sets and other types of sets to the notion of neutrosophic sets. He introduced this concept to deal with imprecise and indeterminate data. Neutrosophic sets are characterized by truth membership function (T), indeterminacy membership function (I) and falsity membership function (F). Many researchers have studied and applied in different fields the neutrosophic sets and its various extensions such as decision making problems (e.g. [39, 41]), image processing (e.g. [8, 44]), educational problem (e.g. [25]), conflict resolution (e.g. [28]), social problems (e.g. [29, 24]), medical diagnosis (e.g. [22, 40, 42]), supply chain management (e.g. [20]), construction projects (e.g. [21]) and to address the conditions of uncertainty and inconsistency (e.g. [23]) and others. In particular, to exercise neutrosophic sets in real life applications suitably, Wang et al. [37] introduced the concept of single valued neutrosophic set as a subclass of a neutrosophic set, and investigated some of its properties. Very recently, Kim et al. [15] studied a single valued neutrosophic (relation/ transitive closure/ equivalence relation class/ partition). The studies, whether theoretical or applied on single valued neutrosophic set have been progressing rapidly. For instance, [2, 7, 14] and more others.

Motivated by recent developments relating to this framework, in this paper, we introduce the notion of single valued neutrosophic mapping defined by single valued neutrosophic relation as a generalization of fuzzy mappings introduced by Ismail and Massa'deh [9] and many properties related to this notion are studied. Also, we generalize the notion of fuzzy topology on fuzzy sets introduced by A. Kandil et al. [16] to the setting of single valued neutrosophic sets to establish the continuity property of single valued neutrosophic mapping. To that end, we investigate relation among various types of single valued neutrosophic continuous mappings.

The contents of the paper are organized as follows. In Section 2, we recall the necessary basic concepts and properties of single valued neutrosophic sets, single valued neutrosophic relations and some related notions that will be needed throughout this paper. In Section 3, the notion of single valued neutrosophic mapping defined by single valued neutrosophic relation is introduced and some properties related to this notion are studied. In Section 4, we establish as an application the single valued neutrosophic continuous mapping in single valued neutrosophic topological space and relationships between various types of single valued neutrosophic continuous mapping are explained. Finally, we present some conclusions and discuss future research in Section 5.

2 Preliminaries

This section contains the basic definitions and properties of single valued neutrosophic sets and some related notions that will be needed throughout this paper.

2.1 Single valued neutrosophic sets

The notion of fuzzy sets was first introduced by Zadeh [43].

Definition 2.1. [43] Let X be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0, 1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for any $x \in X$.

In 1983, Atanassov [3] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 2.2. [3] Let X be a nonempty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$ which satisfy the condition:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \text{ for any } x \in X.$$

In 1998, Smarandache [32] defined the concept of a neutrosophic set as a generalization of Atanassov's intuitionistic fuzzy set. Also, he introduced neutrosophic logic, neutrosophic set and its applications in [33, 34]. In particular, Wang et al. [37] introduced the notion of a single valued neutrosophic set.

Definition 2.3. [33] Let X be a nonempty set. A neutrosophic set (NS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to]^-0, 1^+[$ and an indeterminacy function $\sigma_A : X \to]^-0, 1^+[$ and a non-membership function $\nu_A : X \to]^-0, 1^+[$ which satisfy the condition:

$$^-0 \leq \mu_A(x) + \sigma_A(x) + \nu_A(x) \leq 3^+, \text{ for any } x \in X.$$

Certainly, intuitionistic fuzzy sets are neutrosophic sets by setting $\sigma_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Next, we show the notion of single valued neutrosophic set as an instance of neutrosophic set which can be used in real scientific and engineering applications.

Definition 2.4. [37] Let X be a nonempty set. A single valued neutrosophic set (SVNS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a truth-membership function $\mu_A : X \to [0,1]$, an indeterminacy-membership function $\sigma_A : X \to [0,1]$ and a falsity-membership function $\nu_A : X \to [0,1]$.

The class of single valued neutrosophic sets on X is denoted by $SVN(X)$.

For any two SVNSs A and B on a set X, several operations are defined (see, e.g., [37, 38]). Here we will present only those which are related to the present paper.

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$,

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\sigma_A(x) = \sigma_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$,

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$,

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$,

(v) $\overline{A} = \{\langle x, 1 - \nu_A(x), 1 - \sigma_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$,

(vi) $[A] = \{\langle x, \mu_A(x), \sigma_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$,

(vii) $\langle A\rangle = \{\langle x, 1 - \nu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$.

In the sequel, we need the following definition of level sets (which is also often called (α, β, γ)-cuts) of a single valued neutrosophic set.

Definition 2.5. [2] Let A be a single valued neutrosophic set on a set X. The (α, β, γ)-cut of A is a crisp subset

$$A_{\alpha,\beta,\gamma} = \{x \in X \mid \mu_A(x) \geq \alpha \text{ and } \sigma_A(x) \geq \beta \text{ and } \nu_A(x) \leq \gamma\},$$

where $\alpha, \beta, \gamma \in]0, 1]$.

Definition 2.6. [2] Let A be a single valued neutrosophic set on a set X. The support of A is the crisp subset on X given by

$$Supp(A) = \{x \in X \mid \mu_A(x) \neq 0 \text{ and } \sigma_A(x) \neq 0 \text{ and } \nu_A(x) \neq 0\}.$$

2.2 Single valued neutrosophic relations

Kim et al. [15] introduced the concept of single valued neutrosophic relation as a natural generalization of fuzzy and intuitionistic fuzzy relation.

Definition 2.7. [15] A single valued neutrosophic binary relation (A single valued neutrosophic relation, for short) from a universe X to a universe Y is a single valued neutrosophic subset in $X \times Y$, i.e., is an expression R given by

$$R = \{\langle (x, y), \mu_R(x, y), \sigma_R(x, y), \nu_R(x, y)\rangle \mid (x, y) \in X \times Y\},$$

where $\mu_R : X \times Y \to [0, 1]$, and $\sigma_R : X \times Y \to [0, 1]$, and $\nu_A : X \times Y \to [0, 1]$.
For any $(x, y) \in X \times Y$. The value $\mu_R(x, y)$ is called the degree of a membership of (x, y) in R, $\sigma_R(x, y)$ is called the degree of indeterminacy of (x, y) in R and $\nu_R(x, y)$ is called the degree of non-membership of (x, y) in R.

Example 2.8. Let $X = \{a, b, c, d, e\}$. Then the single valued neutrosophic relation R defined on X by

$$R = \{\langle (x, y), \mu_R(x, y), \sigma_R(x, y), \nu_R(x, y)\rangle \mid x, y \in X\},$$

where μ_R, σ_R and ν_R are given by the following tables:

$\mu_R(.,.)$	a	b	c	d	e
a	0.35	0	0	0.35	0.30
b	0	0.40	0	0.35	0.45
c	0.20	0	0.65	0	0.70
d	0	0	0	1	0
e	0.25	0.35	0	0	0.60

$\sigma_R(.,.)$	a	b	c	d	e
a	0.5	0.5	0.42	0.2	0
b	0.60	0.12	0.40	0.80	0.10
c	0	1	0.02	0.75	0.15
d	0.33	1	0.88	0	0.10
e	0.20	0.55	1	0.55	0.30

$\nu_R(.,.)$	a	b	c	d	e
a	0	1	0.40	0.25	0.25
b	0.30	0.35	0.20	0.35	0.10
c	0.80	1	0	0.85	0.15
d	1	1	1	0	1
e	0.70	0.55	1	0.90	0.30

Next, the following definitions is needed to recall.

Definition 2.9. [30] Let R and P be two single valued neutrosophic relations from a universe X to a universe Y.

(i) The transpose (inverse) R^t of R is the single valued neutrosophic relation from the universe Y to the universe X defined by

$$R^t = \{\langle (x,y), \mu_{R^t}(x,y), \sigma_{R^t}(x,y), \nu_{R^t}(x,y)\rangle \mid (x,y) \in X \times Y\},$$

where

$$\begin{cases} \mu_{R^t}(x,y) = \mu_R(y,x) \\ \quad \text{and} \\ \sigma_{R^t}(x,y) = \sigma_R(y,x) \\ \quad \text{and} \\ \nu_{R^t}(x,y) = \nu_R(y,x)\,, \end{cases}$$

for any $(x,y) \in X \times Y$.

(ii) R is said to be contained in P or we say that P contains R, denoted by $R \subseteq P$, if for all $(x,y) \in X \times Y$ it holds that $\mu_R(x,y) \leq \mu_P(x,y)$, $\sigma_R(x,y) \leq \sigma_P(x,y)$ and $\nu_R(x,y) \geq \nu_P(x,y)$.

(iii) The intersection (resp. *the union*) of two single valued neutrosophic relations R and P from a universe X to a universe Y is a single valued neutrosophic relation defined as

$$\begin{aligned} R \cap P \;=\; & \{\langle (x,y), \min(\mu_R(x,y), \mu_P(x,y)), \min(\sigma_R(x,y), \sigma_P(x,y)), \max(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \\ & \in X \times Y\} \end{aligned}$$

and

$$R \cup P = \{\langle (x,y), \max(\mu_R(x,y), \max(\sigma_R(x,y), \sigma_P(x,y)), \min(\nu_R(x,y), \nu_P(x,y))\rangle \mid (x,y) \in X \times Y\}\,.$$

Definition 2.10. [30, 38] Let R be a single valued neutrosophic relation from a universe X into itself.

(i) *Reflexivity*: $\mu_R(x,x) = \sigma_R(x,x) = 1$ and $\nu_R(x,x) = 0$, for any $x \in X$.

(ii) *Symmetry*: for any $x, y \in X$ then

$$\begin{cases} \mu_R(x,y) = \mu_R(y,x) \\ \sigma_R(x,y) = \sigma_R(y,x) \\ \nu_R(x,y) = \nu_R(y,x) \end{cases},$$

(iii) *Antisymmetry*: for any $x, y \in X$, $x \neq y$ then

$$\begin{cases} \mu_R(x,y) \neq \mu_R(y,x) \\ \sigma_R(x,y) \neq \sigma_R(y,x) \\ \nu_R(x,y) \neq \nu_R(y,x) \end{cases},$$

(iv) *Transitivity*: $R \circ R \subset R$ i.e., $R^2 \subset R$.

3 Single valued neutrosophic mappings defined by single valued neutrosophic relations

In this section, we generalize the notion of fuzzy mapping defined by fuzzy relation introduced by Ismail and Massa'deh [9] to the setting of single valued neutrosophic sets. Also, the main properties related to single valued neutrosophic mapping are studied.

Definition 3.1. Let A be a single valued neutrosophic set on X and B be a single valued neutrosophic set on Y, let $f : Supp\ A \to Supp\ B$ be an ordinary mapping and R be a single valued neutrosophic relation on $X \times Y$. Then f_R is called a single valued neutrosophic mapping if for all $(x, y) \in Supp\ A \times Supp\ B$ the following condition is satisfied:

$$\mu_R(x,y) = \begin{cases} min(\mu_A(x), \mu_B(f(x))) \ , \ if\ y = f(x) \\ 0\ , \ Otherwise\, , \end{cases}$$

and

$$\sigma_R(x,y) = \begin{cases} min(\sigma_A(x), \sigma_B(f(x))) \ , \ if\ y = f(x) \\ 0\ , \ Otherwise\, , \end{cases}$$

and

$$\nu_R(x,y) = \begin{cases} max(\nu_A(x), \nu_B(f(x))) \ , \ if\ y = f(x) \\ 1\ , \ Otherwise\, , \end{cases}$$

Example 3.2. Let $X = \{\alpha, \beta, \gamma\}$, $Y = \{a, b, c\}$, $A \in SVNS(X)$ and $B \in SVNS(Y)$ given by

$$A = \{\langle \alpha, 0.5, 0.2, 0.8\rangle, \langle \beta, 0.1, 0.7, 0.3\rangle, \langle \gamma, 0, 0.9, 1\rangle\}$$

$$B = \{\langle a, 0, 1, 0.3\rangle, \langle b, 0.1, 0.5, 0.2\rangle, \langle c, 0.7, 0.2, 0.4\rangle\}.$$

We will construct the single valued neutrosophic mapping f_R by :

(i) an ordinary mapping $f : \{\alpha, \beta\} \to \{b, c\}$ such that $f(\alpha) = b$ and $f(\beta) = c$,

(ii) a single valued neutrosophic relation R defined by :

$\mu_R(\alpha, f(\alpha)) = \mu_R(\alpha, b) = \mu_A(\alpha) \wedge \mu_B(b) = 0.1$

$\mu_R(\beta, f(\beta)) = \mu_R(\beta, c) = \mu_A(\beta) \wedge \mu_B(c) = 0.1$

$\mu_R(\alpha, a) = \mu_R(\alpha, c) = \mu_R(\beta, a) = \mu_R(\beta, b) = \mu_R(\gamma, a) = \mu_R(\gamma, b) = \mu_R(\gamma, c) = 0$

In similar way, it holds that

$\sigma_R(\alpha, f(\alpha)) = \sigma_R(\alpha, b) = \sigma_A(\alpha) \wedge \sigma_B(b) = 0.2$
$\sigma_R(\beta, f(\beta)) = \sigma_R(\beta, c) = \sigma_A(\beta) \wedge \sigma_B(c) = 0.2$
$\sigma_R(\alpha, a) = \sigma_R(\alpha, c) = \sigma_R(\beta, a) = \sigma_R(\beta, b) = \sigma_R(\gamma, a) = \sigma_R(\gamma, b) = \sigma_R(\gamma, c) = 0$
and
$\nu_R(\alpha, f(\alpha)) = \nu_R(\alpha, b) = \nu_A(\alpha) \vee \nu_B(b) = 0.8$
$\nu_R(\beta, f(\beta)) = \nu_R(\beta, c) = \nu_A(\beta) \vee \nu_B(c) = 0.4$
$\nu_R(\alpha, a) = \nu_R(\alpha, c) = \nu_{R_I}(\beta, a) = \nu_{R_I}(\beta, b) = \sigma_R(\gamma, a) = \sigma_R(\gamma, b) = \sigma_R(\gamma, c) = 1.$
Hence, $\mu_R(x, y) = \{\langle(\alpha, f(\alpha)), 0.1, 0.2, 0.8\rangle, \langle(\beta, f(\beta)), 0.1, 0.2, 0.4\rangle, \langle(\alpha, a), 0, 0, 1\rangle,$
$\langle(\alpha, c), 0, 0, 1\rangle, \langle(\beta, a), 0, 0, 1\rangle, \langle(\beta, b), 0, 0, 1\rangle, \langle(\gamma, a), 0, 0, 1\rangle, \langle(\gamma, b), 0, 0, 1\rangle, \langle(\gamma, c), 0, 0, 1\rangle\}.$

Thus, f_R is a single valued neutrosophic mapping.

Example 3.3. Let $X = \mathbb{Q}$, $Y = \mathbb{R}$, $A \in SVNS(X)$ and $B \in SVNS(Y)$ given by:
$\mu_A(x) = 0.3$, $\sigma_A(x) = 0.25$ and $\nu_A(x) = 0.5$, for any $x \in \mathbb{Q}$.
$\mu_B(x) = \sigma_B(x) = \nu_B(x) = 0.5$, for any $x \in \mathbb{R}$.
We will construct the single valued neutrosophic mapping f_R by :

(i) an ordinary mapping $f : \mathbb{Q} \to \mathbb{R}$ such that $f(x) = x^2$,

(ii) a single valued neutrosophic relation R defined by :

$\mu_R(x, f(x)) = \mu_R(x, x^2) = \mu_A(x) \wedge \mu_B(x^2) = 0.3$
$\sigma_R(x, f(x)) = \sigma_R(x, x^2) = \sigma_A(x) \wedge \mu_B(x^2) = 0.25$
$\nu_R(x, f(x)) = \nu_R(x, x^2) = \nu_A(x) \vee \nu_B(x^2) = 0.5$

Thus, f_R is a single valued neutrosophic mapping.

Remark 3.4. From the above definition, we can construct the single valued neutrosophic mapping by this method

(i) We determine the $Supp\ A$ and $Supp\ B$.

(ii) We determine the ordinary mapping from $Supp\ A$ to $Supp\ B$.

(iii) We determine the single valued neutrosophic relation by its membership function, indeterminacy function and non-membership function.

(iv) Finally, we conclude the construction of the single valued neutrosophic mapping.

Definition 3.5. Let f_R, g_S be two single valued neutrosophic mappings, then f_R and g_S are equal if and only if $f = g$ and $R = S$ i.e., $(\mu_R(x, f(x)) = \mu_S(x, g(x)),\ \sigma_R(x, f(x)) = \sigma_S(x, g(x))$, and $\nu_R(x, f(x)) = \nu_S(x, g(x)))$.

Definition 3.6. Let A be a single valued neutrosophic set on X, let $f : Supp\ A \to Supp\ A$ be an ordinary mapping such that $f(x) = x$ and R be a single valued neutrosophic relation on $X \times X$. Then f_R is called a single valued neutrosophic identity mapping if for all $x, y \in Supp\ A$ the following conditions are satisfied:

$$\mu_R(x,y) = \begin{cases} \mu_A(x)\ ,\ if\ x = y \\ 0\ ,\ Otherwise\ , \end{cases}$$

and

$$\sigma_R(x,y) = \begin{cases} \sigma_A(x)\ ,\ if\ x = y \\ 0\ ,\ Otherwise\ , \end{cases}$$

and

$$\nu_R(x,y) = \begin{cases} \nu_A(x)\ ,\ if\ x = y \\ 1\ ,\ Otherwise\ , \end{cases}$$

Definition 3.7. Let A, B and C are a single valued neutrosophic sets on X, Y and Z respectively, let f : $Supp\ A \to Supp\ B$ and g : $Supp\ B \to Supp\ C$ are an ordinary mappings and R, S are a single valued neutrosophic relations on $X \times Y$ and $Y \times Z$ respectively. Then $(g \circ f)_T$ is called the composition of single valued neutrosophic mappings f_R and g_R such that $g \circ f$: $Supp\ A \to Supp\ C$ and the single valued neutrosophic relation T is defined by

$$\begin{cases} \mu_T(x,z) = \ sup_y(min(\mu_R(x,y), \mu_S(y,z))) \\ \text{and} \\ \sigma_T(x,z) = \ sup_y(min(\sigma_R(x,y), \sigma_S(y,z))) \\ \text{and} \\ \nu_T(x,z) = inf_y(max(\nu_R(x,y), \nu_S(y,z)))\ , \end{cases}$$

for any $(x,z) \in Supp\ A \times Supp\ C$.

Example 3.8. Let $X = \mathbb{N}$, $Y = \mathbb{R}$ and $Z = \mathbb{R}$, and let $A \in SVNS(X)$, $B \in SVNS(Y)$ and $C \in SVNS(Z)$, defined as follows :

$\mu_A(n) = \sigma_A(n) = \frac{1}{1+n}$ and $\nu_A(n) = \frac{n}{2+2n}$, for any $n \in \mathbb{N}$.

$\mu_B(x) = \sigma_B(x) = \begin{cases} 0.25\ ,\ if\ x \in [-1,1] \\ 0\ ,\ Otherwise\ , \end{cases}$ and $\nu_B(x) = \begin{cases} 0.5\ ,\ if\ x \in [-1,1] \\ 1\ ,\ Otherwise\ , \end{cases}$

$\mu_C(x) = \sigma_C(x) = \frac{|cos(x)|}{3}$ and $\nu_C(x) = \frac{|sin(x)|}{3}$, for any $x \in \mathbb{R}$.

We define a single valued neutrosophic mappings $f_R : A \to B$ and $g_S : B \to C$ by :

(i) an ordinary mappings f : $Supp\ A \longrightarrow Supp\ B$, defined for any $n \in Supp\ A$ by :

$f(n) = \begin{cases} 1\ ,\ if\ n\ is\ an\ even\ number, \\ -1\ , if\ n\ is\ an\ odd\ number\ , \end{cases}$

and g : $Supp\ B \longrightarrow Supp\ C$ defined by $g(x) = 2x$, for any $x \in [-1,1]$.

(ii) a single valued neutrosophic relations R and S defined by :

$\mu_R(n, f(n)) = \sigma_R(n, f(n)) = \wedge\{\mu_A(n),\ \mu_B(f(n))\} = \wedge\{\frac{1}{1+n},\ 0.25\}$,

$\nu_R(n, f(n)) = \vee\{\nu_A(n),\ \nu_B(f(n))\} = \vee\{\frac{n}{2+2n},\ 0.5\}$ and

$\mu_S(x, g(x)) = \sigma_S(x, g(x)) = \wedge\{\mu_B(x),\ \mu_C(g(x))\} = \begin{cases} \wedge\{0.25,\ \frac{|cos(2x)|}{3}\}\ ,\ x \in [-1,1], \\ 0\ ,\ otherwise\ , \end{cases}$

and $\nu_S(x, g(x)) = \vee\{\nu_B(x),\ \nu_C(g(x))\} = \begin{cases} \vee\{0.5,\ \frac{|sin(2x)|}{3}\}\ ,\ x \in [-1,1], \\ 1\ ,\ otherwise. \end{cases}$

Then, the composition $g_S \circ f_R = (g \circ f)_T$ is defined by :

(i) an ordinary mapping $f : Supp\ A \longrightarrow Supp\ C$, defined for any $n \in Supp\ A$ by :

$$(g \circ f)(n) = \begin{cases} 2\ ,\ if\ n\ is\ an\ even\ number, \\ -2\ ,\ if\ n\ is\ an\ odd\ number\ , \end{cases}$$

(ii) a single valued neutrosophic relation T defined by :

$$\begin{aligned} \mu_T(n,(g\circ f)(n)) = \sigma_T(n,(g\circ f)(n)) &= \begin{cases} \wedge\{\frac{1}{1+n}, 0.25, \frac{|cos(2)|}{3}\}\ ,\ if\ n\ is\ an\ even\ number \\ \wedge\{\frac{1}{1+n}, 0.25, \frac{|cos(-2)|}{3}\}\ ,\ if\ n\ is\ an\ odd\ number \end{cases} \\ &= \wedge\{\frac{1}{1+n}, 0.25, \frac{|\ cos(2)\ |}{3}\} \\ &= \wedge\{\frac{1}{1+n}, 0.25\}, \end{aligned}$$

$$\begin{aligned} \nu_T(n,(g\circ f)(n)) &= \begin{cases} \vee\{\frac{n}{2+2n}, 0.25, \frac{|sin(2)|}{3}\}\ ,\ if\ n\ is\ an\ even\ number \\ \vee\{\frac{n}{2+2n}, 0.25, \frac{|sin(-2)|}{3}\}\ ,\ if\ n\ is\ an\ odd\ number \end{cases} \\ &= \vee\{\frac{n}{2+2n}, 0.25, \frac{|\ sin(2)\ |}{3}\} \\ &= \vee\{\frac{2}{2+2n}, 0.25\}. \end{aligned}$$

Remark 3.9. The single valued neutrosophic identity mapping Id_R is neutral for the composition of single valued neutrosophic mappings.

In the sequel, we need to introduce the notion of the direct image and the inverse image of a single valued neutrosophic set by a single valued neutrosophic mapping.

Definition 3.10. Let $f_R : A \to B$ be a single valued neutrosophic mapping from a single valued neutrosophic set A to another single valued neutrosophic set B and $C \subseteq A$. The direct image of C by f_R is defined by $f_R(C) = \{\langle y, \mu_{f_R(C)}(y), \sigma_{f_R(C)}(y), \nu_{f_R(C)}(y)\rangle \mid y \in Y\}$, where

$$\mu_{f_R(C)}(y) = \begin{cases} \mu_B(y)\ ,\ if\ y \in f(supp(C)) \\ 0\ ,\ \ Otherwise\ , \end{cases}$$

and

$$\sigma_{f_R(C)}(y) = \begin{cases} \sigma_B(y)\ ,\ if\ y \in f(supp(C)) \\ 0\ ,\ \ Otherwise\ , \end{cases}$$

and

$$\nu_{f_R(C)}(y) = \begin{cases} \nu_B(y)\ ,\ if\ y \in f(supp(C)) \\ 1\ ,\ \ Otherwise. \end{cases}$$

Similarly, if $C' \subseteq B$. The inverse image of C' by f is defined by

$$f_R^{-1}(C') = \{\langle x, \mu_{f_R^{-1}(C')}(x), \sigma_{f_R^{-1}(C')}(x), \nu_{f_R^{-1}(C')}(x)\rangle \mid x \in X\},$$

where

$$\mu_{f_R^{-1}(C')}(x) = \begin{cases} \mu_A(x)\ ,\ if\ x \in f^{-1}(supp(C')) \\ 0\ ,\ Otherwise\ , \end{cases}$$

and

$$\sigma_{f_R^{-1}(C')}(x) = \begin{cases} \sigma_A(x)\ ,\ if\ x \in f^{-1}(supp(C')) \\ 0\ ,\ Otherwise\ , \end{cases}$$

and

$$\nu_{f_R^{-1}(C')}(x) = \begin{cases} \nu_A(x)\ ,\ if\ x \in f^{-1}(supp(C')) \\ 1\ ,\ Otherwise. \end{cases}$$

Example 3.11. Let $X = [0, +\infty[$, $Y = \mathbb{R}$ and $A \in SVNS(X)$ defined for any $x \in X$ by :

$$\mu_A(x) = \sigma_A(x) = \begin{cases} cos(x)\ ,\ if\ x \in [0, \frac{\pi}{2}] \\ 0\ ,\ Otherwise\ , \end{cases} \qquad \nu_A(x) = \begin{cases} 0.9\ ,\ if\ x \in [0, \frac{\pi}{2}] \\ 1\ ,\ Otherwise. \end{cases}$$

Also, let $B \in SVNS(Y)$ given by :

$$\mu_B(y) = \sigma_B(y) = \begin{cases} y\ ,\ if\ y \in [0, 1] \\ 0\ ,\ Otherwise\ , \end{cases} \qquad \nu_B(y) = \begin{cases} 0.2\ ,\ if\ y \in [0, 1] \\ 1\ ,\ Otherwise. \end{cases}$$

We define the single valued neutrosophic mapping $f_R : A \to B$ by:

(i) an ordinary mapping $f : Supp\ A \longrightarrow Supp\ B$, defined for any $x \in [0, \frac{\pi}{2}]$ by $f(x) = \frac{x}{4}$.

(ii) a single valued neutrosophic relation R defined by $\mu_R(x, f(x)) = \sigma_R(x, f(x)) = \mu_A(x) \wedge \mu_B(f(x)) = cos(x) \wedge \frac{1}{4}x$ and $\nu_R(x, f(x)) = \nu_A(x) \vee \nu_B(f(x)) = 0.9$

Now, if we take C an SVNS on X, where $C \subseteq A$ given by :

$$\mu_C(x) = \sigma_C(x) = \begin{cases} -x + 1\ ,\ if\ x \in [0, \frac{1}{2}] \\ 0\ ,\ Otherwise\ , \end{cases} \qquad \nu_C(x) = \begin{cases} 0.99\ ,\ if\ y \in [0, \frac{1}{2}] \\ 1\ ,\ Otherwise\ , \end{cases}$$

Then, the direct image of C by f_R is defined by :

$$\mu_{f_R(C)}(y) = \begin{cases} \mu_B(y)\ ,\ if\ y \in f(supp(C)) \\ 0\ ,\ Otherwise\ , \end{cases} = \begin{cases} y\ ,\ if\ y \in [0, \frac{1}{8}] \\ 0\ ,\ Otherwise\ , \end{cases}$$

$$\sigma_{f_R(C)}(y) = \begin{cases} \sigma_B(y)\ ,\ if\ y \in f(supp(C)) \\ 0\ ,\ Otherwise\ , \end{cases} = \begin{cases} y\ ,\ if\ y \in [0, \frac{1}{8}] \\ 0\ ,\ Otherwise\ , \end{cases}$$

and

$$\nu_{f_R(C)}(y) = \begin{cases} \nu_B(y)\ ,\ if\ y \in f(supp(C)) \\ 0\ ,\ Otherwise\ , \end{cases} = \begin{cases} 0.2\ ,\ if\ y \in [0, \frac{1}{8}] \\ 1\ ,\ Otherwise. \end{cases}$$

Moreover, it is easy to show that $f_R(C) \subseteq B$.

Next, if we take C' an SVNS on Y, where $C' \subseteq B$ given by :

$$\mu_{C'}(y) = \sigma_{C'}(y) = \begin{cases} sin(y)\ ,\ if\ y \in [0, \frac{1}{3}] \\ 0\ ,\ Otherwise\ , \end{cases} \qquad \nu_{C'}(y) = \begin{cases} 0.4\ ,\ if\ y \in [0, \frac{1}{3}] \\ 1\ ,\ Otherwise\ , \end{cases}$$

Then, the inverse image of C' by f is defined by :

$$\mu_{f_R^{-1}(C')}(x) = \begin{cases} \mu_A(x)\ ,\ if\ x \in f^{-1}(supp(C')) \\ 0\ ,\ Otherwise\ , \end{cases} = \begin{cases} cos(x)\ ,\ if\ x \in [0, \frac{4}{3}] \\ 0\ ,\ Otherwise\ , \end{cases}$$

$$\sigma_{f_R^{-1}(C')}(x) = \begin{cases} \sigma_A(x)\ ,\ if\ x \in f^{-1}(supp(C')) \\ 0\ ,\ Otherwise\ , \end{cases} = \begin{cases} cos(x)\ ,\ if\ x \in [0, \frac{4}{3}] \\ 0\ ,\ Otherwise\ , \end{cases}$$

and $\nu_{f_R^{-1}(C')}(x) = \begin{cases} \nu_A(x)\ ,\ if\ x \in f^{-1}(supp(C')) \\ 1\ ,\ Otherwise\ , \end{cases} = \begin{cases} 0.9\ ,\ if\ x \in [0, \frac{4}{3}] \\ 1\ ,\ Otherwise. \end{cases}$

Moreover, it is easy to show that $f_R^{-1}(C') \subseteq A$.

Now, we introduce the product of single valued neutrosophic sets and single valued neutrosophic projection mappings.

Definition 3.12. Let A be a single valued neutrosophic set on X and B be a single valued neutrosophic set on Y. The product of A and B, denoted by $A \times B$ is a single valued neutrosophic set on $X \times Y$ defined by :
$\mu_{X\times Y}(x,y) = min\{\mu_A(x),\, \mu_B(y)\}$, $\sigma_{X\times Y}(x,y) = min\{\sigma_A(x),\, \sigma_B(y)\}$, $\nu_{X\times Y}(x,y) = max\{\nu_A(x),\, \nu_B(y)\}$.
Also, we introduce the first single valued neutrosophic projection mapping $(P_1)_R : A \times B \longrightarrow A$ by:

(i) an ordinary mapping $P_1 : Supp(A\times B) \longrightarrow Supp(A)$ such that $P_1(x,y) = x$ for any $(x,y) \in Supp(A\times B)$,

(i) a single valued neutrosophic relation R defined by :

$$\begin{aligned} \mu_R((x,y),P_1(x,y)) &= min\{\mu_{A\times B}(x,y),\ \mu_A(P_1(x,y))\}\} \\ &= min\{\mu_A(x),\, \mu_B(y),\ \mu_A(x)\}\} \\ &= min\{\mu_A(x),\ \mu_B(y)\} \end{aligned}$$

and

$$\begin{aligned} \sigma_R((x,y),P_1(x,y)) &= min\{\sigma_{A\times B}(x,y),\ \sigma_A(P_1(x,y))\}\} \\ &= min\{\sigma_A(x),\, \sigma_B(y),\ \sigma_A(x)\}\} \\ &= min\{\sigma_A(x),\ \sigma_B(y)\} \end{aligned}$$

and

$$\begin{aligned} \nu_R((x,y),P_1(x,y)) &= max\{\nu_{A\times B}(x,y),\ \nu_A(P_1(x,y))\}\} \\ &= max\{\nu_A(x),\, \nu_B(y),\ \nu_A(x)\}\} \\ &= max\{\nu_A(x),\ \nu_B(y)\} \end{aligned}$$

The second single valued neutrosophic projection mapping is defined analogously.

4 Continuity property in single valued neutrosophic topological space

The aim of the present section, is to introduce and study the notion of single valued neutrosophic continuous mapping in single valued neutrosophic topological spaces. The basic properties, and relationships with some types of continuity are also obtained.

4.1 Single valued neutrosophic topology

In this subsection, we generalize the notion of fuzzy topology on fuzzy sets introduced by Kandil et al. [16] to the setting of single valued neutrosophic sets to establish the continuity property of single valued neutrosophic mapping.

Definition 4.1. Let A be a single valued neutrosophic set on the set X and $O_A = \{U$ is an SVNS on X : $U \subseteq A\}$. We define a single valued neutrosophic topology on single valued neutrosophic set A by the family $T \subseteq O_A$ which satisfies the following conditions :

(i) $A,\ 0_\sim \in T$;

(ii) if $U_1,\ U_2\ \in T$, then $U_1 \cap U_2 \in T$;

(iii) if $U_i \in T$ for all $i \in I$, then $\cup_I U_i \in T$.

T is called a single valued neutrosophic topology of A and the pair (A, T) is a single valued neutrosophic topological space (SVN-TOP, for short). Every element of T is called a single valued neutrosophic open set (SVNOS, for short).

Example 4.2. Let $X = \mathcal{P}(\mathbb{R}^2)$ and $\alpha \in]0,1[$, A be a single valued neutrosophic set on X given by :

$$\mu_A(\theta) = \begin{cases} 1\,,\ if\ \theta = \emptyset \\ \alpha\,,\ \ 0 < |\theta| < \infty\,, \\ 0\,,\ \ Otherwise\,, \end{cases} \quad \sigma_A(\theta) = \begin{cases} 1\,,\ if\ \theta = \emptyset \\ \frac{\alpha}{2}\,,\ \ 0 < |\theta| < \infty\,, \\ 0\,,\ \ Otherwise\,, \end{cases} \quad \nu_A(\theta) = \begin{cases} 0\,,\ if\ \theta = \emptyset \\ 1-\alpha\,,\ \ 0 < |\theta| < \infty\,, \\ 0.5\,,\ \ Otherwise. \end{cases}$$

Then, the family $T = \{A, 0_\sim, U\}$ where:

$$\mu_U(\theta) = \begin{cases} \frac{\alpha}{3}\,,\ \ |\theta| < \infty\,, \\ 0\,,\ \ Otherwise\,, \end{cases} \quad \sigma_U(\theta) = \begin{cases} \frac{\alpha}{4}\,,\ \ |\theta| < \infty\,, \\ 0\,,\ \ Otherwise\,, \end{cases} \quad \nu_U(\theta) = \begin{cases} 1\,,\ \ |\theta| < \infty\,, \\ 0.8\,,\ \ Otherwise\,, \end{cases}$$

is a single valued neutrosophic topology on A.

Inspired by the notion of interior (resp. closure) on intuitionistic fuzzy topological space on a set introduced by Atanassov [4], we generalize these notions in single valued neutrosophic topology on a single valued neutrosophic set.

Definition 4.3. Let (A, T) be a single valued neutrosophic topological space, for every single valued neutrosophic subset G of X we define the interior and closure of G by:

$int(G) = \{\langle x, \max_{x\in X} \mu_U(x), \max_{x\in X} \sigma_U(x), \min_{x\in X} \nu_U(x)\rangle \mid x \in U \subseteq G\}$ and

$cl(G) = \{\langle x, \min_{x\in X} \mu_K(x), \min_{x\in X} \sigma_K(x), \max_{x\in X} \nu_K(x)\rangle \mid x \in A$ and $G \subseteq K\}$

Example 4.4. Let $X = \{a, b, c\}$ and $A, B, C, D \in SVNS(X)$ such that

$A = \{< a, 0.5, 0.7, 0.1 >, < b, 0.7, 0.9, 0.2 >, < c, 0.6, 0.8, 0 >\}$

$B = \{< a, 0.5, 0.6, 0.2 >, < b, 0.5, 0.6, 0.4 >, < c, 0.4, 0.5, 0.4 >\}$

$C = \{< a, 0.4, 0.5, 0.5 >, < b, 0.6, 0.7, 0.3 >, < c, 0.2, 0.3, 0.3 >\}$

$D = \{< a, 0.5, 0.6, 0.2 >, < b, 0.6, 0.7, 0.3 >, < c, 0.4, 0.5, 0.3 >\}$

$E = \{< a, 0.4, 0.5, 0.5 >, < b, 0.5, 0.6, 0.4 >, < c, 0.2, 0.3, 0.4 >\}$

Then the family $T = \{A,\ 0_\sim,\ B, C, D, E\}$ is an SVN-TOP of A.

Now, we suppose that $G \in SVNS(X)$ given by $G = \{< a, 0.41, 0.5, 0.6), < b, 0.3, 0.2, 0.6 >, < c, 0.2, 0.3, 0.7 >\}$. Then, $int(G) = 0_\sim$ and $cl(G) = \overline{E} \cap 1_\sim = \overline{E}$.

Definition 4.5. Let (A, T) be a single valued neutrosophic topological space and $U \in SVNS(A, T)$. Then U is called :

1. a single valued neutrosophic semiopen set (SVNSOS) if $U \subseteq cl(int(U))$;
2. a single valued neutrosophic α-open set (SVNαOS) if $U \subseteq int(cl(int(U)))$;
3. a single valued neutrosophic preopen set (SVNPOS) if $U \subseteq int(cl(U))$;
4. a single valued neutrosophic regular open set (SVNROS) if $U = int(cl(U))$.

4.2 Single valued neutrosophic continuous mappings

In this subsection, we will study some interesting properties of single valued neutrosophic continuous mappings in single valued neutrosophic topological space and relations between various types of single valued neutrosophic continuous mapping. First, we introduce the notion of single valued neutrosophic continuous mapping.

Definition 4.6. Let (A,T) (B,L) be two single valued neutrosophic topological spaces. The mapping f_R : $(A,T) \to (B,L)$ is a single valued neutrosophic continuous if and only if the inverse of each L-open single valued neutrosophic set is T-open single valued neutrosophic set.

Example 4.7. Let (A,T) and (B,T') be two single valued neutrosophic topological spaces, where $\mu_A(x) = 0.8$, $\sigma_A(x) = 0.88$ and $\nu_A(x) = 0.1$, for any $x \in \mathbb{R}_+$ and

$$\mu_B(y) = \begin{cases} 0.5\,, \ if\ y \geq 0 \\ 0.8\,,\ \ Otherwise\,, \end{cases} \quad \sigma_B(y) = \begin{cases} 0.88\,, \ if\ y \geq 0 \\ 0\,,\ \ Otherwise\,, \end{cases} \quad \nu_B(y) = \begin{cases} 0.1\,, \ if\ y \geq 0 \\ 0.3\,,\ \ Otherwise\,, \end{cases}$$

We suppose that $T = \{A, 0_\sim, U_1\}$, where

$$\mu_{U_1}(x) = \begin{cases} 0.8\,, \ if\ x \in [0,\sqrt{2}] \\ 0\,,\ \ Otherwise\,, \end{cases} \quad \sigma_{U_1}(x) = \begin{cases} 0.88\,, \ if\ x \in [0,\sqrt{2}] \\ 0\,,\ \ Otherwise\,, \end{cases} \quad \nu_{U_1}(x) = \begin{cases} 0.1\,, \ if\ x \in [0,\sqrt{2}] \\ 1\,,\ \ Otherwise\,, \end{cases}$$

Also, we suppose that $T' = \{B, 0_\sim, U_1'\}$, where

$$\mu_{U_1'}(y) = \begin{cases} 0.5\,, \ if\ y \in [0,2] \\ 0\,,\ \ Otherwise\,, \end{cases} \quad \sigma_{U_1'}(y) = \begin{cases} 0.8\,, \ if\ y \in [0,2] \\ 0\,,\ \ Otherwise\,, \end{cases} \quad \nu_{U_1'}(y) = \begin{cases} 0.2\,, \ if\ y \in [0,2] \\ 0.4\,,\ \ Otherwise. \end{cases}$$

Then, the single valued neutrosophic mapping $f_R : A \to B$ define by :

(i) an ordinary mapping $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $f(x) = x^2$, for any $x \in \mathbb{R}_+$,

(ii) a single valued neutrosophic relation R defined by :

$\mu_R(x, f(x)) = 0.5$ $\sigma_R(x, f(x)) = 0.88$ and $\nu_R(x, f(x)) = 0.1$.

is a single valued neutrosophic continuous mapping. Indeed, it is easy to show that $f_R^{-1}(B) = A$ and $f_R^{-1}(0_\sim) = 0_\sim$ and we have,

$$\begin{aligned} \mu_{f_R^{-1}(U_1')}(x) &= \begin{cases} \mu_A(x)\,, \ if\ x \in f^{-1}(supp(U_1')) \\ 0\,,\ \ Otherwise\,, \end{cases} \\ &= \begin{cases} 0.8\,, \ if\ x \in [0,\sqrt{2}] \\ 0\,,\ \ Otherwise\,, \end{cases} \\ &= \mu_{U_1}(x), \end{aligned}$$

$$\begin{aligned} \sigma_{f_R^{-1}(U_1')}(x) &= \begin{cases} \sigma_A(x)\,, \ if\ x \in f^{-1}(supp(U_1')) \\ 0\,,\ \ Otherwise\,, \end{cases} \\ &= \begin{cases} 0.88\,, \ if\ x \in [0,\sqrt{2}] \\ 0\,,\ \ Otherwise\,, \end{cases} \\ &= \sigma_{U_1}(x), \end{aligned}$$

and

$$\begin{aligned}\nu_{f_R^{-1}(U_1')}(x) &= \begin{cases} \nu_A(x)\ , \ if\ x \in f^{-1}(supp(U_1')) \\ 1\ , \ \ Otherwise\ , \end{cases} \\ &= \begin{cases} \nu_A(x)\ , \ if\ x \in [0,\sqrt{2}] \\ 1\ , \ \ Otherwise\ , \end{cases} \\ &= \begin{cases} 0.1\ , \ if\ x \in [0,\sqrt{2}] \\ 1\ , \ \ Otherwise\ , \end{cases} \\ &= \nu_{U_1}(x).\end{aligned}$$

Hence, $f_R^{-1}(U_1') = U_1 \in T$. Thus, f_R is a single valued neutrosophic continuous mapping.

Remark 4.8. Let (A, T) be a single valued neutrosophic topological space. Then the single valued neutrosophic identity mapping $Id_R : (A, T) \to (A, T)$ is a single valued neutrosophic continuous mapping.

Next, we provide the relationships between various types of single valued neutrosophic continuous mapping. First, we generalize the notions of precontinuous mapping, α-continuous mapping introduced by Güray et al. [10] to the setting of single valued neutrosophic sets.

Definition 4.9. Let $f_R : (A, T) \to (B, T')$ be a single valued neutrosophic mapping. Then f_R is called :

1. a single valued neutrosophic precontinuous mapping if $f_R^{-1}(U')$ is a SVNPOS on A for every SVNOS U' on B;
2. a single valued neutrosophic α-continuous mapping if $f_R^{-1}(U')$ is a SVNαOS on A for every SVNOS U' on B.

The following proposition shows the relationship between single valued neutrosophic continuous mapping and single valued neutrosophic α-continuous mapping.

Proposition 4.10. *Let $f_R : (A, T) \to (B, T')$ be a single valued neutrosophic mapping. If f_R is a single valued neutrosophic continuous mapping, then f_R is a single valued neutrosophic α-continuous mapping.*

Proof. Let U' be a SVNOS in B and we need to show that $f_R^{-1}(U')$ is an SVNαOS in A. The fact that f_R is a single valued neutrosophic continuous mapping implies that $f_R^{-1}(U')$ is a SVNOS in A. From Definition 3.10, it follows that

$$\mu_{f_R^{-1}(U')}(x) = \begin{cases} \mu_A(x)\ , \ if\ x \in f^{-1}(supp(U')) \\ 0\ , \ \ Otherwise\ , \end{cases} \qquad \sigma_{f_R^{-1}(U')}(x) = \begin{cases} \sigma_A(x)\ , \ if\ x \in f^{-1}(supp(U')) \\ 0\ , \ \ Otherwise\ , \end{cases}$$

and $\nu_{f_R^{-1}(U')}(x) = \begin{cases} \nu_A(x)\ , \ if\ x \in f^{-1}(supp(U')) \\ 1\ , \ \ Otherwise. \end{cases}$

We conclude that, $f_R^{-1}(U')$ is a SVNαOS in A. Hence, f_R is a single valued neutrosophic α-continuous mapping. □

Remark 4.11. The converse of the above implication is not necessarily holds. Indeed, let us consider the single valued neutrosophic mapping f_R given in Example 4.7 and T be a SVN-topology given by $T = \{0_\sim, A, U_1\}$, where: $\mu_A(x) = 1,\ \sigma_A(x) = 0.99,\ \nu_A(x) = 0.001$ and

$$\mu_{U_1}(x) = \begin{cases} 1\ , \ if\ x \in [0,1] \\ 0\ , \ \ Otherwise, \end{cases} \quad \sigma_{U_1}(x) = \begin{cases} 0.99\ , \ if\ x \in [0,1] \\ 0\ , \ \ Otherwise, \end{cases} \quad \nu_{U_1}(x) = \begin{cases} 0.001\ , \ if\ x \in [0,1] \\ 1\ , \ \ Otherwise. \end{cases}$$

Hence, $int(f_R^{-1}(U_1')) = U_1$, $cl(U_1) = 1_\sim$ and $int(1_\sim) = A$. Thus, $f_R^{-1}(U_1') \subseteq int(cl(int(f_R^{-1}(U_1'))))$. We conclude that $f_R^{-1}(U_1')$ is an SVNαS but not SVNOS and f_R is a single valued neutrosophic α-continuous mapping but not a single valued neutrosophic continuous mapping.

The following proposition shows the relationship between single valued neutrosophic α-continuous mapping and single valued neutrosophic pre-continuous mapping.

Proposition 4.12. *Let $f_R : (A,T) \to (B,T')$ be a single valued neutrosophic mapping. If f_R is a single valued neutrosophic α-continuous mapping, then f_R is a single valued neutrosophic pre-continuous mapping.*

Proof. Let U' be an SVNOS in B and we need to show that $f_R^{-1}(U')$ is a SVNPOS in A. The fact that f_R is a single valued neutrosophic α-continuous mapping implies that $f_R^{-1}(U')$ is a SVNαOS in A. From Definition 3.10, it follows that

$$\mu_{f_R^{-1}(U')}(x) = \begin{cases} \mu_A(x)\,,\ if\ x \in f^{-1}(supp(U')) \\ 0\,,\ Otherwise\,, \end{cases} \qquad \sigma_{f_R^{-1}(U')}(x) = \begin{cases} \sigma_A(x)\,,\ if\ x \in f^{-1}(supp(U')) \\ 0\,,\ Otherwise\,, \end{cases}$$

and $\nu_{f_R^{-1}(U')}(x) = \begin{cases} \nu_A(x)\,,\ if\ x \in f^{-1}(supp(U')) \\ 1\,,\ Otherwise. \end{cases}$

We conclude that, $f_R^{-1}(U')$ is an SVNPOS in A. Hence, f_R is a single valued neutrosophic pre-continuous mapping. □

Remark 4.13. The converse of the above implication is not necessarily holds. Indeed, let (A,T) and (B,T') be two single valued neutrosophic topological spaces, where $\mu_A(x) = 1$, $\sigma_A(x) = 1$ and $\nu_A(x) = 0.005$, for any $x \in \mathbb{R}_+$ and

$$\mu_B(y) = \begin{cases} 0.7\,,\ if\ y \geq 0 \\ 0\,,\ Otherwise\,, \end{cases} \qquad \sigma_B(y) = \begin{cases} 0.9\,,\ if\ y \geq 0 \\ 0.8\,,\ Otherwise\,, \end{cases} \qquad \nu_B(y) = \begin{cases} 0.01\,,\ if\ y \geq 0 \\ 0.03\,,\ Otherwise\,, \end{cases}$$

We suppose that $T = \{A, 0_\sim, U_1\}$, where
$\mu_{U_1}(x) = 0$ $\sigma_{U_1}(x) = 1$ and $\nu_{U_1}(x) = 1$.
Also, we suppose that $T' = \{B, 0_\sim, U_1'\}$, where

$$\mu_{U_1'}(y) = \begin{cases} 0.7\,,\ if\ y \in [0,4] \\ 0\,,\ Otherwise\,, \end{cases} \qquad \sigma_{U_1'}(y) = \begin{cases} 0.5\,,\ if\ y \in [0,4] \\ 0\,,\ Otherwise\,, \end{cases} \qquad \nu_{U_1'}(y) = \begin{cases} 0.12\,,\ if\ y \in [0,4] \\ 0.32\,,\ Otherwise. \end{cases}$$

Then, the single valued neutrosophic mapping $f_R : A \to B$ define by :

(i) an ordinary mapping $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $f(x) = \sqrt{x}$, for any $x \in \mathbb{R}_+$,

(ii) a single valued neutrosophic relation R defined by :
$\mu_R(x, f(x)) = 0.7$ $\sigma_R(x, f(x)) = 0.9$ and $\nu_R(x, f(x)) = 0.01$.

$$\begin{aligned} \mu_{f_R^{-1}(U_1')}(x) &= \begin{cases} \mu_A(x)\,,\ if\ x \in f^{-1}(supp(U_1')) \\ 0\,,\ Otherwise\,, \end{cases} \\ &= \begin{cases} 1\,,\ if\ x \in [0,16] \\ 0\,,\ Otherwise\,, \end{cases} \end{aligned}$$

$$\begin{aligned} \sigma_{f_R^{-1}(U_1')}(x) &= \begin{cases} \sigma_A(x)\,,\ if\ x \in f^{-1}(supp(U_1')) \\ 0\,,\ Otherwise\,, \end{cases} \\ &= \begin{cases} 1\,,\ if\ x \in [0,16] \\ 0\,,\ Otherwise\,, \end{cases} \end{aligned}$$

$$\nu_{f_R^{-1}(U_1')}(x) = \begin{cases} \nu_A(x)\ ,\ if\ x \in f^{-1}(supp(U_1')) \\ 1\ ,\ Otherwise\ , \end{cases}$$
$$= \begin{cases} \nu_A(x)\ ,\ if\ x \in [0,16] \\ 1\ ,\ Otherwise\ , \end{cases}$$
$$= \begin{cases} 0.01\ ,\ if\ x \in [0,16] \\ 1\ ,\ Otherwise\ , \end{cases}$$

Hence, $cl(f_R^{-1}(U_1')) = \overline{0_\sim} = 1_\sim$ and $int(1_\sim) = A$. Thus, $f_R^{-1}(U_1') \subseteq int(cl(f_R^{-1}(U_1')))$. We conclude that $f_R^{-1}(U_1')$ is an SVNPOS and f_R is a single valued neutrosophic pre-continuous but not a single valued neutrosophic continuous.

5 Conclusion

In this work, we have generalized the notion of fuzzy mapping defined by fuzzy relation introduced by Ismail and Massa'deh to the setting of single valued neutrosophic sets. Also, the main properties related to the single valued neutrosophic mapping have been studied. Next, as an application we have established the single valued neutrosophic continuous mapping in the single valued neutrosophic topological spaces. Future work will be directed to study the notion of the single valued neutrosophic mapping for other types of topologies based on the single valued neutrosophic sets.

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A Study on Bipolar Single-Valued Neutrosophic Graphs With Novel Application

M. Aslam Malik[1], Hossein Rashmanlou[2,*], Muhammad Shoaib[1], R. A. Borzooei[4] and Morteza Taheri[2] and Said Broumi [5]

[1]Department of Mathematics, University of the Punjab, New Campus, Lahore, Pakistan; aslam.math@pu.edu.pk; muhammadshoaibe14@gmail.com

[2]Department of Mathematics, University of the Mazandaran, Babolsar, Iran; M.taheri@stu.umz.ac.ir

[4]Department of Mathematics, Shahid Beheshti University of the Tehran, Iran; Borzoeei@sbu.ac.ir

[5]Laboratory of Information Processing, Faculty of Science Ben MSik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco; broumisaid78@gmail.com

*Correspondence: h.rashmanlou@stu.umz.ac.ir; Tel.: (989384012972)

Abstract. Unipolar is less fundamental than bipolar cognition based on truth, and composure is a restraint for truth-based worlds. Bipolarity is the most powerful phenomenon that survives when truth disappeared in a black hole due to Hawking radiation or particular / anti-particular emission. The purpose of this research study is to define few four operations, including residue product, rejection, maximal product and symmetric difference of bipolar single-valued neutrosophic graph (BSVNG) and to explore some of their related properties with examples. Bipolar single-valued neutrosophic graph (BSVNG) is the generalization of the single-valued neutrosophic graph (SVNG), intuitionistic fuzzy graph, bipolar intuitionistic fuzzy graph, bipolar fuzzy graph and fuzzy graph. BSVNG plays a significant role in the study of neural networks, daily energy issues, energy systems, and coding. Moreover, we will determine related properties like the degree of a vertex in a BSVNG or total degree of a vertex in a BSVNG. We provide examples of the vertex degree in BSVNG and the total vertex degree in BSVNG. In order to make this useful, we develop an algorithm for our useful method in steps.

Keywords: keyword 1; symmetric difference, residue product, maximal product, rejection of BSVNG, Application, algorithm.

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1. Introduction

In 1965, Zadeh [36] put forward the idea of the one-degree fuzzy set concept that determined the true membership function. Since Zadeh's pioneering work, the fuzzy set theory has been used in various disciplines such as management sciences, engineering, mathematics, social sciences, statistics, signal processing, artificial intelligence, automata theory, medical and life sciences. In the 20th century, Smarandache [31] includes the concept where uncertainty occurs in the form of Neutrosophic set and extend the intuitionistic fuzzy set. There is also a non-membership degree that Atanassove [1] defines in an intuitionistic fuzzy set with two degrees in a set. Abdel-Basset et al. [2–6] studied many concepts on neutrosophic sets. Broumi et al. [7, 9–13, 28, 29] investigated the extension of the fuzzy graph in the form of the single-valued neutrosophic graphs, shortest path problem using bellman algorithm under neutrosophic environment, shortest path problem in fuzzy, intuitionistic fuzzy and neutrosophic environment, single valued neutrosophic coloring, and operations of single valued neutrosophic coloring.

A bipolar fuzzy theory has more scope when we compare to simply a fuzzy theory as compatibility and flexibility. Overall its model is better than the fuzzy model. Borzooei and Rashmanlou [8, 25–27] studied very well on vague graphs and bipolar fuzzy graph. Rashmanlou studied about interval-valued fuzzy graph [22–24]. The neutrosophic set has much scope in neutrosophy and the neutrosophy theory is widely used in graph theory. In this extension, Wang et el. [35] described subclass of a Neutrosophic set known as a single-valued neutrosophic set. In the fields of bio and physics, SVNG has numerous applications. In these days, its purpose evaluates incomplete and uncertainty information. BSVNG has numerous applications in the fields of geometry and operational research. It has been a useful scope in various fields of computer science.Later, Deli et al. [14] described the idea of the bipolar neutrosophic set as the extension of the Neutrosophic set. He also described the concept of the bipolar fuzzy graph with some related properties. One problem of an Fuzzy graph, Intuitionistic fuzzy graph, bipolar fuzzy graph and intuitionistic bipolar fuzzy graph found when uncertainty occurs in the relationship between two vertices. Need for the neutrosophic graph is necessary because these are not suitable properly. Many researchers [32, 33] was famous due to their research work application approach to real-world problems.

The idea of the fuzzy graph is presented by Rosenfeld [30] and [34]. Malik and Hassan [16] both described the classification of the BSVNG together. Later Malik and Naz [21] presented the operations on the SVNG. Gomathi and Keerthika [15] studied neutrosophic labeling graph. Kousik Das et al. [17] defined generalized neutrosophic competition graphs. Mordeson and Peng [18] given some operations on Fuzzy Graphs. Gani et al. [19, 20] defined order, size, and irregular fuzzy graphs. The various application of graph theory in the fields of information technology, operational research, image segmentation, social science, capturing the image, algebra. It is also applicable to bioscience, chemistry, and computer science. The fuzzy is very useful to deduce the unsolved problems in various fields like networking, clustering with a great role in the algorithm. The use of fuzzy graph by which a great extent in a few years and has a scope from 19th century [19, 20]. Neutrosophy is the type of philosophy which studies the nature and scope of neutralities. We will discuss some new properties on a BSVNG. Bipolar fuzzy set has many applications in image processing. It gives more advantages in real problems

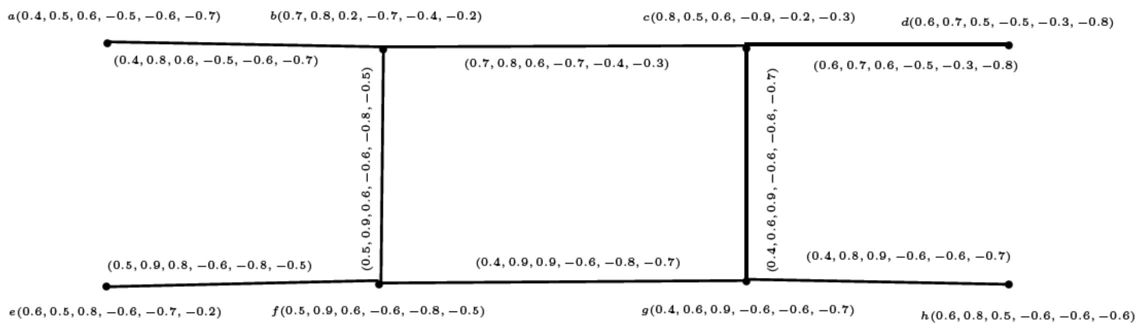

Figure 1. BSVNG

to make it in an easier form. BSVNG is the extension of an Fuzzy graph, Intuitionistic fuzzy graph, interval-valued intuitionistic fuzzy graph and SVNG. Bipolar fuzzy graphs are very useful in the fields of signal processing, computer science, and database theory. The operations we will establish are the symmetric difference and residue product in this paper. Peng [18] defined Some operations which are the join of two graphs, cartesian product of two graphs and the union of two graphs. Also, we discuss examples of these operations. We will find the degree and total degree of BSVNG. In the end, we will make an application on BSVNG with algorithm.

2. Operations on BSVNGs

In this section, we define four operations, including residue product, rejection, maximal product and symmetric difference of bipolar single-valued neutrosophic graph (BSVNG) and to explore some of their related properties with examples.

Definition 2.1. [13] A bipolar single valued neutrosophic graph is such a pair $\mathtt{G} = (X, Y)$ which is of crisp graph G=(V,E) is defined as(i) $\alpha_M : V \to [0,1]$, $\beta_M : V \to [0,1]$, $\gamma_M : V \to [0,1]$, $\delta_M : V \to [-1,0]$, $\eta_M : V \to [-1,0]$, $\theta_M : V \to [-1,0]$. (ii)

$$\alpha_N(mn) \leq \min\{\alpha_M(m), \alpha_M(n)\},\ \beta_N(mn) \geq \max\{\beta_M(m), \beta_M(n)\}$$

$$\gamma_N(mn) \geq \max\{\gamma_M(m), \gamma_M(n)\},\ \delta_N(mn) \geq \max\{\delta_M(m), \delta_M(n)\}$$

$$\eta_N(mn) \leq \min\{\eta_M(m), \eta_M(n)\},\ \theta_N(mn) \leq \min\{\theta_M(m), \theta_M(n)\}.$$

and $0 \leq \alpha_N(mn) + \beta_N(mn) + \gamma_N(mn) \leq 3$ and $-3 \leq \delta_N(mn) + \eta_N(mn) + \theta_N(mn) \leq 0$.

Example 2.2. In Figure 1, we see a graph with eight vertices {a,b,c,d,e,f,g,h} and eight edges {ab, bc, cd ,ef, fg, gh ,bf, cg} that is a bipolar single valued neutrosophic graph. It is easy to see that all conditions of Definition 2.1 is true for this example.

Definition 2.3. The height of a bipolar single valued neutrosophic set (BSVNs) (in universe discourse Y)
$Q = (\alpha_Q(y), \beta_Q(y), \gamma_Q, \delta_Q(y), \eta_Q(y), \theta_Q(y))$ is defined by:

$$\begin{aligned} h(Q) &= (h_1(Q), h_2(Q), h_3(Q), h_4(Q), h_5(Q), h_6(Q)) \\ &= (Sup_{y\in Y}\alpha_Q(y), Inf_{y\in Y}\beta_Q(y), Inf_{y\in Y}\beta_Q(y), Sup_{y\in Y}\delta_Q(y), Inf_{y\in Y}\eta_Q(y), Inf_{y\in Y}\theta_Q(y)) \end{aligned}$$

Example 2.4. Take $Q = \{(a, 0.5, 0.4, 0.5, -0.2, -0.4, -0.5), (b, 0.5, 0.6, 0.4, -0.4, -0.3, -0.6), (c, 0.4, 0.6, 0.4, -0.4, -0.5, -0.3)\}$ be BSVNs then height is defined as $h(Q) = (0.5, 0.4, 0.4, 0.4, 0.3, 0.3)$.

Definition 2.5. let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ are two bipolar single valued neutrosophic fuzzy graphs defined on $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. The symmetric difference of $\mathtt{G}_1$ and $\mathtt{G}_2$ is represented by $\mathtt{G}_1 \oplus \mathtt{G}_2 = (M_1 \oplus M_2, N_1 \oplus N_2)$. Symmetric difference of $\mathtt{G}_1$ *and* $\mathtt{G}_2$ is defined as the following conditions:

(i)

$$\begin{aligned} &(\alpha_{M_1} \oplus \alpha_{M_2})((m_1, m_2)) = \min\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\},\ (\beta_{M_1} \oplus \beta_{M_2})((m_1, m_2)) \\ &= \max\{\beta_{M_1}(m_1), \beta_{M_2}(m_2)\} \\ &(\gamma_{M_1} \oplus \gamma_{M_2})((m_1, m_2)) = \max\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2)\},\ (\delta_{M_1} \oplus \delta_{M_2})((m_1, m_2)) \\ &= \max\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\} \\ &(\eta_{M_1} \oplus \eta_{M_2})((m_1, m_2)) = \min\{\eta_{M_1}(m_1), \eta_{M_2}(m_2)\},\ (\theta_{M_1} \oplus \theta_{M_2})((m_1, m_2)) \\ &= \min\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\} \end{aligned}$$

$\forall (m_1, m_2) \in (V_1 \times V_2)$

(ii)

$$\begin{aligned} &(\alpha_{N_1} \oplus \alpha_{N_2})((m, m_2)(m, n_2)) = \min\{\alpha_{M_1}(m), \alpha_{N_2}(m_2n_2)\},\ (\beta_{N_1} \oplus \beta_{N_2})((m, m_2)(m, n_2)) \\ &= \max\{\beta_{M_1}(m), \beta_{N_2}(m_2n_2)\} \\ &(\gamma_{N_1} \oplus \gamma_{N_2})((m, m_2)(m, n_2)) = \max\{\gamma_{M_1}(m), \gamma_{N_2}(m_2n_2)\},\ (\delta_{N_1} \oplus \delta_{N_2})((m, m_2)(m, n_2)) \\ &= \max\{\delta_{M_1}(m), \delta_{N_2}(m_2n_2)\} \\ &(\eta_{N_1} \oplus \eta_{N_2})((m, m_2)(m, n_2)) = \min\{\eta_{M_1}(m), \eta_{N_2}(m_2n_2)\},\ (\theta_{N_1} \oplus \theta_{N_2})((m, m_2)(m, n_2)) \\ &= \min\{\theta_{M_1}(m), \theta_{N_2}(m_2n_2)\} \end{aligned}$$

$\forall\ m \in V_1$ *and* $m_2n_2 \in E_2$

(iii)

$$(\alpha_{N_1}\oplus\alpha_{N_2})((m_1,m)(n_1,m))=\min\{\alpha_{N_1}(m_1n_1),\alpha_{M_2}(m)\},\ (\beta_{N_1}\oplus\beta_{N_2})((m_1,m)(n_1,m))$$
$$=\max\{\beta_{N_1}(m_1n_1),\beta_{M_2}(m)\}$$
$$(\gamma_{N_1}\oplus\gamma_{N_2})((m_1,m)(n_1,m))=\max\{\gamma_{N_1}(m_1n_1),\gamma_{M_2}(m)\},\ (\delta_{N_1}\oplus\delta_{N_2})((m_1,m)(n_1,m))$$
$$=\max\{\delta_{N_1}(m_1n_1),\delta_{M_2}(m)\}$$
$$(\eta_{N_1}\oplus\eta_{N_2})((m_1,m)(n_1,m))=\min\{\eta_{N_1}(m_1n_1),\eta_{M_2}(m)\},\ (\theta_{N_1}\oplus\theta_{N_2})((m_1,m)(n_1,m))$$
$$=\min\{\theta_{N_1}(m_1n_1),\theta_{M_2}(m)\}$$

$\forall\ z\in V_2\ and\ m_1n_1\in E_1$

$(i\vee)$

$$(\alpha_{N_1}\oplus\alpha_{N_2})((m_1,m_2)(n_1,n_2))=\min\{\alpha_{M_1}(m_1),\alpha_{M_1}(n_1),\alpha_{N_2}(m_2n_2)\}$$
$$for\ all\ m_1n_1\notin E_1\ and\ m_2n_2\in E_2$$
$$or$$
$$=\min\{\alpha_{M_2}(m_2),\alpha_{M_2}(n_2),\alpha_{N_1}(m_1n_1)\} for\ all\ m_1n_1\in E_1\ and\ m_2n_2\notin E_2$$

$$(\beta_{N_1}\oplus\beta_{N_2})((m_1,m_2)(n_1,n_2))=\max\{\beta_{M_1}(m_1),\beta_{M_1}(n_1),\beta_{N_2}(m_2n_2)\}$$
$$forall\ m_1n_1\notin E_1\ and\ m_2n_2\in E_2$$
$$or$$
$$=\max\{\beta_{M_2}(m_2),\beta_{M_2}(n_2),\beta_{N_1}(m_1n_1)\}\ forall\ m_1n_1\in E_1\ and\ m_2n_2\notin E_2$$

$$(\gamma_{N_1}\oplus\gamma_{N_2})((m_1,m_2)(n_1,n_2))=\max\{\gamma_{M_1}(m_1),\gamma_{M_1}(n_1),F_{N_2}(m_2n_2)\}$$
$$forall\ m_1n_1\notin E_1\ and\ m_2n_2\in E_2$$
$$or$$
$$=\max\{\gamma_{M_2}(m_2),\gamma_{M_2}(n_2),\gamma_{N_2}(m_1n_1)\}\ forall\ m_1n_1\in E_1\ and\ m_2n_2\notin E_2$$

$$(\delta_{N_1}\oplus\delta_{N_2})((m_1,m_2)(n_1,n_2))=\max\{\delta_{M_1}(m_1),\delta_{M_1}(n_1),\delta_{N_2}(m_2n_2)\}$$
$$forall\ m_1n_1\notin E_1\ and\ m_2n_2\in E_2$$
$$or$$
$$=\max\{\delta_{M_2}(m_2),\delta_{M_2}(n_2),\delta_{N_1}(m_1n_1)\}\ forall\ m_1n_1\in E_1\ and\ m_2n_2\notin E_2$$

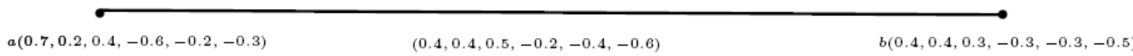

FIGURE 2. G_1

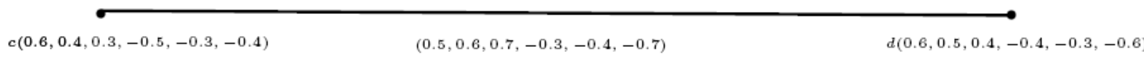

FIGURE 3. G_2

$$(\eta_{N_1} \oplus \eta_{N_2})((m_1, m_2)(n_1, n_2)) = \min\{\eta_{M_1}(m_1), \eta_{M_1}(n_1), \eta_{N_2}(m_2n_2)\}$$
$$forall\ m_1n_1 \notin E_1\ and\ m_2n_2 \in E_2$$
$$or$$
$$= \min\{\eta_{M_2}(m_2), \eta_{M_2}(n_2), \eta_{N_1}(m_1n_1)\}\ forall\ m_1n_1 \in E_1\ and\ m_2n_2 \notin E_2$$

$$(\theta_{N_1} \oplus \theta_{N_2})((m_1, m_2)(n_1, n_2)) = \min\{\theta_{M_1}(m_1), \theta_{M_1}(n_1), F_{N_2}(m_2n_2)\}$$
$$forall\ m_1n_1 \notin E_1\ and\ m_2n_2 \in E_2$$
$$or$$
$$= \min\{\theta_{M_2}(m_2), \theta_{M_2}(n_2), \theta_{N_2}(m_1n_1)\}\ forall\ m_1n_1 \in E_1\ and\ m_2n_2 \notin E_2$$

Example 2.6. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs on $V_1 = \{a, b\}\ and\ V_2 = \{c, d\}$ respectively which shown in Figure 2 and Figure 3. Also symmetric difference shown in Figure 4.

Proposition 2.7. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs of graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Then the symmetric difference $\mathtt{G}_1 \oplus \mathtt{G}_2$ of $\mathtt{G}_1 = (V_1, E_1)$ and $\mathtt{G}_2 = (V_2, E_2)$ is again a BSVNG.

Proof. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs of graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Then the symmetric difference $\mathtt{G}_1 \oplus \mathtt{G}_2$ of $\mathtt{G}_1 = (V_1, E_1)$ and $\mathtt{G}_2 = (V_2, E_2)$ can be proved. Let $(m_1, m_2)(n_1, n_2) \in E_1 \times E_2$

(i) If $m_1 = n_1 = m$

$$\begin{aligned}(\alpha_{N_1} \oplus \alpha_{N_2})((m, m_2)(m, n_2)) &= \min\{\alpha_{M_1}(m), \alpha_{N_2}(m_2n_2)\}\\ &\leq \min\{\alpha_{M_1}(m), \min\{\alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\}\}\\ &= \min\{\min\{\{\alpha_{M_1}(m), \alpha_{M_2}(m_2)\}, \min\{\{\alpha_{M_1}(m), \alpha_{M_2}(n_2)\}\}\\ &= \min\{(\alpha_{M_1} \oplus \alpha_{M_2})(m, m_2), (\alpha_{M_1} \oplus \alpha_{M_2})(m, n_2)\}\end{aligned}$$

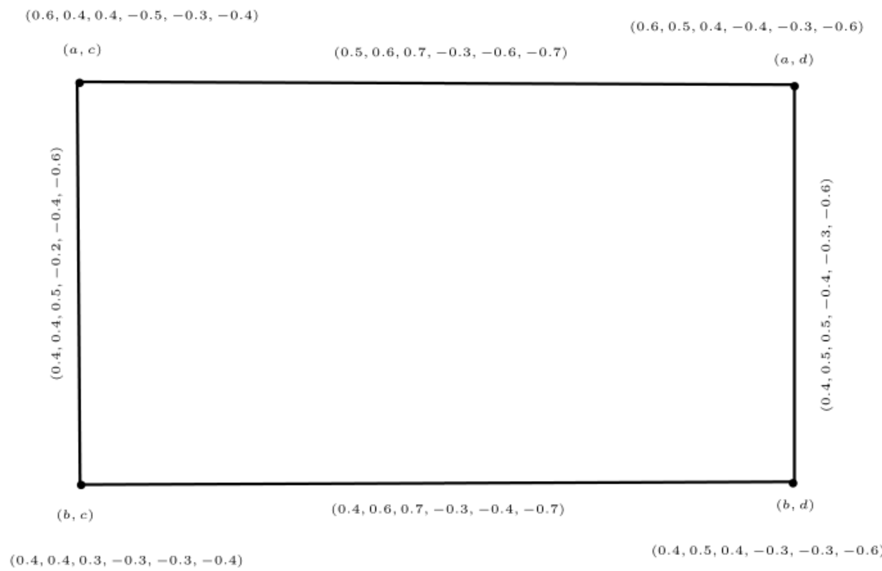

FIGURE 4. $G_1 \oplus G_2$

$$\begin{aligned}(\beta_{N_1} \oplus \beta_{N_2})((m, m_2)(m, n_2)) &= \max\{\beta_{M_1}(m), \beta_{N_2}(m_2 n_2)\} \\ &\geq \max\{\beta_{M_1}(m), \max\{\beta_{M_2}(m_2), \beta_{M_2}(n_2)\}\} \\ &= \max\{\max\{\{\beta_{M_1}(m), \beta_{M_2}(m_2)\}, \max\{\{\beta_{M_1}(m), \beta_{M_2}(n_2)\}\} \\ &= \max\{(\beta_{M_1} \oplus \beta_{M_2})(m, m_2), (\beta_{M_1} \oplus \beta_{M_2})(m, n_2)\}\end{aligned}$$

$$\begin{aligned}(\gamma_{N_1} \oplus \gamma_{N_2})((m, m_2)(m, n_2)) &= \max\{\gamma_{M_1}(m), \gamma_{N_2}(m_2 n_2)\} \\ &\geq \max\{\gamma_{M_1}(m), \max\{\gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\}\} \\ &= \max\{\max\{\{\gamma_{M_1}(m), \gamma_{M_2}(m_2)\}, \max\{\{\gamma_{M_1}(m), \gamma_{M_2}(n_2)\}\} \\ &= \max\{(\gamma_{M_1} \oplus \gamma_{M_2})(m, m_2), (\gamma_{M_1} \oplus \gamma_{M_2})(m, n_2)\}\end{aligned}$$

$$\begin{aligned}(\delta_{N_1} \oplus \delta_{N_2})((m, m_2)(m, n_2)) &= \max\{\delta_{M_1}(m), \delta_{N_2}(m_2 n_2)\} \\ &\geq \max\{\delta_{M_1}(m), \max\{\delta_{M_2}(m_2), \delta_{M_2}(n_2)\}\} \\ &= \max\{\max\{\{\delta_{M_1}(m), \delta_{M_2}(m_2)\}, \min\{\{\delta_{M_1}(m), \delta_{M_2}(n_2)\}\} \\ &= \max\{(\delta_{M_1} \oplus \delta_{M_2})(m, m_2), (\delta_{M_1} \oplus \delta_{M_2})(m, n_2)\}\end{aligned}$$

$$\begin{aligned}(\eta_{N_1} \oplus \eta_{N_2})((m,m_2)(m,n_2)) &= \min\{\eta_{M_1}(m), \eta_{N_2}(m_2n_2)\}\\ &\leq \min\{\eta_{M_1}(m), \min\{\eta_{M_2}(m_2), \eta_{M_2}(n_2)\}\}\\ &= \min\{\min\{\{\eta_{M_1}(m), \eta_{M_2}(m_2)\}, \min\{\{\eta_{M_1}(m), \eta_{M_2}(n_2)\}\}\\ &= \min\{(\eta_{M_1} \oplus \eta_{M_2})(m,m_2), (\eta_{M_1} \oplus \eta_{M_2})(m,n_2)\}\end{aligned}$$

$$\begin{aligned}(\theta_{N_1} \oplus \theta_{N_2})((m,m_2)(m,n_2)) &= \min\{\theta_{M_1}(m), \theta_{N_2}(m_2n_2)\}\\ &\leq \min\{\theta_{M_1}(m), \min\{\theta_{M_2}(m_2), \theta_{M_2}(n_2)\}\}\\ &= \min\{\min\{\{\theta_{M_1}(m), \theta_{M_2}(m_2)\}, \min\{\{\theta_{M_1}(m), \theta_{M_2}(n_2)\}\}\\ &= \min\{(\theta_{M_1} \oplus \theta_{M_2})(m,m_2), (\theta_{M_1} \oplus \theta_{M_2})(m,n_2)\}\end{aligned}$$

(ii) if $m_2 = n_2 = m$

$$\begin{aligned}(\alpha_{N_1} \oplus \alpha_{N_2})((m_1,m)(n_1,m)) &= \min\{\alpha_{N_1}(m_1n_1), \alpha_{M_2}(m)\}\\ &\leq \min\{\min\{\alpha_{N_1}(m_1n_1), \alpha_{M_2}(m)\}\\ &= \min\{\min\{\{\alpha_{M_1}(m_1), \alpha_{M_2}(m)\}, \min\{\{\alpha_{M_1}(n_1), \alpha_{M_2}(m)\}\}\\ &= \min\{(\alpha_{M_1} \oplus \alpha_{M_2})(m_1,m), (\alpha_{M_1} \oplus \alpha_{M_2})(n_1,m)\}\end{aligned}$$

$$\begin{aligned}(\beta_{N_1} \oplus \beta_{N_2})((m_1,m)(n_1,m)) &= \max\{\beta_{N_1}(m_1n_1), \beta_{M_2}(m)\}\\ &\geq \max\{\max\{\beta_{N_1}(m_1n_1), \beta_{M_2}(m)\}\\ &= \max\{\max\{\{\beta_{M_1}(m_1), \beta_{M_2}(m)\}, \max\{\{\beta_{M_1}(n_1), \beta_{M_2}(m)\}\}\\ &= \max\{(\beta_{M_1} \oplus \beta_{M_2})(m_1,m), (\beta_{M_1} \oplus \beta_{M_2})(n_1,m)\}\end{aligned}$$

$$\begin{aligned}(\gamma_{N_1} \oplus \gamma_{N_2})((m_1,m)(n_1,m)) &= \max\{\gamma_{N_1}(m_1n_1), \gamma_{M_2}(m)\}\\ &\geq \max\{\max\{\gamma_{N_1}(m_1n_1), \gamma_{M_2}(m)\}\\ &= \max\{\max\{\{\gamma_{M_1}(m_1), \gamma_{M_2}(m)\}, \max\{\{\gamma_{M_1}(n_1), \gamma_{M_2}(m)\}\}\\ &= \max\{(\gamma_{M_1} \oplus \gamma_{M_2})(m_1,m), (\gamma_{M_1} \oplus \gamma_{M_2})(n_1,m)\}\end{aligned}$$

$$\begin{aligned}(\delta_{N_1}\oplus\delta_{N_2})((m_1,m)(n_1,m)) &= \max\{\delta_{N_1}(m_1n_1),\delta_{M_2}(m)\}\\ &\geq \max\{\max\{\delta_{N_1}(m_1n_1),\delta_{M_2}(m)\}\\ &= \max\{\max\{\{\delta_{M_1}(m_1),\delta_{M_2}(m)\},\max\{\{\delta_{M_1}(n_1),\delta_{M_2}(m)\}\}\\ &= \max\{(\delta_{M_1}\oplus\delta_{M_2})(m_1,m),(\delta_{M_1}\oplus\delta_{M_2})(n_1,m)\}\end{aligned}$$

$$\begin{aligned}(\eta_{N_1}\oplus\eta_{N_2})((m_1,m)(n_1,m)) &= \min\{\eta_{N_1}(m_1n_1),\eta_{M_2}(m)\}\\ &\leq \min\{\min\{\eta_{N_1}(m_1n_1),\eta_{M_2}(m)\}\\ &= \min\{\min\{\{\eta_{M_1}(m_1),\eta_{M_2}(m)\},\min\{\{\eta_{M_1}(n_1),\eta_{M_2}(m)\}\}\\ &= \min\{(\eta_{M_1}\oplus\eta_{M_2})(m_1,m),(\eta_{M_1}\oplus\eta_{M_2})(n_1,m)\}\end{aligned}$$

$$\begin{aligned}(\theta_{N_1}\oplus\theta_{N_2})((m_1,m)(n_1,m)) &= \min\{\theta_{N_1}(m_1n_1),\theta_{M_2}(m)\}\\ &\leq \min\{\min\{\theta_{N_1}(m_1n_1),\theta_{M_2}(m)\}\\ &= \min\{\min\{\{\theta_{M_1}(m_1),\theta_{M_2}(m)\},\min\{\{\theta_{M_1}(n_1),\theta_{M_2}(m)\}\}\\ &= \min\{(\theta_{M_1}\oplus\theta_{M_2})(m_1,m),(\theta_{M_1}\oplus\theta_{M_2})(n_1,m)\}\end{aligned}$$

(iii) If $m_1n_1 \notin E_1$ *and* $m_2n_2 \in E_2$

$$\begin{aligned}(\alpha_{N_1}\oplus\alpha_{N_2})((m_1,m_2)(n_1,n_2)) &= \min\{\alpha_{M_1}(m_1),\alpha_{M_1}(n_1),\alpha_{N_2}(m_2n_2)\}\\ &\leq \min\{\alpha_{M_1}(m_1),\alpha_{M_1}(n_1),\min\{\alpha_{M_2}(m_2)\alpha_{M_2}(n_2)\}\}\\ &= \min\{\min\{\alpha_{M_1}(m_1),\alpha_{M_2}(m_2)\},\{\alpha_{M_1}(m_1),\alpha_{M_2}(n_2)\}\\ &= \min\{(\alpha_{M_1}\oplus\alpha_{M_2})(m_1,m_2),(\alpha_{M_1}\oplus\alpha_{M_2})(n_1,n_2)\}\end{aligned}$$

$$\begin{aligned}(\beta_{N_1}\oplus\beta_{N_2})((m_1,m_2)(n_1,n_2)) &= \max\{\beta_{M_1}(m_1),\beta_{M_1}(n_1),\beta_{N_2}(m_2n_2)\}\\ &\geq \max\{\beta_{M_1}(m_1),\beta_{M_1}(n_1),\max\{\beta_{M_2}(m_2)\beta_{M_2}(n_2)\}\}\\ &= \max\{\max\{\beta_{M_1}(m_1),\beta_{M_2}(m_2)\},\{\beta_{M_1}(m_1),\beta_{M_2}(n_2)\}\\ &= \max\{(\beta_{M_1}\oplus\beta_{M_2})(m_1,m_2),(\beta_{M_1}\oplus\beta_{M_2})(n_1,n_2)\}\end{aligned}$$

$$\begin{aligned}(\gamma_{N_1}\oplus\gamma_{N_2})((m_1,m_2)(n_1,n_2)) &= \max\{\gamma_{M_1}(m_1),\gamma_{M_1}(n_1),\gamma_{N_2}(m_2n_2)\}\\ &\geq \max\{\gamma_{M_1}(m_1),\gamma_{M_1}(n_1),\max\{\gamma_{M_2}(m_2)\gamma_{M_2}(n_2)\}\}\\ &= \max\{\max\{\gamma_{M_1}(m_1),\gamma_{M_2}(m_2)\},\{\gamma_{M_1}(m_1),\gamma_{M_2}(n_2)\}\\ &= \max\{(\gamma_{M_1}\oplus\gamma_{M_2})(m_1,m_2),(\gamma_{M_1}\oplus\gamma_{M_2})(n_1,n_2)\}\end{aligned}$$

$$\begin{aligned}(\delta_{N_1} \oplus \delta_{N_2})((m_1, m_2)(n_1, n_2)) &= \max\{\delta_{M_1}(m_1), \delta_{M_1}(n_1), \delta_{N_2}(m_2 n_2)\} \\ &\geq \max\{\delta_{M_1}(m_1), \delta_{M_1}(n_1), \max\{\delta_{M_2}(m_2)\delta_{M_2}(n_2)\}\} \\ &= \max\{\max\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\}, \{\delta_{M_1}(m_1), \delta_{M_2}(n_2)\} \\ &= \max\{(\delta_{M_1} \oplus \delta_{M_2})(m_1, m_2), (\delta_{M_1} \oplus \delta_{M_2})(n_1, n_2)\}\end{aligned}$$

$$\begin{aligned}(\eta_{N_1} \oplus \eta_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{\eta_{M_1}(m_1), \eta_{M_1}(n_1), \eta_{N_2}(m_2 n_2)\} \\ &\leq \min\{\eta_{M_1}(m_1), \eta_{M_1}(n_1), \min\{\eta_{M_2}(m_2)\eta_{M_2}(n_2)\}\} \\ &= \min\{\min\{\eta_{M_1}(m_1), \eta_{M_2}(m_2)\}, \{\eta_{M_1}(m_1), \eta_{M_2}(n_2)\} \\ &= \min\{(\eta_{M_1} \oplus \eta_{M_2})(m_1, m_2), (\eta_{M_1} \oplus \eta_{M_2})(n_1, n_2)\}\end{aligned}$$

$$\begin{aligned}(\theta_{N_1} \oplus \theta_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{\theta_{M_1}(m_1), \theta_{M_1}(n_1), \theta_{N_2}(m_2 n_2)\} \\ &\leq \min\{\theta_{M_1}(m_1), \theta_{M_1}(n_1), \min\{\theta_{M_2}(m_2)\theta_{M_2}(n_2)\}\} \\ &= \min\{\min\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\}, \{\theta_{M_1}(m_1), \theta_{M_2}(n_2)\} \\ &= \min\{(\theta_{M_1} \oplus \theta_{M_2})(m_1, m_2), (\theta_{M_1} \oplus \theta_{M_2})(n_1, n_2)\}\end{aligned}$$

$(i\vee)$ If $m_1 n_1 \in E_1$ *and* $m_2 n_2 \notin E_2$

$$\begin{aligned}(\alpha_{N_1} \oplus \alpha_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{\alpha_{M_2}(m_2), \alpha_{M_2}(n_2), \alpha_{N_1}(m_1 n_1)\} \\ &\leq \min\{\alpha_{M_2}(m_2), \alpha_{M_2}(n_2), \min\{\alpha_{M_1}(m_1)\alpha_{M_1}(n_1)\}\} \\ &= \min\{\min\{\alpha_{M_2}(m_2), \alpha_{M_1}(m_1)\}, \{\alpha_{M_2}(m_2), \alpha_{M_1}(n_1)\} \\ &= \min\{(\alpha_{M_1} \oplus \alpha_{M_2})(m_1, m_2), (\alpha_{M_1} \oplus \alpha_{M_2})(n_1, n_2)\}\end{aligned}$$

$$\begin{aligned}(\beta_{N_1} \oplus \beta_{N_2})((m_1, m_2)(n_1, n_2)) &= \max\{\beta_{M_2}(m_2), \beta_{M_2}(n_2), \beta_{N_1}(m_1 n_1)\} \\ &\geq \max\{\beta_{M_2}(m_2), \beta_{M_2}(n_2), \max\{\beta_{M_1}(m_1)\beta_{M_1}(n_1)\}\} \\ &= \max\{\max\{\beta_{M_2}(m_2), \beta_{M_1}(m_1)\}, \{\beta_{M_2}(m_2), \beta_{M_1}(n_1)\} \\ &= \max\{(\beta_{M_1} \oplus \beta_{M_2})(m_1, m_2), (\beta_{M_1} \oplus \beta_{M_2})(n_1, n_2)\}\end{aligned}$$

$$\begin{aligned}(\gamma_{N_1} \oplus \gamma_{N_2})((m_1, m_2)(n_1, n_2)) &= \max\{\gamma_{M_2}(m_2), \gamma_{M_2}(n_2), \gamma_{N_1}(m_1 n_1)\} \\ &\geq \max\{\gamma_{M_2}(m_2), \gamma_{M_2}(n_2), \max\{\gamma_{M_1}(m_1)\gamma_{M_1}(n_1)\}\} \\ &= \max\{\max\{\gamma_{M_2}(m_2), \gamma_{M_1}(m_1)\}, \{\gamma_{M_2}(m_2), \gamma_{M_1}(n_1)\} \\ &= \max\{(\gamma_{M_1} \oplus \gamma_{M_2})(m_1, m_2), (\gamma_{M_1} \oplus \gamma_{M_2})(n_1, n_2)\}\end{aligned}$$

$$\begin{aligned}(\delta_{N_1} \oplus \delta_{N_2})((m_1, m_2)(n_1, n_2)) &= \max\{\delta_{M_2}(m_2), \delta_{M_2}(n_2), \delta_{N_1}(m_1 n_1)\} \\ &\geq \max\{\delta_{M_2}(m_2), \delta_{M_2}(n_2), \max\{\delta_{M_1}(m_1)\delta_{M_1}(n_1)\}\} \\ &= \max\{\max\{\delta_{M_2}(m_2), \delta_{M_1}(m_1)\}, \{\delta_{M_2}(m_2), \delta_{M_1}(n_1)\} \\ &= \max\{(\delta_{M_1} \oplus \delta_{M_2})(m_1, m_2), (\delta_{M_1} \oplus \delta_{M_2})(n_1, n_2)\}\end{aligned}$$

$$\begin{aligned}(\eta_{N_1} \oplus \eta_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{\eta_{M_2}(m_2), \eta_{M_2}(n_2), \eta_{N_1}(m_1 n_1)\} \\ &\leq \min\{\eta_{M_2}(m_2), \eta_{M_2}(n_2), \min\{\eta_{M_1}(m_1)\eta_{M_1}(n_1)\}\} \\ &= \min\{\min\{\eta_{M_2}(m_2), \eta_{M_1}(m_1)\}, \{\eta_{M_2}(m_2), \eta_{M_1}(n_1)\} \\ &= \min\{(\eta_{M_1} \oplus \eta_{M_2})(m_1, m_2), (\eta_{M_1} \oplus \eta_{M_2})(n_1, n_2)\}\end{aligned}$$

$$\begin{aligned}(\theta_{N_1} \oplus \theta_{N_2})((m_1, m_2)(n_1, n_2)) &= \min\{\theta_{M_2}(m_2), \theta_{M_2}(n_2), \theta_{N_1}(m_1 n_1)\} \\ &\leq \min\{\theta_{M_2}(m_2), \theta_{M_2}(n_2), \min\{\theta_{M_1}(m_1)\theta_{M_1}(n_1)\}\} \\ &= \min\{\min\{\theta_{M_2}(m_2), \theta_{M_1}(m_1)\}, \{\theta_{M_2}(m_2), \theta_{M_1}(n_1)\} \\ &= \min\{(\theta_{M_1} \oplus \theta_{M_2})(m_1, m_2), (\theta_{M_1} \oplus \theta_{M_2})(n_1, n_2)\}\end{aligned}$$

. Hence $\mathtt{G}_1 \oplus \mathtt{G}_2$ is a BSVNG. □

Definition 2.8. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, Y_2)$ be two BSVNGs. $\forall (m_1, m_2) \in \; V_1 \times V_2$

$$\begin{aligned}(d_\alpha)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1, m_2) = &\sum_{(m_1,m_2)(n_1,n_2) \in E_1 \times E_2.} (\alpha_{N_1} \oplus \alpha_{N_2})((m_1, m_2)(n_1, n_2)) \\ = &\sum_{m_1 = n_1, m_2 n_2 \in E_2} \min\{\alpha_{M_1}(m_1), \alpha_{N_2}(m_2 n_2)\} \\ + &\sum_{m_1 n_1 \in E_1, m_2 = n_2} \min\{\alpha_{N_1}(m_1 n_1, \alpha_{M_2}(m_2)\} \\ + &\sum_{m_1 n_1 \notin E_1 and \; m_2 n_2 \in E_2} \min\{\alpha_{M_1}(m_1), \alpha_{M_1}(n_1), \alpha_{N_2}(m_2 n_2)\} \\ + &\sum_{m_1 n_1 \in E_1 and \; m_2 n_2 \notin E_2} \min\{\alpha_{N_1}(m_1 n_1), \alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\}\end{aligned}$$

$$
\begin{aligned}
(d_\beta)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{N_1} \oplus \beta_{N_2})((m_1, m_2)(n_1, n_2)) \\
&= \sum_{m_1=n_1, m_2n_2\in E_2} \max\{\beta_{M_1}(m_1), \beta_{N_2}(m_2n_2)\} \\
&+ \sum_{m_1n_1\in E_1, m_2=n_2} \max\{\beta_{N_1}(m_1n_1, \beta_{M_2}(m_2)\} \\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \max\{\beta_{M_1}(m_1), \beta_{M_1}(n_1), \beta_{N_2}(m_2n_2)\} \\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \max\{\beta_{N_1}(m_1n_1), \beta_{M_2}(m_2), \beta_{M_2}(n_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(d_\gamma)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{N_1} \oplus \beta_{N_2})((m_1, m_2)(n_1, n_2)) \\
&= \sum_{m_1=n_1, m_2n_2\in E_2} \max\{\gamma_{M_1}(m_1), \gamma_{N_2}(m_2n_2)\} \\
&+ \sum_{m_1n_1\in E_1, m_2=n_2} \max\{\gamma_{N_1}(m_1n_1, \gamma_{M_2}(m_2)\} \\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \max\{\gamma_{M_1}(m_1), \gamma_{M_1}(n_1), \gamma_{N_2}(m_2n_2)\} \\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \max\{\gamma_{N_1}(m_1n_1), \gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(d_\delta)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\delta_{N_1} \oplus \delta_{N_2})((m_1, m_2)(n_1, n_2)) \\
&= \sum_{m_1=n_1, m_2n_2\in E_2} \max\{\delta_{M_1}(m_1), \delta_{N_2}(m_2n_2)\} \\
&+ \sum_{m_1n_1\in E_1, m_2=n_2} \max\{\delta_{N_1}(m_1n_1, \delta_{M_2}(m_2)\} \\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \max\{\delta_{M_1}(m_1), \delta_{M_1}(n_1), \delta_{N_2}(m_2n_2)\} \\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \max\{\delta_{N_1}(m_1n_1), \delta_{M_2}(m_2), \delta_{M_2}(n_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(d_\eta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\eta_{N_1}\oplus\eta_{N_2})((m_1,m_2)(n_1,n_2))\\
&= \sum_{m_1=n_1,m_2n_2\in E_2} \min\{\eta_{M_1}(m_1),\eta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1,m_2=n_2} \min\{\eta_{N_1}(m_1n_1,\eta_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \min\{\eta_{M_1}(m_1),\eta_{M_1}(n_1),\eta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \min\{\eta_{N_1}(m_1n_1),\eta_{M_2}(m_2),\eta_{M_2}(n_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(d_\theta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\theta_{N_1}\oplus\eta_{N_2})((m_1,m_2)(n_1,n_2))\\
&= \sum_{m_1=n_1,m_2n_2\in E_2} \min\{\theta_{M_1}(m_1),\theta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1,m_2=n_2} \min\{\theta_{N_1}(m_1n_1,\theta_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \min\{\theta_{M_1}(m_1),\theta_{M_1}(n_1),\theta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \min\{\theta_{N_1}(m_1n_1),\theta_{M_2}(m_2),\theta_{M_2}(n_2)\}
\end{aligned}
$$

Theorem 2.9. Let $\mathtt{G}_1=(M_1,N_1)$ and $\mathtt{G}_2=(M_2,Y_2)$ be two BSVNGs. If $\alpha_{M_1}\geq\alpha_{N_2},\beta_{M_1}\leq\beta_{N_2},\gamma_{M_1}\leq\gamma_{N_2}$ and $\alpha_{M_2}\geq\alpha_{N_1},\beta_{M_2}\leq\beta_{N_1},\gamma_{M_2}\leq\gamma_{N_1}$. Also if $\delta_{M_1}\leq\delta_{N_2},\eta_{M_1}\geq\eta_{N_2},\theta_{M_1}\geq\theta_{N_2}$ and $\delta_{M_2}\leq\delta_{N_1},\eta_{M_2}\geq\eta_{N_1},\theta_{M_2}\geq\theta_{N_1}$. Then for every $\forall(m_1,m_2)\in\ V_1\times V_2$ $(d)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2)$ =q$(d)_{\mathtt{G}_1}(m_1)+s(d)_{\mathtt{G}_2}(m_2)$ where s=| V_1 | -$(d)_{G_1}(m_1)$ and q=| V_2 | -$(d)_{G_2}(m_2)$
.

Proof.

$$
\begin{aligned}
(d_\alpha)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{N_1}\oplus\alpha_{N_2})((m_1,m_2)(n_1,n_2))\\
&= \sum_{m_1=n_1, m_2n_2\in E_2} \min\{\alpha_{M_1}(m_1), \alpha_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1, m_2=n_2} \min\{\alpha_{N_1}(m_1n_1), \alpha_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \min\{\alpha_{M_1}(m_1), \alpha_{M_1}(n_1), \alpha_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \min\{\alpha_{N_1}(m_1n_1), \alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\}\\
&= \sum_{m_2n_2\in E_2} \alpha_{N_2}(m_2n_2) + \sum_{m_1n_1\in E_1} \alpha_{N_1}(m_1n_1)\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \alpha_{N_2}(m_2n_2)\} + \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \alpha_{N_1}(m_1n_1)\\
&= q(d_\alpha)_{\mathtt{G}_1}(m_1) + s(d_\alpha)_{\mathtt{G}_2}(m_2)
\end{aligned}
$$

$$
\begin{aligned}
(d_\theta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\theta_{N_1}\oplus\theta_{N_2})((m_1,m_2)(n_1,n_2))\\
&= \sum_{m_1=n_1, m_2n_2\in E_2} \min\{\theta_{M_1}(m_1), \theta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1, m_2=n_2} \min\{\theta_{N_1}(m_1n_1), \theta_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \min\{\theta_{M_1}(m_1), \theta_{M_1}(n_1), \theta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \min\{\theta_{N_1}(m_1n_1), \theta_{M_2}(m_2), \theta_{M_2}(n_2)\}\\
&= \sum_{m_2n_2\in E_2} \theta_{N_2}(m_2n_2) + \sum_{m_1n_1\in E_1} \theta_{N_1}(m_1n_1)\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \theta_{N_2}(m_2n_2)\} + \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \theta_{N_1}(m_1n_1)\\
&= q(d_\theta)_{\mathtt{G}_1}(m_1) + s(d_\theta)_{\mathtt{G}_2}(m_2)
\end{aligned}
$$

In a similar way others four will proved obviously.

We conclude that $(d)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2)$ =q$(d)_{\mathtt{G}_1}(m_1) + s(d)_{\mathtt{G}_2}(m_2)$ where s=| V_1 | -$(d)_{G_1}(m_1)$ and q=| V_2 | -$(d)_{G_2}(m_2)$. □

Definition 2.10. Let $\mathtt{G}_1=(M_1,N_1)$ and $\mathtt{G}_2=(M_2,Y_2)$ be two BSVNGs. $\forall (m_1,m_2)\in\ V_1\times V_2$

$$
\begin{aligned}
(td_\alpha)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{N_1}\oplus\alpha_{N_2})((m_1,m_2)(n_1,n_2)) + (\alpha_{M_1}\oplus\alpha_{M_2}(m_1,m_2)\\
&= \sum_{m_1=n_1, m_2n_2\in E_2} \min\{\alpha_{M_1}(m_1),\alpha_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1, m_2=n_2} \min\{\alpha_{N_1}(m_1n_1,\alpha_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \min\{\alpha_{M_1}(m_1),\alpha_{M_1}(n_1),\alpha_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \min\{\alpha_{N_1}(m_1n_1),\alpha_{M_2}(m_2),\alpha_{M_2}(n_2)\}\\
&+ \min\{\alpha_{M_1}(m_1),\alpha_{M_2}(m_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(td_\beta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{N_1}\oplus\beta_{N_2})((m_1,m_2)(n_1,n_2)) + (\beta_{M_1}\oplus\beta_{M_2}(m_1,m_2)\\
&= \sum_{m_1=n_1, m_2n_2\in E_2} \max\{\beta_{M_1}(m_1),\beta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1, m_2=n_2} \max\{\beta_{N_1}(m_1n_1,\beta_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \max\{\beta_{M_1}(m_1),\beta_{M_1}(n_1),\beta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \max\{\beta_{N_1}(m_1n_1),\beta_{M_2}(m_2),\beta_{M_2}(n_2)\}\\
&+ \max\{\beta_{M_1}(m_1),\beta_{M_2}(m_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(td_\gamma)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{N_1}\oplus\gamma_{N_2})((m_1,m_2)(n_1,n_2)) + (\gamma_{M_1}\oplus\gamma_{M_2}(m_1,m_2)\\
&= \sum_{m_1=n_1, m_2n_2\in E_2} \max\{\gamma_{M_1}(m_1),\gamma_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1, m_2=n_2} \max\{\gamma_{N_1}(m_1n_1,\gamma_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \max\{\gamma_{M_1}(m_1),\gamma_{M_1}(n_1),\gamma_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \max\{\gamma_{N_1}(m_1n_1),\gamma_{M_2}(m_2),\gamma_{M_2}(n_2)\}\\
&+ \max\{\gamma_{M_1}(m_1),\gamma_{M_2}(m_2)\}
\end{aligned}
$$

$$\begin{aligned}(td_\delta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\delta_{N_1}\oplus\delta_{N_2})((m_1,m_2)(n_1,n_2)) + (\delta_{M_1}\oplus\delta_{M_2}(m_1,m_2)\\
&= \sum_{m_1=n_1,m_2n_2\in E_2} \max\{\delta_{M_1}(m_1),\delta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1,m_2=n_2} \max\{\delta_{N_1}(m_1n_1,\delta_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \max\{\delta_{M_1}(m_1),\delta_{M_1}(n_1),\delta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \max\{\delta_{N_1}(m_1n_1),\delta_{M_2}(m_2),\delta_{M_2}(n_2)\}\\
&+ \min\{\delta_{M_1}(m_1),\delta_{M_2}(m_2)\}\end{aligned}$$

$$\begin{aligned}(td_\eta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\eta_{N_1}\oplus\eta_{N_2})((m_1,m_2)(n_1,n_2)) + (\eta_{M_1}\oplus\eta_{M_2}(m_1,m_2)\\
&= \sum_{m_1=n_1,m_2n_2\in E_2} \min\{\eta_{M_1}(m_1),\eta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1,m_2=n_2} \min\{\eta_{N_1}(m_1n_1,\eta_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \min\{\eta_{M_1}(m_1),\eta_{M_1}(n_1),\eta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \min\{\eta_{N_1}(m_1n_1),\eta_{M_2}(m_2),\eta_{M_2}(n_2)\}\\
&+ \max\{\eta_{M_1}(m_1),\eta_{M_2}(m_2)\}\end{aligned}$$

$$\begin{aligned}(td_\theta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\theta_{N_1}\oplus\theta_{N_2})((m_1,m_2)(n_1,n_2)) + (\theta_{M_1}\oplus\theta_{M_2}(m_1,m_2)\\
&= \sum_{m_1=n_1,m_2n_2\in E_2} \min\{\theta_{M_1}(m_1),\theta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1,m_2=n_2} \min\{\theta_{N_1}(m_1n_1,\theta_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \min\{\theta_{M_1}(m_1),\theta_{M_1}(n_1),\theta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \min\{\theta_{N_1}(m_1n_1),\theta_{M_2}(m_2),\theta_{M_2}(n_2)\}\\
&+ \max\{\theta_{M_1}(m_1),\theta_{M_2}(m_2)\}\end{aligned}$$

Theorem 2.11. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, Y_2)$ be two BSVNGs. If

(i)

$\alpha_{M_1} \geq \alpha_{N_2}$ *and* $\alpha_{M_2} \geq \alpha_{N_1}$ then $\forall(m_1, m_2) \in V_1 \times V_2$

$$(td_\alpha)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1, m_2) = q(td_\alpha)_{\mathtt{G}_1}(m_1) + s(td_\alpha)_{\mathtt{G}_2}(m_2) - (q-1)T_{\mathtt{G}_1}(m_1) - \max\{T_{\mathtt{G}_1}(m_1), T_{\mathtt{G}_1}(m_1)\}$$

and

$\delta_{M_1} \leq \delta_{N_2}$ *and* $\delta_{M_2} \leq \delta_{N_1}$ then $\forall(m_1, m_2) \in V_1 \times V_2$

$$(td_\delta)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1), m_2) = q(td_\delta)_{\mathtt{G}_1}(m_1) + s(td_\delta)_{\mathtt{G}_2}(m_2) - (q-1)T_{\mathtt{G}_1}(m_1) - \min\{T_{\mathtt{G}_1}(m_1), T_{\mathtt{G}_1}(m_1)\}$$

(ii) $\beta_{M_1} \leq \beta_{N_2}$ *and* $\beta_{M_2} \leq \beta_{N_1}$ then $\forall(m_1, m_2) \in V_1 \times V_2$

$$(td_\beta)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1, m_2) = q(td_\beta)_{\mathtt{G}_1}(m_1) + s(td_\beta)_{\mathtt{G}_2}(m_2) - (q-1)I_{\mathtt{G}_1}(m_1) - \min\{I_{\mathtt{G}_1}(m_1), I_{\mathtt{G}_1}(m_1\}$$

and

$\eta_{M_1} \geq \eta_{N_2}$ *and* $\eta_{M_2} \geq \eta_{N_1}$ then $\forall(m_1, m_2) \in V_1 \times V_2$

$$(td_\eta)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1, m_2) = q(td_\eta)_{\mathtt{G}_1}(m_1) + s(td_\eta)_{\mathtt{G}_2}(m_2) - (q-1)I_{\mathtt{G}_1}(m_1) - \max\{I_{\mathtt{G}_1}(m_1), I_{\mathtt{G}_1}(m_1)\}$$

(iii) $\gamma_{M_1} \leq \gamma_{N_2}$ *and* $\gamma_{M_2} \geq \gamma_{N_1}$ then $\forall(m_1, m_2) \in V_1 \times V_2$

$$(td_\gamma)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1, m_2) = q(td_\gamma)_{\mathtt{G}_1}(m_1) + s(td_\gamma)_{\mathtt{G}_2}(m_2) - (q-1)F_{\mathtt{G}_1}(m_1) - \min\{F_{\mathtt{G}_1}(m_1), F_{\mathtt{G}_1}(m_1)\}$$

and

$\theta_{M_1} \geq \theta_{N_2}$ *and* $\theta_{M_2} \leq \theta_{N_1}$ then $\forall(m_1, m_2) \in V_1 \times V_2$

$$(td_\theta)_{\mathtt{G}_1 \oplus \mathtt{G}_2}(m_1, m_2) = q(td_\theta)_{\mathtt{G}_1}(m_1) + s(td_\theta)_{\mathtt{G}_2}(m_2) - (q-1)F_{\mathtt{G}_1}(m_1) - \max\{F_{\mathtt{G}_1}(m_1), F_{\mathtt{G}_1}(m_1)\}$$

$\forall(m_1, m_2) \in V_1 \times V_2$,s=$\mid V_1 \mid$ -$(d)_{G_1}(m_1)$ and q=$\mid V_2 \mid$ -$(d)_{G_2}(m_2)$.

Proof. $\forall(m_1,m_2)\in V_1\times V_2$

$$\begin{aligned}
(td_\alpha)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{N_1}\oplus\alpha_{N_2})((m_1,m_2)(n_1,n_2)) + (\alpha_{M_1}\oplus\alpha_{M_2})(m_1,m_2)\\
&= \sum_{m_1=n_1,m_2n_2\in E_2} \min\{\alpha_{M_1}(m_1),\alpha_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1,m_2=n_2} \min\{\alpha_{N_1}(m_1n_1),\alpha_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \min\{\alpha_{M_1}(m_1),\alpha_{M_1}(n_1),\alpha_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \min\{\alpha_{N_1}(m_1n_1),\alpha_{M_2}(m_2),\alpha_{M_2}(n_2)\}\\
&+ \max\{\alpha_{M_1}(m_1),\alpha_{M_2}(m_2)\}\\
&= \sum_{m_2n_2\in E_2} \alpha_{N_2}(m_2n_2) + \sum_{m_1n_1\in E_1} \alpha_{N_1}(m_1n_1)\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \alpha_{N_2}(m_2n_2)\} + \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \alpha_{N_1}(m_1n_1)\\
&+ \max\{\alpha_{M_1}(m_1),\alpha_{M_2}(m_2)\}\\
&= \sum_{m_2n_2\in E_2} \alpha_{N_2}(m_2n_2) + \sum_{m_1n_1\in E_1} \alpha_{N_1}(m_1n_1) + \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \alpha_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \alpha_{N_1}(m_1n_1) + \alpha_{M_1}(m_1) + \alpha_{M_2}(m_2) - \max\{\alpha_{M_1}(m_1),\alpha_{M_2}(m_2)\}\\
&= q(td_\alpha)_{\mathtt{G}_1}(m_1) + s(td_\alpha)_{\mathtt{G}_2}(m_2)\\
&- (q-1)T_{\mathtt{G}_1}(m_1) - \max\{T_{\mathtt{G}_1}(m_1),T_{\mathtt{G}_1}(m_1)\}
\end{aligned}$$

$$\begin{aligned}
(td_\delta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\delta_{N_1}\oplus\delta_{N_2})((m_1,m_2)(n_1,n_2)) + (\delta_{M_1}\oplus\delta_{M_2})(m_1,m_2)\\
&= \sum_{m_1=n_1,m_2n_2\in E_2} \max\{\delta_{M_1}(m_1),\delta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1,m_2=n_2} \max\{\delta_{N_1}(m_1n_1),\delta_{M_2}(m_2)\}\\
&+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2} \max\{\delta_{M_1}(m_1),\delta_{M_1}(n_1),\delta_{N_2}(m_2n_2)\}\\
&+ \sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2} \max\{\delta_{N_1}(m_1n_1),\delta_{M_2}(m_2),\delta_{M_2}(n_2)\}\\
&+ \min\{\delta_{M_1}(m_1),\delta_{M_2}(m_2)\}\\
&= \sum_{m_2n_2\in E_2} \delta_{N_2}(m_2n_2) + \sum_{m_1n_1\in E_1} \delta_{N_1}(m_1n_1)
\end{aligned}$$

$$+\sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2}\delta_{N_2}(m_2n_2)\}+\sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2}\delta_{N_1}(m_1n_1)$$
$$+\min\{\delta_{M_1}(m_1),\delta_{M_2}(m_2)\}$$
$$=\sum_{m_2n_2\in E_2}\delta_{N_2}(m_2n_2)+\sum_{m_1n_1\in E_1}\delta_{N_1}(m_1n_1)+\sum_{m_1n_1\notin E_1 and\ m_2n_2\in E_2}\delta_{N_2}(m_2n_2)\}$$
$$+\sum_{m_1n_1\in E_1 and\ m_2n_2\notin E_2}\delta_{N_1}(m_1n_1)+\delta_{M_1}(m_1)+\delta_{M_2}(m_2)-\min\{\delta_{M_1}(m_1),\delta_{M_2}(m_2)\}$$
$$=q(td_\delta)_{\mathtt{G}_1}(m_1)+s(td_\delta)_{\mathtt{G}_2}(m_2)$$
$$-(q-1)T_{\mathtt{G}_1}(m_1)-\min\{T_{\mathtt{G}_1}(m_1),T_{\mathtt{G}_1}(m_1)\}$$

In a similar way others four will proved obviously.
where s=$|V_1|$ -$(d)_{G_1}(m_1)$ and q=$|V_2|$ -$(d)_{G_2}(m_2)$ □

Example 2.12. In Example 2.6 we have to find the degree and total degree of vertices of $\mathtt{G}_1\oplus\mathtt{G}_2$ by using Figure 2, Figure 3, and Figure 4.

$$(d_\alpha)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c)=q(d_\alpha)_{\mathtt{G}_1}(a)+s(d_\alpha)_{\mathtt{G}_2}(c)$$

where s=$|V_1|$ -$(d)_{G_1}(a)$ and q=$|V_2|$ -$(d)_{G_2}(e)$

$$s=|V_1|-(d)_{G_1}(a)=2-1=1,\ q=|V_2|-(d)_{G_2}(e)=2-1=1$$

$$(d_\alpha)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c)=q(d_\alpha)_{\mathtt{G}_1}(a)+s(d_\alpha)_{\mathtt{G}_2}(c)=1(0.4)+1(0.5)=0.4+0.5=0.9$$

$$(d_\beta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c)=q(d_\beta)_{\mathtt{G}_1}(a)+s(d_\beta)_{\mathtt{G}_2}(c)=1(0.2)+1(0.4)=0.2+0.4=0.6$$

$$(d_\gamma)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c)=0.7,\ (d_\delta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c)=-1.1$$

$$(d_\eta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c)=-0.5,\ (d_\theta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c)=-0.7$$

$$So\ (d)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,e)=(0.9,0.6,-1.1,-0.5,-0.7)$$

By applying this technique we can find degree of all vertices in a similar way. Now we will find total degree of vertices. For this select vertex (a,e)

$$(td_\alpha)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c)=q(td_\alpha)_{\mathtt{G}_1}(a)+s(td_\alpha)_{\mathtt{G}_2}(c)$$
$$-(s-1)\alpha_{\mathtt{G}_2}(c)-(q-1)\alpha_{\mathtt{G}_1}(a)-\max\{\alpha_{\mathtt{G}_1}(a),\alpha_{\mathtt{G}_2}(c)\}$$
$$=1(0.7+0.4)+1(0.6+0.5)-(1-1)(0.6)-(1-1)(0.7)$$
$$-\max\{0.6,0.7\}=1(1.1)+1.1-0.7=1.5$$

$$
\begin{aligned}
(td_\delta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c) &= q(td_\delta)_{\mathtt{G}_1}(a) + s(td_\delta)_{\mathtt{G}_2}(c)\\
&- (s-1)\delta_{\mathtt{G}_2}(c) - (q-1)\delta_{\mathtt{G}_1}(a) - \min\{\delta_{\mathtt{G}_1}(a), \delta_{\mathtt{G}_2}(c)\}\\
&= 1(-0.6-0.2) + 1(-0.5-0.3) - (1-1)(-0.5) - (1-1)(-0.6)\\
&- \min\{-0.5,-0.6\} = (-0.8-0.8+0.6 = -1.0
\end{aligned}
$$

$$(td_\beta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c) = 1.0,\ (td_\gamma)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c) = 1.3$$

$$(td_\eta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c) = -1.1,\ (td_\theta)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c) = -1.7$$

$$(td)_{\mathtt{G}_1\oplus\mathtt{G}_2}(a,c) = (1.5, 1.0, 1.3, -1.0, -1.1, -1.7)$$

By applying this technique we can find total degree of all vertices in a similar way.

Definition 2.13. let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ are two bipolar single valued neutrosophic fuzzy graphs defined on $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. The Residue product of $\mathtt{G}_1$ and $\mathtt{G}_2$ is represented by $\mathtt{G}_1 \bullet \mathtt{G}_2 = (M_1 \bullet M_2, N_1 \bullet N_2)$. Residue product of $\mathtt{G}_1$ *and* $\mathtt{G}_2$ is defined as the following conditions: (i)

$$
\begin{aligned}
(\alpha_{M_1}\bullet\alpha_{M_2})((m_1,m_2)) &= \max\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\},\ (\beta_{M_1}\bullet\beta_{M_2})((m_1,m_2))\\
&= \min\{\beta_{M_1}(m_1), \beta_{M_2}(m_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(\gamma_{M_1}\bullet\gamma_{M_2})((m_1,m_2)) &= \min\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2)\},\ (\delta_{M_1}\bullet\delta_{M_2})((m_1,m_2))\\
&= \min\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(\eta_{M_1}\bullet\eta_{M_2})((m_1,m_2)) &= \max\{\eta_{M_1}(m_1), \eta_{M_2}(m_2)\},\ (\theta_{M_1}\bullet\theta_{M_2})((m_1,m_2))\\
&= \max\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\}
\end{aligned}
$$

$\forall (m_1, m_2) \in (V_1 \times V_2)$

(ii)

$$(\alpha_{N_1}\bullet\alpha_{N_2})((m_1,m_2)(n_1,n_2)) = \alpha_{N_1}(m_1n_1),\ (\beta_{N_1}\bullet\beta_{N_2})((m_1,m_2)(n_1,n_2)) = \beta_{N_1}(m_1n_1)$$

$$(\gamma_{N_1}\bullet\gamma_{N_2})((m_1,m_2)(n_1,n_2)) = \gamma_{N_1}(m_1n_1),\ (\delta_{N_1}\bullet\delta_{N_2})((m_1,m_2)(n_1,n_2)) = \delta_{N_1}(m_1n_1)$$

$$(\eta_{N_1}\bullet\eta_{N_2})((m_1,m_2)(n_1,n_2)) = \eta_{N_1}(m_1n_1),\ (\theta_{N_1}\bullet\theta_{N_2})((m_1,m_2)(n_1,n_2)) = \theta_{N_1}(m_1n_1)$$

$\forall m_1n_1 \in E_1, m_2 \neq n_2.$

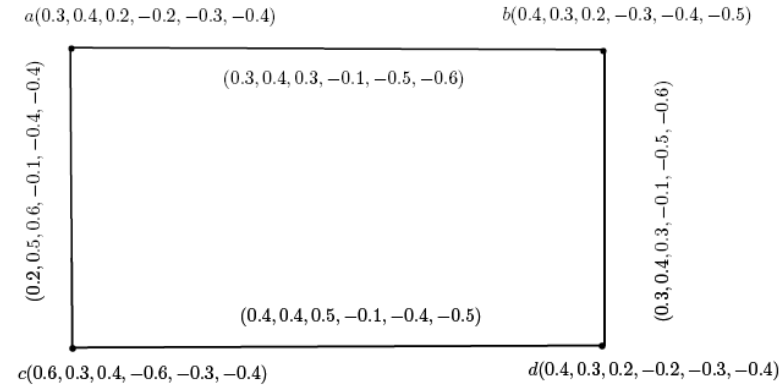

FIGURE 5. $\mathtt{G}_1$

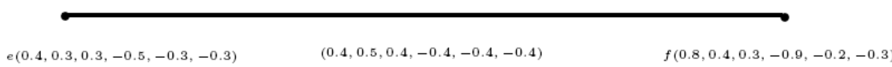

FIGURE 6. $\mathtt{G}_2$

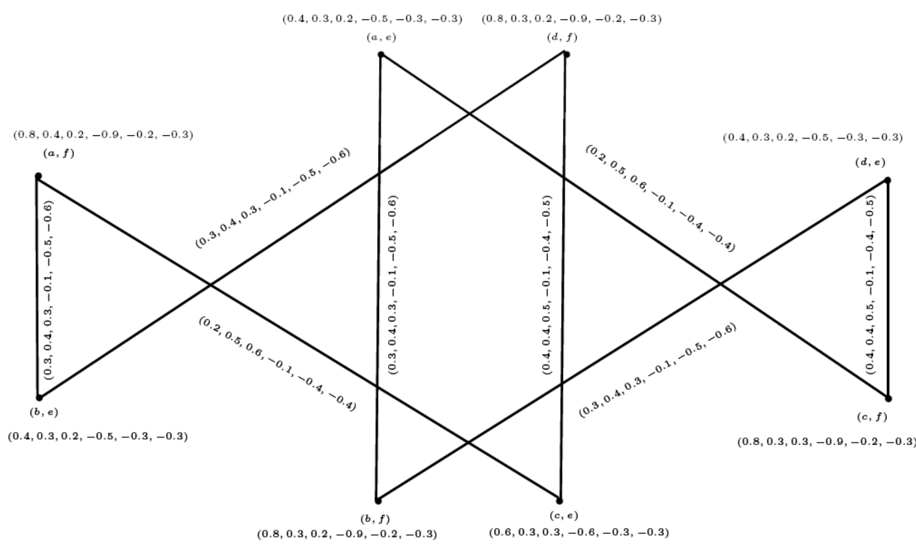

FIGURE 7. $\mathtt{G}_1 \bullet \mathtt{G}_2$

Example 2.14. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs on $V_1 = \{a, b, c, d\}$ *and* $V_2 = \{e, f\}$ respectively which shown in Figure 5 and Figure 6. Also Residue product is shown in Figure 7.

Proposition 2.15. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs of graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Then the Residue product $\mathtt{G}_1 \bullet \mathtt{G}_2$ of $\mathtt{G}_1 = (V_1, E_1)$ and $\mathtt{G}_2 = (V_2, E_2)$ is a BSVNG.

Proof. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs of graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Let $(m_1, m_2)(n_1, n_2) \in E_1 \times E_2$ If $m_1n_1 \in E_1$ *and* $m_2 \neq n_2$ then

$$\begin{aligned}
(\alpha_{N_1} \bullet \alpha_{N_2})((m_1, m_2)(n_1, n_2)) &= \alpha_{N_1}(m_1n_1) \\
&\leq \min\{\alpha_{M_1}(m_1), \alpha_{M_1}(n_1)\} \\
&\leq \max\{\min\{\alpha_{M_1}(m_1), \alpha_{M_1}(n_1)\}, \min\{\alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\}\} \\
&= \min\{\max\{\alpha_{M_1}(m_1), \alpha_{M_1}(n_1)\}, \max\{\alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\}\} \\
&= \min\{(\alpha_{M_1} \bullet \alpha_{M_2})(m_1, m_2), (\alpha_{M_1} \bullet \alpha_{M_2})(n_1, n_2)\}
\end{aligned}$$

$$\begin{aligned}
(\beta_{N_1} \bullet \beta_{N_2})((m_1, m_2)(n_1, n_2)) &= \beta_{N_1}(m_1n_1) \\
&\geq \max\{\beta_{M_1}(m_1), \beta_{M_1}(n_1)\} \\
&\geq \min\{\max\{\beta_{M_1}(m_1), \beta_{M_1}(n_1)\}, \max\{\beta_{M_2}(m_2), \beta_{M_2}(n_2)\}\} \\
&= \max\{\min\{\beta_{M_1}(m_1), \beta_{M_1}(n_1)\}, \min\{\beta_{M_2}(m_2), \beta_{M_2}(n_2)\}\} \\
&= \max\{(\beta_{M_1} \bullet \beta_{M_2})(m_1, m_2), (\beta_{M_1} \bullet \beta_{M_2})(n_1, n_2)\}
\end{aligned}$$

$$\begin{aligned}
(\gamma_{N_1} \bullet \gamma_{N_2})((m_1, m_2)(n_1, n_2)) &= \gamma_{N_1}(m_1n_1) \\
&\geq \max\{\gamma_{M_1}(m_1), \gamma_{M_1}(n_1)\} \\
&\geq \min\{\max\{\gamma_{M_1}(m_1), \gamma_{M_1}(n_1)\}, \max\{\gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\}\} \\
&= \max\{\min\{\gamma_{M_1}(m_1), \gamma_{M_1}(n_1)\}, \min\{\gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\}\} \\
&= \max\{(\gamma_{M_1} \bullet \gamma_{M_2})(m_1, m_2), (\gamma_{M_1} \bullet \gamma_{M_2})(n_1, n_2)\}
\end{aligned}$$

$$\begin{aligned}
(\delta_{N_1} \bullet \delta_{N_2})((m_1, m_2)(n_1, n_2)) &= \delta_{N_1}(m_1n_1) \\
&\geq \max\{\delta_{M_1}(m_1), \delta_{M_1}(n_1)\} \\
&\geq \min\{\max\{\delta_{M_1}(m_1), \delta_{M_1}(n_1)\}, \max\{\delta_{M_2}(m_2), \delta_{M_2}(n_2)\}\} \\
&= \max\{\min\{\delta_{M_1}(m_1), \delta_{M_1}(n_1)\}, \min\{\delta_{M_2}(m_2), \delta_{M_2}(n_2)\}\} \\
&= \max\{(\delta_{M_1} \bullet \delta_{M_2})(m_1, m_2), (\delta_{M_1} \bullet \delta_{M_2})(n_1, n_2)\}
\end{aligned}$$

$$\begin{aligned}
(\eta_{N_1} \bullet \eta_{N_2})((m_1, m_2)(n_1, n_2)) &= \eta_{N_1}(m_1n_1) \\
&\leq \min\{\eta_{M_1}(m_1), \eta_{M_1}(n_1)\} \\
&\leq \max\{\min\{\eta_{M_1}(m_1), \eta_{M_1}(n_1)\}, \min\{\eta_{M_2}(m_2), \eta_{M_2}(n_2)\}\} \\
&= \min\{\max\{\eta_{M_1}(m_1), \eta_{M_1}(n_1)\}, \max\{\eta_{M_2}(m_2), \eta_{M_2}(n_2)\}\} \\
&= \min\{(\eta_{M_1} \bullet \eta_{M_2})(m_1, m_2), (\eta_{M_1} \bullet \eta_{M_2})(n_1, n_2)\}
\end{aligned}$$

$$\begin{aligned}(\theta_{N_1} \bullet \theta_{N_2})((m_1, m_2)(n_1, n_2)) &= \theta_{N_1}(m_1 n_1)\\ &\leq \min\{\theta_{M_1}(m_1), \theta_{M_1}(n_1)\}\\ &\leq \max\{\min\{\theta_{M_1}(m_1), \theta_{M_1}(n_1)\}, \min\{\theta_{M_2}(m_2), \theta_{M_2}(n_2)\}\}\\ &= \min\{\max\{\theta_{M_1}(m_1), \theta_{M_1}(n_1)\}, \max\{\theta_{M_2}(m_2), \theta_{M_2}(n_2)\}\}\\ &= \min\{(\theta_{M_1} \bullet \theta_{M_2})(m_1, m_2), (\theta_{M_1} \bullet \theta_{M_2})(n_1, n_2)\}\end{aligned}$$

□

Definition 2.16. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs.For any vertex$(m_1, m_2) \in V_1 \times V_2$

$$\begin{aligned}(d_\alpha)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{N_1} \bullet \alpha_{N_2})((m_1, m_2)(n_1, n_2))\\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} \alpha_{N_1}(m_1 n_1) = (d_\alpha)_{\mathtt{G}_1}(m_1)\end{aligned}$$

$$\begin{aligned}(d_\beta)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{N_1} \bullet \beta_{N_2})((m_1, m_2)(n_1, n_2))\\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} \beta_{N_1}(m_1 n_1) = (d_\beta)_{\mathtt{G}_1}(m_1)\end{aligned}$$

$$\begin{aligned}(d_\gamma)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{N_1} \bullet \gamma_{N_2})((m_1, m_2)(n_1, n_2))\\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} \gamma_{N_1}(m_1 n_1) = (d_\gamma)_{\mathtt{G}_1}(m_1)\end{aligned}$$

$$\begin{aligned}(d_\delta)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\delta_{N_1} \bullet \delta_{N_2})((m_1, m_2)(n_1, n_2))\\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} \delta_{N_1}(m_1 n_1) = (d_\delta)_{\mathtt{G}_1}(m_1)\end{aligned}$$

$$\begin{aligned}(d_\eta)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\eta_{N_1} \bullet \eta_{N_2})((m_1, m_2)(n_1, n_2))\\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} \eta_{N_1}(m_1 n_1) = (d_\eta)_{\mathtt{G}_1}(m_1)\end{aligned}$$

$$\begin{aligned}(d_\theta)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\theta_{N_1} \bullet \theta_{N_2})((m_1, m_2)(n_1, n_2))\\ &= \sum_{m_1 n_1 \in E_1, m_2 \neq n_2} \theta_{N_1}(m_1 n_1) = (d_\theta)_{\mathtt{G}_1}(m_1)\end{aligned}$$

Definition 2.17. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs. For any vertex$(m_1, m_2) \in V_1 \times V_2$

$$\begin{aligned}
(td_\alpha)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{N_1} \bullet \alpha_{N_2})((m_1, m_2)(n_1, n_2)) + (\alpha_{M_1} \bullet \alpha_{M_2})(m_1, m_2)) \\
&= \sum_{m_1n_1\in E_1, m_2\neq n_2} \alpha_{N_1}(m_1n_1) + \min\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\} \\
&= \sum_{m_1n_1\in E_1, m_2\neq n_2} \alpha_{N_1}(m_1n_1) + \alpha_{M_1}(m_1) + \alpha_{M_2}(m_2) - \max\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\} \\
&= (td_\alpha)_{\mathtt{G}_1}(m_1) + \alpha_{M_2}(m_2) - \max\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\}
\end{aligned}$$

$$\begin{aligned}
(td_\beta)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{N_1} \bullet \beta_{N_2})((m_1, m_2)(n_1, n_2)) + (\beta_{M_1} \bullet \beta_{M_2}(m_1, m_2)) \\
&= \sum_{m_1n_1\in E_1, m_2\neq n_2} \beta_{N_1}(m_1n_1) + \max\{\beta_{M_1}(m_1), \beta_{M_2}(m_2)\} \\
&= \sum_{m_1n_1\in E_1, m_2\neq n_2} \beta_{N_1}(m_1n_1) + \beta_{M_1}(m_1) + \beta_{M_2}(m_2) - \min\{\beta_{M_1}(M_1), \beta_{M_2}(m_2)\} \\
&= (td_\beta)_{\mathtt{G}_1}(m_1) + \beta_{M_2}(m_2) - \min\{\beta_{M_1}(m_1), \beta_{M_2}(m_2)\}
\end{aligned}$$

$$\begin{aligned}
(td_\gamma)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{N_1} \bullet \gamma_{N_2})((m_1, m_2)(n_1, n_2)) + (\gamma_{M_1} \bullet \gamma_{M_2}(m_1, m_2)) \\
&= \sum_{m_1n_1\in E_1, m_2\neq n_2} \gamma_{N_1}(m_1n_1) + \max\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2)\} \\
&= \sum_{m_1n_1\in E_1, m_2\neq n_2} \gamma_{N_1}(m_1n_1) + \gamma_{M_1}(m_1) + \gamma_{M_2}(m_2) - \min\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2)\} \\
&= (td_\gamma)_{\mathtt{G}_1}(m_1) + \gamma_{M_2}(m_2) - \min\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2)\}
\end{aligned}$$

$$\begin{aligned}
(td_\delta)_{\mathtt{G}_1 \bullet \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\delta_{N_1} \bullet \delta_{N_2})((m_1, m_2)(n_1, n_2)) + (\delta_{M_1} \bullet \delta_{M_2})(m_1, m_2)) \\
&= \sum_{m_1n_1\in E_1, m_2\neq n_2} \delta_{N_1}(m_1n_1) + \max\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\} \\
&= \sum_{m_1n_1\in E_1, m_2\neq n_2} \delta_{N_1}(m_1n_1) + \delta_{M_1}(m_1) + \delta_{M_2}(m_2) - \min\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\} \\
&= (td_\delta)_{\mathtt{G}_1}(m_1) + \delta_{M_2}(m_2) - \min\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\}
\end{aligned}$$

$$(td_\eta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\eta_{N_1}\bullet\eta_{N_2})((m_1,m_2)(n_1,n_2)) + (\eta_{M_1}\bullet\eta_{M_2}(m_1,m_2))$$
$$= \sum_{m_1n_1\in E_1, m_2\neq n_2} \eta_{N_1}(m_1n_1) + \min\{\eta_{M_1}(m_1), \eta_{M_2}(m_2)\}$$
$$= \sum_{m_1n_1\in E_1, m_2\neq n_2} \eta_{N_1}(m_1n_1) + \eta_{M_1}(m_1) + \eta_{M_2}(m_2) - \max\{I^-_{M_1}(m_1), \eta_{M_2}(m_2)\}$$
$$= (td_\eta)_{\mathtt{G}_1}(m_1) + \eta_{M_2}(m_2) - \max\{\eta_{M_1}(m_1), \eta_{M_2}(m_2)\}$$

$$(td_\theta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\theta_{N_1}\bullet\theta_{N_2})((m_1,m_2)(n_1,n_2)) + (\theta_{M_1}\bullet\theta_{M_2}(m_1,m_2))$$
$$= \sum_{m_1n_1\in E_1, m_2\neq n_2} \theta_{N_1}(m_1n_1) + \min\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\}$$
$$= \sum_{m_1n_1\in E_1, m_2\neq n_2} \theta_{N_1}(m_1n_1) + \theta_{M_1}(m_1) + \theta_{M_2}(m_2) - \max\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\}$$
$$= (td_\theta)_{\mathtt{G}_1}(m_1) + \theta_{M_2}(m_2) - \max\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\}$$

Example 2.18. In Example 2.14 we have to find the degree and total degree of vertices of $\mathtt{G}_1 \bullet \mathtt{G}_2$ by using Figure 5, Figure 6, and Figure 7.

$$(d_\beta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = (d_\beta)_{\mathtt{G}_1}(a) = 0.5 + 0.4 = 0.9$$

$$(d_\eta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = (d_\eta)_{\mathtt{G}_1}(a) = -0.4 - 0.5 = -0.9$$

$$(d_\alpha)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = 0.5,\ (d_\gamma)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = 0.9$$

$$(d_\delta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = -0.2,\ (d_\theta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = -1.0$$

$$(d)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = (0.5, 0.9, 0.9, -0.2, -0.9, -1.0)$$

By applying same method we can find degree of all vertices. Now we are to find total degree of vertices. For this select vertices (a,f)

$$(td_\beta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = (td_\beta)_{\mathtt{G}_1}(a) + \beta_{M_2}(f) - \min\{\beta_{M_1}(a), \beta_{M_2}(f)\}$$
$$= (0.5 + 0.4 + 0.4) + 0.8 - \min(0.3, 0.8)$$
$$= 1.3 + 0.8 - 0.3 = 1.8$$

$$(td_\eta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = (td_\eta)_{\mathtt{G}_1}(a) + \eta_{M_2}(f) - \max\{\eta_{M_1}(a), \eta_{M_2}(f)\}$$
$$= (-0.4 - 0.3 - 0.5) + (-0.2) - \max(-0.3, -0.2)$$
$$= -1.2 - 0.2 + 0.2 = -1.2$$

$$(td_\gamma)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = 1.1,\ (td_\delta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = -0.4$$

$$(td_\theta)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = -1.4,\ (td_\alpha)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = 0.8$$

$$So\ (td)_{\mathtt{G}_1\bullet\mathtt{G}_2}(a,f) = (0.8, 1.8, 1.1 - 0.4, -1.2, -1.4)$$

by applying similar method we can find total degree of all others vertices in a similar way.

Definition 2.19. let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ are bipolar single valued neutrosophic fuzzy graphs defined on $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_2)$ respectively. The maximal product of $\mathtt{G}_1$ and $\mathtt{G}_2$ is represented by $\mathtt{G}_1 * \mathtt{G}_2 = (M_1 * M_2, N_1 \oplus N_2)$. The Maximal product of $\mathtt{G}_1$ *and* $\mathtt{G}_2$ is defined as the following conditions (i)

$$(\alpha_{M_1} * \alpha_{M_2})((m_1, m_2)) = \max\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\},\ (\beta_{M_1} * \beta_{M_2})((m_1, m_2)) = \min\{\beta_{M_1}(m_1), \beta_{M_2}(m_2)\}$$

$$(\gamma_{M_1} * \gamma_{M_2})((m_1, m_2)) = \min\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2)\},\ (\delta_{M_1} * \delta_{M_2})((m_1, m_2)) = \min\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\}$$

$$(\eta_{M_1} * \eta_{M_2})((m_1, m_2)) = \max\{\eta_{M_1}(m_1), \eta_{M_2}(m_2)\},\ (\theta_{M_1} * \theta_{M_2})((m_1, m_2)) = \max\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\}$$

$\forall\ (m_1, m_2) \in (V_1 \times V_2)$

(ii)

$$(\alpha_{M_1} * \alpha_{M_2})((m, m_2)(m, n_2)) = \max\{\alpha_{M_1}(m), \alpha_{N_2}(m_2n_2)\},\ (\beta_{M_1} * \beta_{M_2})((m, m_2)(m, n_2)) = \min\{\beta_{M_1}(m), \beta_{N_2}(m_2n_2)\}$$

$$(\gamma_{M_1} * \gamma_{M_2})((m, m_2)(m, n_2)) = \min\{\gamma_{M_{m_1}}(m), \gamma_{N_2}(m_2n_2)\},\ (\delta_{M_1} * \delta_{M_2})((m, m_2)(m, n_2)) = \min\{\delta_{M_1}(m), \delta_{N_2}(m_2n_2)\}$$

$$(\eta_{M_1} * \eta_{M_2})((m, m_2)(m, n_2)) = \max\{\eta_{M_1}(m), \eta_{N_2}(m_2n_2)\},\ (\theta_{M_1} * \theta_{M_2})((m, m_2)(m, n_2)) = \max\{\theta_{M_{m_1}}(m), \theta_{N_2}(m_2n_2)\}$$

$\forall\ m\ \in\ V_1\ and\ m_2n_2\ \in\ E_2$

(iii)

$$(\alpha_{M_1} * \alpha_{M_2})((m_1, m)(n_1, m)) = \max\{\alpha_{N_1}(m_1n_1), \alpha_{M_2}(m)\},\ (\beta_{M_1} * \beta_{M_2})((m_1, m)(n_1, m)) = \min\{\beta_{N_1}(m_1n_1), \beta_{M_2}(m)\}$$

$$(\gamma_{M_1} * \gamma_{M_2})((m_1, m)(n_1, m)) = \min\{\gamma_{N_1}(m_1n_1), \gamma_{M_2}(m)\},\ (\delta_{M_1} * \delta_{M_2})((m_1, m)(n_1, m)) = \min\{\delta_{N_1}(m_1n_1), \delta_{M_2}(m)\}$$

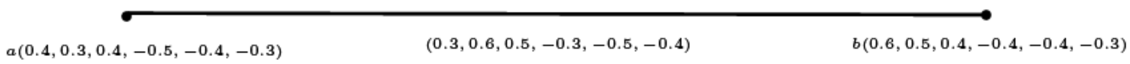

FIGURE 8. $\mathtt{G}_1$

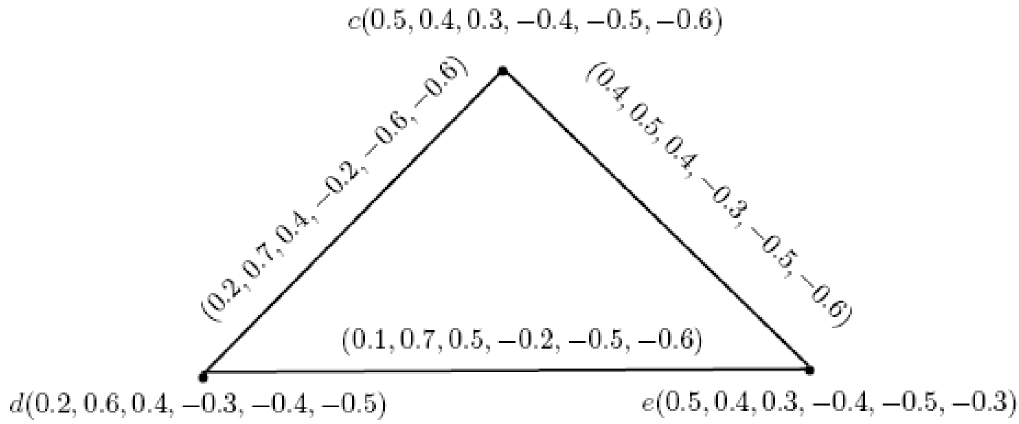

FIGURE 9. $\mathtt{G}_2$

$$(\eta_{M_1} * \eta_{M_2})((m_1, m)(n_1, m)) = \max\{\eta_{N_1}(m_1 n_1), \eta_{M_2}(m)\},\ (\theta_{M_1} * \theta_{M_2})((m_1, m)(n_1, m)) = \max\{\theta_{N_1}(m_1 n_1), \theta_{M_2}(m)\}$$

$\forall\ m\ \in\ V_2\ and\ m_1 n_1\ \in\ E_1$

Example 2.20. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs on $V_1 = \{a, b\}$ *and* $V_2 = \{c, d, e\}$ respectively which shown in Figure 8 and Figure 9. Also maximal product is shown in Figure 10.

Proposition 2.21. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs of graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Then then maximal product $\mathtt{G}_1 * \mathtt{G}_2$ of $\mathtt{G}_1 = (V_1, E_1)$ and $\mathtt{G}_2 = (V_2, E_2)$ is a BSVNG.

Proof. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs of graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Then the Maximal product $\mathtt{G}_1 * \mathtt{G}_2$ of $\mathtt{G}_1 = (V_1, E_1)$ and $\mathtt{G}_2 = (V_2, E_2)$ can be proved. Let $(m_1, m_2)(n_1, n_2) \in E_1 \times E_2$

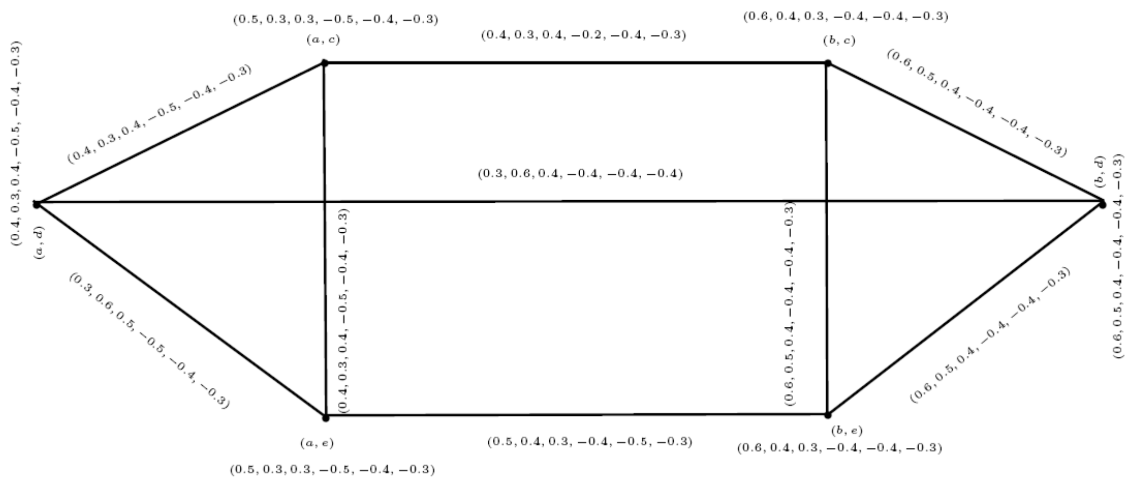

FIGURE 10. $\mathtt{G}_1 * \mathtt{G}_2$

(i) If $m_1 = n_1 = m$

$$\begin{aligned}(\alpha_{N_1} * \alpha_{N_2})((m, m_2)(m, n_2)) &= \max\{\alpha_{M_1}(m), \alpha_{N_2}(m_2 n_2)\} \\ &\leq \max\{\alpha_{M_1}(m), \min\{\alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\}\} \\ &= \min\{\max\{\{\alpha_{M_1}(m), \alpha_{M_2}(m_2)\}, \max\{\{\alpha_{M_1}(m), \alpha_{M_2}(n_2)\}\} \\ &= \min\{(\alpha_{M_1} * \alpha_{M_2})(m, m_2), (\alpha_{M_1} * \alpha_{M_2})(m, n_2)\}\end{aligned}$$

$$\begin{aligned}(\beta_{N_1} * \beta_{N_2})((m, m_2)(m, n_2)) &= \min\{\beta_{M_1}(m), \beta_{N_2}(m_2 n_2)\} \\ &\geq \min\{\beta_{M_1}(m), \max\{\beta_{M_2}(m_2), \beta_{M_2}(n_2)\}\} \\ &= \max\{\min\{\{\beta_{M_1}(m), \beta_{M_2}(m_2)\}, \min\{\{\beta_{M_1}(m), \beta_{M_2}(n_2)\}\} \\ &= \max\{(\beta_{M_1} * \beta_{M_2})(m, m_2), (\beta_{M_1} * \beta_{M_2})(m, n_2)\}\end{aligned}$$

$$\begin{aligned}(\gamma_{N_1} * \gamma_{N_2})((m, m_2)(m, n_2)) &= \min\{\gamma_{M_1}(m), \gamma_{N_2}(m_2 n_2)\} \\ &\geq \min\{\gamma_{M_1}(m), \max\{\gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\}\} \\ &= \max\{\min\{\{\gamma_{M_1}(m), \gamma_{M_2}(m_2)\}, \min\{\{\gamma_{M_1}(m), \gamma_{M_2}(n_2)\}\} \\ &= \max\{(\gamma_{M_1} * \gamma_{M_2})(m, m_2), (\gamma_{M_1} * \gamma_{M_2})(m, n_2)\}\end{aligned}$$

$$\begin{aligned}(\delta_{N_1} * \delta_{N_2})((m, m_2)(m, n_2)) &= \min\{\delta_{M_1}(m), \delta_{N_2}(m_2 n_2)\} \\ &\geq \min\{\delta_{M_1}(m), \max\{\delta_{M_2}(m_2), \delta_{M_2}(n_2)\}\} \\ &= \max\{\min\{\{\delta_{M_1}(m), \delta_{M_2}(m_2)\}, \min\{\{\delta_{M_1}(m), \delta_{M_2}(n_2)\}\} \\ &= \max\{(\delta_{M_1} * \delta_{M_2})(m, m_2), (\delta_{M_1} * \delta_{M_2})(m, n_2)\}\end{aligned}$$

$$\begin{aligned}(\eta_{N_1} * \eta_{N_2})((m, m_2)(m, n_2)) &= \max\{\eta_{M_1}(m), \eta_{N_2}(m_2 n_2)\} \\ &\leq \max\{\eta_{M_1}(m), \min\{\eta_{M_2}(m_2), \eta_{M_2}(n_2)\}\} \\ &= \min\{\max\{\{\eta_{M_1}(m), \eta_{M_2}(m_2)\}, \max\{\{\eta_{M_1}(m), \eta_{M_2}(n_2)\}\} \\ &= \min\{(\eta_{M_1} * \eta_{M_2})(m, m_2), (\eta_{M_1} * \eta_{M_2})(m, n_2)\}\end{aligned}$$

$$\begin{aligned}(\theta_{N_1} * \theta_{N_2})((m, m_2)(m, n_2)) &= \max\{\theta_{M_1}(m), \theta_{N_2}(m_2 n_2)\} \\ &\leq \max\{\theta_{M_1}(m), \min\{\theta_{M_2}(m_2), \theta_{M_2}(n_2)\}\} \\ &= \min\{\max\{\{\theta_{M_1}(m), \theta_{M_2}(m_2)\}, \max\{\{\theta_{M_1}(m), \theta_{M_2}(n_2)\}\} \\ &= \min\{(\theta_{M_1} * \theta_{M_2})(m, m_2), (\theta_{M_1} * \theta_{M_2})(m, n_2)\}\end{aligned}$$

(ii) If $m_2 = n_2 = m$

$$\begin{aligned}(\alpha_{N_1} * \alpha_{N_2})((m_1, m)(n_1, m)) &= \max\{\alpha_{N_1}(m_1 n_1), \alpha_{M_2}(m)\} \\ &\leq \max\{\min\{\alpha_{N_1}(m_1 n_1), \alpha_{M_2}(m)\} \\ &= \min\{\max\{\{\alpha_{N_1}(m_1), \alpha_{M_2}(m)\}, \max\{\{\alpha_{M_1}(n_1), \alpha_{M_2}(m)\}\} \\ &= \min\{(\alpha_{M_1} * \alpha_{M_2})(m_1, m), (\alpha_{M_1} * \alpha_{M_2})(n_1, m)\}\end{aligned}$$

$$\begin{aligned}(\beta_{N_1} * \beta_{N_2})((m_1, m)(n_1, m)) &= \min\{\beta_{N_1}(m_1 n_1), \beta_{M_2}(m)\} \\ &\geq \min\{\max\{\beta_{N_1}(m_1 n_1), \beta_{M_2}(m)\} \\ &= \max\{\min\{\{\beta_{N_1}(m_1), \beta_{M_2}(m)\}, \min\{\{\beta_{M_1}(n_1), \beta_{M_2}(m)\}\} \\ &= \max\{(\beta_{M_1} * \beta_{M_2})(m_1, m), (\beta_{M_1} * \beta_{M_2})(n_1, m)\}\end{aligned}$$

$$\begin{aligned}(\gamma_{N_1} * \gamma_{N_2})((m_1,m)(n_1,m)) &= \min\{\gamma_{N_1}(m_1n_1), \gamma_{M_2}(m)\}\\ &\geq \min\{\max\{\gamma_{N_1}(m_1n_1), \gamma_{M_2}(m)\}\\ &= \max\{\min\{\{\gamma_{N_1}(m_1), \gamma_{M_2}(m)\}, \min\{\{\gamma_{M_1}(n_1), \gamma_{M_2}(m)\}\}\\ &= \max\{(\gamma_{M_1} * \gamma_{M_2})(m_1,m), (\gamma_{M_1} * \gamma_{M_2})(n_1,m)\}\end{aligned}$$

$$\begin{aligned}(\delta_{N_1} * \delta_{N_2})((m_1,m)(n_1,m)) &= \min\{\delta_{N_1}(m_1n_1), \delta_{M_2}(m)\}\\ &\geq \min\{\max\{\delta_{N_1}(m_1n_1), \delta_{M_2}(m)\}\\ &= \max\{\min\{\{\delta_{N_1}(m_1), \delta_{M_2}(m)\}, \min\{\{\delta_{M_1}(n_1), \delta_{M_2}(m)\}\}\\ &= \max\{(\delta_{M_1} * \delta_{M_2})(m_1,m), (\delta_{M_1} * \delta_{M_2})(n_1,m)\}\end{aligned}$$

$$\begin{aligned}(\eta_{N_1} * \eta_{N_2})((m_1,m)(n_1,m)) &= \max\{\eta_{N_1}(m_1n_1), \eta_{M_2}(m)\}\\ &\leq \max\{\min\{\eta_{N_1}(m_1n_1), \eta_{M_2}(m)\}\\ &= \min\{\max\{\{\eta_{N_1}(m_1), \eta_{M_2}(m)\}, \max\{\{\eta_{M_1}(n_1), \eta_{M_2}(m)\}\}\\ &= \min\{(\eta_{M_1} * \eta_{M_2})(m_1,m), (\eta_{M_1} * \eta_{M_2})(n_1,m)\}\end{aligned}$$

$$\begin{aligned}(\theta_{N_1} * \theta_{N_2})((m_1,m)(n_1,m)) &= \max\{\theta_{N_1}(m_1n_1), \theta_{M_2}(m)\}\\ &\leq \max\{\min\{\theta_{N_1}(m_1n_1), \theta_{M_2}(m)\}\\ &= \min\{\max\{\{\theta_{N_1}(m_1), \theta_{M_2}(m)\}, \max\{\{\theta_{M_1}(n_1), \theta_{M_2}(m)\}\}\\ &= \min\{(\theta_{M_1} * \theta_{M_2})(m_1,m), (\theta_{M_1} * \theta_{M_2})(n_1,m)\}\end{aligned}$$

□

Definition 2.22. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs. $\forall (m_1, m_2) \in V_1 \times V_2$

$$\begin{aligned}(d_\alpha)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1,m_2) = &\sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{N_1} * \alpha_{N_2})((m_1,m_2)(n_1,n_2))\\ = &\sum_{m_1=n_1, m_2n_2\in E_2} \max\{\alpha_{M_1}(m_1), \alpha_{N_2}(m_2n_2)\}\\ + &\sum_{m_1n_1\in E_1, m_2=n_2} \max\{\alpha_{N_1}(m_1n_1), \alpha_{M_2}(m_2)\}\end{aligned}$$

$$(d_\beta)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{N_1}*\beta_{N_2})((m_1,m_2)(n_1,n_2))$$
$$= \sum_{m_1=n_1,m_2n_2\in E_2} \min\{\beta_{M_1}(m_1),\beta_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1,m_2=n_2} \min\{\beta_{N_1}(m_1n_1),\beta_{M_2}(m_2)\}$$

$$(d_\gamma)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{N_1}*\gamma_{N_2})((m_1,m_2)(n_1,n_2))$$
$$= \sum_{m_1=n_1,m_2n_2\in E_2} \min\{\gamma_{M_1}(m_1),\gamma_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1,m_2=n_2} \min\{\gamma_{N_1}(m_1n_1),\gamma_{M_2}(m_2)\}$$

$$(d_\delta)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\delta_{N_1}*\delta_{N_2})((m_1,m_2)(n_1,n_2))$$
$$= \sum_{m_1=n_1,m_2n_2\in E_2} \min\{\delta_{M_1}(m_1),\delta_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1,m_2=n_2} \min\{\delta_{N_1}(m_1n_1),\delta_{M_2}(m_2)\}$$

$$(d_\eta)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\eta_{N_1}*\eta_{N_2})((m_1,m_2)(n_1,n_2))$$
$$= \sum_{m_1=n_1,m_2n_2\in E_2} \max\{\eta_{M_1}(m_1),\eta_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1,m_2=n_2} \max\{\eta_{N_1}(m_1n_1),\eta_{M_2}(m_2)\}$$

$$(d_\theta)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\theta_{N_1}*\theta_{N_2})((m_1,m_2)(n_1,n_2))$$
$$= \sum_{m_1=n_1,m_2n_2\in E_2} \max\{\theta_{M_1}(m_1),\theta_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1,m_2=n_2} \max\{\theta_{N_1}(m_1n_1),\theta_{M_2}(m_2)\}$$

Theorem 2.23. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ are two BSVNGs. If $\alpha_{M_1} \geq \alpha_{N_2}, \beta_{M_1} \leq \beta_{N_2}, \gamma_{M_1} \leq \gamma_{N_2}$ and $\alpha_{M_2} \geq \alpha_{N_1}, \beta_{M_2} \leq \beta_{N_1}, \gamma_{M_2} \leq \gamma_{N_1}$. Also If $\delta_{M_1} \leq \delta_{N_2}, \eta_{M_1} \geq \eta_{N_2}, \theta_{M_1} \geq \theta_{N_2}$ and $\delta_{M_2} \leq \delta_{N_1}, \eta_{M_2} \geq \eta_{N_1}, \theta_{M_2} \geq \theta_{N_1}$ Then for every $\forall(m_1, m_2) \in V_1 \times V_2$
$(d_\alpha)_{\mathtt{G}_1*\mathtt{G}_2}(m_1, m_2) = (d)_{G_2}(m_2)\alpha_{M_1}(m_1) + (d)_{G_1}(m_1)\alpha_{M_2}(m_2)$
$(d_\beta)_{\mathtt{G}_1*\mathtt{G}_2}(m_1, m_2) = (d)_{\mathtt{G}_2}(m_2)\beta_{M_1}(m_1) + (d)_{G_1}(m_1)\beta_{M_2}(m_2)$
$(d_\gamma)_{\mathtt{G}_1*\mathtt{G}_2}(m_1, m_2) = (d)_{G_2}(m_2)\gamma_{M_1}(m_1) + (d)_{G_1}(m_1)\gamma_{M_2}(m_2)$
$(d_\delta)_{\mathtt{G}_1*\mathtt{G}_2}(m_1, m_2) = (d)_{G_2}(m_2)\delta_{M_1}(m_1) + (d)_{G_1}(m_1)\delta_{M_2}(m_2)$

$(d_\eta)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2)=(d)_{\mathtt{G}_2}(m_2)\eta_{M_1}(m_1)+(d)_{G_1}(m_1)\eta_{M_2}(m_2)$
$(d_\theta)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2)=(d)_{G_2}(m_2)\theta_{M_1}(m_1)+(d)_{G_1}(m_1)\theta_{M_2}(m_2)$

Proof.

$$\begin{aligned}(d_\alpha)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.}(\alpha_{N_1}*\alpha_{N_2})((m_1,m_2)(n_1,n_2))\\ &= \sum_{m_1=n_1,m_2n_2\in E_2}\max\{\alpha_{M_1}(m_1),\alpha_{N_2}(m_2n_2)\}\\ &+ \sum_{m_1n_1\in E_1,m_2=n_2}\max\{\alpha_{N_1}(m_1n_1),\alpha_{M_2}(m_2)\}\\ &= \sum_{m_2n_2\in E_2,m_1=n_1}\alpha_{N_2}(m_2n_2)+\sum_{m_1n_1\in E_1,m_2=n_2}\alpha_{N_1}(m_1n_1)\\ &= (d)_{G_2}(m_2)\alpha_{M_1}(m_1)+(d)_{G_1}(m_1)\alpha_{M_2}(m_2)\end{aligned}$$

$$\begin{aligned}(d_\delta)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.}(\delta_{N_1}*\delta_{N_2})((m_1,m_2)(n_1,n_2))\\ &= \sum_{m_1=n_1,m_2n_2\in E_2}\min\{\delta_{M_1}(m_1),\delta_{N_2}(m_2n_2)\}\\ &+ \sum_{m_1n_1\in E_1,m_2=n_2}\min\{\delta_{N_1}(m_1n_1),\delta_{M_2}(m_2)\}\\ &= \sum_{m_2n_2\in E_2,m_1=n_1}\delta_{N_2}(m_2n_2)+\sum_{m_1n_1\in E_1,m_2=n_2}\delta_{N_1}(m_1n_1)\\ &= (d)_{G_2}(m_2)\delta_{M_1}(m_1)+(d)_{G_1}(m_1)\delta_{M_2}(m_2)\end{aligned}$$

In a similar way others four will proved obviously. □

Definition 2.24. Let $\mathtt{G}_1=(M_1,N_1)$ and $\mathtt{G}_2=(M_2,N_2)$ be two BSVNGs. $\forall(m_1,m_2)\in\ V_1\times V_2$

$$\begin{aligned}(td_\alpha)_{\mathtt{G}_1*\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.}(\alpha_{N_1}*\alpha_{N_2})((m_1,m_2)(n_1,n_2))+(\alpha_{M_1}*\alpha_{M_2}(m_1,m_2)\\ &= \sum_{m_1=n_1,m_2n_2\in E_2}\max\{\alpha_{M_1}(m_1),\alpha_{N_2}(m_2n_2)\}\\ &+ \sum_{m_1n_1\in E_1,m_2=n_2}\max\{\alpha_{N_1}(m_1n_1),\alpha_{M_2}(m_2)\}\\ &+\max\{\alpha_{M_1}(m_1),\alpha_{M_2}(m_2)\}\end{aligned}$$

$$(td_\beta)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{N_1} * \beta_{N_2})((m_1,m_2)(n_1,n_2)) + (\beta_{M_1} * \beta_{M_2}(m_1,m_2)$$
$$= \sum_{m_1=n_1, m_2n_2\in E_2} \min\{\beta_{M_1}(m_1), \beta_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1, m_2=n_2} \min\{\beta_{N_1}(m_1n_1), \beta_{M_2}(m_2)\}$$
$$+ \min\{\beta_{M_1}(m_1), \beta_{M_2}(m_2)\}$$

$$(td_\gamma)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{N_1} * \gamma_{N_2})((m_1,m_2)(n_1,n_2)) + (\gamma_{M_1} * \gamma_{M_2}(m_1,m_2)$$
$$= \sum_{m_1=n_1, m_2n_2\in E_2} \min\{\gamma_{M_1}(m_1), \gamma_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1, m_2=n_2} \min\{\gamma_{N_1}(m_1n_1, \gamma_{M_2}(m_2)\}$$
$$+ \max\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2)\}$$

$$(td_\delta)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\delta_{N_1} * \delta_{N_2})((m_1,m_2)(n_1,n_2)) + (\delta_{M_1} * \delta_{M_2}(m_1,m_2)$$
$$= \sum_{m_1=n_1, m_2n_2\in E_2} \min\{\delta_{M_1}(m_1), \delta_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1, m_2=n_2} \min\{\delta_{N_1}(m_1n_1), \delta_{M_2}(m_2)\}$$
$$+ \min\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\}$$

$$(td_\eta)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\eta_{N_1} * \eta_{N_2})((m_1,m_2)(n_1,n_2)) + (\eta_{M_1} * \eta_{M_2}(m_1,m_2)$$
$$= \sum_{m_1=n_1, m_2n_2\in E_2} \max\{\eta_{M_1}(m_1), \eta_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1, m_2=n_2} \max\{\eta_{N_1}(m_1n_1), \eta_{M_2}(m_2)\}$$
$$+ \max\{\eta_{M_1}(m_1), \eta_{M_2}(m_2)\}$$

$$(td_\theta)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1,m_2) = \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\theta_{N_1} * \theta_{N_2})((m_1,m_2)(n_1,n_2)) + (\theta_{M_1} * \theta_{M_2}(m_1,m_2)$$
$$= \sum_{m_1=n_1, m_2n_2\in E_2} \max\{\theta_{M_1}(m_1), \theta_{N_2}(m_2n_2)\}$$
$$+ \sum_{m_1n_1\in E_1, m_2=n_2} \max\{\theta_{N_1}(m_1n_1, \theta_{M_2}(m_2)\}$$
$$+ \max\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\}$$

Theorem 2.25. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs. If $\alpha_{M_1} \geq \alpha_{N_2}, \beta_{M_1} \leq \beta_{N_2}, \gamma_{M_1} \leq \gamma_{N_2}$ and $\alpha_{M_2} \geq \alpha_{N_1}, \beta_{M_2} \leq \beta_{N_1}, \gamma_{M_2} \leq \gamma_{N_1}$. Also If $\delta_{M_1} \leq \delta_{N_2}, \eta_{M_1} \geq \eta_{N_2}, \theta_{M_1} \geq \theta_{N_2}$ and $\delta_{M_2} \leq \delta_{N_1}, \eta_{M_2} \geq \eta_{N_1}, \theta_{M_2} \geq \theta_{N_1}$ Then for every $\forall (m_1, m_2) \in V_1 \times V_2$

$(d_\alpha)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1, m_2) = (d)_{G_2}(m_2)\alpha_{M_1}(m_1) + (d)_{G_1}(m_1)\alpha_{M_2}(m_2)$

$(d_\beta)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1, m_2) = (d)_{\mathtt{G}_2}(m_2)\beta_{M_1}(m_1) + (d)_{G_1}(m_1)\beta_{M_2}(m_2)$

$(d_\gamma)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1, m_2) = (d)_{G_2}(m_2)\gamma_{M_1}(m_1) + (d)_{G_1}(m_1)\gamma_{M_2}(m_2)$

$(d_\delta)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1, m_2) = (d)_{G_2}(m_2)\delta_{M_1}(m_1) + (d)_{G_1}(m_1)\delta_{M_2}(m_2)$

$(d_\eta)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1, m_2) = (d)_{\mathtt{G}_2}(m_2)\eta_{M_1}(m_1) + (d)_{G_1}(m_1)\eta_{M_2}(m_2)$

$(d_\theta)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1, m_2) = (d)_{G_2}(m_2)\theta_{M_1}(m_1) + (d)_{G_1}(m_1)\theta_{M_2}(m_2)$

Proof.

$$
\begin{aligned}
(td_\alpha)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2) \in E_1 \times E_2.} (\alpha_{N_1} * \alpha_{N_2})((m_1, m_2)(n_1, n_2)) + (\alpha_{M_1} * \alpha_{M_2})(m_1, m_2) \\
&= \sum_{m_1 = n_1, m_2 n_2 \in E_2} \max\{\alpha_{M_1}(m_1), \alpha_{N_2}(m_2 n_2)\} \\
&+ \sum_{m_1 n_1 \in E_1, m_2 = n_2} \max\{\alpha_{N_1}(m_1 n_1), \alpha_{M_2}(m_2)\} \\
&+ \max\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\} \\
&= \sum_{m_2 n_2 \in E_2, m_1 = n_1} \alpha_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1, m_2 = n_2} \alpha_{N_1}(m_1 n_1) \\
&+ \max\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\} \\
&= (d)_{G_2}(m_2)\alpha_{M_1}(m_1) + (d)_{G_1}(m_1)\alpha_{M_2}(m_2) + max\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\}
\end{aligned}
$$

$$
\begin{aligned}
(td_\delta)_{\mathtt{G}_1 * \mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2) \in E_1 \times E_2.} (\delta_{N_1} * \delta_{N_2})((m_1, m_2)(n_1, n_2)) + (\delta_{M_1} * \delta_{M_2})(m_1, m_2) \\
&= \sum_{m_1 = n_1, m_2 n_2 \in E_2} \min\{\delta_{M_1}(m_1), \delta_{N_2}(m_2 n_2)\} \\
&+ \sum_{m_1 n_1 \in E_1, m_2 = n_2} \min\{\delta_{N_1}(m_1 n_1), \delta_{M_2}(m_2)\} \\
&+ \min\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\} \\
&= \sum_{m_2 n_2 \in E_2, m_1 = n_1} \delta_{N_2}(m_2 n_2) + \sum_{m_1 n_1 \in E_1, m_2 = n_2} \delta_{N_1}(m_1 n_1) \\
&+ min\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\} \\
&= (d)_{G_2}(m_2)\delta_{M_1}(m_1) + (d)_{G_1}(m_1)\delta_{M_2}(m_2) + min\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\}
\end{aligned}
$$

In a similar way others four will proved obviously. □

Example 2.26. In Example 2.20 we have to find the degree and total degree of vertices of $\mathtt{G}_1 * \mathtt{G}_2$ by using Figure 8, Figure 9, and Figure 10. Select the vertex (e,a).

$$(d_\alpha)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = (d)_{G_2}(c)\alpha_{M_1}(a) + (d)_{G_1}(a)\alpha_{M_2}(c)$$
$$= 2(0.4) + 1(0.5) = 0.8 + 0.5 = 1.3$$

$$(td_\delta)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = (d)_{G_2}(c)\delta_{M_1}(a) + (d)_{G_1}(a)\delta_{M_2}(c)$$
$$= 2(-0.5) + 1(-0.4) = -1.0 - 0.4 = -1.4$$

, $(d_\beta)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = 1.0$, $(d_\gamma)_{\mathtt{G}_1*\mathtt{G}_2}(e,a) = 1.1$, $(td_\eta)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = -1.3$, $(td_\theta)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = -1.2$. By applying the same method we can find the degree of all vertices.now we are find the total degree of vertices in maximal product. For this select the same vertex (e,a).

$$(td_\alpha)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = (d)_{G_2}(c)\alpha_{M_1}(a) + (d)_{G_1}(a)\alpha_{M_2}(c) + \max\{\alpha_{M_1}(a), \alpha_{M_2}(c)\}$$
$$= 2(0.4) + 1(0.5) + \max(0.4, 0.5) = 0.8 + 0.5 + 0.5 = 1.8$$

$$(td_\theta)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = (d)_{G_2}(c)\theta_{M_1}(a) + (d)_{G_1}(a)\theta_{M_2}(c) + \min\{\theta_{M_1}(a), \theta_{M_2}(c)\}$$
$$= 2(-0.3) + 1(-0.6) + \min(-0.3, -0.6) = -0.6 - 0.6 - 0.6 = -1.8$$

$(td_\beta)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = 1.3$,$(td_\gamma)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = 1.4$, $(td_\delta)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = -1.8$,$(td_\eta)_{\mathtt{G}_1*\mathtt{G}_2}(a,c) = -1.8$. By applying same method or technique we can find all other vertices total degree.

Definition 2.27. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ are two bipolar single valued neutrosophic fuzzy graphs defined on $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. The rejection of $\mathtt{G}_1$ and $\mathtt{G}_2$ is represented by $\mathtt{G}_1|\mathtt{G}_2 = (M_1|M_2, N_1|N_2)$. Rejection of $\mathtt{G}_1 and$ $\mathtt{G}_2$ is defined as the following conditions:

(i)

$$(\alpha_{M_1}|\alpha_{M_2})((m_1, m_2)) = \min\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\},\ (\beta_{M_1}|\beta_{M_2})((m_1, m_2)) = \max\{\beta_{M_1}(m_1), \beta_{M_2}(m_2)\}$$

$$(\gamma_{M_1}|\gamma_{M_2})((m_1, m_2)) = \max\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2)\},\ (\delta_{M_1}|\delta_{M_2})((m_1, m_2)) = \max\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\}$$

$$(\eta_{M_1}|\eta_{M_2})((m_1, m_2)) = \min\{\eta_{M_1}(m_1), \eta_{M_2}(m_2)\},\ (\theta_{M_1}|\theta_{M_2})((m_1, m_2)) = \min\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\}$$

$\forall\ (m_1, m_2) \in (V_1 \times V_2)$.

(ii)

$$(\alpha_{N_1}|\alpha_{N_2})((m, m_2)(m, n_2)) = \min\{\alpha_{M_1}(m), \alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\},\ (\beta_{N_1}|\beta_{N_2})((m, m_2)(m, n_2))$$
$$= \max\{\beta_{M_1}(m), \beta_{M_2}(m_2), \beta_{M_2}(n_2)\}$$

$(\gamma_{N_1}|\gamma_{N_2})((m, m_2)(m, n_2)) = \max\{\gamma_{M_1}(m), \gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\}$, $(\delta_{N_1}|\delta_{N_2})((m, m_2)(m, n_2))$
$= \max\{\delta_{M_1}(m), \delta_{M_2}(m_2), \delta_{M_2}(n_2)\}$

$(\eta_{N_1}|\eta_{N_2})((m, m_2)(m, n_2)) = \min\{\eta_{M_1}(m), \eta_{M_2}(m_2), \eta_{M_2}(n_2)\}$, $(\theta_{N_1}|\theta_{N_2})((m, m_2)(m, n_2))$
$= \min\{\theta_{M_1}(m), \theta_{M_2}(m_2), \theta_{M_2}(n_2)\}$

$\forall\ m \in V_2$ *and* $m_2n_2 \notin E_2$.

(iii)

$(\alpha_{N_1}|\alpha_{N_2})((m, m_2)(m, n_2)) = \min\{\alpha_{M_1}(m), \alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\}$, $(\beta_{N_1}|\beta_{N_2})((m, m_2)(m, n_2))$
$= \max\{\beta_{M_1}(m), \beta_{M_2}(m_2), \beta_{M_2}(n_2)\}$

$(\gamma_{N_1}|\gamma_{N_2})((m, m_2)(m, n_2)) = \max\{\gamma_{M_1}(m), \gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\}$, $(\delta_{N_1}|\delta_{N_2})((m, m_2)(m, n_2))$
$= \max\{\delta_{M_1}(m), \delta_{M_2}(m_2), \delta_{M_2}(n_2)\}$

$(\eta_{N_1}|\eta_{N_2})((m, m_2)(m, n_2)) = \min\{\eta_{M_1}(m), \eta_{M_2}(m_2), \eta_{M_2}(n_2)\}$, $(\theta_{N_1}|\theta_{N_2})((m, m_2)(m, n_2))$
$= \min\{\theta_{M_1}(m), \theta_{M_2}(m_2), \theta_{M_2}(n_2)\}$

$\forall\ z \in V_2$ *and* $m_1n_1 \notin E_1$.

(i∨) $(\alpha_{N_1}|\alpha_{N_2})((m_1, m_2)(n_1, n_2)) = \min\{\alpha_{M_1}(m_1), \alpha_{M_1}(n_1), \alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\}$,
$(\beta_{N_1}|\beta_{N_2})((m_1, m_2)(n_1, n_2)) = \max\{\beta_{M_1}(m_1), \beta_{M_1}(n_1), \beta_{M_2}(m_2), \alpha_{N_2}(n_2)\}$, $(\gamma_{N_1}|\gamma_{N_2})((m_1, m_2)(n_1, n_2)) = \max\{\gamma_{M_1}(m_1), \gamma_{M_1}(n_1), \gamma_{M_2}(m_2), \alpha_{M_2}(n_2)\}$,
$(\delta_{N_1}|\delta_{N_2})((m_1, m_2)(n_1, n_2)) = \max\{\delta_{M_1}(m_1), \delta_{M_1}(n_1), \delta_{M_2}(m_2), \delta_{M_2}(n_2)\}$
,
$(\eta_{N_1}|\eta_{N_2})((m_1, m_2)(n_1, n_2)) = \min\{\eta_{M_1}(m_1), \eta_{M_1}(n_1), \eta_{M_2}(m_2), \delta_{N_2}(n_2)\}$
,
$(\theta_{N_1}|\theta_{N_2})((m_1, m_2)(n_1, n_2)) = \min\{\theta_{M_1}(m_1), \theta_{M_1}(n_1), \theta_{M_2}(m_2), \delta_{M_2}(n_2)\}$

$\forall\ m_1n_1 \notin E_1$ and $m_2n_2 \notin E_2$.

Example 2.28. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs on $V_1 = \{a, b, c, d\}$ *and* $V_2 = \{e, f\}$, respectively which shown in Figure 11 and Figure 12. Also rejection shown in Figure 13.

Proposition 2.29. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs of graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, respectively. Then the rejection $\mathtt{G}_1|\mathtt{G}_2$ of $\mathtt{G}_1 = (V_1, E_1)$ and $\mathtt{G}_2 = (V_2, E_2)$ is a BSVNG.

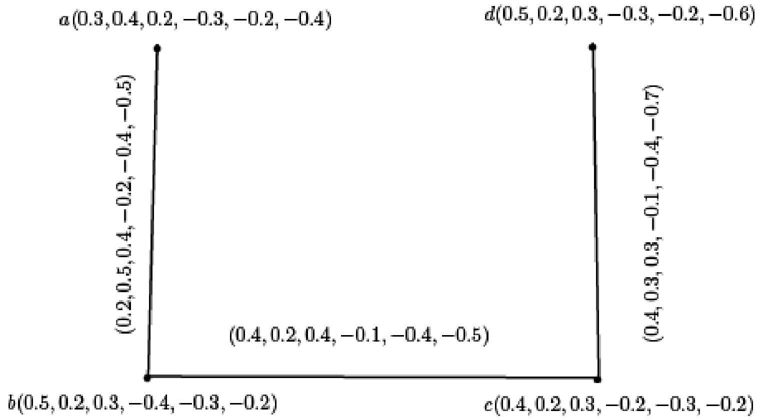

FIGURE 11. $\mathtt{G}_1$

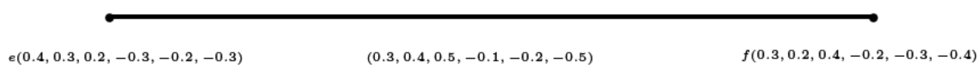

FIGURE 12. $\mathtt{G}_2$

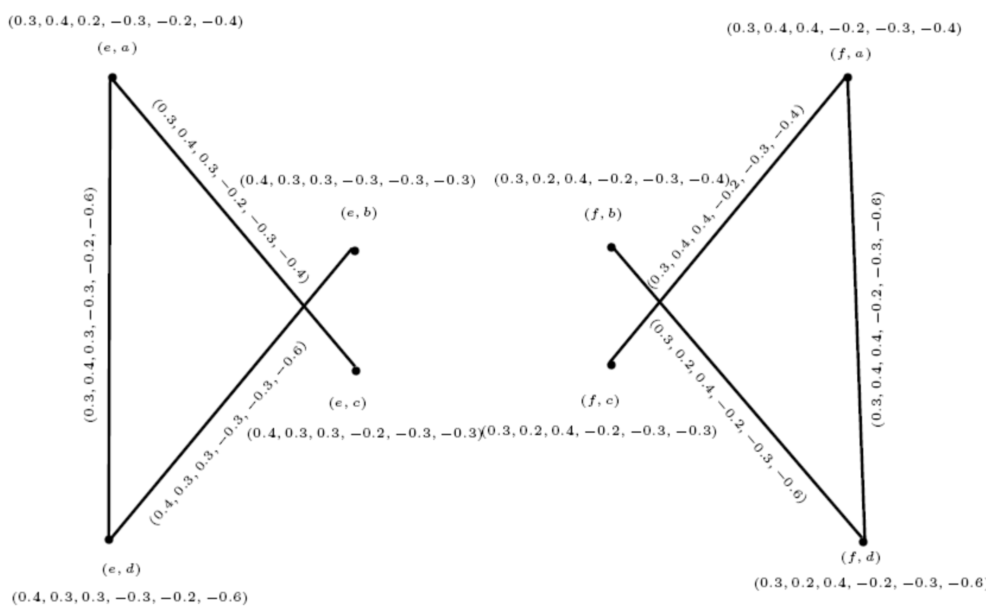

FIGURE 13. $\mathtt{G}_1 \mid \mathtt{G}_2$

Proof. Suppose that $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs of graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ respectively. Then for $(m_1, m_2)(n_1, n_2) \in E_1 \times E_2$.

(i) If $m_1 = n_1, m_2n_2 \notin E_2$

$$\begin{aligned}(\beta_{N_1}|\beta_{N_2})((m_1,m_2)(n_1,n_2)) &= \max\{\beta_{M_1}(m_1), \beta_{M_2}(m_2), \beta_{M_2}(n_2)\}\\ &= \max\{\max\{\beta_{M_1}(m_1), \beta_{M_2}(m_2)\}, \max\{\beta_{M_1}(n_1), \beta_{M_2}(n_2)\}\}\\ &= \max\{(\beta_{M_1}|\beta_{M_2})(m_1,m_2), (\beta_{M_1}|\beta_{M_2})(n_1,n_2)\}\end{aligned}$$

$$\begin{aligned}(\eta_{N_1}|\eta_{N_2})((m_1,m_2)(n_1,n_2)) &= \min\{\eta_{M_1}(m_1), \eta_{M_2}(m_2), \eta_{M_2}(n_2)\}\\ &= \min\{\min\{\eta_{M_1}(m_1), \eta_{M_2}(m_2)\}, \min\{\eta_{M_1}(n_1), \eta_{M_2}(n_2)\}\}\\ &= \min\{(\eta_{M_1}|\eta_{M_2})(m_1,m_2), (\eta_{M_1}|\eta_{M_2})(n_1,n_2)\}\end{aligned}$$

In a similar way others four will proved obviously.

(ii) If $m_2 = n_2, m_1n_1 \notin E_1$

$$\begin{aligned}(\alpha_{N_1}|\alpha_{N_2})((m_1,m_2)(n_1,n_2)) &= \min\{\alpha_{M_1}(m_1), \alpha_{M_1}(n_1), \alpha_{M_2}(m_2)\}\\ &= \min\{\min\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2)\}, \min\{\alpha_{M_1}(n_1), \alpha_{M_2}(n_2)\}\}\\ &= \min\{(\alpha_{M_1}|\alpha_{M_2})(m_1,m_2), (\alpha_{M_1}|\alpha_{M_2})(n_1,n_2)\}\end{aligned}$$

$$\begin{aligned}(\delta_{N_1}|\delta_{N_2})((m_1,m_2)(n_1,n_2)) &= \max\{\delta_{M_1}(m_1), \delta_{M_1}(n_1), \delta_{M_2}(m_2)\}\\ &= \max\{\max\{\delta_{M_1}(m_1), \delta_{M_2}(m_2)\}, \max\{\delta_{M_1}(n_1), \delta_{M_2}(n_2)\}\}\\ &= \max\{(\delta_{M_1}|\delta_{M_2})(m_1,m_2), (\delta_{M_1}|\delta_{M_2})(n_1,n_2)\}\end{aligned}$$

In a similar way others four will proved obviously.
(iii) If $m_1n_1 \notin E_1 and\ m_2n_2 \notin E_2$

$$\begin{aligned}(\gamma_{N_1}|\gamma_{N_2})((m_1,m_2)(n_1,n_2)) &= \max\{\gamma_{M_1}(m_1), \gamma_{M_1}(n_1), \gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\}\\ &= \max\{\max\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2)\}, \max\{\gamma_{M_1}(n_1), \gamma_{M_2}(n_2)\}\}\\ &= \max\{(\gamma_{M_1}|\gamma_{M_2})(m_1,m_2), (\gamma_{M_1}|\gamma_{M_2})(n_1,n_2)\}.\end{aligned}$$

$$\begin{aligned}(\theta_{N_1}|\theta_{N_2})((m_1,m_2)(n_1,n_2)) &= \min\{\theta_{M_1}(m_1), \theta_{M_1}(n_1), \theta_{M_2}(m_2), \theta_{M_2}(n_2)\}\\ &= \min\{\min\{\theta_{M_1}(m_1), \theta_{M_2}(m_2)\}, \min\{\theta_{M_1}(n_1), \theta_{M_2}(n_2)\}\}\\ &= \min\{(\theta_{M_1}|\theta_{M_2})(m_1,m_2), (\theta_{M_1}|\theta_{M_2})(n_1,n_2)\}.\end{aligned}$$

In a similar way others four will proved obviously.
Hence all properties are satisfied truly, so in all cases $N_1|N_2$ is a BSVNG on $M_1|M_2$. Therefore we can say $\mathtt{G}_1|\mathtt{G}_2 = (M_1|M_2, N_1|N_2)$ is a BSVNG. □

Definition 2.30. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, Y_2)$ be two BSVNGs. $\forall (m_1, m_2) \in V_1 \times V_2$

$$\begin{aligned}(d_\alpha)_{\mathtt{G}_1|\mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{N_1}|\alpha_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2n_2\notin E_2} \min\{\alpha_{M_1}(m_1), \alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\} \\ &+ \sum_{m_2=n_2, m_1n_1\notin E_1} \min\{\alpha_{M_1}(m_1), \alpha_{M_1}(n_1), \alpha_{M_2}(m_2)\} \\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \min\{\alpha_{M_1}(m_1), \alpha_{M_1}(n_1), \alpha_{M_2}(m_2), \alpha_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(d_\beta)_{\mathtt{G}_1|\mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{N_1}|\beta_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2n_2\notin E_2} \max\{\beta_{M_1}(m_1), \beta_{M_2}(m_2), \beta_{M_2}(n_2)\} \\ &+ \sum_{m_2=n_2, m_1n_1\notin E_1} \max\{\beta_{M_1}(m_1), \beta_{M_1}(n_1), \beta_{M_2}(m_2)\} \\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \max\{\beta_{M_1}(m_1), \beta_{M_1}(n_1), \beta_{M_2}(m_2), \beta_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(d_\gamma)_{\mathtt{G}_1|\mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{N_1}|\gamma_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2n_2\notin E_2} \max\{\gamma_{M_1}(m_1), \gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\} \\ &+ \sum_{m_2=n_2, m_1n_1\notin E_1} \max\{\gamma_{M_1}(m_1), \gamma_{M_1}(n_1), \gamma_{M_2}(m_2)\} \\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \max\{\gamma_{M_1}(m_1), \gamma_{M_1}(n_1), \gamma_{M_2}(m_2), \gamma_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(d_\delta)_{\mathtt{G}_1|\mathtt{G}_2}(m_1, m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\delta_{N_1}|\delta_{N_2})((m_1, m_2)(n_1, n_2)) \\ &= \sum_{m_1=n_1, m_2n_2\notin E_2} \max\{\delta_{M_1}(m_1), \delta_{M_2}(m_2), \delta_{M_2}(n_2)\} \\ &+ \sum_{m_2=n_2, m_1n_1\notin E_1} \max\{\delta_{M_1}(m_1), \delta_{M_1}(n_1), \delta_{M_2}(m_2)\} \\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \max\{\delta_{M_1}(m_1), \delta_{M_1}(n_1), \delta_{M_2}(m_2), \delta_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(d_\eta)_{\mathtt{G}_1|\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\eta_{N_1}|\eta_{N_2})((m_1,m_2)(n_1,n_2))\\ &= \sum_{m_1=n_1, m_2n_2\notin E_2} \min\{\eta_{M_1}(m_1),\eta_{M_2}(m_2),\eta_{M_2}(n_2)\}\\ &+ \sum_{m_2=n_2, m_1n_1\notin E_1} \min\{\eta_{M_1}(m_1),\eta_{M_1}(n_1),\eta_{M_2}(m_2)\}\\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \min\{\eta_{M_1}(m_1),\eta_{M_1}(n_1),\eta_{M_2}(m_2),\eta_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(d_\theta)_{\mathtt{G}_1|\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\theta_{N_1}|\theta_{N_2})((m_1,m_2)(n_1,n_2))\\ &= \sum_{m_1=n_1, m_2n_2\notin E_2} \min\{\theta_{M_1}(m_1),\theta_{M_2}(m_2),\theta_{M_2}(n_2)\}\\ &+ \sum_{m_2=n_2, m_1n_1\notin E_1} \min\{\theta_{M_1}(m_1),\theta_{M_1}(n_1),\theta_{M_2}(m_2)\}\\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \min\{\theta_{M_1}(m_1),\theta_{M_1}(n_1),\theta_{M_2}(m_2),\theta_{M_2}(n_2)\}\end{aligned}$$

Definition 2.31. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, Y_2)$ be two BSVNGs. $\forall (m_1, m_2) \in\ V_1 \times V_2$

$$\begin{aligned}(td_\alpha)_{\mathtt{G}_1|\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\alpha_{N_1}|\alpha_{N_2})((m_1,m_2)(n_1,n_2)) + (\alpha_{M_1}|\alpha_{M_2})(m_1,m_2)\\ &= \sum_{m_1=n_1, m_2n_2\notin E_2} \min\{\alpha_{M_1}(m_1),\alpha_{M_2}(m_2),\alpha_{M_2}(n_2)\}\\ &+ \sum_{m_2=n_2, m_1n_1\notin E_1} \min\{\alpha_{M_1}(m_1),\alpha_{M_1}(n_1),\alpha_{M_2}(m_2)\}\\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \min\{\alpha_{M_1}(m_1),\alpha_{M_1}(n_1),\alpha_{M_2}(m_2),\alpha_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(td_\beta)_{\mathtt{G}_1|\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\beta_{N_1}|\beta_{N_2})((m_1,m_2)(n_1,n_2)) + (\beta_{M_1}|\beta_{M_2})(m_1,m_2)\\ &= \sum_{m_1=n_1, m_2n_2\notin E_2} \max\{\beta_{M_1}(m_1),\beta_{M_2}(m_2),\beta_{M_2}(n_2)\}\\ &+ \sum_{m_2=n_2, m_1n_1\notin E_1} \max\{\beta_{M_1}(m_1),\beta_{M_1}(n_1),\beta_{M_2}(m_2)\}\\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \max\{\beta_{M_1}(m_1),\beta_{M_1}(n_1),\beta_{M_2}(m_2),\beta_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(td_{\gamma})_{\mathtt{G}_1|\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\gamma_{N_1}|\gamma_{N_2})((m_1,m_2)(n_1,n_2)) + (\gamma_{M_1}|\gamma_{M_2})(m_1,m_2)\\ &= \sum_{m_1=n_1,m_2n_2\notin E_2} \max\{\gamma_{M_1}(m_1),\gamma_{M_2}(m_2),\gamma_{M_2}(n_2)\}\\ &+ \sum_{m_2=n_2,m_1n_1\notin E_1} \max\{\gamma_{M_1}(m_1),\gamma_{M_1}(n_1),\gamma_{M_2}(m_2)\}\\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \max\{\gamma_{M_1}(m_1),\gamma_{M_1}(n_1),\gamma_{M_2}(m_2),\gamma_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(td_{\delta})_{\mathtt{G}_1|\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\delta_{N_1}|\delta_{N_2})((m_1,m_2)(n_1,n_2)) + (\delta_{M_1}|\delta_{M_2})(m_1,m_2)\\ &= \sum_{m_1=n_1,m_2n_2\notin E_2} \max\{\delta_{M_1}(m_1),\delta_{M_2}(m_2),\delta_{M_2}(n_2)\}\\ &+ \sum_{m_2=n_2,m_1n_1\notin E_1} \max\{\delta_{M_1}(m_1),\delta_{M_1}(n_1),\delta_{M_2}(m_2)\}\\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \max\{\delta_{M_1}(m_1),\delta_{M_1}(n_1),\delta_{M_2}(m_2),\delta_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(td_{\eta})_{\mathtt{G}_1|\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\eta_{N_1}|\eta_{N_2})((m_1,m_2)(n_1,n_2)) + (\eta_{M_1}|\eta_{M_2})(m_1,m_2)\\ &= \sum_{m_1=n_1,m_2n_2\notin E_2} \min\{\eta_{M_1}(m_1),\eta_{M_2}(m_2),\eta_{M_2}(n_2)\}\\ &+ \sum_{m_2=n_2,m_1n_1\notin E_1} \min\{\eta_{M_1}(m_1),\eta_{M_1}(n_1),\eta_{M_2}(m_2)\}\\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \min\{\eta_{M_1}(m_1),\eta_{M_1}(n_1),\eta_{M_2}(m_2),\eta_{M_2}(n_2)\}\end{aligned}$$

$$\begin{aligned}(td_{\theta})_{\mathtt{G}_1|\mathtt{G}_2}(m_1,m_2) &= \sum_{(m_1,m_2)(n_1,n_2)\in E_1\times E_2.} (\theta_{N_1}|\theta_{N_2})((m_1,m_2)(n_1,n_2)) + (\theta_{M_1}|\theta_{M_2})(m_1,m_2)\\ &= \sum_{m_1=n_1,m_2n_2\notin E_2} \min\{\theta_{M_1}(m_1),\theta_{M_2}(m_2),\theta_{M_2}(n_2)\}\\ &+ \sum_{m_2=n_2,m_1n_1\notin E_1} \min\{\theta_{M_1}(m_1),\theta_{M_1}(n_1),\theta_{M_2}(m_2)\}\\ &+ \sum_{m_1n_1\notin E_1 and\ m_2n_2\notin E_2} \min\{\theta_{M_1}(m_1),\theta_{M_1}(n_1),\theta_{M_2}(m_2),\theta_{M_2}(n_2)\}\end{aligned}$$

Example 2.32. Let $\mathtt{G}_1 = (M_1, N_1)$ and $\mathtt{G}_2 = (M_2, N_2)$ be two BSVNGs as in Example 2.28. Their rejection is also shown in Figure 13. We will find the vertex degree in rejection. Consider the vertex (d,a) here:

$$\begin{aligned}(d_\gamma)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) &= \max\{\gamma_{M_2}(e),\gamma_{M_1}(a),\gamma_{M_1}(d)\} + \max\{\gamma_{M_2}(a),\gamma_{M_1}(a),\gamma_{M_1}(c)\}\\ &= \max\{0.2,0.2,0.3\} + \max\{0.2,0.2,0.3\}\\ &= 0.3+0.3\\ &= 0.6\end{aligned}$$

$$\begin{aligned}(d_\theta)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) &= \min\{\theta_{M_2}(e),\theta_{M_1}(a),\theta_{M_1}(d)\} + \min\{\theta_{M_2}(a),\theta_{M_1}(a),\theta_{M_1}(c)\}\\ &= \min\{-0.3,-0.4,-0.6\} + \min\{-0.3,-0.4,-0.2\}\\ &= -0.6-0.4\\ &= -1.0\end{aligned}$$

$$(d_\alpha)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) = 0.6,\ (d_\beta)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) = 0.8$$

$$(d_\delta)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) = -0.5,\ (d_\eta)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) = -0.5$$

In a similar way, we can find degree of all vertices of a graph in rejection. Now we will find out the total vertex degree of graph in rejection. Consider the same vertex (d,a) here:

$$\begin{aligned}(td_\gamma)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) &= \max\{\gamma_{M_2}(e),\gamma_{M_1}(a),\gamma_{M_1}(d)\} + \max\{\gamma_{M_2}(a),\gamma_{M_1}(a),\gamma_{M_1}(c)\} + \min\{\gamma_{M_2}(e),\gamma_{M_1}(a)\}\\ &= \max\{0.2,0.2,0.3\} + \max\{0.2,0.2,0.3\} + \min\{0.2,0.2\}\\ &= 0.3+0.3+0.2\\ &= 0.8\end{aligned}$$

$$\begin{aligned}(td_\theta)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) &= \min\{\theta_{M_2}(e),\theta_{M_1}(a),\theta_{M_1}(d)\} + \min\{\theta_{M_2}(a),\theta_{M_1}(a),\theta_{M_1}(c) + \min\{\theta_{M_2}(e),\theta_{M_1}(a)\}\}\\ &= \min\{-0.3,-0.4,-0.6\} + \min\{-0.3,-0.4,-0.2\} + \min\{-0.3,-0.4\}\\ &= -0.6-0.4-0.4\\ &= -1.4\end{aligned}$$

$$(td_\alpha)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) = 0.9,\ (td_\beta)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) = 1.1$$

$$(td_\delta)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) = -0.8,\ (td_\eta)_{\mathtt{G}_1|\mathtt{G}_2}(e,a) = -0.7$$

In a similar way we can find total vertex degree in rejection.

3. Application of bipolar single valued neutrosophic graph (BSVNG)

3.1. *Educational Designation participation*

Let {Bilal, Asif, Shoaib, Ijaz } be the set of four applicants for designations {Head of department(HOD),Director of Department(DOD),Assistant director of department(ADOD)}. For this purpose p=4 (say) be number of applicants and d=3 be number of designations. Consider bipolar single valued-neutrosophic diagraph which is shown in figure **??** representing the competition between applicants for designation in organization. $\alpha(y)$ is the positive degree of membership for every applicants denote the percentage of ability toward the purpose of organization , $\beta(y)$ and $\gamma(y)$ are indeterminacy and false in percentage. $\delta(y)$ is the is the negative degree of membership for every applicants denote the percentage of non ability toward the purpose of organization, $\eta(y)$ and $\theta(y)$ are represents the indeterminacy and false in percentage. $\alpha(y)$ of every directed edge between both designations and applicants denote the eligibility or positive response from designation in organization , $\beta(y)$ and $\gamma(y)$ are indeterminacy and false in this percentage. $\delta(y)$ of every directed edge between both designations and applicants denote the non-eligibility or negative response from designation in organization , $\eta(y)$ and $\theta(y)$ are indeterminacy and false in this percentage. Edge membership degree of

TABLE 1

y∈ Y	N(y)
Bilal	$\{(ADOD,0.5,0.3,0.4,-0.4,-0.5,-0.8),(HOD,0.6,0.4,0.2,-0.4,-0.6,-0.5)\}$
Asif	$\{(ADOD,0.8,0.6,0.5,-0.1,-0.4,-0.5),(HOD,0.5,0.6,0.6,-0.3,-0.4,-0.7),(DOD,0.4,0.6,0.4,-0.2,-0.3,-0.5)\}$
Shoaib	$\{(DOD,0.5,0.4,0.5,-0.5,-0.4,-0.4)\})$
Ijaz	$\{(HOD,0.7,0.5,0.6,-0.3,-0.5,-0.4),(DOD,0.7,0.4,0.5,-0.4,-0.3,-0.2)\})$

graph is also determined by the following

$$N(Bilal)\cap N(Asif) = \{(ADOD,0.5,0.6,0.5,-0.1,-0.5,-0.8),(HOD,0.5,0.6,0.6,$$
$$-0.3,-0.6,-0.7)\}$$
$$N(Bilal)\cap N(Shoaib) = \emptyset$$
$$N(Bilal)\cap N(Ijaz) = \{(HOD,0.6,0.5,0.6,-0.3,-0.6,-0.5)\}$$
$$N(Asif)\cap N(Shoaib) = \{(DOD,0.4,0.6,0.5,-0.2,-0.4,-0.5)\}$$
$$N(Asif)\cap N(Ijaz) = \{(HOD,0.5,0.6,0.6,-0.3,-0.5,-0.7),(DOD,0.4,0.6,0.5,-0.2,$$
$$-0.3,-0.5)\}$$
$$N(Shoaib)\cap N(Ijaz) = \{(DOD,0.5,0.4,0.5,-0.4,-0.4,-0.4)\}$$

There is no edge between Shoaib and Bilal because there is no common designation.

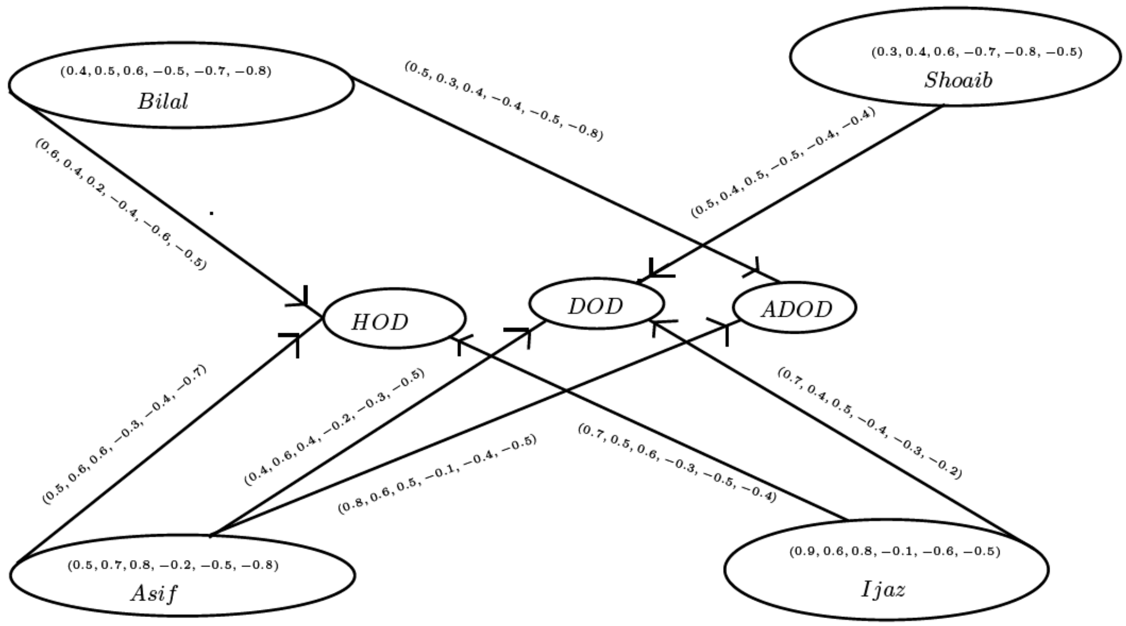

FIGURE 14. Bipolar single valued neutrosophic digraph

$$\begin{aligned}
(Bilal, Asif) &= (0.4, 0.7, 0.8, -0.2, -0.7, -0.8)(0.5, 0.6, 0.5, 0.3, 0.6, 0.7) \\
&= (0.20, 0.42, 0.40, -0.06, -0.42, -0.56) \\
(Bilal, Shoaib) &= \emptyset \\
(Bilal, Ijaz) &= (0.4, 0.6, 0.8, -0.1, -0.7, -0.8)(0.6, 0.5, 0.6, 0.3, 0.6, 0.5) \\
&= (0.24, 0.30, 0.48, -0.03, -0.42, -0.40) \\
(Asif, Shoaib) &= (0.3, 0.7, 0.8, -0.2, -0.8, -0.8)(0.4, 0.6, 0.5, 0.2, 0.4, 0.5) \\
&= (0.12, 0.42, 0.40, -0.04, -0.32, -0.40) \\
(Asif, Ijaz) &= (0.5, 0.7, 0.8, -0.1, -0.6, -0.8)(0.5, 0.6, 0.5, 0.3, 0.3, 0.5) \\
&= (0.25, 0.42, 0.40, -0.03, -0.18, -0.40) \\
(Shoaib, Ijaz) &= (0.3, 0.6, 0.8, -0.1, -0.8, -0.5)(0.5, 0.4, 0.5, 0.4, 0.4, 0.4) \\
&= (0.15, 0.24, 0.40, -0.04, -0.32, -0.20)
\end{aligned}$$

Bipolar single-valued neurotrophic graph for competition of all participant is shown in figure 15. Competition between two individually applicants and when applicant competing for designation is also given in graph 15.

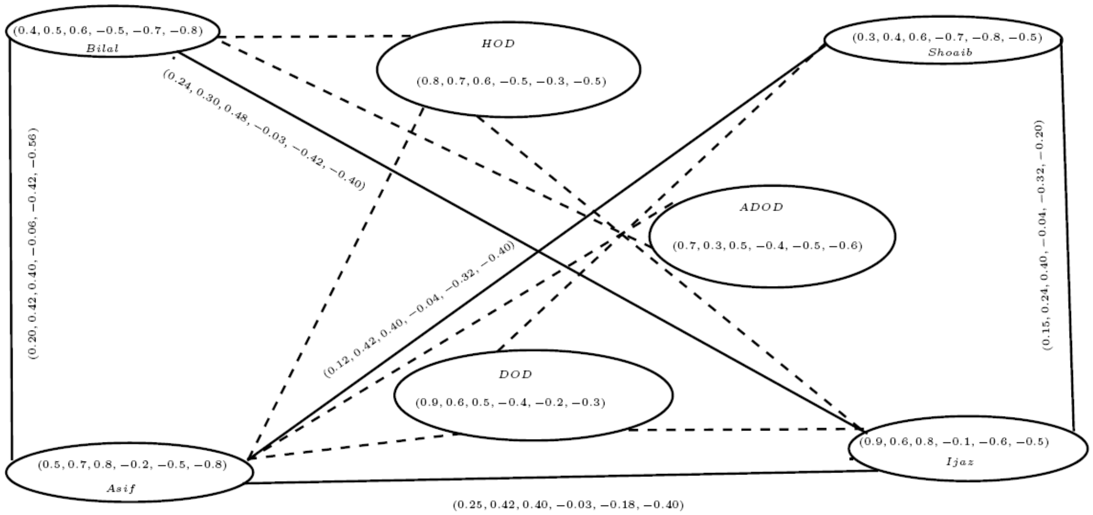

FIGURE 15. Bipolar single valued neutrosophic competition graph

$$R(Bilal, HOD) = (\frac{0.20+0.24}{2}, \frac{0.20+0.24}{2}, \frac{0.42+0.30}{2}, \frac{0.40+0.48}{2}, \frac{-0.06-0.03}{2}$$
$$, \frac{-0.42-0.42}{2}, \frac{-0.56-0.40}{2}) = (0.22, 0.36, 0.44, -0.045, -0.42, -0.48)$$

Similarly we will find others R(applicant,Designation).

$$S(Bilal, HOD) = 1 + 0.22 - 0.045 - (0.36 + 0.44 - 0.42 - 0.48) = 1.275$$
$$S(Asif, HOD) = 1 + 0.225 - 0.045 - (0.42 + 0.40 - 0.30 - 0.48) = 1.14$$
$$S(Ijaz, HOD) = 1 + 0.245 - 0.03 - (0.36 + 0.44 - 0.225 - 0.29) = 0.93$$
$$S(Bilal, ADOD) = 1 + 0.20 - 0.06 - (0.42 + 0.40 - 0.42 - 0.56) = 1.30$$

$$S(Asif, ADOD) = 1 + 0.20 - 0.06 - (0.42 + 0.40 - 0.42 - 0.56) = 1.30$$
$$S(Asif, DOD) = 1 + 0.185 - 0.035 - (0.42 + 0.40 - 0.25 - 0.40) = 0.98$$
$$S(Shoaib, DOD) = 1 + 0.135 - 0.04 - (0.33 + 0.40 - 0.32 - 0.30) = 0.985$$
$$S(Ijaz, DOD) = 1 + 0.20 - 0.035 - (0.33 + 0.40 - 0.25 - 0.30) = 0.985$$

Black solid lines show comparison between two applicants and dot line means applicant compete for designation. From above table, applicants compete other if it has a more strength. For example, in HOD designation Bilal has more strength from all. Its eligibility is strong than other. In ADOD designation Asif and Bilal are in equal position. In DOD designation Shoaib and Ijaz compete the others but equally compete to each other. [H]In this algorithm these are the steps

Step 1: Start. **Step 2:** Input $\alpha(y), \beta(y)$ *and* $\gamma(y)$ membership values for set p applicants.

Table 2

(Applicant,designation)	in competition	R(applicant,Designation)	S(applicant,Designation)
(Bilal,HOD)	Asif, Ijaz	(0.22,0.36,0.44,-0.045,-0.42,-0.48)	1.275
(Asif,HOD)	Bilal,Ijaz	(0.225,0.42,0.40,-0.045,-0.30,-0.48)	1.14
(Ijaz,HOD)	Bilal,Asif	(0.245,0.36,0.44,-0.03,-0.225,-0.29)	0.93
(Bilal,ADOD)	Asif	(0.20,0.42,0.40,-0.06,-0.42,-0.56)	1.30
(Asif,ADOD)	Bilal	(0.20,0.42,0.40,-0.06,-0.42,-0.56)	1.30
(Asif,DOD)	Shoaib,Ijaz	(0.185,0.42,0.40,-0.035,-0.25,-0.40)	0.98
(Shoaib,DOD)	Asif,Ijaz	(0.135,0.33,0.40,-0.04,-0.32,-0.30)	0.985
(Ijaz,DOD)	Asif,Shoaib	(0.20,0.33,0.40,-0.035,-0.25,-0.30)	0.985

Step3: For any two vertices x_i *and* xj *taking* $\alpha(x_ix_j), \beta(x_ix_j)$ *and* $\gamma(x_ix_j)$ are positive but $\delta(x_ix_j), \eta(x_ix_j)$ *and* $\theta(x_ix_j)$ are negative. Then $(x_i, \alpha(x_ix_j), \beta(x_ix_j), \gamma(x_ix_j), \delta(x_ix_j), \eta(x_ix_j), \theta(x_ix_j))$
Step4: To obtain bipolar single valued neutrosohic out-neighbourhoods $N(x_i)$ Repeat step 3 for all vertices x_i *and* x_j.

Step5: Find out $N(x_i) \cap N(x_j)$. **Step6:** Calculate height $h(N(x_i) \cap N(x_j))$. **Step7:** Draw all edge where $N(x_i) \cap N(x_j)$ is non empty. **Step8:** Give a membership value to every edge x_ix_j by using the following conditions
$\alpha(x_ix_j = (\min\{x_i \cap x_j\})[N(x_i \cap N(x_j)]$, $\beta(x_ix_j = (\max\{x_i \cap x_j\})[N(x_i \cap N(x_j)]$
$\gamma(x_ix_j = (\max\{x_i \cap x_j\})[N(x_i \cap N(x_j)]$, $\delta(x_ix_j = (\max\{x_i \cap x_j\})[N(x_i \cap N(x_j)]$
$\eta(x_ix_j = (\min\{x_i \cap x_j\})[N(x_i \cap N(x_j)]$, $\theta(x_ix_j = (\min\{x_i \cap x_j\})[N(x_i \cap N(x_j)]$
Step9: If x, $z_1, z_2, z_3, ..., z_p$ are applicants for designations d, then strength of applicants competition is R(x,d)=$(\alpha(x,d), \beta(x,d), \gamma(x,d), \delta(x,d), \eta(x,d), \theta(x,d))$ of every applicants x and designation d is given by the following R(x,d)=$(\frac{\alpha(xz_1)+...\alpha(xz_p)}{p}, \frac{\beta(xz_1)+...\beta(xz_p)}{p}, \frac{\gamma(xz_1)+...\gamma(xz_p)}{p}, \frac{\delta(xz_1)+...\delta(xz_p)}{p}, \frac{\eta(xz_1)+...\eta(xz_p)}{p}, \frac{\theta(xz_1)+...\theta(xz_p)}{p})$
Step10: Find out $S(x, d) = 1 + \alpha(x, d) + \delta(x, d) - (\beta(x, d) + \gamma(x, d) + \eta(x, d) + \theta(x, d))$. **Step11:** End

4. Conclusion

There are more advantages of a bipolar fuzzy set than fuzzy set in real life phenomenon. A BSVNG has many applications in the field of economics, medical science as well as in scientific engineering. The flexibility and compatibility of BSVNG are higher than SVNG. We presented the new properties on a bipolar single-valued neutrosophic graph known as Residue product, maximal product, Symmetric difference and Rejection of a graph. These all graph products are suggestive of some aspects of network design. They can be applicable for the configuration processing of space structures. The repeated application of these operations in constructing a network generates graphs that display fractal properties. We also discussed the idea with examples to find the degree and total degree of vertices of some graphs. We have established some related theorems of these graphs. We have also proved the theorems which are related to these properties. In the future, our goal is to extend this work on the (1) complex neutrosophic graphs and some (2) bipolar complex neutrosophic graph.

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Neutrosophic Geometric Programming (NGP) with (Max, Product) Operator; An Innovative Model

Huda E. Khalid
Telafer University, Head of the Scientific Affairs and Cultural Relations Department, Telafer, Iraq.
E-mail: hodaesmail@yahoo.com

Abstract. In this paper, a neutrosophic optimization model has been first constructed for the neutrosophic geometric programming subject to (max-product) neutrosophic relation constraints. For finding the maximum solution, two new operations (i.e. $\bowtie$, Θ) between a_{ij} and b_i have been defined, which have a key role in the structure of the maximum solution. Also, two new theorems and some propositions are introduced that discussed the cases of the incompatibility in the relational equations $A o x = b$, with some properties of the operation Θ. Numerical examples have been solved to illustrate new concepts.

Keyword: Neutrosophic Geometric Programming (NGP); (max-product) Operator; Neutrosophic Relation Constraints; Maximum Solution; Incompatible Problem; Pre-Maximum Solution; Relational Neutrosophic Geometric Programming (RNGP).

1. Introduction

The first scientist who put forward the fuzzy relational equations was Elie Sanchez, a famous fuzzy biology mathematician in 1976 [2], while the theoretical concept of the neutrosophic logic has been put by the popular polymath Florentin Smarandache at 1995 [11]. B. Y. Cao constructed the mathematical models of fuzzy relation geometric programming (FRGP) at 2005 [1], his works include the structuring of the maximum and minimum solution of the (FRGP) depending upon the original model for the maximum solution and the minimum solution for the fuzzy relation equations that was put by Elie Sanchez. At 2015, Huda E. Khalid introduced an original structure of the maximum solution for the fuzzy neutrosophic relation geometric programming (FNRGP) [6], Also at 2016, she put a novel algorithm for finding the minimum solution for the same (FNRGP) problems [7]. As of 2016 so far Huda E. Khalid et al [3-10] introducing a big qualitative shift in the concept of neutrosophic geometric programming (NGP) by establishing new concepts for the notion of (over, off, under) in the same (NGP), as well as she introduced and for the first time, a new type of the neutrosophic geometric programming using (over, off, under) neutrosophic less than or equal which contained a new version of the convex condition, furthermore, new decomposition theorems of neutrosophic sets were presented, and new representations for the neutrosophic sets using (α, β, γ)-cuts, with strong (α, β, γ)-cuts had been defined.

In this article, section 2 contains the preliminaries which are necessary for the sake of this paper, while in section 3, a max- product neutrosophic relation geometric programming model has been proposed with an innovative investigation of the maximum solution for this model and two new theorems with some propositions, section 4 presents numerical examples to illustrate the proposed method. The final section was dedicated to the conclusion.

2. Basic Concepts

Without loss of generality, the elements of b must be rearranged in decreasing or increasing order and the elements of the matrix A are correspondingly rearranged.

2.1 Definition [7]

In this definition, the author proposed the following axioms:

a- decreasing partial order

1-The greatest element in $[0,1) \cup \mathrm{I}$ is equal to I, $\max(\mathrm{I}, \mathrm{x}) = \mathrm{I} \quad \forall\ \mathrm{x} \in [0,1)$

2- The fuzzy values in a decreasing order will be rearranged as follows: $1 > \mathrm{x}_1 > \mathrm{x}_2 > \mathrm{x}_3 > \cdots > \mathrm{x}_\mathrm{n} \geq 0$

3- One is the greatest element in $[0,1] \cup \mathrm{I}$, $\max(\mathrm{I}, 1) = 1$

b- Increasing partial order

1- the smallest element in $(0,1] \cup \mathrm{I}$ is I , $\min(\mathrm{I}, \mathrm{x}) = \mathrm{I} \quad \forall\ \mathrm{x} \in (0,1]$

2- The fuzzy values in increasing order will be rearranged as follows: $0 < \mathrm{x}_1 < \mathrm{x}_2 < \mathrm{x}_3 < \cdots < \mathrm{x}_\mathrm{n} \leq 1$

3- Zero is the smallest element in $[0,1] \cup \mathrm{I}$, $\min(\mathrm{I}, 0) = 0$

2.2 Definition [7]

If there exists a solution to $Aox = b$ it's called compatible. Suppose $X(A, b) = \{(x_1, x_2, \dots, x_n)^T \in [0,1]^n \cup I,\ I^n = I, n > 0 \,|Aox = b, x_j \in [0,1] \cup I\, \}$ is a solution set of $Aox = b$ we define $x^1 \leq x^2 \Leftrightarrow x_j^1 \leq x_j^2\ (1 \leq j \leq n), \forall\, x^1, x^2 \in \mathrm{X(A,b)}$. Where " $\leq$ " is a partial order relation on $\mathrm{X(A,b)}$.

2.3 Corollary [1]

If $\mathrm{X(A,b)} \neq \emptyset$. Then $\hat{x} \in X(A, b)$.

Similar to fuzzy relation equations, the above corollary works on neutrosophic relation equations.

2.4 Basic Notes [3, 10]

1. A component I to the zero power is undefined value, (i.e. I^{0} is undefined), since. $I^{0} = I^{1+(-1)} = I^{1} * I^{-1} = \frac{I}{I}$, which is an impossible case (avoid to divide by I).

2. The value of I to the negative power is undefined (i.e. $I^{-n}, n > 0$ is undefined).

3. The Innovative Structure of the Maximum Solution.

We call

$$\left.\begin{array}{l} \min f(x) = \left(c_1.x_1^{\gamma_1}\right) \vee \left(c_2.x_2^{\gamma_2}\right) \vee \dots \vee \left(c_n.x_n^{\gamma_n}\right) \\ s.t. \qquad\qquad Aox = b \\ x_j \in [0,1] \cup I, \quad 1 \le j \le n \end{array}\right\} \qquad (1)$$

A $(\vee, .)$ (max- product) neutrosophic geometric programming, where $A = (a_{ij})$, $1 \le i \le m, 1 \le j \le n$, is $(m \times n)$ dimensional neutrosophic matrix, $x = (x_1, x_2, \dots, x_n)^T$ an n-dimensional variable vector, $b = (b_1, b_2, \dots, b_m)^T$ $(b_i \in [0,1] \cup I)$ an m- dimensional constant vector, $c = (c_1, c_2, \dots, c_n)^T$ $(c_j \ge 0)$ an n- dimensional constant vector, γ_j is an arbitrary real number, and the composition operator $''o''$ is $(\vee, .)$, i.e. $\bigvee_{j=1}^{n}(a_{ij}.x_j) = b_i$.

Note that the program (1) is undefined and has no minimal solution in the case of $\gamma_j < 0$ with some $x_j's$ taking indeterminacy value. Therefore, if $\gamma_j < 0$ with indeterminacy value in some $x_j's$, then the greatest solution $\hat{x}_j$ is an optimal solution for problem (1), the author introduced theorem 3.4 to treat this issue.

3.1 The Shape of the Maximum Solution $\hat{\mathrm{x}}$.

Since 1976, the biological mathematician Elie Sanchez put the formula of the maximum solution in both composite fuzzy relation equations of type $(\vee, \wedge)$ operator and $(\vee, .)$ operator [2], these definitions won't be adequate with neutrosophic relation equations especially neutrosophic geometric programming type, therefore and for the importance of relational neutrosophic geometric programming (RNGP) in real-world problems, the author established a new structure for the maximum solution of (RNGP) with the $(\vee, \wedge)$operator in ref. [6], while this article was dedicated to set up the maximum solution of (RNGP) with the $(\vee, .)$ operator.

Every mathematician who works with neutrosophic theory know that the generality which characterizes the neutrosophic theory are determined in many ways of which,

$\max(I, x) = \min(I, x) = I \quad \forall\, x \in (0,1)$

This property gives some vague and difficulty for determining the maximum solution of the relation equations $Aox = b$, the author still searches about the answer of the following question.

How will be the shape of the greatest solution $\hat{x}$?

Actually, any single solution (the same solution that suggested by Elie Sanchez 1976) would not be accepted and won't be appropriate for the program (1), unless there are two integrated pre-maximum solutions gathered to get the final shape of $\hat{x}$, as follow:

1. The first integrated pre-maximum solution named $\hat{x}_{v1}$ which supports the fuzzy part of the problem, this solution has an adjoint matrix named A_{v1}, this adjoint matrix is derived from the matrix A.
2. The second integrated pre-maximum solution named $\hat{x}_{v2}$ which supports the neutrosophic part of the problem, this solution has an adjoint matrix named A_{v2}, which is derived from the matrix A too.
   The following definition describes the mathematical formula of $\hat{x}_{v1}$ and $\hat{x}_{v2}$.

3.2 Definition

$$a_{ij} \bowtie b_i = \begin{cases} \frac{b_i}{a_{ij}}, & if\ \ a_{ij} > b_i\ ,\ a_{ij} \in [0,1],\ b_i \in [0,1] \\ 1, & if\ \ a_{ij} \le b_i\ ,\ a_{ij} \in [0,1],\ b_i \in [0,1] \\ 1\ , & if\ \ a_{ij} \in [0,1],\ b_i = nI, n \in (0,1] \end{cases} \quad (2)$$

$$a_{ij}\Theta b_i = \begin{cases} \frac{nI}{a_{ij}}, & if\ \ a_{ij} > n\ ,\ a_{ij} \in [0,1], b_i = nI, n \in (0,1] \\ 1, & if\ \ a_{ij} \le n\ ,\ a_{ij} \in [0,1], b_i = nI, n \in (0,1] \\ not\ comp. & if\ \ a_{ij} = mI\ , m \in (0,1]\ , b_i \in [0,1] \cup I \\ 1 & if\ \ a_{ij}, b_{ij} \in [0,1] \end{cases} \quad (3)$$

Where $\bowtie$ is an operator defined at [0,1], while the operator Θ is defined at $[0,1] \cup I$.

Let $\hat{x}_j = \wedge_{i=1}^{m}(a_{ij} \bowtie b_i), \quad (1 \le j \le n)\ ,$ (4)

be the components of the pre-maximum solution $\hat{x}_{v1}$, (i.e. $\hat{x}_{v1} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$).

Let $\hat{x}_j = \wedge_{i=1}^{m}(a_{ij}\Theta b_i), \quad (1 \le j \le n)\ ,$ (5)

be the components of the pre maximum solution $\hat{x}_{v2}$, (i.e. $\hat{x}_{v2} = (\hat{x}_1, \hat{x}_2, \dots, \hat{x}_n)$).

Now the following question will be raised,

Which one $\hat{x}_{v1}$ or $\hat{x}_{v2}$ should be the exact maximum solution?

Neither $\hat{x}_{v1}$ nor $\hat{x}_{v2}$ will be the exact solution! the exact solution is the integration between them. Before solving $Ao\hat{x} = b$, we first define the matrices A_{v1}, A_{v2}.

Let A_{v1} be a matrix has the same dimension and the same rows elements of A except for those rows of the indexes $i = i_o$ corresponding to those indexes of $b_{i_o} = nI$, those special rows of A_{v1} will be zeros.

Let A_{v2} be a matrix has the same dimension and the same rows elements of A except for those rows of the indexes $i = i_o$ corresponding to those indexes of $b_{i_o} \in [0,1]$, those special rows of A_{v2} will be zeros.

Consequently,

$$Ao\hat{x} = b = (A_{v1}o\hat{x}_{v1}) + (A_{v2}o\hat{x}_{v2}) \quad (6)$$

The formula (6) is the greatest solution in $X(A,b)$.

The maximum value of the objective function $f(\hat{x}) = f(\hat{x}_{v1}) \vee f(\hat{x}_{v2})$.

3.3 Theorem

If $a_{ij} = mI,\ m \in (0,1], b_i \in [0,1] \cup I$ then $Aox = b$, is not compatible.

Proof

Let $a_{ij} = mI$, $b_i \in [0,1] \cup I$, the essential question in this case is

What is the value of $x_j \in [0,1] \cup I$ satisfying

$$\vee_{1 \le j \le n}(a_{ij}.x_j) = b_i \ ? \quad (7)$$

It is well known that the equation (7) can be written as an upper-bound constraint and a lower-bound constraint, that is,

$$\vee_{1 \le j \le n}(a_{ij}.x_j) \le b_i \quad (8)$$

$$\vee_{1 \le j \le n}(a_{ij}.x_j) \ge b_i \quad (9)$$

First,

The inequality (8) can be written in n constraints:

$a_{ij}.x_j \leq b_i$, $i.e.\ x_j \leq \frac{b_i}{a_{ij}}$,$1 \leq j \leq n$.

Hence $x_j \leq \wedge\left(\frac{b_i}{a_{ij}}\right)$, where the notation '' $\wedge$'' denotes the minimum operator.

So, we have $x_j \in [0, \wedge\left(\frac{b_i}{a_{ij}}\right)] \cup I$, but $a_{ij} = mI$, this is a contradict for the fact that the variables of the system $Aox = b$ are being in the interval $[0,1] \cup I$.

Second,

The inequality (9) can be written in n constraints:

$(a_{ij}.x_j) \geq b_i$, $i.e.\ x_j \geq \frac{b_i}{a_{ij}}$,$1 \leq j \leq n$.

Hence, $x_j \geq \vee\left(\frac{b_i}{a_{ij}}\right)$, where the notation '' $\vee$'' denotes the maximum operator.

Thus, we have $x_j \in [\vee\left(\frac{b_i}{a_{ij}}\right), 1] \cup I$, but $a_{ij} = mI$, in this proof we faced the division on the indeterminate component (I) which is prohibited behavior. Consequently the variable x_j will either belong to the interval $[0,\wedge (b_i/I)] \cup I$ or belong to the interval$[\vee (b_i/I),1] \cup I$, this implies that the system of the relation equation $Aox = b$ will be not compatible.
Therefore, the system of the relative equations $Aox = b$ is incompatible at $a_{ij} = mI, m \in (0,1]$. So, the restriction of $Aox = b$ for being compatible is that all elements of the matrix A $(i.e. a_{ij})$ are belonging to the interval [0,1]. □

3.4 Theorem

If $\gamma_j < 0$ $(1 \leq j \leq n)$, then the greatest solution to the problem (1) is an optimal solution.

Proof

Since $\gamma_j < 0$ $(1 \leq j \leq n)$, with $x_j \in [0,1] \cup I$, then $\frac{d(x_j^{\gamma_j})}{dx_j} = \gamma_j x_j^{\gamma_j - 1} < 0$ for each $x_j \in [0,1] \cup I$, this means that ${x_j}^{\gamma_j}$ is monotone decreasing function of x_j. It is clear that $c_j x_j^{\gamma_j}$ is also a monotone decreasing function about x_j. Therefore, $\forall\, x \in X(A, b)$, when $x \leq \hat{x}$, then $c_j.x_j^{\gamma_j} \geq c_j.\hat{x}_j^{\gamma_j}$ $(1 \leq j \leq n)$, such that $f(x) \geq f(\hat{x})$, so $\hat{x}$ is an optimal solution to the problem (1).

It remains to study the case that if $\gamma_j < 0$ with the component $\hat{x}_j$ in $\hat{x}_{v2}$ equal to I, we know that I^n is undefined for $n \leq 0$, in this case, the component $x_j = I$ that has a power $\gamma_j < 0$ will be replaced by that corresponding x_j in the $\hat{x}_{v1}$. □

3.5 Proposition

Let $a \in (0,1), b = mI$ & $c = nI, n, m \in (0,1],\ if\ m \geq n$, then $a\ \Theta b \geq a\ \Theta\ c$.

Proof

1) Let $a > m \implies a > n,$

But we have $m \geq n \implies b \geq c \implies \frac{b}{a} \geq \frac{c}{a} \implies a\ \Theta b \geq a\ \Theta\ c$.

2) Let $a \leq m \implies a\ \Theta\ b = 1$, since $m \geq n \implies a\ \Theta\ c \leq 1$

Hence, $a\ \Theta\ c \leq a\ \Theta\ b$.

3.6 Corollary

Let $a \in (0,1), b = mI,\ \ c = nI, m, n \in (0,1]$, if $m \geq n$ then $a\ \Theta\ (b \vee c) \geq a\ \Theta\ c$

Proof

Since $m \geq n \implies b \geq c \implies b \vee c = b$, from proposition 2.5, we have

$a\ \Theta\ b \geq a\ \Theta\ c \qquad (replacing\ b \vee c\ instead\ of\ b) \implies a\ \Theta\ (b \vee c) \geq a\ \Theta\ c.$

3.7 Proposition

Let $a \in (0,1), b = mI, m \in (0,1]$, then $a. (a\ \Theta\ b) = a \wedge b$.

Proof

1) Let $a > m \implies \frac{mI}{a} = \frac{b}{a} = a\ \Theta\ b$ [multiply both sides by a] $\implies$

$$b = a. (a\ \Theta\ b) \tag{10}$$

2) Let $a \leq m \implies a\ \Theta\ b = 1$ [multiply both sides by a] $\implies$

$$a = a. (a\ \Theta\ b) \tag{11}$$

From (10) & (11) we have $a. (a\ \Theta\ b) = a \wedge b$.

3.8 Proposition

Let $a \in (0,1), b = mI, m \in (0,1]$, then $a.(a \,\Theta\, b) = \begin{cases} b & a > am \\ 1 & a \le am \end{cases}$.

Proof

1) Let $a > am$, from definition (3.2) we have $a \,\Theta\, (a.m) = \frac{a.mI}{a} = mI = b$.
2) Let $a \le am$, again from definition (3.2) we have $a \,\Theta\, (a.b) = 1$.

Hence, $a \,\Theta\, (a.b) = \begin{cases} b & a > am \\ 1 & a \le am \end{cases}$

4 Numerical examples

In the upcoming examples, the (max- product) neutrosophic geometric problem is considered.

4.1 Example

Let $\min f(x) = (0.3.x_1^2) \vee (1.8I\,.x_2^{\frac{1}{3}}) \vee (I\,.x_3^{\frac{1}{4}})$

s. t. $Aox = b$

$x_j \in [0,1] \cup I \quad (1 \le j \le n)$

Where $b = (1, \frac{1}{3}I, \frac{1}{5}I)^T$, $A = \begin{pmatrix} .6 & 1 & .2 \\ .5 & .2 & .1 \\ .3 & .5 & .1 \end{pmatrix}_{3\times 3}$

Using the formula (2), we can find the components of $\hat{x}_{v1}$ as follows

$$\hat{x}_1 = \bigwedge_{i=1}^{3} (a_{i1} \bowtie b_i) = (a_{11} \bowtie b_1) \wedge (a_{21} \bowtie b_2) \wedge (a_{31} \bowtie b_3)$$

$$= (0.6 \bowtie 1) \wedge \left(0.5 \bowtie \frac{1}{3}I\right) \wedge (0.3 \bowtie 0.2I) = 1 \wedge 1 \wedge 1 = 1$$

$$\hat{x}_2 = \bigwedge_{i=1}^{3} (a_{i2} \bowtie b_i) = (a_{12} \bowtie b_1) \wedge (a_{22} \bowtie b_2) \wedge (a_{32} \bowtie b_3)$$

$$= (1 \bowtie 1) \wedge \left(0.2 \bowtie \frac{1}{3}I\right) \wedge (0.5 \bowtie 0.2I) = 1 \wedge 1 \wedge 1 = 1$$

$$\hat{x}_3 = \bigwedge_{i=1}^{3} (a_{i3} \bowtie b_i) = (a_{13} \bowtie b_1) \wedge (a_{23} \bowtie b_2) \wedge (a_{33} \bowtie b_3)$$

$$= (0.2 \bowtie 1) \wedge \left(0.1 \bowtie \frac{1}{3}I\right) \wedge (0.1 \bowtie 0.2I) = 1 \wedge 1 \wedge 1 = 1$$

$\therefore\ \hat{x}_{v1} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)^T = (1,1,1)^T$

Using the formula (3), we can find the components of $\hat{x}_{v2}$ as follows

$$\hat{x}_1 = \bigwedge_{i=1}^{3} (a_{i1}\Theta\, b_i) = (a_{11}\Theta\, b_1) \wedge (a_{21}\Theta\, b_2) \wedge (a_{31}\Theta\, b_3)$$

$$= (0.6\ \Theta\ 1) \wedge \left(0.5\ \Theta\ \frac{1}{3}I\right) \wedge (0.3\ \Theta\ 0.2I) = 1 \wedge \frac{1/3}{0.5}I \wedge \frac{0.2}{0.3}I = \frac{2}{3}I$$

$$\hat{x}_2 = \bigwedge_{i=1}^{3} (a_{i2}\Theta\, b_i) = (a_{12}\Theta\, b_1) \wedge (a_{22}\Theta\, b_2) \wedge (a_{32}\Theta\, b_3)$$

$$= (1\ \Theta\ 1) \wedge \left(0.2\ \Theta\ \frac{1}{3}I\right) \wedge (0.5\ \Theta\ 0.2I) = 1 \wedge 1 \wedge \frac{2}{5}I = \frac{2}{5}I$$

$$\hat{x}_3 = \bigwedge_{i=1}^{3} (a_{i3}\Theta b_i) = (a_{13}\Theta\, b_1) \wedge (a_{23}\Theta\, b_2) \wedge (a_{33}\Theta\, b_3)$$

$$= (0.2\ \Theta\ 1) \wedge \left(0.1\ \Theta\ \frac{1}{3}I\right) \wedge (0.1\ \Theta\ 0.2I) = 1 \wedge 1 \wedge 1 = 1$$

$\therefore\ \hat{x}_{v2} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)^T = \left(\frac{2}{3}I, \frac{2}{5}I, 1\right)^T$

In this example, $A_{v1} = \begin{pmatrix} .6 & 1 & .2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A_{v2} = \begin{pmatrix} 0 & 0 & 0 \\ .5 & .2 & .1 \\ .3 & .5 & .1 \end{pmatrix}$,

$$Ao\hat{x} = (A_{v1}o\hat{x}_{v1}) + (A_{v2}o\hat{x}_{v2}) = \begin{pmatrix} .6 & 1 & .2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} o \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ .5 & .2 & .1 \\ .3 & .5 & .1 \end{pmatrix} o \begin{bmatrix} \frac{2}{3}I \\ \frac{2}{5}I \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ \frac{1}{3}I \\ \frac{1}{5}I \end{bmatrix} = b$$

Since $Ao\hat{x} = b$, then there is a solution in $X(A, b)$ and $\hat{x}$ is the greatest solution to $Aox = b$. The value of $f(\hat{x})$ is calculated as follow,

$f(\hat{x}) = f(\hat{x}_{v1}) \vee f(\hat{x}_{v2})$

$f(\hat{x}) = \langle (0.3\,.\,(1)^2) \vee \left(1.8I\,.\,(1)^{\frac{1}{3}}\right) \vee \left(I\,.\,(1)^{\frac{1}{4}}\right) \rangle \vee \langle \left(0.3.\,(\frac{2}{3}I)^2\right) \vee \left(1.8I\,.\left(\frac{2}{5}I\right)^{\frac{1}{3}}\right) \vee$

$\left(I\,.\,(1)^{\frac{1}{4}}\right) \rangle = \langle (0.3\,) \vee (1.8I) \vee (I\,) \rangle \vee \langle (0.133I) \vee (1.33I) \vee (I) \rangle = 1.8I$

Do not forget that the indeterminate component I to the power n where $n > 0$ is equal to I (i.e. $I^n = I \;\; for\, n > 0$).

4.2 Example

Let $A = \begin{pmatrix} 0.1 & 1 & 0.4 \\ I & 0.9 & 0 \\ 0.5 & 0.2I & 0.7 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0.3I \\ 0.6 \end{pmatrix},$

It easy to see that some components of the matrix A are of the form $a_{ij} = mI, m \in (0,1]$, while $b_i \in [0,1] \cup I$, in this case, and by theorem (3.2), the system of the relation equation $Aox = b$ is incompatible.

4.3 Example

Let $\min f(x) = (0.2I.\,x_1^{-\frac{2}{3}}) \vee (1.3.\,x_2^{\frac{1}{3}}) \vee (I\,.\,x_3^{\frac{1}{2}}) \vee (0.35.\,x_4^{-2})$

s. t. $\;\; Aox = b$

$x_j \in [0,1] \cup I \quad (1 \le j \le n)$

Where $\;\; b = (0.3, 0.7I, 0.5, 0.2I)^T \;,\; A = \begin{pmatrix} .2 & .3 & .4 & .6 \\ .3 & .2 & .9 & .8 \\ 1 & 0 & .1 & 1 \\ 0 & .5 & 1 & 0 \end{pmatrix}_{4\times 4}$

Using the formula (2), the components of $\hat{x}_{v1}$ are

$$\hat{x}_1 = \bigwedge_{i=1}^{4} (a_{i1} \bowtie b_i) = 0.5$$

$$\hat{x}_2 = \bigwedge_{i=1}^{4} (a_{i2} \bowtie b_i) = 1$$

$$\hat{x}_3 = \bigwedge_{i=1}^{4} (a_{i3} \bowtie b_i) = \frac{3}{4}$$

$$\hat{x}_4 = \bigwedge_{i=1}^{4} (a_{i4} \bowtie b_i) = \frac{1}{2}$$

$$\therefore \hat{x}_{v1} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)^T = \left(0.5, 1, \frac{3}{4}, 0.5\right)^T$$

Using the formula (3), the components of $\hat{x}_{v2}$ are

$$\hat{x}_1 = \bigwedge_{i=1}^{4} (a_{i1} \Theta\, b_i) = 1$$

$$\hat{x}_2 = \bigwedge_{i=1}^{4} (a_{i2} \Theta\, b_i) = \frac{2}{5} I$$

$$\hat{x}_3 = \bigwedge_{i=1}^{4} (a_{i3} \Theta b_i) = 0.2I$$

$$\hat{x}_4 = \bigwedge_{i=1}^{4} (a_{i4}\, \Theta b_i) = 0.875I$$

$$\therefore \hat{x}_{v2} = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4)^T = \left(\frac{2}{5} I, 1, 0.2I, 0.875I\right)^T$$

In this example, $A_{v1} = \begin{pmatrix} .2 & .3 & .4 & .6 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & .1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $A_{v2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ .3 & .2 & .9 & .8 \\ 0 & 0 & 0 & 0 \\ 0 & .5 & 1 & 0 \end{pmatrix}$,

$$A o \hat{x} = (A_{v1} o \hat{x}_{v1}) + (A_{v2} o \hat{x}_{v2})$$

$$= \begin{pmatrix} .2 & .3 & .4 & .6 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & .1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} o \begin{bmatrix} 0.5 \\ 1 \\ \frac{3}{4} \\ 0.5 \end{bmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ .3 & .2 & .9 & .8 \\ 0 & 0 & 0 & 0 \\ 0 & .5 & 1 & 0 \end{pmatrix} o \begin{bmatrix} \frac{2}{5} I \\ 1 \\ 0.2I \\ 0.875I \end{bmatrix}$$

$$= \begin{bmatrix} 0.3 \\ 0.7I \\ 0.5 \\ 0.2I \end{bmatrix} = b$$

Since $Ao\hat{x} = b$, then there is a solution in $X(A, b)$ and $\hat{x}$ is the greatest solution to $Aox = b$. The value of $f(\hat{x})$ is calculated as follow,

$$f(\hat{x}) = f(\hat{x}_{v1}) \vee f(\hat{x}_{v2})$$

$$f(\hat{x}) = \langle \left(0.2I \,.\, (\tfrac{1}{2})^{-\frac{3}{2}}\right) \vee \left(1.3.\,(1)^{\frac{1}{3}}\right) \vee \left(I \,.\, \left(\tfrac{3}{4}\right)^{\frac{1}{2}}\right) \vee (0.35.\,(0.5)^{-2})\rangle \vee$$

$$\langle \left(0.2I \,.\, (1)^{-\frac{3}{2}}\right) \vee \left(1.3.\,(0.4I)^{\frac{1}{3}}\right) \vee \left(I \,.\, (0.2I)^{\frac{1}{2}}\right) \vee (0.35.\,(0.5)^{-2})\rangle = \langle (0.57I\,) \vee (1.3) \vee (0.87I\,) \vee (0.5I\,)\rangle \vee \langle (0.2I) \vee (0.96I) \vee (0.45I) \vee (0.5I)\rangle = 1.3$$

5 Conclusion

It is important to know that the fuzzy geometric programming problems (FGPP) have wide applications in the business management, communication system, civil engineering, mechanical engineering, structural design and optimization, chemical engineering, optimal control, decision making, and electrical engineering, unfortunately, the fuzzy logic lacks to cover the indeterminate solution of any real-world problems, this pushed the author to construct a new branch of the neutrosophic geometric programming (NGP) problems subject to neutrosophic relation equations (NRE) and made a series of articles in an attempt to cover the theoretical sides of (NGP) problems. This paper contains a new (NGP) model subject to (NRE) with setting up a definition for the maximum solution of this program as well as some new theorems dealt with the consistency of the problem and some propositions of the new operation Θ. The future prospects are to make a deep study for the above-mentioned applications from the point of view of relational neutrosophic geometric programming (RNGP) problems.

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Neutrosophic Soft Sets Applied on Incomplete Data

Abhijit Saha [1], **Said Broumi** [2], **and Florentin Smarandache** [3]

[1] Faculty of Mathematics, Techno College of Engineering Agartala, , Tripura, India, Pin-799004; Email: abhijit84.math@gmail.com
[2] Faculty of Science, University of Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco; Email: broumisaid78@gmail.com
[3] Faculty of Mathematics ,University of New Mexico, Gallup, New Mexico 87301, USA; Email: fsmarandache@gmail.com

* **Correspondence:** abhijit84.math@gmail.com

Abstract: A neutrosophic set is a part of neutrosophy that studies the origin, nature and scope of neutralities as well as their interactions with different ideational spectra. In this present paper first we have introduced the concept of a neutrosophic soft set having incomplete data with suitable examples. Then we have tried to explain the consistent and inconsistent association between the parameters. We have introduced few new definitions, namely- consistent association number between the parameters, consistent association degree, inconsistent association number between the parameters and inconsistent association degree to measure these associations. Lastly we have presented a data filling algorithm. An illustrative example is employed to show the feasibility and validity of our algorithm in practical situation.

Keywords: Soft set, neutrosophic set, neutrosophic soft set, data filling.

1. Introduction

In 1999, Molodstov **[01]** initiated the concept of soft set theory as a new mathematical tool for modelling uncertainty, vague concepts and not clearly defined objects. Although various traditional tools, including but not limited to rough set theory **[02]**, fuzzy set theory **[03]**, intuitionistic fuzzy set theory **[04]** etc. have been used by many researchers to extract useful information hidden in the uncertain data, but there are immanent complications connected with each of these theories. Additionally, all these approaches lack in parameterizations of the tools and hence they couldn't be applied effectively in real life problems, especially in areas like environmental, economic and social problems. Soft set theory is standing uniquely in the sense that it is free from the above mentioned impediments and obliges approximate illustration of an object from the beginning, which makes this theory a natural mathematical formalism for approximate reasoning.

The Theory of soft set has excellent potential for application in various directions some of which are reported by Molodtsov in his pioneer work. Later on Maji et al. **[05]** introduced some new annotations on soft sets such as subset, complement, union and intersection of soft sets and discussed in detail its applications in decision making problems. Ali et al. **[06]** defined some new operations on soft sets and shown that De Morgan's laws holds in soft set theory with respect to these newly defined operations. Atkas and Cagman **[07]** compared soft sets with fuzzy sets and rough sets to show that every fuzzy set and every rough set may be considered as a soft set. Jun **[08]** connected soft sets to the theory of BCK/BCI-algebra and introduced the concept of soft BCK/BCI-algebras. Feng et al. **[09]** characterized soft semi rings and a few related notions to establish a relation between soft sets and semi rings. In 2001, Maji et al. **[10]** defined the concept of fuzzy soft set by combining of fuzzy sets and soft sets . Roy and Maji **[11]** proposed a fuzzy soft set based decision making method. Xiao et al. **[12]** presented a combined forecasting method based on fuzzy soft set. Feng et al. **[13]** discussed the validity of the

Roy-Maji method and presented an adjustable decision-making method based on fuzzy soft set. Yang et al. **[14]** initiated the idea of interval valued fuzzy soft set (IVFS-set) and analyzed a decision making method using the IVFS-sets. The notion of intuitionistic fuzzy set (IFS) was initiated by Atanassov as a significant generalization of fuzzy set. Intuitionistic fuzzy sets are very useful in situations when description of a problem by a linguistic variable, given in terms of a membership function only, seems too complicated. Recently intuitionistic fuzzy sets have been applied to many fields such as logic programming, medical diagnosis, decision making problems etc. Smarandache **[15]** introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Thao and Smaran **[16]** proposed the concept of divergence measure on neutrosophic sets with an application to medical problem. Song et al. **[17]** applied neutrosophic sets to ideals in BCK/BCI algebras. Some recent applications of neutrosophic sets can be found in **[18]**, **[19]**, **[20]**, **[21]**, **[22]**, **[23]** and **[24]**. Maji **[25]** introduced the concept of neutrosophic soft set and established some operations on these sets. Mukherjee et al **[26]** introduced the concept of interval valued neutrosophic soft sets and studied their basic properties. In 2013, Broumi and Smarandache **[27, 28]** combined the intuitionistic neutrosophic and soft set which lead to a new mathematical model called "intuitionistic neutrosophic soft set". They studied the notions of intuitionistic neutrosophic soft set union, intuitionistic neutrosophic soft set intersection, complement of intuitionistic neutrosophic soft set and several other properties of intuitionistic neutrosophic soft set along with examples and proofs of certain results. Also, in **[29]** S. Broumi presented the concept of "generalized neutrosophic soft set" by combining the generalized neutrosophic sets and soft set models, studied some properties on it, and presented an application of generalized neutrosophic soft set in decision making problem. Recently, Deli **[30]** introduced the concept of interval valued neutrosophic soft set as a combination of interval neutrosophic set and soft set. In 2014, S. Broumi et al. **[31]** initiated the concept of relations on interval valued neutrosophic soft sets.

The soft sets mentioned above are based on complete information. However, incomplete information widely exists in various real life problems. Soft sets under incomplete information become incomplete soft sets. H. Qin et al **[32]** studied the data filling approach of incomplete soft sets. Y. Zou et al **[33]** investigated data analysis approaches of soft sets under incomplete information. In this paper first we have introduced the concept of a neutrosophic soft set with incomplete data supported by examples. Then we have introduced few new definitions to measure the consistent and inconsistent association between the parameters. Lastly we have presented a data filling algorithm supported by an illustrative example to show the feasibility and validity of our algorithm.

2. Preliminaries:

2.1 Definition: [03] Let U be a non empty set. Then a fuzzy set τ on U is a set having the form $\tau = \{(x, \mu_\tau(x)) : x \in U\}$ where the function $\mu_\tau : U \to [0, 1]$ is called the membership function and $\mu_\tau(x)$ represents the degree of membership of each element $x \in U$.

2.2 Definition: [04] Let U be a non empty set. Then an intuitionistic fuzzy set (IFS for short) τ is an object having the form $\tau = \{\langle x, \mu_\tau(x), \gamma_\tau(x)\rangle : x \in U\}$ where the functions $\mu_\tau : U \to [0, 1]$ and $\gamma_\tau : U \to [0, 1]$ are called membership function and non-membership function respectively.

$\mu_\tau(x)$ and $\gamma_\tau(x)$ represent the degree of membership and the degree of non-membership respectively of each element $x \in U$ and $0 \le \mu_\tau(x) + \gamma_\tau(x) \le 1$ for each $x \in U$. We denote the class of all intuitionistic fuzzy sets on U by IFS^U.

2.3 Definition: [01] Let U be a universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subseteq E$. Then the pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \rightarrow P(U)$.

In other words, the soft set is not a kind of set, but a parameterized family of subsets of U. For $e \in A$, $F(e) \subseteq U$ may be considered as the set of e-approximate elements of the soft set (F, A).

2.4 Definition: [10] Let U be a universe set, E be a set of parameters and $A \subseteq E$. Then the pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F: A \rightarrow FS^U$.

2.5 Definition: [34] Let U be a universe set, E be a set of parameters and $A \subseteq E$. Then the pair (F, A) is called an intuitionistic fuzzy soft set over U, where F is a mapping given by $F: A \rightarrow IFS^U$.

For $e \in A$, $F(e)$ is an intuitionistic fuzzy subset of U and is called the intuitionistic fuzzy value set of the parameter 'e'.

Let us denote $\mu_{F(e)}(x)$ by the membership degree that object 'x' holds parameter 'e' and $\gamma_{F(e)}(x)$ by the membership degree that object 'x' doesn't hold parameter 'e', where $e \in A$ and $x \in U$. Then $F(e)$ can be written as an intuitionistic fuzzy set such that $F(e) = \{(x, \mu_{F(e)}(x), \gamma_{F(e)}(x)) : x \in U\}$.

2.6 Definition: [15] A neutrosophic set A on the universe of discourse U is defined as $A = \{\langle x, \mu_A(x), \gamma_A(x), \delta_A(x)\rangle : x \in U\}$, where $\mu_A, \gamma_A, \delta_A : U \rightarrow]^-0, 1^+[$ are functions such that the condition: $\forall x \in U, ^-0 \leq \mu_A(x) + \gamma_A(x) + \delta_A(x) \leq 3^+$ is satisfied.

Here $\mu_A(x), \gamma_A(x), \delta_A(x)$ represent the truth-membership, indeterminacy-membership and falsity-membership respectively of the element $x \in U$.

Smarandache **[15]** applied neutrosophic sets in many directions after giving examples of neutrosophic sets. Then he introduced the neutrosophic set operations namely-complement, union, intersection, difference, Cartesian product etc.

2.7 Definition: [21] Let U be an initial universe, E be a set of parameters and $A \subseteq E$. Let $NP(U)$ denotes the set of all neutrosophic sets of U. Then the pair (f, A) is termed to be the neutrosophic soft set over U, where f is a mapping given by $f : A \rightarrow NP(U)$.

2.8 Example: Let us consider a neutrosophic soft set (f, A) which describes the "attractiveness of the house". Suppose $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be the set of six houses under consideration and $E = \{e_1$ (beautiful), e_2 (expensive), e_3 (cheap), e_4 (good location), e_5 (wooden)$\}$ be the set of parameters. Then a neutrosophic soft set (f, A) over U can be given by:

U	e_1	e_2	e_3	e_4	e_5
u_1	(0.8,0.5,0.2)	(0.3,0.4,0.6)	(0.1,0.6,0.4)	(0.7,0.3,0.6)	(0.3,0.4,0.6)
u_2	(0.4,0.1,0.7)	(0.8,0.2,0.4)	(0.4,0.1,0.7)	(0.2,0.4,0.4)	(0.1,0.1,0.3)
u_3	(0.2,0.6,0.4)	(0.5,0.5,0.5)	(0.8,0.1,0.7)	(0.5,0.3,0.5)	(0.5,0.5,0.5)
u_4	(0.3,0.4,0.4)	(0.1,0.3,0.3)	(0.3,0.4,0.4)	(0.6,0.6,0.6)	(0.1,0.1,0.5)

u_5	(0.1,0.1,0.7)	(0.2,0.6,0.7)	(0.4,0.2,0.1)	(0.8,0.6,0.1)	(0.6,0.7,0.7)
u_6	(0.5,0.3,0.9)	(0.3,0.6,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.5)	(0.4,0.4,0.4)

3. Neutrosophic soft sets with incomplete (missing) data:

Suppose that (f,E) is a neutrosophic soft set over U, such that $\exists\, x_i \in U$ and $e_j \in E$ so that none of $m_{f(e_j)}(x_i), g_{f(e_j)}(x_i)$ and $d_{f(e_j)}(x_i)$ is known. In this case, in the tabular representation of the neutrosophic soft set (f,E), we write $\left(m_{f(e_j)}(x_i), g_{f(e_j)}(x_i), d_{f(e_j)}(x_i)\right) = *$. Here we say that the data for $f(e_j)$ is missing and the neutrosophic soft set (f,E) over U has incomplete data.

3.1 Example: Suppose Tech Mahindra is recruiting some new Graduate Trainee for the session 2019-2020 and suppose that eight candidates have applied for the job. Assume that $U = \{u_1, u_2, u_3, ..., u_8\}$ be the set of candidates and $E = \{e_1$ (communication skill), e_2 (domain knowledge), e_3 (experienced), e_4 (young), e_5 (highest academic degree), e_6 (professional attitute)$\}$ be the set of parameters. Then a neutrosophic soft set over U having missing data can be given by Table-1.

Table-1

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	(0.8,0.5,0.2)	(0.3,0.4,0.6)	(0.1,0.6,0.4)	(0.7,0.3,0.6)	(0.3,0.4,0.6)	(0.2,0.5,0.5)
u_2	(0.4,0.1,0.7)	(0.8,0.2,0.4)	(0.4,0.1,0.7)	(0.2,0.4,0.4)	*	(0.6,0.6,0.4)
u_3	(0.2,0.6,0.4)	(0.5,0.5,0.5)	*	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.3,0.4,0.6)
u_4	(0.3,0.4,0.4)	(0.1,0.3,0.3)	(0.3,0.4,0.4)	(0.6,0.6,0.6)	(0.1,0.1,0.5)	(0.3,0.4,0.4)
u_5	(0.1,0.1,0.7)	*	(0.4,0.2,0.1)	(0.8,0.6,0.1)	(0.6,0.7,0.7)	(0.3,0.4,0.3)
u_6	(0.5,0.3,0.9)	(0.3,0.6,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.6)	(0.4,0.4,0.4)	(0.3,0.6,0.6)
u_7	(0.2,0.4,0.6)	(0.4,0.4,0.5)	(0.5,0.5,0.6)	*	(0.7,0.5,0.8)	(0.4,0.4,0.5)
u_8	(0.2,0.3,0.1)	(0.6,0.6,0.1)	(0.8,0.3,0.8)	(0.4,0.3,0.4)	(0.5,0.6,0.3)	(0.9,0.3,0.3)

In case of soft set theory, there always exist some obvious or hidden associations between parameters. Let us focus on this to find the associations between the parameters of a neutrosophic soft set.

In example **2.8**, one can easily find that if a house is expensive, the house is not cheap and vice versa. Thus there is an inconsistent association between the parameters 'expensive' and 'cheap'. Generally, if a house is beautiful or situated in a good location, the house is expensive. Thus there is a consistent association between the parameters 'beautiful' and 'expensive' or the parameters 'good location' and 'expensive'.

In example **3.1**, we find that if a candidate is experienced or have highest academic degree, he/she is not young. Thus there is an inconsistent association between parameters 'experienced' and 'young' or between 'highest academic degree' and 'young'.

The above two examples reveal the interior relations of parameters. In a neutrosophic soft set, these associations between parameters will be very useful for filling incomplete data. If it is found that the parameters e_i and e_j are associated and the data for $f(e_i)$ is missing, then we can fill the missing data according to the corresponding data in $f(e_j)$. To measure these associations, let us define the notion of association degree and some relevant concepts.

For the rest of the paper we shall assume that *U* be the universe set and *E* be the set of parameters.

Let U_{ij} denotes the set of objects that have specified values in the form of an ordered triplet (*a*, *b*, *c*) where *a*, *b*, *c*∈[0, 1] on both parameters e_i and e_j such that

$$U_{ij} = \left\{ x \in U : \left(m_{f(e_i)}(x), g_{f(e_i)}(x), d_{f(e_i)}(x)\right) \neq *, \left(m_{f(e_j)}(x), g_{f(e_j)}(x), d_{f(e_j)}(x)\right) \neq * \right\}$$

In other words U_{ij} is the collection of those objects that have known data both on e_i and e_j .

3.2 Definition: Let $e_i, e_j \in E$. Then the consistent association number between the parameters e_i and e_j is denoted by CAN_{ij} and is defined as:

$$CAN_{ij} = \left| \left\{ x \in U_{ij} : m_{f(e_i)}(x) = m_{f(e_j)}(x), g_{f(e_i)}(x) = g_{f(e_j)}(x), d_{f(e_i)}(x) = d_{f(e_j)}(x) \right\} \right|$$ where $|.|$ denotes the cardinality of a set.

3.3 Definition: Let $e_i, e_j \in E$. Then the consistent association degree between the parameters e_i and e_j is denoted by CAD_{ij} and is defined as: $CAD_{ij} = \dfrac{CAN_{ij}}{|U_{ij}|}$ where $|.|$ denotes the cardinality of a set.

It can be easily verified that the value of CAD_{ij} lies in [0, 1]. Actually CAD_{ij} measures the extent to which the value of parameter e_i keeps consistent with that of parameter e_j over U_{ij} . Next we define inconsistent association number and inconsistent association degree as follows:

3.4 Definition: Let $e_i, e_j \in E$. Then the inconsistent association number between the parameters e_i and e_j is denoted by $ICAN_{ij}$ and is defined as

$$ICAN_{ij} = \left| \left\{ x \in U_{ij} : m_{f(e_i)}(x) \neq m_{f(e_j)}(x) \text{ or } g_{f(e_i)}(x) \neq g_{f(e_j)}(x) \text{ or } d_{f(e_i)}(x) \neq d_{f(e_j)}(x) \right\} \right|$$

where $|.|$ denotes the cardinality of a set.

3.5 Definition: Let $e_i, e_j \in E$. Then the inconsistent association degree between the parameters e_i and e_j is denoted by $ICAD_{ij}$ and is defined as: $ICAD_{ij} = \dfrac{ICAN_{ij}}{|U_{ij}|}$ where $|.|$ denotes the cardinality of a set.

It can be easily verified that the value of $ICAD_{ij}$ lies in [0, 1]. Actually $ICAD_{ij}$ measures the extent to which the parameters e_i and e_j is inconsistent.

3.6 Definition: Let $e_i, e_j \in E$. Then the association degree between the parameters e_i and e_j is denoted by AD_{ij} and is defined by $AD_{ij} = \max\left\{CAD_{ij}, ICAD_{ij}\right\}$.

If $CAD_{ij} > ICAD_{ij}$, then $AD_{ij} = CAD_{ij}$, which means that most of the objects over U_{ij} have consistent values on parameters e_i and e_j. If $CAD_{ij} < ICAD_{ij}$, then $AD_{ij} = ICAD_{ij}$, which means that most of the objects over U_{ij} have inconsistent values on parameters e_i and e_j. Again if $CAD_{ij} = ICAD_{ij}$, then it means that there is the lowest association degree between the parameters e_i and e_j.

3.7 Theorem: For parameters e_i and e_j, $AD_{ij} \geq 0.5$ for all *i, j*.

Proof: Follows from the fact that $CAD_{ij} + ICAD_{ij} = 1$.

3.8 Definition: If $e_i \in E$, then the maximal association degree of parameter e_i is denoted by MAD_i and is defined by $MAD_i = \max_j AD_{ij}$.

4. DATA Filling Algorithm for a neutrosophic soft set:

Step-1: Input the neutrosophic soft set (f, E) which has incomplete data.

Step-2: Find all parameters e_i for which data is missing.

Step-3: Compute AD_{ij} for *j*=1,2,3….,*m* (where '*m*' is the number of parameters in *E*).

Step-4: Compute MAD_i.

Step-5: Find out all parameters e_j which have the maximal association degree MAD_i with the parameter e_i.

Step-6: In case of consistent association between the parameter e_i and e_j 's (*j*=1,2,3,….) $\left(\mu_{f(e_i)}(x), \gamma_{f(e_i)}(x), \delta_{f(e_i)}(x)\right) = \left(\max_j \mu_{f(e_j)}(x), \max_j \gamma_{f(e_j)}(x), \max_j \delta_{f(e_j)}(x)\right)$. In case of inconsistent association between the parameter e_i and e_j 's (*j*=1,2,3,….) $\left(\mu_{f(e_i)}(x), \gamma_{f(e_i)}(x), \delta_{f(e_i)}(x)\right) = \left(1 - \max_j \mu_{f(e_j)}(x), 1 - \max_j \gamma_{f(e_j)}(x), 1 - \max_j \delta_{f(e_j)}(x)\right)$.

Step-7: If all the missing data are filled then stop else go to step-2.

- **An Illustrative example:** Consider the neutrosophic soft set given in example 3.1.

Step-1:

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	(0.8,0.5,0.2)	(0.3,0.4,0.6)	(0.1,0.6,0.4)	(0.7,0.3,0.6)	(0.3,0.4,0.6)	(0.2,0.5,0.5)
u_2	(0.4,0.1,0.7)	(0.8,0.2,0.4)	(0.4,0.1,0.7)	(0.2,0.4,0.4)	*	(0.6,0.6,0.4)
u_3	(0.2,0.6,0.4)	(0.5,0.5,0.5)	*	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.3,0.4,0.6)
u_4	(0.3,0.4,0.4)	(0.1,0.3,0.3)	(0.3,0.4,0.4)	(0.6,0.6,0.6)	(0.1,0.1,0.5)	(0.3,0.4,0.4)
u_5	(0.1,0.1,0.7)	*	(0.4,0.2,0.1)	(0.8,0.6,0.1)	(0.6,0.7,0.7)	(0.3,0.4,0.3)
u_6	(0.5,0.3,0.9)	(0.3,0.6,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.6)	(0.4,0.4,0.4)	(0.3,0.6,0.6)
u_7	(0.2,0.4,0.6)	(0.4,0.4,0.5)	(0.5,0.5,0.6)	*	(0.7,0.5,0.8)	(0.4,0.4,0.5)
u_8	(0.2,0.3,0.1)	(0.6,0.6,0.1)	(0.8,0.3,0.8)	(0.4,0.3,0.4)	(0.5,0.6,0.3)	(0.9,0.3,0.3)

Step-2: Clearly there are missing data in $f(e_2), f(e_3), f(e_4), f(e_5)$. We shall fill these missing data.

Step-3:

(a) **For the parameter** e_2 **.**

\ $U_{21} = \{u_1, u_2, u_3, u_4, u_6, u_7, u_8\}, U_{23} = \{u_1, u_2, u_4, u_6, u_7, u_8\}, U_{24} = \{u_1, u_2, u_3, u_4, u_6, u_8\},$
$U_{25} = \{u_1, u_3, u_4, u_6, u_7, u_8\}, U_{26} = \{u_1, u_2, u_3, u_4, u_6, u_7, u_8\}.$

Now $CAN_{21} = |\{\ \}| = 0$ and so $CAD_{21} = 0$. Again $ICAN_{21} = |\{u_1, u_2, u_3, u_4, u_6, u_7, u_8\}| = 7$ and so $ICAD_{21} = \frac{ICAN_{21}}{|U_{21}|} = \frac{7}{7} = 1$. Hence $AD_{21} = \max\{CAD_{21}, ICAD_{21}\} = \max\{0,1\} = 1.$

$CAN_{23} = |\{\ \}| = 0$ and so $CAD_{23} = 0$. Again $ICAN_{23} = |\{u_1, u_2, u_4, u_6, u_7, u_8\}| = 6$ and so $ICAD_{23} = \frac{ICAN_{23}}{|U_{23}|} = \frac{6}{6} = 1$. Hence $AD_{23} = \max\{CAD_{23}, ICAD_{23}\} = \max\{0,1\} = 1.$

$CAN_{24} = |\{u_3, u_6\}| = 2$ and so $CAD_{24} = \frac{2}{6} = 0.33$. Again $ICAN_{24} = |\{u_1, u_2, u_4, u_8\}| = 4$ and so $ICAD_{24} = \frac{ICAN_{24}}{|U_{24}|} = \frac{4}{6} = 0.66$. Hence $AD_{24} = \max\{CAD_{24}, ICAD_{24}\} = \max\{0.33, 0.66\} = 0.66.$

$CAN_{25} = |\{u_3, u_1\}| = 2$ and so $CAD_{25} = \frac{2}{6} = 0.33$. Again $ICAN_{25} = |\{u_4, u_6, u_7, u_8\}| = 4$ and so $ICAD_{25} = \frac{ICAN_{24}}{|U_{24}|} = \frac{4}{6} = 0.66$. Hence $AD_{25} = \max\{CAD_{25}, ICAD_{25}\} = \max\{0.33, 0.66\} = 0.66.$

$CAN_{26} = |\{u_4\}| = 1$ and so $CAD_{26} = \frac{1}{7} = 0.14$. Again $ICAN_{26} = |\{u_1, u_2, u_3, u_6, u_7, u_8\}| = 6$ and so $ICAD_{26} = \frac{ICAN_{26}}{|U_{26}|} = \frac{6}{7} = 0.85$. Hence $AD_{26} = \max\{CAD_{26}, ICAD_{26}\} = \max\{0.14, 0.85\} = 0.85.$

Thus $MAD_2 = \max_j AD_{2j} = \max\{AD_{21}, AD_{23}, AD_{24}, AD_{25}, AD_{26}\} = \max\{1, 1, 0.66, 0.66, 0.85\} = 1.$.

(b) **For the parameter** e_3 **.**

\ $U_{31} = \{u_1, u_2, u_4, u_6, u_7, u_8\}, U_{32} = \{u_1, u_2, u_4, u_6, u_7, u_8\}, U_{34} = \{u_1, u_2, u_4, u_5, u_6, u_8\},$
$U_{35} = \{u_1, u_4, u_5, u_6, u_7, u_8\}, U_{36} = \{u_1, u_2, u_4, u_5, u_6, u_7, u_8\}.$

Now $CAN_{31} = |\{u_2, u_4\}| = 2$ and so $CAD_{31} = \frac{2}{6} = 0.33$. Again $ICAN_{31} = |\{u_1, u_6, u_7, u_8\}| = 4$ and so $ICAD_{31} = \frac{ICAN_{31}}{|U_{31}|} = \frac{4}{6} = 0.66$. Hence $AD_{31} = \max\{CAD_{31}, ICAD_{31}\} = \max\{0.33, 0.66\} = 0.66.$

$CAN_{32} = |\{\ \}| = 0$ and so $CAD_{32} = 0$. Again $ICAN_{32} = |\{u_1, u_2, u_4, u_6, u_7, u_8\}| = 6$ and so $ICAD_{32} = \frac{ICAN_{32}}{|U_{32}|} = \frac{6}{6} = 1$. Hence $AD_{32} = \max\{CAD_{32}, ICAD_{32}\} = \max\{0,1\} = 1.$

$CAN_{34} = |\{\ \}| = 0$ and so $CAD_{34} = 0$. Again $ICAN_{34} = |\{u_1, u_2, u_4, u_5, u_6, u_8\}| = 6$ and so $ICAD_{34} = \frac{ICAN_{34}}{|U_{34}|} = \frac{4}{6} = 0.66$. Hence $AD_{34} = \max\{CAD_{34}, ICAD_{34}\} = \max\{0, 0.66\} = 0.66.$

$CAN_{35} = |\{\ \}| = 0$ and so $CAD_{35} = 0$. Again $ICAN_{35} = |\{u_1, u_4, u_5, u_6, u_7, u_8\}| = 6$ and so $ICAD_{35} = \frac{ICAN_{35}}{|U_{35}|} = \frac{6}{6} = 1$. Hence $AD_{35} = \max\{CAD_{35}, ICAD_{35}\} = \max\{0,1\} = 1.$

$CAN_{36} = |\{u_4\}| = 1$ and so $CAD_{36} = \frac{1}{7} = 0.14$. Again $ICAN_{36} = |\{u_1, u_2, u_5, u_6, u_7, u_8\}| = 6$ and so $ICAD_{36} = \frac{ICAN_{36}}{|U_{36}|} = \frac{6}{7} = 0.85$. Hence $AD_{36} = \max\{CAD_{36}, ICAD_{36}\} = \max\{0.14, 0.85\} = 0.85.$

Thus $MAD_3 = \max_j AD_{3j} = \max\{AD_{31}, AD_{32}, AD_{34}, AD_{35}, AD_{36}\} = \max\{0.66, 1, 0.66, 1, 0.85\} = 1.$

(c) **For the parameter** e_4 **.**

\ $U_{41} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_8\}, U_{42} = \{u_1, u_2, u_3, u_4, u_6, u_8\}, U_{43} = \{u_1, u_2, u_4, u_5, u_6, u_8\},$
$U_{45} = \{u_1, u_3, u_4, u_5, u_6, u_8\}, U_{46} = \{u_1, u_2, u_3, u_4, u_5, u_6, u_8\}.$

Now $CAN_{41} = |\{\ \}| = 0$ and so $CAD_{41} = 0$. Again $ICAN_{41} = |\{u_1, u_2, u_3, u_4, u_5, u_6, u_8\}| = 7$ and so $ICAD_{41} = \frac{ICAN_{41}}{|U_{41}|} = \frac{7}{7} = 1$. Hence $AD_{41} = \max\{CAD_{41}, ICAD_{41}\} = \max\{0,1\} = 1.$

$CAN_{42} = |\{u_3, u_6\}| = 2$ and so $CAD_{42} = \frac{2}{6} = 0.33$. Again $ICAN_{42} = |\{u_1, u_2, u_4, u_8\}| = 4$ and so $ICAD_{42} = \frac{ICAN_{42}}{|U_{42}|} = \frac{4}{6} = 0.66$. Hence $AD_{42} = \max\{CAD_{42}, ICAD_{42}\} = \max\{0.33, 0.66\} = 0.66.$

$CAN_{43} = |\{\ \}| = 0$ and so $CAD_{43} = 0$. Again $ICAN_{43} = |\{u_1, u_2, u_4, u_5, u_6, u_8\}| = 6$ and so $ICAD_{43} = \frac{ICAN_{43}}{|U_{43}|} = \frac{6}{6} = 1$. Hence $AD_{43} = \max\{CAD_{43}, ICAD_{43}\} = \max\{0,1\} = 1.$

$CAN_{45} = |\{u_3\}| = 1$ and so $CAD_{45} = \frac{1}{6} = 0.16$. Again $ICAN_{45} = |\{u_1, u_4, u_5, u_6, u_8\}| = 5$ and so $ICAD_{45} = \frac{ICAN_{45}}{|U_{45}|} = \frac{5}{6} = 0.83$. Hence $AD_{45} = \max\{CAD_{45}, ICAD_{35}\} = \max\{0.16, 0.83\} = 0.83.$

$CAN_{46} = |\{u_6\}| = 1$ and so $CAD_{46} = \frac{1}{7} = 0.14$. Again $ICAN_{46} = |\{u_1, u_2, u_3, u_4, u_5, u_8\}| = 6$ and so $ICAD_{46} = \frac{ICAN_{46}}{|U_{46}|} = \frac{6}{7} = 0.85$. Hence $AD_{46} = \max\{CAD_{46}, ICAD_{46}\} = \max\{0.14, 0.85\} = 0.85.$

Thus $MAD_4 = \max_j AD_{4j} = \max\{AD_{41}, AD_{42}, AD_{43}, AD_{45}, AD_{46}\} = \max\{1, 0.66, 1, 0.83, 0.85\} = 1.$

(d) **For the parameter** e_5 **.**

\ $U_{51} = \{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}, U_{52} = \{u_1, u_3, u_4, u_6, u_7, u_8\}, U_{53} = \{u_1, u_4, u_5, u_6, u_7, u_8\},$
$U_{54} = \{u_1, u_3, u_4, u_5, u_6, u_8\}, U_{56} = \{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}.$

Now $CAN_{51} = |\{\ \}| = 0$ and so $CAD_{51} = 0$. Again $ICAN_{51} = |\{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}| = 7$ and so $ICAD_{51} = \frac{ICAN_{51}}{|U_{51}|} = \frac{7}{7} = 1$. Hence $AD_{51} = \max\{CAD_{51}, ICAD_{51}\} = \max\{0,1\} = 1.$

$CAN_{52} = |\{u_1, u_3\}| = 2$ and so $CAD_{52} = \frac{2}{6} = 0.33$. Again $ICAN_{52} = |\{u_4, u_6, u_7, u_8\}| = 4$ and so $ICAD_{52} = \frac{ICAN_{52}}{|U_{52}|} = \frac{4}{6} = 0.66$. Hence $AD_{52} = \max\{CAD_{52}, ICAD_{52}\} = \max\{0.33, 0.66\} = 0.66.$

$CAN_{53} = |\{\ \}| = 0$ and so $CAD_{53} = 0$. Again $ICAN_{53} = |\{u_1, u_4, u_5, u_6, u_7, u_8\}| = 6$ and so $ICAD_{53} = \frac{ICAN_{53}}{|U_{53}|} = \frac{6}{6} = 1$. Hence $AD_{53} = \max\{CAD_{53}, ICAD_{53}\} = \max\{0,1\} = 1.$

$CAN_{54} = |\{u_3\}| = 1$ and so $CAD_{54} = \frac{1}{6} = 0.16$. Again $ICAN_{54} = |\{u_1, u_4, u_5, u_6, u_8\}| = 5$ and so $ICAD_{54} = \frac{ICAN_{54}}{|U_{54}|} = \frac{5}{6} = 0.83$. Hence $AD_{54} = \max\{CAD_{54}, ICAD_{54}\} = \max\{0.16, 0.83\} = 0.83.$

$CAN_{56} = |\{\ \}| = 0$ and so $CAD_{56} = 0$. Again $ICAN_{56} = |\{u_1, u_3, u_4, u_5, u_6, u_7, u_8\}| = 7$ and so $ICAD_{56} = \frac{ICAN_{56}}{|U_{56}|} = \frac{7}{7} = 1$. Hence $AD_{56} = \max\{CAD_{56}, ICAD_{56}\} = \max\{0,1\} = 1.$

Thus $MAD_5 = \max_j AD_{5j} = \max\{AD_{51}, AD_{52}, AD_{53}, AD_{54}, AD_{56}\} = \max\{1, 0.66, 1, 0.83, 1\} = 1.$

The association degree table for the neutrosophic soft set (f, E) is given below:

	e_1	e_2	e_3	e_4	e_5	e_6
e_2	1	_	1	0.66	0.66	0.85
e_3	0.66	1	_	0.66	1	0.85
e_4	1	0.66	1	_	0.83	0.85
e_5	1	0.66	1	0.83	_	1

Step-4: From step-3, we have, $MAD_2 = 1, MAD_3 = 1, MAD_4 = 1, MAD_5 = 1$.

Step-5: The parameters e_1 and e_3 have the maximal association degree AD_{21} and AD_{23} respectively with the parameter e_2.

The parameters e_2 and e_5 have the maximal association degree AD_{32} and AD_{35} respectively with the parameter e_3.

The parameters e_1 and e_3 have the maximal association degree AD_{41} and AD_{43} respectively with the parameter e_4.

The parameters e_1, e_3 and e_6 have the maximal association degree AD_{51}, AD_{53} and AD_{56} respectively with the parameter e_5.

Step-6: There is a consistent association between the parameters e_2 and e_1, e_2 and e_3, e_5 and e_1, e_3 and e_5; while there is an inconsistent association between the parameters e_4 and e_1, e_4 and e_3 .So we have,

$\left(m_{f(e_2)}(u_5), g_{f(e_2)}(u_5), d_{f(e_2)}(u_5)\right)$

$= \left(\max\left(m_{f(e_1)}(u_5), m_{f(e_3)}(u_5)\right), \max\left(g_{f(e_1)}(u_5), g_{f(e_3)}(u_5)\right), \max\left(d_{f(e_1)}(u_5), d_{f(e_3)}(u_5)\right)\right)$

$= (\max(0.1, 0.4), \max(0.1, 0.2), \max(0.7, 0.1)) = (0.4, 0.2, 0.7)$,

$\left(m_{f(e_3)}(u_3), g_{f(e_3)}(u_3), d_{f(e_3)}(u_3)\right)$

$= \left(\max\left(m_{f(e_2)}(u_3), m_{f(e_5)}(u_3)\right), \max\left(g_{f(e_2)}(u_3), g_{f(e_5)}(u_3)\right), \max\left(d_{f(e_2)}(u_3), d_{f(e_5)}(u_3)\right)\right)$

$= (\max(0.5, 0.5), \max(0.5, 0.5), \max(0.5, 0.5)) = (0.5, 0.5, 0.5)$,

$\left(m_{f(e_4)}(u_7), g_{f(e_4)}(u_7), d_{f(e_4)}(u_7)\right)$

$= \left(1 - \max\left(m_{f(e_1)}(u_7), m_{f(e_3)}(u_7)\right), 1 - \max\left(g_{f(e_1)}(u_7), g_{f(e_3)}(u_7)\right), 1 - \max\left(d_{f(e_1)}(u_7), d_{f(e_3)}(u_7)\right)\right)$

$= (\max(0.2, 0.5), \max(0.4, 0.5), \max(0.6, 0.6)) = (0.5, 0.5, 0.6)$,

$\left(m_{f(e_5)}(u_2), g_{f(e_5)}(u_2), d_{f(e_5)}(u_2)\right)$

$= \left(\max\left(m_{f(e_1)}(u_2), m_{f(e_3)}(u_2)\right), \max\left(g_{f(e_1)}(u_2), g_{f(e_3)}(u_2)\right), \max\left(d_{f(e_1)}(u_2), d_{f(e_3)}(u_2)\right)\right)$

$= (\max(0.4, 0.4), \max(0.1, 0.1), \max(0.7, 0.7)) = (0.4, 0.1, 0.7)$.

Thus we have the following table which gives the tabular representation of the filled neutrosophic soft set:

U	e_1	e_2	e_3	e_4	e_5	e_6
u_1	(0.8,0.5,0.2)	(0.3,0.4,0.6)	(0.1,0.6,0.4)	(0.7,0.3,0.6)	(0.3,0.4,0.6)	(0.2,0.5,0.5)
u_2	(0.4,0.1,0.7)	(0.8,0.2,0.4)	(0.4,0.1,0.7)	(0.2,0.4,0.4)	**(0.4,0.1,0.7)**	(0.6,0.6,0.4)
u_3	(0.2,0.6,0.4)	(0.5,0.5,0.5)	**(0.5,0.5,0.5)**	(0.5,0.5,0.5)	(0.5,0.5,0.5)	(0.3,0.4,0.6)
u_4	(0.3,0.4,0.4)	(0.1,0.3,0.3)	(0.3,0.4,0.4)	(0.6,0.6,0.6)	(0.1,0.1,0.5)	(0.3,0.4,0.4)
u_5	(0.1,0.1,0.7)	**(0.4,0.2,0.7)**	(0.4,0.2,0.1)	(0.8,0.6,0.1)	(0.6,0.7,0.7)	(0.3,0.4,0.3)
u_6	(0.5,0.3,0.9)	(0.3,0.6,0.6)	(0.1,0.5,0.5)	(0.3,0.6,0.6)	(0.4,0.4,0.4)	(0.3,0.6,0.6)
u_7	(0.2,0.4,0.6)	(0.4,0.4,0.5)	(0.5,0.5,0.6)	**(0.5,0.5,0.6)**	(0.7,0.5,0.8)	(0.4,0.4,0.5)
u_8	(0.2,0.3,0.1)	(0.6,0.6,0.1)	(0.8,0.3,0.8)	(0.4,0.3,0.4)	(0.5,0.6,0.3)	(0.9,0.3,0.3)

Conclusion: Incomplete information or missing data in a neutrosophic soft set restricts the usage of the neutrosophic soft set. To make the neutrosophic soft set (with missing / incomplete data) more useful, in this paper, we have proposed a data filling approach, where missing data is filled in terms of the association degree between the parameters. We have validated the proposed algorithm by an example and drawn the conclusion that relation between parameters can be applied to fill the missing data.

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Aggregate Operators of Neutrosophic Hypersoft Set

Muhammad Saqlain [1], Sana Moin[2], Muhammad Naveed Jafar [3], Muhammad Saeed[3] and Florentin Smarandache[4]

1 Lahore Garrison University, DHA Phase-VI, Sector C, Lahore, 54000, Pakistan. E-mail: msaqlain@lgu.edu.pk
2 Lahore Garrison University, DHA Phase-VI, Sector C, Lahore, 54000, Pakistan. E-mail: moinsana64@gmail.com
3 Lahore Garrison University, DHA Phase-VI, Sector C, Lahore, 54000, Pakistan. E-mail: naveedjafar@lgu.edu.pk
3 University of Management and Technology, C-II Johar Town, Lahore, 54000, Pakistan. E-mail: Muhammad.saeed@umt.edu.pk
4 Department of Mathematics, University of New Mexico, Gallup, NM 87301, USA. E-mail: smarand@unm.edu

Abstract: Multi-criteria decision making (MCDM) is concerned about organizing and taking care of choice and planning issues including multi-criteria. When attributes are more than one, and further bifurcated, neutrosophic softset environment cannot be used to tackle such type of issues. Therefore, there was a dire need to define a new approach to solve such type of problems, So, for this purpose a new environment namely, Neutrosophic Hypersoft set (NHSS) is defined. This paper includes basics operator's like union, intersection, complement, subset, null set, equal set etc., of Neutrosophic Hypersoft set (NHSS). The validity and the implementation are presented along with suitable examples. For more precision and accuracy, in future, proposed operations will play a vital role is decision-makings like personal selection, management problems and many others.

Keywords: MCDM, Uncertainty, Soft set, Neutrosophic soft set, Hyper soft set.

1. Introduction

The idea of fuzzy sets was presented by Lotfi A. Zadeh in 1965 [1]. From that point the fuzzy sets and fuzzy logic have been connected in numerous genuine issues in questionable and uncertain conditions. The conventional fuzzy sets are based on the membership value or the level of membership value. A few times it might be hard to allot the membership values for fuzzy sets. Therefore, the idea of interval valued fuzzy sets was proposed [2] to catch the uncertainty for membership values. In some genuine issues like real life problems, master framework, conviction framework, data combination, etc., we should consider membership just as the non- membership values for appropriate depiction of an object in questionable and uncertain condition. Neither the fuzzy sets nor the interval valued fuzzy sets is convenient for such a circumstance. Intuitionistic fuzzy sets proposed by Atanassov [3] is convenient for such a circumstance. The intuitionistic fuzzy sets can just deal with the inadequate data considering both the membership and non-membership values. It doesn't deal with the vague and conflicting data which exists in conviction framework.
Smarandache [4] presented the idea of Neutrosophic set which is a scientific apparatus for taking care of issues including uncertain, indeterminacy and conflicting information. Neutrosophic set indicate truth membership value (T), indeterminacy membership value (I) and falsity membership value (F). This idea is significant in numerous application regions since indeterminacy is evaluated exceptionally and the truth membership values, indeterminacy membership values and falsity membership values are independent.

The idea of soft sets was first defined by Molodtsov [5] as a totally new numerical device for taking care of issues with uncertain conditions. He defines a soft set as a parameterized family of subsets of universal set. Soft sets are useful in various regions including artificial insight, game hypothesis and basic decision-making problems [6] and it serves to define various functions for various parameters and utilize values against defined parameters. These functions help us to oversee various issues and choices throughout everyday life.

In the previous couple of years, the essentials of soft set theory have been considered by different researchers. Maji et al. [7] gives a hypothetical study of soft sets which covers subset and super set of a soft set, equality of soft sets and operations on soft sets, for Example, union, intersection, AND and OR-Operations between different sets. Ali at el. [8] presented new operations in soft set theory which includes restricted union, intersection and difference. Cagman and Enginoglu [9, 10] present soft matrix theory which substantiated itself a very significant measurement in taking care of issues while making various choices. Singh and Onyeozili [11] come up with the research that operations on soft set is equivalent to the corresponding soft matrices. From Molodsov [9, 6, 5, 12] up to present, numerous handy applications identified with soft set theory have been presented and connected in numerous fields of sciences and data innovation.

Maji [13] come up with Neutrosophic soft set portrayed by truth, indeterminacy, and falsity membership values which are autonomous in nature. Neutrosophic soft set can deal with inadequate, uncertain, and inconsistence data, while intuitionistic fuzzy soft set and fuzzy soft set can just deal with partial data.

Smarandache [14] presented a new technique to deal with uncertainty. He generalized the soft to hyper soft set by converting the function into multi-decision function. Smarandache, [15, 16, 17, 18, 19, 20] also discuss the various extension of neutrosophic sets in TOPSIS and MCDM. Saqlain *et.al*. [21] proposed a new algorithm along with a new decision-making environment. Many other novel approaches are also used by many researches [22-39] in decision makings.

1.1 Contribution

Since uncertainty is human sense which for the most part surrounds a man while taking any significant choice. Let's say if we get a chance to pick one best competitor out of numerous applicants, we originally set a few characteristics and choices that what we need in our chose up-and-comer. based on these objectives we choose the best one. To make our decision easy we use different techniques. The purpose of this paper is to overcome the uncertainty problem in more precise way by combing Neutrosophic set with Hypersoft set. This combination will produce a new mathematical tool "Neutrosophic Hypersoft Set" and will play a vital role in future decision-making research.

2.Preliminaries

Definition 2.1: Soft Set

Let ξ be the universal set and € be the set of attributes with respect to ξ. Let P(ξ) be the power set of ξ and Ą ⊆ € . A pair (F, Ą) is called a soft set over ξ and its mapping is given as

$$₣: Ą \rightarrow P(\xi)$$

It is also defined as:

$$(₣, Ą) = \{₣(e) \in P(\xi): e \in €, ₣(e) = \emptyset \; if \; e \neq Ą\}$$

Definition 2.2: Neutrosophic Soft Set

Let ξ be the universal set and € be the set of attributes with respect to ξ. Let P(ξ) be the set of Neutrosophic values of ξ and Ą ⊆ € . A pair (F, Ą) is called a Neutrosophic soft set over ξ and its mapping is given as

$$₣: Ą \rightarrow P(\xi)$$

Definition 2.3: Hyper Soft Set:

Let ξ be the universal set and $P(\xi)$ be the power set of ξ. Consider $l^1, l^2, l^3 \dots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \dots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$, then the pair $(₣, L^1 \times L^2 \times L^3 \dots L^n)$ is said to be Hypersoft set over ξ where

$$₣: L^1 \times L^2 \times L^3 \dots L^n \rightarrow P(\xi)$$

3. Calculations

Definition 3.1: Neutrosophic Hypersoft Set (NHSS)

Let ξ be the universal set and $P(\xi)$ be the power set of ξ. Consider $l^1, l^2, l^3 \dots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \dots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $L^1 \times L^2 \times L^3 \dots L^n = \$$, then the pair $(₣, \$)$ is said to be Neutrosophic Hypersoft set (NHSS) over ξ where

$₣: L^1 \times L^2 \times L^3 \dots L^n \rightarrow P(\xi)$ and

$₣(L^1 \times L^2 \times L^3 \dots L^n) = \{< x, T(₣(\$)), I(₣(\$)), F(₣(\$)) >, x \in \xi \}$ where T is the membership value of truthiness, I is the membership value of indeterminacy and F is the membership value of falsity such that $T, I, F: \xi \rightarrow [0,1]$ also $0 \leq T(₣(\$)) + I(₣(\$)) + F(₣(\$)) \leq 3$.

Example 3.1:

Let ξ be the set of decision makers to decide best mobile phone given as

$$\xi = \{m^1, m^2, m^3, m^4, m^5\}$$

also consider the set of attributes as

$$s^1 = Mobile\ type, s^2 = RAM, s^3 = Sim\ Card, s^4 = Resolution, s^5 = Camera, s^6 = Battery\ Power$$

And their respective attributes are given as

$S^1 = Mobile\ type = \{Iphone, Samsung, Oppo, lenovo\}$

$S^2 = RAM = \{8\ GB, 4\ GB, 6\ GB, 2\ GB\}$

$S^3 = Sim\ Card = \{Single, Dual\}$

$S^4 = Resolution = \{1440 \times 3040\ pixels, 1080 \times 780\ pixels, 2600 \times 4010\ pixels\}$

$S^5 = Camera = \{12\ MP, 10MP, 15MP\}$

$S^6 = Battery\ Power = \{4100\ mAh, 1000\ mAh, 2050\ mAh\}$

Let the function be $₣: S^1 \times S^2 \times S^3 \times S^4 \times S^5 \times S^6 \rightarrow P(\xi)$

Below are the tables of their Neutrosophic values

Table 1: Decision maker Neutrosophic values for mobile type

$S^1(Mobile\ type)$	m^1	m^2	m^3	m^4	m^5
Iphone	(0.3, 0.6, 0.7)	(0.7, 0.6, 0.4)	(0.4, 0.5, 0.7)	(0.6, 0.5, 0.3)	(0.5, 0.3, 0.8)
Samsung	(0.7, 0.5, 0.6)	(0.3, 0.2, 0.1)	(0.3, 0.6, 0.2)	(0.8, 0.1, 0.2)	(0.5, 0.4, 0.5)
Oppo	(0.5, 0.2, 0.1)	(0.9, 0.5, 0.3)	(0.9, 0.4, 0.1)	(0.9, 0.3, 0.1)	(0.6, 0.1, 0.2)
Lenovo	(0.5, 0.3, 0.2)	(0.5, 0.2, 0.1)	(0.8, 0.5, 0.2)	(0.6, 0.4, 0.3)	(0.7, 0.4, 0.2)

Table 2: Decision maker Neutrosophic values for RAM

$S^2(RAM)$	m^1	m^2	m^3	m^4	m^5
8 GB	(0.3, 0.4, 0.7)	(0.4, 0.5, 0.7)	(0.5, 0.6, 0.8)	(0.5, 0.3, 0.8)	(0.3, 0.6, 0.7)
4 GB	(0.4, 0.2, 0.5)	(0.3, 0.6, 0.2)	(0.4, 0.7, 0.3)	(0.5, 0.4, 0.5)	(0.7, 0.5, 0.6)
6 GB	(0.7, 0.2, 0.3)	(0.9, 0.4, 0.1)	(0.8, 0.3, 0.2)	(0.6, 0.1, 0.2)	(0.5, 0.2, 0.1)
2 GB	(0.8, 0.2, 0.1)	(0.8, 0.5, 0.2)	(0.9 0.4, 0.1)	(0.7, 0.4, 0.2)	(0.5, 0.3, 0.2)

Table 3: Decision maker Neutrosophic values for sim card

$S^3(Sim\ Card)$	m^1	m^2	m^3	m^4	m^5
Single	(0.6, 0.4, 0.3)	(0.6, 0.5, 0.3)	(0.5, 0.4, 0.3)	(0.7, 0.8, 0.3)	(0.9, 0.2, 0.1)
Dual	(0.8, 0.2, 0.1)	(0.4, 0.8, 0.7)	(0.7, 0.3, 0.2)	(0.3, 0.6, 0.4)	(0.8, 0.4, 0.2)

Table 4: Decision maker Neutrosophic values for resolution

$S^4(Resolution)$	m^1	m^2	m^3	m^4	m^5
1440×3040	(0.7, 0.8, 0.3)	(0.7, 0.5, 0.3)	(0.6, 0.4, 0.3)	(0.5, 0.6, 0.9)	(0.4, 0.5, 0.3)
1080×780	(0.3, 0.6, 0.4)	(0.7, 0.3, 0.2)	(0.8, 0.3, 0.1)	(0.6, 0.4, 0.7)	(0.3, 0.5, 0.8)
2600×4010	(0.5, 0.2, 0.1)	(0.6, 0.3, 0.4)	(0.5, 0.7, 0.2)	(0.9, 0.3, 0.1)	(0.7, 0.4, 0.3)

Table 5: Decision maker Neutrosophic values for camera

$S^5(Camera)$	m^1	m^2	m^3	m^4	m^5
12 MP	(0.6, 0.4, 0.3)	(0.7, 0.8, 0.3)	(0.6, 0.4, 0.3)	(0.4, 0.5, 0.3)	(0.9, 0.2, 0.1)
10 MP	(0.8, 0.3, 0.1)	(0.3, 0.6, 0.4)	(0.8, 0.2, 0.1)	(0.3, 0.5, 0.8)	(0.8, 0.4, 0.2)
15 MP	(0.5, 0.7, 0.2)	(0.5, 0.2, 0.1)	(0.8, 0.5, 0.2)	(0.7, 0.4, 0.3)	(0.7, 0.4, 0.2)

Table 6: Decision maker Neutrosophic values for battery power

$S^6(Battery\ Power)$	m^1	m^2	m^3	m^4	m^5
4100 mAh	(0.7, 0.8, 0.3)	(0.7, 0.6, 0.4)	(0.4, 0.5, 0.7)	(0.9, 0.2, 0.1)	(0.5, 0.3, 0.8)
1000 mAh	(0.3, 0.6, 0.4)	(0.3, 0.2, 0.1)	(0.3, 0.6, 0.2)	(0.8, 0.4, 0.2)	(0.5, 0.4, 0.5)
2050 mAh	(0.5, 0.2, 0.1)	(0.9, 0.5, 0.3)	(0.9, 0.4, 0.1)	(0.7, 0.4, 0.2)	(0.6, 0.1, 0.2)

Neutrosophic Hypersoft set is define as,

$$₣: (S^1 \times S^2 \times S^3 \times S^4 \times S^5 \times S^6) \rightarrow P(\xi)$$

Let's assume $₣(\$) = ₣(samsung, 6\ GB, Dual) = \{m^1, m^4\}$

Then Neutrosophic Hypersoft set of above assumed relation is

$$₣(\$) = ₣(samsung, 6\ GB, Dual\ ) = \{$$
$$< m^1, (samsung\{0.7, 0.5, 0.6\}, 6\ GB\{0.7, 0.2, 0.3\}, Dual\{0.8,0.2,0.1\}) >$$
$$< m^4(samsung\{0.8,0.1,0.2\}, 6\ GB\{0.6, 0.1, 0.2\}, Dual\{0.3, 0.6,0.4\}) >\}$$

Its tabular form is given as

Table 7: Tabular Representation of Neutrosophic Hypersoft Set

$₣(\$) = ₣(\boldsymbol{samsung, 6\ GB, Dual}\ )$	$\boldsymbol{m^1}$	$\boldsymbol{m^4}$
Samsung	(0.7,0.5, 0.6)	(0.8, 0.1, 0.2)
6 GB	(0.7, 0.2, 0.3)	(0.6, 0.1, 0.2)
Dual	(0.8, 0.2, 0.1)	(0.3, 0.6, 0.4)

Definition 3.2: Neutrosophic Hypersoft Subset

Let $₣(\$^1)$ and $₣(\$^2)$ be two Neutrosophic Hypersoft set over ξ. Consider $l^1, l^2, l^3 \dots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \dots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $L^1 \times L^2 \times L^3 \dots L^n = \$$ then $₣(\$^1)$ is the Neutrosophic Hypersoft subset of $₣(\$^2)$ if

$$T(₣(\$^1)) \leq T(₣(\$^2))$$
$$I(₣(\$^1)) \leq I(₣(\$^2))$$

$$F(₣(\$^1)) \geq F\big(₣(\$^2)\big)$$

Numerical Example of Subset

Consider the two NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$ over the same universe $\xi = \{m^1, m^2, m^3, m^4, m^5\}$. The NHSS $₣(\$) = ₣(samsung, 6\ GB, Dual\ ) = \{m^1, m^4\}$ is the subset of NHSS $₣(\$^2) = ₣(Samsung, 6GB) = \{m^1\}$ if $T(₣(\$^1)) \leq T(₣(\$^2))$, $I(₣(\$^1)) \leq I(₣(\$^2))$, $F(₣(\$^1)) \geq F(₣(\$^2))$. Its tabular form is given below

Table 8: Tabular Representation of NHSS $₣(\$^1)$

$₣(\$^1) = ₣(\boldsymbol{samsung, 6\ GB, Dual}\ )$	$\boldsymbol{m^1}$	$\boldsymbol{m^4}$
Samsung	(0.7,0.5, 0.6)	(0.8, 0.1, 0.2)
6 GB	(0.7, 0.2, 0.3)	(0.6, 0.1, 0.2)
Dual	(0.8, 0.2, 0.1)	(0.3, 0.6, 0.4)

Table 9: Tabular Representation of NHSS $₣(\$^2)$

$₣(\$^2) = ₣(\boldsymbol{samsung, 6\ GB})$	$\boldsymbol{m^1}$
Samsung	(0.9, 0.6, 0.3)
6 GB	(0.8, 0.4, 0.1)

This can also be written as

$$₣(\$^1) \subset ₣(\$^2) = ₣(samsung, 6\ GB, Dual\ ) \subset ₣(samsung, 6\ GB)$$

$$= \begin{Bmatrix} < m^1, (samsung\{0.7, 0.5, 0.6\}, 6\ GB\{0.7, 0.2, 0.3\}, Dual\{0.8,0.2,0.1\}) >, \\ < m^4(samsung\{0.8,0.1,0.2\}, 6\ GB\{0.6, 0.1, 0.2\}, Dual\{0.3, 0.6,0.4\}) > \end{Bmatrix}$$

$$\subset \{< m^1, (samsung\{0.9, 0.6, 0.3\}, 6\ GB\{0.8, 0.4, 0.1\})>\}$$

Here we can see that membership value of Samsung for m^1 in both sets is $(0.7, 0.5, 0.6)$ and $(0.9, 0.6, 0.3)$ which satisfy the Definition of Neutrosophic Hypersoft subset as $0.7 < 0.9, 0.5 < 0.6$, and $0.6 > 0.3$. This shows that $(0.7, 0.5, 0.6) \subset (0.9, 0.6, 0.3)$ and same was the case with the rest of the attributes of NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$.

Definition 3.3: Neutrosophic Equal Hypersoft Set

Let $₣(\$^1)$ and $₣(\$^2)$ be two Neutrosophic Hypersoft set over ξ. Consider $l^1, l^2, l^3 \ldots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \ldots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \ldots n\}$ and their relation $L^1 \times L^2 \times L^3 \ldots L^n = \$$ then $₣(\$^1)$ is the Neutrosophic equal Hypersoft subset of $₣(\$^2)$ if

$$T\big(₣(\$^1)\big) = T\big(₣(\$^2)\big)$$

$$I\big(₣(\$^1)\big) = I\big(₣(\$^2)\big)$$

$$F\big(₣(\$^1)\big) = F\big(₣(\$^2)\big)$$

Numerical Example of Equal Neutrosophic Hypersoft Set

Consider the two NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$ over the same universe $\xi = \{m^1, m^2, m^3, m^4, m^5\}$. The NHSS $₣(\$^1) = ₣(samsung, 6\ GB, Dual\ ) = \{m^1, m^4\}$ is the equal to NHSS $₣(\$^2) = ₣(samsung, 6\ GB) = \{m^1\}$ if $T(₣(\$^1)) = T(₣(\$^2))$, $I(₣(\$^1)) = I(₣(\$^2))$, $F(₣(\$^1)) = F(₣(\$^2))$. Its tabular form is given below

Table 10: Tabular Representation of NHSS $₣(\$^1)$

$₣(\$^1) = ₣(samsung, 6\ GB, Dual\ )$	m^1	m^4
Samsung	(0.7,0.5, 0.6)	(0.8, 0.1, 0.2)
6 GB	(0.7, 0.2, 0.3)	(0.6, 0.1, 0.2)
Dual	(0.8, 0.2, 0.1)	(0.3, 0.6, 0.4)

Table 11: Tabular Representation of NHSS $₣(\$^2)$

$₣(\$^2) = ₣(\boldsymbol{samsung, 6\ GB})$	$\boldsymbol{m^1}$
Samsung	(0.7,0.5, 0.6)
6 GB	(0.7, 0.2, 0.3)

This can also be written as

$$\big(₣(\$^1) = ₣(\$^2)\big) = \big(₣(samsung, 6\ GB, Dual\ ) = ₣(samsung, 6\ GB)\big)$$
$$= \big((\{< m^1, (samsung\{0.7, 0.5, 0.6\}, 6\ GB\{0.7, 0.2, 0.3\}, Dual\{0.8,0.2,0.1\}) >,$$
$$< m^4(samsung\{0.8,0.1,0.2\}, 6\ GB\{0.6, 0.1, 0.2\}, Dual\{0.3, 0.6,0.4\}) >\}$$
$$= \{< m^1, (samsung\{0.7, 0.5, 0.6\}, 6\ GB\{0.7, 0.2, 0.3\}) >\})\big)$$

Here we can see that membership value of Samsung for m^1 in both sets is $(0.7, 0.5, 0.6)$ and $(0.7, 0.5, 0.6)$ which satisfy the Definition of Neutrosophic Equal Hypersoft set as $0.7 = 0.7, 0.5 = 0.5$ and $0.6 = 0.6$. This shows that $(0.7, 0.5, 0.6) = (0.7, 0.5, 0.6)$ and same was the case with the rest of the attributes of NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$.

Definition 3.4: Null Neutrosophic Hypersoft Set

Let $₣(\$^1)$ be the Neutrosophic Hypersoft set over ξ. Consider $l^1, l^2, l^3 \dots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \dots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $L^1 \times L^2 \times L^3 \dots L^n = \$$ then $₣(\$^1)$ is Null Neutrosophic Hypersoft set if

$$T\big(₣(\$^1)\big) = 0$$
$$I\big(₣(\$^1)\big) = 0$$
$$F\big(₣(\$^1)\big) = 0$$

Numerical Example of Null Neutrosophic Hypersoft Set

Consider the NHSS $₣(\$^1)$ over the universe $\xi = \{m^1, m^2, m^3, m^4, m^5\}$. The NHSS $₣(\$^1) = ₣(samsung, 6\ GB, Dual\ ) = \{m^1, m^4\}$ is said to be null NHSS if its Neutrosophic values are 0. Its tabular form is given below

Table 12: Tabular Representation of NHSS $₣(\$^1)$

$₣(\$^1) = ₣(samsung, 6\ GB, Dual\ )$	m^1	m^4
Samsung	(0, 0, 0)	(0, 0, 0)
6 GB	(0, 0, 0)	(0, 0, 0)
Dual	(0, 0, 0)	(0, 0, 0)

This can also be written as

$$₣(\$^1) = ₣(samsung, 6\ GB, Dual\ )$$
$$= \{< m^1, (samsung\{0, 0, 0\}, 6\ GB\{0, 0, 0\}, Dual\{0,0,0\}) >,$$
$$< m^4(samsung\{0,0,0\}, 6\ GB\{0, 0, 0\}, Dual\{0, 0,0\}) >\}$$

Definition 3.5: Compliment of Neutrosophic Hypersoft Set

Let $₣(\$^1)$ be the Neutrosophic Hypersoft set over ξ. Consider $l^1, l^2, l^3 \dots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \dots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $L^1 \times L^2 \times L^3 \dots L^n = \$$ then $₣^c(\$^1)$ is the Compliment of Neutrosophic Hypersoft set of $₣(\$^1)$ if

$$₣^c(\$^1): (\neg L^1 \times \neg L^2 \times \neg L^3 \dots \neg L^n) \rightarrow P(\xi)$$

Such that

$$T^C(₣(\$^1)) = F(₣(\$^1))$$
$$I^C(₣(\$^1)) = I(₣(\$^1))$$
$$F^C(₣(\$^1)) = T(₣(\$^1))$$

Numerical Example of Compliment of NHSS

Consider the NHSS $₣(\$^1)$ over the universe $\xi = \{m^1, m^2, m^3, m^4, m^5\}$. The compliment of NHSS $₣(\$^1) = ₣(samsung, 6\ GB, Dual) = \{m^1, m^4\}$ is given as $T^C(₣(\$^1)) = F(₣(\$^1))$, $I^C(₣(\$^1)) = I(₣(\$^1))$, $F^C(₣(\$^1)) = T(₣(\$^1))$.Its tabular form is given below

Table 13: Tabular Representation of NHSS $₣(\$^1)$

$₣^C(\$^1) = ₣(Not\ samsung, Not\ 6\ GB, Not\ Dual)$	m^1	m^4
Not Samsung	(0.6, 0.5, 0.7)	(0.2, 0.1, 0.8)
Not 6 GB	(0.3, 0.2, 0.7)	(0.2, 0.1, 0.6)
Not Dual	(0.1, 0.2, 0.8)	(0.4, 0.6, 0.3)

This can also be written as

$$₣^c(\$^1) = ₣(not\ samsung, not\ 6\ GB, not\ Dual)$$
$$= \{< m^1, (not\ samsung\{0.6, 0.5, 0.7\}, not\ 6\ GB\{0.3, 0.2, 0.7\}, not\ Dual\{0.1,0.2,0.8\}) >,$$
$$< m^4(not\ samsung\{0.2,0.1,0.8\}, not\ 6\ GB\{0.2, 0.1, 0.6\}, not\ Dual\{0.4, 0.6,0.3\}) >\}$$

Here we can see that membership value of Samsung for m^1 in $₣(\$^1)$ is $(0.7, 0.5, 0.6)$ and its compliment is $(0.6, 0.5, 0.7)$ which satisfy the Definition of compliment of Neutrosophic Hypersoft set. This shows that $(0.6, 0.5, 0.7)$ is the compliment of $(0.7, 0.5, 0.6)$ and same was the case with the rest of the attributes of NHSS $₣(\$^1)$.

Definition 3.6: Union of Two Neutrosophic Hypersoft Set

Let $₣(\$^1)$ and $₣(\$^2)$ be two Neutrosophic Hypersoft set over ξ. Consider $l^1, l^2, l^3 \dots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \dots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $L^1 \times L^2 \times L^3 \dots L^n = \$$ then $₣(\$^1) \cup ₣(\$^2)$ is given as

$$T\big(₣(\$^1) \cup ₣(\$^2)\big) = \begin{cases} T\big(₣(\$^1)\big) & if\ x \in \$^1 \\ T\big(₣(\$^2)\big) & if\ x \in \$^2 \\ \max\Big(T\big(₣(\$^1)\big), T\big(₣(\$^2)\big)\Big) & if\ x \in \$^1 \cap \$^2 \end{cases}$$

$$I\big(₣(\$^1) \cup ₣(\$^2)\big) = \begin{cases} I\big(₣(\$^1)\big) & if\ x \in \$^1 \\ I\big(₣(\$^2)\big) & if\ x \in \$^2 \\ \frac{\left(I\left(₣(\$^1)\right) + I\left(₣(\$^2)\right)\right)}{2} & if\ x \in \$^1 \cap \$^2 \end{cases}$$

$$F(₣(\$^1) \cup ₣(\$^2)) = \begin{cases} F(₣(\$^1)) & if\ x \in \$^1 \\ F(₣(\$^2)) & if\ x \in \$^2 \\ \min\left(F(₣(\$^1)), F(₣(\$^2))\right) & if\ x \in \$^1 \cap \$^2 \end{cases}$$

Numerical Example of Union

Consider the two NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$ over the same universe $\xi = \{m^1, m^2, m^3, m^4, m^5\}$. Tabular representation of NHSS $₣(\$^1) = ₣(samsung, 6\ GB, Dual\ ) = \{m^1, m^4\}$ and NHSS $₣(\$^2) = ₣(samsung, 6\ GB) = \{m^1\}$ is given below,

Table 14: Tabular Representation of NHSS $₣(\$^1)$

$₣(\$^1) = ₣(samsung, 6\ GB, Dual\ )$	m^1	m^4
Samsung	(0.7,0.5, 0.6)	(0.8, 0.1, 0.2)
6 GB	(0.7, 0.2, 0.3)	(0.6, 0.1, 0.2)
Dual	(0.8, 0.2, 0.1)	(0.3, 0.6, 0.4)

Table 15: Tabular Representation of NHSS $₣(\$^2)$

$₣(\$^2) = ₣(samsung, 6\ GB)$	m^1
Samsung	(0.9, 0.5, 0.3)
6 GB	(0.8, 0.4, 0.1)

Then the union of above NHSS is given as

Table 16: Union of NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$

$₣(\$^1) \cup ₣(\$^2)$	m^1	m^4
Samsung	(0.9, 0.5, 0.3)	(0.8, 0.1, 0.2)
6 GB	(0.8, 0.3, 0.1)	(0.6, 0.1, 0.2)
Dual	(0.8, 0.1, 0.0)	(0.3, 0.6, 0.4)

This can also be written as

$$₣(\$^1) \cup ₣(\$^2) = ₣(samsung, 6\ GB, Dual\ ) \cup ₣(samsung, 6\ GB)$$
$$= \{< m^1, (samsung\{0.9, 0.5, 0.3\}, 6\ GB\{0.8, 0.3, 0.1\}, Dual\{0.8,0.1,0.0\}) >,$$
$$< m^4(samsung\{0.8,0.1,0.2\}, 6\ GB\{0.6, 0.1, 0.2\}, Dual\{0.3, 0.6,0.4\}) >\}$$

Definition 3.7: Intersection of Two Neutrosophic Hypersoft Set

Let $₣(\$^1)$ and $₣(\$^2)$ be two Neutrosophic Hypersoft set over ξ. Consider $l^1, l^2, l^3 \dots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \dots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $L^1 \times L^2 \times L^3 \dots L^n = \$$ then $₣(\$^1) \cap ₣(\$^2)$ is given as

$$T(₣(\$^1) \cap ₣(\$^2)) = \begin{cases} T(₣(\$^1)) & if\ x \in \$^1 \\ T(₣(\$^2)) & if\ x \in \$^2 \\ \min\left(T(₣(\$^1)), T(₣(\$^2))\right) & if\ x \in \$^1 \cap \$^2 \end{cases}$$

$$I(₣(\$^1) \cap ₣(\$^2)) = \begin{cases} I(₣(\$^1)) & if\ x \in \$^1 \\ I(₣(\$^2)) & if\ x \in \$^2 \\ \frac{\left(I(₣(\$^1)) + I(₣(\$^2))\right)}{2} & if\ x \in \$^1 \cap \$^2 \end{cases}$$

$$F(₣(\$^1) \cap ₣(\$^2)) = \begin{cases} F(₣(\$^1)) & if\ x \in \$^1 \\ F(₣(\$^2)) & if\ x \in \$^2 \\ \max\left(F(₣(\$^1)), F(₣(\$^2))\right) & if\ x \in \$^1 \cap \$^2 \end{cases}$$

Numerical Example of Intersection

Consider the two NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$ over the same universe $\xi = \{m^1, m^2, m^3, m^4, m^5\}$. Tabular representation of NHSS $₣(\$^1) = ₣(samsung, 6\ GB, Dual) = \{m^1, m^4\}$ and NHSS $₣(\$^2) = ₣(samsung, 6\ GB) = \{m^1\}$ is given below

Table 17: Tabular Representation of NHSS $₣(\$^1)$

$₣(\$^1) = ₣(samsung, 6\ GB, Dual)$	m^1	m^4
Samsung	(0.7,0.5, 0.6)	(0.8, 0.1, 0.2)
6 GB	(0.7, 0.2, 0.3)	(0.6, 0.1, 0.2)
Dual	(0.8, 0.2, 0.1)	(0.3, 0.6, 0.4)

Table 18: Tabular Representation of NHSS $₣(\$^2)$

$₣(\$^2) = ₣(samsung, 6\ GB)$	m^1
Samsung	(0.9, 0.5, 0.3)
6 GB	(0.8, 0.4, 0.1)

Then the intersection of above NHSS is given as

Table 19: Intersection of NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$

$₣(\$^1) \cap ₣(\$^2)$	m^1
Samsung	(0.7, 0.5, 0.6)
6 GB	(0.7, 0.3, 0.3)
Dual	(0.0, 0.1, 0.1)

This can also be written as

$$₣(\$^1) \cap ₣(\$^2) = ₣(samsung, 6\ GB, Dual) \cap ₣(samsung, 6\ GB)$$
$$= \{< m^1, (samsung\{0.7, 0.5, 0.6\}, 6\ GB\{0.7, 0.3, 0.3\}, Dual\{0.0,0.1,0.1\}) >\}$$

Definition 3.8: AND Operation on Two Neutrosophic Hypersoft Set

Let $₣(\$^1)$ and $₣(\$^2)$ be two Neutrosophic Hypersoft set over ξ. Consider $l^1, l^2, l^3 \ldots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \ldots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \ldots n\}$ and their relation $L^1 \times L^2 \times L^3 \ldots L^n = \$$ then $₣(\$^1) \wedge ₣(\$^2) = ₣(\$^1 \times \$^2)$ is given as

$$T(\$^1 \times \$^2) = min\left(T\big(₣(\$^1)\big), T\big(₣(\$^2)\big)\right)$$

$$I(\$^1 \times \$^2) = \frac{\left(I\big(₣(\$^1)\big), I\big(₣(\$^2)\big)\right)}{2}$$

$$F(\$^1 \times \$^2) = max\left(F\big(₣(\$^1)\big), F\big(₣(\$^2)\big)\right)$$

Numerical Example of AND-Operation

Consider the two NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$ over the same universe $\xi = \{m^1, m^2, m^3, m^4, m^5\}$. Tabular representation of NHSS $₣(\$^1) = ₣(samsung, 6\ GB, Dual) = \{m^1, m^4\}$ and NHSS $₣(\$^2) = ₣(samsung, 6\ GB,) = \{m^1\}$ is given below

Table 20: Tabular representation of NHSS $₣(\$^1)$

$₣(\$^1) = ₣(samsung, 6\ GB, Dual)$	m^1	m^4
Samsung	(0.7,0.5, 0.6)	(0.8, 0.1, 0.2)
6 GB	(0.7, 0.2, 0.3)	(0.6, 0.1, 0.2)
Dual	(0.8, 0.2, 0.1)	(0.3, 0.6, 0.4)

Table 21: Tabular representation of NHSS $₣(\$^2)$

$₣(\$^2) = ₣(\boldsymbol{samsung, 6\ GB})$	$\boldsymbol{m^1}$
Samsung	(0.9, 0.5, 0.3)
6 GB	(0.8, 0.4, 0.1)

Then the AND Operation of above NHSS is given as

Table 22: AND of NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$

$₣(\$^1) \wedge ₣(\$^2)$	$\boldsymbol{m^1}$	$\boldsymbol{m^4}$
$Samsung \times\ Samsung$	(0.7,0.5,0.6)	(0.0,0.1,0.2)
$Samsung \times\ 6\ GB$	(0.7, 0.45,0.6)	(0.0,0.1,0.2)
$6\ GB\ \times Samsung$	(0.7, 0.35,0.3)	(0.0,0.1,0.2)
$6\ GB \times\ 6\ GB$	(0.7,0.3, 0.3)	(0.0,0,1,0.2)
$Dual\ \times Samsung$	(0.8,0.35,0.3)	(0.0,0.6,0.4)
$Dual \times\ 6\ GB$	(0.8, 0.3, 0.1)	(0.0,0.6,0.4)

Definition 3.9: OR Operation on Two Neutrosophic Hypersoft Set

Let $₣(\$^1)$ and $₣(\$^2)$ be two Neutrosophic Hypersoft set over ξ. Consider $l^1, l^2, l^3 \dots l^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $L^1, L^2, L^3 \dots L^n$ with $L^i \cap L^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $L^1 \times L^2 \times L^3 \dots L^n = \$$ then $₣(\$^1) \vee ₣(\$^2) = ₣(\$^1 \times \$^2)$ is given as

$$T(\$^1 \times \$^2) = max\left(T\big(₣(\$^1)\big), T\big(₣(\$^2)\big)\right)$$

$$I(\$^1 \times \$^2) = \frac{\left(I\big(₣(\$^1)\big), I\big(₣(\$^2)\big)\right)}{2}$$

$$F(\$^1 \times \$^2) = min\left(F\big(₣(\$^1)\big), F\big(₣(\$^2)\big)\right)$$

Numerical Example of OR-Operation

Consider the two NHSS $₣(\$^1)$ and NHSS $₣(\$^2)$ over the same universe $\xi = \{m^1, m^2, m^3, m^4, m^5\}$. Tabular representation of NHSS $₣(\$^1) = ₣(samsung, 6\ GB, Dual\ ) = \{m^1, m^4\}$ and NHSS $₣(\$^2) = ₣(samsung, 6\ GB,) = \{m^1\}$ is given below

Table 23: Tabular representation of NHSS $₣(\$^1)$

$₣(\$^1)$ $= ₣(\boldsymbol{samsung, 6\ GB, Dual}\ )$	$\boldsymbol{m^1}$	$\boldsymbol{m^4}$
Samsung	(0.7,0.5, 0.6)	(0.8, 0.1, 0.2)
6 GB	(0.7, 0.2, 0.3)	(0.6, 0.1, 0.2)
Dual	(0.8, 0.2, 0.1)	(0.3, 0.6, 0.4)

$₣(\$^2) = ₣(\boldsymbol{samsung, 6\ GB})$	$\boldsymbol{m^1}$
Samsung	(0.9, 0.5, 0.3)
6 GB	(0.8, 0.4, 0.1)

Table 24: Tabular representation of NHSS $₣(\$^2)$

Then the OR Operation of above NHSS is given as

Table 25: OR of NHSS $₣(\$^{1})$ and NHSS $₣(\$^{2})$

$₣(\$^{1}) \vee ₣(\$^{2})$	$\boldsymbol{m^1}$	$\boldsymbol{m^4}$
$Samsung \times\ Samsung$	(0.9,0.5,0.3)	(0.8,0.1,0.0)
$Samsung \times\ 6\ GB$	(0.8, 0.45,0.1)	(0.8,0.1,0.0)
$6\ GB\ \times Samsung$	(0.9, 0.35,0.3)	(0.6,0.1,0.0)
$6\ GB \times\ 6\ GB$	(0.8,0.3, 0.1)	(0.6,0,1,0.0)
$Dual\ \times Samsung$	(0.9,0.35,0.1)	(0.3,0.6,0.0)
$Dual \times\ 6\ GB$	(0.8, 0.3, 0.1)	(0.3,0.6,0.0)

4. Result Discussion

Decision-making is a complex issue due to vague, imprecise and indeterminate environment specially, when attributes are more than one, and further bifurcated. Neutrosophic softset environment cannot be used to tackle such type of issues. Therefore, there was a dire need to define a new approach to solve such type of problems, So, for this purpose neutrosophic hypersoft set environment is defined along with necessary operations and elaborated with examples.

5. Conclusions

In this paper, operations of Neutrosophic Hypersoft set like union, intersection, compliment, AND OR operations are presented. The validity and implementation of the proposed operations and definitions are verified by presenting suitable example. Neutrosophic hypersoft set NHSS will be a new tool in decision-making problems for suitable selection. In future, many decision-makings like personal selection, office management, industrial equipment and many other problems can be solved with the proposed operations [23]. Properties of Union and Intersection operations, cardinality and functions on NHSS are to be defined in future.

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Conflicts of Interest

The authors declare that they have no conflict of interest.

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A New Approach of Neutrosophic Soft Set with Generalized Fuzzy TOPSIS in Application of Smart Phone Selection

Muhammad Saqlain [1], Muhammad Naveed Jafar [2] and Muhammad Riaz [2]

[1] Lahore Garrison University, DHA Phase-VI, Sector C, Lahore, 54000, Pakistan. E-mail: msaqlain@lgu.edu.pk
[2] Lahore Garrison University, DHA Phase-VI, Sector C, Lahore, 54000, Pakistan. E-mail: naveedjafar@lgu.edu.pk
[2] Department of Mathematics, University of Punjab, Lahore, 54000, Pakistan. E-mail: mriaz.math@pu.edu.pk

Abstract: With the invention of new technologies, the competition elevates in market. Therefore, it creates more difficulties for consumer to select the right smart phone. In this paper, a new approach is proposed to select smart phone, in which environment of decision-making is MCDM. Firstly, an algorithm is proposed in which problem is formulated in the form of neutrosophic soft set and then solved with generalized fuzzy TOPSIS (GFT). Secondly, rankings are compared with [10]. Finally, it is concluded that proposed approach is applicable in decision-making where uncertainty and imprecise information-based environment is confronted. In future, this evolutionary algorithm can be used along with other methodologies to solve MCDM problems.

Keywords: Accuracy Function, MCDM, TOPSIS, Mobile Phone, Soft set, Neutrosophic Numbers NNs, Neutrosophic Soft set, Linguistic Variable.

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1. Introduction

Mobile / cell phones are widely used for making call, SMS, MMS, email or to access internet. The first portable cell phone was manifest by Martin in 1973 [8], using a handset weighing 4.4 IBS. In the advance world, smart-phone have currently overtaken the usage of earlier telecommunication system. There may be an outstanding doubt and complications concerning the reputation of cellular technologies by decision makers, provider, trader, and clients alike. To help this selection process amongst different available options for technology evaluation, multi-standards decision-making approach appears to be suitable. Due to brutal market competition by inventions of different models with innovative designs and characteristics have made the buying decision making more complex [10]. It is typically tough for a decision-maker to assign a particular performance rating to another for the attributes into consideration. The advantage of employing a fuzzy approach is to assign the relative importance of attributes victimization fuzzy ranges rather than a particular number for textile the $64000 world during a fuzzy atmosphere. MCDM approach [9] with cluster deciding is employed to judge smartphones as another per client preferences [6]. TOPSIS methodology is especially appropriate for finding the cluster call –making drawback beneath fuzzy atmosphere. TOPSIS methodology [22] is predicated on the idea that the chosen various ought to have the shortest distance from the positive ideal solution. In decision making problems TOPSIS method have been studied by many researchers: Adeel et al. [3-5, 7 ,11, 13, 18, 21, 24]. This technique of MCDM is used by Saqlain et. al. [16] to predict CWC 2019. Maji [12] introduced the idea of Neutrosophic soft set. Riaz and

Naeem [14, 15] presented some essential ideas of soft sets together with soft sigma algebra. Neutrosophic set could be a terribly powerful tool to agitate incomplete and indeterminate data planned by F. Smarandache [20] and has attracted the eye of the many students [1], which might offer the credibleness of the given linguistic analysis worth and linguistic set can offer qualitative analysis values. At the primary, soft set theory was planned by a Russian scientist [2] that was used as a standard mathematical mean to come back across the difficulty of hesitant and uncertainty [19]. He additionally argues that however, the same theory of sentimental set is free from the parameterization inadequacy syndrome of fuzzy set theory [23], rough set theory, and applied mathematics. Nowadays, researchers are focusing to present new theories to deal with uncertainty, imprecision and vagueness [25-35], along with suitable examples to elaborate their theories. Neutrosophic soft sets along with TOPSIS technique is widely used in decision making problems, every day many researchers are working in this era [36-45] to discuss the validity of Neutrosophy in decision problems.

1.1 Novelties

It is a very complicated decision to select the utmost suitable phone. In this condition Neutrosophic soft-set-environment is considered and simplified with Generalized TOPSIS. An algorithm is proposed to tackle uncertain, vague and imprecise environment in selection problems.

1.2 Contribution

Cell phone selection is a challenging problem in current generation. To solve this complexity, a few methods regarding the usage of fuzzy ideas has been proposed. For the few kinds of uncertainty within the selection method fuzzy linguistic method is used. The objective of the study is to investigate the uncertainty in selection criteria of cell phone with respect to the consumer's choice under Neutrosophic softset environment by applying Generalized fuzzy TOPSIS.

2.Preliminaries

Definition 2.1: Neutrosophic Set [2]

Let U be a universe of discourse then the neutrosophic set A is an object having the form

$$A = \{< x:\ T_A\ (x), I_A(x), F_A\ (x), >; x \in U\}$$

where the functions $T, I, F : U \rightarrow [0,1]$ define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element $x \in X$ to the set A with the condition. $\leq T_A\ (x) + I_A\ (x) + F_A\ (x)\ \leq 3$.

Definition 2.2: Soft Set [2]

Let $\mho$ be a universe of discourse, $P(\mho)$the power set of $\mho$, and A set of parameters. Then, the pair (F, $\mho$), where

$$F :\ A \longrightarrow P(\mho)$$

is called a softset over $\mho$.

Definition 2.3: Neutrosophic Soft Set [12]

Let $\mho$ be an initial universal set and E be a set of parameters. Assume, $A \subset E$. Let $P(\mho)$denotes the set of all neutrosophic sets over $\mho$, where F is a mapping given by

$$F :\ A \longrightarrow P(\mho)$$

Definition 2.4: Accuracy Function [17]

Accuracy function is used to convert neutrosophic number NFN into fuzzy number (Deneutrosophication using $\boldsymbol{A_F}$). $A(F) = \{ x = \frac{[T_x + I_x + F_x]}{3} \}$

$\boldsymbol{A_F}$ represents the De-Neutrosophication of neutrosophic number into Fuzzy Number.

3. Calculations

In this section an algorithm is proposed to solve MCDM problem under neutrosophic environment.

3.1 Algorithm

Cell phone selection is a challenging problem in current generation. To solve this complexity, a few methods regarding the usage of neutrosophic fuzzy TOPSIS ideas have been proposed. For the few kinds of uncertainty within the selection method fuzzy linguistic method is used. The objective of the study is to investigate the uncertainty in selection criteria of cell phone.

To solve this problem following algorithm is applied as in sequence.

Step 1: defining a problem

Step 2: Consideration of problem as MCDM (alternatives and attributes)

Step 3: Assigning linguistic variables to alternatives and criteria's / attributes

Step 4: Substitution of NNs to linguistic variables

Step 5: Conversion of NNs to fuzzy numbers by using accuracy function [?] defined as,

$$A(F) = \{ x = \frac{[T_x + I_x + F_x]}{3} \}$$

$Where\ T_x, I_x, F_x \epsilon\ NNs\ assigned\ by\ decision\ makers\ to\ each\ criteria\ individually$

Step 6: Apply TOPSIS technique

Step 7: Arrange by ascending order and rank accordingly.

Step 8: Discussion

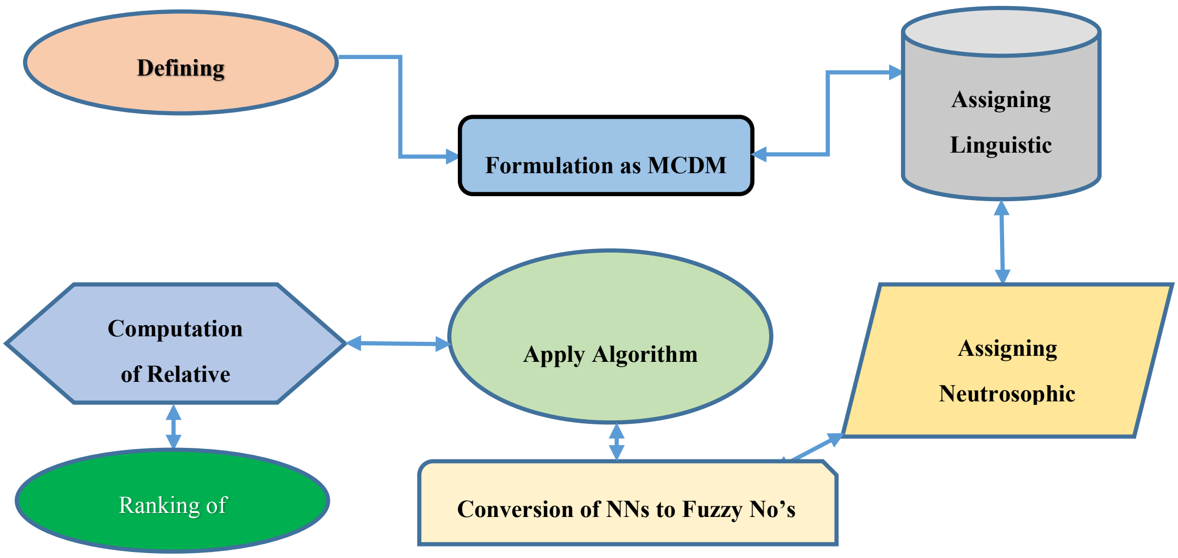

Figure 1: Algorithm used in mobile selection, under neutrosophic softset environment

3.2: Case Study

To discuss the;

- Validity
- Applicability

of the proposed algorithm, mobile selection is considered as a MCDM problem.

3.2.1 Problem Formulation

The mobile phone has been identified for choosing criterion and after that the criterion is depending upon the public choice. The result gets from criterion, some mobile phone has been selected according to their criterion. With invention of new technologies, the competition is raised upon in market it makes more difficult for consumer to select the right phone. In fast growing market, we think that the result got from fuzzy idea has been improved, so we applied Neutrosophic set to get more accuracy in result. The aim of the study is to explore the accuracy in the selection of criteria of mobile phone.

3.2.2 Parameters

Selection is a complex issue, to resolve this problem criteria and alternative plays an important role. Following criteria and alternatives are considered in this problem formulation.

Criteria's						
$\acute{C}_1$	$\acute{C}_2$	$\acute{C}_3$	$\acute{C}_4$	$\acute{C}_5$	$\acute{C}_6$	$\acute{C}_7$
Ram	Rom	Processor	Camera	Display Size	Model	Price

Mobiles as Alternatives					
$\underset{.}{M}_1$	$\underset{.}{M}_2$	$\underset{.}{M}_3$	$\underset{.}{M}_4$	$\underset{.}{M}_5$	$\underset{.}{M}_6$
SAMSUNG	NOKIA	HTC	HUAWEI	Q-MOBILE	RIVO

3.2.3 Assumptions

The decision makers $\{Ɗ_1, Ɗ_2, Ɗ_3, Ɗ_4\}$ will assign linguistic values from Table .1 according to his own interest, knowledge and experience, to the above-mentioned criteria and alternatives and shown in Table.2.

Table 1: Linguistic variables, codes and neutrosophic numbers obtained by expert opinion

Sr # No	Linguistic variable	Code	Neutrosophic Number
1	Very Low	$\tilde{V}\bar{\underset{.}{L}}$	(0.1, 0.3,0.7)
2	Low	$\bar{\underset{.}{L}}$	(0.3,0.5,0.6)
3	Satisfactory	$\underset{,}{S}$	(0.5,0.5,0.5)
4	High	$\underset{\smile}{H}$	(0.7,0.3,0.4)
5	Very High	$\tilde{V}\,\underset{\smile}{H}$	(1.0,0.1,0.2)

3.3 Application of Proposed Algorithm

Step 1: Problem consideration 3.2.

Step 2: Formulation and assumptions 3.2.1 and 3.2.2.

Step 3: Assigning linguistic variables to each alternatives and criteria's / attributes.

Table 2: Each decision maker, will assign linguistic values to each attribute, from Table .1

	Strategies	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{D}_4$
C_1= RAM	Ṃ$_1$	ṼḸ	Ṣ	Ḫ	Ṣ
	Ṃ$_2$	Ḹ	Ḫ	Ṽ Ḫ	Ḫ
	Ṃ$_3$	Ṣ	Ṽ Ḫ	ṼḸ	Ṽ Ḫ
	Ṃ$_4$	Ḫ	Ṣ	ṼḸ	ṼḸ
	Ṃ$_5$	Ṽ Ḫ	ṼḸ	Ḹ	Ḹ
	Ṃ$_6$	ṼḸ	Ḹ	Ṣ	Ṣ
C_2= ROM	Ṃ$_1$	Ḹ	Ṣ	Ḫ	Ḫ
	Ṃ$_2$	Ṣ	Ḫ	Ṽ Ḫ	Ṽ Ḫ
	Ṃ$_3$	Ḫ	Ṽ Ḫ	ṼḸ	Ṣ
	Ṃ$_4$	Ṽ Ḫ	Ṣ	Ḹ	Ḫ
	Ṃ$_5$	ṼḸ	Ḫ	Ṣ	Ṽ Ḫ
	Ṃ$_6$	Ḹ	Ṽ Ḫ	Ḫ	Ṣ
C_3= PROCESSOR	Ṃ$_1$	Ṣ	ṼḸ	Ṽ Ḫ	Ḫ
	Ṃ$_2$	Ḫ	Ḹ	Ṣ	Ṽ Ḫ
	Ṃ$_3$	Ṽ Ḫ	Ṣ	Ḫ	ṼḸ
	Ṃ$_4$	Ṣ	Ḫ	Ṽ Ḫ	Ḹ
	Ṃ$_5$	Ḫ	Ṽ Ḫ	Ḹ	Ṣ
	Ṃ$_6$	Ṽ Ḫ	Ṣ	Ḫ	ṼḸ
C_4= CAMERA	Ṃ$_1$	ṼḸ	Ḫ	Ṽ Ḫ	Ḹ
	Ṃ$_2$	Ḹ	Ṽ Ḫ	ṼḸ	Ḫ
	Ṃ$_3$	Ṣ	Ḫ		Ṽ Ḫ
	Ṃ$_4$	Ḫ	Ṽ Ḫ	ṼḸ	Ḹ
	Ṃ$_5$	Ṽ Ḫ	ṼḸ	Ḹ	Ḫ
	Ṃ$_6$	ṼḸ	Ṣ	Ḹ	Ṣ
C_5= DISPLAY SIZE	Ṃ$_1$	Ḹ	Ḫ	Ḫ	Ḫ
	Ṃ$_2$	Ṣ	Ṽ Ḫ	Ḹ	Ṽ Ḫ
	Ṃ$_3$	Ḫ	Ṣ	Ṽ Ḫ	ṼḸ
	Ṃ$_4$	Ṽ Ḫ	Ḫ	Ḹ	Ṽ Ḫ
	Ṃ$_5$	Ṣ	Ṽ Ḫ	Ḫ	ṼḸ
	Ṃ$_6$	Ṽ Ḫ	ṼḸ	Ḹ	Ḫ

Step 4: Substitution of Neutrosophic Numbers (NNs) to each linguistic variable.

Table3: Assign neutrosophic number to each linguistic value from table 1.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
M_1	(0.1, 0.3,0.7)	(1,0.1,0.2)	(0.7,0.3,0.4)	(0.7,0.3,0.4)	(0.5,0.5,0.5)	(0.1, 0.3,0.7)	(0.7,0.3,0.4)
M_2	(0.3,0.5,0.6)	(0.5,0.5,0.5)	(0.1, 0.3,0.7)	(1,0.1,0.2)	(0.7,0.3,0.4)	(0.3,0.5,0.6)	(0.1, 0.3,0.7)
M_3	(0.5,0.5,0.5)	(0.1, 0.3,0.7)	(0.3,0.5,0.6)	(1,0.1,0.2)	(0.7,0.3,0.4)	(0.5,0.5,0.5)	(1,0.1,0.2)
M_4	(0.7,0.3,0.4)	(1,0.1,0.2)	(0.5,0.5,0.5)	(0.3,0.5,0.6)	(1,0.1,0.2)	(0.7,0.3,0.4)	(0.1, 0.3,0.7)
M_5	(1,0.1,0.2)	(0.3,0.5,0.6)	(0.7,0.3,0.4)	(0.5,0.5,0.5)	(0.1, 0.3,0.7)	(1,0.1,0.2)	(0.5,0.5,0.5)
M_6	(0.5,0.5,0.5)	(0.1, 0.3,0.7)	(1,0.1,0.2)	(0.7,0.3,0.4)	(0.1, 0.3,0.7)	(0.5,0.5,0.5)	(0.7,0.3,0.4)

Step 5: Conversion of fuzzy neutrosophic numbers NNs of step 4, into fuzzy numbers by using accuracy function.

$$\mathrm{A(F)} = \{\, x = \frac{[T_x + I_x + F_x]}{3} \,\}$$

Table: 4 After applied accuracy function the obtain result converted into fuzzy value

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
M_1	0.367	0.433	0.467	0.467	0.5	0.367	0.467
M_2	0.467	0.5	0.367	0.433	0.467	0.467	0.367
M_3	0.5	0.367	0.467	0.433	0.467	0.5	0.433
M_4	0.467	0.433	0.5	0.467	0.433	0.467	0.367
M_5	0.433	0.467	0.467	0.5	0.367	0.433	0.5
M_6	0.5	0.367	0.433	0.467	0.367	0.5	0.467

Step 6: Now we apply algorithm of TOPSIS to obtain relative closeness.

Table 5: Normalized decision matrices

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
M_1	0.327	0.410	0.422	0.413	0.468	0.327	0.437
M_2	0.416	0.474	0.332	0.383	0.437	0.416	0.343
M_3	0.446	0.348	0.422	0.383	0.437	0.446	0.405
M_4	0.416	0.410	0.452	0.413	0.405	0.416	0.343
M_5	0.386	0.443	0.422	0.442	0.343	0.386	0.468
M_6	0.446	0.348	0.391	0.413	0.343	0.446	0.437

Step 6.1: Calculation of weighted normalized matrix

Table6: Weighted normalized decision matrices

weight	0.2	0.3	0.17	0.02	0.25	0.05	0.01
	C_1	C_2	C_3	C_4	C_5	C_6	C_7
M_1	0.0654	0.123	0.07174	0.00826	0.117	0.01635	0.00437
M_2	0.0832	0.1422	0.05644	0.00766	0.10925	0.0208	0.00343
M_3	0.0892	0.1044	0.07174	0.00766	0.10925	0.0223	0.00405
M_4	0.0832	0.123	0.07684	0.00826	0.1015	0.0208	0.00343
M_5	0.0772	0.1329	0.07174	0.00884	0.08575	0.0193	0.00468
M_6	0.0892	0.1044	0.06647	0.00826	0.08575	0.0223	0.00437

Step 6.2: Calculation of the ideal best and ideal worst value,

v_j^+ =Indicates the ideal (best)

v_j^- = Indicates the ideal (worst)

Table 7: Ideal worst and Ideal best values

	C_1	C_2	C_3	C_4	C_5	C_6	C_7
M_1	0.0654	0.123	0.07174	0.00826	0.117	0.01635	0.00437
M_2	0.0832	0.1422	0.05644	0.00766	0.10925	0.0208	0.00343
M_3	0.0892	0.1044	0.07174	0.00766	0.10925	0.0223	0.00405
M_4	0.0832	0.123	0.07684	0.00826	0.1015	0.0208	0.00343
M_5	0.0772	0.1329	0.07174	0.00884	0.08575	0.0193	0.00468
M_6	0.0892	0.1044	0.06647	0.00826	0.08575	0.0223	0.00437
v_j^+	0.0892	0.1422	0.07684	0.0084	0.117	0.0223	0.00343
v_j^-	0.0654	0.1044	0.05644	0.00766	0.08575	0.01635	0.00437

Step 6.3: Calculation of rank.

$$p_i = \frac{s_{ij}^-}{s_{ij}^+ + s_{ij}^-}$$

Table 8: Calculation of rank by relative closeness

	s_j^+	s_j^-	$s_{ij}^+ + s_{ij}^-$	p	Rank
M_1	0.0316	0.0400	0.0716	0.5587	3
M_2	0.0245	0.0843	0.1088	0. 3402	6
M_3	0.0400	0.0374	0.0774	0.4832	4
M_4	0.0249	0.0374	0.0623	0.6003	2
M_5	0.0671	0.0346	0.1017	0.7748	1
M_6	0.0500	0.0271	0.0771	0.3515	5

Step 7: Calculation of rank and discussion.

4. Result Discussion

Firstly, the generalized neutrosophic TOPSIS approach is used to simplify mobile selection MCDM problem. In this calculation, the ranking of each mobile with respect to each criterion is represented below in Table 8 and Figure 2. To test the validity and the implementation of the technique proposed by Saqlain *et. al.* [17], in neutrosophic soft set environment and multi-criteria decision making, mobile selection problem is considered. Result shows that generalized neutrosophic TOPSIS along with proposed algorithm can be used to find best alternative.

Secondly, results are compared with [10], in which fuzzy multi-criteria group decision making approach was used by considering same alternative and attributes. Graphical and tabular comparison is presented in Table 8 and Figure 2, which shows that under Generalized TOPSIS and Fuzzy TOPSIS M_5 and M_5 are best alternative whereas, M_2 and M_3 is the worst selection respectively.

If we compare the results of Generalized fuzzy TOPSIS and Fuzzy TOPSIS M_1, M_4, M_5 has same raking whereas, M_2, M_3, M_6.

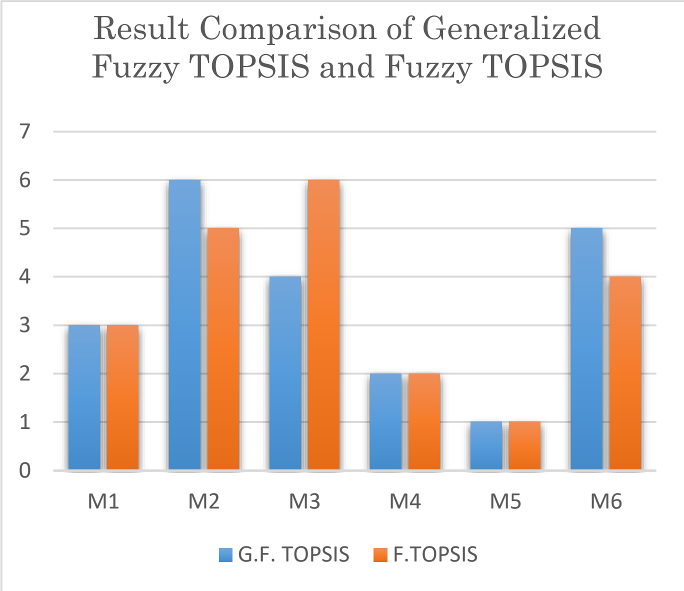

Figure 2: Ranking comparison of alternatives

Table 9: Ranking comparison of alternatives using G.F. TOPSIS and F. TOPSIS

Strategy	Generalized Fuzzy TOPSIS-Result Ranking	Fuzzy TOPSIS Ranking
M_1	3	3
M_2	6	5
M_3	4	6
M_4	2	2
M_5	1	1
M_6	5	4

5. Conclusions

In MCDM problems, TOPSIS is widely used to find the best alternative, whereas, due to the vague and imprecise information in fuzzy environment, ranking of alternatives may not be accurate. Thus, neutrosophic soft set environment plays a vital role in selection problem. In this article, firstly, an algorithm is proposed based on accuracy function under neutrosophic soft set environment and to check the validity of the proposed technique in this environment, mobile selection problem is considered. Secondly, results are compared with same problem under FMCGDM [10] environment. However, the article may open a new avenue of research in competitive Neutrosophic decision-making arena. Thus, this proposed technique can be used in decision-makings such as supplier selection, personal selection in academia and many other areas of management system.

Conflicts of Interest

The authors declare no conflict of interest.

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Single and Multi-valued Neutrosophic Hypersoft set and Tangent Similarity Measure of Single valued Neutrosophic Hypersoft Sets

Muhammad Saqlain [1], Naveed Jafar[2] , Sana Moin[2] , Muhammad Saeed[3] and Said Broumi[4]

[1] Lahore Garrison University, DHA Phase-VI, Sector C, Lahore, 54000, Pakistan. E-mail: msaqlain@lgu.edu.pk
[2] Lahore Garrison University, DHA Phase-VI, Sector C, Lahore, 54000, Pakistan. E-mail: naveedjafar@lgu.edu.pk
[2] Lahore Garrison University, DHA Phase-VI, Sector C, Lahore, 54000, Pakistan. E-mail: moinsana64@gmail.com
[3] University of Management and Technology, Township, Lahore, 54000, Pakistan. E-mail: Muhammad.saeed@umt.edu.pk
[4] Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco. E-mail: broumisaid78@gmail.com

Abstract: In this paper, we present a single-valued Neutrosophic Hypersoft set, multi-valued Neutrosophic Hypersoft set and tangent similarity measure for single-valued neutrosophic hypersoft sets and its properties. Then we use this technique in an application namely selection of cricket players for different types of matches (ODI, T20, and test) based on Neutrosophic Hypersoft set in decision making of single-valued neutrosophic hypersoft sets. This technique will help us to decide the best option for the players.

Keywords: Neutrosophic hypersoft set (NHSS), single-valued neutrosophic hypersoft set (SVNHSS), multi-valued Neutrosophic Hypersoft set (MVNHSS), tangent similarity measure (TSM), multiple attribute decision making, cricket player

1. Introduction

As the analysis of classical sets, fuzzy set [1] and intuitionistic fuzzy set [2], the neutrosophic set was introduced by Smarandache [3, 4] to capture the insufficient, indicate, uncertain and conflicting information. The neutrosophic set has three free parts, which are truth, indeterminacy and falsity membership degree; subsequently, it is applied in a wide range, for example, basic decision-making problems [5-20].

By accomplishing that the neutrosophic sets are difficult to be applied in some genuine issues on account of truth, indeterminacy and falsity membership degree, Wang, Smarandache, Zhang, and Sunderraman [21] presented the idea of a single-valued neutrosophic set. The single-valued neutrosophic set can freely express truth-membership degree, indeterminacy-membership degree, and falsity-membership degree and manages inadequate, uncertain and conflicting data. All the aspects of the elements depicted by the single-valued neutrosophic set are entirely appropriate for human intuition because of the flaw of information that human gets or sees from the surrounding. The single-valued neutrosophic set has been growing quickly because of its wide scope of hypothetical distinction and application zones, as discussed in [22-30].

The idea of similarity is significant in examining approximately every logical field. Literature audit indicates that numerous strategies have been proposed for estimating the degree of similarity between fuzzy sets has been examined by Chen [32], Chen, et al., [33], Hyung et al. [34], Pappis and Karacapilidis [35] and Wang [36]. It is also a powerful instrument in building multi-criteria decision-making techniques in numerous regions, for example, therapeutic diagnosis, design acknowledgment, grouping investigation, decision making, etc. But these strategies are not fit for managing the similarity measures including indeterminacy. In the literature, few investigations have studied to similarity measures for neutrosophic sets and single-valued neutrosophic sets [37-46].

Ye [47] present the distance-based similarity measure of single-valued neutrosophic sets and applied it to the group decision-making problems with single-valued neutrosophic data. Broumi and Smarandache [48] invent another similarity measure known as cosine similarity measure of interval-valued neutrosophic sets. Ye [49] further considered and found that there exist a few flaws in existing cosine similarity measure characterized in vector space [50] in certain circumstances. He [49] referenced that they may deliver an unreasonable outcome in some real cases. To conquer these problems, Ye [49] proposed improved cosine similarity measure dependent on cosine function, including single-valued neutrosophic cosine similarity measures and interval neutrosophic cosine similarity measures.

Working on the similarity measures Pramanik and Mondal [51] also present a cotangent similarity measure of rough neutrosophic sets and their application to the medical field. Pramanik and Mondal [52] also give tangent similarity measures between intuitionistic fuzzy sets and some of its properties and applications.

Smarandache [53] presented a new technique to deal with uncertainty. He generalized the soft set to hypersoft set by converting the function into a multi-decision function. In the same way, we convert hypersoft set to neutrosophic Hypersoft set to overcome the uncertainty problems. [54] introduced the TOPSIS by using accuracy function in his work and an application of MCDM is proposed. Application of fuzzy numbers in mobile selection in metros like Lahore is proposed by [55]. In medical the application of fuzzy numbers is proposed by Naveed et.al [56]. TOPSIS technique of MCDM can also be used for the prediction of games, and it's applied in FIFA 2018 by [57]. prediction of games is a very complex topic and this game is also predicted by [58]. Many researches presented theories along with application in neutrosophic environment [59-66].

1.1 Novelties

In this paper, we have continued the idea of intuitionistic tangent similarity measure to neutrosophic class. We have characterized another similarity measure known as Tangent similarity measure for neutrosophic Hypersoft set and its properties with the application.

2.Preliminaries

Definition 2.1: Neutrosophic Soft Set

Let Ů be the universal set and the set for respective attributes is given by Ë. Let P(Ů) be the set of Neutrosophic values of Ů and Ǻ ⊆ Ë. A pair (F, Ǻ) is called a Neutrosophic soft set over Ů and its mapping is given as

$$₣: Ǻ \rightarrow P(Ů)$$

Definition 2.2: Hyper Soft Set

Let Ů be the universal set and $P(Ů)$ be the power set of Ů. Consider $p^1, p^2, p^3 \dots p^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $P^1, P^2, P^3 \dots P^n$ with $P^i \cap P^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$, then the pair (₣, $P^1 \times P^2 \times P^3 \dots P^n$) is said to be Hypersoft set over Ů where

₣: $P^1 \times P^2 \times P^3 \dots P^n \rightarrow P(Ů)$

Definition 2.3: Neutrosophic Hypersoft Set

Let Ů be the universal set and $P(Ů)$ be the power set of Ů. Consider $p^1, p^2, p^3 \dots p^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $P^1, P^2, P^3 \dots P^n$ with $P^i \cap P^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $P^1 \times P^2 \times P^3 \dots P^n = ß$, then the pair (₣, ß) is said to be Neutrosophic Hypersoft set (NHSS) over Ů where

₣: $P^1 \times P^2 \times P^3 \dots P^n \rightarrow P(Ů)$ and

₣$(P^1 \times P^2 \times P^3 \dots P^n) = \{< x, T(₣(ß)), I(₣(ß)), F(₣(ß)) >, x \in Ů\}$ where T is the membership value of truthiness, I is the membership value of indeterminacy and F is the membership value of falsity such that T, I, F: Ů $\rightarrow [0,1]$ also $0 \leq T(₣(ß)) + I(₣(ß)) + F(₣(ß)) \leq 3$.

3. Calculations

Definition 3.1: Single valued Neutrosophic Hypersoft Set

Let Ů be the universal set and $P(Ů)$ be the power set of Ů. Consider $p^1, p^2, p^3 \dots p^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $P^1, P^2, P^3 \dots P^n$ with $P^i \cap P^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $P^1 \times P^2 \times P^3 \dots P^n = ß$, then the pair (₣, ß) is said to be Single valued Neutrosophic Hypersoft set (SVNHSS) over Ů where

₣: $P^1 \times P^2 \times P^3 \dots P^n \rightarrow P(Ů)$ and this mapping to $P(Ů)$ is single-valued.

₣$(P^1 \times P^2 \times P^3 \dots P^n) = \{< x, T(₣(ß)), I(₣(ß)), F(₣(ß)) >, x \in Ů\}$ where T is the membership value of truthiness, I is the membership value of indeterminacy and F is the membership value of falsity such that T, I, F: Ů $\rightarrow [0,1]$ also $0 \leq T(₣(ß)) + I(₣(ß)) + F(₣(ß)) \leq 3$.

Example 3.1:

Let ξ be the set of doctors under consideration given as

$$\xi = \{d^1, d^2, d^3, d^4, d^5\}$$

also consider the set of attributes as

$$l^1 = Qualification, l^2 = Experience, l^3 = Gender, l^4 = Skills$$

And their respective attributes are given as

$L^1 = Qualification$

$= \{MBBS, MS\ diploma, Diploma\ of\ national\ board(DNB), Diploma\ in\ clinical\ research(DCR)\}$

$L^2 = Experience = \{5yr, 8yr, 10yr, 15yr\}$

$L^3 = Gender = \{Male, Female\}$

$L^4 = Skills = \{Compassionate, Problem\ solving, Communicative, leadership\}$

Let the function be ₣: $L^1 \times L^2 \times L^3 \times L^4 \rightarrow P(\xi)$

Below are the tables of their Neutrosophic values from different decision makers

Table 1: Decision maker Neutrosophic values for Qualification

$L^1(Qualification)$	d^1	d^2	d^3	d^4	d^5
MBBS	(0.4, 0.5, 0.8)	(0.7, 0.6, 0.4)	(0.4, 0.5, 0.7)	(0.5, 0.3, 0.7)	(0.5, 0.3, 0.8)
MS diploma	(0.5, 0.3, 0.6)	(0.3, 0.2, 0.1)	(0.3, 0.6, 0.2)	(0.7, 0.3, 0.6)	(0.5, 0.4, 0.5)
DNB	(0.8, 0.2, 0.4)	(0.9, 0.5, 0.3)	(0.9, 0.4, 0.1)	(0.6, 0.3, 0.2)	(0.6, 0.1, 0.2)
DCR	(0.9, 0.3, 0.1)	(0.5, 0.2, 0.1)	(0.8, 0.5, 0.2)	(0.8, 0.2, 0.1)	(0.7, 0.4, 0.2)

Table 2: Decision maker Neutrosophic values for Experience

$L^2(Experience)$	d^1	d^2	d^3	d^4	d^5
5 yr.	(0.3, 0.4, 0.7)	(0.6, 0.5, 0.3)	(0.5, 0.6, 0.8)	(0.6, 0.4, 0.8)	(0.3, 0.6, 0.7)
8 yr.	(0.4, 0.2, 0.5)	(0.8, 0.1, 0.2)	(0.4, 0.7, 0.3)	(0.4, 0.8, 0.7)	(0.7, 0.5, 0.6)
10 yr.	(0.7, 0.2, 0.3)	(0.9, 0.3, 0.1)	(0.8, 0.3, 0.2)	(0.5, 0.4, 0.3)	(0.5, 0.2, 0.1)
15 yr.	(0.8, 0.2, 0.1)	(0.6, 0.4, 0.3)	(0.9 0.4, 0.1)	(0.6, 0.2, 0.3)	(0.5, 0.3, 0.2)

Table 3: Decision maker Neutrosophic values for Gender

$L^3(Gender)$	d^1	d^2	d^3	d^4	d^5
Male	(0.5, 0.6, 0.9)	(0.7, 0.8, 0.3)	(0.6, 0.4, 0.3)	(0.8, 0.5, 0.4)	(0.9, 0.2, 0.1)
Female	(0.6, 0.4, 0.7)	(0.3, 0.6, 0.4)	(0.8, 0.2, 0.1)	(0.4, 0.5, 0.6)	(0.8, 0.4, 0.2)

Table 4: Decision maker Neutrosophic values for Skills

$L^4(Skills)$	d^1	d^2	d^3	d^4	d^5
Compassionate	(0.6, 0.4, 0.5)	(0.7, 0.5, 0.3)	(0.6, 0.4, 0.3)	(0.6, 0.2, 0.1)	(0.4, 0.5, 0.3)
Problem solving	(0.8, 0.2, 0.4)	(0.7, 0.3, 0.2)	(0.8, 0.3, 0.1)	(0.3, 0.4, 0.5)	(0.3, 0.5, 0.8)
Communicative	(0.5, 0.3, 0.4)	(0.6, 0.3, 0.4)	(0.5, 0.7, 0.2)	(0.8, 0.4, 0.1)	(0.7, 0.4, 0.3)
Leadership	(0.4, 0.9, 0.6)	(0.8, 0.4, 0.2)	(0.2, 0.6, 0.5)	(0.7, 0.5, 0.2)	(0.6, 0.4, 0.7)

Single valued neutrosophic hypersoft set is define as $₣: (L^1 \times L^2 \times L^3 \times L^4) \rightarrow P(\xi)$

Let's assume $₣(£) = ₣(DNB, 10\ yr, male, compassionate) = \{d^1\}$

Then the single-valued neutrosophic hypersoft set of above-assumed relation is

$₣(£) = ₣(DNB, 10\ yr, male, compassionate) = \{$

$\ll d^1, (DNB\{0.8, 0.2, 0.4\}, 10\ yr\{0.7, 0.2, 0.3\}, male\{0.5, 0.6, 0.9\}, compassionate\{0.6, 0.4, 0.5\}) \gg\}$

Its tabular form is given as

Table 5: Tabular Representation of Single Valued Neutrosophic Hypersoft Set

$₣(£) = ₣(\boldsymbol{DNB}, \boldsymbol{10\ yr}, \boldsymbol{male}, \boldsymbol{compassionate})$	$\boldsymbol{d^1}$
DNB	(0.8, 0.2, 0.4)
10 yr.	(0.7, 0.2, 0.3)
Male	(0.5, 0.6, 0.9)
Compassionate	(0.6, 0.4, 0.5)

Definition 3.2: Multi-valued Neutrosophic Hypersoft Set

Let Ů be the universal set and $P(\mathring{U})$ be the power set of Ů. Consider $p^1, p^2, p^3 \dots p^n$ for $n \geq 1$, be n well-defined attributes, whose corresponding attributive values are respectively the set $P^1, P^2, P^3 \dots P^n$ with $P^i \cap P^j = \emptyset$, for $i \neq j$ and $i, j \epsilon \{1,2,3 \dots n\}$ and their relation $P^1 \times P^2 \times P^3 \dots P^n =$ ß, then the pair (₣, ß) is said to be Single valued Neutrosophic Hypersoft set (SVNHSS) over Ů where

₣: $P^1 \times P^2 \times P^3 \dots P^n \rightarrow P(\mathring{U})$ and this mapping to $P(\mathring{U})$ is multi-valued.

₣$(P^1 \times P^2 \times P^3 \dots P^n) = \{< x, T($₣(ß)$), I($₣(ß)$), F($₣(ß)$) >, x \in \mathring{U}\}$ where T is the membership value of truthiness, I is the membership value of indeterminacy and F is the membership value of falsity such that $T, I, F: \mathring{U} \rightarrow [0,1]$ also $0 \leq T($₣(ß)$) + I($₣(ß)$) + F($₣(ß)$) \leq 3$.

Example 3.2:

Let ξ be the set of doctors under consideration given as $\xi = \{d^1, d^2, d^3, d^4, d^5\}$

also consider the set of attributes as

$$l^1 = Qualification, l^2 = Experience, l^3 = Gender, l^4 = Skills$$

And their respective attributes are given as

$L^1 = Qualification$

$= \{MBBS, MS\ diploma, Diploma\ of\ national\ board(DNB), Diploma\ in\ clinical\ research(DCR)\}$

$L^2 = Experience = \{5yr, 8yr, 10yr, 15yr\}$

$L^3 = Gender = \{Male, Female\}$

$L^4 = Skills = \{Compassionate, Problem\ solving, Communicative, leadership\}$

Let the function be ₣: $L^1 \times L^2 \times L^3 \times L^4 \rightarrow P(\xi)$

Below are the tables of their Neutrosophic values from different decision makers

Table 6: Decision maker Neutrosophic values for Qualification

$L^1(Qualification)$	d^1	d^2	d^3	d^4	d^5
MBBS	(0.4, 0.5, 0.8)	(0.7, 0.6, 0.4)	(0.4, 0.5, 0.7)	(0.5, 0.3, 0.7)	(0.5, 0.3, 0.8)
MS diploma	(0.5, 0.3, 0.6)	(0.3, 0.2, 0.1)	(0.3, 0.6, 0.2)	(0.7, 0.3, 0.6)	(0.5, 0.4, 0.5)
DNB	(0.8, 0.2, 0.4)	(0.9, 0.5, 0.3)	(0.9, 0.4, 0.1)	(0.6, 0.3, 0.2)	(0.6, 0.1, 0.2)
DCR	(0.9, 0.3, 0.1)	(0.5, 0.2, 0.1)	(0.8, 0.5, 0.2)	(0.8, 0.2, 0.1)	(0.7, 0.4, 0.2)

Table 7: Decision maker Neutrosophic values for Experience

$L^2(Experience)$	d^1	d^2	d^3	d^4	d^5
5 yr.	(0.3, 0.4, 0.7)	(0.6, 0.5, 0.3)	(0.5, 0.6, 0.8)	(0.6, 0.4, 0.8)	(0.3, 0.6, 0.7)
8 yr.	(0.4, 0.2, 0.5)	(0.8, 0.1, 0.2)	(0.4, 0.7, 0.3)	(0.4, 0.8, 0.7)	(0.7, 0.5, 0.6)
10 yr.	(0.7, 0.2, 0.3)	(0.9, 0.3, 0.1)	(0.8, 0.3, 0.2)	(0.5, 0.4, 0.3)	(0.5, 0.2, 0.1)
15 yr.	(0.8, 0.2, 0.1)	(0.6, 0.4, 0.3)	(0.9 0.4, 0.1)	(0.6, 0.2, 0.3)	(0.5, 0.3, 0.2)

Table 8: Decision maker Neutrosophic values for Gender

$L^3(Gender)$	d^1	d^2	d^3	d^4	d^5
Male	(0.5, 0.6, 0.9)	(0.7, 0.8, 0.3)	(0.6, 0.4, 0.3)	(0.8, 0.5, 0.4)	(0.9, 0.2, 0.1)
Female	(0.6, 0.4, 0.7)	(0.3, 0.6, 0.4)	(0.8, 0.2, 0.1)	(0.4, 0.5, 0.6)	(0.8, 0.4, 0.2)

Table 9: Decision maker Neutrosophic values for Skills

$L^4(Skills)$	d^1	d^2	d^3	d^4	d^5
Compassionate	(0.6, 0.4, 0.5)	(0.7, 0.5, 0.3)	(0.6, 0.4, 0.3)	(0.6, 0.2, 0.1)	(0.4, 0.5, 0.3)
Problem solving	(0.8, 0.2, 0.4)	(0.7, 0.3, 0.2)	(0.8, 0.3, 0.1)	(0.3, 0.4, 0.5)	(0.3, 0.5, 0.8)
Communicative	(0.5, 0.3, 0.4)	(0.6, 0.3, 0.4)	(0.5, 0.7, 0.2)	(0.8, 0.4, 0.1)	(0.7, 0.4, 0.3)
Leadership	(0.4, 0.9, 0.6)	(0.8, 0.4, 0.2)	(0.2, 0.6, 0.5)	(0.7, 0.5, 0.2)	(0.6, 0.4, 0.7)

Multi-valued neutrosophic hyper soft set is define as

$$₣: (L^1 \times L^2 \times L^3 \times L^4) \to P(\xi)$$

Let's assume $₣(£) = ₣(DNB, 10\ yr, male, compassionate) = \{d^1, d^4\}$

Then multi-valued neutrosophic hyper soft set of above assumed relation is

$₣(£) = ₣(DNB, 10\ yr, male, compassionate) = \{$

$\ll d^1, (DNB\{0.8, 0.2, 0.4\}, 10\ yr\{0.7, 0.2, 0.3\}, male\{0.5, 0.6, 0.9\}, compassionate\{0.6, 0.4, 0.5\}) \gg,$

$\ll d^4(DNB\{0.6, 0.3, 0.2\}, 10\ yr\{0.5, 0.4, 0.3\}, male\{0.8, 0.5, 0.4\}, compassionate\{0.6, 0.2, 0.1\}) \gg\}$

Its tabular form is given as

Table 10: Tabular Representation of Multi-valued Neutrosophic Hypersoft Set

$₣(£) = ₣(DNB, 10\ yr, male, compassionate)$	d^1	d^4
DNB	(0.8, 0.2, 0.4)	(0.6, 0.3, 0.2)
10 yr.	(0.7, 0.2, 0.3)	(0.5, 0.4, 0.3)
Male	(0.5, 0.6, 0.9)	(0.8, 0.5, 0.4)
Compassionate	(0.6, 0.4, 0.5)	(0.6, 0.2, 0.1)

3.3: Tangent similarity measures for single valued neutrosophic hypersoft set

Let $Ŕ =< x, T^{Ŕ}(₣(ß)), I^{Ŕ}(₣(ß)), F^{Ŕ}(₣(ß)) >$ and $Ś =< x, T^{Ś}(₣(ß)), I^{Ś}(₣(ß)), F^{Ś}(₣(ß)) >$ be two single valued neutrosophic hypersoft set(SVNHSS) for ₣(ß). Tangent similarity measure for these sets to measure the similarity between them is presented as

$$T_{SVNHSS}(Ŕ, Ś) =< x, \frac{1}{n}\sum_{i=1}^{n}\left[1 - \tan\left(\frac{\pi\left(\left|T^{Ŕ}(₣(ß)_i) - T^{Ś}(₣(ß)_i)\right| + \left|I^{Ŕ}(₣(ß)_i) - I^{Ś}(₣(ß)_i)\right| + \left|F^{Ŕ}(₣(ß)_i) - F^{Ś}(₣(ß)_i)\right|\right)}{12}\right)\right] > \quad, \quad x \in ₣(ß)$$

3.3.1: Proposition

Tangent similarity measure between two single valued Neutrosophic hypersoft set $T_{SVNHSS}(Ŕ, Ś)$ satisfies the following properties.

1. $0 \leq T_{SVNHSS}(Ŕ, Ś) \leq 1$
2. $T_{SVNHSS}(Ŕ, Ś) = 1\ if\ and\ only\ if\ Ŕ = Ś$
3. $T_{SVNHSS}(Ŕ, Ś) = T_{SVNHSS}(Ś, Ŕ)$
4. If Ő is a SVNHSS and $Ŕ \subset Ś \subset Ő$ then $T_{SVNHSS}(Ŕ, Ő) \leq T_{SVNHSS}(Ŕ, Ś)$ and $T_{SVNHSS}(Ŕ, Ő) \leq T_{SVNHSS}(Ś, Ő)$.

It is easy to see that the define similarity measure satisfies the above properties easily so the proofs are left for the reader.

3.4: Decision making using single-valued neutrosophic hypersoft set based on the tangent similarity measure

Let $L^1, L^2, L^3 \dots L^n$ be the distinct set of participants, $M^1, M^2, M^3 \dots M^n$ by the set of norms for participants and $N^1, N^2, N^3 \dots N^n$ be the set of options for each participant. By using a decision-making technique, the decision-makers add ranking of options concerning each participant. This ranking gives the effectiveness of participants L against the norms of participants M then theses values associated with the options for multiple attribute decision making. Algorithm of this procedure are given below

3.4.1: Algorithm

Step 1: Determine the association between participants and the norms.

The association between participants and the norms is given by the below decision matrix in terms of single-valued Neutrosophic hyper soft sets.

Table 21: Association between participants and the norms in term of SVNHSS

	M^1	M^2	...	M^n
L^1	$\langle T_{11}, I_{11}, F_{11} \rangle$	$\langle T_{12}, I_{12}, F_{12} \rangle$	...	$\langle T_{1n}, I_{1n}, F_{1n} \rangle$
L^2	$\langle T_{21}, I_{21}, F_{21} \rangle$	$\langle T_{22}, I_{22}, F_{22} \rangle$	...	$\langle T_{2n}, I_{2n}, F_{2n} \rangle$
...	...	...	...	...
L^m	$\langle T_{m1}, I_{m1}, F_{m1} \rangle$	$\langle T_{m2}, I_{m2}, F_{m2} \rangle$	...	$\langle T_{mn}, I_{mn}, F_{mn} \rangle$

Step 2: Determine the association between norms and options.

The association between the norms and the options is given by the below decision matrix in terms of single-valued Neutrosophic hypersoft sets.

Table 22: Association between the norms and the options in term of SVNHSS

	N^1	N^2	...	N^k
M^1	$\langle T_{11}, I_{11}, F_{11} \rangle$	$\langle T_{12}, I_{12}, F_{12} \rangle$	...	$\langle T_{1k}, I_{1k}, F_{1k} \rangle$
M^2	$\langle T_{21}, I_{21}, F_{21} \rangle$	$\langle T_{22}, I_{22}, F_{22} \rangle$	...	$\langle T_{2k}, I_{2k}, F_{2k} \rangle$
...	...	...	...	...
M^n	$\langle T_{n1}, I_{n1}, F_{n1} \rangle$	$\langle T_{n2}, I_{n2}, F_{n2} \rangle$	...	$\langle T_{nk}, I_{nk}, F_{nk} \rangle$

Step 3: Determine the association between participants and options.

The association between participants and the options is determined with the help of tangent similarity measures for single-valued neutrosophic hypersoft numbers.

Step 4: Decision of best option

The best option is decided by arranging the results in the descending orders and choosing the highest value as the highest value represents the best option for the participants.

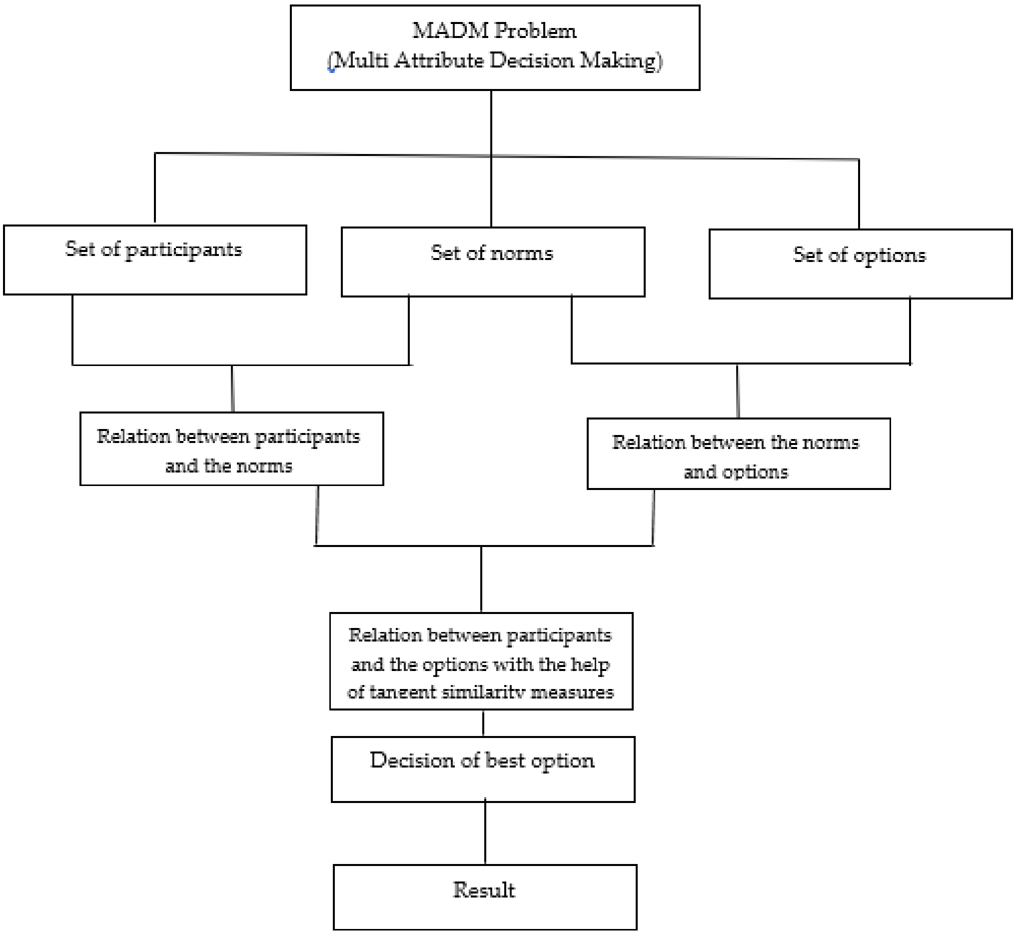

Figure 1: Algorithm design for the proposed technique

4. Example

We have seen a large number of the matches that a team loses because of improper selection of players. we can't choose which player is perfect for which sort of matches like the test, ODI and T20 due to the presence of the huge amount of uncertainties and a large volume of information about the players. With such a piece of vast information, we are unable to focus on every aspect because we may have the cases in which we have the same truth membership, indeterminate membership, and falsity membership values.

To overcome this issue, let us consider an illustrative example by using proposed method for the selection of the players in any type of match which is significant for cricket board as cricket board is the administering body for cricket in the state and the selection of cricket crew is likewise a key duty of cricket board. For this purpose, let us consider two sets, μ, and η. μ be the set of players and η be the set of type of matches played by players i.e.

$\mu = \{ P^1, P^2, P^3, P^4, P^5, P^6, P^7, P^8, P^9, P^{10}, P^{11}, P^{12}, P^{13}\}$ and

$\eta = \{ \text{Test match, ODI match, T20 match}\}$.

ζ be the set of attributes corresponding to μ and η.

$$\zeta^1 = Players\ Strike\ Rate, \zeta^2 = Players\ Average, \zeta^3 = Players\ Economy, \zeta^4 = Players\ attitude,$$
$$\zeta^5 = Players\ Fitness\ \ test$$

And respective attributes for the above-mentioned attributes are given as

$$\varsigma^1 = Players\ Strike\ Rate(PSR) = \{below\ 40\ , 40-60, 60-80, 80-100, 100-150, 150\ above\}$$

$$\varsigma^2 = Players\ Average(PAv) = \{below\ 30, 30-50, 50-70, 70\ above\}$$

$$\varsigma^3 = Players\ Economy(PE) = \{below\ 3, 3-7, 7-13, above\ 13\}$$

$$\varsigma^4 = Players\ attitude(PA) = \{coperative, rude, emotional, moody\}$$

$$\varsigma^5 = Players\ Fitness\ \ test(PFT) = \{passed, not\ passed\}$$

Then Neutrosophic Hypersoft set is given as

$$₣: (\varsigma^1 \times \varsigma^2 \times \varsigma^3 \times \varsigma^4 \times \varsigma^5) \rightarrow P(\mu)$$

And $₣: (\varsigma^1 \times \varsigma^2 \times \varsigma^3 \times \varsigma^4 \times \varsigma^5) \rightarrow P(\eta)$

Let's assume $₣(\alpha) = ₣(100-150, 30-50, above\ 13, cooperative, passed) = \{P^1, P^3, P^6, P^8, P^9\}$ and

$₣(\beta) = ₣(100-150, 30-50, above\ 13, cooperative, passed) = \{\text{Test match, ODI match, T20 match}\}$

Now using the proposed tangent similarity measures for single-valued neutrosophic hypersoft sets, we will decide which player is best for which type of match. For this purpose first we will provide ranking between $\{100-150, 30-50, above\ 13, cooperative, passed\}$ and $\{P^1, P^3, P^6, P^8, P^9\}$ in terms of the single-valued neutrosophic hypersoft sets. In the 2nd step we will provide ranking between $\{100-150, 30-50, above\ 13, cooperative, passed\}$ and $\{\text{Test match, ODI match, T20 match}\}$. In the 3rd step, we will find a correlation between $\{P^1, P^3, P^6, P^8, P^9\}$ and $\{\text{Test match, ODI match, T20 match}\}$ using T_{SVNHSS}. In the last step, we will decide by arranging the results in the descending order and selecting the highest value.

Step 1: Determine the association between $\{\mathbf{P^1, P^3, P^6, P^8, P^9}\}$ and $\{\mathbf{100-150, 30-50, above\ 13, coperative, passed}\}$.

The association between $\{100-150, 30-50, above\ 13, cooperative, passed\}$ and $\{P^1, P^3, P^6, P^8, P^9\}$ is given by the below decision matrix in terms of single-valued Neutrosophic hypersoft sets.

Table 13: Association between $\{P^1, P^3, P^6, P^8, P^9\}$ and $\{100-150, 30-50, above\ 13, coperative, passed\}$ in term of SVNHSS

	$\mathbf{100-150(PSR)}$	$\mathbf{30-50(PAv)}$	$\boldsymbol{Above\ 13(PE)}$	$\boldsymbol{Cooperative\ (PA)}$	$\boldsymbol{Passed\ (PFT)}$
P^1	(0.7,0.3,0,2)	(0.4, 0.5, 0.7)	(0.5, 0.3, 0.8)	(0.7, 0.6, 0.4)	(0.5, 0.3, 0.7)
P^3	(0.5,0.4,0.7)	(0.3, 0.6, 0.2)	(0.5, 0.4, 0.5)	(0.3, 0.2, 0.1)	(0.7, 0.3, 0.6)
P^6	(0.8,0.2,0.1)	(0.9, 0.4, 0.1)	(0.6, 0.1, 0.2)	(0.9, 0.5, 0.3)	(0.6, 0.3, 0.2)
P^8	(0.9,0.1,0.3)	(0.8, 0.5, 0.2)	(0.7, 0.4, 0.2)	(0.5, 0.2, 0.1)	(0.8, 0.2, 0.1)
P^9	(0.6,0.3,0.3)	(0.5, 0.4, 0.3)	(0.8, 0.3, 0.2)	(0.9, 0.2, 0.1)	(0.4, 0.5, 0.7)

Step 2: Determine the association between $\{\textbf{Test match, ODI match, T20 match}\}$ and $\{\mathbf{100-150, 30-50,}\ \boldsymbol{above}\ \mathbf{13,}\ \boldsymbol{coperative, passed}\}$.

The association between $\{100-150, 30-50, above\ 13, cooperative, passed\}$ and $\{\text{Test match, ODI match, T20 match}\}$ is given by the below decision matrix in terms of single-valued Neutrosophic hypersoft sets.

Table 14: Association between $\{100-150, 30-50, above\ 13, coperative, passed\}$ and $\{Test\ match, ODI\ match, T20\ match\}$ in term of SVNHSS

	$Test\ match$	$ODI\ match$	$T20\ match$
$100-150(PSR)$	(0.7, 0.5, 0.3)	(0.6, 0.4, 0.3)	(0.4, 0.5, 0.3)
$30-50(PAv)$	(0.7, 0.3, 0.2)	(0.8, 0.3, 0.1)	(0.3, 0.5, 0.8)
$Above\ 13(PE)$	(0.6, 0.3, 0.4)	(0.5, 0.7, 0.2)	(0.7, 0.4, 0.3)
$Cooperative\ (PA)$	(0.5, 0.4, 0.5)	(0.9, 0.2, 0.1)	(0.5, 0.2, 0.1)
$Passed\ (PFT)$	(0.6, 0.4, 0.7)	(0.3, 0.6, 0.4)	(0.8, 0.2, 0.1)

Step 3: Determine the association between $\{\textbf{Test match}, \textbf{ODI match}, \textbf{T20 match}\}$ and $\{\boldsymbol{P^1}, \boldsymbol{P^3}, \boldsymbol{P^6}, \boldsymbol{P^8}, \boldsymbol{P^9}\}$.

The association between $\{P^1, P^3, P^6, P^8, P^9\}$ and $\{\text{Test match}, \text{ODI match}, \text{T20 match}\}$ is determined with the help of tangent similarity measures for single-valued neutrosophic hypersoft numbers.

Table 14: Association between $\{P^1, P^3, P^6, P^8, P^9\}$ and $\{Test\ match, ODI\ match, T20\ match\}$ using tangent similarity measure for SVNHSS

	$Test\ match$	$ODI\ match$	$T20\ match$
P^1	**0.8728**	0.7752	0.8137
P^3	0.8513	0.8143	**0.8627**
P^6	**0.8786**	0.8519	0.7798
P^8	0.8463	0.8402	**0.8875**
P^9	0.8729	**0.8997**	0.8289

Step 4: Decision of best option

The best option is decided by choosing the highest value as the highest value represents the best match type for the players. The table shows that player P^1 should be selected for a test match, player P^3 should be selected for the T20 match, player P^6 should be selected for a test match, player P^8 should be selected for T20 match and player P^9 should be selected for ODI match.

5. Conclusions

Decision-making is a complex issue due to vague, imprecise and indeterminate environment specially, when attributes are more than one, and further bifurcated. Neutrosophic softset environment cannot be used to tackle such type of issues. Therefore, there was a dire need to define a new approach to solve such type of problems.

In this paper, we have proposed a single-valued Neutrosophic hypersoft set and multi-valued neutrosophic hypersoft set, then using a single-valued Neutrosophic hypersoft set we present a tangent similarity measure and some of its properties. We have also presented an application namely selection of cricket team players for any type of match based on multi-attribute decision making using tangent similarity measure. The concept of this paper is to make our decision more precise.

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Conflicts of Interest

The authors declare that they have no conflict of interest.

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On Optimizing Neutrosophic Complex Programming Using Lexicographic Order

Hamiden Abd El- Wahed Khalifa[1], Pavan Kumar[2,*] and Florentin Smarandache[3]

1 Operations Research Department, Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt, Email: hamidenkhalifa@cu.edu.eg

1 Present Address: Mathematics Department, College of Science and Arts, Al- Badaya, Qassim University, Qassim, Saudi Arabia, Email: Ha.Ahmed@qu.edu.sa

2 Mathematics Department, Koneru Lakshmaiah Education Foundation, Vaddeswaram, AP, INDIA-522502, Email: pavankmaths@kluniversity.in

3 Mathematics Department, University of New Mexico, 705 Gurley Ave., Gallup, NM 87301, USA, Email: fsmarandache@gmail.com

Abstract: Neutrosophic sets are a generalization of the crisp set, fuzzy set, and intuitionistic fuzzy set for representing the uncertainty, inconsistency, and incomplete knowledge about the real world problems. This paper aims to characterize the solution of complex programming (CP) problem with imprecise data instead of its prices information. The neutrosophic complex programming (NCP) problem is considered by incorporating single valued trapezoidal neutrosophic numbers in all the parameters of objective function and constraints. The score function corresponding to the neutrosophic number is used to transform the problem into the corresponding crisp CP. Here, Lexicographic order is applied for the comparison between any two complex numbers. The comparison is developed between the real and imaginary parts separately. Through this manner, the CP problem is divided into two real sub-problems. In the last, a numerical example is solved for the illustration that shows the applicability of the proposed approach. The advantage of this approach is more flexible and makes a real-world situation more realistic.

Keywords: Complex programming; Neutrosophic numbers; Score function; Lexicographic order; Lingo software; Kuhn- Tucker conditions; Neutrosophic optimal solution

1. Introduction

In many earlier works in complex programming, the researchers considered the real part only of the complex objective function as the objective function. The constraints of the problem are considered as a cone in complex space $\mathbb{C}^n$. Since the concept of complex fuzzy numbers was first introduced [17], many researchers studied the problems of the concept of fuzzy complex numbers. This branch subject will be widely applied in fuzzy system theory, especially in fuzzy mathematical programming, and in complex programming too.

Complex programming problem was studied first by Levinson who studied the linear programming (LP) in complex space [39]. The duality theorem has extended to the quadratic complex programming by an adaption of the technique, which introduced by Dorn [27, 22]. The linear fractional programming in complex space has proposed [45]. Linear and nonlinear complex programming problems were treated by numerous authors [24, 33- 37, 41]. In applications, many practical problems related to complex variables, for instance, electrical engineering, filter theory, statistical signal processing, etc., were studied.

Some more general minimax fractional programming problem with complex variables was proposed with the establishment of the necessary and sufficient optimality conditions [36, 37]. A certain kind of linear programming with fuzzy complex numbers in the objective function coefficients also considered as complex fuzzy numbers [52]. The hyper complex neutrosophic similarity measure was proposed by numerous authors [29]. Also, they discussed its application in multicriteria decision making problem. There was proposed an interval neutrosophic multiple attribute decision-making method with credibility information [50]. Later, the multiple attribute group decision making based on interval neutrosophic uncertain linguistic variables was studied [51].

An extended TOPSIS for multi-attribute decision making problems with neutrosophic cubic information was proposed [42]. A single valued neutrosophic hesitant fuzzy computational algorithm was developed for multiple objective nonlinear optimization problem [9]. A computational algorithm was developed for the neutrosophic optimization model with an application to determine the optimal shale gas water management under uncertainty [10]. The interval complex neutrosophic set was studied by the formulation and applications in decision-making [11]. A group decision-making method was proposed under hesitant interval neutrosophic uncertain linguistic environment [40]. The neutrosophic complex topological spaces was studied, and introduced the concept of neutrosophic complex $\alpha\psi$ connectedness in neutrosophic complex topological spaces [30].

A computational algorithm based on the single-valued neutrosophic hesitant fuzzy was developed for multiple objective nonlinear optimization problems [9]. A neutrosophic optimization model was formulated and presented a computational algorithm for optimal shale gas water management under uncertainty [10]. A multiple objective programming approach was proposed to solve integer valued neutrosophic shortest path problems [32]. Some linguistic approaches were developed to study the interval complex neutrosophic sets in decision making applications [39].

Neutrosophic sets were studied to search some applications in the area of transportations and logistics. A multi-objective transportation model was studied under neutrosophic environment [43]. The multi-criteria decision making based on generalized prioritized aggregation operators was presented under simplified neutrosophic uncertain linguistic environment [46]. Some dynamic interval valued neutrosophic set were proposed by modeling decision making in dynamic environments [48]. A hybrid plithogenic decision-making approach was proposed with quality function deployment for selecting supply chain sustainability metrics [1]. Some applications of neutrosophic theory were studied to solve transition difficulties of IT-based enterprises [2].

Based on plithogenic sets, a novel model for the evaluation of hospital medical care systems was presented [3]. Some decision making applications of soft computing and IoT were proposed for a novel intelligent medical decision support model [4]. A novel neutrosophic approach was applied to evaluate the green supply chain management practices [5]. Numerous researchers studied the under type-2 neutrosophic numbers. An application of under type-2 neutrosophic number was presented for developing supplier selection with group decision making by using TOPSIS [6]. An application of hybrid neutrosophic multiple criteria group decision making approach for project selection was presented [7]. The Resource levelling problem was studied in construction projects under neutrosophic environment [8].

The N-valued interval neutrosophic sets with their applications in the field of medical diagnosis was presented [16]. Based on the pentagonal neutrosophic numbers, the de-neutrosophication technique was proposed with some applications in determining the minimal spanning tree [18]. The pentagonal fuzzy numbers were studied with their different representations, properties, ranking, defuzzification. The concept of pentagonal fuzzy neutrosophic numbers was proposed with some applications in game and transportation models [19- 20]. Various forms of linear as well as non-linear form of trapezoidal neutrosophic numbers, de-neutrosophication techniques were studied. Their application were also presented in time cost optimization technique and sequencing problems [21]. The parametric divergence measure of neutrosophic sets was studied with its application in decision-making situations [25]. A technique for reducing dimensionality of data in decision-making utilizing neutrosophic soft matrices was proposed [26].

In this paper, we aim to characterize the solution of complex programming (NCP) neutrosophic numbers. The score function corresponding to the neutrosophic number is used to convert the problem into the corresponding crisp CP, and hence lexicographic order used for comparing between any two complex numbers. The comparison developed between the real and imaginary parts separately. Through this manner, the CP problem is divided into two real sub-problems.

The outlay of the paper is organized as follows: In section 2; some preliminaries are presented. In section 3, a NCP problem is formulated. Section 4 characterizes a solution to the NCP problem to obtain neutrosophic optimal solution. In section 5, two numerical examples are given for illustration. Finally some concluding remarks are reported in section6.

2. Preliminaries

In order to discuss our problem conveniently, basic concepts and results related to fuzzy numbers, trapezoidal fuzzy numbers, intuitionistic trapezoidal fuzzy numbers, neutrosophic set, and complex mathematical programming are recalled.

Definition 1. (Trapezoidal fuzzy numbers, Kaur and Kumar [31]). A fuzzy number $\tilde{A} = (r, s, t, u)$ is a trapezoidal fuzzy numbers where $r, s, t, u \in \mathbb{R}$ and its membership function is defined as:

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-r}{s-r}, & r \le x \le s, \\ 1, & s \le x \le t, \\ \frac{u-x}{u-t}, & t \le x \le u, \\ 0, & \text{otherwise}, \end{cases}$$

Definition 2 (Intuitionistic fuzzy set, Atanassov, [12]). A fuzzy set $\tilde{A}$ is said to be an intuitionistic fuzzy set $\tilde{A}^{IN}$ of a non empty set X if $\tilde{A}^{IN} = \{\langle x, \mu_{\tilde{B}^{IN}}, \rho_{\tilde{B}^{IN}} \rangle : x \in X\}$, where $\mu_{\tilde{A}^{IN}}$, and $\rho_{\tilde{B}^{IN}}$ are membership and nonmembership functions such that $\mu_{\tilde{A}^{IN}}$, $\rho_{\tilde{A}^{IN}}: X \to [0, 1]$ and $0 \le \mu_{\tilde{A}^{IN}} + \rho_{\tilde{A}^{IN}} \le 1$, for all $x \in X$.

Definition 3 (Intuitionistic fuzzy number, Atanassov, [13]). An intuitionistic fuzzy set $\widetilde{A}^{IN}$ of $\mathbb{R}$ is called an Intuitionistic fuzzy number if the following conditions hold:

1. There exists $c \in \mathbb{R}: \mu_{\widetilde{A}^{IN}}(c) = 1$, and $\rho_{\widetilde{A}^{IN}}(c) = 0$.
2. $\mu_{\widetilde{A}^{IN}}: \mathbb{R} \to [0,1]$ is continuous function such that
$$0 \le \mu_{\widetilde{A}^{IN}} + \rho_{\widetilde{B}^{IN}} \le 1, \text{ for all } x \in X.$$

3. The membership and non-membership functions of $\widetilde{B}^{IN}$ are:

$$\mu_{\widetilde{B}^{IN}}(x) = \begin{cases} 0, & -\infty < x < r \\ h(x), & r \le x \le s \\ 1, & x = s \\ l(x), & s \le x \le t \\ 0, & t \le x < \infty, \end{cases}$$

$$\rho_{\widetilde{B}^{IN}}(x) = \begin{cases} 0, & -\infty < x < a \\ f(x), & a \le x \le s \\ 1, & x = s \\ g(x), & s \le x \le b \\ 0, & b \le x < \infty, \end{cases}$$

Where $f, g, h, l: \mathbb{R} \to [0,1]$, h and g are strictly increasing functions, l and f are strictly decreasing functions with the conditions $0 \le f(x) + f(x) \le 1$, and $0 \le l(x) + g(x) \le 1$.

Definition 4 (Trapezoidal intuitionistic fuzzy number, Jianqiang and Zhong, [28]).

A trapezoidal intuitionistic fuzzy number is denoted by $\widetilde{B}^{IN} = (r, s, t, u), (a, s, t, b)$, where $a \le r \le s \le t \le u \le b$ with membership and nonmembership functions are defined as:

$$\mu_{\widetilde{B}^{INT}}(x) = \begin{cases} \frac{x-r}{s-r}, & r \le x < s, \\ 1, & s \le x \le t, \\ \frac{u-x}{u-t}, & t \le x \le u, \\ 0, & \text{otherwise}, \end{cases}$$

$$\rho_{\widetilde{B}^{INT}}(x) = \begin{cases} \frac{s-x}{s-a}, & a \le x < s, \\ 0, & s \le x \le t, \\ \frac{x-t}{b-t}, & t \le x \le b, \\ 1, & \text{otherwise}, \end{cases}$$

Definition 5 (Neutrosophic set, Smarandache, [44]). A neutrosophic set $\overline{B}^{N}$ of non-empty set X is defined as:

$\overline{B}^{N} = \left\{\langle x, I_{\overline{B}^N}(x), J_{\overline{B}^N}(x), V_{\overline{B}^N}(x)\rangle : x \in X, I_{\overline{B}^N}(x), J_{\overline{B}^N}(x), V_{\overline{B}^N}(x) \in \left]0_{-}, 1^{+}\right[\right\}$, where

$I_{\overline{B}^N}(x), J_{\overline{B}^N}(x)$, and $V_{\overline{B}^N}(x)$ are truth membership function, an indeterminacy- membership function, and a falsity- membership function and there is no restriction on the sum of

$I_{\bar{B}^N}(x), J_{\bar{B}^N}(x)$, and $V_{\bar{B}^N}(x)$, so $0^- \leq I_{\bar{B}^N}(x) + J_{\bar{B}^N}(x) + V_{\bar{B}^N}(x) \leq 3^+$, and $]0_-, 1^+[$ is a nonstandard unit interval.

Definition 6 (Single-valued neutrosophic set, Wang et al., [49]). A Single-valued neutrosophic set $\bar{B}^{SVN}$ of a non empty set X is defined as: $\bar{B}^{SVN} = \{\langle x, I_{\bar{B}^N}(x), J_{\bar{B}^N}(x), V_{\bar{B}^N}(x)\rangle : x \in X\}$, where

$I_{\bar{B}^N}(x), J_{\bar{B}^N}(x)$, and $V_{\bar{B}^N}(x) \in [0,1]$ for each $x \in X$ and $0 \leq I_{\bar{B}^N}(x) + J_{\bar{B}^N}(x) + V_{\bar{B}^N}(x) \leq 3.$

Definition 7 (Single-valued neutrosophic number, Thamariselvi and Santhi, [47]). Let $\tau_{\tilde{b}}, \varphi_{\tilde{b}}, \omega_{\tilde{b}} \in [0,1]$ and $r, s, t, u \in \mathbb{R}$ such that $r \leq s \leq t \leq u$. Then a single valued trapezoidal neutrosophic number, $\tilde{b}^N = \langle (r,s,t,u): \tau_{\tilde{b}}, \varphi_{\tilde{b}}, \omega_{\tilde{b}} \rangle$ is a special neutrosophic set on $\mathbb{R}$, whose truth-membership, indeterminacy-membership, and falsity- membership functions are

$$\mu_{\tilde{b}}^N(x) = \begin{cases} \tau_{\tilde{b}^N}\left(\frac{x-r}{s-r}\right), & r \leq x < s \\ \tau_{\tilde{b}}, & s \leq x \leq t \\ \tau_{\tilde{b}^N}\left(\frac{u-x}{u-t}\right), & t \leq x \leq u \\ 0, & \text{otherwise}, \end{cases}$$

$$\rho_{\tilde{b}}^N(x) = \begin{cases} \frac{s-x+\varphi_{\tilde{b}^N}(x-r)}{s-r}, & r \leq x < s \\ \varphi_{\tilde{b}^N}, & s \leq x \leq t \\ \frac{x-t+\varphi_{\tilde{b}^N}(u-x)}{u-t}, & t \leq x \leq u \\ 1, & \text{otherwise}, \end{cases}$$

$$\sigma_{\tilde{b}}^N(x) = \begin{cases} \frac{s-x+\omega_{\tilde{b}^N}(x-r)}{s-r}, & r \leq x < s \\ \omega_{\tilde{b}^N}, & s \leq x \leq t \\ \frac{x-t+\omega_{\tilde{b}^N}(u-x)}{u-t}, & t \leq x \leq u \\ 1, & \text{otherwise}. \end{cases}$$

Where $\tau_{\tilde{b}}, \varphi_{\tilde{b}}$, and $\omega_{\tilde{b}}$ denote the maximum truth, minimum-indeterminacy, and minimum falsity membership degrees, respectively. A single-valued trapezoidal neutrosophic number $\tilde{b}^N = \langle (r,s,t,u): \tau_{\tilde{b}^N}, \varphi_{\tilde{b}^N}, \omega_{\tilde{b}^N} \rangle$ may express in ill- defined quantity about b , which is approximately equal to $[s,t]$.

Definition 8. Let $\tilde{b}^N = \langle (r,s,t,u): \tau_{\tilde{b}^N}, \varphi_{\tilde{b}^N}, \omega_{\tilde{b}^N} \rangle$, and $\tilde{d}^N = \langle (r', s', t', u'): \tau_{\tilde{d}^N}, \varphi_{\tilde{d}^N}, \omega_{\tilde{d}^N} \rangle$ be two single-valued trapezoidal neutrosophic numbers and $v \neq 0$. The arithematic operations on $\tilde{b}^N$, and $\tilde{d}^N$ are

1. $\tilde{b}^N \oplus \tilde{d}^N = \langle (r+r', s+s', t+t', u+u');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle$,
2. $\tilde{b}^N \ominus \tilde{d}^N = \langle (r-u', s-t', t-s', u'-r);\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle$,

3. $$\tilde{b}^N \otimes \tilde{d}^N = \begin{cases} \langle (rr', ss', tt', uu');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u,\ u' > 0 \\ \langle (ru', st', st', ru');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' > 0 \\ \langle (uu', ss', tt', rr');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' < 0, \end{cases}$$

4. $$\tilde{b}^N \oslash \tilde{d}^N = \begin{cases} \langle (r/u', s/t', t/s', u/r');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u,\ u' > 0, \\ \langle (u/u', t/t', s/s', r/r');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' > 0, \\ \langle (u/r', t/s', s/t', r/u');\ \tau_{\tilde{b}^N} \wedge \tau_{\tilde{d}^N}, \varphi_{\tilde{b}^N} \vee \varphi_{\tilde{d}^N}, \omega_{\tilde{b}^N} \vee \omega_{\tilde{d}^N} \rangle, u < 0,\ u' < 0, \end{cases}$$

5. $$k\tilde{d}^N = f(x) = \begin{cases} (kr, ks,\ kt,\ k);\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}},\ \omega_{\tau_{\tilde{d}^N}}, k > 0, \\ (ku,\ kt,\ ks, k\,r);\ \tau_{\tau_{\tilde{d}^N}},\ \varphi_{\tau_{\tilde{d}^N}},\ \omega_{\tau_{\tilde{d}^N}}, k < 0, \end{cases}$$

Definition 9 (Score function of single-valued trapezoidal neutrosophic number, Thamaraiselvi and Santhi [47]). A two single-valued trapezoidal neutrosophic numbers $\tilde{b}$, and $\tilde{d}$ can be compared based on the score function as

Score function $$SC\left(\tilde{b}^N\right) = \left(\frac{1}{16}\right)[r+s+t+u] \times [\mu_{\tilde{b}^N} + (1-\rho_{\tilde{b}^N}(x)) + \left(1-\sigma_{\tilde{b}^N}(x)\right)].$$

Definition 10. (Thamaraiselvi and Santhi, [47]). The order relations between $\tilde{b}^N$ and $\tilde{d}^N$ based on $SC\left(\tilde{b}^N\right)$ are defined as:

1. If $SC\left(\tilde{b}^N\right) < SC\left(\tilde{d}^N\right)$ then $\tilde{b}^N \prec \tilde{d}^N$
2. If $SC\left(\tilde{b}^N\right) = SC\left(\tilde{d}^N\right)$ then $\tilde{b}^N \approx \tilde{d}^N$, and
3. If $SC\left(\tilde{b}^N\right) > SC\left(\tilde{d}^N\right)$ then $\tilde{b}^N \succ \tilde{d}^N$

3. Problem definition and solution concepts

Consider the following single -valued trapezoidal neutrosophic (NCP) problem

(NCP) $\quad \min \ \tilde{F}^N(x) = \tilde{v}^N(x) + i\,\tilde{w}^N(x)$

Subject to (1)

$$x \in \tilde{X}^N = \left\{ x \in \Re^n : \tilde{f}_r^{\ N}(x) = \tilde{p}_r^{\ N}(x) + i\,\tilde{q}_r^{\ N}(x) \le \tilde{l}_r^{\ N} + i\,\tilde{h}_r^{\ N}, \ r = 1, 2, \ldots, m \right\}$$

where

$\tilde{v}^N(x) = \sum_{j=1}^{n} \tilde{c}_j^{\ N} x_j$, $\tilde{w}^N(x) = \sum_{j=1}^{n} \tilde{d}_j^{\ N} x_j$, $\tilde{p}_r^{\ N}(x) = \sum_{j=1}^{n} x_j^{\ T} \tilde{a}_{rj}^{\ N} x_j$, $\tilde{q}_r^{\ N}(x) = \sum_{j=1}^{n} x_j^{\ T} \tilde{e}_{rj}^{\ N} x_j$ are convex functions on $\tilde{X}^N$, $\tilde{c}_j^{\ N}, \tilde{d}_j^{\ N}$, $\tilde{a}_{rj}^{\ N}$, $\tilde{e}_{rj}^{\ N}$ $\tilde{l}_r^{\ N} = \left(\tilde{l}_1^{\ N}, \tilde{l}_2^{\ N}, \ldots, \tilde{l}_m^{\ N}\right)^T$, $\tilde{h}_r^{\ N} = \left(\tilde{h}_1^{\ N}, \tilde{h}_2^{\ N}, \ldots, \tilde{h}_m^{\ N}\right)^T$ are single-valued trapezoidal neutrosophic numbers.

Definition 11. Lexicographic order of two complex numbers $z_1 = a + ib$, and $z_2 = c + id$ is defined as $z_1 \leq z_2 \Leftrightarrow a \leq c$ and $b \leq d$.

Definition 12. A neutrosophic feasible point x° is called single-valued trapezoidal neutrosophic optimal solution to NCP problem if:

$\tilde{v}^N(x^{\circ}) \leq \tilde{v}^N(x)$, and $\tilde{w}^N(x^{\circ}) \leq \tilde{w}^N(x)$ for each $x \in \tilde{X}^N$.

According to the score function in Definition 9, the NCP problem is converted into the following crisp CP problem as

$$(\text{CP}) \quad \text{Min } F(x) = v(x) + i\, w(x)$$

Subject to (2)

$$x \in X = \{x \in \mathbb{R}^n : f_r(x) = p_r(x) + i\, q_r(x) \leq l_r + i\, h_r,\ r = 1, 2, \ldots, m\}$$

4. Characterization of neutrosophic optimal solution for NCP problem

To characterize the neutrosophic optimal solution of NCP problem, let us divide the CP problem into the following two subprobems

$$(P_v) \quad \text{Min } (v)$$

Subject to (3)

$$x \in X = \{x \in \mathbb{R}^n : f_r(x) = p_r(x) + i\, q_r(x) \leq l_r + i\, h_r,\ r = 1, 2, \ldots, m\}, \text{ and}$$

$$(P_w) \quad \text{Min } (w)$$

Subject to (4)

$$x \in X = \{x \in \mathbb{R}^n : f_r(x) = p_r(x) + i\, q_r(x) \leq l_r + i\, h_r,\ r = 1, 2, \ldots, m\}.$$

Definition 13. $x^{\circ} \in X$ is said to be an optimal solution for CP problem if and only if $v(x^{\circ}) \leq v(x)$ and $w(x^{\circ}) \leq w(x)$ for each $x \in X$.

Let us denote S_v and S_w be the set of solution for P_v and P_w respectively, i.e.,

$$S_v = \{x^{\circ} \in X : v(x^{\circ}) \leq v(x);\ \text{for all } x \in X\}, \text{ and} \quad (5)$$

$$S_w = \{x^{*} \in X : v(x^{*}) \leq v(x);\ \text{for all } x \in X\}. \quad (6)$$

Lemma 1. For $S_v \cap S_w \neq \varnothing$, the solution of the CP problem is embedded into $S_v \cap S_w$.

Proof. Assume that $\hat{x}$ be a solution of CP problem this leads to $v(\hat{x}) \leq v(x); \forall x \in X$ i. e., $\hat{x} \in S_v)$. Similarly, $w(\hat{x}) \leq w(x); \forall x \in X$ (i. e., $\hat{x} \in S_w)$ Then, $\hat{x} \in S_v \cap S_w$.

Lemma 2. If S_v and S_w are open, $S_v \cap S_w = \varnothing$, and v, w are strictly convex functions on X then $x^\circ \in S_v$ is a solution of a conjugate function $\bar{F}(x) = v(x) - i\,w(x)$.

Proof. Since $x^\circ \in S_v$, then $v(x^\circ) \leq v(x); \forall x \in X$. Also,

$$v(x^\circ) \leq v(x^*); \forall x^* \in S_v \subset X \tag{7}$$

But $x^* \in S_w$ which means that $w(x^*) \leq v(x^\circ); \forall x^\circ \in S_v \subset X$ and $-i\,w\,(x^*) \geq -i\,w\,(x^\circ)$

i. e.,

$$-i\,w\,(x^\circ) \leq -i\,w\,(x^*) \tag{8}$$

From (7) and (8), we get

$v(x^\circ) - i\,w\,(x^\circ) \leq v(x^*) - i\,w\,(x^*); \forall x^* \in S_w$, i. e.,

$x^\circ \in S_v$ is a solution of a conjugate function $\bar{F}(x) = v(x) - i\,w(x)$. Now we will prove that there is no $\hat{x} \in X$ and $\hat{x} \in S_v$ such that:

$$\bar{F}(\hat{x}) = v(\hat{x}) - i\,w(\hat{x}) \leq \bar{F}(x^\circ) = v(x^\circ) - i\,w(x^\circ). \tag{9}$$

There are two cases:

Case 1: Assume that $x \in X$ $\hat{x} \notin S_v$, $x' \in S_w$ and $v(\hat{x}) - i\,w(\hat{x}) \leq v(x^\circ) - i\,w(x^\circ)$ i.e.,

$w(x^\circ) \leq w(\hat{x})$. From the strictly convexity of the function $w(x)$ and S_w is open, then

$w(\tau\hat{x} + (1-\tau)x^\circ) < \tau w\,(\hat{x}) + (1-\tau)w(x^\circ), 0 < \tau < 1$, this leads to

$w(\tau\hat{x} + (1-\tau)x^\circ) < \tau w\,(\hat{x}) + (1-\tau)w(\hat{x})$ i. e.,

For certain τ such that $\tau\hat{x} + (1-\tau)x^\circ \in S_w$, we have

$w\left(\tau \hat{x}+(1-\tau)x^{\circ}\right)<w\left(\hat{x}\right)$ .Which contradicts to $\hat{x}\in S_w$ i.e., there is no $\hat{x}\in X,\ \hat{x}\notin S_v,\ \hat{x}\in S_w$ such that:

$$\overline{F}(\hat{x})=v(\hat{x})-i\,w(\hat{x})\le\overline{F}\left(x^{\circ}\right)=v\left(x^{\circ}\right)-i\,w\left(x^{\circ}\right).$$

Case 2: Assume that $\hat{x}\in X,\hat{x}\notin S_v,\hat{x}\notin S_w$ and $v(\hat{x})-i\,w(\hat{x})<v\left(x^{\circ}\right)-i\,w\left(x^{\circ}\right)$ i.e., $v\left(\hat{x}\right)<v\left(x^{\circ}\right)$, and $w\left(x^{\circ}\right)<w\left(\hat{x}\right)$. Since the function $v(x)$ is strictly convex and S_v is open, then

$v\left(\tau x^{\circ}+(1-\tau)\hat{x}\right)<\tau v\left(x^{\circ}\right)+(1-\tau)v(\hat{x}),\ 0\le\tau\le1$ This leads to

$v\left(\tau x^{\circ}+(1-\tau)\hat{x}\right)<\tau v\left(x^{\circ}\right)+(1-\tau)v(\hat{x})$, i.e., for certain τ, we have

$\tau x^{\circ}+(1-\tau)\hat{x}\in S_v$, such that $\tau\breve{x}+(1-\tau)\hat{x}\in S_v$,we have

$v\left(\tau x^{\circ}+(1-\tau)\hat{x}\right)<v\left(x^{\circ}\right)$. Contradicts that $x^{\circ}\in S_v$.

Thus, there is no $\hat{x}\in X$ such that:

$v(\hat{x})-i\,w(\hat{x})<v\left(x^{\circ}\right)-i\,w\left(x^{\circ}\right)$

5. Numerical examples

Example1. (Illustration of Lemma1)

Consider the following complex problem

$$\min\left(\cos x+i\,\sin x\right)$$

Subject to (10)

$$x\in X=\{x\in R:0\le x\le\pi\}$$

Problem (10) is divided into the following two problems as:

(P_v) $\min cos x$

Subject to (11)

$$x\in X=\{x\in\mathbb{R}:0\le x\le\pi\},\ \text{and}$$

(P_w) $\text{Min}\sin x$

Subject to (12)

$$x \in X = \{x \in \mathbb{R}: 0 \le x \le \pi\},$$

The optimal solution of problem (11) is $x = \pi \in S_v$, i. e., $S_v = \{\pi\}$. Also, the optimal solution of problem (12) is $x = 0, \pi$, i. e., $S_w = \{0, \pi\}$ Thus, the optimal solution of problem (10) is $x = \pi \in S_v \cap S_w$.

Example2. (Illustration of Lemma2)

Consider the following NCP problem:

$$\text{Min} \quad \tilde{F}^N(x) = \left(\tilde{c}_1{}^N x_1 + \tilde{c}_2{}^N x_2\right) + i\left(\tilde{d}_1{}^N x_1 - \tilde{d}_2{}^N x_2\right)$$

Subject to (13)

$$\left(\tilde{p}_{11}{}^N x_1^2 + \tilde{p}_{22}{}^N x_2^2\right) + i\left(\tilde{q}_1{}^N x_1 + \tilde{q}_2{}^N x_2\right) = \tilde{e}^N + i\tilde{g}^N.$$

Where,

$$\tilde{c}_1{}^N = \langle 5, 8, 10, 14; 0.3, 0.6, 0.6 \rangle,$$

$$\tilde{c}_2{}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3 \rangle,$$

$$\tilde{d}_1{}^N = \langle 4, 8, 11, 15; 0.6, 0.3, 0.2 \rangle,$$

$$\tilde{d}_2{}^N = \langle 16, 18, 22, 30; 0.6, 0.2, 0.4 \rangle,$$

$$\tilde{p}_{11}{}^N = \tilde{p}_{22}{}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3 \rangle,$$

$$\tilde{q}_1{}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3 \rangle = \tilde{q}_2{}^N,$$

$$\tilde{e}^N = \langle 4, 8, 11, 15; 0.6, 0.3, 0.2 \rangle,$$

$$\tilde{g}^N = \langle 0, 1, 3, 6; 0.7, 0.5, 0.3 \rangle.$$

Using the score function of the single- valued trapezoidal neutrosophic number introduced in definition9, problem (13) becomes:

$$\text{Min} \quad F(x) = (3x_1 + x_2) + i(5x_1 - 11x_2)$$

Subject to (14)

$$x_1^2 + x_2^2 + i(x_1 - x_2) \le 5 + i.$$

According to the Lexicographic order, the problem is divided into the following two subproblems as:

$$(P_v) \qquad \text{Min} \quad v(x) = 3x_1 + x_2$$

Subject to (15)

$x_1^2 + x_2^2 \leq 5,\ x_1 - x_2 \leq 1,$ and

$$(P_w) \quad \text{Min} \quad w(x) = 5x_1 - 11x_2$$

Subject to (16)

$$x_1^2 + x_2^2 \leq 5, x_1 - x_2 \leq 1.$$

By applying the Kuhn- Tucker conditions [14, 22], the optimal solutions of problems (15), (16) and problem (9) are illustrated in the following tables.

Table 1. The set of solution of (P_v)

S_v	Optimum value
$\{(-2, -1)\}$	$P_v = -7$ $\tilde{P}_v{}^N = \langle -34, -23, -17, -10; 0.3, 0.6, .06 \rangle$

Table 2. The set of solution of (P_w)

S_w	Optimum value
$\{(-2, 1)\}$	$P_w = -21$ $\tilde{P}_w{}^N = \langle -60, -44, -34, -24; 0.6, 0.3, 0.4 \rangle$

Therefore, $S_v \cap S_w = \emptyset$ and the solution of problem S_v is not a solution of the conjugate function $v(x) - i\, w(x)$, because of $v(x)$, and $w(x)$ are not strictly convex functions.

6. Concluding Remarks

In this paper, the solution of complex programming (NCP) with single valued trapezoidal neutrosophic numbers in all the parameters of objective function and constraints has characterized. Based on the score function definition, the NCP has converted into the corresponding crisp CP problem and hence Lexicographic order has used for comparing between any two complex numbers. The comparison has developed between the real and imaginary parts separately. Through this manner, the CP problem has divided into two real sub-problems. The main contribution of this approach is more flexible and makes a situation realistic to real world application. The obtained results are more significant to enhance the applicability of single-valued trapezoidal neutrosophic number in various new fields of decision-making situations. The future research scope is to apply the proposed approach to more complex and new applications. Another possibility is to work on the interval type of complex neutrosophic sets for the applications in forecasting filed.

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Conflicts and Interest

The authors declare no conflict of interest.

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A New Model for the Selection of Information Technology Project in a Neutrosophic Environment

Maikel Leyva-Vázquez[1], Miguel A. Quiroz-Martínez[2], Yoenia Portilla-Castell[3], Jesús R. Hechavarría-Hernández[4], and Erick González-Caballero[5]

[1] Universidad Politécnica Salesiana, Carrera de Ingeniería en Sistema/Instituto Superior Bolivariano de Tecnología, Guayaquil, Guayas, Ecuador, E-mail: mleyvaz@ups.edu.ec

[2] Universidad Politécnica Salesiana, Carrera de Ingeniería en Sistema/Instituto Superior Bolivariano de Tecnología, Guayaquil, Guayas, Ecuador, E-mail: mquiroz@ups.edu.ec

[3] Higher Technological Institute of Administrative and Commercial Professional Training, Ecuador, E-mail: yoenia.portilla@formacion.edu.ec

[4] Universidad Católica de Santiago de Guayaquil, Guayaquil, Guayas, Ecuador, E-mail: jesusr2h@ups.edu.ec

[5] Universidad Tecnológica de La Habana, La Habana, Cuba. E-mail: erickgc@yandex.com

Abstract. Usually, companies confront the difficulty to make the best decision about the way to invest their recourses in different project alternatives. The company acquires competitive advantages when their software development projects are well evaluated and correctly selected. Selecting projects in the Information Technology field presents challenges in many senses; e.g., the difficulty that entails assessing intangible benefits, projects are interdependent and companies impose self-constraints. In addition, the framework to make the decision is generally uncertain with many unknown factors. This paper aims to propose a model that integrates methods, techniques and tools such as the Balanced Scorecard Model, neutrosophic Analytic Hierarchy Process and zero-one linear programming. The proposed model is designed to select the best portfolio of Information Technology projects, it overcomes the obstacles mentioned above and can be coherently incorporated in the strategic plan process of any company. In addition, it eases the course of experts' decision making, because it is based on Neutrosophy and hence incorporates the indeterminacy term.

Keywords: Information Technology Project, Balanced Scorecard Model, Neutrosophic Analytic Hierarchy Process, zero-one linear programming.

1. Introduction

According to the guide to the project management body of knowledge (PMBOK) [1], "project management is the application of knowledge, skills, tools and techniques to projects activities to meet project requirements". The guide to the PMBOK also makes reference to the multiple project management. Some authors acknowledge that sometimes exist missing or vaguely defined processes in any commercial corporations; some of them are the coordination in a multi-project environment and the strategic processes [2].

Later on, Project Management Institute published in detail additional standards for the Programs and Portfolio management [1, 3, 4]. A Program is defined as a related group of projects, which are coordinately managed to obtain benefits and controls, under the constraint that these benefits and controls would not be available, in the case they were managed individually.

On the other hand, a Project Portfolio is a group of projects performed during a certain time span and which share common resources. Some kinds of relationships that can exist among the projects are complementariness, incompatibility and synergies, which are derived from the division of costs and benefits obtained from the performance of more than one project simultaneously [5]. See schematized of an example in Fig. 1.

The foundations of project portfolio management have been developing since the seventies. Its roots can be found in the theory of Harry Markowitz, which deserved the Nobel Prize in Economic Sciences. He shared this award with Merton H. Miller and William F. Sharpe, for their work in the field of financial economics theory. Its basic contribution is the "portfolio choice theory". He proposed a model for the choice of a portfolio of securities in conditions of uncertainty in which it reduced it to a two-dimensional dilemma: the expected income and the variance.

Nevertheless, some authors point out that significant differences exist between the theory of project portfolio management and Markowitz's theory [6, 7].

Four of the six responsibilities in project portfolios management, which were emphasized by Kendall and Rollins, are the following, [8]:

- To determine a suitable combination of projects such that the company's goal could be achieved.
- To attain an adequate balance in the portfolio, where the combination of projects has an adequate balance between risks and rewards, research and development and so on.
- To assess the possible existence of new opportunities for the present portfolio, taking into account the company's capacity for execution.
- To provide information and recommendations for decision makers at every level.

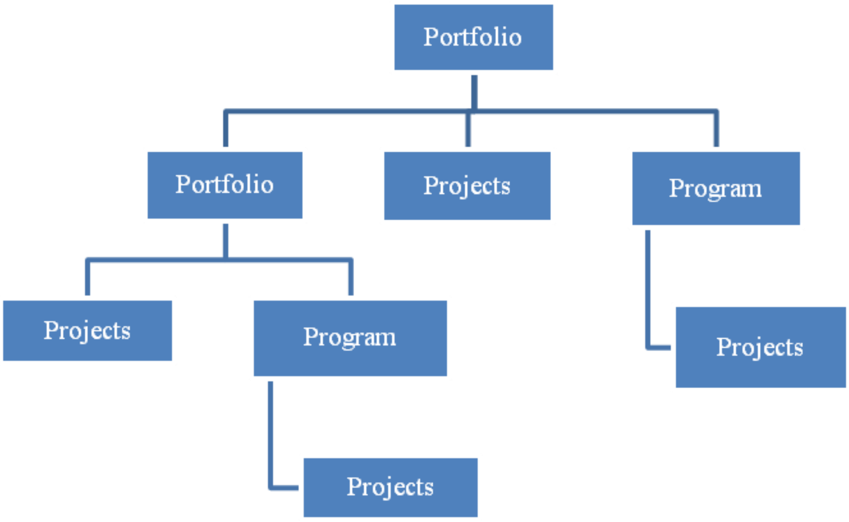

Figure 1: Scheme of a possible Portfolio-Program-Project relationship

The project portfolio management is inherently strategic, it is more related to efficacy (to perform the adequate project) than the efficiency (to execute the project correctly). It should avail a framework of work for assessing decisions about to invest, maintain and remove [9].

According to the reports of A. T. Kearney, which is an American global management consulting firm that focuses on strategic and operational CEO-agenda issues, the plan in investment projects have barely changed in enterprises since the 1920s, see [10]. The forthcoming necessities of the company are not forecasted, instead, decision makers assign the budget that they consider sufficient to carry out each project individually, no doubt this is a drawback, see [11, 12]. The second drawback is when decision makers do not identify potential synergies that could exist among the projects and therefore, unexpected increases in project costs could arise.

Kaplan and Norton introduced a framework of work to measure the effectiveness of a company; they called it Balanced Scorecard (BSC). This model integrates four perspectives, namely, financial, customer, business process and learning and growth [13]. Additionally, this is a way to display the strategies inside the company. Particularly, BSC is useful to select measures that guarantee the balance in project portfolios of Information Technologies [6].

The relationship existing between strategy and Project Management is a subject that has considerably evolved during the years pass. One example is project portfolio management, consisting of a close relationship that connects strategy with Project Management by selecting and prioritizing those projects which satisfy strategic objectives. Both selection and prioritization are based on criteria that could perfectly coincide with indicators proper of the Balanced Scorecard model designed for this company [5, 8].

The economic importance of Information Technology projects is evident. Frequently, Information Technology projects represent a significant portion of the set of projects inside a company [2]. In the present-day, the hardware is considered as a commodity, whereas software provides the major part of a computational system [14].

Information Technology (IT) management is a subject that has quickly grown since the very near past. Pells in [15] presented the factors which have repercussions on the growth of the IT projects management, they are the following:

- The massive investment in IT all over the world.
- The natural orientation of the project management toward the IT industry.
- The fast change of technologies.
- Failures in IT projects.
- The arrival of the Information Era.
- IT embraces every industry, company and project.

When these factors are taken into consideration as a whole, they conduce to other important trends and developments in the fields of project management, project portfolio management and complex project management.

In this present research, the authors used a balanced scorecard model as a tool to determine the coherence of the project with company's strategy, particularly considering their perspectives. Moreover, the criteria to determine the project feasibility have been included. The proposed model is based on the balanced scorecard model, neutrosophic analytic hierarchy process and zero-one linear programming.

The analytic hierarchy process (AHP) was created by Aczél et al. [16]. It is a well-known multicriteria decision-making technique founded on mathematics and cognitive psychology. This technique has been widely applied to make decisions in complex situations.

Buckley in [17, 18] designed a fuzzy hierarchical analysis, where the crisp decision ratio of the classical AHP is substituted by a fuzzy ratio represented by a trapezoidal membership function. This approach introduces uncertainty and imprecision from the fuzzy viewpoint.

Abdel-Basset et al. in [19] designed a neutrosophic AHP-SWOT model, based on neutrosophic sets, where a neutrosophic set is a part of neutrosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra [20]. The neutrosophy included for the first time the notion of indeterminacy in the fuzzy set theory, which is also part of real-world situations. Neutrosophic AHP permits that experts could express their criteria more realistically, by indicating the truthfulness, falseness and indeterminacy of the decision ratio.

This paper aims to present a new mathematical model to select the best information technology projects. In the first step, a balanced scorecard model is applied to establish the criteria selection. The second stage consists in applying a neutrosophic AHP technique, where crisp weights of project importance are output. During this step neutrosophic triangular numbers and the operations among them are used for calculating. These weights of each project's importance are inputs to the third stage. The third stage consists of a zero-one linear programming model for selecting the best projects that satisfy the feasible constraints.

Hybridizing different Multicriteria Decision-Making (MCDM) methods for creating new project selection models have become recurrent in the literature that is why the model proposed in this paper can also be of interest to researches and decision makers. In [21] the state of the art in project selection problem is studied for 60 papers published in the period from 1980 to 2017 and it is concluded that the most popular techniques to perform hybridizations are the Order of Preference by Similarity to Ideal Solution (TOPSIS) and the analytic hierarchy process / analytic network process followed by the VI-

KOR method. For example, in [22] the AHP technique is hybridized with PROMETHEE with the goal of urban renewal project selection. Papers in [23-30] introduce the hybridization of methods and techniques of MCDM within the framework of neutrosophy, obtaining more complete models than those based on fuzzy logic theory because uncertainty in decision-making also incorporates indeterminacy.

In addition that the hybridization of MCDM methods seems to be an inexhaustible source of creating new models for project selection, the model proposed in this paper differs from the rest of the similar ones. This is specifically designed to select information technology projects, which is why the Balanced Scorecard is included to guide the managers on which aspects to test in decision-making. BSC is so far infrequent in the published papers on hybridization. The AHP technique avoids bias in decision making due to the use of the consistency index. zero-one linear programming is the tool used to make the final decision, while neutrosophy is used to model the indeterminacy that decision makers might have. Another advantage of the model is that it allows decision makers to rate based on linguistic terms. To the best of the authors' knowledge, this seems to be the first model for selecting information technology projects by using the hybridization of Balanced Scorecard, neutrosophic AHP and zero-one linear programming, where a scale of linguistic terms serves to evaluate.

This paper is distributed as follows; section 2 contains the main theories used as the basis of this document. The proposed mathematical model is developed in section 3. In section 4 the application of the model is illustrated with an example. Section 5 states the conclusions.

2 Preliminaries

This section exposes the theories used to design the model. It is started with part of the theory of the project portfolio. Further, the authors summarize the AHP technique and neutrosophic set theory. Finally, the main concepts of zero-one linear programming are written.

2.1 Approaches to Portfolio IT Project

An important part of IT projects is related to software development. The difference of software development projects with respect to other engineerings, e.g., electronic engineering, is that the former one imposes additional challenges to project management, mainly due to the particular characteristics of software [30] and these characteristics are the following:

- The software is an intangible product.
- The standard software processes do not exist.
- The uniqueness of the large scale projects of software developments.

When a computer product will be developed, or an information system, or any other modifications, in that case, the elaboration of an innovative project is needed for planning and executing the introduction of this product inside the company. Technological innovation projects are elaborated to introduce scientific results obtained from scientific creation. This is related to applied researches, technological developments; and the commercialization of novel technologies, products, systems and processes. This is the final stage in the cycle of science-technology-production [31].

Literature had paid attention to project selection, see [2, 21-34], especially for research and development projects (R&D), see [35, 36]. One main difference exists between IT and (R&D) projects, it is that projects interdependence in the former has elevated importance [1, 3, 4]. Moreover, two IT projects can share identical code sections or hardware.

The project selection process in general, including IT projects, is a very complex process that is influenced by several factors. One key aspect of IT control is the prioritization of investments. Projects have to be assessed as an investment viewpoint, by having as a goal to analyze the project capacity for maximizing the company's value [32].

One of the criteria to approve the start of one project would be to determine its possibility of success and impact; evidently, most companies cannot start simultaneously every project. The project assessment consists of gathering pertinent information in the end to facilitate the project selection process and to determine the value of every project [8, 37]. The closing phases assessment allows us to build a base of knowledge that shall be communicated during the organization's continuous learning

[6].

One of the goals in portfolio management is to maximize the portfolio value, by carefully assessing those projects and programs which could be included in the portfolio and also to opportunely exclude those of them which do not fulfill the portfolio strategic objectives [38]. IT portfolio management is basically a selection process to locate resources to develop/maintain those projects that better satisfy strategic objectives [39].

There exist a number of difficulties in evaluating projects. Rebaza points out, referring to computer projects that in most cases the projects are evaluated according to cost-benefit criteria [40]. The task of evaluating projects is not simple and involves many difficulties, some of them are methodological. These difficulties include the following:

- Lack of information availability,
- Lack of qualified staff for evaluation,
- Lack of evaluation processes in the company.
- Use of limited criteria for evaluation.

Project selection methods are used to determine which project the organization will select. Generally, these methods are divided into four major categories according to Bonham, see [5]:

A. Mathematical programming—Integer programming, linear programming, nonlinear programming, goal programming and dynamic programming
B. Economic models—IRR, NPV, PB period, ROI, cost-benefit analysis, option pricing theory, the average rate of return and profitability index;
C. Decision analysis—Multiattribute utility theory, decision trees, risk analysis, analytic hierarchy process, unweighted 0–1 factor model, unweighted (1 – n) factor scoring model and weighted factor scoring model;
D. Interactive comparative models—Delphi, Q-sort, behavioral decision aids and decentralized hierarchical modeling.

A relatively recent trend in the information technology area is value-based software engineering (VBSE) [41]. VBSE is considered as part of the life cycle of software engineering management activities such as the development of the Business Case, project evaluation, project planning etc, which have so far been considered peripheral. The VBSE aims to guide proposals and solutions based on the maximization of the value provided.

Any decision to construct (or re-engineering) a software system should be guided by its "value" ([42]). Thus, a system brings more "value" to their users if it provides greater benefits, either in terms of return on investment (ROI), social benefits, reduced management costs, strategic advantages, or any other aspect. As can be assumed, the quantification of all these types of benefits is complex [42].

Sometimes intangible benefits, such as learning and opportunity for growth, are the fundamental sources of value. As a result, other indicators to be taken into consideration for investment have emerged. An example of this is the social return on investment [42], which seeks to capture social values by translating social goals into financial and non-financial measures. Kendal and Rolling ([8]) claim that the more projects that are initiated with insufficient resources, the fewer projects that are completed and the longer each project takes to complete. Surveys indicate that companies with the highest number of project selection criteria are associated with better performance ([6]).

Bonham [5] proposes a model for project selection based on three phases, viz., strategic analysis, individual project analysis (maximization) and portfolio selection (balance). He also noted the importance of analyzing the interdependence between projects.

Bergman and Mark ([2]) present a way to issue the problem of project selection using the requirement analysis to better inform each project option. As a project option develops through the selection process, its specification of requirements is detailed and refined. Project requirements provide a better technical, economic and organizational understanding of each project.

Value Measuring Methodology (VMM) ([4]) is a methodology for evaluating and selecting initiatives that offer the greatest benefits. Moreover, Rapid Economic Justification ([39]) is a framework developed by Microsoft to decide the value of investments in information technology.

Wibowo notes that existing approaches present the following limitations, see [43]:

- The inability to deal with the subjectivity and the imprecision of the evaluation processes and the selection of information systems projects.
- Failure to properly manage the multidimensional nature of the problem.
- It is very cognitively demanding for the decision-maker.

The model proposed in this paper overcomes all the difficulties specified above, as can be further seen.

2.2 AHP Technique

AHP consists first in designing a hierarchical structure, where the upper elements are more generic than those situated below. The layer on top contains a single leaf, representing the decision goal, the second layer that connected with the goal emerges as a set of leaves representing the criteria and the followed third layer is containing subcriteria and so on. The last bottom layer of this tree contains leaves representing the alternatives. See, Fig. 2.

Consequently, square matrices represent the expert or experts' decision, containing the pair-wise comparison of criteria, subcriteria or alternatives assessment. Aczél et al. in [16] proposed the scale that they considered is the better to evaluate decisions, as can be seen in Tab. 1.

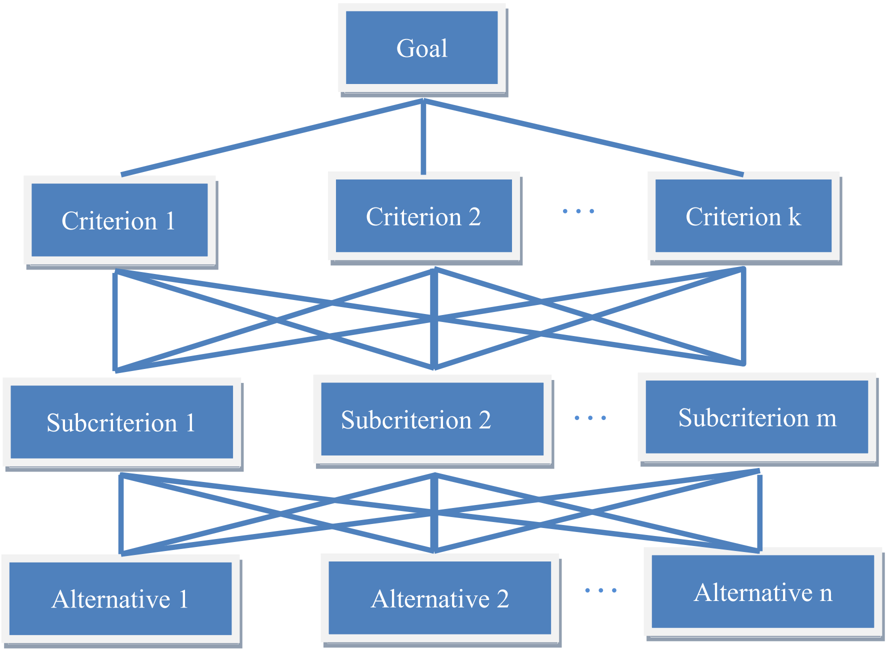

Figure 2: Scheme of a generic tree representing an Analytic Hierarchy Process

Table 1: Intensity of importance according to the classical AHP

The intensity of importance on an absolute scale	Definition	Explanation
1	Equal importance	Two activities contribute equally to the objective
3	Moderate importance of one over another	Experience and judgment moderately favor one activity over another
5	Essential or strong importance	Experience and judgment strongly favor one activity over another
7	Very strong importance	Activity is strongly favored and its dominance demonstrated in practice
9	Extreme importance	The evidence favoring one activity over another is of the highest possible order of affirmation
2, 4, 6, 8	Intermediate values between the two adjacent judgments.	When comprise is needed
Reciprocals	If activity *i* has one of the above numbers assigned to it when compared with activity *j*, i.e., number $a \in \{1,2,\cdots,9\}$, then *j* has the reciprocal value when compared with *i*, i.e., value $1/a$.	

On the other hand, Aczél et al. established that the *Consistency Index* (CI) should depend on λ_{max}, the maximum eigenvalue of the matrix. They defined the equation $CI = \frac{\lambda_{max}-n}{n-1}$, where n is the order of the matrix. Additionally, they defined the *Consistency Ratio* (CR) with equation CR = CI/RI, where the Random Index or RI is given in Tab. 2.

Table 2: RI associated with every order.

Order (n)	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.52	0.89	1.11	1.25	1.35	1.40	1.45	1.49

Each RI value is an average random consistency index computed for n ≤ 10 for very large samples. Randomly generated reciprocal matrices were created using the scale 1/9, 1/8, …,1/2, …, 8, 9 and the average of their eigenvalues were calculated. This average is used to form the RI.

If CR≤10% it is considered that experts' evaluation is consistent enough and hence, proceed to use AHP.

AHP aims to score criteria, subcriteria and alternatives and to rank every alternative according to these scores.

AHP can also be used in group assessment. In such a case, the final value is calculated by the weighted geometric mean, which satisfies the inverse requirements [44], see Eq. 1 and 2. The weights are utilized to measure the importance of each expert's criteria, where some factors are taken into consideration like expert's authority, knowledge, effort, among others

$$\bar{x} = \left(\prod_{i=1}^{n} x_i^{w_i}\right)^{1/\sum_{i=1}^{n} w_i} \quad (1)$$

If $\sum_{i=1}^{n} w_i = 1$, i.e., when expert's weights sum one, Eq. 1 transforms in Eq. 2,

$$\bar{x} = \prod_{i=1}^{n} x_i^{w_i} \quad (2)$$

2.3 Neutrosophic sets

Neutrosophic sets extend classical sets, fuzzy sets and intuitionistic fuzzy sets.. Fuzzy set models are based on the degree of membership of an element to a set. It has been applied in many areas of knowledge, including decision making.

Fuzzy set theory was introduced by Lotfi A. Zadeh for the first time at 1965. A fuzzy set consists of the following manners [45, 46]:

Given a Universe of Discourse U containing a set of objects and A being its subset, a membership function is a function $T_A: U \to [0,1]$, defined for every $x \in U$, where $T_A(x)$ is the degree of truth for which x belongs to A.

The intuitionistic fuzzy set theory was introduced by Krassimir T. Atanassov at 1986. An intuitionistic fuzzy set is defined by two membership functions, T_A meaning that x belongs to U and F_A meaning that x does not belong to A. They must satisfy the restriction $T_A(x) + F_A(x) \leq 1$, [47].

On the other hand, Neutrosophic set includes a third membership function I_A, meaning indeterminacy. Thus, a neutrosophic set is a triple of membership functions, T_A, I_A and F_A with no restriction. The inclusion of indeterminacy is a contribution made by Florentin Smarandache [20], which agreed that neutrality and ignorance are also part of the uncertainty. Moreover, he accepts the possibility that truthfulness, indeterminacy and falseness can be simultaneously maximal. Also, he uses the idea of non-standard analysis of Abraham Robinson and he utilizes hyperreal numbers in calculations.

Let us define formally the concept of neutrosophic set.

Definition 2.3.1([20]): The neutrosophic set N is characterized by three membership functions, which are the truth-membership function T_A, indeterminacy-membership function I_A and falsity-membership function F_A, where U is the Universe of Discourse and $\forall x \in U$, $T_A(x), I_A(x), F_A(x) \subseteq]^-0, 1^+[$ and $^-0 \leq inf\, T_A(x) + inf\, I_A(x) + inf\, F_A(x) \leq sup\, T_A(x) + sup\, I_A(x) + sup\, F_A(x) \leq 3^+$.

See that according to the definition, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$ and hence, $T_A(x)$, $I_A(x)$ and $F_A(x)$ can be subintervals of [0, 1]. $^-0$ and 1^+ belong to the set of hyperreal numbers.

Definition 2.3.2([20]): The Single Valued Neutrosophic Set (SVN) N over U is $A = \{< x, T_A(x), I_A(x), F_A(x) > : x \in U\}$, where $T_A: U \to [0, 1]$, $I_A: U \to [0, 1]$ and $F_A: U \to [0, 1]$. $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

The Single Valued Neutrosophic (SVN) number is symbolized by

N = (t, i, f), such that $0 \leq t, i, f \leq 1$ and $0 \leq t + i + f \leq 3$.

Definition 3.2.3 ([19, 48]): The single valued triangular neutrosophic number,

$\tilde{a} = \langle (a_1, a_2. a_3); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle$, is a neutrosophic set on $\mathbb{R}$, whose truth, indeterminacy and falsity membership functions are defined as follows:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}}\left(\frac{x-a_1}{a_2-a_1}\right), & a_1 \leq x \leq a_2 \\ \alpha_{\tilde{a}}, & x = a_2 \\ \alpha_{\tilde{a}}\left(\frac{a_3-x}{a_3-a_2}\right), & a_2 < x \leq a_3 \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

$$I_{\tilde{a}}(x) = \begin{cases} \frac{\left(a_2 - x + \beta_{\tilde{a}}(x - a_1)\right)}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \beta_{\tilde{a}}, & x = a_2 \\ \frac{\left(x - a_2 + \beta_{\tilde{a}}(a_3 - x)\right)}{a_3 - a_2}, & a_2 < x \leq a_3 \\ 1, & \text{otherwise} \end{cases} \quad (4)$$

$$F_{\tilde{a}}(x)=\begin{cases}\frac{\left(a_2-x+\gamma_{\tilde{a}}(x-a_1)\right)}{a_2-a_1}, & a_1\le x\le a_2\\ \gamma_{\tilde{a}}, & x=a_2\\ \frac{\left(x-a_2+\gamma_{\tilde{a}}(a_3-x)\right)}{a_3-a_2}, & a_2<x\le a_3\\ 1, & \text{otherwise}\end{cases} \quad (5)$$

Where $\alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \in [0,1]$, $a_1, a_2, a_3 \in \mathbb{R}$ and $a_1 \le a_2 \le a_3$.

Definition 2.3.4 ([19, 48]): Given $\tilde{a} = \langle (a_1, a_2, a_3); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle$ and $\tilde{b} = \langle (b_1, b_2, b_3); \alpha_{\tilde{b}}, \beta_{\tilde{b}}, \gamma_{\tilde{b}} \rangle$ two single-valued triangular neutrosophic numbers and λ any non-null number in the real line. Then, the following operations are defined:

1. Addition: $\tilde{a} + \tilde{b} = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle$
2. Subtraction: $\tilde{a} - \tilde{b} = \langle (a_1 - b_3, a_2 - b_2, a_3 - b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle$
3. Inversion: $\tilde{a}^{-1} = \langle ({a_3}^{-1}, {a_2}^{-1}, {a_1}^{-1}); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle$, where $a_1, a_2, a_3 \neq 0$.
4. Multiplication by a scalar number:
$$\lambda\tilde{a} = \begin{cases}\langle (\lambda a_1, \lambda a_2, \lambda a_3); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle, & \lambda > 0\\ \langle (\lambda a_3, \lambda a_2, \lambda a_1); \alpha_{\tilde{a}}, \beta_{\tilde{a}}, \gamma_{\tilde{a}} \rangle, & \lambda < 0\end{cases}$$
5. Division of two triangular neutrosophic numbers:
$$\frac{\tilde{a}}{\tilde{b}} = \begin{cases}\langle \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1}\right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle, a_3 > 0 \text{ and } b_3 > 0\\ \langle \left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}\right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle, a_3 < 0 \text{ and } b_3 > 0\\ \langle \left(\frac{a_3}{b_1}, \frac{a_2}{b_2}, \frac{a_1}{b_3}\right); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle, a_3 < 0 \text{ and } b_3 < 0\end{cases}$$
6. Multiplication of two triangular neutrosophic numbers:
$$\tilde{a}\tilde{b} = \begin{cases}\langle (a_1 b_1, a_2 b_2, a_3 b_3); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle, & a_3 > 0 \text{ and } b_3 > 0\\ \langle (a_1 b_3, a_2 b_2, a_3 b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle, & a_3 < 0 \text{ and } b_3 > 0\\ \langle (a_3 b_3, a_2 b_2, a_1 b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}, \gamma_{\tilde{a}} \vee \gamma_{\tilde{b}} \rangle, & a_3 < 0 \text{ and } b_3 < 0\end{cases}$$

Where $\wedge$ is a t-norm and $\vee$ is a t-conorm.

2.4 Zero-one linear programming

A zero-one linear programming theory solves problems like the following:

$$\text{Max(Min) } f(\boldsymbol{x}) = c_1 x_1 + c_2 x_2 + \cdots + c_I x_I \quad (6)$$
$$\text{Subject to: } x_i \in B$$

Where, $\boldsymbol{x} = (x_1, x_2, \ldots, x_I)^T$, $x_i \in \{0,1\}$ and $c_i \in \mathbb{R}$, i = 1, 2, …, I; B is the feasible set of solutions. B can be defined with equalities like $Ax = b$, inequalities like $Ax \le b$ or $Ax \ge b$, a combination of them, or simply an empty set. Where A is an mxI matrix and b is an m-column vector.

This theory solves decision problems, where only two alternatives exist, 1 represents to make the decision and 0 to not make the decision.

Zero-one linear programming problems are part of the Integer programming problems, when $x_i \in \mathbb{Z}$. Despite their seeming simplicity, these problems are NP-complete [49, 50], thus, a good universal algorithm cannot be found to solve them during a rational time of execution. This subject is out of the scope of this paper.

To solve the zero-one linear programming problem let us consider the following equivalent problem:

$$\text{Max } f(\boldsymbol{x}) = c_1 x_1 + c_2 x_2 + \cdots + c_I x_I$$
$$\text{Subject to: } x_i \in B$$

Where, $\boldsymbol{x} = (x_1, x_2, \ldots, x_I)^T$, $x_i \in \mathbb{Z}$, $x_i \le 1$ and $c_i \in \mathbb{R}$, i = 1, 2, …, I.

3 Neutrosophic model for IT project assessment

The model consists of three main processes, criteria selection, assessment and project portfolio selection. These processes are integrated by means of a Balanced Scorecard Model (BSC), a Neutrosophic Analytic Hierarchy Process (NAHP) and zero-one linear programming, see Fig. 3.

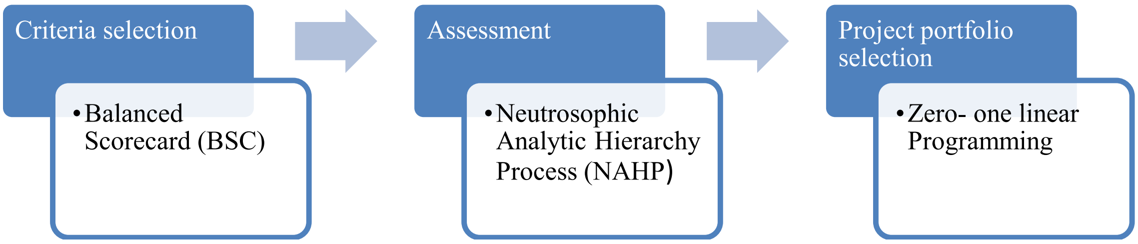

Figure 3: General structure of the model

The first step is to identify a potential group of projects. Next, a criteria selection is made. Some possible criteria are schematized in Fig. 4. This step is based on the BSC, which is an unusual tool for use in project selection. This tool could be incorporated because the proposed model is designed to solve the specific problem of information technology project selection. Fig. 4 can serve as a guide for decision makers on which aspects are the most important for evaluating information technology projects. The second stage of the model is to apply the NAHP. The proposed linguistic scale is based on triangular neutrosophic numbers summarized in Tab. 3, according to the scale defined in [19].

The hybridization of AHP with neutrosophic set theory was used in [19]. This is a more flexible approach to a model of uncertainty in decision making. The indeterminacy is an essential component to be assumed in real-world organizational decisions.

The neutrosophic pair-wise comparison matrix is defined in Eq. 7.

$$\tilde{A} = \begin{bmatrix} \tilde{1} & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \vdots & & \ddots & \vdots \\ \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{1} \end{bmatrix} \quad (7)$$

$\tilde{A}$ satisfies the condition $\tilde{a}_{ji} = \tilde{a}_{ij}^{-1}$, according to the inversion operator defined in Def. 4.

Abdel-Basset et al. in [19] defined two indices to convert a neutrosophic triangular number in a crisp number. Eqs. 8 and 9 indicate the score and the accuracy respectively as follow:

$$S(\tilde{a}) = \frac{1}{8}[a_1 + a_2 + a_3](2 + \alpha_{\tilde{a}} - \beta_{\tilde{a}} - \gamma_{\tilde{a}}) \quad (8)$$

$$A(\tilde{a}) = \frac{1}{8}[a_1 + a_2 + a_3](2 + \alpha_{\tilde{a}} - \beta_{\tilde{a}} + \gamma_{\tilde{a}}) \quad (9)$$

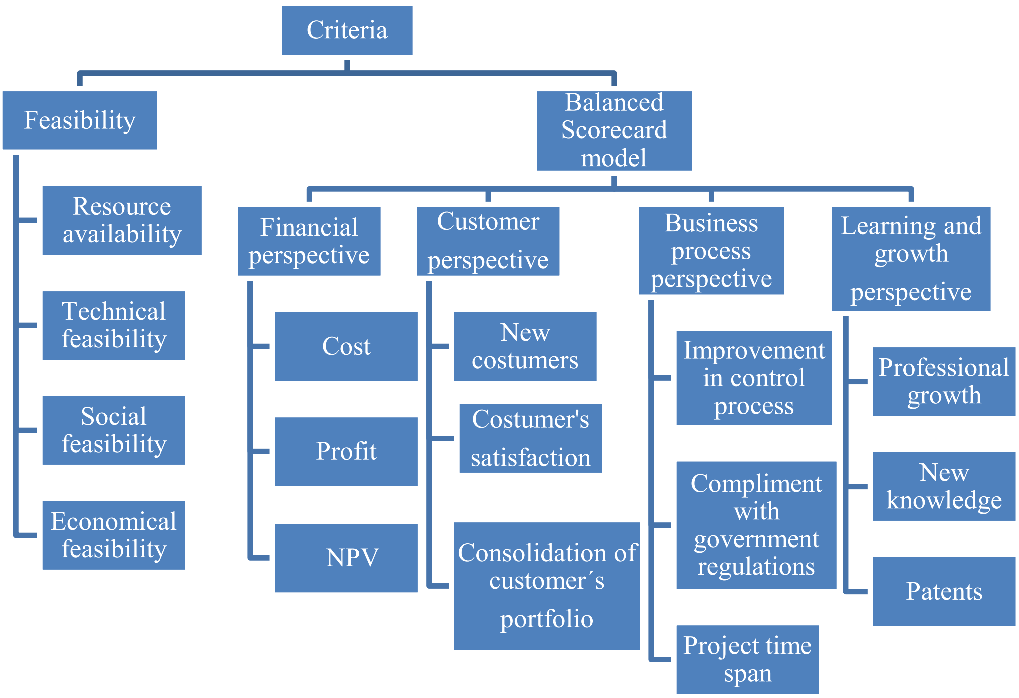

Figure 4: Example of possible project selection criteria

Table 3: Aczél et al.'s scale translated to a neutrosophic triangular scale.

Original scale	Definition	Neutrosophic Triangular Scale
1	Equally influential	$\tilde{1} = \langle(1,1,1); 0.50, 0.50, 0.50\rangle$
3	Slightly influential	$\tilde{3} = \langle(2,3,4); 0.30, 0.75, 0.70\rangle$
5	Strongly influential	$\tilde{5} = \langle(4,5,6); 0.80, 0.15, 0.20\rangle$
7	Very strongly influential	$\tilde{7} = \langle(6,7,8); 0.90, 0.10, 0.10\rangle$
9	Absolutely influential	$\tilde{9} = \langle(9,9,9); 1.00, 1.00, 1.00\rangle$
2, 4, 6, 8	Sporadic values between two close scales	$\tilde{2} = \langle(1,2,3); 0.40, 0.65, 0.60\rangle$ $\tilde{4} = \langle(3,4,5); 0.60, 0.35, 0.40\rangle$ $\tilde{6} = \langle(5,6,7); 0.70, 0.25, 0.30\rangle$ $\tilde{8} = \langle(7,8,9); 0.85, 0.10, 0.15\rangle$

Suppose that the criteria in Fig. 4 and the neutrosophic triangular scale in Table 3 are given, then the steps to apply the NAHP are as follow:

1. To design an AHP tree. This contains the selected criteria, subcriteria and alternatives from the first stage.
2. To create the matrices per level from the AHP tree, according to experts' criteria expressed in neutrosophic triangular scales and respecting the matrix scheme in Eq. 7.
3. To evaluate the consistency of these matrices. Abdel-Basset et al. make reference to Buckley, who demonstrated that if the crisp matrix $A = [a_{ij}]$ is consistent, then the neutrosophic matrix $\widetilde{A} = [\tilde{a}_{ij}]$ is consistent.
4. To follow the other steps of a classical AHP. Here, operations among neutrosophic triangular numbers substitute equivalent operations among crisp numbers in classical AHP.
5. The results obtained from step 4 are the project weights expressed in form of neutrosophic triangular numbers. Now, Eq. 8 is applied to convert, $w_1, w_2, \ldots, w_n$ to crisp weights.
6. If more than one expert make the assessment, then $w_1, w_2, \ldots, w_n$ are replaced by $\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_n$, which are their corresponding weighted geometric mean values, see Eq.1. and Eq. 2.

The obtained weights are not necessarily expressed in normal form, accordingly, there exists the choice to calculate equivalent normalized weights $w_1', w_2', \cdots, w_n'$ or $\overline{w}_1', \overline{w}_2', \cdots, \overline{w}_n'$, such that $\sum_{i=1}^{n} w_i' = 1$ or $\sum_{i=1}^{n} \overline{w}_i' = 1$. The precedent algorithm can be seen in the form of a flow chart in Fig. 5.

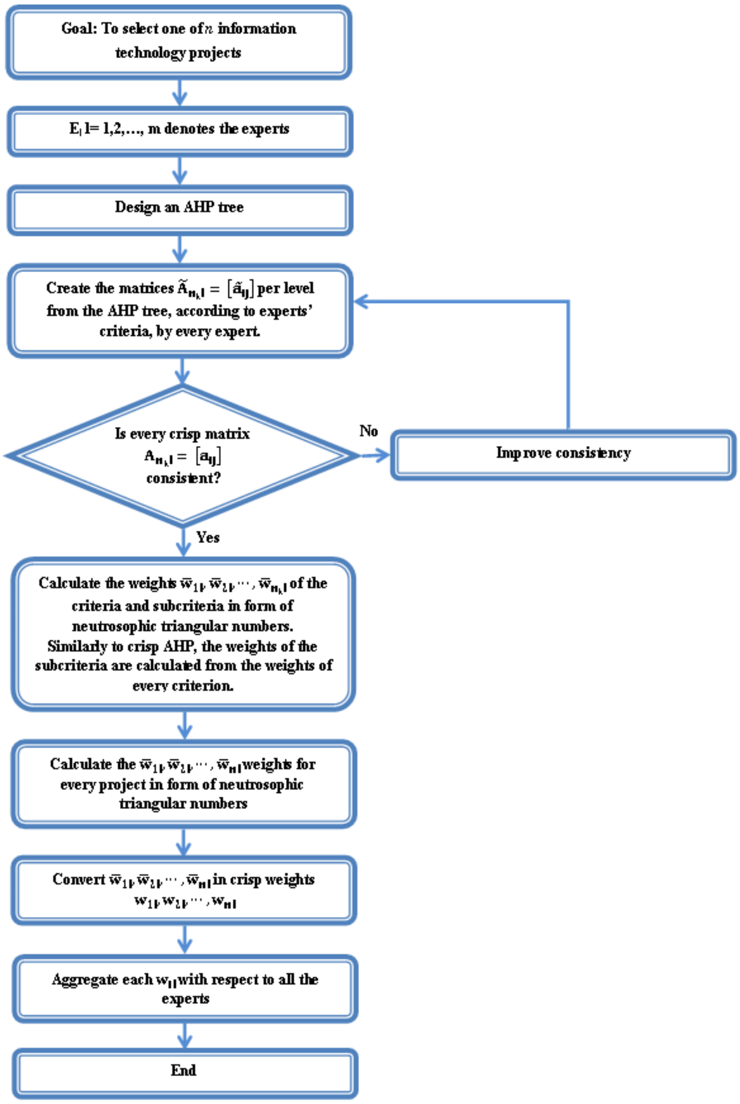

Figure 5: Flow chart of the NAHP algorithm.

Let us remark that in Abdel-Basset's method, $\tilde{A}$ is converted in A and later they continue applying classical AHP to A. In contrast, in the proposed model, data is converted to numeric value only in the last step. This way seems to be more acceptable because imprecision is kept throughout all the calculations.

The third stage consists of the application of a zero-one linear programming problem defined as follows:

$$\text{Max } f(\boldsymbol{x}) = w_1x_1 + w_2x_2 + \cdots + w_nx_n \quad (10)$$
$$\text{Subject to: } x_i \in B$$

See that the problem defined in Eq. 10 is a particular case of that appeared in Eq. 6.

Where, $x_i = \begin{cases} 1 & \text{, if Project i is selected} \\ 0 & \text{, otherwise} \end{cases}$ and w_i are the weights per project obtained from stage 2.

The purpose of this stage is to select the best projects, which optimally satisfy the constraints imposed by B, considering the weights obtained from NAHP.

4 Application of the model to an example

This section contains an example to illustrate the application of the model to a particular case of project selection. The authors simplified this example significantly for the sake of facilitating readers' comprehension.

Once the BSC model and the first stage are concluded, suppose that two project assessment criteria have been chosen; they are financial perspectives and internal processes, see Fig. 6.

To apply the AHP technique in the second stage, the elements of the problem were hierarchically structured. The goal appears on top of the tree, criteria to evaluate the goal were situated in the intermediate level and alternatives to reach that goal are on the bottom. Where, the goal is to assess IT projects, the intermediate level contains three criteria, viz., cost, project time span and profit and the bottom contain the three potential projects, called Project 1, Project 2 and Project 3. The tree is depicted in Fig. 7.

The expert expresses its criteria by means of the linguistic terms summarized in Tab. 3. The criteria defined in the intermediate level are pair-wise linguistically compared to determine their relative importance to achieve the objective.

Later, neutrosophic evaluations in the third column of Tab. 3 substitute their equivalent linguistic terms. Experts' evaluations can be seen in Tab. 4.

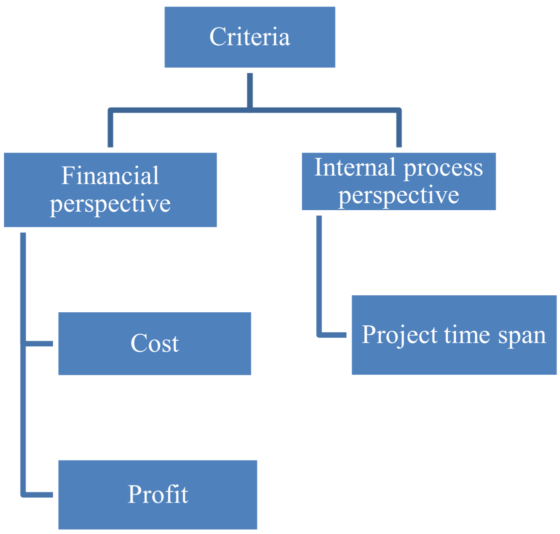

Figure 6: Selected criteria for the example

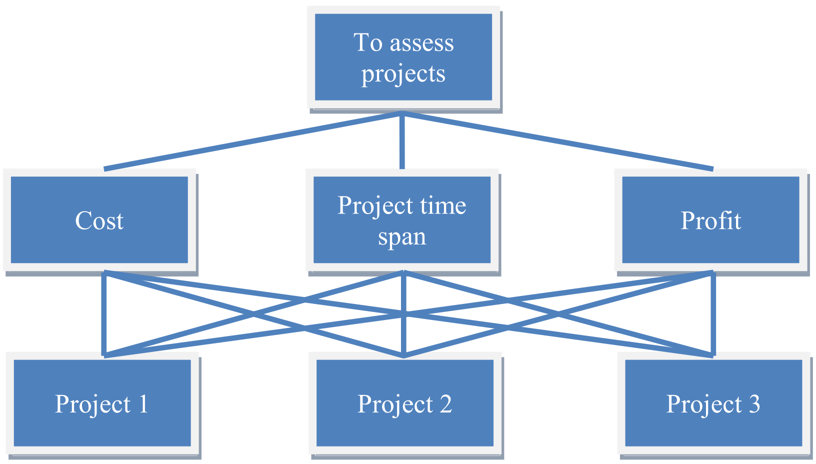

Figure 7: AHP tree of the example

Table 4: Reciprocal matrix corresponding to the second level

	Cost	Project Time span	Profit
Cost	$\tilde{1}$	$\tilde{2}$	$\tilde{5}^{-1}$
Project time span	$\tilde{2}^{-1}$	$\tilde{1}$	$\tilde{4}^{-1}$
Profit	$\tilde{5}$	$\tilde{4}$	$\tilde{1}$

See that evaluations contain the uncertainty and imprecision proper of neutrosophic set theory and hence the results are more realistic than those obtained from the classical Aczél et. al.'s AHP technique, now experts can include the indeterminacy term. Also, let us observe that the inverse of the single-valued triangular neutrosophic numbers can be calculated by using the inversion operator defined in Def. 4.

In this example, Cost is assessed with a value between equally and slightly more influential than Project time span, Profit is strongly more influential than Cost and Profit is evaluated between slightly and strongly more influential than Project time span. When the last three criteria comparisons are analyzed, let us note a certain degree of inconsistency, where it is expected that Profit is at least strongly more influential than the Project time span.

To measure the neutrosophic reciprocal matrix consistency, it is sufficient to calculate the CI of the crisp matrix, where $\tilde{a}_{ij}$ is substituted by a_{ij}, according to the theorem proved in [9], which says that given a fuzzy reciprocal matrix of fuzzy numbers $\bar{a}_{ij} = (\alpha_{ij}/\beta_{ij}/\gamma_{ij}/\delta_{ij})$, when choosing $a_{ij} \in [\beta_{ij}, \gamma_{ij}]$, if the matrix $(a_{ij})_{ij}$ is consistent then $(\bar{a}_{ij})_{ij}$ is also consistent.

Now on, the *eig* function coded in Octave 4.2.1 shall be used for estimating λ_{max}, in this case, CI = 9.0404%<10%, i.e., the matrix is consistent.

The values per row are summed and the weights are calculated. The results were summarized in Tab. 5.

Table 5: Sum per row and neutrosophic triangular weights in the second level criteria

	Row sum	Weight
Cost	<(2.17, 3.20, 4.25); 0.40, 0.65, 0.60>	<(0.12, 0.21, 0.36); 0.40, 0.65, 0.60>
Project time span	<(1.53, 1.75, 2.33); 0.40, 0.65, 0.60>	<(0.08 , 0.12, 0.12); 0.40, 0.65, 0.60>
Profit	<(8.00, 10.0, 12.0); 0.50, 0.50, 0.50>	<(0.43, 0.67, 1.03); 0.40, 0.65, 0.60>
Total	<(11.70, 14.95, 18.58); 0.40, 0.65, 0.60>	<(0.63, 1.00, 1.59); 0.40, 0.65, 0.60>

Tabs. 6, 7 and 8 contain reciprocal matrices for the third level and their weights. Where, Tab. 6 is related to the Cost, Tab. 7 with Project time span and Tab. 8 with Profit. The CIs of these matrices are, 5.1558%, 0.53269% and 0.53269%, respectively.

Table 6: Reciprocal matrix of the third level related to Cost and their weights.

	Project 1	Project 2	Project3	Weight
Project 1	$\tilde{1}$	$\tilde{2}$	$\tilde{5}$	<(0.31, 0.50, 0.79); 0.40, 0.65, 0.60>
Project 2	$\tilde{2}^{-1}$	$\tilde{1}$	$\tilde{5}$	<(0.27, 0.41, 0.63); 0.40, 0.65, 0.60>
Project 3	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{1}$	<(0.07, 0.09, 0.12); 0.40, 0.65, 0.60>

Table 7: Reciprocal matrix of the third level related to Project time span and their weights.

	Project 1	Project 2	Project3	Weight
Project 1	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{2}^{-1}$	<(0.09, 0.13, 0.23); 0.40, 0.65, 0.60>
Project 2	$\tilde{5}$	$\tilde{1}$	$\tilde{2}$	<(0.35, 0.61, 1.02); 0.40, 0.65, 0.60>
Project 3	$\tilde{2}$	$\tilde{2}^{-1}$	$\tilde{1}$	<(0.14, 0.26, 0.51); 0.40, 0.65, 0.60>

Table 8 Reciprocal matrix of the third level related to Profit and their weights.

	Project 1	Project 2	Project3	Weight
Project 1	$\tilde{1}$	$\tilde{5}$	$\tilde{2}$	<(0.35, 0.61,1.02); 0.40, 0.65, 0.60>
Project 2	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{2}^{-1}$	<(0.09, 0.13, 0.23); 0.40, 0.65, 0.60>
Project 3	$\tilde{2}^{-1}$	$\tilde{2}$	$\tilde{1}$	<(0.14, 0.26, 0.51); 0.40, 0.65, 0.60>

Table 9: Global weight matrix

	Costs	Project time span	Profits	Global Weight
Project 1	<(0.31, 0.50, 0.79); 0.40, 0.65, 0.60>	<(0.09, 0.13, 0.23); 0.40, 0.65, 0.60>	<(0.35, 0.61, 1.02); 0.40, 0.65, 0.60>	<(0.19, 0.53, 1.36); 0.40, 0.65, 0.60>
Project 2	<(0.27, 0.41, 0.63); 0.40, 0.65, 0.60>	<(0.35, 0.61, 1.02); 0.40, 0.65, 0.60>	<(0.09, 0.13, 0.23); 0.40, 0.65, 0.60>	<(0.10, 0.25, 0.59); 0.40, 0.65, 0.60>
Project 3	<(0.07, 0.09, 0.12); 0.40, 0.65, 0.60>	<(0.14, 0.26, 0.51); 0.40, 0.65, 0.60>	<(0.14, 0.26, 0.51); 0.40, 0.65, 0.60>	<(0.08, 0.22, 0.63); 0.40, 0.65, 0.60>
Criterion Weight	<(0.12, 0.21, 0.36); 0.40, 0.65, 0.60>	<(0.08, 0.12, 0.12); 0.40, 0.65, 0.60>	<(0.43, 0.67, 1.03); 0.40, 0.65, 0.60>	

Tab. 9 contains the global weight matrix, which is calculated similarly to the crisp case, where the algebra of crisp values is substituted by its equivalent neutrosophic one.

Now, let us calculate crisp global weights of projects applying Eq. 8 to elements in Tab. 9 and normalizing, they are 0.52658 for Project 1, 0.23797 for Project 2 and 0.23545 for Project 3.

Evidently, according to the obtained weights, the projects can be ranked in the following order, Project 1 ≻ Project 2 ≻ Project 3.

Additionally, in the third stage, if the decision-makers have to make the choice about what projects should be carried out, which satisfies some constraints, the precedent weights can be used as inputs in the optimization problem.

Suppose the manager counts on a total budget of $9000. In case of approval, $3000 must be spent in Project 1, $3500 in Project 2 and $5000 in Project 3. As well, the total possible number of man-hour is 1100 and it is known that Project 1 needs 1000, Project 2 needs 200 and Project 3 needs 700.

Then, none, one, two or all of the three projects can be selected, always that they satisfy the restrictions imposed on the problem. Our goal is to optimize this selection, i.e., the project or projects which can be simultaneously carried out have to be selected and then to maximize the benefits.

Formally, let us define three variables x_i, i = 1, 2, 3 as follows:

$$x_i = \begin{cases} 1 & \text{, if Project i is selected} \\ 0 & \text{, otherwise} \end{cases}$$

Let us divide the data by their upper bounds for calculating with dimensionless magnitudes. Hence, the mathematical problem is the following:

Max $f(\boldsymbol{x}) = w_1x_1 + w_2x_2 + w_3x_3$

Subject to:

$(3000/9000)x_1 + (3500/9000)x_2 + (5000/9000)x_3 \leq 1$ (Budget constraint)

$(1000/1100)x_1 + (200/1100)x_2 + (700/1100)x_3 \leq 1$ (Man-hour constraint)

$w_1 = 0.52658$, $w_2 = 0.23797$ and $w_3 = 0.23545$ are the previously calculated project weights.

This is a problem of zero-one linear programming. The best solution is **x** = (1, 0, 0), i.e., the best option is to only select Project 1.

Conclusion

To select appropriately an information technology project is generally a complex task and at the same time an unavoidable one because this kind of project is essential for many companies. One of the difficulties arisen by decision makers is the environmental uncertainty and limitations of the existent assessment systems. In this paper, the neutrosophy theory was chosen, which allows us to deal with uncertainty and imprecision for IT project selection. Analytic hierarchy process is the technique for making complex decisions. Then, the proposed model is based on a neutrosophic analytic hierarchy process. This technique was complemented with a balanced scorecard model for determining the IT selection criteria and zero-one linear programming to make the best feasible choice of projects. Finally, an example was used for illustrating the advantages that were obtained from integrating these four tools. It is necessary to emphasize that this model is unique to the set of information technology project selection models, as it was reviewed by the authors in the literature on that subject and it is particularly adjusted for solving the problem of IT project selection.

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Analyzing Age Group and Time of the Day Using Interval Valued Neutrosophic Sets

S. Broumi[1], M.Lathamaheswari[2], A. Bakali[3], M. Talea[1], F. Smarandache[4], D. Nagarajan[2], Kavikumar[5] and Guennoun Asmae[1]

[1]Laboratory of Information Processing, Faculty of Science Ben M'Sik, University Hassan II, B.P 7955, Sidi Othman, Casablanca, Morocco, E-mail: broumisaid78@gmail.com, E-mail: taleamohamed@yahoo.fr

[2]Department of Mathematics, Hindustan Institute of Technology & Science, Chennai-603 103, India, E-mail: mlatham@hindustanuniv.ac.in, E-mail: dnagarajan@hindustanuniv.ac.in

[3]Ecole Royale Navale, Boulevard Sour Jdid, B. P 16303 Casablanca, Morocco, E-mail: assiabakali@yahoo.fr,

[4]Department of Mathematics, University of New Mexico, 705 Gurley Avenue, Gallup, NM 87301, USA, E-mail: fsmarandache@gmail.com

[5]Department of Mathematics and Statistics, Faculty of Applied Science and Technology, Universiti Tun Hussein Onn, Malaysia, E-mail:kavi@uthm.edu.my

Abstract: Human psychological behavior is always uncertain in nature with the truth, indeterminacy and falsity of the information and hence neutrosophic logic is able to deal with this kind of real world problems as it resembles human's attitude very closely. In this paper, age group analysis and time (day or night) analysis have been carried out using interval valued neutrosophic sets. Further, the impact of the present work is presented.

Keywords: Neutrosophic Logic; Human Psychological Behavior; Age Group; Day; Interval Valued Neutrosophic Set.

1. Introduction

Uncertainty saturates our daily lives and period the entire range from index fluctuations of stock market to prediction of weather and car parking in a congested area to traffic control management. Hence almost all the area contains ambiguity or impression. For various real world problems, intelligent models with many types of mathematical designs of different logics have been modeled by the researchers. In the area of computational intelligence, fuzzy logic is one of the superior logic that provides appropriate representation of real world information and permits reasoning that are almost accurate in nature [1].

Generally the inputs conquered by the fuzzy logic are determinate and complete. Humans can able to take knowledgeable decisions in those situations, however it is difficult to express in proper terms. But fuzzy models need complete information. Due to basic non-linearity, huge erratic substantial disturbances, time varying nature, difficulties to find precise and predictable measurements, incompleteness and indeterminacy may arise in the data. All these problems can be dealt by neutrosophic logic proposed by Smarandache in the year 1999 [2-10]. Also this logic can able to represent mathematical structure of uncertainty, ambiguity, vagueness, imprecision, inconsistency, incompleteness and contradiction.

Also it is efficient in characterizing various attributes of data such as incompleteness and inaccuracy and hence gives proper estimation about the authenticity of the information. This approach proposes extending the proficiencies of representation of fuzzy logic and system of reasoning by introducing neutrosophic representation of the information and system of neutrosophic reasoning. Neutrosophic logic can exhibit various logical behaviors according to the nature of the problem to be solved and hence it influences its chance to be utilized and experimented for real world performance and simulations in human psychology [15].

Due to computational complexity of the neutrosophic sets, single valued neutrosophic sets have been introduced. It can deal with only exact numerical value of the three components truth, indeterminacy and falsity. While the data in the form of interval, then single valued neutrosophic sets unable to scope up and hence interval valued neutrosophic sets have been introduced. As it has lower and upper membership functions it can deal more uncertainty with less computational complexity than other types [25]. Neutrosophic set has been used in several areas like traffic control management, solving minimum spanning tree problem, analyzing failure modes and effect analysis, blockchain technology, resource leveling problem, medical diagnostic system, evaluating time-cost tradeoffs, analysis of criminal behavior, petal analysis, decision making problem etc. [26-40].

The major advantage of neutrosophic set and its types namely single valued neutrosophic sets and interval valued neutrosophic sets overrule other sets namely conventional set, fuzzy set, type-2 fuzzy, intuitionistic fuzzy and type-2 intuinistic fuzzy by their capability of dealing with indeterminacy which is missing with other types of sets. Since there is a possibility of having interval number than the exact number we consider interval valued neutrosophic set in this study of analyzing age group and time. Prediction of future trend is one of the interesting areas in the research field. Hence, in this paper, age group analysis and time (day or night) analysis have been done using interval valued neutrosophic sets. The remaining part of the paper is organized as follows. In section 2, review of literature is given. In section 3, preliminaries are given for better understanding of the paper. In section 4, age group and day and night time have been analyzed using the concept of interval valued neutrosophic sets. In section 5, impact of the present work is given. In section 6, concluded the present work with the future direction.

2. Review of Literature

The author in, [1] analyzed uncertainty exists in the project schedule using fuzzy logic. And the authors of, [2] analyzed power flow using fuzzy logic. [3] Examined specific seasonal prediction spatially under fuzzy environment for the group of long-term daily rainfall and temperature data spatiotemporally. [4] examined about the prediction of temperature flow of the atmosphere based on fuzzy knowledge–rule base for interior cities in India. [5] proposed a novel approach for intuitionistic fuzzy sets and its applications in the prediction area.

[6] proposed single-valued neutrosophic minimum spanning tree and its aggregation method. [7] proposed a new approach for the advisory of weather using fuzzy logic. [8] Proposed a method for prediction of weather under fuzzy neural network environment and Hierarchy particle swarm optimization algorithm. [9] Proposed various types of neutrosophic graphs and algebraic model and applied in the field of technology. [10] proposed single valued neutrosophic graphs (SVNGs).

[11] examined bipolar single valued neutrosophic graphs. [12] Proposed interval valued neutrosophic graphs. [13] proposed isolated SVNGs. [14] provided an introduction to the theory bipolar SVNG. [15] proposed the degree, size and order of SVNGs. 16] applied Dijkstra algorithm to solve shortest path problem under IVN environment. [17] solved minimum spanning tree problem under trapezoidal fuzzy neutrosophic environment.

[18] applied minimum spanning tree algorithm for shortest path (SP) problem using bipolar neutrosophic numbers. [19] proposed a novel matrix algorithm for solving MST for undirected interval value NG. [20] solved a spanning tree problem with neutrosophic edge weights. [21] proposed a new algorithm to solve MST problem with undirected NGs. [22] analyzed the role of SVNSs and rough sets with imperfect and incomplete information systems.

[23] Studied about neutrosophic set and its development . [24] studied about the prediction of long-term weather elements using adaptive neuro-fuzzy system using GIS approach in Jordan. [25] have done overview of neutrosophic sets. [26] proposed a methodology of traffic control management using triangular interval type-2 fuzzy sets and interval neutrosophic sets. [27] Solved MST problem using single valued trapezoidal neutrosophic numbers.

[28] estimated risk priority number in design failure modes and effect analysis using factor analysis. [29] have done edge detection on DICOM image using type-2 fuzzy logic. [30] made a review on the applications of type-2 fuzzy in the field of biomedicine. [31] have done image extraction on DICOM image usingtype-2 fuzzy. [32] made a review on application of type-2 fuzzy in control system. [33] proposed single and interval valued neutrosophic graphs using blockchain technology. [34] introduced interval valued neutrosophic graphs using Dombi triangular norms. [35] solved resource leveling problem under neutrosophic environment.

[36] introduced cosine similarity measures of bipolar neutrosophic sets and applied in diagnosis of disorder diseases. [37] introduced a methodology for petal analysis using neutrosophic cognitive maps. [38] analyzed criminal behavior using neutrosophic model. [39] presented assessments of linear time-cost tradeoffs using neutrosophic sets. [40] solved sustainable supply chain risk management problem using plithogenic TOPSIS-CRITIC methodology. In view of the literature, prediction of age group and day or night time under interval neutrosophic set are yet to be studied and which is the reason of the present study.

3. Preliminaries

In this section, preliminaries of the proposed concept are given

3. 1. Neutrosophic Set (NS) [25]

Consider the space X consists of universal elements characterized by e . The NS A is a phenomenon which has the structure $A=\left\{\left(T_A(e), I_A(e), F_A(e)\right)/ e \in X\right\}$ where the three grades of memberships are from X to]−0,1+[of the element $e \in X$ to the set A, with the criterion:

$$^{-}0 \le T_A(e) + I_A(e) + F_A(e) \le 3^{+} \quad (1)$$

The functions $T_A(e)$, $I_A(e)$ and $F_A(e)$ are the truth, indeterminate and falsity grades lies in real standard/non-standard subsets of] −0, 1+ [.
Since there is a complication of applying NSs to real issues, Samarandache and Wang et al. [11-12] proposed the notion of SVNS, which is a specimen of NS and it is useful for realistic applications of all the fields.

3.2. Single Valued Neutrosophic Set (SVNS) [25]

For the space X of objects contains global elements e. A SVNS is represented by degrees of bership grades mentioned in Def. 2.8. For all e in X, $T_A(e)$, $I_A(e)$, $F_A(e) \in [0, 1]$. A SVNS can be written as

$$A = \{\langle e : T_A(e), I_A(e), F_A(e)\rangle / e \in X\} \quad (2)$$

3.3. Interval Valued Neutrosophic Set [12]

Let X be a space of objects with generic elements in X denoted by e. An interval valued neutrosophic set (IVNS) A in X is characterized by truth-membership function, $T_A(e)$, indeterminacy-membership function $I_A(e)$ and falsity membership function $F_A(e)$. For each point e in X, $T_A(e)$, $I_A(e), F_A(e) \in [0,1]$, and an IVNS A is defined by

$$A = \{\langle [T_A^L(e), T_A^U(e)], [I_A^L(e), I_A^U(e)], [F_A^L(e), F_A^U(e)]\rangle \mid e \in X\} \quad (3)$$

Where, $T_A(e) = [T_A^L(e), T_A^U(e)]$, $I_A(e) = [I_A^L(e), I_A^U(e)]$ and $F_A(e) = [F_A^L(e), F_A^U(e)]$

Fig 1 shows the Pictorial Representation of the neutrosophic set [5]

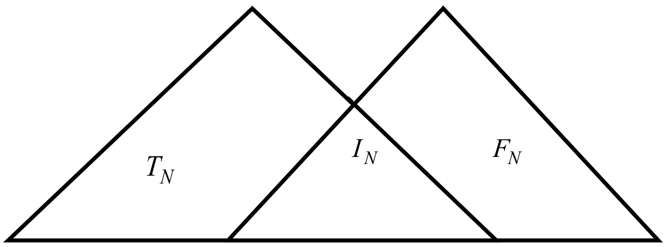

Fig.1. Neutrosophic set

4. Proposed Methodology

In this section, age group and time (day or night) have been analyzed using interval valued neutrosophic set.

4.1 Application of Interval Valued Neutrosophic Set in Age Group Analysis

As per our convenience, the age group is divided into three groups: young people, middle aged people and old people. Assume young people are a truth membership function, middle aged people are indeterminate membership function and old people are a falsity membership function. Here, the degree of middle aged people may provide either degree of old people or young people or both. Let us consider the age group is definitely young at and below 18-40, it is definitely old at and beyond 51-100 and in between the age group is middle. i.e., the level of the young age people decreases and the level of old age people increases. The age group is represented pictorially for young people, middle aged people and old people as in Fig. 2.

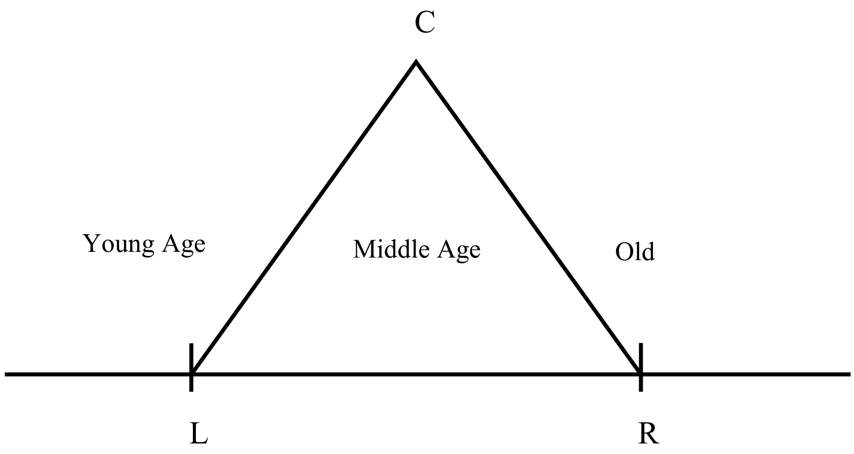

Fig.2. The degrees of 'young age', 'middle age' and 'old age' people.

Let A be the different age groups of the people and N be an interval valued neutrosophic set defined in the set A. Let $T_N(a)$ be the membership degree of the age group 'young age people' at a, here a denotes a numerical value. For example, $a = 20$. Similarly, indeterminate degree of 'middle age people' can be denoted by $I_N(a)$ and the falsity degree of 'old age people' denoted by $F_N(a)$ at a.

Consider $A = \{\langle [18,40],[41,50],[51,100] \rangle\}$ and

$$N = \{\langle T_N([18,40]), I_N([18,40]), F_N([18,40]) \rangle,$$

$$\langle T_N([41,50]), I_N([41,50]), F_N([41,50]) \rangle, \langle T_N([51,100]), I_N([51,100]), F_N([51,100]) \rangle\}$$

Case (i). At and below [18, 40], there is no middle age people and old age people but there exist only young age people. Therefore the following values are obtained.

$\left[T_N^L, T_N^U\right]([18,40]) = [1,1]$, $\left[I_N^L, I_N^U\right]([18,40]) = [0,0]$ and

$\left[F_N^L, F_N^U\right]([18,40]) = [0,0]$

i.e., the membership function of the interval valued neutrosophic set is $([1,1],[0,0],[0,0])$

Case (ii). At age [41, 50] (at the point C)

$\left[T_N^L, T_N^U\right]([41,50]) = [0,0]$, $\left[I_N^L, I_N^U\right]([41,50]) = [1,1]$ and

$\left[F_N^L, F_N^U\right]([41,50]) = [0,0]$

i.e., the membership function of the interval valued neutrosophic set is $([0,0],[1,1],[0,0])$

Case (iii). At and above [51,100], there are no young age people and middle age people, but there exist only old age people.

$\left[T_N^L, T_N^U\right]([51,100]) = [0,0]$, $\left[I_N^L, I_N^U\right]([51,100]) = [0,0]$ and $\left[F_N^L, F_N^U\right]([51,100]) = [1,1]$

i.e., the membership function of the interval valued neutrosophic set is $([0,0],[0,0],[1,1])$

Hence, $N = \left\{\langle[1,1],[0,0],[0,0]\rangle, \langle[0,0],[1,1],[0,0]\rangle, \langle[0,0],[0,0],[1,1]\rangle\right\}$

Also, young age people decreases and middle age people increases in between L and C.

i.e., $[1,1] > \left[T_N^L, T_N^U\right] > [0,0]$ and $[0,0] < \left[I_N^L, I_N^U\right] < [1,1]$

Further, middle age people decreases and old age people increases in between C and R.

i.e., $[1,1] > \left[I_N^L, I_N^U\right] > [0,0]$ and $[0,0] < \left[F_N^L, F_N^U\right] < [1,1]$

4.2 Application of Interval Valued Neutrosophic Set in Day and Night Time Analysis

As per our convenience, time of the day is divided into three groups: day, day or night (or both) and night. Assume day time is a truth membership function, day or night (or both) is an indeterminate membership function and night time is a falsity membership function. Here, the degree of day or night time may provide either degree of day time or night time or both. Let us consider the time of the day is definitely day time at and below 7 AM to 6 PM, it is definitely night at and beyond 7 PM and 5 AM and in between time is day or night. i.e., the level of the day time decreases and the level of night time increases. The time of the day is represented pictorially for day, day or night people and night as in Fig. 3.

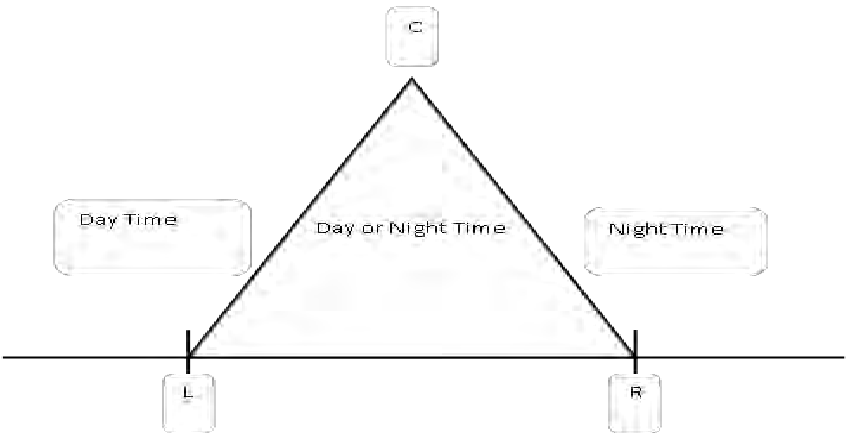

Fig.3. The degrees of time for 'day', 'day or night' and 'night'

Let B be the different times of the day, M an interval valued neutrosophic set defined in the set B. Let $T_M(b)$ be the membership degree of the time 'day' at b, here, b denotes a numerical value.

For example $b = 8$ AM or PM. Similarly, the indeterminate degree of the time $I_N(b)$ and the falsity degree of the time $F_M(b)$ can be represented by b.

Consider two cases.

$B = \{\langle [7AM, 6PM], [5AM, 6AM], [7PM, 5AM] \rangle\}$ and

$$M = \{\langle T_N([7AM,6PM]), I_N([7AM,6PM]), F_N([7AM,6PM]) \rangle,$$

$$\langle T_N([5AM,6AM]), I_N([5AM,6AM]), F_N([5AM,6AM]) \rangle,$$

$$\langle T_N([7PM,5AM]), I_N([7PM,5AM]), F_N([7PM,5AM]) \rangle\}.$$

Also we can consider, $B = \{\langle [7AM, 6PM], [6PM, 7PM], [7PM, 5AM] \rangle\}$ and

$$M = \{\langle T_N([7AM,6PM]), I_N([7AM,6PM]), F_N([7AM,6PM]) \rangle,$$

$$\langle T_N([6PM,7PM]), I_N([6PM,7PM]), F_N([6PM,7PM]) \rangle,$$

$$\langle T_N([7PM,5AM]), I_N([7PM,5AM]), F_N([7PM,5AM]) \rangle\}.$$

Case (i). At and below [7AM, 6 PM], there is no hesitation of day or night time and no night time but there exist only day time. Therefore the following values are obtained.

$[T_N^L, T_N^U]([7AM, 6PM]) = [1,1]$

$[I_N^L, I_N^U]([7AM, 6PM]) = [0,0]$ and

$\left[F_N^L, F_N^U\right]\left(\left[7AM, 6PM\right]\right) = \left[0,0\right]$

i.e., the membership function of the interval valued neutrosophic set is $\left(\left[1,1\right],\left[0,0\right],\left[0,0\right]\right)$

Case (ii). At [5AM, 6AM] (at the point C) and at [6 PM, 7PM]

$\left[T_N^L, T_N^U\right]\left(\left[5AM, 6AM\right]\right) = \left[0,0\right]$and $\left[T_N^L, T_N^U\right]\left(\left[6PM, 7PM\right]\right) = \left[0,0\right]$

$\left[I_N^L, I_N^U\right]\left(\left[5AM, 6AM\right]\right) = \left[1,1\right]$and $\left[I_N^L, I_N^U\right]\left(\left[6PM, 7PM\right]\right) = \left[1,1\right]$

$\left[F_N^L, F_N^U\right]\left(\left[5AM, 6AM\right]\right) = \left[0,0\right]$and $\left[F_N^L, F_N^U\right]\left(\left[6PM, 7PM\right]\right) = \left[0,0\right]$

i.e., the membership function of the interval valued neutrosophic set is $\left(\left[0,0\right],\left[1,1\right],\left[0,0\right]\right)$

Case (iii). At and above [7 PM, 5 PM], there is no day time and no hesitation of day or night time, but there exist only night time.

$\left[T_N^L, T_N^U\right]\left(\left[7PM, 5AM\right]\right) = \left[0,0\right]$

$\left[I_N^L, I_N^U\right]\left(\left[7PM, 5AM\right]\right) = \left[0,0\right]$and

$\left[F_N^L, F_N^U\right]\left(\left[7PM, 5AM\right]\right) = \left[1,1\right]$

i.e., the membership function of the interval valued neutrosophic set is $\left(\left[0,0\right],\left[0,0\right],\left[1,1\right]\right)$

Hence, $M = \left\{\left\langle\left[1,1\right],\left[0,0\right],\left[0,0\right]\right\rangle, \left\langle\left[0,0\right],\left[1,1\right],\left[0,0\right]\right\rangle, \left\langle\left[0,0\right],\left[0,0\right],\left[1,1\right]\right\rangle\right\}$

Also, day time decreases and day or night time increases in between L and C.

i.e., $\left[1,1\right] > \left[T_N^L, T_N^U\right] > \left[0,0\right]$ and $\left[0,0\right] < \left[I_N^L, I_N^U\right] < \left[1,1\right]$

Further, day or night time decreases and night time increases in between C and R.

i.e., $\left[1,1\right] > \left[I_N^L, I_N^U\right] > \left[0,0\right]$ and $\left[0,0\right] < \left[F_N^L, F_N^U\right] < \left[1,1\right]$

5. Impacts of the work

i). The proposed approach is the effective one in determining age group forecasting while the data is in the form of interval data with indeterminate information too.
ii). Time (day or night) analysis under interval neutrosophic environment will be very useful as it is the major scientific and technical problems.
iii). Analysing any future trend can be done easily by inferring the existing information into the future using interval neutrosophic sets as it has the capacity of addressing with the set of numbers in the real unit interval which is not just a determined number, it is efficient to deal with real world problems with various possible interval values

iv). The proposed methodology of age group analysis can be used in facial image analysis as age detection system.
v). The proposed methodology of time analysis can be utilized in time series analysis.

6. Conclusion

Since neutrosophic logic resembles human behavior for predicting age and time (day or night), it is suitable for this study. According to the knowledge of human, membership values of the truth, indeterminacy and falsity may be exact numbers or interval numbers. In this paper, analysis of age group and time(day or night) have been done using interval valued neutrosophic set with the detailed description and pictorial representation. Also the impact of the present work has been given. In future, the proposed concept can be done based on the concept of neutrosophic rough and soft sets.

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Air Pollution Model using Neutrosophic Cubic Einstein Averaging Operators

Majid Khan[1], Muhammad Gulistan[1], Nasruddin Hassan[2,]* and Abdul Muhaimin Nasruddin[3]

[1] Department of Mathematics and Statistics, Hazara University, Mansehra 21130 Pakistan; majid_swati@yahoo.com, gulistanmath@hu.edu.pk

[2] School of Mathematical Sciences, Faculty of Science and Technology, Universiti Kebangsaan Malaysia, Bangi 43600, Selangor Malaysia; nas@ukm.edu.my

[3] Department of Management and Marketing, Faculty of Economics and Management, Universiti Putra Malaysia, Serdang 43400, Selangor Malaysia; abdulmuhaimin085@gmail.com

* Correspondence: nas@ukm.edu.my; Tel.: (+60192145750)

Abstract: The neutrosophic cubic averaging and Einstein averaging aggregation operators are presented and applied to the air pollution model of the city of Peshawar, Pakistan. Neutrosophic cubic set (NCS) is a more generalized version of the neutrosophic set (NS) and an interval neutrosophic set (INS). It is in a better position to express consistent, indeterminant and incomplete information, thus it is able to be applied to aggregate the air pollution model. Aggregation operators have a key role in science and engineering problems. Firstly, the neutrosophic cubic weighted averaging (NCWA) operator, neutrosophic cubic ordered weighted averaging (NCOWA) operator, neutrosophic cubic hybrid aggregation (NCHA) operator, neutrosophic cubic Einstine weighted averaging (NCEWA) operator, neutrosophic cubic Einstine ordered weighted averaging (NCEOWA) operator and neutrosophic cubic Einstine hybrid aggregation (NCEHA) operator are defined. Secondly, these operators are applied to the air pollution model of particulate matter with the size of less than 10 micron (PM10) in Peshawar. Subsequently, the results are compared with the World Health Organization (WHO) standards using score/accuracy function. The pollution of PM10 is found to be very much higher than WHO standards. Hence, strong measures are required to control air pollution.

Keywords: Air pollution; neutrosophic cubic weighted averaging; neutrosophic cubic hybrid averaging; neutrosophic cubic Einstein weighted averaging; neutrosophic cubic Einstein hybrid averaging.

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1. Introduction

The uncertainty is a complex phenomenon that occurs in the real world. Since uncertainty is inevitably involved in problems, it occurs in different areas of life such that conventional methods have failed to cope with such problems. The big task is to deal with uncertain information. Many models have been introduced to incorporate uncertainty into the description of the system. The fuzzy set was initiated by Zadeh [1]. Henceforth, it is applied in different fields of sciences like artificial intelligence, information sciences, medical sciences, decision making theory and much more. Due to its applicability in sciences and daily life problem, fuzzy set has been extended into interval valued fuzzy sets (IVFS) [2,3], intuitionistic fuzzy set (IFS) [4], interval valued fuzzy set (IVIFS) [5] and cubic set [6] among others, besides Q-fuzzy [7-11] and vague soft set [12]. IFS consists of two components, membership and non-membership whereas the hesitant component is considered under the condition that sum of these components is one. Smarandache presented the idea of neutrosophic sets (NS) [13], which provides a more general form to extend the ideas of classic theory and fuzzy set theory. The NS expresses three components namely truth, indeterminacy and falsity and all these components are independent, which makes NS more general than IFS such that NS can be seen as a generalization of IFS [14]. For sciences and engineering problems Wang *et al.* [15] presented a single valued neutrosophic set (SVNS), while Wang *et al*. [16] introduced the interval neutrosophic set (INS). Jun *et al.* [17] combined INS and NS to form a neutrosophic cubic set (NCS) which enables us to choose both interval value and single value membership, indeterminacy and falsehood components, hence presenting a more general form for uncertain and vague data.

Aggregation operators are an imperative part of decision making. The lack of data or knowledge makes it difficult for decision maker to give the exact decision. This uncertain situation can be minimized due to the vague nature of NS and its extensions. Researchers [18-27] introduced different aggregation operators and multicriteria decision making methods in NS and INS. Khan *et al.* [28] presented neutrosophic cubic Einstein geometric aggregation operators. Zhan *et al.* [29] worked on multi criteria decision making on neutrosophic cubic sets. Banerjee *et al.* [30] used grey rational analysis (GRA) techniques to neutrosophic cubic sets. Lu and Ye [31] defined a cosine measure to neutrosophic cubic set. Pramanik *et al.* [32] used similarity measure to neutrosophic cubic set. Shi and Ji [33] defined Dombi aggregation operators on neutrosophic cubic sets. Ye [34] defined aggregation operators over the neutrosophic cubic numbers. Alhazaymeh *et al*. [35] presented a hybrid geometric aggregation operator with application to multiple attribute decision making method on neutrosophic cubic sets.

According to WHO, air pollution causes millions of premature deaths every year globally. 90% of these deaths are caused by air pollution in middle and low income countries, mainly in Africa and Asia. Indeed it is a great threat to the environment. Inhaling polluted air may cause different types of diseases like lung cancer, respiratory diseases etc. In the last few years Pakistan witnessed a significant increased in cancer, asthma and chronic lung disease. The particulate matter (PM) is one of the major factors that cause such types of diseases. The data extracted from Alam *et al.* [36] consists of particulate matter with the size of less than 10 micron (PM10) in Peshawar, Pakistan.

The collection of accurate data has always been a tough job which may cause some uncertain results. That is why the need was felt to analyze the data using vague set. The neutrosophic cubic set is one of the better choices to deal with vague and inconsistent data. For this purpose, firstly the neutrosophic cubic averaging and Einstein averaging operators are defined. Then these operators are used to analyze the air pollution of PM10 model for the city of Peshawar, Pakistan with WHO standards. In this paper, the NCWA, NCOWA, NCHA, NCEWA, NCEOWA and NCEHA are defined. Both algebraic and Einstein operators are applied to an air pollution model [36] and compared. The goal of this work is to analyze the PM10 in the city of Peshawar and compare it with WHO standards.

The methodology to measure the aggregate value of neutrosophic cubic values is as follows. Firstly, the data is extracted from [36] and converted to neutrosophic cubic values so that the aggregated value can be measured. Secondly, the data is analyzed using the WHO standard. It is to be noted that the neutrosophic cubic set is the combination of both interval neutrosophic and neutrosophic set, which enable us to deal with both interval-valued neutrosophic and neutrosophic set at the same time.

This paper is structured as follows. In section 2, some preliminaries are reviewed. In section 3, the air pollution data is formed. In section 4, neutrosophic cubic averaging operators are defined and applied over the data of section 3. In section 5, neutrosophic cubic Einstein averaging operators are defined and applied over the data of section 3. The analyzing results are concluded by both numerically and graphically. We hope to expand the study further to numerical analysis [37-42], construction management [43-45], Q-neutrosophic soft environment [46], geometric programming [47] and binomial factorial problem [48].

2. Preliminaries

This section consists of some definitions and results which provide the foundation of the work.

Definition 2.1 [13] A structure $N=\left\{\left(T_N(u), I_N(u), F_N(u)\right) \mid u \in U\right\}$ is neutrosophic set (NS), where $\left\{T_N(u), I_N(u), F_N(u) \in \left]0^-, 1^+\right[\right\}$ *and* $T_N(u), I_N(u), F_N(u)$ are truth, indeterminancy and falsity function respectively.

Definition 2.2 [15] A structure $N=\left\{\left(T_N(u), I_N(u), F_N(u)\right) \mid u \in U\right\}$ is single value neutrosophic set (SVNS), where $\left\{T_N(u), I_N(u), F_N(u) \in [0,1]\right\}$ respectively called truth, indeterminancy and falsity functions.,simply denoted by $N=\left(T_N, I_N, F_N\right)$.

Definition 2.3 [16] An interval neutrosophic set (INS) in U is a structure

$$N=\left\{\left(\tilde{T}_N(u), \tilde{I}_N(u), \tilde{F}_N(u)\right) \mid u \in U\right\} \text{ where}$$

$\left\{\tilde{T}_N(u), \tilde{I}_N(u), \tilde{F}_N(u) \in D[0,1]\right\}$ are respectively called truth, indeterminacy an falsity function in U.

Simply denoted by $N=\left(\tilde{T}_N, \tilde{I}_N, \tilde{F}_N\right)$ for convenience being actually

$$N=\left(\tilde{T}_N=\left[T_N^L, T_N^U\right], \tilde{I}N=\left[I_N^L, I_N^U\right], \tilde{F}_N=\left[F_N^L, F_N^U\right]\right).$$

Definition 2.4 [17] A structure $N=\left\{\left(u, \tilde{T}_N(u), \tilde{I}_N(u), \tilde{F}_N(u), T_N(u), I_N(u), F_N(u)\right) \mid u \in U\right\}$ is neutrosophic cubic set (NCS) in U in which $\left(\tilde{T}_N=\left[T_N^L, T_N^U\right], \tilde{I}_N=\left[I_N^L, I_N^U\right], \tilde{F}_N=\left[F_N^L, F_N^U\right]\right)$ is an interval neutrosophic set and $\left(T_N, I_N, F_N\right)$ is neutrosophic set in U, where

$$N=\left(\tilde{T}_N, \tilde{I}_N, \tilde{F}_N, T_N, I_N, F_N\right),\ [0,0] \le \tilde{T}_N+\tilde{I}_N+\tilde{F}_N \le [3,3] \text{ and } 0 \le T_N+I_N+F_N \le 3$$

such that N^U denotes the collection of neutrosophic cubic sets in U.

Definition 2.5 [22] The t-operators are basically union and intersection in the fuzzy sets which are denoted by t-conorm $\left(\Gamma^*\right)$ and t-norm (Γ). The role of t-operators is very important in fuzzy theory and its applications.

Definition 2.6 [22] $\Gamma^{*}:[0,1]\times[0,1]\to[0,1]$ is t-conorm if the following axioms hold.

Axiom 1 $\Gamma^{*}(1,u)=1$ and $\Gamma^{*}(0,u)=0$

Axiom 2 $\Gamma^{*}(u,v)=\Gamma^{*}(v,u)$ for all *u* and *v*.

Axiom 3 $\Gamma^{*}(u,\Gamma^{*}(v,w))=\Gamma^{*}(\Gamma^{*}(u,v),w)$ for all *u*, *v* and *w*.

Axiom 4 If $u\le u'$ and $v\le v'$, then $\Gamma^{*}(u,v)\le\Gamma^{*}(u',v')$

Definition 2.7 [22] $\Gamma:[0,1]\times[0,1]\to[0,1]$ is t-norm if it the following axioms hold.

Axiom 1 $\Gamma(1,u)=u$ and $\Gamma(0,u)=0$

Axiom 2 $\Gamma(u,v)=\Gamma(v,u)$ for all *u* and *v*.

Axiom 3 $\Gamma(u,\Gamma(v,w))=\Gamma(\Gamma(u,v),w)$ for all *u*, *v* and *w*.

Axiom 4 If $u\le u'$ and $v\le v'$, then $\Gamma(u,v)\le\Gamma(u',v')$

The t-conorms and t-norms families have a vast range, which correspond to unions and intersections, among these Einstein sum and Einstein product are good choices since they give the smooth approximation like algebraic sum and algebraic product, respectively. Einstein sums $\oplus_E$ and Einstein products $\otimes_E$ are the examples of t-conorm and t-norm respectively:

$$\Gamma_E^{*}(u,v)=\frac{u+v}{1+uv},\quad \Gamma_E(u,v)=\frac{uv}{1+(1-u)(1-v)}$$

Definition 2.8 [28] The sum of two neutrosophic cubic sets (NCS),

$$A=(\tilde{T}_A,\tilde{I}_A,\tilde{F}_A,T_A,I_A,F_A) \text{ and } B=(\tilde{T}_B,\tilde{I}_B,\tilde{F}_B,T_B,I_B,F_B), \text{where}$$

$$\tilde{T}_A=[T_A^L,T_A^U],\tilde{I}_A=[I_A^L,I_A^U],\tilde{F}_A=[F_A^L,F_A^U] \text{ and } \tilde{T}_B=[T_B^L,T_B^U],\tilde{I}_B=[I_B^L,I_B^U],\tilde{F}_B=[F_B^L,F_B^U]$$

is defined as

$$A\oplus B=\left([T_A^L+T_B^L-T_A^LT_B^L,T_A^U+T_B^U-T_A^UT_B^U],[I_A^L+I_B^L-I_A^LI_B^L,I_A^U+I_B^U-I_A^UI_B^U],[F_A^LF_B^L,F_A^UF_B^U],T_AT_B,I_AI_B,F_A+F_B-F_AF_B\right).$$

Definition 2.9 [28] The product between two neutrosophic cubic sets (NCS),

$$A=(\tilde{T}_A,\tilde{I}_A,\tilde{F}_A,T_A,I_A,F_A) \text{ and } B=(\tilde{T}_B,\tilde{I}_B,\tilde{F}_B,T_B,I_B,F_B), \text{ where}$$

$$\tilde{T}_A=[T_A^L,T_A^U],\tilde{I}_A=[I_A^L,I_A^U],\tilde{F}_A=[F_A^L,F_A^U] \text{ and } \tilde{T}_B=[T_B^L,T_B^U],\tilde{I}_B=[I_B^L,I_B^U],\tilde{F}_B=[F_B^L,F_B^U]$$

is defined as

$$A\otimes B=\left([T_A^LT_B^L,T_A^UT_B^U],[I_A^LI_B^L,I_A^UI_B^U],[F_A^L+F_B^L-F_A^LF_B^L,F_A^U+F_B^U-F_A^UF_B^U],T_A+T_B-T_AT_B,I_A+I_B-I_AI_B,F_AF_B\right)$$

Definition 2.10 [28] The scalar multiplication on a neutrosophic cubic set (NCS)

$$A=(\tilde{T}_A,\tilde{I}_A,\tilde{F}_A,T_A,I_A,F_A) \text{ and a scalar } k \text{ where}$$

$$\tilde{T}_A=[T_A^L,T_A^U],\tilde{I}_A=[I_A^L,I_A^U],\tilde{F}_A=[F_A^L,F_A^U]$$

is defined as

$$kA=\left(\left[1-(1-T_A^L)^k,1-(1-T_A^U)^k\right],\left[1-(1-I_A^L)^k,1-(1-I_A^U)^k\right],\left[\left(F_A^L\right)^k,\left(F_A^U\right)^k\right],\left(T_A\right)^k,\left(I_A\right)^k,1-\left(1-F_A\right)^k\right)$$

Definition 2.11 [18] The Einstein sum between two neutrosophic cubic sets (NCS),

$A=\left(\tilde{T}_A,\tilde{I}_A,\tilde{F}_A,T_A,I_A,F_A\right)$ and $B=\left(\tilde{T}_B,\tilde{I}_B,\tilde{F}_B,T_B,I_B,F_B\right)$, where

$\tilde{T}_A=\left[T_A^L,T_A^U\right],\tilde{I}_A=\left[I_A^L,I_A^U\right],\tilde{F}_A=\left[F_A^L,F_A^U\right]$ and $\tilde{T}_B=\left[T_B^L,T_B^U\right],\tilde{I}_B=\left[I_B^L,I_B^U\right],\tilde{F}_B=\left[F_B^L,F_B^U\right]$

is defined as

$$A\oplus_E B=\left(\left[\frac{T_A^L+T_B^L}{1+T_A^LT_B^L},\frac{T_A^U+T_B^U}{1+T_A^UT_B^U}\right],\left[\frac{I_A^L+I_B^L}{1+I_A^LI_B^L},\frac{I_A^U+I_B^U}{1+I_A^UI_B^U}\right],\left[\frac{F_A^LF_B^L}{1+(1-F_A^L)\left(1-F_B^L\right)},\frac{F_A^UF_B^U}{1+(1-F_A^U)\left(1-F_B^U\right)}\right],\frac{T_AT_B}{1+(1-T_A)(1-T_B)},\frac{I_AI_B}{1+(1-I_A)(1-I_B)},\frac{F_A+F_B}{1+F_AF_B}\right)$$

Definition 2.12 [28] The Einstein product between two neutrosophic cubic sets (NCS),

$A=\left(\tilde{T}_A,\tilde{I}_A,\tilde{F}_A,T_A,I_A,F_A\right)$ and $B=\left(\tilde{T}_B,\tilde{I}_B,\tilde{F}_B,T_B,I_B,F_B\right)$, where

$\tilde{T}_A=\left[T_A^L,T_A^U\right],\tilde{I}_A=\left[I_A^L,I_A^U\right],\tilde{F}_A=\left[F_A^L,F_A^U\right]$ and $\tilde{T}_B=\left[T_B^L,T_B^U\right],\tilde{I}_B=\left[I_B^L,I_B^U\right],\tilde{F}_B=\left[F_B^L,F_B^U\right]$

is defined as

$$A\otimes_E B=\left(\left[\frac{T_A^LT_B^L}{1+(1-T_A^L)\left(1-T_B^L\right)},\frac{T_A^UT_B^U}{1+(1-T_A^U)\left(1-T_B^U\right)}\right],\left[\frac{I_A^LI_B^L}{1+(1-I_A^L)\left(1-I_B^L\right)},\frac{I_A^UI_B^U}{1+(1-I_A^U)\left(1-I_B^U\right)}\right],\left[\frac{F_A^L+F_B^L}{1+F_A^LF_B^L},\frac{F_A^U+F_B^U}{1+F_A^UF_B^U}\right],\frac{T_A+T_B}{1+T_AT_B},\frac{I_A+I_B}{1+I_AI_B},\frac{F_AF_B}{1+(1-F_A)(1-F_B)}\right)$$

Definition 2.13 [28] The Einstein scalar multiplication on a neutrosophic cubic set (NCS),

$A=\left(\tilde{T}_A,\tilde{I}_A,\tilde{F}_A,T_A,I_A,F_A\right)$, and a scalar k where

$\tilde{T}_A=\left[T_A^L,T_A^U\right],\tilde{I}_A=\left[I_A^L,I_A^U\right],\tilde{F}_A=\left[F_A^L,F_A^U\right]$

is defined as

$$k_EA=$$

$$\left(\left[\frac{(1+T_A^L)^k-(1-T_A^L)^k}{(1+T_A^L)^k+(1-T_A^L)^k},\frac{(1+T_A^U)^k-(1-T_A^U)^k}{(1+T_A^U)^k+(1-T_A^U)^k}\right],\left[\frac{(1+I_A^L)^k-(1-I_A^L)^k}{(1+I_A^L)^k+(1-I_A^L)^k},\frac{(1+I_A^U)^k-(1-I_A^U)^k}{(1+I_A^U)^k+(1-I_A^U)^k}\right],\left[\frac{2\left(F_A^L\right)^k}{\left(2-F_A^L\right)^k+\left(F_A^L\right)^k},\frac{2\left(F_A^U\right)^k}{\left(2-F_A^U\right)^k+\left(F_A^U\right)^k}\right],\frac{2\left(T_A\right)^k}{\left(2-T_A\right)^k+\left(T_A\right)^k},\frac{2\left(I_A\right)^k}{\left(2-I_A\right)^k+\left(I_A\right)^k},\frac{(1+F_A)^k-(1-F_A)^k}{(1+F_A)^k+(1-F_A)^k}\right)$$

Definition 2.14 [28] Let $N=\left(\tilde{T}_N,\tilde{I}_N,\tilde{F}_N,T_N,I_N,F_N\right)$, where $\tilde{T}_N=\left[T_N^L,T_N^U\right],\tilde{I}_N=\left[I_N^L,I_N^U\right],\tilde{F}_N=\left[F_N^L,F_N^U\right]$ be a neutrosophic cubic value. The score function is defined as

$$Scr\left(N\right)=\left[T_N^L-F_N^L+T_N^U-F_N^U+T_N-F_N\right] \quad (1)$$

If the score function of two values are equal, the accuracy function is used to compare the neutrosophic cubic values.

Definition 2.15 [28] Let $N=\left(\tilde{T}_N,\tilde{I}_N,\tilde{F}_N,T_N,I_N,F_N\right)$, where $\tilde{T}_N=\left[T_N^L,T_N^U\right],\tilde{I}_N=\left[I_N^L,I_N^U\right],\tilde{F}_N=\left[F_N^L,F_N^U\right]$ be a neutrosophic cubic vlaue. The accuracy function is defined as

$$Acu(u) = \frac{1}{9}\left\{T_N^L + I_N^L + F_N^L + T_N^U + I_N^U + F_N^U + T_N + I_N + F_N\right\} \quad (2)$$

The following definition describes the comparison relation between two neutrosophic cubic values.

Definition 2.16 [28] Let N_1, N_2 be two neutrosophic cubic values, with score functions $Scr(N_1), Scr(N_2)$ and accuracy functions $Acu(N_1), Acu(N_2)$. Then

1). $Scr(N_1) > Scr(N_2) \Rightarrow N_1 > N_2$

2). If $Scr(N_1) = Scr(N_2)$, then

(i). $Acu(N_1) > Acu(N_2) \Rightarrow N_1 > N_2$, (ii). $Acu(N_1) = Acu(N_2) \Rightarrow N_1 = N_2$

3. Model Formulation of Air Pollution

Air pollution is a great threat to the environment. It causes different diseases to the human being. Inhaling polluted air may cause different types of disease like lung cancer and other respiratory diseases. According to WHO, air pollution causes 7 million premature deaths globally in 2016. Ambient air pollution alone caused 4.2 million deaths, while the atmospheric contamination of households from the kitchen with fuels and contaminating technologies led to an estimated 3.8 million deaths in the same year. More than 90% of deaths related to air pollution occur in middle- and low-income countries, mainly in Africa and Asia. In the last few years Pakistan witnessed a significant increase in cancer, asthma and chronic lung diseases. The particulate matter (PM) cause such type of diseases. The PM size is categorized as PM25, PM10 and PM2.5. The recommendation of the WHO for air quality call the countries to reduce their annual air pollution to the annual mean value of 20ug/m^3 for PM10. In this model, the data for PM10 was considered.

The collection of data is a hard task to do since most of the time we are unable to collect the correct and appropriate data. The problems may arise due to unskilled data collectors, inappropriate methods of collecting data and others. These obstacles can be minimized by using neutrosophic cubic sets which provide a vast variety to choose and decide. In this paper, a problem regarding PM10 in Peshawar, Pakistan is considered and their values aggregated using a neutrosophic cubic environment. Data is taken from [36] and converted to neutrosophic cubic form. To consider overall values, data aggregation operators are being proposed so that its value can be compared with WHO standards. According to WHO recommendation, the neutrosophic cubic value for PM10 is calculated as

$$N_{WHO} = ([0.15, 0.30], [0.10, 0.30], [0.70, 0.85], 0.20, 0.40, 0.75) \quad (3)$$

The neutrosophic cubic data for 1st , 5th, 10th, 15th and 20th April 2014 are respectively shown as follows.

$$N_A = ([0.82, 0.92], [0.39, 0.66], [0.18, 0.38], 0.88, 0.7, 0.42),\ N_B = ([0.59, 0.78], [0.68, 0.73], [0.22, 0.41], 0.68, 0.78, 0.32),$$

$$N_C = ([0.86, 0.96], [0.8, 0.85], [0.24, 0.36], 0.17, 0.8, 0.4),\ N_D = ([0.8, 0.93], [0.11, 0.41], [0.5, 0.9], 0.9, 0.5, 0.4)$$

and $N_E = \left([0.61, 0.79], [0.41, 0.53], [0.39, 0.56], 0.45, 0.30, 0.45\right)$.

The accuracy function is used to rank the air pollution in their relevant dates, in which day the air is most polluted by PM10.

$$Acu(N_A) = 0.5944, Acu(N_B) = 0.5766, Acu(N_C) = 0.6044, Acu(N_D) = 0.6055, \text{ and } Acu(N_E) = 0.4988.$$

We observe that

$$Acu\left(N_D\right) > Acu\left(N_C\right) > Acu\left(N_A\right) > Acu\left(N_B\right) > Acu\left(N_E\right).$$

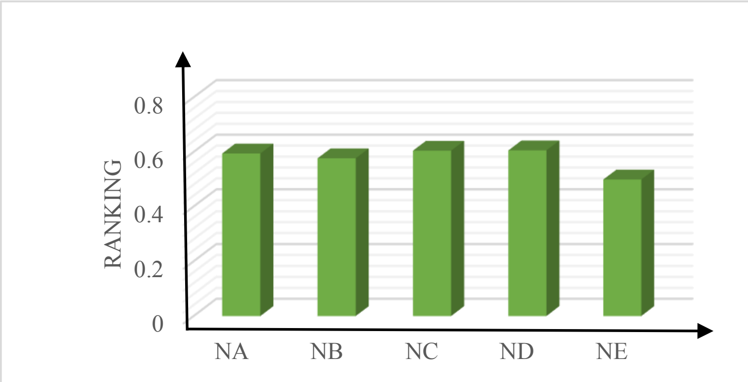


Figure 1. Pollution Graph in Peshawar City in April 2014

The graphical analysis can be seen in Figure 1. To analyze the overall pollution of PM10, the aggregation operators are needed. To fulfill this desire, the notion of neutrosophic cubic aggregation operators and neutrosophic cubic Einstein aggregation operators are proposed.

4. Neutrosophic Cubic Weighted Averaging Aggregation Operator

This section consist of some fundamental definitions of neutrosophic cubic weighted averaging (NCWA), neutrosophic cubic ordered weighted averaging (NCOWA) and neutrosophic cubic Einstein hybrid avregaing (NCEHA) aggregation operator, which are defined as follows.

Definition 4.1 The neutrosophic cubic weighted averaging is a function, $NCWA : R^n \to R$ defined by

$$NCWA_w(N_1, N_2, ..., N_n) = \sum_{k=1}^{n} w_k N_k \text{, where} \tag{4}$$

$W = (w_1, w_2, ..., w_n)^T$ of $N_k (k = 1, 2, 3, ..., n)$, be the weight such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

Note that in NCWA, the neutrosophic values are weighted first and then aggregated.

Definition 4.2 The neutrosophic cubic ordered weighted averaging is a function, $NCOWA : R^n \to R$ defined by

$$NCOWA_w(N_1, N_2, ..., N_n) = \sum_{k=1}^{n} w_k S_k \text{, where} \tag{5}$$

S_k denotes the ordered position of neutrosophic cubic (NC) values whereby the NC values are ordered in descending order, $W = (w_1, w_2, ..., w_n)^T$ of $N_k (k = 1, 2, 3, ..., n)$, be the weight such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

Note that in NCOWA, the neutrosophic cubic values are first ordered and then aggregated. The basic concept of NCOWA is to rearrange the neutrosophic cubic values in descending order and then aggregate them.

Theorem 4.3 Let $N_k = \left(\tilde{T}_{N_k}, \tilde{I}_{N_k}, \tilde{F}_{N_k}, T_{N_k}, I_{N_k}, F_{N_k}\right)$, where $\tilde{T}_{N_k} = \left[T^L_{N_k}, T^U_{N_k}\right], \tilde{I}_{N_k} = \left[I^L_{N_k}, I^U_{N_k}\right], \tilde{F}_{N_k} = \left[F^L_{N_k}, F^U_{N_k}\right]$ $(k = 1, 2, ..., n)$ be a collection of neutrosophic cubic values, then the neutrosophic cubic weighted average operator (NCWA) operator of N_k is also a neutrosophic cubic value and

$$NCWA(N_k) = \left(\left[1-\prod_{k=1}^{n}(1-T^L_{N_k})^{w_k}, 1-\prod_{k=1}^{n}(1-T^U_{N_k})^{w_k}\right], \left[1-\prod_{k=1}^{n}(1-I^L_{N_k})^{w_k}, 1-\prod_{k=1}^{n}(1-I^U_{N_k})^{w_k}\right], \left[\prod_{k=1}^{n}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{n}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{n}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{n}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{n}\left(1-\left(F_{N_k}\right)\right)^{w_k}\right)$$

where $W = (w_1, w_2, ..., w_n)^T$ of $N_k (k = 1, 2, 3, ..., n)$, be the weight such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

Proof: By mathematical induction for $n = 2$,

$$w_1N_1 \oplus w_2N_2 = \left(\left[1-\prod_{k=1}^{2}(1-T^L_{N_k})^{w_k}, 1-\prod_{k=1}^{2}(1-T^U_{N_k})^{w_k}\right], \left[1-\prod_{k=1}^{2}(1-I^L_{N_k})^{w_k}, 1-\prod_{k=1}^{2}(1-I^U_{N_k})^{w_k}\right], \left[\prod_{k=1}^{2}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{2}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{2}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{2}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{2}\left(1-\left(F_{N_k}\right)\right)^{w_k}\right)$$

Assume that, the result holds for $n = m$. That is

$$\sum_{k=1}^{m} w_kN_k = \left(\left[1-\prod_{k=1}^{m}(1-T^L_{N_k})^{w_k}, 1-\prod_{k=1}^{m}(1-T^U_{N_k})^{w_k}\right], \left[1-\prod_{k=1}^{m}(1-I^L_{N_k})^{w_k}, 1-\prod_{k=1}^{m}(1-I^U_{N_k})^{w_k}\right], \left[\prod_{k=1}^{m}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{m}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m}\left(1-\left(F_{N_k}\right)\right)^{w_k}\right)$$

Consider $n = m + 1$, the following result will be proven.

$$\sum_{k=1}^{m} w_kN_k \oplus w_{k+1}N_{k+1} = \left\{\begin{array}{l} \left(\left[1-\prod_{k=1}^{m+1}(1-T^L_{N_k})^{w_k}, 1-\prod_{k=1}^{m}(1-T^U_{N_k})^{w_k}\right], \left[1-\prod_{k=1}^{m+1}(1-I^L_{N_k})^{w_k}, 1-\prod_{k=1}^{m+1}(1-I^U_{N_k})^{w_k}\right], \left[\prod_{k=1}^{m+1}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{m}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m}\left(1-\left(F_{N_k}\right)\right)^{w_k}\right) \\ \oplus\left(\left[1-(1-T^L_{N_{k+1}})^{w_{k+1}}, 1-(1-T^U_{N_{k+1}})^{w_{k+1}}\right], \left[1-(1-I^L_{N_{k+1}})^{w_{k+1}}, 1-(1-I^U_{N_{k+1}})^{w_{k+1}}\right], \left[\left(F^L_{N_{k+1}}\right)^{w_{k+1}}, \left(F^U_{N_{k+1}}\right)^{w_{k+1}}\right], \left(T_{N_{k+1}}\right)^{w_{k+1}}, \left(I_{N_{k+1}}\right)^{w_{k+1}}, 1-\left(1-\left(F_{N_{k+1}}\right)\right)^{w_{k+1}}\right) \end{array}\right\}$$

$$= \left(\left[1-\prod_{k=1}^{m+1}\left(1-T^L_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m+1}\left(1-T^U_{N_k}\right)^{w_k}\right], \left[1-\prod_{k=1}^{m+1}\left(1-I^L_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m+1}\left(1-I^U_{N_k}\right)^{w_k}\right], \left[\prod_{k=1}^{m+1}\left(F^L_{N_k}\right)^{w_k}, \prod_{k=1}^{m+1}\left(F^U_{N_k}\right)^{w_k}\right], \prod_{k=1}^{m+1}\left(T_{N_k}\right)^{w_k}, \prod_{k=1}^{m+1}\left(I_{N_k}\right)^{w_k}, 1-\prod_{k=1}^{m+1}\left(1-F_{N_k}\right)^{w_k}\right)$$

Hence proved.

Example 4.4 The NCWA operator is applied on the data as stated in section 3 with corresponding weight, $w=(0.21,0.14,0.25,0.29,0.11)^T$. This weight calculated by Xu and Yager [27] is an essential part of aggregation operators and will be used throughout this paper.

The value of $NCWA=\left(\left[0.7871,0.9171\right],\left[0.5311,0.7292\right],\left[0.2912,0.5075\right],0.5216,0.6071,0.3995\right)$.

Theorem 4.5 Let $N_k=\left(\tilde{T}_{N_k},\tilde{I}_{N_k},\tilde{F}_{N_k},T_{N_k},I_{N_k},F_{N_k}\right)$, where $\tilde{T}_{N_k}=\left[T^L_{N_k},T^U_{N_k}\right],\tilde{I}_{N_k}=\left[I^L_{N_k},I^U_{N_k}\right],\tilde{F}_{N_k}=\left[F^L_{N_k},F^U_{N_k}\right]$, $(k=1,2,...,n)$ be collection of neutrosophic cubic values with weight $W=(w_1,w_2,...,w_n)^T$ of $N_k\,(k=1,2,3,...,n)$, such that $w_k\in[0,1]$ and $\sum_{k=1}^{n}w_k=1$. The following properties are true.

1. Idempotence: If for all $N_k=\left(\tilde{T}_{N_k},\tilde{I}_{N_k},\tilde{F}_{N_k},T_{N_k},I_{N_k},F_{N_k}\right)$, *where* $\tilde{T}_{N_k}=\left[T^L_{N_k},T^U_{N_k}\right],\tilde{I}_{N_k}=\left[I^L_{N_k},I^U_{N_k}\right],\tilde{F}_{N_k}=\left[F^L_{N_k},F^U_{N_k}\right]$

   $(k=1,2,...,n)$ are equal, i.e. $N_k=N$ for all k, then $\text{NCWA}_w(N_1,N_2,...,N_n)=N$

2. Monotonicity: Let $B_k=\left(\tilde{T}_{B_k},\tilde{I}_{B_k},\tilde{F}_{B_k},T_{B_k},I_{B_k},F_{B_k}\right)$, *where* $\tilde{T}_{B_k}=\left[T^L_{B_k},T^U_{B_k}\right],\tilde{I}_{B_k}=\left[I^L_{B_k},I^U_{B_k}\right],\tilde{F}_{B_k}=\left[F^L_{B_k},F^U_{B_k}\right]$

   be the collection of neutrosophic cubic values. If $S_B(u)\geq S_N(u)$ *and* $B_k(u)\geq N_k(u)$, where $u\in U$, then

$$\text{NCWA}_w(N_1,N_2,...,N_n)\leq\text{NCWA}_w(B_1,B_2,...,B_n)$$

3. Boundary: $N^-\leq NCWA_w\left\{(N_1),(N_2),...,(N_n)\right\}\leq N^+$, where

$$N^-=\left\{\min_k T^L_{N_k},\min_k I^L_{N_k},1-\max_k F^L_{N_k},\min_k T_{N_k},\min_k I_{N_k},1-\max_k F^L_{N_k},\min_k T_{N_k},\min_k I_{N_k},1-\max_k F_{N_k}\right\}$$

$$N^+=\left\{\max_k T^L_{N_k},\max_k I^L_{N_k},1-\min_k F^L_{N_k},\max_k T_{N_k},\max_k I_{N_k},1-\min_k F^L_{N_k},\max_k T_{N_k},\max_k I_{N_k},1-\min_k F_{N_k}\right\}$$

Proof.

1. Idempotence: Since $N_k=N$ so

$$NCWA\left(N_k\right)=NCWA(N)$$

$$\left(\left[1-\prod_{k=1}^{n}(1-T^L_N)^{w_k},1-\prod_{k=1}^{n}(1-T^U_N)^{w_k}\right],\left[1-\prod_{k=1}^{n}(1-I^L_N)^{w_k},1-\prod_{k=1}^{n}(1-I^U_N)^{w_k}\right],\left[\prod_{k=1}^{n}\left(F^L_N\right)^{w_k},\prod_{k=1}^{n}\left(F^U_N\right)^{w_k}\right],\prod_{k=1}^{n}(T_N)^{w_k},\prod_{k=1}^{n}(I_N)^{w_k},1-\prod_{k=1}^{n}\left(1-(F_N)\right)^{w_k}\right)$$

$$=\left(\tilde{T}_N,\tilde{I}_N,\tilde{F}_N,T_N,I_N,F_N\right)$$

2. Monotonicity: Since neutrosophic cubic ordered weighted average operator (NCOWA) is strictly monotone function, hence the proof is trivial.

3. Boundary: Let $u = \min N^{-}$ and $y = \max N^{+}$, then by the idempotent law we have

$$u \leq NCOWA(N_k) \leq y \Rightarrow N^{-} \leq NCOWA(N_k) \leq N^{+}$$

Theorem 4.6 Let $N_k = \left(\tilde{T}_{N_k}, \tilde{I}_{N_k}, \tilde{F}_{N_k}, T_{N_k}, I_{N_k}, F_{N_k}\right)$, where $\tilde{T}_{N_k} = \left[T^L_{N_k}, T^U_{N_k}\right], \tilde{I}_{N_k} = \left[I^L_{N_k}, I^U_{N_k}\right], \tilde{F}_{N_k} = \left[F^L_{N_k}, F^U_{N_k}\right]$, $(k = 1, 2, ..., n)$ be the collection of neutrosophic cubic values and $W = (w_1, w_2, ..., w_n)^T$ is weight of the NCOWA with $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$. The following properties will hold, where N_k is the largest kth of $(N_1, N_2, ..., N_n)$.

1. If $W = (1, 0, ..., 0)^T$, then NCOWA $(N_1, N_2, ..., N_n) = \max N_k$
2. If $W = (0, 0, ..., 1)^T$, then NCOWA $(N_1, N_2, ..., N_n) = \min N_k$
3. If $w_k = 1, w_l = 0$, and $k \neq l$, then NCOWA $(N_1, N_2, ..., N_n) = N_k$.

Proof: Since in NCOWA, the neutrosophic values are ordered in descending order, hence NCWA operator aggregates the weighted values. On the other hand NCOWA weights only the ordering positions.

The idea of neutrosophic cubic hybrid aggregation operators (NCHA) is developed to not only weigh the values but also weigh their ordering position as well.

Definition 4.7 $NCHA : \Omega^n \rightarrow \Omega$ is a mapping of n-dimension, which has associated weight

$W = (w_1, w_2, ..., w_n)^T$, where $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$, such that

$NCHA_W(N_1, N_2, ..., N_n) = w_1 N^{\varpi}_{\sigma(1)} \oplus w_2 N^{\varpi}_{\sigma(2)} \oplus ... \oplus w_n N^{\varpi}_{\sigma(n)}$ where

N_k is the largest kth of the weighted neutrosophic cubic values N^{ϖ}_k. The N^{ϖ}_k can be calculated by the following formula $N^{\varpi}_k = n w_k N_k, k = 1, 2, 3, , , n,$ $W = (w_1, w_2, ..., w_n)^T, w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$, where n is the balancing coefficient.

Theorem 4.8 Let $N^{\varpi}_{\sigma(k)} = \left(\tilde{T}_{N_k}, \tilde{I}_{N_k}, \tilde{F}_{N_k}, T_{N_k}, I_{N_k}, F_{N_k}\right)$, where $\tilde{T}_{N_k} = \left[T^L_{N_k}, T^U_{N_k}\right], \tilde{I}_{N_k} = \left[I^L_{N_k}, I^U_{N_k}\right], \tilde{F}_{N_k} = \left[F^L_{N_k}, F^U_{N_k}\right]$ $(k = 1, 2, n.$ be a collection of neutrosophic cubic values. Then the aggregated value by NCHA is also a neutrosophic cubic value and

$$NCHA_w(N_k)$$

$$=\left(\left[1-\prod_{k=1}^{n}\left(1-\left(T_{\sigma(k)}^{\varpi L}\right)_T\right)^{w_k},1-\prod_{k=1}^{n}\left(1-\left(T_{\sigma(k)}^{\varpi U}\right)_T\right)^{w_k}\right],\left[1-\prod_{k=1}^{n}\left(1-\left(I_{\sigma(k)}^{\varpi L}\right)_I\right)^{w_k},1-\prod_{k=1}^{n}\left(1-\left(I_{\sigma(k)}^{\varpi U}\right)_I\right)^{w_k}\right],\left[\prod_{k=1}^{n}\left(F_{\sigma(k)}^{\varpi L}\right)_F^{w_k},\prod_{k=1}^{n}\left(F_{\sigma(k)}^{\varpi U}\right)_F^{w_k}\right],\prod_{k=1}^{n}\left(T_{\sigma(k)}^{\varpi}\right)^{w_k},\prod_{k=1}^{n}\left(I_{\sigma(k)}^{\varpi}\right)^{w_k},1-\prod_{k=1}^{n}\left(1-\left(F_{\sigma(k)}^{\varpi}\right)\right)^{w_k}\right)$$

where the weight $W=(w_1,w_2,...,w_n)^T$ is such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

Proof: The proof is directly concluded by Theorem 4.3.

Theorem 4.9 The NCWA operator is a special case of NCHA operator when all the components of *w* are equal, i.e. $w_1 = w_2 = ... = w_n$.

Proof. Let $W=(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$.

Then NCHA $_w w(N_1,N_2,...,N_n)$

$= w_1 N_{\sigma(1)}^{\varpi} \oplus w_2 N_{\sigma(2)}^{\varpi} \oplus ... \oplus w_n N_{\sigma(n)}^{\varpi}$

$= \frac{1}{n}(N_{\sigma(1)}^{\varpi} \oplus N_{\sigma(2)}^{\varpi} \oplus ... \oplus N_{\sigma(n)}^{\varpi}) = \frac{1}{n}(N_1,N_2,...,N_n)$

$= w_1 N_1, w_2 N_2, ..., w_n N_n = NCWA(N_1,N_2,...,N_n)$.

Theorem 4.10 The NCOWA is a special case of NCHA when all the components of *w* are equal, i.e. $w_1 = w_2 = ... = w_n$.

Proof. Let $W=(\frac{1}{n},\frac{1}{n},...,\frac{1}{n})^T$.

Then $NCHA_w w(N_1,N_2,...,N_n)$

$= w_1 N_{\sigma(1)}^{\varpi} \oplus w_2 N_{\sigma(2)}^{\varpi} \oplus ... \oplus w_n N_{\sigma(n)}^{\varpi}$

$= w_1 N_{\sigma(1)} \oplus w_2 N_{\sigma(2)} \oplus ... \oplus w_n N_{\sigma(n)} = NCOWA(N_1, N_2, ., N_n$

Example 4.11 The NCHA is applied to the data as stated in section 3 with corresponding weight $w=(0.21,0.14,0.25,0.29,0.11)^T$ of Xu and Yager [27].

Solution The weighted values are

$$N_A^{\varpi} = ([0.8384, 0.9294],[0.4049, 0.6778],[0.1652, 0.3620], 0.8744, 0.6876, 0.4355)$$

$$N_B^{\varpi} = ([0.4643, 0.6535],[0.5496, 0.6001],[0.3465, 0.5357], 0.7635, 0.8404, 0.3170)$$

$$N_C^{\varpi} = \left(\left[0.9143, 0.9821\right], \left[0.8662, 0.9066\right], \left[0.1679, 0.2788\right], 0.1091, 0.7566, 0.4719\right)$$

$$N_D^{\varpi} = \left(\left[0.9030, 0.9788\right], \left[0.1555, 0.5347\right], \left[0.3660, 0.8583\right], 0.8583, 0.3660, 0.5232\right)$$

$$N_E^{\varpi} = \left(\left[0.4042, 0.5761\right], \left[0.2519, 0.3398\right], \left[0.5958, 0.7269\right], 0.6446, 0.5157, 0.2802\right)$$

$$Scr\left(N_A\right) = 1.6759,\ Scr\left(N_B\right) = 0.6821,\ Scr\left(N_C\right) = 1.0856,\ Scr\left(N_D\right) = 0.9926,\ Scr\left(N_E\right) = 0.0220.,$$

Here, $Scr\left(N_A\right) > Scr\left(N_C\right) > Scr\left(N_D\right) > Scr\left(N_B\right) > Scr\left(N_E\right)$.

According to their ranking, the values are

$$N_{\sigma(1)}^{\varpi} = \left(\left[0.8384, 0.9294\right], \left[0.4049, 0.6778\right], \left[0.1652, 0.3620\right], 0.8744, 0.6876, 0.4355\right)$$

$$N_{\sigma(2)}^{\varpi} = \left(\left[0.9143, 0.9821\right], \left[0.8662, 0.9066\right], \left[0.1679, 0.2788\right], 0.1091, 0.7566, 0.4719\right)$$

$$N_{\sigma(3)}^{\varpi} = \left(\left[0.9030, 0.9788\right], \left[0.1555, 0.5347\right], \left[0.3660, 0.8583\right], 0.8583, 0.3660, 0.5232\right)$$

$$N_{\sigma(4)}^{\varpi} = \left(\left[0.4643, 0.6535\right], \left[0.5496, 0.6001\right], \left[0.3465, 0.5357\right], 0.7635, 0.8404, 0.3170\right)$$

$$N_{\sigma(5)}^{\varpi} = \left(\left[0.4042, 0.5761\right], \left[0.2519, 0.3398\right], \left[0.5958, 0.7269\right], 0.6446, 0.5157, 0.2802\right)$$

The new associated weight is derived by the normal distribution method [19]. Here the associated weight $W = (0.110, 0.237, 0.303, 0.235, 0.115)^T$ is the weighting of the NCHA operator.

$$NCHA_w(N_{\sigma(1)}^{\varpi}, N_{\sigma(2)}^{\varpi}, N_{\sigma(3)}^{\varpi}, N_{\sigma(4)}^{\varpi}, N_{\sigma(5)}^{\varpi})$$

$$= \left(\left[1-\prod_{i=1}^{5}\left(1-\left(T_{\sigma(i)}^{\varpi L}\right)_T\right)^{w_i}, 1-\prod_{i=1}^{5}\left(1-\left(T_{\sigma(i)}^{\varpi U}\right)_T\right)^{w_i} \right], \left[1-\prod_{i=1}^{5}\left(1-\left(I_{\sigma(i)}^{\varpi L}\right)_I\right)^{w_i}, 1-\prod_{i=1}^{5}\left(1-\left(I_{\sigma(i)}^{\varpi U}\right)_I\right)^{w_i} \right], \left[\prod_{i=1}^{5}\left(F_{\sigma(i)}^{\varpi L}\right)_F^{w_i}, \prod_{i=1}^{5}\left(F_{\sigma(i)}^{\varpi U}\right)_F^{w_i} \right], \prod_{i=1}^{5}\left(T_{\sigma(i)}^{\varpi}\right)^{w_i}, \prod_{i=1}^{5}\left(I_{\sigma(i)}^{\varpi}\right)^{w_i}, 1-\prod_{i=1}^{5}\left(1-\left(F_{\sigma(i)}^{\varpi}\right)\right)^{w_i} \right)$$

$$= \left(\left[0.8165, 0.9367\right], \left[0.5533, 0.6932\right], \left[0.2911, 0.5251\right], 0.4966, 0.5893, 0.4322\right)$$

In order to analyze these results with WHO standard, the scores of NCWA and NCHA operators are calculated and indicated as follows.

$Ac(N_{NCWA}) = 0.5879$, $Ac(N_{NCHA}) = 0.5927$ and $Ac(N_{WHO}) = 0.4166$

The graphical analysis is illustrated in Figure 2. It is observed that in both scores that Peshawar has highly polluted air.

Figure 2. Comparison of aggregations with WHO standard

5. Neutrosophic Cubic Einstein Aggregation Operators

This section consist of some fundamental definitions of neutrosophic cubic Einstein weighted averaging (NCEWA), neutrosophic cubic Einstein ordered weighted averaging (NCEOWA) and neutrosophic cubic Einstein hybrid avregaing (NCEHA) aggregation operator, which are defined as follows. These are defined using the Einstein addition, Einstein multiplication and Einstein scalar multiplication.

Definition 5.1 The neutrosophic cubic Einstein weighted averaging is a function, NCEWA $: R^n \to R$ defined by

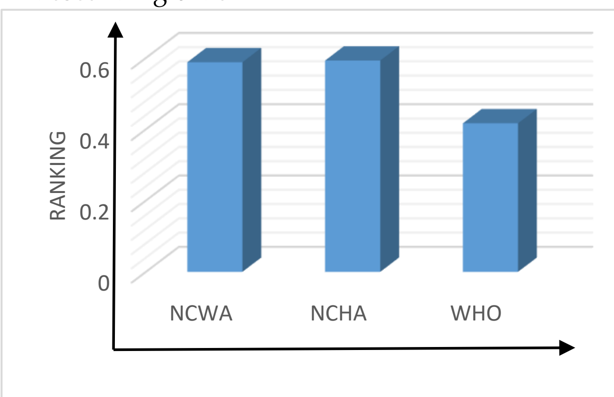
$$NCEWA_w(N_1, N_2, ..., N_n) = \sum_{k=1}^{n} \left(w_k N_k\right)_E \tag{6}$$

where $W = (w_1, w_2, ..., w_n)^T$ is the weight of $N_k (k = 1,2,3,...,n), w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1.$

This implies that the neutrosophic cubic values are weighted and then aggregated using Einstein operations.

Definition 5.2 Order neutrosophic cubic Einstein weighted average operator (NCEOWA) is defined as NCEOWA $: R^n \to R$ by $NCEOWA_w(N_1, N_2, ..., N_n) = \sum_{k=1}^{n} \left(w_k B_k\right)_E$ where, B_k denotes the ordered position of neutrosophic cubic (NC) values in descending order, $W = (w_1, w_2, ..., w_n)^T$ is the weight of $N_k (k = 1, 2, 3, ..., n)$, be such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1.$

Note that, NCEOWA values are ordered and then weighted. Thereafter, the ordering values are aggregated using Einstein operations. The basic concept of ordered weighted operator is to rearrange the values in descending order.



Theorem 5.3 Let $N_k = \left(T_{N_k}, \tilde{I}_{N_k}, F_{N_k}, T_{N_k}, I_{N_k}, F_{N_k}\right)$, where $T_{N_k} = \left[T_{N_k}^L, T_{N_k}^U\right], \tilde{I}_{N_k} = \left[I_{N_k}^L, I_{N_k}^U\right], F_{N_k} = \left[F_{N_k}^L, F_{N_k}^U\right]$, $(k = 1, 2, ..., n)$ be the collection of neutrosophic cubic values. Then their NCEWA operator is also a neutrosophic cubic value where $W = (w_1, w_2, ..., w_n)^T$ is the weight vector of $N_k\,(k = 1, 2, 3, ..., n)$, such that $w_k \in [0,1]$ and $\sum_{k=1}^{n} w_k = 1$.

Proof. By mathematical induction for $n = 2$, using Einstein's addition and ascalar multiplication, we will have the following.

$$(w_1N_1)_E \oplus (w_2N_2)_E = \left\{\begin{array}{l} \left(\begin{array}{l} \left[\frac{\left(1+T_{N_1}^L\right)^{w_1} - \left(1-T_{N_1}^L\right)^{w_1}}{\left(1+T_{N_1}^L\right)^{w_1} + \left(1-T_{N_1}^L\right)^{w_1}}, \frac{\left(1+T_{N_1}^U\right)^{w_1} - \left(1-T_{N_1}^U\right)^{w_1}}{\left(1+T_{N_1}^U\right)^{w_1} + \left(1-T_{N_1}^U\right)^{w_1}}\right], \left[\frac{\left(1+I_{N_1}^L\right)^{w_1} - \left(1-I_{N_1}^L\right)^{w_1}}{\left(1+I_{N_1}^L\right)^{w_1} + \left(1-I_{N_1}^L\right)^{w_1}}, \frac{\left(1+I_{N_1}^U\right)^{w_1} - \left(1-I_{N_1}^U\right)^{w_1}}{\left(1+I_{N_1}^U\right)^{w_1} + \left(1-I_{N_1}^U\right)^{w_1}}\right], \\ \left[\frac{2\left(F_{N_1}^L\right)^{w_1}}{\left(2-F_{N_1}^L\right)^{w_1} + \left(F_{N_1}^L\right)^{w_1}}, \frac{2\left(F_{N_1}^U\right)^{w_1}}{\left(2-F_{N_1}^U\right)^{w_1} + \left(F_{N_1}^U\right)^{w_1}}\right], \frac{2\left(T_{N_1}\right)^{w_1}}{\left(2-T_{N_1}\right)^{w_1} + \left(T_{N_1}\right)^{w_1}}, \frac{2\left(I_{N_1}\right)^{w_1}}{\left(2-I_{N_1}\right)^{w_1} + \left(I_{N_1}\right)^{w_1}}, \frac{\left(1+F_{N_1}\right)^{w_1} - \left(1-F_{N_1}\right)^{w_1}}{\left(1+F_{N_1}\right)^{w_1} + \left(1-F_{N_1}\right)^{w_1}} \end{array}\right) \\ \oplus \left(\begin{array}{l} \left[\frac{\left(1+T_{N_2}^L\right)^{w_2} - \left(1-T_{N_1}^L\right)^{w_2}}{\left(1+T_{N_2}^L\right)^{w_2} + \left(1-T_{N_2}^L\right)^{w_2}}, \frac{\left(1+T_{N_2}^U\right)^{w_2} - \left(1-T_{N_2}^U\right)^{w_2}}{\left(1+T_{N_2}^U\right)^{w_2} + \left(1-T_{N_2}^U\right)^{w_2}}\right], \left[\frac{\left(1+I_{N_2}^L\right)^{w_2} - \left(1-I_{N_1}^L\right)^{w_2}}{\left(1+I_{N_1}^L\right)^{w_2} + \left(1-I_{N_2}^L\right)^{w_1}}, \frac{\left(1+I_{N_2}^U\right)^{w_2} - \left(1-I_{N_2}^U\right)^{w_2}}{\left(1+I_{N_2}^U\right)^{w_2} + \left(1-I_{N_2}^U\right)^{w_2}}\right], \\ \left[\frac{2\left(F_{N_2}^L\right)^{w_2}}{\left(2-F_{N_2}^L\right)^{w_2} + \left(F_{N_2}^L\right)^{w_2}}, \frac{2\left(F_{N_1}^U\right)^{w_2}}{\left(2-F_{N_1}^U\right)^{w_2} + \left(F_{N_2}^U\right)^{w_2}}\right], \frac{2\left(T_{N_2}\right)^{w_2}}{\left(2-T_{N_2}\right)^{w_1} + \left(T_{N_2}\right)^{w_2}}, \frac{2\left(I_{N_2}\right)^{w_2}}{\left(2-I_{N_2}\right)^{w_2} + \left(I_{N_2}\right)^{w_2}}, \frac{\left(1+F_{N_2}\right)^{w_2} - \left(1-F_{N_2}\right)^{w_2}}{\left(1+F_{N_2}\right)^{w_2} + \left(1-F_{N_2}\right)^{w_2}} \end{array}\right) \end{array}\right\}$$

$$\sum_{k=1}^{2}(w_kN_k)_E = \left(\begin{array}{l} \left[\frac{\prod_{k=1}^{2}\left(1+T_{N_k}^L\right)^{w_k} - \prod_{k=1}^{2}\left(1-T_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{2}\left(1+T_{N_k}^L\right)^{w_k} + \prod_{k=1}^{2}\left(1-T_{N_k}^L\right)^{w_k}}, \frac{\prod_{k=1}^{2}\left(1+T_{N_k}^U\right)^{w_k} - \prod_{k=1}^{2}\left(1-T_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{2}\left(1+T_{N_k}^U\right)^{w_k} + \prod_{k=1}^{2}\left(1-T_{N_k}^U\right)^{w_k}}\right], \left[\frac{\prod_{k=1}^{2}\left(1+I_{N_k}^L\right)^{w_k} - \prod_{k=1}^{2}\left(1-I_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{2}\left(1+I_{N_k}^L\right)^{w_k} + \prod_{k=1}^{2}\left(1-I_{N_k}^L\right)^{w_k}}, \frac{\prod_{k=1}^{2}\left(1+I_{N_k}^U\right)^{w_k} - \prod_{k=1}^{2}\left(1-I_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{2}\left(1+I_{N_k}^U\right)^{w_k} + \prod_{k=1}^{2}\left(1-I_{N_k}^U\right)^{w_k}}\right], \\ \left[\frac{2\prod_{k=1}^{2}\left(F_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{2}\left(2-F_{N_k}^L\right)^{w_k} + \prod_{k=1}^{2}\left(F_{N_k}^L\right)^{w_k}}, \frac{2\prod_{k=1}^{2}\left(F_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{2}\left(2-F_{N_k}^U\right)^{w_k} + \prod_{k=1}^{2}\left(F_{N_k}^U\right)^{w_k}}\right], \frac{2\prod_{k=1}^{2}\left(T_{N_k}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-T_{N_k}\right)^{w_k} + \prod_{k=1}^{2}\left(T_{N_k}\right)^{w_k}}, \frac{2\prod_{k=1}^{2}\left(I_{N_k}\right)^{w_k}}{\prod_{k=1}^{2}\left(2-I_{N_k}\right)^{w_k} + \prod_{k=1}^{2}\left(I_{N_k}\right)^{w_k}}, \frac{\prod_{k=1}^{2}\left(1+F_{N_k}\right)^{w_k} - \prod_{k=1}^{2}\left(1-F_{N_k}\right)^{w_k}}{\prod_{k=1}^{2}\left(1+F_{N_k}\right)^{w_k} + \prod_{k=1}^{2}\left(1-F_{N_k}\right)^{w_k}} \end{array}\right)$$

Assuming that for $n = m$ the result holds true, that is

$$\sum_{k=1}^{m}(w_kN_k)_E = \left(\begin{array}{l} \left[\frac{\prod_{k=1}^{m}\left(1+T_{N_k}^L\right)^{w_k} - \prod_{k=1}^{m}\left(1-T_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{m}\left(1+T_{N_k}^L\right)^{w_k} + \prod_{k=1}^{m}\left(1-T_{N_k}^L\right)^{w_k}}, \frac{\prod_{k=1}^{m}\left(1+T_{N_k}^U\right)^{w_k} - \prod_{k=1}^{m}\left(1-T_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{m}\left(1+T_{N_k}^U\right)^{w_k} + \prod_{k=1}^{m}\left(1-T_{N_k}^U\right)^{w_k}}\right], \left[\frac{\prod_{k=1}^{m}\left(1+I_{N_k}^L\right)^{w_k} - \prod_{k=1}^{m}\left(1-I_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{m}\left(1+I_{N_k}^L\right)^{w_k} + \prod_{k=1}^{m}\left(1-I_{N_k}^L\right)^{w_k}}, \frac{\prod_{k=1}^{m}\left(1+I_{N_k}^U\right)^{w_k} - \prod_{k=1}^{m}\left(1-I_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{m}\left(1+I_{N_k}^U\right)^{w_k} + \prod_{k=1}^{m}\left(1-I_{N_k}^U\right)^{w_k}}\right] \\ \left[\frac{2\prod_{k=1}^{m}\left(F_{N_k}^L\right)^{w_k}}{\prod_{k=1}^{m}\left(2-F_{N_k}^L\right)^{w_k} + \prod_{k=1}^{m}\left(F_{N_k}^L\right)^{w_k}}, \frac{2\prod_{k=1}^{m}\left(F_{N_k}^U\right)^{w_k}}{\prod_{k=1}^{m}\left(2-F_{N_k}^U\right)^{w_k} + \prod_{k=1}^{m}\left(F_{N_k}^U\right)^{w_k}}\right], \frac{2\prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-T_{N_k}\right)^{w_k} + \prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}}, \frac{2\prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-I_{N_k}\right)^{w_k} + \prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}}, \frac{\prod_{k=1}^{m}(1+F_{N_k})^{w_k} - \prod_{k=1}^{m}(1-F_{N_k})^{w_k}}{\prod_{k=1}^{m}(1+F_{N_k})^{w_k} + \prod_{k=1}^{m}(1-F_{N_k})^{w_k}} \end{array}\right)$$

The result is proven for $n = m+1$, since

$$w_{m+1}N_{m+1}=\left(\begin{array}{l}\left[\frac{\left(1+T^L_{N_{m+1}}\right)^{w_{m+1}}-\left(1-T^L_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+T^L_{N_{m+1}}\right)^{w_{m+1}}+\left(1-T^L_{N_{m+1}}\right)^{w_{m+1}}},\frac{\left(1+T^U_{N_{m+1}}\right)^{w_{m+1}}-\left(1-T^U_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+T^U_{N_{m+1}}\right)^{w_{m+1}}+\left(1-T^U_{N_{m+1}}\right)^{w_{m+1}}}\right],\left[\frac{\left(1+I^L_{N_{m+1}}\right)^{w_{m+1}}-\left(1-I^L_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+I^L_{N_{m+1}}\right)^{w_{m+1}}+\left(1-I^L_{N_{m+1}}\right)^{w_{m+1}}},\frac{\left(1+I^U_{N_{m+1}}\right)^{w_{m+1}}-\left(1-I^U_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+I^U_{N_{m+1}}\right)^{w_{m+1}}+\left(1-I^U_{N_{m+1}}\right)^{w_{m+1}}}\right]\\ \left[\frac{2\left(F^L_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-F^L_{N_{m+1}}\right)^{w_{m+1}}+F^L_{N_{m+1}}},\frac{2\left(F^U_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-F^U_{N_{m+1}}\right)^{w_{m+1}}+F^U_{N_{m+1}}}\right],\frac{2\left(T_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-T_{N_{m+1}}\right)^{w_{m+1}}+T_{N_{m+1}}},\frac{2\left(I_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-I_{N_{m+1}}\right)^{w_{m+1}}+I_{N_{m+1}}},\frac{\left(1+F_{N_{m+1}}\right)^{w_{m+1}}-\left(1-F_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+F_{N_{m+1}}\right)^{w_{m+1}}+\left(1-F_{N_{m+1}}\right)^{w_{m+1}}}\end{array}\right)$$

$$\sum_{k=1}^{m}\left(w_kN_k\right)_E\oplus\left(w_{m+1}N_{m+1}\right)_E=$$

$$\left\{\begin{array}{l}\left(\begin{array}{l}\left[\frac{\prod_{k=1}^{m}\left(1+T^L_{N_k}\right)^{w_k}-\prod_{k=1}^{m}\left(1-T^L_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(1+T^L_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(1-T^L_{N_k}\right)^{w_k}},\frac{\prod_{k=1}^{m}\left(1+T^U_{N_k}\right)^{w_k}-\prod_{k=1}^{m}\left(1-T^U_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(1+T^U_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(1-T^U_{N_k}\right)^{w_k}}\right],\left[\frac{\prod_{k=1}^{m}\left(1+I^L_{N_k}\right)^{w_k}-\prod_{k=1}^{m}\left(1-I^L_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(1+I^L_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(1-I^L_{N_k}\right)^{w_k}},\frac{\prod_{k=1}^{m}\left(1+I^U_{N_k}\right)^{w_k}-\prod_{k=1}^{m}\left(1-I^U_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(1+I^U_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(1-I^U_{N_k}\right)^{w_k}}\right]\\ \left[\frac{2\prod_{k=1}^{m}\left(F^L_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-F^L_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(F^L_{N_k}\right)^{w_k}},\frac{2\prod_{k=1}^{m}\left(F^U_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-F^U_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(F^U_{N_k}\right)^{w_k}}\right],\frac{2\prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-T_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(T_{N_k}\right)^{w_k}},\frac{2\prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}}{\prod_{k=1}^{m}\left(2-I_{N_k}\right)^{w_k}+\prod_{k=1}^{m}\left(I_{N_k}\right)^{w_k}},\frac{\prod_{k=1}^{m}(1+F_{N_k})^{w_k}-\prod_{k=1}^{m}(1-F_{N_k})^{w_k}}{\prod_{k=1}^{m}(1+F_{N_k})^{w_k}+\prod_{k=1}^{m}(1-F_{N_k})^{w_k}}\end{array}\right)\\ \oplus\left(\begin{array}{l}\left[\frac{\left(1+T^L_{N_{m+1}}\right)^{w_{m+1}}-\left(1-T^L_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+T^L_{N_{m+1}}\right)^{w_{m+1}}+\left(1-T^L_{N_{m+1}}\right)^{w_{m+1}}},\frac{\left(1+T^U_{N_{m+1}}\right)^{w_{m+1}}-\left(1-T^U_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+T^U_{N_{m+1}}\right)^{w_{m+1}}+\left(1-T^U_{N_{m+1}}\right)^{w_{m+1}}}\right],\left[\frac{\left(1+I^L_{N_{m+1}}\right)^{w_{m+1}}-\left(1-I^L_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+I^L_{N_{m+1}}\right)^{w_{m+1}}+\left(1-I^L_{N_{m+1}}\right)^{w_{m+1}}},\frac{\left(1+I^U_{N_{m+1}}\right)^{w_{m+1}}-\left(1-I^U_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+I^U_{N_{m+1}}\right)^{w_{m+1}}+\left(1-I^U_{N_{m+1}}\right)^{w_{m+1}}}\right]\\ \left[\frac{2\left(F^L_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-F^L_{N_{m+1}}\right)^{w_{m+1}}+F^L_{N_{m+1}}},\frac{2\left(F^U_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-F^U_{N_{m+1}}\right)^{w_{m+1}}+F^U_{N_{m+1}}}\right],\frac{2\left(T_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-T_{N_{m+1}}\right)^{w_{m+1}}+T_{N_{m+1}}},\frac{2\left(I_{N_{m+1}}\right)^{w_{m+1}}}{\left(2-I_{N_{m+1}}\right)^{w_{m+1}}+I_{N_{m+1}}},\frac{\left(1+F_{N_{m+1}}\right)^{w_{m+1}}-\left(1-F_{N_{m+1}}\right)^{w_{m+1}}}{\left(1+F_{N_{m+1}}\right)^{w_{m+1}}+\left(1-F_{N_{m+1}}\right)^{w_{m+1}}}\end{array}\right)\end{array}\right\}$$

$$\sum_{k=1}^{m+1}\left(w_kN_k\right)_E=$$

$$\left(\begin{array}{l}\left[\frac{\prod_{k=1}^{m+1}\left(1+T^L_{N_k}\right)^{w_k}-\prod_{k=1}^{m+1}\left(1-T^L_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(1+T^L_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(1-T^L_{N_k}\right)^{w_k}},\frac{\prod_{k=1}^{m+1}\left(1+T^U_{N_k}\right)^{w_k}-\prod_{k=1}^{m+1}\left(1-T^U_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(1+T^U_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(1-T^U_{N_k}\right)^{w_k}}\right],\left[\frac{\prod_{k=1}^{m+1}\left(1+I^L_{N_k}\right)^{w_k}-\prod_{k=1}^{m+1}\left(1-I^L_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(1+I^L_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(1-I^L_{N_k}\right)^{w_k}},\frac{\prod_{k=1}^{m+1}\left(1+I^U_{N_k}\right)^{w_k}-\prod_{k=1}^{m+1}\left(1-I^U_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(1+I^U_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(1-I^U_{N_k}\right)^{w_k}}\right]\\ \left[\frac{2\prod_{k=1}^{m+1}\left(F^L_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(2-F^L_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(F^L_{N_k}\right)^{w_k}},\frac{2\prod_{k=1}^{m+1}\left(F^U_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(2-F^U_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(F^U_{N_k}\right)^{w_k}}\right],\frac{2\prod_{k=1}^{m+1}\left(T_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(2-T_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(T_{N_k}\right)^{w_k}},\frac{2\prod_{k=1}^{m+1}\left(I_{N_k}\right)^{w_k}}{\prod_{k=1}^{m+1}\left(2-I_{N_k}\right)^{w_k}+\prod_{k=1}^{m+1}\left(I_{N_k}\right)^{w_k}},\frac{\prod_{k=1}^{m+1}(1+F_{N_k})^{w_k}-\prod_{k=1}^{m+1}(1-F_{N_k})^{w_k}}{\prod_{k=1}^{m+1}(1+F_{N_k})^{w_k}+\prod_{k=1}^{m+1}(1-F_{N_k})^{w_k}}\end{array}\right)$$

Hence proved.

Example 5.4 The NCEWA is applied to data in section 3 with corresponding weight of Xu and Yager [27], $w=(0.21,0.14,0.25,0.29,0.11)^T$.

Then $NCEWA=\left(\left[0.7848,0.9163\right],\left[0.5058,0.6650\right],\left[0.2957,0.5241\right],0.5652,0.6187,0.3990\right)$

Note that the NCEWA operator aggregates the weighted value whereas the NCEOWA operator weight the ordering position and then aggregates the values. The idea of NCEHA is developed to overcome to not only weight the neutrosophic cubic values but their order positioning as well.

In order to analyze these results with WHO standard, the score of NCEWA and NCEHA operators are compared in terms of the accuracy functions as follows as illustrated in Figure 3.

$$Ac(N_{NCEWA}) = 0.5861,, \quad Ac(N_{NCEHA}) = 0.6099 \quad \text{and} \quad Ac(N_{WHO}) = 0.4166.$$

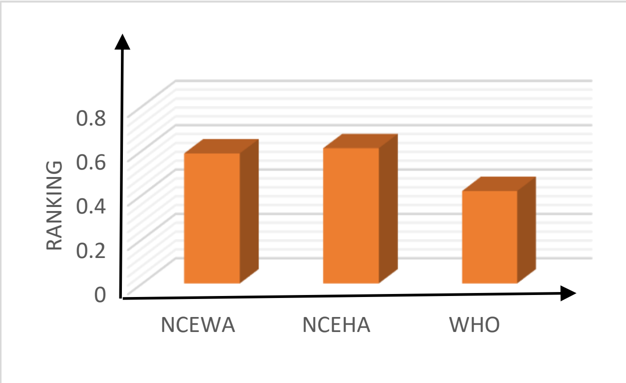


Figure 3. The comparison of Einstein aggregation with WHO

Observe that using both operators NCEWA and NCEHA, Peshawar has highly polluted air as shown in Figure 3. The overall graphical presentation is illustrated in Figure 4. Hence we conclude that serious measures by the relevant government agencies are needed to overcome the situation.

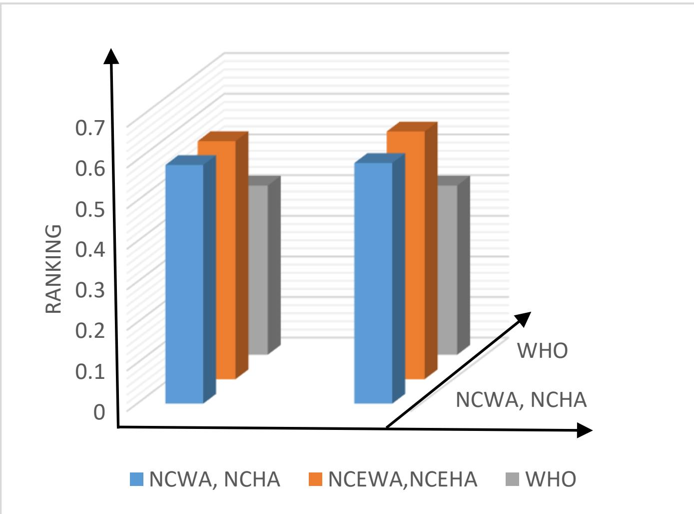


Figure 4. Comparison of neutrosophic cubic aggregation operators with WHO

6. Conclusions

In this research, NCWA, NCHA, NCEWA, NCEHA operators are compared with WHO standards. The aggregation operators are applied to the numerical data of PM10. These aggregation operators enabled us to analyze the air pollution model in the city of Peshawar, Pakistan. We computed the accuracy functions of all of these aggregation operators and WHO standard. The analysis is then presented graphically to illustrate the comparison. It is observed that in the month of April 2014, the pollution of PM10 is very much higher than WHO standards. Strong measures are thus required to control air pollution. Our future research will be to apply further the NCWA, NCHA, NCEWA, NCEHA operators to construction management, geometric programming, binomial factorial problem, and numerical convergence of polynomial roots [49-50].

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Neutrosophic α-Irresolute Multifunction in Neutrosophic Topological Spaces

T.RajeshKannan[1] and S.Chandrasekar[2]
[1,2] PG and Research Department of Mathematics, Arignar Anna Government Arts College, Namakkal(DT),Tamil Nadu, India, rajeshkannan03@yahoo.co.in, chandrumat@gmail.com

* Correspondence: Author (chandrumat@gmail.com)

Abstract: Aim of this present paper is, we define some new type of irresolute multifunction between the two spaces. We obtain some characterization and some properties between such as Lower & Upper α- irresolute multifunction
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Keywords: Neutrosophic α -irresolute lower; Neutrosophic αirresolute upper; Neutrosophic α -closed sets; Neutrosophic topological spaces

1. Introduction

C.L. Chang [3] was introduced fuzzy topological space by using .Zadeh's L.A [18] (uncertain) fuzzy sets. Further Coker [4] was developed the notion of Intuitionistic fuzzy topological spaces by using Atanassov's[1] Intuitionistic fuzzy set. Neutrality the degree of indeterminacy, as an independent concept was introduced by Smarandache [7]. He also defined the Neutrosophic set of three component Neutrosophic topological spaces (t, f, i) =(Truth, Falsehood, Indeterminacy),The Neutrosophic crisp set concept converted to Neutrosophic topological spaces by A.A.Salama [13]. I.Arokiarani.[2] et al, introduced Neutrosophic α -closed sets. T Rajesh kannan[10] et.al introduced and investigated a new class of continuous multivalued function is called Neutrosophic α- continuous multivalued function in Neutrosophic topological spaces.

Aim of this present paper is, we define some new type of irresolute multifunction between the two spaces. we obtain some characterization and some properties between such as Lower & Upper α- irresolute multifunction.

2. PRELIMINARIES

In this section, we introduce the basic definition for Neutrosophic sets and its operations.
Throughout this presentation, $(R^C{}_1,\tau_{R^C{}_1})$ is namely as classical topological spaces on $R^C{}_1$ (represent as CTS$R^C{}_1$) , $(R^N{}_2.,\tau_{N_{R^N{}_2,}})$ is namely as an Neutrosophic topological spaces on $R^N{}_2$.(represent as NUTS$R^N{}_2$,),The family of all open set in $R^C{}_1$ (α −Open in$R^C{}_1$, semi-openin $R^C{}_1$ and pre-open in $R^C{}_1$ respectively) is denoted by O(CTS$R^C{}_1$)(αO(CTS$R^C{}_1$) , SO(CTS$R^C{}_1$) and PO(CTS$R^C{}_1$) respectively). The family of all Neutrosophic open set in $R^N{}_2$,(α −Open in $R^N{}_2$,, semi-open in $R^N{}_2$, and pre-open in $R^N{}_2$, respectively) is denoted by O(NUTS$R^N{}_2$,).(αO(NUTS$R^N{}_2$,) , SO(NUTS$R^N{}_2$,) and PO(NUTS$R^N{}_2$,) respectively). The family of all closed set in $R^C{}_1$ (α −closed in$R^C{}_1$, semi-closed in $R^C{}_1$ and pre-Closed in $R^C{}_1$ respectively) is denoted by C(CTS$R^C{}_1$).(αC(CTS$R^C{}_1$) , SC(CTS$R^C{}_1$) and PS(CTS$R^C{}_1$) respectively). The family of all Neutrosophic Closed in $R^N{}_2$(α −closed in $R^N{}_2$, , semi-closed in $R^N{}_2$, and pre-closed in $R^N{}_2$, respectively) is denoted by C(NUTS$R^N{}_2$,).(αC(NUTS$R^N{}_2$,) , SC(NUTS$R^N{}_2$,) and PC(NUTS$R^N{}_2$,) respectively)

Definition 2.1 [7]

Let ${R^N}_1$ be a non-empty fixed set. A Neutrosophic set $A_{{R^N}_1}$ is an object having the form$A_{{R^N}_1}$= {<ξ, $\mu_{A_{{R^N}_1}}(\xi)$,$\sigma_{A_{{R^N}_1}}(\xi)$,$\gamma_{A_{{R^N}_1}}(\xi)$> : $\xi \in {R^N}_1$}.Where $\mu_{{R^N}_1}(\xi)$:${R^N}_1 \to [0,1]$,$\sigma_{{R^N}_1}(\xi)$):${R^N}_1 \to [0,1]$, ,$\gamma_{A_{{R^N}_1}}(\xi)$):${R^N}_1 \to [0,1]$, are represent Neutrosophic of the degree of membership function, the degree indeterminacy and the degree of non membership function respectively of each element $\xi \in {R^C}_1$ to the set $A_{{R^C}_1}$with $0 \le \mu_{A_{{R^N}_1}}(\xi)$+$\sigma_{A_{{R^N}_1}}(\xi)$+$\gamma_{A_{{R^N}_1}}(\xi) \le 1$.This is called standard form generalized fuzzy sets. But also Neutrsophic set may be $0 \le \mu_{A_{{R^N}_1}}(\xi)$+$\sigma_{A_{{R^N}_1}}(\xi)$+$\gamma_{A_{{R^N}_1}}(\xi) \le 3$

Remark 2.2[7]

we denote$A_{{R^N}_1}$= {<ξ, $\mu_{A_{{R^N}_1}}$,$\sigma_{A_{{R^N}_1}}$,$\gamma_{A_{{R^N}_1}}$ > } for the Neutrosophic set

$A_{{R^N}_1}$= {<ξ, $\mu_{A_{{R^N}_1}}(\xi)$,$\sigma_{A_{{R^N}_1}}(\xi)$,$\gamma_{A_{{R^N}_1}}(\xi)$> : $\xi \in {R^C}_1$}.

Example 2.3 [7]

Each Intuitionistic fuzzy set $A_{{R^N}_1}$ is a non-empty set in ${R^N}_1$ is obviously on Neutrosophic set having the form $A_{{R^N}_1}$= {<ξ, $\mu_{A_{{R^N}_1}}(\xi)$,$(1-(\mu_{A_{{R^N}_1}}+\gamma_{A_{{R^N}_1}}(\xi)))$,$\gamma_{A_{{R^N}_1}}(\xi)$> : $\xi \in {R^C}_1$}

Definition 2.4 [7]

We must introduce the Neutrosophic set 0_N and 1_N in ${R^N}_1$ as follows: :

0_N= {<ξ, 0, 0, 1>: $\xi \in {R^N}_1$} & 1_N= {< ξ, 1, 0, 0>: $\xi \in {R^N}_1$}

Definition 2.5 [7]

Let ${R^N}_1$ be a non-empty set and Neutrosophic sets $A_{{R^N}_1}$ and $B_{{R^N}_1}$ in the form NS $A_{{R^N}_1}$= {<ξ, $\mu_{A_{{R^N}_1}}(\xi)$,$\sigma_{A_{{R^N}_1}}(\xi)$,$\gamma_{A_{{R^N}_1}}(\xi)$)> : $\xi \in {R^C}_1$}&$B_{{R^N}_1}$= {<ξ, $\mu_{B_{{R^N}_1}}(\xi)$,$\sigma_{B_{{R^N}_1}}(\xi)$,$\gamma_{B_{{R^N}_1}}(\xi)$> : $\xi \in {R^C}_1$} defined as:

(1)$A_{{R^N}_1} \subseteq B_{{R^N}_1} \Leftrightarrow \mu_{A_{{R^N}_1}}(\xi) \le \mu_{B_{{R^N}_1}}(\xi)$, $\sigma_{A_{{R^N}_1}}(\xi)$, $\le \sigma_{B_{{R^N}_1}}(\xi)$, and $\gamma_{B_{{R^N}_1}}(\xi) \ge \gamma_{B_{{R^N}_1}}(\xi)$

(2)${A_{{R^N}_1}}^C$ = {<ξ , $\gamma_{B_{{R^N}_1}}(\xi)$, $\sigma_{A_{{R^N}_1}}(\xi)$,$\mu_{B_{{R^N}_1}}(\xi)$>: $\xi \in {R^C}_1$}

(3)$A_{{R^N}_1} \cap B_{{R^N}_1}$ ={<ξ, $\mu_{A_{{R^N}_1}}(\xi)$)$\wedge \mu_{B_{{R^N}_1}}(\xi)$, $\sigma_{A_{{R^N}_1}}(\xi)$)$\wedge$ $\sigma_{B_{{R^N}_1}}(\xi)$,$\gamma_{A_{{R^N}_1}}(\xi) \vee \gamma_{B_{{R^N}_1}}(\xi)$>: $\xi \in {R^N}_1$

(4)$A_{{R^N}_1} \cup B_{{R^N}_1}$ ={<x, $\mu_{A_{{R^N}_1}}(\xi) \vee \mu_{B_{{R^N}_1}}(\xi)$, $\sigma_{A_{{R^N}_1}}(\xi) \vee \sigma_{B_{{R^N}_1}}(\xi)$, $\gamma_{A_{{R^N}_1}}(\xi) \wedge \gamma_{B_{{R^N}_1}}(\xi)$> : $\xi \in {R^N}_1$}

(5) $\cap Aj_{{R_1}^c}$={<ξ, $\wedge_j$ $\mu_{Aj_{{R^N}_1}}(\xi)$, $\wedge_j$ $\sigma_{Aj_{{R^N}_1}}(\xi)$, $\vee_j$ $\gamma_{Aj_{{R^N}_1}}(\xi)$> : $\xi \in {R^N}_1$ }

(6) $\cup Aj_{{R^N}_1}$= {<ξ, $\vee_j$ $\mu_{Aj_{{R^N}_1}}(\xi)$, $\vee_j$ $\sigma_{Aj_{{R^N}_1}}(\xi)$, $\wedge_j$ $\gamma_{Aj_{{R^N}_1}}(\xi)$> : $\xi \in {R^N}_1$} for all $\xi \in {R^C}_1$

Proposition 2.6 [9]

For all $A_{{R^N}_1}$ and$B_{{R^N}_1}$are two Neutrosophic sets then the following condition are true:

(1) $(A_{{R^N}_1} \cap B_{{R^N}_1})^C$= $(A_{{R^N}_1})^C \cup (B_{{R^N}_1})^C$

(2) $(A_{{R^N}_1} \cup B_{{R^N}_1})^C$= $(A_{{R^N}_1})^C \cap (B_{A_{{R^N}_1}})^C$

Definition 2.7 [10]

A Neutrosophic topology is a non -empty set ${R^N}_1$ is a family $\tau_{N_{{R^N}_1}}$ of Neutrosophic subsets in ${R^N}_1$ satisfying the following axioms:

(i) 0_N, $1_N \in \tau_{N_{{R^N}_1}}$

(ii) $G_{{R^N}_1} \cap H_{{R^N}_1} \in \tau_{N_{{R^N}_1}}$ for any $G_{{R^N}_1}$,$H_{{R^N}_1} \in \tau_{N_{{R^N}_1}}$

(iii) $\cup_i Gi_{{R^N}_1} \in \tau_{N_{{R^N}_1}}$ for every $Gi_{{R^N}_1} \in \tau_{N_{{R^N}_1}}$, I ∈ J

The pair (${R^N}_1$,$\tau_{N_{{R^N}_1}}$) is called a Neutrosophic topological space.

The element Neutrosophic topological spaces of $\tau_{N_{{R^N}_1}}$are called Neutrosophic open sets.

A Neutrosophic set $A_{{R^N}_1}$ is closed if and only if ${A_{{R^N}_1}}^C$ is Neutrosophic open.

Definition 2.8[10]

Let (${R^N}_1$, $\tau_{N_{{R^N}_1}}$) be Neutrosophic topological spaces.

$A_{R^N{}_1}$= {<ξ,$\mu_{A_{R^N{}_1}}(\xi)$,$\sigma_{A_{R^N{}_1}}(\xi)$,$\gamma_{A_{R^N{}_1}}(\xi)$)> : $\xi \in R^N{}_1$} be a Neutrosophic set in $R^N{}_1$

.1.Neu-Cl($A_{R^N{}_1}$) =∩{ $K_{A_{R^N{}_1}}$:$K_{A_{R^N{}_1}}$ is a Neutrosophic closed set in $R^N{}_1$and $A_{R^N{}_1} \subseteq K_{A_{R^N{}_1}}$}

2.Neu-Int($A_{R^N{}_1}$) = ∪{$G_{A_{R^N{}_1}}$:$G_{A_{R^N{}_1}}$ is a Neutrosophic open set in $R^N{}_1$ and $G_{A_{R^N{}_1}} \subseteq A_{R^N{}_1}$}.

3.Neutrosophic Semi-open if $A_{R^N{}_1} \subseteq$ Neu-Cl(Neu-Int($A_{R^N{}_1}$)).

4.The complement of Neutrosophic Semi-open set is called Neutrosophic semi-closed.

5.Neu-sCl($A_{R^N{}_1}$) =∩ { $K_{A_{R^N{}_1}}$:$K_{A_{R^N{}_1}}$ is a Neutrosophic Semi closed set in $R^N{}_1$ and $A_{R^N{}_1} \subseteq K_{A_{R^N{}_1}}$}

6.Neu-sInt($A_{R^N{}_1}$) = ∪{$G_{A_{R^N{}_1}}$:$G_{A_{R^N{}_1}}$ is a Neutrosophic Semi open set in $R^C{}_1$ and $G_{A_{R^N{}_1}} \subseteq A_{R^N{}_1}$}.

7.Neutrosophic α-open set if $A_{R^N{}_1} \subseteq$ Neu-Int(Neu-Cl(Neu-Int($A_{R^N{}_1}$))).

8.The complement of Neutrosophic α-open set is called Neutrosophic α-closed.

9.Neuα- Cl($A_{R^N{}_1}$) =∩{ $K_{A_{R^N{}_1}}$:$K_{A_{R^N{}_1}}$ is a Neutrosophic α - closed set in $R^N{}_1$ and $A_{R^N{}_1} \subseteq K_{A_{R^N{}_1}}$}

10.Neu α -Int($A_{R^N{}_1}$) = ∪ {$G_{A_{R^N{}_1}}$:$G_{A_{R^N{}_1}}$ is a Neutrosophic α - open set in $R^N{}_1$ and $G_{A_{R^N{}_1}} \subseteq A_{R^N{}_1}$}.

11.Neutrosophicpre open set if $A_{R^N{}_1} \subseteq$ Neu-Int(Neu-Cl$A_{R^N{}_1}$)).

12.The complement of Neutrosophic Pre-open set is called Neutrosophic pre-closed.

13.Neu- pCl($A_{R^N{}_1}$) =∩{ $K_{A_{R^N{}_1}}$:$K_{A_{R^N{}_1}}$ is a Neutrosophic P- closed set in $R^N{}_1$ and $A_{R^N{}_1} \subseteq K_{A_{R^N{}_1}}$}

14.Neu- pInt($A_{R^N{}_1}$) = ∪{$G_{A_{R^N{}_1}}$:$G_{A_{R^N{}_1}}$ is a Neutrosophic P - open set in $R^N{}_1$ and $G_{A_{R^N{}_1}} \subseteq A_{R^N{}_1}$}.

Remark:2.9[11]

Let $A_{R^N{}_1}$ be an Neutrosophic topological space ($R^N{}_1$, $\tau_{N_{R^C{}_1}}$).Then

(i) Neu α-Cl($A_{R^N{}_1}$) = $A_{R^N{}_1}$ ∪ Neu-Cl(Neu-Int(Neu-Cl($A_{R^N{}_1}$))).

(ii) Neu α-Int($A_{R^N{}_1}$) = $A_{R^N{}_1}$ ∩Neu-Int(Neu-Cl(Neu-Int($A_{R^N{}_1}$))).

Definition 2.10[9]

Take ξ_1,ξ_2,ξ_3 are belongs to real numbers 0 to 1 such that $0 \le \xi_1+\xi_2+\xi_3 \le 1$ .An Neutrosophic point $\wp(\xi_1,\xi_2,\xi_3)$is Neutrosophic set defined by

$$\wp(\xi_1,\xi_2,\xi_3) = \begin{cases} (\xi_1,\xi_2,\xi_3) & if\ \xi = \wp \\ (0,0,1) & if\ \xi \neq \wp \end{cases}$$

Take $\wp(\xi_1,\xi_2,\xi_3)$ =<$\wp_{\xi_1}\wp_{\xi_2}.\wp_{\xi_3}$> Where $\wp_{\xi_1}\wp_{\xi_2}.\wp_{\xi_3}$are represent Neutrosophic the degree of membership function, the degree indeterminacy and the degree of non-membership function respectively of each element $\xi \in R^N{}_1$ to the set $A_{R^N{}_1}$

Definition:2.11

A Neutrosophic set $A_{R^N{}_1}$ in $R^N{}_1$ is said to be quasi-coincident (q-coincident) with a Neutrosophic set $B_{R^N{}_1}$ denoted by $A_{R^N{}_1}$q$B_{R^N{}_1}$ if and only if there exists $\xi \in R^N{}_1$ such that $A_{R^N{}_1}(\xi)$ + $B_{R^N{}_1}(\xi)$ >1.

Remark: 2.12

$A_{R^N{}_1}$q $B_{R^N{}_1} \Leftrightarrow A_{R^N{}_1} \nsubseteq B_{R^N{}_1}{}^C$

Definition 2.13[9]

let$R^N{}_1$ and $R^N{}_2$ be two finite sets. Define ψ_1:$R^N{}_1 \to R^N{}_2$.

If$A_{R^N{}_2}$= {<θ, $\mu_{A_{R^N{}_2}}(\theta)$,$\sigma_{A_{R^N{}_2}}(\theta)$,$\gamma_{A_{R^N{}_2}}(\theta)$)> : $\theta \in R^C{}_2$}.is an NS in $R^N{}_2$, then the inverse image(pre image) $A_{R^N{}_2}$ under ψ_1 is an NS defined by $\psi_1{}^{-1}(A_{R^N{}_2})$=< ξ, $\psi_1{}^{-1}\mu_{A_{R^N{}_2}}(\xi)$, $\psi_1{}^{-1}\sigma_{A_{R^N{}_2}}(\xi)$, $\psi_1{}^{-1}\gamma_{A_{R^N{}_2}}(\xi)$: $\xi \in R^N{}_1$>. Also define image NS U=<ξ, $\mu_U(\xi)$, $\sigma_U(\xi)$, $\gamma_U(\xi)$: $\xi \in R^N{}_1$:> under ψ_1 is an NS defined by ψ_1 (U)=< θ, $\psi_1(\mu_{A_{R^N{}_2}}(\theta))$, $\psi_1(\sigma_{A_{R^N{}_2}}(\theta))$, $\psi_1\gamma_{A_{R^N{}_2}}(\theta)$: $\theta \in R^N{}_2$>

where

$\psi_1(\mu_{A_{R^N{}_2}}(\theta))$, ={ sup $\mu_{A_{R^N{}_2}}(\xi)$, if $\psi_1{}^{-1}(\theta) \neq \phi$, $\xi \in \psi_1{}^{-1}(\theta)$

0, elsewhere

$\psi_1(\sigma_{A_{R^N{}_2}}(\theta))$= { sup$\sigma_{A_{R^N{}_2}}(\xi)$ if $\psi_1{}^{-1}(\theta) \neq \phi$, $\xi \in \psi_1{}^{-1}(\theta)$

0, elsewhere

$\psi_1(\gamma_{A_{R^N{}_2}}(\theta))$= { inf $(\gamma_{A_{R^N{}_2}}(\xi)$ if $\psi_1{}^{-1}(\theta) \neq \phi$, $\xi \in \psi_1{}^{-1}(\theta)$

0, Elsewhere

Definition 2.14[2]

A mapping ψ_1:$(R^N{}_1,\tau_{N_{R^N{}_1}}) \rightarrow (R^N{}_2., \tau_{N_{R^N{}_2}})$ is called a

(1) Neutrosophic continuous(Neu-continuous) if$\psi_1{}^{-1}(A_{R^N{}_2}) \in$ C(CTS$R^C{}_1$)whenever $A_{R^N{}_2} \in$ C(NUTS$R^N{}_2$)

(2) Neutrosophic α-continuous(Neu α - continuous) if $\psi_1{}^{-1}(A_{R^N{}_2}) \in$ αC(CTS$R^C{}_1$) whenever$A_{R^N{}_2} \in$ C(NUTS$R^N{}_2$)

(3) Neutrosophic Semi-continuous(Neu Semi - continuous) if $\psi_1{}^{-1}(A_{R^N{}_2}) \in$ sC(CTS$R^C{}_1$) whenever$A_{R^N{}_2} \in$ C(NUTS$R^N{}_2$)

Definition 2.15.

Let $(R^C{}_1,\tau_{N_{R^C{}_1}})$ be a topological space in the classical sense and $(R^N{}_2.,\tau_{N_{R^N{}_2}}$ be an Neutrosophic topological space. $\Psi : (R^C{}_1,\tau_{R^C{}_1}) \rightarrow (R^N{}_2.,\tau_{N_{R^N{}_2}})$ is called a Neutrosophic multifunction if and only if for each $\xi \in R^C{}_1$, $\Psi(\xi)$ is a Neutrosophic set in $R^N{}_2$.

Definition 2.16

For a Neutrosophicmultifunction : $\Psi : (R^C{}_1,\tau_{R^C{}_1}) \rightarrow (R^N{}_2.,\tau_{N_{R^N{}_2}})$,the upper inverse$\Psi^+(\Gamma)$ and lower inverse $\Psi^-(\Gamma)$ of a Neutrosophic set $\Gamma_{R^N{}_2.}$ in $R^N{}_2$ are defined as follows:

$\Psi^+(\Gamma_{R^N{}_2.})$ ={ $\xi \in R^C{}_1 \setminus \Psi(\xi) \leq \Gamma_{R^N{}_2.}$} and

$\Psi^-(\Gamma_{R^N{}_2.})$ = {$\xi \in R^C{}_1 \setminus \Psi(\xi)$q $\Gamma_{R^N{}_2.}$}.

Lemma 2.17.

For a Neutrosophicmultifunction Ψ: $(R^C{}_1,\tau_{R^C{}_1}) \rightarrow (R^N{}_2.,\tau_{N_{R^N{}_2}})$,

we have $\Psi^-(1- \Gamma_{R^N{}_2}) = R^C{}_1 - \Psi^+(\Gamma_{R^N{}_2.})$, for any Neutrosophic set $\Gamma_{R^N{}_2.}$ in $R^N{}_2$.

Lemma:2.18

Let $\Gamma_{R^N{}_2}$ be a subset of Neutrosophic topology $\tau_{N_{R^N{}_2}}$.then

1.$\Gamma_{R^N{}_2}$isα-closed in$R^N{}_2$iffNeu-SInt (Neu-Cl($\Gamma_{R^N{}_2}$) $\subset \Gamma_{R^N{}_2}$

2.Neu- SInt(Neu-Cl($\Gamma_{R^N{}_2}$) $= Neu-Cl(Neu-Int(Neu-Cl(\Gamma_{R^N{}_2}))$

Lemma:2.19

Let $\Gamma_{R^N{}_2}$ be a subset of Neutrosophic topology $\tau_{N_{R^N{}_2}}$.then below are equivalent

1$\Gamma_{R^N{}_2}$isNeuα-open in$R^N{}_2$

2.$U_{R^N{}_2} \subset \Gamma_{R^N{}_2} \subset Neu-Int(Neu-Cl(U_{R^N{}_2}))$ for some $U_{R^N{}_2}$of $R^N{}_2$.

3.$U_{R^N{}_2} \subset \Gamma_{R^N{}_2} \subset Neu-S(Cl(U_{R^N{}_2}))$ for some $U_{R^N{}_2}$of $R^N{}_2$

4.$\Gamma_{R^N{}_2} \subset Neu-SCl(Neu-Int(\Gamma_{R^N{}_2}))$

Definition 2.19[6]

A Neutrosophicmultifunction :$\Psi : (R^C{}_1,\tau_{R^C{}_1}) \rightarrow (R^N{}_2,\tau_{N_{R^N{}_2}})$ is said to be 1.Neutrosophic upper semi continuous at a point $\xi \in R^C{}_1$ if for any $\Gamma_{R^N{}_2} \in$ O(NUTS$R^N{}_2$), $\Gamma_{R^N{}_2}$. containing$\Psi(\xi)$,there exist $\xi \in U_{R^C{}_1} \in$ O(CTS$R^C{}_1$) such that $\Psi(U_{R^C{}_1}) \subset \Gamma_{R^N{}_2}$.

2.Neutrosophic lower semi continuous at a point $\xi \in R^C{}_1$ if for any$\Gamma_{R^N{}_2} \in$ O(NUTS$R^N{}_2$), with $\Psi(\xi)$q$\Gamma_{R^N{}_2}$, there exist x $\in U_{R^C{}_1} \in$ O(CTS$R^C{}_1$) such that $\Psi(U_{R^C{}_1})$q$\Gamma_{R^N{}_2}$

3.Neutrosophic upper semi continuous (Neutrosophic lower semi continuous) if it is Neutrosophic upper semi continuous (Neutrosophic lower semi continuous) at each point $\xi \in R^C{}_1$.

4.Neutrosophic upper pre -continuous at a point $\xi \in R^C{}_1$ if for any $\Gamma_{R^N{}_2} \in$ O(NUTS$R^N{}_2$), Γ containing

$\Psi(\xi)$,there exist $\xi \in U_{R^C_1} \in$PO(CTS$R^C{}_1$) such that $\Psi(U_{R^C_1}) \subset \Gamma_{R^N_2}$

5.Neutrosophic lower pre- continuous at a point $\xi \in R^C{}_1$ if for any$\Gamma_{R^N{}_2} \in$ O(NUTS$R^N{}_2$), with$\Psi(\xi)$q$\Gamma_{R^N{}_2}$, there exist $\xi \in U_{R^C_1} \in$PO(CTS$R^C{}_1$) such that $\Psi(U_{R^C_1})$q$\Gamma_{R^N{}_2}$

6.Neutrosophic upper pre-continuous (Neutrosophic lower pre-continuous) if it is Neutrosophic upper pre-continuous (Neutrosophic lower pre-continuous) at each point $\xi \in R^C{}_1$.

7.Neutrosophic upper α -continuous at a point $\xi \in R^C{}_1$ if for any $\Gamma_{R^N{}_2} \in$ O(NUTS$R^N{}_2$), Γ containing $\Psi(\xi)$ (that is , F $(\xi) \subset \Gamma$), there exist $\xi \in U_{R^C_1} \in \alpha$ O(CTS$R^C{}_1$) such that $\Psi(U_{R^C_1}) \subset \Gamma_{R^N{}_2}$

8.Neutrosophic lower α - continuous at a point $\xi \in R^C{}_1$ if for any$\Gamma_{R^N{}_2} \in$ O(NUTS$R^N{}_2$), with$\Psi(\xi)$q$\Gamma_{R^N{}_2}$, there exist x $\in U_{R^C_1} \in \alpha$ O(CTS$R^C{}_1$) such that $\Psi(U_{R^C_1})$q$\Gamma_{R^N{}_2}$

9.Neutrosophic upper α -continuous (Neutrosophic lower α -continuous) if it is Neutrosophic upper α -continuous (Neutrosophic lower α -continuous) at each point $\xi \in R^C{}_1$.

10.Neutrosophic upper quasi-continuous at a point $\xi \in R^C{}_1$ if for any $\Gamma_{R^N{}_2} \in$ O(NUTS$R^N{}_2$),$\Gamma_{R^N{}_2}$containing $\Psi(\xi)$,there exist $\xi \in U_{R^C_1} \in S$ O(CTS$R^C{}_1$) such that $\Psi(U_{R^C_1}) \subset \Gamma_{R^N{}_2}$

11.Neutrosophic lower quasi semi continuous at a point $\xi \in R^C{}_1$ if for any$\Gamma_{R^N{}_2} \in$ O(NUTS$R^N{}_2$), with $\Psi(\xi)$q$\Gamma_{R^N{}_2}$, there exist $\xi \in U_{R^C_1} \in$S O(CTS$R^C{}_1$) such that $\Psi(U_{R^C_1})$q$\Gamma_{R^N{}_2}$

12.Neutrosophic upper quasi semi continuous (Neutrosophic lower quasi semi continuous) if it is Neutrosophic upper quasi semi continuous (Neutrosophic lower quasi semi continuous) at each point $\xi \in R^C{}_1$.

III. Lower α-Irresolute Neutrosophic Multifunctions

In this section, we introduce the Definition for Neutrosophic Lower α- irresolute multifunction and its properties

Definition 3.1.

An Neutrosophic multifunction $\Psi : (R^C{}_1, \tau_{R^C_1}) \to (R^N{}_2., \tau_{N_{R^N{}_2}})$ is said to be

(1) Neutrosophic lower α-irresolute at a point $x_0 \in R^C{}_1$, if for any $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$ such that $\Psi(x_0) q\, \Gamma_{R^N{}_2}$ there exists $U_{R^C_1} \in \alpha$O(CTS$R^C{}_1$) containing x_0 such that $\Psi(\xi) q \Gamma_{R^N{}_2}, \forall \xi \in U_{R^C_1}$

(2) Neutrosophic lower α-irresolute if it is Neutrosophic lower α-irresolute at each point of $R^C{}_1$.

Theorem 3.2

Every Neutrosophic lowerα-irresolute multifunction is Neutrosophic lower α-continuous multifunction.

Proof:

Letting$x_0 \in R^C{}_1$, Ψ:$(R^C{}_1, \tau_{R^C_1}) \to (R^N{}_2.\tau_{N_{R^N{}_2}})$ and $\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$ such that $\Psi(x_0) q\, \Gamma_{R^N{}_2}$. But we know that , Every$\Gamma_{R^N{}_2}$,$\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$ is $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$,.Therefore $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$.By our assumption ,Neutrosophic lowerα −irresolute multifunction, there exists $U_{R^C_1} \in \alpha$O(CTS$R^C{}_1$) containing x_0 such that $\Psi(\xi) q \Gamma_{R^N{}_2}, \forall \xi \in U_{R^C_1}$.Hence$\Psi$ is Neutrosophic lower α-continuous multifunction at x_0.

Theorem 3.3

Every Neutrosophic lower$\alpha -$ irresolute multifunction is Neutrosophic lower Pre continuous multifunction.

Proof:

Letting $x_0 \in R^C{}_1$, $\Psi : (R^C{}_1, \tau_{R^C_1}) \to (R^N{}_2., \tau_{N_{R^N{}_2}})$ and $\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$ such that $\Psi(x_0) q\, \Gamma_{R^N{}_2}. But$ we know that , Every$\Gamma_{R^N{}_2}$,$\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$is $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$.Therefore $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$.By our assumption ,Neutrosophic lower α-irresolute multifunction, there exists $U_{R^C_1} \in \alpha$O(CTS$R^C{}_1$) containing x_0 such that $\Psi(x_0) q \Gamma_{R^N{}_2}, \forall x \in U_{R^C_1}$. every$U_{R^C_1}$,$U_{R^C_1} \in \alpha O(NUTSR^N{}_2)$ is $U_{R^C_1} \in PO(CTTSR^N{}_2)$.

There exists $U_{R^C_1} \in PO(\text{CTS}R^C{}_1)$ containing x_0 such that $\Psi(\xi)q\Gamma_{R^N{}_2}, \forall\xi \in U_{R^C{}_1}$.Hence Ψ is Neutrosophic lower Pre-continuous multifunction at x_0.

Theorem 3.4

Every Neutrosophic lower α-irresolute multifunction is Neutrosophic lower quasi semi continuous multifunction.

Proof:

Letting $x_0 \in R^C{}_1$, $\Psi : (R^C{}_1, \tau_{R^C{}_1}) \to (R^N{}_2, \tau_{N_{R^N{}_2}})$ and $\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$ such that $\Psi(x_0)q\,\Gamma_{R^N{}_2}$, But we know that , Every $\Gamma_{R^N{}_2}$, $\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$is $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$,Therefore $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$.By our assumption ,Neutrosophic lower $\alpha-$ irresolute multifunction, There exists $U_{R^C{}_1} \in \alpha O(\text{CTS}R^C{}_1)$ containing x_0 such that $\Psi(\xi)q\Gamma_{R^N{}_2}, \forall\xi \in U_{R^C{}_1}$Here every $U_{R^C{}_1}$, $U_{R^C{}_1} \in \alpha O(NUTSR^N{}_2)$ is $U_{R^C{}_1} \in SO(CTTSR^N{}_2)$. Finally we get , There exists $U_{R^C{}_1} \in SO(\text{CTS}R^C{}_1)$ containing x_0 such that $\Psi(\xi)q\Gamma_{R^N{}_2}, \forall\xi \in U_{R^C{}_1}$hence Ψ is Neutrosophic lower quasi semi continuous multifunction at x_0.

Theorem 3.5

Let $\Psi : (R^C{}_1, \tau_{R^C{}_1}) \to (R^N{}_2, \tau_{N_{R^N{}_2}})$, be an Neutrosophic multifunction and letting $x_0 \in R^C{}_1$. Then the following statements are equivalent:

(a) Ψ is Neutrosophic lower α-irresolute at x_0.

(b) For any $\Gamma_{R^N{}_2}, \Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$ with $(x_0)q\Gamma_{R^N{}_2}$, $\Longrightarrow x_0 \in sCl(Int(\Psi^-(\Gamma_{R^N{}_2})))$.

(c) For any $U_{R^C{}_1}$, $U_{R^C{}_1} \in SO(CTSU_{R^C{}_1})$,$x_0 \epsilon U_{R^C{}_1}$ and for each $\Gamma_{R^N{}_2}, \Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$ with $\Psi(x_0)q\Gamma_{R^N{}_2}$, there exists a $V_{R^C{}_1} \in O(CTSR^C{}_1)$, $V_{R^C{}_1} \subset U_{R^C{}_1}$ such that $\Psi(\xi)qV_{R^C{}_1}, \forall\ \xi \in V_{R^C{}_1}$

Proof.

(a) ⇒ (b). Let $x_0 \in R^C{}_1$ and $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$ such that $\Psi(x_0)q\Gamma_{R^N{}_2}$ Then by our assumption (a) , we get there exists $U_{R^C{}_1} \in \alpha O(\text{CTS}R^C{}_1)$ such that $x_0 \in U_{R^C{}_1}$ and $F(\xi)q\Gamma_{R^N{}_2}, \forall\ \xi \in U_{R^C{}_1}$.Thus $x_0 \in U_{R^C{}_1} \subset \Psi^-(\Gamma_{R^N{}_2})$ …… (1) Here $U_{R^C{}_1} \in \alpha O(\text{CTS}R^C{}_1)$ .we know that for any set $A_{R^C{}_1}$, $A_{R^C{}_1} \in \alpha O(\text{CTS}R^C{}_1) \Leftrightarrow A_{R^C{}_1} \subset sCl\left(Int\left(A_{R^C{}_1}\right)\right)$. Therefore, $U_{R^C{}_1} \subset sCl\left(Int\left(U_{R^C{}_1}\right)\right)$ … (2). from(1) and(2), we get $x_0 \in sCl\left(Int\Psi^-\left(\Gamma_{R^N{}_2}\right)\right)$.Hence (b).

(b) ⇒ (c). Let $\Gamma_{R^N{}_2} \in \alpha O\left(NUTSR^N{}_2\right)$ such that $(x_0)q\Gamma_{R^N{}_2}$, then $x_0 \in sCl\left(Int\Psi^-\left(\Gamma_{R^N{}_2}\right)\right)$. Let $U_{R^C{}_1} \in sO(\text{CTS}R^C{}_1)$ and $x_0 \in U_{R^C{}_1}$.Then $U_{R^C{}_1} \cap Int\left(\Psi^-\left(\Gamma_{R^N{}_2}\right)\right) \neq \varphi$ and $U_{R^C{}_1} \cap Int\left(\Psi^-\left(\Gamma_{R^N{}_2}\right)\right)$ is semi-open in $R^C{}_1$. Put $V_{R^C{}_1} = Int(U_{R^C{}_1} \cap Int(\Psi^-(\Gamma_{R^N{}_2})))$, Then $V_{R^C{}_1}$ is an open set of $R^C{}_1$, $V_{R^C{}_1} \subset U_{R^C{}_1}$, $V_{R^C{}_1} \neq \varphi$and $\Psi(v)q\Gamma_{R^N{}_2}, \forall v \in V_{R^C{}_1}$. (c) ⇒ (a).Let $\{U_\xi\}$ be the system of the sO(CTS$R^C{}_1$) containing ξ.

Let $U_{R^C{}_1} \in SO(\text{CTS}R^C{}_1)$ and $x_0 \in U_{R^C{}_1}$ and Any $\Gamma_{R^N{}_2} \in \alpha O(NUTSY)$ such that $\Psi(x_0)q\Gamma_{R^N{}_2}$, there exists a nonempty open set $B_U \subset U_{R^C{}_1}$Such that $\Psi(v)q\Gamma_{R^N{}_2} \forall v \in B_U$. Let $W_{R^C{}_1} = \cup B_U : U \in \{U_{x_0}\}$, then $W_{R^C{}_1} \in O(\text{CTS}R^C{}_1)$,and $x_0 \in sCl(W_{R^C{}_1})$ and $\Psi(v)q\Gamma_{R^N{}_2}, \forall v \in W_{R^C{}_1}$.Put $S_{R^C{}_1} = W_{R^C{}_1} \cup \{x_0\}$, then $W_{R^C{}_1} \subset S_{R^C{}_1} \subset sCl(W_{R^C{}_1})$. Thus $S_{R^C{}_1} \in \alpha O(CTSR^C{}_1)$), $x_0 \in S_{R^C{}_1}$ and $\Psi(v)q\Gamma_{R^N{}_2}, \forall v \in S_{R^C{}_1}$. Hence $\Psi$is Neutrosophic lower α-irresolute at x_0.

Theorem 3.6

Let $\Psi : (R^C{}_1, \tau_{R^C{}_1}) \to (R^N{}_2., \tau_{N_{R^N{}_2}})$, be an Neutrosophic multifunction. Then the following statements are equivalent:

(a) Ψ is Neutrosophic lower α-irresolute.

(b)$\Psi^-(\lambda_{R^N{}_2}) \in \alpha O(CTSR^C{}_1)$,for every Neutrosophic α-open set $\lambda_{R^N{}_2}$ of $R^N{}_2$.

(c) $\Psi^+(\beta_{R^N{}_2}) \in \alpha C(CTSR^C{}_1)$,for every Neutrosophic α-closed set $\beta_{R^N{}_2}$ of $R^N{}_2$.

(d) $sInt(Cl(\Psi^{+}(\Gamma_{R^{N}_{2}}))) \subset \Psi^{+}(Neu - \alpha Cl(\Gamma_{R^{N}_{2}}))$, for each Neutrosophic set $\Gamma_{R^{N}_{2}}$ of R^{N}_{2}.

(e) $\Psi\left(sInt\left(Cl\left(V_{R^{C}_{1}}\right)\right)\right) \subset Neu - \alpha Cl(\Psi(V_{R^{C}_{1}}))$, for each subset $V_{R^{C}_{1}}$ of R^{C}_{1}.

(f) $\Psi\left(\alpha Cl\left(V_{R^{C}_{1}}\right)\right) \subset Neu - \alpha Cl(\Psi(V_{R^{C}_{1}}))$, for each subset $V_{R^{C}_{1}}$ of R^{C}_{1},

(g) $\alpha Cl(\Psi^{+}(\Gamma_{R^{N}_{2}})) \subset \Psi^{+}(Neu - \alpha Cl(\Gamma_{R^{N}_{2}}))$, for each Neutrosophic set $\Gamma_{R^{N}_{2}}$ of R^{N}_{2}.

(h) $\Psi\left(Cl\left(Int\left(Cl\left(A_{R^{C}_{1}}\right)\right)\right)\right) \subset Neu - \alpha Cl(\Psi(A_{R^{C}_{1}}))$, for each subset $A_{R^{C}_{1}}$ of R^{C}_{1}.

Proof.

(a)⇒(b). Let $\lambda_{R^{N}_{2}} \in \alpha O(NUTSR^{N}_{2})$ and $x_0 \in \Psi^{-}(\lambda_{R^{N}_{2}})$ such that $\Psi(x_0) q \lambda_{R^{N}_{2}}$, since Ψ is Neutrosophic lower α-irresolute, Applying previous theorem, it follows that $x_0 \in sCl(Int(\Psi^{-}(\lambda_{R^{N}_{2}})))$. As x_0 is chosen arbitray in $\Psi^{-}(\lambda_{R^{N}_{2}})$, we have $\Psi^{-}(\lambda_{R^{N}_{2}}) \subset sCl(Int\Psi^{-}(\lambda_{R^{N}_{2}}))$ and thus $\Psi^{-}(\lambda_{R^{N}_{2}}) \in \alpha O(CTSR^{C}_{1})$. Hence $\Psi^{-}(\lambda_{R^{N}_{2}})$ is an α −open in R^{C}_{1}.(b)⇒(a). Let $x_0 \in R^{C}_{1}$ and $\lambda_{R^{N}_{2}} \in \alpha O(NUTSR^{N}_{2})$ such that $\Psi(x_0) q \lambda_{R^{N}_{2}}$, so that $x_0 \in \Psi^{-}(\lambda_{R^{N}_{2}})$. By hypothesis $\Psi^{-}(\lambda_{R^{N}_{2}}) \in \alpha O(CTSR^{C}_{1})$. We have $x_0 \in \Psi^{-}(\lambda_{R^{N}_{2}}) \subset sCl(Int(\Psi^{-}(\lambda_{R^{N}_{2}})))$ and we get Ψ is Neutrosophic lower α-irresolute at x_0. As x_0 was arbitrarily chosen, Ψ is Neutrosophic lower α-irresolute.

(b)⇔(c). From the definition, both are equivalent.

(c)⇒ (d). Let $\Gamma_{R^{N}_{2}} \in \left(NUTS\Gamma_{R^{N}_{2}}\right)$. taking closure , Neu- $\alpha Cl\left(\Gamma_{R^{N}_{2}}\right)$ is Neutrosophic α-closed set in R^{N}_{2}. By our assumption, $\Psi^{+}\left(Neu - \alpha Cl\left(\Gamma_{R^{N}_{2}}\right)\right) \in \alpha C(CTSR^{C}_{1})$.

We know that sIntCl($A_{R^{C}_{1}}$) $\subset A_{R^{C}_{1}}$ iff $A_{R^{C}_{1}} \in \alpha C(CTSR^{C}_{1})$.

we obtain $\Psi^{+}\left(Neu - \alpha Cl\left(\Gamma_{R^{N}_{2}}\right)\right) \supset sInt\left(Cl\left(\Psi^{+}\left(Neu - Cl\left(\Gamma_{R^{N}_{2}}\right)\right)\right)\right) \supset sInt\left(Cl\left(\Psi^{+}\left(\Gamma_{R^{N}_{2}}\right)\right)\right)$.

(d) ⇒ (e) Suppose that (d) is satisfied and let $V_{R^{C}_{1}}$ be an arbitrary subset of R^{C}_{1}. Let us Take $\Gamma_{R^{N}_{2}} = \Psi(V_{R^{C}_{1}})$, Then $V_{R^{C}_{1}} \subset \Psi^{+}(\Gamma_{R^{N}_{2}})$. Therefore, by hypothesis, we have

$$sInt(Cl(V_{R^{C}_{1}})) \subset sInt(Cl(\Psi^{+}(\Gamma_{R^{N}_{2}}))) \subset \Psi^{+}(Neu - \alpha Cl(\Gamma_{R^{N}_{2}})).$$

Therefore, $\Psi\left(sInt\left(Cl\left(V_{R^{C}_{1}}\right)\right)\right) \subset \Psi\left(\Psi^{+}\left(Neu - \alpha Cl\left(\Gamma_{R^{N}_{2}}\right)\right)\right) \subset Neu - \alpha Cl\left(\Gamma_{R^{N}_{2}}\right) = Neu - \alpha Cl\left(\Psi\left(V_{R^{C}_{1}}\right)\right)$.

(e) ⇒(c). Suppose that (e) is true. and let $\Gamma_{R^{N}_{2}} \in \alpha C(NUTSR^{N}_{2})$. Put $V_{R^{C}_{1}} = \Psi^{+}(\Gamma)$, Then $\Psi(V_{R^{C}_{1}}) \subset \Gamma_{R^{N}_{2}}$. Therefore, by our hypothesis, we have $\Psi\left(sInt\left(Cl\left(V_{R^{C}_{1}}\right)\right)\right) \subset Neu - \alpha Cl\left(\Psi\left(V_{R^{C}_{1}}\right)\right) \subset Neu - \alpha Cl(\Gamma_{R^{N}_{2}}) = \Gamma_{R^{N}_{2}}$.And $\Psi^{+}(\Psi(sInt(Cl(V_{R^{C}_{1}})))) \subset \Psi^{+}(\Gamma_{R^{N}_{2}})$. Since we always have $\Psi^{+}(\Psi(sInt(Cl(V_{R^{C}_{1}})))) \supset sInt(Cl(V_{R^{C}_{1}}))$, Then must verify $\Psi^{+}(\Gamma_{R^{N}_{2}}) \supset sInt\left(Cl\left(\Psi^{+}\left(\Gamma_{R^{N}_{2}}\right)\right)\right)$. We know that sIntCl$V_{R^{C}_{1}} \subset V_{R^{C}_{1}}$ iff $V_{R^{C}_{1}} \in \alpha C(CTSR^{C}_{1})$, Finally we get $F^{+}(\Gamma_{R^{N}_{2}}) \in \alpha C(CTSR^{C}_{1})$.

(c)⇒ (f). Here $V_{R^{C}_{1}} \subset \Psi^{+}(\Psi(V_{R^{C}_{1}}))$, we have $V_{R^{C}_{1}} \subset \Psi^{+}(Neu - Cl(\Psi(V_{R^{C}_{1}})))$. NowNeu- $\alpha Cl(\Psi(V_{R^{C}_{1}}))$ is an Neutrosophic α-closed set in R^{N}_{2} and so by our assumption, $\Psi^{+}(Neu - Cl(\Psi(V_{R^{C}_{1}}))) \in \alpha C(CTSR^{C}_{1})$.Thus $\alpha Cl(V_{R^{C}_{1}}) \subset \Psi\Psi^{+}(Neu - \alpha Cl(\Psi(V_{R^{C}_{1}})))$.

Consequently, $\Psi\left(\alpha Cl\left(V_{R^{C}_{1}}\right)\right) \subset \Psi\left(\Psi^{+}\left(Neu - \alpha Cl\left(\Psi\left(V_{R^{C}_{1}}\right)\right)\right)\right) \subset Neu - \alpha Cl(\Psi(V_{R^{C}_{1}}))$.

(f)⇒ (c). Let $\Gamma_{R^{N}_{2}} \in \alpha CO(NUTSR^{N}_{2})$. Replacing $V_{R^{C}_{1}}$ by Ψ^{+} we get by(f), $\Psi(\alpha Cl(\Psi^{+}(\Gamma_{R^{N}_{2}}))) \subset Neu - \alpha Cl(\Psi(\Psi^{+}(\Gamma_{R^{N}_{2}}))) \subset Neu - \alpha Cl(\Gamma_{R^{N}_{2}}) = \Gamma_{R^{N}_{2}}$.Consequently, $\alpha Cl(\Psi^{+}(\Gamma_{R^{N}_{2}})) \subset \Psi^{+}(\Gamma_{R^{N}_{2}})$. But $\Psi^{+}(\Gamma_{R^{N}_{2}}) \subset \alpha Cl(\Psi^{+}(\Gamma_{R^{N}_{2}}))$ and so, $\alpha Cl(\Psi^{+}(\Gamma_{R^{N}_{2}})) = \Psi^{+}(\Gamma_{R^{N}_{2}})$.

Thus $\Psi^{+}(\Gamma_{R^{N}_{2}}) \in \alpha C(CTSR^{C}_{1})$.

(f) ⇒ (g). Let$\Gamma_{R^N{}_2}$ be any Neutrosophic set of $R^N{}_2$. Replacing $V_{R^C{}_1}$ by $\Psi^+(\Gamma_{R^N{}_2})$ we get by (f),$\Psi\left(\alpha Cl\left(\Psi^+\left(\Gamma_{R^N{}_2}\right)\right)\right) \subset NEU - \alpha Cl(\Psi(\Psi^+(\Gamma_{R^N{}_2}))) \subset Neu - \alpha Cl(\Gamma_{R^N{}_2})$.Therefore we get $\alpha Cl(\Psi^+(\Gamma_{R^N{}_2})) \subset \Psi^+(Neu - \alpha Cl(\Gamma_{R^N{}_2}))$.

(g)⇒ (f). Replacing $\Gamma_{R^N{}_2}$ by $\Psi(V_{R^C{}_1})$, where $V_{R^C{}_1}$ is a subset of $R^C{}_1$, we get by our result (g),$\alpha Cl(V_{R^C{}_1}) \subset \alpha Cl(\Psi^+(\Psi(V_{R^C{}_1}))) = \alpha Cl(\Psi^+(\Gamma_{R^N{}_2})) = \Psi^+(\alpha Cl(\Gamma_{R^N{}_2})) = \Psi^+(\alpha Cl(\Psi(V_{R^C{}_1})))$.Thus $\Psi(\alpha Cl(V_{R^C{}_1})) \subset \Psi(\Psi^+(\alpha Cl(\Psi(V_{R^C{}_1}))) \subset Neu\,\alpha Cl(\Psi(V_{R^C{}_1}))$.

(e)⇒ (h). Clearly is true from the above result.

(h)⇒(a). Let $\xi \in R^C{}_1$and $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$such that $(\xi)q\Gamma_{R^N{}_2}$. Then $\xi \in \Psi^-(\Gamma_{R^N{}_2})$.We shall show that $\Psi^-(\Gamma_{R^N{}_2}) \in \alpha O(CTSR^C{}_1)$. By the hypothesis, We have $\Psi(Cl(Int(Cl(\Psi^+(\Gamma_{R^N{}_2}{}^c))))) \subset Neu - \alpha Cl(\Psi(\Psi^+(\Gamma_{R^N{}_2}{}^c))) \subset (\Gamma_{R^N{}_2}{}^c)$, Which implies $Cl(Int(Cl(\Psi^+(\Gamma_{R^N{}_2}{}^+))))) \subset \Psi^+ (\Gamma_{R^N{}_2}{}^c) \subset (\Psi^-(\Gamma_{R^N{}_2}))^c$.Therefore, we obtain $\Psi^-(\Gamma_{R^N{}_2}) \subset Int(Cl(Int(\Psi^-(\Gamma_{R^N{}_2}))))$. Hence $\Psi^-(\Gamma_{R^N{}_2}) \in \alpha O(CTSR^C{}_1)$. Put $U_{R^C{}_1} = \Psi^-(\Gamma_{R^N{}_2})$. Then $\xi \in U_{R^C{}_1} \in \alpha O(CTSR^C{}_1)$ and $\Psi(u)q\Gamma_{R^N{}_2}$ for every $u \in U_{R^C{}_1}$. Therefore Ψ is Neutrosophic lower α-irresolute.

IV. Upper α-Irresolute Neutrosophic Multifunctions

In this section, we introduce the Definition for Neutrosophic upper α- irresolute multifunction and its properties

Definition 4.1.

An Neutrosophicmultifunction $\Psi: (R^C{}_1,\tau_{R^C{}_1}) \to (R^N{}_2.,\tau_{N_{R^N{}_2}})$, is called

(a) Neutrosophic upper α-irresolute at a point $x_0 \in R^C{}_1$, if for any $\Gamma_{R^N{}_2},\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2.)$ such that $\Psi(x_0) \subset \Gamma_{R^N{}_2}$ there exists $U_{R^C{}_1} \in \alpha O(CTSR^C{}_1)$ containing $x_0$such that $\Psi(U_{R^C{}_1}) \subset \Gamma_{R^N{}_2}$.

(b) Neutrosophic upper α-irresolute if it is satisfied that property at each point of $R^C{}_1$.

Theorem 4.2

Every Neutrosophic upper α-irresolute multifunction is Neutrosophic upperα-continuous multifunction.

Proof:

Letting$x_0 \in R^C{}_1$, $\Psi : (R^C{}_1,\tau_{R^C{}_1}) \to (R^N{}_2.,\tau_{N_{R^N{}_2}})$ and $\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$ such that $\Psi(x_0) \subset \Gamma_{R^N{}_2}$, But we know that , every$\Gamma_{R^N{}_2}$,$\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$ is $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$, Therefore $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$,By our assumption ,Neutrosophic lower $\alpha -$ irresolute multifunction, There exists $U_{R^C{}_1} \in \alpha$O(CTS$R^C{}_1$) containing x_0 such that $\Psi(\xi) \subset \Gamma_{R^N{}_2}, \forall\xi \in U_{R^C{}_1}$Hence Ψ is Neutrosophic lower α-continuous multifunction at x_0.

Theorem 4.3

Every Neutrosophic upper α-irresolute multifunction is Neutrosophic upper Pre-continuous multifunction.

Proof:

Letting $x_0 \in R^C{}_1$, $\Psi : (R^C{}_1,\tau_{R^C{}_1}) \to (R^N{}_2.,\tau_{N_{R^N{}_2}})$ and $\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$ such that $\Psi(x_0) \subset \Gamma_{R^N{}_2}$.But we know that , Every$\Gamma_{R^N{}_2}$,$\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$is $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$.Therefore $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$.By our assumption ,Neutrosophic upperα −irresolute multifunction, There exists $U_{R^C{}_1} \in \alpha$O(CTS$R^C{}_1$) containing x_0 such that $\Psi(\xi) \subset \Gamma_{R^N{}_2}, \forall\xi \in U_{R^C{}_1}$, every$U_{R^C{}_1}$,$U_{R^C{}_1} \in \alpha O(NUTSR^N{}_2)$ is $U_{R^C{}_1} \in PO(CTTSR^N{}_2)$.There exists $U_{R^C{}_1} \in$ PO(CTS$R^C{}_1$) containing x_0 such that $\Psi(\xi) \subset \Gamma_{R^N{}_2}, \forall\xi \in U_{R^C{}_1}$hence Ψ is Neutrosophic upperPre-continuous multifunction at x_0.

Theorem 4.4

Every Neutrosophic upper α-irresolute multifunction is Neutrosophic upper quasi semi continuous multifunction.

Proof:

Letting $x_0 \in R^C{}_1$, Ψ : $(R^C{}_1,\tau_{R^C{}_1}) \to (R^N{}_2.,\tau_{N_{R^N{}_2}})$ and $\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$ such that $\Psi(x_0) \subset \Gamma_{R^N{}_2}$,.But we know that , Every$\Gamma_{R^N{}_2}$,$\Gamma_{R^N{}_2} \in O(NUTSR^N{}_2)$is $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$,
Therefore $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$, By our assumption ,Neutrosophic upperα -irresolute multifunction, there exists $U_{R^C{}_1} \in \alpha$O(CTS$R^C{}_1$) containing x_0 such that $\Psi(\xi) \subset \Gamma_{R^N{}_2}, \forall \xi \in U_{R^C{}_1}$. Every$U_{R^C{}_1}$, $U_{R^C{}_1} \in \alpha O(NUTSR^N{}_2)$ is $U_{R^C{}_1} \in SO(CTTSR^N{}_2)$. Their exists $U_{R^C{}_1} \in$ SO(CTS$R^C{}_1$) containing x_0 such that $\Psi(\xi) \subset \Gamma_{R^N{}_2}, \forall \xi \in U_{R^C{}_1}$Hence Ψ is Neutrosophic upper quasi semi continuous multifunction at x_0.

Theorem 4.5

Let : $(R^C{}_1,\tau_{R^C{}_1}) \to (R^N{}_2.,\tau_{N_{R^N{}_2}})$, be an Neutrosophic multifunction and let $\xi \in R^C{}_1$. Then the following statements are equivalent:

(a) $\Psi$is Neutrosophic Upper α-irresolute at ξ.

(b) For each $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$ with $(\xi) \subset \Gamma_{R^N{}_2}$, Implies $\xi \in sCl(Int(\Psi^-(\Gamma)))$.

(c) For any ξ , $\xi \in U_{R^C{}_1} \in SO(CTSR^C{}_1)$ and for any $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$ with $(\xi) \subset \Gamma_{R^N{}_2}$, there exists a nonempty open set $V_{R^C{}_1} \subset U_{R^C{}_1}$ such that $\Psi(V_{R^C{}_1}) \subset \Gamma_{R^N{}_2}$.

Proof.

(a)⇒ (b) Let $\xi \in R^C{}_1$ and $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$ Such that $\Psi(\xi) \subset \Gamma_{R^N{}_2}$ Then by our assumption (a), we get there exists $U_{R^C{}_1} \in \alpha$O(CTS$R^C{}_1$) such that $\xi \in U_{R^C{}_1}$ and $F(U_{R^C{}_1}) \subset \Gamma_{R^N{}_2}$, Thus $\xi \in U_{R^C{}_1} \subset \Psi^+(\Gamma_{R^N{}_2})$. here $U_{R^C{}_1} \in \alpha$O(CTS$R^C{}_1$) .We know that for any set $A_{R^C{}_1}$, $A_{R^C{}_1} \in \alpha$O(CTS$R^C{}_1$)⇔ $A_{R^C{}_1} \subset sCl(Int(A_{R^C{}_1}))$. Therefore, $U_{R^C{}_1} \subset sCl(Int(U_{R^C{}_1}))$. Finally we get $\xi \in sCl(Int\Psi^+(\Gamma_{R^N{}_2}))$.hence(b).

(b)⇒ (c). Let $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$ such that $\Psi(\xi) \subset \Gamma_{R^N{}_2}$, then $\xi \in sCl(Int\Psi^-(\Gamma_{R^N{}_2}))$. Let $U_{R^C{}_1} \in S$O(CTS$R^C{}_1$) and $\xi \in U_{R^C{}_1}$.Then$U_{R^C{}_1} \cap Int(\Psi^-(\Gamma_{R^N{}_2})) \neq \varphi$ and $U_{R^C{}_1} \cap Int(\Psi^-(\Gamma_{R^N{}_2}))$ is semi-open in $R^C{}_1$.Put $V_{R^C{}_1} = Int(U_{R^C{}_1} \cap Int(\Psi^-(\Gamma_{R^N{}_2}))$, Then $V_{R^C{}_1}$ is an open set of $R^C{}_1$, $V_{R^C{}_1} \subset U_{R^C{}_1}$, $V_{R^C{}_1} \neq \varphi$ and $\Psi(V_{R^C{}_1}) \subset \Gamma_{R^N{}_2}$,

(c) ⇒(a). Let $\{U_\xi\}$ be the system of the SO(CTS$R^C{}_1$) containingξ. Let $U_{R^C{}_1} \in S$O(CTS$R^C{}_1$) and $\xi \in U_{R^C{}_1}$ and Let $\Gamma_{R^N{}_2} \in \alpha O(NUTSR^N{}_2)$ such that $\Psi(\xi) \subset \Gamma_{R^N{}_2}$, there exists a nonempty open set $B_U \subset U_{R^C{}_1}$ Such that $\Psi(v) \subset \Gamma_{R^N{}_2}, \forall v \in B_U$. Let $W_{R^C{}_1} = \cup B_U : U_{R^C{}_1} \in \{U_\xi\}$, then $W_{R^C{}_1} \in$O(CTS$R^C{}_1$) and $\xi \in sCl(W_{R^C{}_1})$ and $\Psi(v) \subset \Gamma_{R^N{}_2}, \forall v \in W_{R^C{}_1}$ .Put $S_{R^C{}_1} = W_{R^C{}_1} \cup \xi$.Then $W_{R^C{}_1} \subset S_{R^C{}_1} \subset sCl(W_{R^C{}_1})$. Thus $S_{R^C{}_1} \in \alpha O(CTSR^C{}_1)), \xi \in S_{R^C{}_1}$ and $\Psi(v) \subset \Gamma_{R^N{}_2}, \forall v \in S$. Hence Ψ is Neutrosophic Upper α-irresolute at ξ.

Theorem 4.6

For an Neutrosophicmultifunction : $(R^C{}_1,\tau_{R^C{}_1}) \to (R^N{}_2,\tau_{N_{R^N{}_2}})$the following statements are equivalent:

(a) Ψ is Neutrosophic upper α-irresolute.

(b) $\Psi^+(\Gamma_{R^N{}_2}) \in \alpha O(CTSR^C{}_1)$,for every Neutrosophic α-open set $\Gamma_{R^N{}_2} of\ R^N{}_2$

(c) $\Psi^-(\lambda_{R^N{}_2.}) \in \alpha C(CTSR^C{}_1)$,for each Neutrosophic α-closed set $\lambda_{R^N{}_2.} of\ R^N{}_2$.

(d) For each point $\xi \in R^C{}_1$ and for each α-neighborhood $V_{R^N{}_2.}$ of $\Psi(\xi)\ in\ R^N{}_2.\ F^+(V_{R^N{}_2.})$ is an α-neighborhood of ξ.

(e) For each point $\xi \in R^C{}_1$ and for each α-neighborhood $V_{R^N{}_2.}$of Ψ (ξ) in $R^N{}_2$., there is an α-neighborhood $U_{R^C{}_1}$ of ξ such that $\Psi(U_{R^C{}_1}) \subset V_{R^N{}_2.}$.

(f) $\alpha Cl(\Psi^-(\lambda_{R^N{}_2.})) \subset \Psi^-(Neu - \alpha Cl(\lambda_{R^N{}_2.}))\ for$ each Neutrosophic set $\lambda_{R^N{}_2.} of\ R^N{}_2$.

(g) $sInt(Cl(\Psi^-(\lambda_{R^N{}_2.}))) \subset \Psi^-(Neu - \alpha Cl(\lambda_{R^N{}_2.}))$for any Neutrosophic set $\lambda\ of\ R^N{}_2$..

Proof.

(a)⇒ (b). Let $\Gamma_{R^N{}_2}\epsilon\, \alpha O(NUTSR^N{}_2.)$ and $\xi \in \Psi^+(\Gamma_{R^N{}_2})$.Applying previous theorem, we get $\xi \in sCl(Int\Psi^+(\Gamma_{R^N{}_2}))$.Therefore, we obtain $\Psi^+(\Gamma_{R^N{}_2}) \subset sCl(Int\Psi^+(\Gamma_{R^N{}_2}))$., Finally we get $\Psi^+(\Gamma_{R^N{}_2}) \in \alpha O(CTSR^C{}_1)$.

(b)⇒ (a). Let ξbe arbitrarily point in $R^C{}_1$ and $\Gamma_{R^N{}_2}\epsilon\alpha O(NUTSR^N{}_2.)$ such that $\Psi(\xi) \subset \Gamma_{R^N{}_2}$so $\in \Psi^+(\Gamma_{R^N{}_2})$. By hypothesis $\Psi^+(\Gamma_{R^N{}_2}) \in \alpha O(CTSR^C{}_1)$, we get$\xi \in \Psi^+(\Gamma_{R^N{}_2}) \subset sCl(Int(\Psi^+(\Gamma_{R^N{}_2})))$ and hence F is Neutrosophic upper α-irresolute at ξ.As ξ is arbitrarily chosen, Ψ is Neutrosophic upper α-irresolute.

(b)⇒ (c). This implies easily get from that $[\Psi^-(\Gamma_{R^N{}_2})]^C = [\Psi^+(\Gamma_{R^N{}_2})^C]$.where $\Gamma_{R^N{}_2}\epsilon\alpha O(NUTSY)$

(c)⇒ (f). Let $\lambda_{R^N{}_2.}\,\epsilon\alpha O(NUTSR^N{}_2.)$ .Then by our assumption (c), $\Psi^-(Neu - \alpha Cl(\lambda_{R^N{}_2.}))$ is an α-closed set in $R^C{}_1$. We have $\Psi^-(Neu - \alpha Cl(\lambda_{R^N{}_2.})) \supset sInt(Cl(\Psi^-(Neu - Cl(\lambda_{R^N{}_2.})))) \supset sInt(Cl(\Psi^-(\lambda_{R^N{}_2.}))) \supset \Psi^-(\lambda) \cup sInt(Cl(\Psi^-(\lambda_{R^N{}_2.}))) \supset \alpha Cl(\Psi^-(\lambda_{R^N{}_2.}))$.Hence the result.

(f)⇒(g). Let $\lambda_{R^N{}_2.}\epsilon\alpha O(NUTSR^N{}_2.)$. we have $\alpha Cl(\Psi^-(\lambda_{R^N{}_2.})) = \Psi^-(\lambda_{R^N{}_2.}) \cup sInt(Cl(\Psi^-(\lambda_{R^N{}_2.}))) \subset \Psi^-(Neu - \alpha Cl(\lambda_{R^N{}_2.}))$. Hence (g).

(g)⇒(c). Let $\lambda_{R^N{}_2.}{}^C\epsilon\, \alpha C(NUTSR^N{}_2.)$Then by (g) we have,

$$sInt\left(Cl\left(\Psi^-(\lambda_{R^N{}_2.}{}^C)\right)\right) \subset \Psi^-(\lambda_{R^N{}_2.}{}^C) \cup sInt\left(Cl\left(\Psi^-(\lambda_{R^N{}_2.}{}^C)\right)\right) \subset \Psi^-(\alpha Cl(\lambda_{R^N{}_2.}{}^C) = \Psi^-(\lambda_{R^N{}_2.}{}^C).$$

Hence By our result, $\Psi^-(\lambda_{R^N{}_2.}{}^C) \in \alpha C(CTSR^C{}_1)$.

(b)⇒(d).Let $\xi \in R^C{}_1$ and $V_{R^N{}_2.}$ be an α-neighborhood of $\Psi(\xi)$ in $R^N{}_2$.Then there is an $\lambda_{R^N{}_2.}\,\epsilon\alpha O(NUTSR^N{}_2.)$ such that $\Psi(\xi) \subset \lambda_{R^N{}_2.} \subset V_{R^N{}_2.}$. Hence, $\xi \in \Psi^+(\lambda_{R^N{}_2}) \subset \Psi^+(V_{R^N{}_2})$. Now by hypothesis $\Psi^+(\lambda_{R^N{}_2}) \in \alpha O(CTSR^C{}_1)$, and Thus $\Psi^+(V_{R^N{}_2.})$ is an α-neighborhood of ξ.

(d)⇒(e). Let $\xi \in R^C{}_1$and $V_{R^N{}_2}$ be an α-neighborhood of $\Psi(\xi)$ in $R^N{}_2$.Put $U_{R^C{}_1} = \Psi^+(V_{R^N{}_2.})$.Then $U_{R^C{}_1}$is an α-neighborhood of ξ and $\Psi(U) \subset V_{R^N{}_2.}$

(e)⇒(a). Let$\xi \in R^C{}_1$ and$V_{R^N{}_2}$be an Neutrosophic set in $R^N{}_2$ such that $\Psi(\xi) \subset V_{R^N{}_2.}$. $V_{R^N{}_2.}$ being an Neutrosophic α-open set in $R^N{}_2$. , is an α-neighborhood of $\Psi(\xi)$ and according to the hypothesis there is an α-neighborhood $U_{R^C{}_1}$ of ξ such that $\Psi(U_{R^C{}_1}) \subset V_{R^N{}_2.}$ Therefore $V_{R^N{}_2.} \in \alpha O(CTSR^C{}_1)$ such that $\xi \in A_{R^C{}_1} \subset U_{R^C{}_1}$ and hence $\Psi(A) \subset \Psi(U_{R^C{}_1}) \subset V_{R^N{}_2.}$. Hence Ψ is Neutrosophic upper α-irresolute at ξ.

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Neutrosophic Fuzzy Matrices and Some Algebraic Operations

Rakhal Das[1], Florentin Smarandache[2] and Binod Chandra Tripathy[3,*]
[1] Department of Mathematics, Tripura University Agartala -799022; Tripura, INDIA; E mail: rakhaldas95@gmail.com and rakhal.mathematics@tripurauniv.in;
[2] Department of Mathematics, University of New Mexico, 705 Gurley Avenue, Gallup, NM 87301, USA.; E mail : fsmarandache@gmail.com
[3] Department of Mathematics, Tripura University Agartala -799022; Tripura, INDIA; E mail binodtripathy@tripurauniv.in, tripathybc@yahoo.com and tripathybc@gmail.com
*Correspondence: e-mail tripathybc@yahoo.com; Tel.: (+91 9864087231)

Abstract: In this article, we study neutrosophic fuzzy set and define the subtraction and multiplication of two rectangular and square neutrosophic fuzzy matrices. Some properties of subtraction, addition and multiplication of these matrices and commutative property, distributive property have been examined.
Keywords: Neutrosophic fuzzy matrix, Neutrosophic set. Commutativity, Distributive, Subtraction of neutrosophic matrices.

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1. Introduction

Neutrosophic set was introduced by Florentin Smarandache [1] in 1998, where each element had three associated defining functions, namely the membership function (T), the non-membership (F) function and the indeterminacy function (I) defined on the universe of discourse X, the three functions are completely independent. Relative to the natural problems sometimes one may not be able to decide. After the development of the Neutrosophic set theory, one can easily take decision and indeterminacy function of the set is the nondeterministic part of the situation. The applications of the theory has been found in various field for dealing with indeterminate and inconsistent information in real world one may refer to [2,3,4]. Neutrosophic set is a part of neutrosophy which studied the origin, nature and scope of neutralities, as well as their interactions with ideational spectra. The neutrosophic set generalizes the concept of classical fuzzy set [10, 11], interval valued fuzzy set, intuitionistic fuzzy set and so on. In the recent years, the concept of neutrosophic set has been applied successfully by Broumi et al. [12, 13, 14] and Abdel-Basset et al. [15, 16, 17, 18]

The single-valued neutrosophic number which is a generalization of fuzzy numbers and intuitionistic fuzzy numbers. A single-valued neutrosophic number is simply an ordinary number whose precise value is somewhat uncertain from a philosophical point of view. There are two special forms of single-valued neutrosophic numbers such as single-valued trapezoidal neutrosophic numbers and single-valued triangular neutrosophic numbers.

The neutrosophic interval matrices have been defined by Vasantha Kandasamy and Florentin Smarandache in their book "Fuzzy interval matrices, Neutrosophic interval matrices, and applications". A neutrosophic fuzzy matrix $[a_{ij}]_{nxm}$, whose entries are of the form $a + Ib$ (neutrosophic number), where a, b are the elements of the interval [0,1] and I is an indeterminate such that $I^n = I$, n being a positive integer.

So the difference between the neutrosophic number of the form $a + Ib$ and the single-valued neutrosophic numbers is that the generalization of fuzzy number and the single-valued neutrosophic components <*T*, *I*, *F*> is the generalization of fuzzy numbers and intuitionistic fuzzy numbers. Since fuzzy number lies between 0 to 1 so the component neutrosophic fuzzy number a and b lies in [0,1]. In the case of single-valued neutrosophic matrix components will be the true value, indeterminacy and fails value with three components in each element of a matrix [3, 4, 8].

We know the important role of matrices in science and technology. However, the classical matrix theory sometimes fails to solve the problems involving uncertainties, occurring in an imprecise environment. Kandasamy and Smarandache [7] introduced fuzzy relational maps and neutrosophic relational maps. Thomason [8], introduced the fuzzy matrices to represent fuzzy relation in a system based on fuzzy set theory and discussed about the convergence of powers of fuzzy matrix. Dhar, Broumi and Smarandache [2] define Square Neutrosophic Fuzzy Matrices whose entries are of the form $a+Ib$, where a and b are fuzzy number from [0, 1] gives the definition of Neutrosophic Fuzzy Matrices multiplication.

In this paper our ambition is to define the subtraction of fuzzy neutrosophic matrices, rectangular fuzzy neutrosophic matrices and study some algebraic properties. We shall focus on all types of neutrosophic fuzzy matrices. The paper unfolds as follows. The next section briefly introduces some definitions related to neutrosophic set, neutrosophic matrices, Fuzzy integral neutrosophic matices and fuzzy matrix. Section 3 presents a new type of fuzzy neutrosophic matrices and investigated some properties such as subtraction, commutative property and distributive property.

2. Materials and Methods (proposed work with more details)

In this section we recall some concepts of neutrosophic set, neutrosophic matrices and fuzzy neutrosophic matrices proposed by Kandasamy and Smarandache in their monograph [3], and also the concept of fuzzy matrix (One may refer to [2])

Definition 2.1 (Smarandache [1]). Let U be an universe of discourse then the neutrosophic set A is an object having the form $A = \{< x{:}T_A(x), I_A(x), F_A(x)>, x \in U\}$, where the functions $T, I, F : U \to\ ]^-0, 1^+[$ define respectively the degree of membership (or Truthness), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $x \in U$ to the set A with the condition.

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+.$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. So instead of $]^-0, 1^+[$ we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$will be difficult to apply in the real applications such as in scientific and engineering problems.

Definition 2.2 (Dhar et al. [3]). Let $M_{mxn}= \{(a_{ij}) : a_{ij} \in K(I)\}$, where $K(I)$, is a neutrosophic field. We call M_{mxn} to be the neutrosophic matrix.

Example 2.1: Let $R(I) = \langle R \cup I \rangle$be the neutrosophic field

$$M_{4x3} = \begin{pmatrix} 5 & 0 & 2.1I \\ 3.5I & 3 & 5 \\ 7 & 4I & 0 \\ 8 & -5I & I \end{pmatrix}$$

M_{4x3} denotes the neutrosophic matrix, with entries from real and the indeterminacy.

Definition 2.3 (Kandasamy and Smarandache [5])

Let $N = [0, 1] \cup I$ where I is the indeterminacy. The $m \times n$ matrices $M_{mxn} = \{(a_{ij}) : a_{ij} \in [0, 1] \cup I\}$ is called the fuzzy integral neutrosophic matrices. Clearly the class of $m \times n$ matrices is contained in the class of fuzzy integral neutrosophic matrices.

The row vector $1 \times n$ and column vector $m \times 1$ are the fuzzy neutrosophic row matrices and fuzzy neutrosophic column matrices respectively.

Example 2.2: Let $M_{4x3} = \begin{pmatrix} 0.5 & 0 & 0.1I \\ I & 0.3 & 0.5 \\ 0.7 & 0.4I & 0 \\ 0.8 & 0.5I & I \end{pmatrix}$ be a 4 ×3 integral fuzzy neutrosophic matrix

Definition 2.5 (Kandasamy and Smarandache [5]).

Let $N_s = [0, 1] \cup \{bI : b \in [0, 1]\}$; we call the set N_s to be the fuzzy neutrosophic set. Let N_s be the fuzzy neutrosophic set. $M_{mxn} = \{(a_{ij}): a_{ij} \in N_s\ i = 1$ to m and $j = 1$ to $n\}$ we call the matrices with entries from N_s to be the fuzzy neutrosophic matrices.

Example 2.3: Let $N_s = [0,1] \cup \{bI: b \in [0,1]\}$ be the fuzzy neutrosophic set and

$$P = \begin{pmatrix} 0.5 & 0 & 0.1I \\ I & 0.3 & 0.5 \\ 0 & I & 0.01 \end{pmatrix}$$

be a 3 ×3 fuzzy neutrosophic matrix.

Definition 2.6 (Thomas [9]). A fuzzy matrix is a matrix which has its elements from the interval [0, 1], called the unit fuzzy interval. A_{mxn} fuzzy matrix for which $m = n$ (i.e. the number of rows is equal to the number of columns) and whose elements belong to the unit interval [0, 1] is called a fuzzy square matrix of order n. A fuzzy square matrix of order two is expressed in the following way

$A = \begin{pmatrix} x & y \\ t & z \end{pmatrix}$,

where the entries x, y, t, z all belongs to the interval [0,1].

Definition 2.7 (Kandasamy and Smarandache [5]). Let A be a neutrosophic fuzzy matrix, whose entries is of the form $a + Ib$ (neutrosophic number), where a, b are the elements of [0,1] and I is an indeterminate such that $I^n = I$, n being a positive integer.

$$A = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}$$

Definition 2.8 Multiplication Operation of two Neutrosophic Fuzzy Matrices

Consider two neutrosophic fuzzy matrices, whose entries are of the form $a + Ib$ (neutrosophic number), where a, b are the elements of [0,1] and I is an indeterminate such that $I^n = I$, n being a positive integer, given by

$$A = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix}, \quad B = \begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$$

The Multiplication Operation of two Neutrosophic Fuzzy Matrices is given by

$$AB = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \end{pmatrix} \begin{pmatrix} m_1 + In_1 & m_2 + In_2 \\ m_3 + In_3 & m_4 + In_4 \end{pmatrix}$$

$$D_{11} = [\max\{ \min(x_1, m_1), \min(x_2, m_3)\} + I \max\{ \min\{(y_1, n_1), \min(y_2, n_3)\}]$$

$$D_{21} = [\max \{\min(x_1, m_2), \min (x_2, m_4)\} + I \max \{\min(y_1, n_2), \min(y_2, n_4)\}]$$

$$D_{21} = [\max \{\min\{(x_3, m_1), \min (x_4, m_3)\} + I \max \{\min\{(y_3, n_1), \min (y_4, n_3)\}]$$

$$D_{22} = [\max \{\min\{(x_3, m_2), \min (x_4, m_4)\} + I \max\{ \min\{(y_3, n_2), \min (y_4, n_4)\}]$$

Hence, $AB = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}$.

3. Results (examples / case studies related to the proposed work)

In this section we define the subtraction and distributive property of neutrosophic fuzzy matrices along with some properties associated with such matrices.

3.1 Subtraction Operation of two Neutrosophic Fuzzy Matrices

Consider two neutrosophic fuzzy matrices given by

$$A = \begin{pmatrix} x_1 + Iy_1 & x_2 + Iy_2 \\ x_3 + Iy_3 & x_4 + Iy_4 \\ x_5 + Iy_5 & x_6 + Iy_6 \end{pmatrix}$$

and $$B = \begin{pmatrix} t_1 + Iz_1 & t_2 + Iz_2 \\ t_3 + Iz_3 & t_4 + Iz_4 \\ t_5 + Iz_5 & t_6 + Iz_6 \end{pmatrix}.$$

Addition and multiplication between two neutrosophic fuzzy matrices have been defined in Smarandache [2]. We would like to define the subtraction of these two matrices as follows.

$A- B = C$,

where c_{ij} are as follows

$c_{11} = \min\{x_1, t_1\} + I \min\{y_1, z_1\}$

$c_{12} = \min\{x_2, t_2\} + I \min\{y_2, z_2\}$

$c_{21} = \min\{x_3, t_3\} + I \min\{y_3, z_3\}$

$c_{21} = \min\{x_4, t_4\} + I \min\{y_4, z_4\}$

$c_{31} = \min\{x_5, t_5\} + I \min\{y_5, z_5\}$

$c_{32} = \min\{x_6, t_6\} + I \min\{y_6, z_6\}$

Since $\min\{a, b\} = \min\{b, a\}$ so based on this we have the following properties.

Proposition 3.1. The following properties hold in the case of neutrosophic fuzzy matrix for subtraction

(*i*) $A\text{-}B = B\text{-}A$

(*ii*) $(A - B) - C = A - (B - C) = (B - C) - A = (C - B) - A$.

Proof. Consider three neutrosophic fuzzy matrices A, B and C as follows.

$$A = \begin{pmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I \end{pmatrix}, B = \begin{pmatrix} c_{11} + d_{11}I & c_{12} + d_{12}I \\ c_{21} + d_{21}I & c_{22} + d_{22}I \\ c_{31} + d_{31}I & c_{32} + d_{32}I \end{pmatrix}$$

and $$C = \begin{pmatrix} l_{11} + m_{11}I & l_{12} + m_{12}I \\ l_{21} + m_{21}I & l_{22} + m_{22}I \\ l_{31} + m_{31}I & l_{32} + m_{32}I \end{pmatrix}$$

$$A - B = \begin{pmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I \end{pmatrix} - \begin{pmatrix} c_{11} + d_{11}I & c_{12} + d_{12}I \\ c_{21} + d_{21}I & c_{22} + d_{22}I \\ c_{31} + d_{31}I & c_{32} + d_{32}I \end{pmatrix} = D \text{ (say)},$$

where,

$D_{11} = \min\{a_{11}, c_{11}\} + I\min\{b_{11}, d_{11}\} = x_{11} + Iy_{11}$

$D_{12} = \min\{a_{12}, c_{12}\} + I\min\{b_{12}, d_{12}\} = x_{12} + Iy_{12}$

$D_{21} = \min\{a_{21}, c_{21}\} + I\min\{b_{21}, d_{21}\} = x_{21} + Iy_{21}$

$D_{22} = \min\{a_{22}, c_{22}\} + I\min\{b_{22}, d_{22}\} = x_{22} + Iy_{22}$

$D_{31} = \min\{a_{31}, c_{31}\} + I\min\{b_{31}, d_{31}\} = x_{31} + Iy_{31}$

$D_{32} = \min\{a_{32}, c_{32}\} + I\min\{b_{32}, d_{32}\} = x_{32} + Iy_{32}$

$$D = \begin{pmatrix} x_{11} + Iy_{11} & x_{12} + Iy_{12} \\ x_{21} + Iy_{21} & x_{22} + Iy_{22} \\ x_{31} + Iy_{31} & x_{32} + Iy_{32} \end{pmatrix} \text{ and } B - A = \begin{pmatrix} x_{11} + Iy_{11} & x_{12} + Iy_{12} \\ x_{21} + Iy_{21} & x_{22} + Iy_{22} \\ x_{31} + Iy_{31} & x_{32} + Iy_{32} \end{pmatrix} = D,$$

[$\therefore$ $\min(a, c) = \min(c, a)$]

Hence, $A - B = B - A$.

Now we have,

$D - C = (A - B) - C$

$$= \begin{pmatrix} x_{11} + Iy_{11} & x_{12} + Iy_{12} \\ x_{21} + Iy_{21} & x_{22} + Iy_{22} \\ x_{31} + Iy_{31} & x_{32} + Iy_{32} \end{pmatrix} - \begin{pmatrix} l_{11} + m_{11}I & l_{12} + m_{12}I \\ l_{21} + m_{21}I & l_{22} + m_{22}I \\ l_{31} + m_{31}I & l_{32} + m_{32}I \end{pmatrix}$$

$= F$ (say),

where,

$F_{11} = \min\{x_{11}, l_{11}\} + I\min\{y_{11}, m_{11}\} = \min\{a_{11}, c_{11}, l_{11}\} + I\min\{b_{11}, d_{11}, m_{11}\} = n_{11} + Ik_{11}$

$F_{12} = \min\{x_{12}, l_{12}\} + I\min\{y_{12}, m_{12}\} = \min\{a_{12}, c_{12}, l_{12}\} + I\min\{b_{11}, d_{12}, m_{12}\} = n_{12} + Ik_{12}$

$F_{21} = \min\{x_{21}, l_{21}\} + I\min\{y_{21}, m_{21}\} = \min\{a_{21}, c_{21}, l_{21}\} + I\min\{b_{21}, d_{21}, m_{21}\} = n_{21} + Ik_{21}$

$F_{22} = \min\{x_{22}, l_{22}\} + I\min\{y_{22}, m_{22}\} = \min\{a_{22}, c_{22}, l_{22}\} + I\min\{b_{22}, d_{22}, m_{22}\} = n_{22} + Ik_{22}$

$F_{31} = \min\{x_{31}, l_{31}\} + I\min\{y_{31}, m_{31}\} = \min\{a_{31}, c_{31}, l_{31}\} + I\min\{b_{31}, d_{31}, m_{31}\} = n_{31} + Ik_{31}$

$F_{32} = \min\{x_{32}, l_{32}\} + I\min\{y_{32}, m_{32}\} = \min\{a_{31}, c_{31}, l_{31}\} + I\min\{b_{31}, d_{31}, m_{31}\} = n_{32} + Ik_{32}$

$$(A - B) - C = F = \begin{pmatrix} n_{11} + Ik_{11} & n_{12} + Ik_{12} \\ n_{21} + Ik_{21} & n_{22} + Ik_{22} \\ n_{31} + Ik_{31} & n_{32} + Ik_{32} \end{pmatrix}.$$

Next we have,

$$B - C = \begin{pmatrix} c_{11} + d_{11}I & c_{12} + d_{12}I \\ c_{21} + d_{21}I & c_{22} + d_{22}I \\ c_{31} + d_{31}I & c_{32} + d_{32}I \end{pmatrix} - \begin{pmatrix} l_{11} + m_{11}I & l_{12} + m_{12}I \\ l_{21} + m_{21}I & l_{22} + m_{22}I \\ l_{31} + m_{31}I & l_{32} + m_{32}I \end{pmatrix} = E \text{ (say)},$$

where

$E_{11} = \min\{c_{11}, l_{11}\} + I\min\{d_{11}, m_{11}\} = p_{11} + Iq_{11}$

$E_{12} = \min\{c_{12}, l_{12}\} + I\min\{d_{12}, m_{12}\} = p_{12} + Iq_{12}$

$E_{21} = \min\{c_{21}, l_{21}\} + I\min\{d_{21}, m_{21}\} = p_{21} + Iq_{21}$

$E_{22} = \min\{c_{22}, l_{22}\} + I\min\{d_{22}, m_{22}\} = p_{22} + Iq_{22}$

$E_{31} = \min\{c_{31}, l_{31}\} + I\min\{d_{31}, m_{31}\} = p_{31} + Iq_{31}$

$E_{32} = \min\{c_{32}, l_{32}\} + I\min\{d_{32}, m_{32}\} = p_{32} + Iq_{32}.$

We have

$$B - C = E = \begin{pmatrix} p_{11} + Iq_{11} & p_{12} + Iq_{12} \\ p_{21} + Iq_{21} & p_{22} + Iq_{22} \\ p_{31} + Iq_{31} & p_{32} + Iq_{32} \end{pmatrix}$$

$$A - (B - C) = \begin{pmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I \end{pmatrix} - \begin{pmatrix} p_{11} + Iq_{11} & p_{12} + Iq_{12} \\ p_{21} + Iq_{21} & p_{22} + Iq_{22} \\ p_{31} + Iq_{31} & p_{32} + Iq_{32} \end{pmatrix},$$

where

$\min\{a_{11}, p_{11}\} + I\min\{b_{11}, q_{11}\} = \min\{a_{11}, c_{11}, l_{11}\} + I\min\{b_{11}, d_{11}, m_{11}\}$

min{a_{12}, p_{12}}+Imin{b_{12}, q_{12}} = min{a_{12}, c_{12}, l_{12}}+Imin{b_{11}, d_{12}, m_{12}}

min{a_{21}, p_{21}}+Imin{b_{21}, q_{21}} = min{a_{21}, c_{21}, l_{21}}+Imin{b_{21}, d_{21}, m_{21}}

min{a_{22}, p_{22}}+Imin{b_{22}, q_{22}}= min{a_{22}, c_{22}, l_{22}}+Imin{b_{22}, d_{22}, m_{22}}

min{a_{31}, p_{31}}+Imin{b_{31}, q_{31}} = min{a_{31}, c_{31}, l_{31}}+Imin{b_{31}, d_{31}, m_{31}}

min{a_{32}, p_{32}}+Imin{b_{32}, q_{32}} = min{a_{31}, c_{31}, l_{31}}+Imin{b_{31}, d_{31}, m_{31}}

$$F = \begin{pmatrix} n_{11} + Ik_{11} & n_{12} + Ik_{12} \\ n_{21} + Ik_{21} & n_{22} + Ik_{22} \\ n_{31} + Ik_{31} & n_{32} + Ik_{32} \end{pmatrix}$$

Therefore, $A - (B - C) = F = (A - B) - C$.

3.2 Identity element for subtraction

In the group theory under the operation "*" the identity element I_N of a set is an element such that $I_N * A = A * I_N = A$.

Specially the identity element of neutrosophic set is $I_N = \{[a_{ij} + b_{ij}I]_{m \times n}: \quad a_{ij} = 1 = b_{ij}$ for all $i, j\}$.

Result 3.1. For a neutrosophic fuzzy matrix, I_N is the identity matrix for subtraction.

Let $A= \begin{pmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I \end{pmatrix}$, and $I_N = \begin{pmatrix} 1+I & 1+I \\ 1+I & 1+I \\ 1+I & 1+I \end{pmatrix}$ be the neutrosophic identity matrix of order 3x2.

Then we have the following

$$A - I_N = \begin{pmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I \end{pmatrix} - \begin{pmatrix} 1+I & 1+I \\ 1+I & 1+I \\ 1+I & 1+I \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I \end{pmatrix} = I_N - A = A,$$

where

min{a_{11}, 1}+Imin{b_{11},1} = $a_{11} + b_{11}I$

min{a_{12}, 1}+Imin{b_{12},1}= $a_{12} + b_{12}I$

min{a_{21}, 1}+Imin{b_{21},1} = $a_{21} + b_{21}I$

min{a_{22}, 1}+Imin{b_{22},1} = $a_{22} + b_{22}I$

min{a_{31}, 1}+Imin{b_{31},1} = $a_{31} + b_{31}I$

$\min\{a_{32}, 1\}+I\min\{b_{32},1\} = a_{32} + b_{32}I$

3.3 Identity element for addition

In neutrosophic matrix addition we can define a identity element I_N such that $I_N = \{[a_{ij} + b_{ij}I]_{m\times n}: \quad a_{ij} = 0 = b_{ij}$ for all $i, j\}$

Let $A = \begin{pmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I \end{pmatrix}$, and $I_N = \begin{pmatrix} 0 + 0I & 0 + 0I \\ 0 + 0I & 0 + 0I \\ 0 + 0I & 0 + 0I \end{pmatrix}$ be the neutrosophic identity matrix of order 3x2.

Then we have the following

$$A - I_N = \begin{pmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I \end{pmatrix} - \begin{pmatrix} 1 + I & 1 + I \\ 1 + I & 1 + I \\ 1 + I & 1 + I \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11}I & a_{12} + b_{12}I \\ a_{21} + b_{21}I & a_{22} + b_{22}I \\ a_{31} + b_{31}I & a_{32} + b_{32}I \end{pmatrix}$$

$$= I_N - A = A,$$

where

$\max\{a_{11}, 0\}+I\max\{b_{11},0\} = a_{11} + b_{11}I$
$\max\{a_{12}, 0\}+I\max\{b_{12},0\} = a_{12} + b_{12}I$
$\max\{a_{21}, 0\}+I\max\{b_{21},0\} = a_{21} + b_{21}I$
$\max\{a_{22}, 0\}+I\max\{b_{22},0\} = a_{22} + b_{22}I$
$\max\{a_{31}, 0\}+I\max\{b_{31},0\} = a_{31} + b_{31}I$
$\max\{a_{32}, 0\}+I\max\{b_{32},0\} = a_{32} + b_{32}I.$

Result 3.2. The neutrosophic set forms a groupoid, semigroup, monaid and is commutative under the neutrosophic matrix operation of subtraction. The distributive law also holds for subtraction, i.e. $A(B - C) = AB - AC$.

Result 3.3. The neutrosophic set forms a groupoid, semigroup, monaid and commutative under the operation of addition. The distributive law also holds for addition, i.e. $A(B + C) = AB + AC$.

Thus we have, $A(B \pm C) = AB \pm AC$.

4. Applications

The formation of neutrosophic group structure, neutrosophic matrix set and algebraic structure on this set, the results are applicable

5. Conclusions

In this paper we have established some neutrosophic algebraic property, and subtraction operation addition and multiplication of these matrices and commutative property, distributive property had been examine. This result can be applied further application of neutrosophic fuzzy matric theory. For the development of neutrosophic group and its algebraic property the results of this paper would be helpful.

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Neutrosophic Bipolar Vague Soft Set and Its Application to Decision Making Problems

Anjan Mukherjee[1] and Rakhal Das[2]
[1] Department of Mathematics, Tripura University Agartala -799022; Tripura, INDIA; Email: [1]mukherjee123anjan@gmail.com
[2] Department of Mathematics, Tripura University Agartala -799022; Tripura, INDIA; Email [2]rakhaldas95@gmail.com.

Abstract: In this paper we study the concept of neutrosophic bipolar vague soft sets and some of its operations. It is the combination of neutrosophic bipolar vague sets and soft sets. Further we develop a decision making method based on neutrosophic bipolar vague soft set. A numerical example has been shown. Some new operations on neutrosophic bipolar vague soft set have also been designed.

Keywords: Neutrosophic set, Neutrosophic bipolar vague set, Soft set, vague set, Neutrosophic bipolar vague soft set.

1. Introduction

Most real life problems involve data with a high level of uncertainty and imprecision. Traditionally, classical mathematical theories such as fuzzy mathematics, probability theories and interval mathematics are used to deal with uncertain and fuzziness. However all these theories have their difficulties and weakness as pointed out by Molodtsov. This led to the introduction of the theory of soft sets by Molodtsov [18] in 1999. Among the significant milestones in the development of the theory of soft sets and its generalizations in the introduction of the possibility value which indicates the degree of possibility of belongingness of the elements in the universal set as well as the elements of each sets which enables the users to know the opinion of the experts in one model without the need for any operation. However, in order to handle the indeterminate and inconsistent information, neutrosophic set is defined [23,24] as a new mathematical tool for dealing with problems involving incomplete, indeterminacy and inconsistent knowledge. The theory of vague set was first proposed by Gau and Buehrer [13] as an extension of fuzzy set theory[29] and vague sets are regarded as a special case of content-dependent fuzzy sets.

In, [23] ,Samerandeche talked about neutrosophic set theory, one of the most important new mathematical tools for handling problems involving imprecise, indeterminacy and inconsistent data. Neutrosophic vague set was defined by S. Allehezaleh [2] in 2015. Lee [17] introduced bipolar-valued fuzzy sets and their operations in 2000. It an extension of fuzzy set [29]. Ali et al.[1] introduced the notion of bipolar neutrosophic soft set in 2017. Hassain et al. [16] introduced the concept of neutrosophic bipolar vague set and its application to neutrosophic bipolar vague graphs. For real life problems see the following ([3] to [12],[14],[15],[19]to[22],[25]to[28],[30]to[32]).

In this paper, we first introduce the concept of neutrosophic bipolar vague soft set and some of its operations. It is the combination of neutrosophic bipolar vague set and soft set. We develop a decision making method based on neutrosophic bipolar vague soft set. A numerical example has been shown. Some new operations on neutrosophic bipolar vague soft set have been designed. Finally we present an application of this concept in solving a decision making problem.

2. Materials and Methods (proposed work with more details)

In this section we recall some definitions and results for our future work.

Definition2.1:[8] Let *U* be an initial universal set and let *E* be a set of parameters. Let P(*U*) denote the power set of all subsets of *U* and let $A \subseteq E$. *A* collection of pairs (*f*, *A*) is called a soft set over *U*, where f is a mapping given by $f: A \to P(U)$.

Definition2.2:[17] Let *U* be the universe. Then a bipolar fuzzy set *A* on *U* is defined by

$$A = \{< x, \mu_A^+(x), \mu_A^-(x) >; x \in U\}.$$

Here $\mu_A^+: U \to [0,1]$ the positive membership function.

$\mu_A^-: U \to [-1,0]$the negative membership function.

Definition2.3:[17] If *A* and *B* be two bipolar fuzzy sets then their union, intersection and complement are defined as follows:

(i) $\mu_{A\cup B}^+(x) = \max\{\mu_A^+(x), \mu_B^+(x)\}$

(ii) $\mu_{A\cup B}^-(x) = \max\{\mu_A^-(x), \mu_B^-(x)\}$

(iii) $\mu_{A\cap B}^+(x) = \min\{\mu_A^+(x), \mu_B^+(x)\}$

(iv) $\mu_{A\cup B}^-(x) = \max\ \{\mu_A^-(x), \mu_B^-(x)\}$

(v) $\mu_{A^c}^+(x) = 1 - \mu_A^+(x)$ and $\mu_{A^c}^-(x) = -1 - \mu_A^-(x), \forall x \in U$.

Definition2.4: [13] A vague set *A* in the universe of discourse *U* is a pair

(t_A, f_A) where t_A, F_A:$U \to [0, 1]$ such that $t_A + f_A \leq 1$ for all U. The function t_A and f_A are called the true membership function and the false membership function respectively. The interval $[t_A, 1 - f_A]$ is called the value of *u* in *A* and is denoted by $V_A = [t_A, 1 - f_A]$.

Definition2.5:[13] Let *X* be a non-empty set. Let *A* and *B* be two vague sets in the form $A = \{< x, t_A, 1 - f_A >: x \in X\}$, $B = \{< x, t_B, 1 - f_B >: x \in X\}$. Then

(i) $A \subseteq B\ if\ and\ only\ if\ t_A \leq t_B$ and $1 - f_A \leq 1 - f_B$.

(ii) $A \cup B = \{< x, \max(t_A(x), t_B(x)), \max(1 - f_A(x), 1 - f_B(x)) >: x \in X\}$

(iii) $A \cap B = \{< x, \min(t_A(x), t_B(x)), \min(1 - f_A(x), 1 - f_B(x)) >: x \in X\}$

(iv) $A^c = \{< x, f_A, 1 - t_A >: x \in X\}$.

Definition2.6: [23,24] A neutrosophic set A on the universe of discourse U is defined as $A = \{\langle x, \mu_A(x), \gamma_A(x), \delta_A(x)\rangle : x \in U\}$, where $\mu_A, \gamma_A, \delta_A : U \to]^-0, 1^+[$ are functions such that the condition: $\forall x \in U,\ {}^-0 \le \mu_A(x) + \gamma_A(x) + \delta_A(x) \le 3^+$ is satisfied.

Here $\mu_A(x), \gamma_A(x), \delta_A(x)$ represent the truth-membership, indeterminacy-membership and falsity-membership respectively of the element $x \in U$ .From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. But in real life application in scientific and engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0, 1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of $[0,1]$.

Definition2.7: [2] A neutrosophic vague set A_{NV} on the universe of discourse U written as A_{NV} = {<x; $\hat{T}A_{NV}(x)$; $\hat{I}A_{NV}(x)$; $\hat{F}A_{NV}(x)$>; $x \in U$ } whose truth-membership, indeterminacy-membership, and falsity-membership functions is defined as $\hat{T}A_{NV}(x) = [T^-, T^+]$, $\hat{I}A_{NV}(x) = [I^-, I^+]$ and $\hat{F}A_{NV}(x) = [F^-, F^+]$, where (1) $T^+ = 1 - F^-$, (2) $F^+ = 1 - T^-$ and (3) ${}^-0 \le T^- + I^- + F^- \le 2^+$.

Definition2.8:[16] Let U be the universe of discourse. The neutrosophic bipolar vague set defined as A_{NBV} where

$$A_{NBV} = \{< x, \tilde{T}^p_{A_{NBV}}(x), \tilde{I}^p_{A_{NBV}}(x), \tilde{F}^p_{A_{NBV}}(x), \tilde{T}^n_{A_{NBV}}(x), \tilde{I}^n_{A_{NBV}}(x), \tilde{F}^n_{A_{NBV}}(x) >: x \in V\}$$

Here

$$\tilde{T}^p_{A_{NBV}}(x) = [(T^-)^p(x),\ (T^+)^p(x)],$$

$$\tilde{I}^p_{A_{NBV}}(x) = [(I^-)^p(x),\ (I^+)^p(x)],$$

$\tilde{F}^p_{A_{NBV}}(x) = [(F^-)^p(x),\ (F^+)^p(x)]$, Where $(T^+)^p(x) = 1 - (F^-)^p(x)$,

$$(F^+)^p(x) = 1 - (T^-)^p(x)$$

The condition is $0 \le (T^-)^p(x) + (I^-)^p(x) + (F^-)^p(x) \le 2$.

Also, $\tilde{T}^n_{A_{NBV}}(x) = [(T^-)^n(x),\ (T^+)^n(x)]$,

$\tilde{I}^n_{A_{NBV}}(x) = [(I^-)^n(x),\ (I^+)^n(x)]$,

$\tilde{F}^n_{A_{NBV}}(x) = [(F^-)^n(x),\ (F^+)^n(x)]$,

where $(T^+)^n(x) = -1 - (F^-)^n(x)$, $(F^+)^n(x) = -1 - (T^-)^n(x)$

The condition is $0 \geq (T^-)^n(x) + (I^-)^n(x) + (F^-)^n(x) \geq -2$.

Example 2.9: Let $U = \{u_1, u_2, u_3\}$ be a set of universe then the NBV set A_{NBV} as follows:

$A_{NBV} = \left(\frac{u_1}{[0.3,0.5]^p,[0.5,0.5]^p,[0.5,0.7]^p,[-0.3,-0.4]^n,[-0.4,-0.4]^n,[-0.6,-0.7]^n},\right.$

$\frac{u_2}{[0.3,0.7]^p,[0.4,0.6]^p,[0.3,0.7]^p,[-0.3,-0.3]^n,[-0.4,-0.4]^n,[-0.7,-0.7]^n},$

$\left.\frac{u_3}{[0.4,0.7]^p,[0.4,0.6]^p,[0.3,0.6]^p,[-0.2,-0.6]^n,[-0.5,-0.6]^n,[-0.4,-0.8]^n}\right).$

Definition 2.10: [16] The compliment A^c_{NBV} of A_{NBV} is as

$(\tilde{T}^c_{A_{NBV}}(x))^p = \{(1 - T^+(x))^p, (1 - T^-(x))^p\}$,

$(\tilde{T}^c_{A_{NBV}}(x))^n = \{(-1 - T^+(x))^n, (-1 - T^-(x))^n\}$,

$(\tilde{I}^c_{A_{NBV}}(x))^p = \{(1 - I^+(x))^p, (1 - I^-(x))^p\}$,

$(\tilde{I}^c_{A_{NBV}}(x))^n = \{(-1 - I^+(x))^n, (-1 - I^-(x))^n\}$,

$(\tilde{F}^c_{A_{NBV}}(x))^p = \{(1 - F^+(x))^p, (1 - F^-(x))^p\}$,

$(\tilde{F}^c_{A_{NBV}}(x))^n = \{(-1 - F^+(x))^n, (-1 - F^-(x))^n\}$.

Example 2.11: Considering the example 2.9, we have

$A^c_{NBV} = \left(\frac{u_1}{[0.7,0.5]^p,[0.5,0.5]^p,[0.5,0.3]^p,[-0.7,-0.6]^n,[-0.6,-0.6]^n,[-0.4,-0.3]^n},\right.$

$\frac{u_2}{[0.7,0.3]^p,[0.6,0.4]^p,[0.7,0.3]^p,[-0.7,-0.7]^n,[-0.6,-0.6]^n,[-0.3,-0.3]^n},$

$\left.\frac{u_3}{[0.6,0.3]^p,[0.6,0.4]^p,[0.7,0.4]^p,[-0.8,-0.4]^n,[-0.5,-0.4]^n,[-0.6,-0.2]^n}\right).$

Definition 2.12: [16] Two NBV sets A_{NBV} and B_{NBV} of the universe U are said to be equal if for all $u_i \in U$,

$\left(\tilde{T}_{A_{NBV}}\right)^{p}(u_i) = \left(\tilde{T}_{B_{NBV}}\right)^{p}(u_i)$, $\left(\tilde{I}_{A_{NBV}}\right)^{p}(u_i) = \left(\tilde{I}_{B_{NBV}}\right)^{p}(u_i)$,

$\left(\tilde{F}_{A_{NBV}}\right)^{p}(u_i) = \left(\tilde{F}_{B_{NBV}}\right)^{p}(u_i)$ and $\left(\tilde{T}_{A_{NBV}}\right)^{n}(u_i) = \left(\tilde{T}_{B_{NBV}}\right)^{n}(u_i)$,

$\left(\tilde{I}_{A_{NBV}}\right)^{n}(u_i) = \left(\tilde{I}_{B_{NBV}}\right)^{n}(u_i)$, $\left(\tilde{F}_{A_{NBV}}\right)^{n}(u_i) = \left(\tilde{F}_{B_{NBV}}\right)^{n}(u_i)$.

Where $1 \le i \le m$(say).

Definition 2.13 [16] If in the universe U, two *NBV* sets A_{NBV} and B_{NBV} be given as

$\left(\tilde{T}_{A_{NBV}}\right)^{p}(u_i) \le \left(\tilde{T}_{B_{NBV}}\right)^{p}(u_i)$, $\left(\tilde{I}_{A_{NBV}}\right)^{p}(u_i) \ge \left(\tilde{I}_{B_{NBV}}\right)^{p}(u_i)$,

$\left(\tilde{F}_{A_{NBV}}\right)^{p}(u_i) \ge \left(\tilde{F}_{B_{NBV}}\right)^{p}(u_i)$ and $\left(\tilde{T}_{A_{NBV}}\right)^{n}(u_i) \ge \left(\tilde{T}_{B_{NBV}}\right)^{n}(u_i)$,

$\left(\tilde{I}_{A_{NBV}}\right)^{n}(u_i) \le \left(\tilde{I}_{B_{NBV}}\right)^{n}(u_i)$, $\left(\tilde{F}_{A_{NBV}}\right)^{n}(u_i) \le \left(\tilde{F}_{B_{NBV}}\right)^{n}(u_i)$

Then $(A_{NBV})^{p} \subseteq (B_{NBV})^{p}$, and $(A_{NBV})^{n} \subseteq (B_{NBV})^{n}$ for $1 \le i \le m$

Definition 2.14: [16] The union and intersection of two *NBV* sets A_{NBV} and B_{NBV} are given as

(i) $A_{NBV} \cup B_{NBV} = C_{NBV}$ where

$\left(\tilde{T}_{C_{NBV}}\right)^{p}(x) = \{\left(\left(\tilde{T}^{-}_{A_{NBV}}\right)^{p}(x) \vee \left(\tilde{T}^{-}_{B_{NBV}}\right)^{p}(x), \left(\left(\tilde{T}^{+}_{A_{NBV}}\right)^{p}(x) \vee \left(\tilde{T}^{+}_{B_{NBV}}\right)^{p}(x)\right)\right)\}$

$\left(\tilde{I}_{C_{NBV}}\right)^{p}(x) = \{\left(\left(\tilde{I}^{-}_{A_{NBV}}\right)^{p}(x) \wedge \left(\tilde{I}^{-}_{B_{NBV}}\right)^{p}(x), \left(\left(\tilde{I}^{+}_{A_{NBV}}\right)^{p}(x) \wedge \left(\tilde{I}^{+}_{B_{NBV}}\right)^{p}(x)\right)\right)\}$

$\left(\tilde{F}_{C_{NBV}}\right)^{p}(x) = \{\left(\left(\tilde{F}^{-}_{A_{NBV}}\right)^{p}(x) \wedge \left(\tilde{F}^{-}_{B_{NBV}}\right)^{p}(x), \left(\left(\tilde{F}^{+}_{A_{NBV}}\right)^{p}(x) \wedge \left(\tilde{F}^{+}_{B_{NBV}}\right)^{p}(x)\right)\right)\}$, and

$\left(\tilde{T}_{C_{NBV}}\right)^{n}(x) = \{\left(\left(\tilde{T}^{-}_{A_{NBV}}\right)^{n}(x) \wedge \left(\tilde{T}^{-}_{B_{NBV}}\right)^{n}(x), \left(\left(\tilde{T}^{+}_{A_{NBV}}\right)^{n}(x) \wedge \left(\tilde{T}^{+}_{B_{NBV}}\right)^{n}(x)\right)\right)\}$

$\left(\tilde{I}_{C_{NBV}}\right)^{n}(x) = \{\left(\left(\tilde{I}^{-}_{A_{NBV}}\right)^{n}(x) \vee \left(\tilde{I}^{-}_{B_{NBV}}\right)^{n}(x), \left(\left(\tilde{I}^{+}_{A_{NBV}}\right)^{n}(x) \vee \left(\tilde{I}^{+}_{B_{NBV}}\right)^{n}(x)\right)\right)\}$

$\left(\tilde{F}_{C_{NBV}}\right)^{n}(x) = \{\left(\left(\tilde{F}^{-}_{A_{NBV}}\right)^{n}(x) \vee \left(\tilde{F}^{-}_{B_{NBV}}\right)^{n}(x), \left(\left(\tilde{F}^{+}_{A_{NBV}}\right)^{n}(x) \vee \left(\tilde{F}^{+}_{B_{NBV}}\right)^{n}(x)\right)\right)\}$.

(ii) $A_{NBV} \cap B_{NBV} = D_{NBV}$ is given by

$\left(\tilde{T}_{D_{NBV}}\right)^{p}(x) = \{\left(\left(\tilde{T}^{-}_{A_{NBV}}\right)^{p}(x) \wedge \left(\tilde{T}^{-}_{B_{NBV}}\right)^{p}(x), \left(\left(\tilde{T}^{+}_{A_{NBV}}\right)^{p}(x) \wedge \left(\tilde{T}^{+}_{B_{NBV}}\right)^{p}(x)\right)\right)\}$

$\left(\tilde{I}_{D_{NBV}}\right)^{p}(x) = \{\left(\left(\tilde{I}^{-}_{A_{NBV}}\right)^{p}(x) \vee \left(\tilde{I}^{-}_{B_{NBV}}\right)^{p}(x), \left(\left(\tilde{I}^{+}_{A_{NBV}}\right)^{p}(x) \vee \left(\tilde{I}^{+}_{B_{NBV}}\right)^{p}(x)\right)\right)\}$

$\left(\tilde{F}_{D_{NBV}}\right)^{p}(x) = \{\left(\left(\tilde{F}^{-}_{A_{NBV}}\right)^{p}(x) \vee \left(\tilde{F}^{-}_{B_{NBV}}\right)^{p}(x), \left(\left(\tilde{F}^{+}_{A_{NBV}}\right)^{p}(x) \vee \left(\tilde{F}^{+}_{B_{NBV}}\right)^{p}(x)\right)\right)\}$, and

$$\left(\tilde{T}_{D_{NBV}}\right)^n(x) = \{\left(\left(\tilde{T}^-_{A_{NBV}}\right)^n(x) \vee \left(\tilde{T}^-_{B_{NBV}}\right)^n(x), \left(\left(\tilde{T}^+_{A_{NBV}}\right)^n(x) \vee \left(\tilde{T}^+_{B_{NBV}}\right)^n(x)\right)\right)\}$$

$$\left(\tilde{I}_{D_{NBV}}\right)^n(x) = \{\left(\left(\tilde{I}^-_{A_{NBV}}\right)^n(x) \wedge \left(\tilde{I}^-_{B_{NBV}}\right)^n(x), \left(\left(\tilde{I}^+_{A_{NBV}}\right)^n(x) \wedge \left(\tilde{I}^+_{B_{NBV}}\right)^n(x)\right)\right)\}$$

$$\left(\tilde{F}_{D_{NBV}}\right)^n(x) = \{\left(\left(\tilde{F}^-_{A_{NBV}}\right)^n(x) \wedge \left(\tilde{F}^-_{B_{NBV}}\right)^n(x), \left(\left(\tilde{F}^+_{A_{NBV}}\right)^n(x) \wedge \left(\tilde{F}^+_{B_{NBV}}\right)^n(x)\right)\right)\}.$$

3. Neutrosophic Bipolar Vague Soft Set.

In this section we study the concept of Neutrosophic bipolar vague soft set. It is a combination of neutrosophic vague set & the soft set. Further we study some of its operation and properties.

Definition 3.1: Let U be a universal set. E be a set of parameters and $A \subseteq E$. Let NBVset(U) denotes the set of all neutrosophic bipolar vague set of U. Then the pair (f, A) is called an neutrosophic bipolar vague soft set (*NBVS* set in short) over U. Here f is a mapping $f: A \to NBV\ set(u)$. The collection of all neutrosophic bipolar vague soft sets over U is denoted by $NBVS\ set(U)$.

Example 3.2: Let $U = \{u_1, u_2, u_3\}$, $E = \{e_1, e_2\}$. Then neutrosophic bipolar vague soft sets A_1 and A_2 over U are as follows:

A_1
$= [(e_1, \{(u_1, [0.3, 0.5]^p, [0.5, 0.5]^p, [0.5, 0.7]^p, [-0.3, -0.4]^n, [-0.4, -0.4]^n, [-0.6, -0.7]^n),$

$(u_2, [0.2, 0.6]^p, [0.6, 0.7]^p, [0.4, 0.8]^p, [-0.2, -0.5]^n, [-0.3, -0.5]^n, [-0.5, -0.8]^n),$

$(u_3, [0.4, 0.6]^p, [0.3, 0.4]^p, [0.4, 0.6]^p, [-0.3, -0.5]^n, [-0.4, -0.5]^n, [-0.5, -0.7]^n)\ \}),$

$(e_2, \{(u_1, [0.5, 0.6]^p, [0.3, 0.4]^p, [0.4, 0.5]^p, [-0.4, -0.5]^n, [-0.6, -0.7]^n, [-0.5, -0.6]^n),$

$(u_2, [0.3, 0.4]^p, [0.6, 0.8]^p, [0.6, 0.7]^p, [-0.4, -0.7]^n, [-0.6, -0.8]^n, [-0.3, -0.6]^n),$

$(u_3, [0.5, 0.6]^p, [0.7, 0.8]^p, [0.4, 0.5]^p, [-0.2, -0.4]^n, [-0.5, -0.6]^n, [-0.6, -0.8]^n)\ ]$

A_2
$= [(e_1, \{(u_1, [0.4, 0.5]^p, [0.3, 0.4]^p, [0.5, 0.6]^p, [-0.4, -0.5]^n, [-0.3, -0.4]^n, [-0.5, -0.6]^n),$

$(u_2, [0.3, 0.7]^p, [0.5, 0.6]^p, [0.3, 0.7]^p, [-0.3, -0.6]^n, [-0.2, -0.4]^n, [-0.4, -0.7]^n),$

$(u_3, [0.5, 0.7]^p, [0.2, 0.3]^p, [0.3, 0.5]^p, [-0.4, -0.6]^n, [-0.3, -0.4]^n, [-0.4, -0.6]^n)\ \}),$

$(e_2,\{(u_1,[0.6,0.7]^p,[0.2,0.4]^p,[0.3,0.4]^p,[-0.5,-0.6]^n,[-0.5,-0.6]^n,[-0.4,-0.5]^n),$

$(u_2,[0.4,0.5]^p,[0.5,0.7]^p,[0.5,0.6]^p,[-0.5,-0.8]^n,[-0.5,-0.7]^n,[-0.2,-0.5]^n),$

$(u_3,[0.6,0.7]^p,[0.5,0.7]^p,[0.3,0.4]^p,[-0.3,-0.5]^n,[-0.4,-0.5]^n,[-0.5,-0.7]^n)\ ].$

Definition 3.3: An empty neutrosophic bipolar vague soft set $\emptyset$ in U is defined as

$\emptyset=\{(e,\{(u,[0,0]^p,[0,0]^p,[1,1]^p,[-1,-1]^n,[0,0]^n,[0,0]^n)\}: e\in E \text{ and } u\in U\}.$

Definition 3.4: An absolute neutrosophic bipolar vague soft set I in U is defined as

$I=\{(e,\{(u,[1,1]^p,[1,1]^p,[0,0]^p,[0,0]^n,[-1,-1]^n,[-1,-1]^n)\}: e\in E \text{ and } u\in U\}.$

Example 3.5: Let $U=\{u_1,u_2,u_3\}, E=\{e_1,e_2\}$ then

$\emptyset=\{(e_1,(u_1,[0,0]^p,[0,0]^p,[1,1]^p,[-1,-1]^n,[0,0]^n,[0,0]^n),$

$(u_2,[0,0]^p,[0,0]^p,[1,1]^p,[-1,-1]^n,[0,0]^n,[0,0]^n),$

$(u_3,[0,0]^p,[0,0]^p,[1,1]^p,[-1,-1]^n,[0,0]^n,[0,0]^n)$,

$(e_2,(u_1,[0,0]^p,[0,0]^p,[1,1]^p,[-1,-1]^n,[0,0]^n,[0,0]^n),$

$(u_2,[0,0]^p,[0,0]^p,[1,1]^p,[-1,-1]^n,[0,0]^n,[0,0]^n),$

$(u_3,[0,0]^p,[0,0]^p,[1,1]^p,[-1,-1]^n,[0,0]^n,[0,0]^n)$

(a) Absolute neutrosophic bipolar vague soft set I in U is defined as

$I=\{(e_1,(u_1,[1,1]^p,[1,1]^p,[0,0]^p,[0,0]^n,[-1,-1]^n,[-1,-1]^n),$

$(u_2,[1,1]^p,[1,1]^p,[0,0]^p,[0,0]^n,[-1,-1]^n,[-1,-1]^n)$,

$(u_3,[1,1]^p,[1,1]^p,[0,0]^p,[0,0]^n,[-1,-1]^n,[-1,-1]^n),$

$(e_2,(u_1,[1,1]^p,[1,1]^p,[0,0]^p,[0,0]^n,[-1,-1]^n,[-1,-1]^n),$

$(u_2, [1,1]^p, [1,1]^p, [0,0]^p, [0,0]^n, [-1,-1]^n, [-1,-1]^n)$,

$(u_3, [1,1]^p, [1,1]^p, [0,0]^p, [0,0]^n, [-1,-1]^n, [-1,-1]^n)\}$

Definition 3.6: $C^i = \{(e, (u, (\widetilde{T}^{\xi}_{C^i_{NBVS}})^p(u), (\widetilde{I}^{\xi}_{C^i_{NBVS}})^p(u), (\widetilde{F}^{\xi}_{C^i_{NBVS}})^p(u),$

$(\widetilde{T}^{\xi}_{C^i_{NBVS}})^n(u), (\widetilde{I}^{\xi}_{C^i_{NBVS}})^n(u), (\widetilde{F}^{\xi}_{C^i_{NBVS}})^n(u) >: u \in U, e \in E\}$. Where $i = 1, 2$ be two neutrosophic bipolar vague soft set over U. then C^1 is neutrosophic bipolar vague soft subset of C^2 is denoted by $C^1 \subseteq C^2$ if

$(\tilde{T}_{C^1_{NBVS}})^p(u) \leq (\tilde{T}_{C^2_{NBVS}})^p(u)$, And $(\tilde{T}_{C^1_{NBVS}})^n(u) \geq (\tilde{T}_{C^2_{NBVS}})^n(u)$,

$(\tilde{I}_{C^1_{NBVS}})^p(u) \geq (\tilde{I}_{C^2_{NBVS}})^p(u)$, $(\tilde{I}_{C^1_{NBVS}})^n(u) \leq (\tilde{I}_{C^2_{NBVS}})^n(u)$,

$(\tilde{F}_{C^1_{NBVS}})^p(u) \geq (\tilde{F}_{C^2_{NBVS}})^p(u)$ $(\tilde{F}_{C^1_{NBVS}})^n(u) \leq (\tilde{F}_{C^2_{NBVS}})^n(u)$.

Example 3.7: Consider the example 3.2. In this case $A_1 \subseteq A_2$ as per our definition 3.6.

Definition 3.8: Let *A* be a neutrosophic bipolar vague soft set over$U$. Then the complement of a neutrosophic bipolar vague soft set *A* is denoted by A^c is defined as

$A^c = \{(e, (u, (\tilde{T}_{A^c_{NBVS}})^p(u), (\tilde{I}_{A^c_{NBVS}})^p(u), (\tilde{F}_{A^c_{NBVS}})^p(u),$

$(\tilde{T}_{A^c_{NBVS}})^n(u), (\tilde{I}_{A^c_{NBVS}})^n(u), (\tilde{F}_{A^c_{NBVS}})^n(u)\}$.

$(\tilde{T}_{A^c_{NBVS}})^p(u) = \{(1 - T^+(u))^p, (1 - T^-(u))^p\}$,

$(\tilde{I}_{A^c_{NBVS}})^p(u) = \{(1 - I^+(u))^p, (1 - I^-(u))^p\}$,

$(\tilde{F}_{A^c_{NBVS}})^p(u) = \{(1 - F^+(u))^p, (1 - F^-(u))^p\}$ and

$(\tilde{T}_{A^c_{NBVS}})^n(u) = \{(-1 - T^+(u))^n, (-1 - T^-(u))^n\}$,

$(\tilde{I}_{A^c_{NBVS}})^n(u) = \{(-1 - I^+(u))^n, (-1 - I^-(u))^n\}$,

$(\tilde{F}_{A^c_{NBVS}})^n(u) = \{(-1 - F^+(u))^n, (-1 - F^-(u))^n\}$.

Example 3.9: Let $U = \{u_1, u_2\}, E = \{e_1, e_2\}$ then the neutrosophic bipolar vague soft set A is

$$A = [(e_1,\{(u_1,[0.1,0.3]^p,[0.2,0.4]^p,[0.7,0.9]^p,[-0.5,-0.2]^n,[-0.6,-0.4]^n,[-0.8,-0.5]^n)\},$$

$$\{(u_2,[0.7,0.9]^p,[0.2,0.4]^p,[0.1,0.3]^p,[-0.6,-0.2]^n,[-0.7,-0.3]^n,[-0.8,-0.4]^n),$$

$$(e_2,\{(u_1,[0.7,0.9]^p,[0.2,0.5]^p,[0.1,0.3]^p,[-0.9,-0.8]^n,[-0.4,-0.2]^n,[-0.2,-0.1]^n)\},$$

$$\{(u_2,[0.8,0.9]^p,[0.5,0.6]^p,[0.1,0.2]^p,[-0.7,-0.5]^n,[-0.7,-0.4]^n,[-0.5,-0.3]^n)$$

Then the complement of A is A^c is as

$$A^c = [(e_1,\{(u_1,[0.7,0.9]^p,[0.6,0.8]^p,[0.1,0.3]^p,[-0.8,-0.5]^n,[-0.6,-0.4]^n,[-0.5,-0.2]^n)\},$$

$$\{(u_2,[0.1,0.3]^p,[0.6,0.8]^p,[0.7,0.9]^p,[-0.8,-0.4]^n,[-0.7,-0.3]^n,[-0.6,-0.2]^n),$$

$$(e_2,\{(u_1,[0.1,0.3]^p,[0.5,0.8]^p,[0.7,0.9]^p,[-0.2,-0.1]^n,[-0.8,-0.6]^n,[-0.9,-0.8]^n)\},$$

$$\{(u_2,[0.1,0.2]^p,[0.4,0.5]^p,[0.8,0.9]^p,[-0.5,-0.3]^n,[-0.6,-0.3]^n,[-0.7,-0.5]^n)$$

Definition 3.10: Let

$A^i = \{(e,(u,(\tilde{T}_{A^i_{NBVS}})^p(u),(\tilde{I}_{A^i_{NBVS}})^p(u),(\tilde{F}_{A^i_{NBVS}})^p(u),$

$(\tilde{T}_{A^i_{NBVS}})^n(u),(\tilde{I}_{A^i_{NBVS}})^n(u),(\tilde{F}_{A^i_{NBVS}})^n(u)\}\ e \in E, u \in U, i = 1,2$. Then the union and intersection of A^1 and A^2 of two neutrosophic bipolar vague soft set are defined as follows:

(a) $A^1 \cup A^2 = A^3$

$$= \{(e,(u,(\tilde{T}_{A^3_{NBVS}})^p(u),(\tilde{I}_{A^3_{NBVS}})^p(u),(\tilde{F}_{A^3_{NBVS}})^p(u),$$

$$(\tilde{T}_{A^3_{NBVS}})^n(u),(\tilde{I}_{A^3_{NBVS}})^n(u),(\tilde{F}_{A^3_{NBVS}})^n(u)\}$$

Where

$$(\tilde{T}_{A^3_{NBVS}})^p(u) =$$

$$\{((\tilde{T}^-_{A^1_{NBVS}})^p(u) \vee \left(\tilde{T}^-_{A^2_{NBVS}}\right)^p(u)), \left(\tilde{T}^+_{A^1_{NBVS}}\right)^p(u) \vee \left(\tilde{T}^+_{A^2_{NBVS}}\right)^p(u))\}$$

$$(\tilde{I}_{A^3_{NBVS}})^p(u) = \{((\tilde{I}^-_{A^1_{NBVS}})^p(u) \wedge \left(\tilde{I}^-_{A^2_{NBVS}}\right)^p(u)),\ \left(\tilde{I}^+_{A^1_{NBVS}}\right)^p(u) \wedge \left(\tilde{I}^+_{A^2_{NBVS}}\right)^p(u))\}$$

$$(\tilde{F}_{A^3_{NBVS}})^p(u) = \{((\tilde{F}^-_{A^1_{NBVS}})^p(u) \wedge \left(\tilde{F}^-_{A^2_{NBVS}}\right)^p(u)),\ \left(\tilde{F}^+_{A^1_{NBVS}}\right)^p(u) \wedge \left(\tilde{F}^+_{A^2_{NBVS}}\right)^p(u))\}$$

And

$$(\tilde{T}_{A^3_{NBVS}})^n(u) = \{((\tilde{T}^-_{A^1_{NBVS}})^n(u) \wedge \left(\tilde{T}^-_{A^2_{NBVS}}\right)^n(u)),\ \left(\tilde{T}^+_{A^1_{NBVS}}\right)^n(u) \wedge \left(\tilde{T}^+_{A^2_{NBVS}}\right)^n(u))\}$$

$$(\tilde{I}_{A^3_{NBVS}})^n(u) = \{((\tilde{I}^-_{A^1_{NBVS}})^n(u) \vee \left(\tilde{I}^-_{A^2_{NBVS}}\right)^n(u)),\ \left(\tilde{I}^+_{A^1_{NBVS}}\right)^n(u) \vee \left(\tilde{I}^+_{A^2_{NBVS}}\right)^n(u))\}$$

$$(\tilde{F}_{A^3_{NBVS}})^n(u) = \{((\tilde{F}^-_{A^1_{NBVS}})^n(u) \vee \left(\tilde{F}^-_{A^2_{NBVS}}\right)^n(u)),\ \left(\tilde{F}^+_{A^1_{NBVS}}\right)^n(u) \vee \left(\tilde{F}^+_{A^2_{NBVS}}\right)^n(u))\}$$

(b) $A^1 \cap A^2 = A^4$

$$= \{(e, (u, (\tilde{T}_{A^4_{NBVS}})^p(u), (\tilde{I}_{A^4_{NBVS}})^p(u), (\tilde{F}_{A^4_{NBVS}})^p(u),$$

$$(\tilde{T}_{A^4_{NBVS}})^n(u), (\tilde{I}_{A^4_{NBVS}})^n(u), (\tilde{F}_{A^4_{NBVS}})^n(u)\}$$

Where

$$(\tilde{T}_{A^4_{NBVS}})^p(u) = \{((\tilde{T}^-_{A^1_{NBVS}})^p(u) \wedge \left(\tilde{T}^-_{A^2_{NBVS}}\right)^p(u)),\ \left(\tilde{T}^+_{A^1_{NBVS}}\right)^p(u) \wedge \left(\tilde{T}^+_{A^2_{NBVS}}\right)^p(u))\}$$

$$(\tilde{I}_{A^4_{NBVS}})^p(u) = \{((\tilde{I}^-_{A^1_{NBVS}})^p(u) \vee \left(\tilde{I}^-_{A^2_{NBVS}}\right)^p(u)),\ \left(\tilde{I}^+_{A^1_{NBVS}}\right)^p(u) \vee \left(\tilde{I}^+_{A^2_{NBVS}}\right)^p(u))\}$$

$$(\tilde{F}_{A^4_{NBVS}})^p(u) = \{((\tilde{F}^-_{A^1_{NBVS}})^p(u) \vee \left(\tilde{F}^-_{A^2_{NBVS}}\right)^p(u)),\ \left(\tilde{F}^+_{A^1_{NBVS}}\right)^p(u) \vee \left(\tilde{F}^+_{A^2_{NBVS}}\right)^p(u))\}$$

And

$$(\tilde{T}_{A^4_{NBVS}})^n(u) = \{((\tilde{T}^-_{A^1_{NBVS}})^n(u) \vee \left(\tilde{T}^-_{A^2_{NBVS}}\right)^n(u)),\ \left(\tilde{T}^+_{A^1_{NBVS}}\right)^n(u) \vee \left(\tilde{T}^+_{A^2_{NBVS}}\right)^n(u))\}$$

$$(\tilde{I}_{A^4_{NBVS}})^n(u) = \{((\tilde{I}^-_{A^1_{NBVS}})^n(u) \wedge \left(\tilde{I}^-_{A^2_{NBVS}}\right)^n(u)),\ \left(\tilde{I}^+_{A^1_{NBVS}}\right)^n(u) \wedge \left(\tilde{I}^+_{A^2_{NBVS}}\right)^n(u))\}$$

$$(\tilde{F}_{A^4_{NBVS}})^n(u) = \{((\tilde{F}^-_{A^1_{NBVS}})^n(u) \wedge \left(\tilde{F}^-_{A^2_{NBVS}}\right)^n(u)),\ \left(\tilde{F}^+_{A^1_{NBVS}}\right)^n(u) \wedge \left(\tilde{F}^+_{A^2_{NBVS}}\right)^n(u))\}$$

Example 3.11. Consider the example 3.2 then

$A_1 \cup A_2 =$
$[(e_1, \{(u_1, [0.4, 0.5]^p, [0.3, 0.4]^p, [0.5, 0.6]^p, [-0.5, -0.4]^n, [-0.4, -0.3]^n, [-0.6, -0.5]^n),$

$(u_2, [0.3, 0.7]^p, [0.5, 0.6]^p, [0.3, 0.7]^p, [-0.6, -0.3]^n, [-0.1, -0.2]^n, [-0.7, -0.4]^n),$

$(u_3, [0.5, 0.7]^p, [0.2, 0.3]^p, [0.3, 0.5]^p, [-0.6, -0.4]^n, [-0.4, -0.3]^n, [-0.6, -0.4]^n) \}),$

$(e_2, \{(u_1, [0.6, 0.7]^p, [0.2, 0.4]^p, [0.3, 0.4]^p, [-0.6, -0.5]^n, [-0.6, -0.5]^n, [-0.6, -0.4]^n),$

$(u_2, [0.4, 0.5]^p, [0.5, 0.7]^p, [0.5, 0.6]^p, [-0.8, -0.5]^n, [-0.7, -0.5]^n, [-0.5, -0.2]^n),$

$(u_3, [0.6, 0.7]^p, [0.5, 0.7]^p, [0.3, 0.4]^p, [-0.5, -0.3]^n, [-0.5, -0.4]^n, [-0.7, -0.5]^n)] = A_2$

Similarly $A_1 \cap A_2 = A_1$.

Definition 3.12 Let $A = \{(e, \{((\tilde{T}_{C_{NBVS}})^p(u), (\tilde{I}_{C_{NBVS}})^p(u), (\tilde{F}_{C_{NBVS}})^p(u), (\tilde{T}_{C_{NBVS}})^n(u),$ $(\tilde{I}_{C_{NBVS}})^n(u), (\tilde{F}_{C_{NBVS}})^n(u)\}: : u \in U \text{ and } e \in E\}$ be a neutrosophic bipolar vague soft set over U. then aggregation neutrosophic bipolar vague soft operator denoted by A_{agg} is defined as

$$A_{agg} = \left\{\frac{[\mu_A^+(u), \mu_A^-(u)]}{u} : u \in U\right\}.$$

Where $[\mu_A^+(u), \mu_A^-(u)]$

$$= \frac{1}{2|E \times U|}\left[\sum_{e \in E}([1,1] - (\tilde{I}_e)^p(u), [(\tilde{T}_e)^p(u) - (\tilde{F}_e)^p(u)]\right.$$

$$\left. + ((\tilde{I}_e)^n(u)[(\tilde{T}_e)^p(u) - (\tilde{F}_e)^p(u)]\right]$$

Where

$$(\tilde{I}_e)^p(u) = [(I_e^+)^p(u) - (I_e^-)^p(u)]$$

$$(\tilde{T}_e)^p(u) = [(T_e^+)^p(u) - (T_e^-)^p(u)]$$

$$(\tilde{F}_e)^p(u) = [(F_e^+)^p(u) - (F_e^-)^p(u)]$$

$$(\tilde{I}_e)^n(u) = [(I_e^+)^n(u) - (I_e^-)^n(u)]$$

$$\left(\tilde{T}_e\right)^n(u) = [(T_e^+)^n(u) - (T_e^-)^n(u)]$$

$$\left(\tilde{F}_e\right)^n(u) = [(F_e^+)^n(u) - (F_e^-)^n(u)]$$

Where $|E \times U|$ is the cardinality of $E \times U$.

4. Application of neutrosophic bipolar vague soft set.

We develop an algorithm based on neutrosophic bipolar vague soft sets and give numerical example to show the fossibility and effectiveness of the approaches in definition 3.12.

Algorithm

1. First we construct the neutrosophic bipolar vague soft set on U.
2. Then we compute the neutrosophic bipolar vague soft set aggregation operator.
3. Average of each intervals and find $|A_{agg}|$.
4. Find the optimum value on U.

Assume that a farm wants to fill a position in the office. There are three candidates for the post. The selection committee use the neutrosophic bipolar vague soft decision making method. Assume that the set of candidate $U = \{u_1, u_2, u_3\}$ which may be characterized by a set of parameters $E = \{e_1, e_2, e_3\}$. Where e_1= "experience", e_2 = "technical knowledge", e_3= "age".

(a) The selection committee construct a neutrosophic bipolar vague soft set *A* over the set U as

A_2
$= \{(e_1, \{(u_1, [0.8, 0.9]^p, [0.5, 0.7]^p, [0.1, 0.2]^p, [-0.5, -0.3]^n, [-0.7, -0.5]^n, [-0.7, -0.5]^n),$

$(u_2, [0.5, 0.7]^p, [0.4, 0.6]^p, [0.3, 0.5]^p, [-0.5, -0.4]^n, [-0.7, -0.5]^n, [-0.6, -0.5]^n),$

$(u_3, [0.5, 0.7]^p, [0.5, 0.6]^p, [0.3, 0.5]^p, [-0.8, -0.6]^n, [-0.5, -0.3]^n, [-0.4, -0.2]^n) \}),$

$(e_2, \{(u_1, [0.5, 0.7]^p, [0.4, 0.6]^p, [0.3, 0.5]^p, [-0.4, -0.2]^n, [-0.2, -0.1]^n, [-0.8, -0.6]^n),$

$(u_2, [0.7, 0.9]^p, [0.4, 0.6]^p, [0.1, 0.3]^p, [-0.6, -0.4]^n, [-0.3, -0.2]^n, [-0.6, -0.4]^n),$

$(u_3, [0.2, 0.4]^p, [0.8, 0.9]^p, [0.6, 0.8]^p, [-0.3, -0.1]^n, [-0.5, -0.3]^n, [-0.9, -0.7]^n) \},$

$(e_3, \{(u_1, [0.7, 0.9]^p, [0.2, 0.4]^p, [0.1, 0.3]^p, [-0.3, -0.2]^n, [-0.5, -0.3]^n, [-0.8, -0.7]^n),$

$(u_2, [0.6, 0.8]^p, [0.4, 0.6]^p, [0.2, 0.4]^p, [-0.4, -0.2]^n, [-0.5, -0.4]^n, [-0.8, -0.6]^n),$

$(u_3, [0.3, 0.5]^p, [0.5, 0.7]^p, [0.5, 0.7]^p, [-0.3, -0.2]^n, [-0.6, -0.4]^n, [-0.8, -0.7]^n)$ }.

(b) Then we find the neutrosophic vague soft set aggregation operator A_{agg} of A_2 as

For u_1 ,

$\frac{1}{18}[[1,1]-[0.5,0.7]([0.8,0.9]-[0.1,0.2])+[-0.7,-0.5]([-0.5,-0.3]-$
$[-0.7,-0.5])+[1,1]-[0.4,0.6]([0.5,0.7]-[0.3,0.5])+$
$[-0.2,-0.1]([-0.4,-0.2]-[-0.8,-0.6])$

$+[1,1]-[0.2,0.4]([0.7,0.9]-[0.1,0.3])+$

$[-0.5,-0.3]([-0.3,-0.2]-[-0.8,-0.7])]$

For u_2 ,

$\frac{1}{18}[[1,1]-[0.4,0.6]([0.5,0.7]-[0.3,0.5])+[-0.7,-0.5]([-0.5,-0.4]-$
$[-0.6,-0.5])+[1,1]-[0.4,0.6]([0.7,0.9]-[0.1,0.3])+$
$[-0.3,-0.2]([-0.6,-0.4]-[-0.6,-0.4])$

$+[1,1]-[0.4,0.6]([0.6,0.8]-[0.2,0.4])+$

$[-0.5,-0.4]([-0.4,-0.2]-[-0.8,-0.6])]$

For u_3 ,

$\frac{1}{18}[[1,1]-[0.5,0.6]([0.5,0.7]-[0.3,0.5])+[-0.7,-0.5]([-0.8,-0.6]-$
$[-0.4,-0.2])+[1,1]-[0.8,0.9]([0.2,0.4]-[0.6,0.8])+$
$[-0.5,-0.3]([-0.3,-0.1]-[-0.9,-0.7])$

$+[1,1]-[0.5,0.7]([0.3,0.5]-[0.5,0.7])+$

$[-0.6,-0.4]([-0.3,-0.2]-[-0.8,-0.7])]$

(c) We take the average of each interval.
i.e. [1,1]=1,

$(\tilde{T}_e)^p(x) = [(T_e^-)^p(x), (T_e^+)^p(x)]$

$$(\tilde{I}_e)^p(x) = [(I_e^-)^p(x), (I_e^+)^p(x)]$$

$$(\tilde{F}_e)^p(x) = [(F_e^-)^p(x), (F_e^+)^p(x)]$$

$$(\tilde{T}_e)^n(x) = [(T_e^-)^n(x), (T_e^+)^n(x)]$$

$$(\tilde{I}_e)^n(x) = [(I_e^-)^n(x), (I_e^+)^n(x)]$$

$$(\tilde{F}_e)^n(x) = [(F_e^-)^n(x), (F_e^+)^n(x)]$$

(d) The $|A_{agg}| = \frac{0.1006}{u_1}, \frac{0.1311}{u_2}, \frac{0.1455}{u_3}$

(e) Finally the selection committee choose u_3 for the post since $|A_{agg}|$ has the maximum degree 0.1455 among them.

5. Conclusion

In this paper, we introduce the neutrosophic bipolar vague soft set. It is a combination of soft set and the neutrosophic bipolar vague set. We develop a decision making method based on neutrosophic bipolar vague soft set. A numerical example has beengiven. Some new operations on neutrosophic bipolar vague soft set have been designed. For further study, it may be applied to real world problems with realistic data and extend proposed algorithm to other decision making problem with vagueness and uncertainty. Here we require less calculations and few steps to get our result.

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Conflicts of Interest

The authors declare no conflict of interest.

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On Some Types of Neutrosophic Topological Groups with Respect to Neutrosophic Alpha Open Sets

Qays Hatem Imran[1]*, Ali Hussein Mahmood Al-Obaidi[2], and Florentin Smarandache[3]

[1]Department of Mathematics, College of Education for Pure Science, Al-Muthanna University, Samawah, Iraq.
E-mail: qays.imran@mu.edu.iq
[2]Department of Mathematics, College of Education for Pure Science, University of Babylon, Hillah, Iraq.
E-mail: aalobaidi@uobabylon.edu.iq
[3]Department of Mathematics, University of New Mexico 705 Gurley Ave. Gallup, NM 87301, USA.
E-mail: smarand@unm.edu
* Correspondence: qays.imran@mu.edu.iq

Abstract: In this article, we presented eight different types of neutrosophic topological groups, each of which depends on the conceptions of neutrosophic α-open sets and neutrosophic α-continuous functions. Also, we found the relation between these types, and we gave some properties on the other side.

Keywords: Neutrosophic α-open sets, neutrosophic α-continuous functions, neutrosophic topological groups, and neutrosophic topological groups of type (R), $R = 1,2,3, \dots ,8$.

1. Introduction

Smarandache [1,2] originally handed the theory of "neutrosophic set". Recently, Abdel-Basset et al. discussed a novel neutrosophic approach [3-6]. Salama et al. [7] gave the clue of neutrosophic topological space. Arokiarani et al. [8] added the view of neutrosophic α-open subsets of neutrosophic topological spaces. Dhavaseelan et al. [9] presented the idea of neutrosophic α^m-continuity. Banupriya et al. [10] investigated the notion of neutrosophic αgs continuity and neutrosophic αgs irresolute maps. Nandhini et al. [11] presented Nαg#ψ-open map, Nαg#ψ-closed map, and Nαg#ψ-homomorphism in neutrosophic topological spaces. Sumathi et al. [12] submitted the perception of neutrosophic topological groups. The target of this article is to perform eight different types of neutrosophic topological groups, each of which depends on the notions of neutrosophic α-open sets and neutrosophic α-continuous functions and also we found the relation between these types.

2. Preliminaries

In all this paper, $(\mathcal{G},\tau)$ and $(\mathcal{H},\sigma)$ (or briefly $\mathcal{G}$ and $\mathcal{H}$) frequently refer to neutrosophic topological spaces (or shortly NTSs). Suppose $\mathcal{A}$ be a neutrosophic open subset (or shortly Ne-OS) of $\mathcal{G}$, then its complement $\mathcal{A}^c$ is closed (or shortly Ne-CS). In addition, its interior and closure are denoted by $Nint(\mathcal{A})$and $Ncl(\mathcal{A})$, correspondingly.

Definition 2.1 [8]: Let $\boldsymbol{\mathcal{A}}$ be a Ne-OS in NTS $\boldsymbol{\mathcal{G}}$, then it is said that a neutrosophic **α**-open subset (or briefly Ne-**α**OS) if $\boldsymbol{\mathcal{A} \subseteq Nint(Ncl(Nint(\mathcal{A})))}$. Then $\boldsymbol{\mathcal{A}^c}$ is the so-called a neutrosophic **α**-closed (or briefly Ne-**α**CS). The collection of all such these Ne-**α**OSs (resp. Ne-**α**CSs) of $\boldsymbol{\mathcal{G}}$ is denoted by $\boldsymbol{N\alpha O(\mathcal{G})}$ (resp. $\boldsymbol{N\alpha C(\mathcal{G})}$).

Definition 2.2 [8]: Let $\mathcal{A}$ be a neurrosophic set in NTS $\mathcal{G}$. Then the union of all such these Ne-αOSs involved in $\mathcal{A}$(symbolized by $\alpha Nint(\mathcal{A})$) is said to be the neutrosophic α-interior of $\mathcal{A}$.

Definition 2.3 [8]: Let $\mathcal{A}$ be a neurrosophic set in NTS $\mathcal{G}$. Then the intersection of all such these Ne-αCSs that contain $\mathcal{A}$ (symbolized by $\alpha Ncl(\mathcal{A})$) is said to be the neutrosophic α-closure of $\mathcal{A}$.

Proposition 2.4 [13]: Let $\mathcal{A}$ be a neutrosophic set in NTS $\mathcal{G}$. Then $\mathcal{A} \in N\alpha O(\mathcal{B})$ iff there exists a Ne-αOS $\mathcal{B}$ where $\mathcal{B} \subseteq \mathcal{A} \subseteq Nint(Ncl(\mathcal{B}))$.

Proposition 2.5 [8]: In any NTS, the following claims hold, and not vice versa:

(i) For each, Ne-OS is a Ne-αOS.

(ii) For each, Ne-CS is a Ne-αCS.

Definition 2.6: Let $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ be a function, then $\hbar$ is called:

(i) a neutrosophic continuous (in short Ne-continuous) iff for each $\mathcal{A}$ Ne-OS in $\mathcal{H}$, then $\hbar^{-1}(\mathcal{A})$ is a Ne-OS in $\mathcal{G}$ [14].

(ii) a neutrosophic α-continuous (in short Ne-α-continuous) iff for each $\mathcal{A}$ Ne-OS in $\mathcal{H}$, then $\hbar^{-1}(\mathcal{A})$ is a Ne-αOS in $\mathcal{G}$ [8].

(iii) a neutrosophic α-irresolute (in short Ne-α-irresolute) iff for each $\mathcal{A}$ Ne-αOS in $\mathcal{H}$, then $\hbar^{-1}(\mathcal{A})$ is a Ne-αOS in $\mathcal{G}$.

Proposition 2.7 [8]: Every Ne-continuous function is a Ne-α-continuous, but the opposite is not valid in general.

Proposition 2.8: Every Ne-α-irresolute function is a Ne-α-continuous, but the opposite is not exact in general.

Proof: Let $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ be a Ne-α-irresolute function and let $\mathcal{A}$ be any Ne-OS in $\mathcal{H}$. From proposition 2.5, we get $\mathcal{A}$ is a Ne-αOS in $\mathcal{H}$. Since $\hbar$ is a Ne-α-irresolute, then $\hbar^{-1}(\mathcal{A})$ is a Ne-αOS in $\mathcal{G}$. Therefore $\hbar$ is a Ne-α-continuous. ■

Example 2.9: Let $\mathcal{G} = \{p, q\}$. Suppose the neutrosophic sets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ be in $\mathcal{G}$ as follows:

$\mathcal{A} = \langle x, \left(\frac{p}{0.5}, \frac{q}{0.3}\right), \left(\frac{p}{0.5}, \frac{q}{0.3}\right), \left(\frac{p}{0.5}, \frac{q}{0.7}\right) \rangle$, $\mathcal{B} = \langle x, \left(\frac{p}{0.5}, \frac{q}{0.6}\right), \left(\frac{p}{0.5}, \frac{q}{0.6}\right), \left(\frac{p}{0.5}, \frac{q}{0.4}\right) \rangle$,

$\mathcal{C} = \langle x, \left(\frac{p}{0.6}, \frac{q}{0.3}\right), \left(\frac{p}{0.6}, \frac{q}{0.3}\right), \left(\frac{p}{0.4}, \frac{q}{0.7}\right) \rangle$ and $\mathcal{D} = \langle x, \left(\frac{p}{0.6}, \frac{q}{0.7}\right), \left(\frac{p}{0.6}, \frac{q}{0.7}\right), \left(\frac{p}{0.4}, \frac{q}{0.3}\right) \rangle$.

Then the families $\tau = \{0_N, \mathcal{A}, 1_N\}$ and $\sigma = \{0_N, \mathcal{D}, 1_N\}$ are neutrosophic topologies on $\mathcal{G}$.

Thus, $(\mathcal{G}, \tau)$ and $(\mathcal{G}, \sigma)$ are NTSs. Define $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{G}, \sigma)$ as $(p) = p$, $\hbar(q) = q$. Hence $\hbar$ is a Ne-α-continuous function, but not Ne-α-irresolute.

Definition 2.10: A function $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is said to be $\mathcal{M}$-function iff $\hbar^{-1}(Nint(Ncl(\mathcal{B}))) \subseteq Nint(Ncl(\hbar^{-1}(\mathcal{B})))$, for every Ne-αOS $\mathcal{B}$ of $\mathcal{H}$.

Theorem 2.11: If $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is a Ne-α-continuous function and $\mathcal{M}$-function, then $\hbar$ is a Ne-α-irresolute.

Proof: Let $\mathcal{A}$ be any Ne-αOS of $\mathcal{H}$, there exists a Ne-OS $\mathcal{B}$ of $\mathcal{H}$ where $\mathcal{B} \subseteq \mathcal{A} \subseteq Nint(Ncl(\mathcal{B}))$.

Since $\hbar$ is $\mathcal{M}$-function, we have $\hbar^{-1}(\mathcal{B}) \subseteq \hbar^{-1}(\mathcal{A}) \subseteq \hbar^{-1}(Nint(Ncl(\mathcal{B}))) \subseteq Nint(Ncl(\hbar^{-1}(\mathcal{B})))$.

By proposition 2.4, we have $\hbar^{-1}(\mathcal{A})$ is a Ne-αOS. Hence, $\hbar$ is a Ne-α-irresolute. ■

Definition 2.12 [8]: A function $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is called a neutrosophic α-open (resp. neutrosophic α-closed) iff for each $\mathcal{A} \in N\alpha O(\mathcal{G})$ (resp. $\mathcal{A} \in N\alpha C(\mathcal{G})$), $\hbar(\mathcal{A}) \in N\alpha O(\mathcal{H})$ (resp. $\hbar(\mathcal{A}) \in N\alpha C(\mathcal{H})$).

Definition 2.13 [15]: A bijective function $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is called a neutrosophic homeomorphism iff $\hbar$ and $\hbar^{-1}$ are Ne-continuous.

Definition 2.14 [12]: A neutrosophic topological group (briefly NTG) is a set $\mathcal{G}$ which carries a group structure and a neutrosophic topology with the following two postulates:
(i) The operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, given as $\mu(g, h) = g \cdot h$ is a Ne-continuous.
(ii) The inversion function $I: \mathcal{G} \to \mathcal{G}$, given as $I(g) = g^{-1}$ is a Ne-continuous.

Remark 2.15 [12]:
(i) The function $\gamma: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, given as $\gamma(g, h) = g \cdot h$ is a Ne-continuous iff for each Ne-OS $\mathcal{C}$ and $g \cdot h \in \mathcal{C}$, there exist Ne-OS $\mathcal{A}, \mathcal{B}$ such that $g \in \mathcal{A}, h \in \mathcal{B}$, and $\mathcal{A} \cdot \mathcal{B} \subseteq \mathcal{C}$.
(ii) The function $inv: \mathcal{G} \to \mathcal{G}$ is a Ne-continuous iff for each Ne-OS $\mathcal{A}$ and $g^{-1} \in \mathcal{A}$, there exists a Ne-OS $\mathcal{B}$ and $g \in \mathcal{B}$ where $\mathcal{B}^{-1} \subseteq \mathcal{A}$.

Definition 2.16 [16]: A group $\mathcal{G}$ is nice iff its operation is nice.

3. Different Types of Neutrosophic Topological Groups

In this section, we introduce eight types of neutrosophic topological groups, each of which depends on the notions of neutrosophic α-open sets and neutrosophic α-continuous functions.

Definition 3.1: Let $\mathcal{G}$ be a set that equips with a group structure and a neutrosophic topology. Then $\mathcal{G}$ is called:
(i) NTG of type (1) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and the inversion function $I: \mathcal{G} \to \mathcal{G}$ are both Ne-α-continuous.
(ii) NTG of type (2) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and the inversion function $I: \mathcal{G} \to \mathcal{G}$ are both Ne-α-irresolute.
(iii) NTG of type (3) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-α-continuous and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-continuous.
(iv) NTG of type (4) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-α-irresolute and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-continuous.
(v) NTG of type (5) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-α-irresolute and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-α-continuous.
(vi) NTG of type (6) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-α-continuous and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-α-irresolute.
(vii) NTG of type (7) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-continuous, and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-α-continuous.

(viii) NTG of type (8) iff the operation function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is Ne-continuous, and the inversion function $I: \mathcal{G} \to \mathcal{G}$ is Ne-α-irresolute.

Proposition 3.2:
(i) Every NTG is a NTG of type (R), where $R = 1,3,7$.
(ii) Every NTG of type (2) is a NTG of type (5).
(iii) Every NTG of type (2) is a NTG of type (6).
(iv) Every NTG of type (4) is a NTG of type (3).
(v) Every NTG of type (4) is a NTG of type (5).
(vi) Every NTG of type (R) is a NTG of type (1), where $R = 2,3,\dots,8$.
Proof:
(i) Let $\mathcal{G}$ be a NTG, then the operation function μ and the inversion function I are both Ne-continuous. By proposition 2.7, we have that the operation function μ and the inversion function I are both Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (R), where $R = 1,3,7$.
(ii) Let $\mathcal{G}$ be a NTG of type (2), then the operation function μ and the inversion function I are both Ne-α-irresolute. By proposition 2.8, we have that the inversion function I is a Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (5).
(iii) Let $\mathcal{G}$ be a NTG of type (2), then the operation function μ and the inversion function I are both Ne-α-irresolute. By proposition 2.8, we have that the operation function μ is a Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (6).
(iv) Let $\mathcal{G}$ be a NTG of type (4), then the operation function μ is a Ne-α-irresolute and the inversion function I is a Ne-continuous. By proposition 2.8, we have that the operation function μ is a Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (3).
(v) Let $\mathcal{G}$ be a NTG of type (4), then the operation function μ is a Ne-α-irresolute and the inversion function I is a Ne-continuous. By proposition 2.7, we have that the inversion function I is a Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (5).
(vi) Let $\mathcal{G}$ be a NTG of type (R), where $R = 2,3,\dots,8$. By proposition 2.7 and proposition 2.8, we have that the operation function μ and the inversion function I are both Ne-α-continuous. Hence, $\mathcal{G}$ is a NTG of type (1).

Proposition 3.3:
(i) A NTG of type (3) with $\mathcal{M}$-function operation μ is a NTG of type (4).
(ii) A NTG of type (1) with $\mathcal{M}$-function inversion I and $\mathcal{M}$-function operation μ is a NTG of type (2).
(iii) A NTG of type (1) with $\mathcal{M}$-function operation μ is a NTG of type (5).
(iv) A NTG of type (1) with $\mathcal{M}$-function inversion I is a NTG of type (6).
(v) A NTG of type (5) with $\mathcal{M}$-function inversion I is a NTG of type (2).
(vi) A NTG of type (6) with $\mathcal{M}$-function operation μ is a NTG of type (2).
(vii) A NTG of type (7) with $\mathcal{M}$-function inversion I is a NTG of type (8).
Proof:
(i) Let $\mathcal{G}$ be a NTG of type (3), then the operation function μ is a Ne-α-continuous and the inversion function I is a Ne-continuous. Since μ is $\mathcal{M}$-function. So by Theorem 2.11, we get that operation μ

is a Ne-α-irresolute. Hence, $\mathcal{G}$ is a NTG of type (4).

(ii) Let $\mathcal{G}$ be a NTG of type (1), then the operation function μ and the inversion function I are both Ne-α-continuous. Since μ, I are $\mathcal{M}$-function. So by Theorem 2.11, we get that the operation function μ and the inversion function I are both Ne-α-irresolute. Hence, $\mathcal{G}$ is a NTG of type (2). The proof is evident for others.

Remark 3.4: The next illustration displays relationship among different kinds of neutrosophic topological groups mentioned in this section and the neutrosophic topological group:

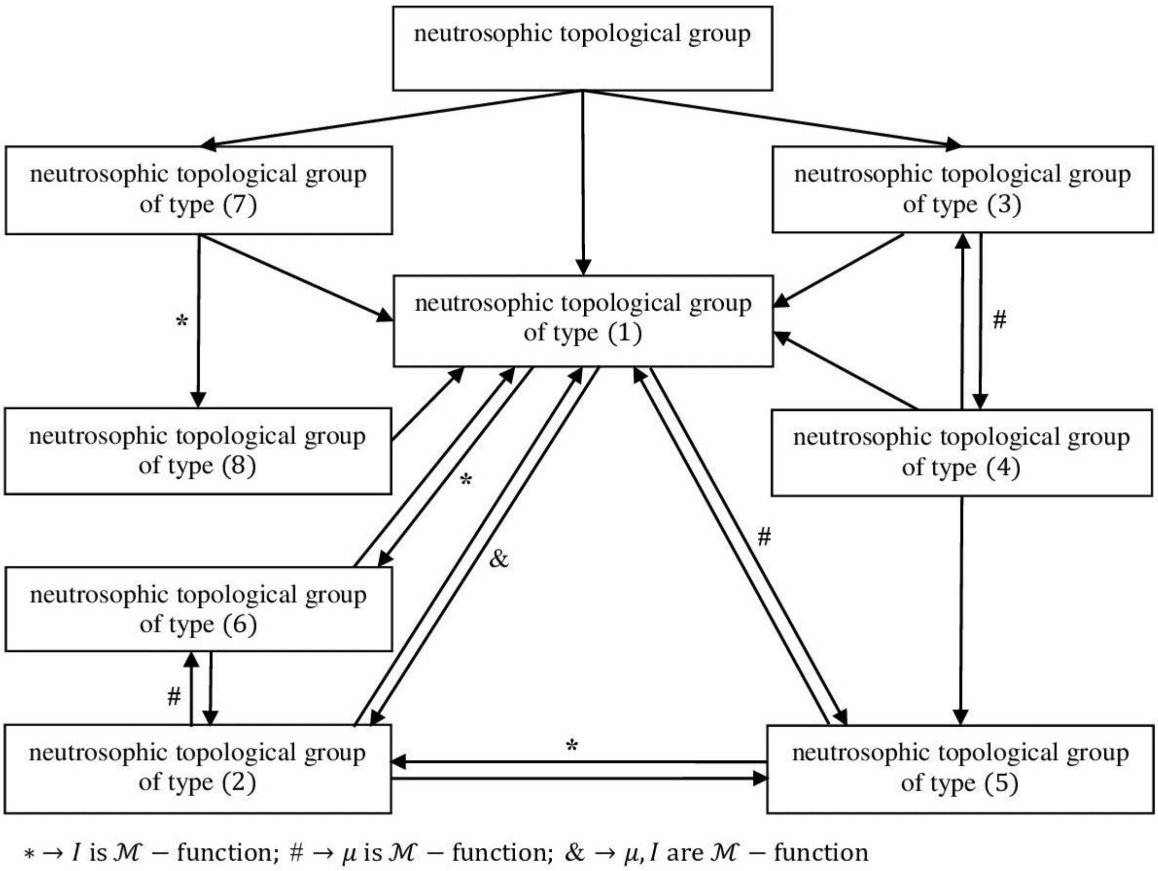


Fig. 3.1

Definition 3.5: A bijective function $\hbar: (\mathcal{G}, \tau) \longrightarrow (\mathcal{H}, \sigma)$ is said to be:

(i) Neutrosophic α-homeomorphism iff $\hbar$ and $\hbar^{-1}$ are Ne-α-continuous.

(ii) Neutrosophic α-irresolute – homeomorphism iff $\hbar$ and $\hbar^{-1}$ are Ne-α-irresolute.

Definition 3.6: Let $(\mathcal{G}, \tau)$ be a NTS, then $\mathcal{G}$ is called neutrosophic α-homogeneous (resp. neutrosophic α-irresolute – homogeneous) iff for any two elements $g, h \in \mathcal{G}$, there exists a neutrosophic α-homeomorphism (resp. neutrosophic α-irresolute – homeomorphism) from $\mathcal{G}$ onto $\mathcal{G}$ which transforms g into h.

Proposition 3.7: The inversion function I in a NTG of type (R), where $R = 1,2, \dots \dots ,8$ is a neutrosophic α-homeomorphism.

Proof: Let $\mathcal{G}$ be a NTG of type (1). Since $\mathcal{G}$ is a group, $I(\mathcal{G}) = \mathcal{G}^{-1} = \mathcal{G}$ which implies I is onto, also for any $g \in \mathcal{G}$, there exists a unique inverse which is equal to $I(g)$ which implies, I is one-to-one. Now; we have I is a Ne-α-continuous and $I^{-1}: \mathcal{G} \to \mathcal{G}$ such that $I^{-1}(g) = g$, i.e $I^{-1}(g) = I(g)$ for each $g \in \mathcal{G}$, so, I^{-1} is a Ne-α-continuous. Thus, I is a neutrosophic α-homeomorphism. In the case of type (R), we have a similar proof, where $R = 2,3, \dots, 8$.

Corollary 3.8: Let $\mathcal{G}$ be a NTG of type (1) and $\mathcal{A} \subseteq \mathcal{G}$. If $\mathcal{A} \in \tau$, then $\mathcal{A}^{-1} \in N\alpha O(\mathcal{G})$.
Proof: Since the inversion function I is a neutrosophic α-homeomorphism, then $I(\mathcal{A}) = \mathcal{A}^{-1}$ is a Ne-αOS in $\mathcal{G}$ for each $\mathcal{A} \in \tau$.

Proposition 3.9: The inversion function I in a NTG of type (3) [and type (4)] is a neutrosophic homeomorphism.
Proof: Suppose $\mathcal{G}$ be a NTG of type (3). Since $\mathcal{G}$ is a group, $I(\mathcal{G}) = \mathcal{G}^{-1} = \mathcal{G}$ which implies I is onto, also for any $g \in \mathcal{G}$, there exists a unique inverse which is equal to $I(g)$ which implies, I is one-to-one. Now; we have I is a Ne-continuous and $I^{-1}: \mathcal{G} \to \mathcal{G}$ such that $I^{-1}(g) = g$, i.e $I^{-1}(g) = I(g)$ for each $g \in \mathcal{G}$, so, I^{-1} is a Ne-continuous. Thus, I is a neutrosophic homeomorphism. In the case of type (4), we have similar proof.

Proposition 3.10: The inversion function I in a NTG of type (R), where $R = 2,6,8$ is a neutrosophic α-irresolute – homeomorphism.
Proof: Suppose $\mathcal{G}$ be a NTG of type (2). Since $\mathcal{G}$ is a group, $I(\mathcal{G}) = \mathcal{G}^{-1} = \mathcal{G}$ which implies I is onto, also for any $g \in \mathcal{G}$, there exists a unique inverse which is equal to $I(g)$ which implies, I is one-to-one. Now; we have I is a Ne-α-irresolute and $I^{-1}: \mathcal{G} \to \mathcal{G}$ such that $I^{-1}(g) = g$, i.e $I^{-1}(g) = I(g)$ for each $g \in \mathcal{G}$, so, I^{-1} is a Ne-α-irresolute. Thus, I is a neutrosophic α-irresolute – homeomorphism. In the case of type (6) and type (8), we have a similar proof.

Proposition 3.11: Let $\mathcal{G}$ be a set which carries a group structure and a neutrosophic topology, let $k_1, k_2 \in \mathcal{G}$. Then for each $g \in \mathcal{G}$ if one of the following functions:
(i) $l_{k_1}(g) = k_1 \cdot g$
(ii) $r_{k_1}(g) = g \cdot k_1$
(iii) $\hbar_{k_1 k_2}(g) = k_1 \cdot g \cdot k_2$
is a neutrosophic α-homeomorphism (resp. neutrosophic α-irresolute – homeomorphism), then so the others.
Proof: Since $k_1$and k_2 are arbitrary elements in $\mathcal{G}$, clear that $l_{k_1}$and r_{k_1}come from $\hbar_{k_1 k_2}$ by taking $k_2 = e$ or $k_1 = e$ respectively. Hence, when $\hbar_{k_1 k_2}$ is a neutrosophic α-homeomorphism, both l_{k_1} and r_{k_2} are neutrosophic α-homeomorphisms. Now; when l_{k_1} is a neutrosophic α-homeomorphism. Since $\mathcal{G}$ is a group, $\mathcal{G} \cdot k = \mathcal{G}$ for each $k \in \mathcal{G}$ then $\mathcal{G} \cdot k_2 = \mathcal{G}$. Hence, for each $h \in \mathcal{G} \cdot k_2$, $l_{k_1}(h) = k_1 \cdot h$, l_{k_1} is a neutrosophic α-homeomorphism. But $h = g \cdot k_2$ for some $g \in \mathcal{G}$, then for each $g \in \mathcal{G}$, $l_{k_1}(h) = l_{k_1}(g \cdot k_2) = k_1 \cdot g \cdot k_2 = \hbar_{k_1 k_2}(g)$, $\hbar_{k_1 k_2}$ is a neutrosophic α-homeomorphism. Then by the first part of the proof, r_{k_1}. And we have a similar proof if we are beginning with r_{k_1} is a neutrosophic α-homeomorphism. In the case of neutrosophic α-irresolute – homeomorphism, we have a similar proof.

Theorem 3.12: Let $\mathcal{G}$ be a nice NTG of type (R), where $R = 1,2,3, \dots ,8$ and let $k_1, k_2 \in \mathcal{G}$. Then for each $g \in \mathcal{G}$ the following functions:

(i) $l_{k_1}(g) = k_1 \cdot g$

(ii) $r_{k_1}(g) = g \cdot k_1$

(iii) $\hbar_{k_1 k_2}(g) = k_1 \cdot g \cdot k_2$

are neutrosophic α-homeomorphisms.

Proof: Let $\mathcal{G}$ be a nice NTG of type (1). It is clear that each of the functions l_{k_1}, r_{k_1} and $\hbar_{k_1 k_2}$ is a bijective function. Let $\hbar$ be the operation of $\mathcal{G}$, then $\hbar$ is a Ne-α-continuous. Since $\mathcal{G}$ is a nice, so $l_{k_1} = \hbar/\{k_1\} \times \mathcal{G}$ is a Ne-α-continuous. Similarly, $l_{k_1}{}^{-1}(g) = k_1{}^{-1} \cdot g$, $l_{k_1}{}^{-1}$ is a Ne-α-continuous. Hence, l_{k_1} is a neutrosophic α-homeomorphism. Thus, because of the preceding proposition, r_{k_1} and $\hbar_{k_1 k_2}$ are neutrosophic α-homeomorphisms. The case of type (R) has a similar proof, where $R = 2,3, \dots ,8$.

Theorem 3.13: Let $\mathcal{G}$ be a nice NTG of type (R), where $R = 2,4,5$ and let $k_1, k_2 \in \mathcal{G}$. Then for each $g \in \mathcal{G}$ the following functions:

(i) $l_{k_1}(g) = k_1 \cdot g$

(ii) $r_{k_1}(g) = g \cdot k_1$

(iii) $\hbar_{k_1 k_2}(g) = k_1 \cdot g \cdot k_2$

are neutrosophic α-irresolute – homeomorphisms.

Proof: Let $\mathcal{G}$ be a nice NTG of type (2). It is clear that each of the functions l_{k_1}, r_{k_1} and $\hbar_{k_1 k_2}$ is a bijective function. Let $\hbar$ be the operation of $\mathcal{G}$, then $\hbar$ is a Ne-α-irresolute. Since $\mathcal{G}$ is a nice, so $l_{k_1} = \hbar/\{k_1\} \times \mathcal{G}$ is a Ne-α-irresolute. Similarly, $l_{k_1}{}^{-1}(g) = k_1{}^{-1} \cdot g$, $l_{k_1}{}^{-1}$ is a Ne-α-irresolute. Hence, l_{k_1} is a neutrosophic α-irresolute – homeomorphism. Thus, given the preceding proposition, r_{k_1} and $\hbar_{k_1 k_2}$ are neutrosophic α-irresolute – homeomorphisms. The case of type (R) has a similar proof, where $R = 4,5$.

Corollary 3.14: Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be subsets of a nice NTG $\mathcal{G}$ of type (1) (resp. of type (4)) such that $\mathcal{A}$ is a Ne-CS (resp. Ne-αCS), and $\mathcal{B}$ is a Ne-OS (resp. Ne-αOS). Then for each $k \in \mathcal{G}, k \cdot \mathcal{A}$ and $\mathcal{A} \cdot k$ are Ne-α-CSs also $k \cdot \mathcal{B}, \mathcal{B} \cdot k, \mathcal{C} \cdot \mathcal{B}$ and $\mathcal{B} \cdot \mathcal{C}$ are Ne-αOSs.

Proof: Since $\mathcal{A}$ is a Ne-CS so in view of the theorem 3.12, $l_k(\mathcal{A}) = k \cdot \mathcal{A}$ and $r_k(\mathcal{A}) = \mathcal{A} \cdot k$ are Ne-αCSs.

Similarly, since $\mathcal{B}$ is a Ne-OS so in view of the theorem 3.12, $l_k(\mathcal{B}) = k \cdot \mathcal{B}$ and $r_k(\mathcal{B}) = \mathcal{B} \cdot k$ are Ne-αOSs. Also, $\mathcal{C} \cdot \mathcal{B} = \bigcup_{c \in \mathcal{C}} c \cdot \mathcal{B}$ but $c \cdot \mathcal{B}$ is a Ne-αOS for each $c \in \mathcal{C}$. Hence, $\mathcal{C} \cdot \mathcal{B}$ is a Ne-αOS. Similarly, $\mathcal{B} \cdot \mathcal{C}$ is a Ne-αOS. In the case of type (4), we have a similar proof.

Corollary 3.15: A nice NTG of type (R), where $R = 1,2,3, \dots ,8$ is neutrosophic α-homogeneous.

Proof: Let $\mathcal{G}$ be a nice NTG of type (1) and $a, b \in \mathcal{G}$. Then for any fixed element $k \in \mathcal{G}, r_k$ is a neutrosophic α-homeomorphism, therefore, it is true when $k = a^{-1} \cdot b$. Thus, $r_{a^{-1}b}(g) = g \cdot a^{-1} \cdot b$ is a neutrosophic α-homeomorphism we need because $r_{a^{-1}b}(a) = b$. Therefore, $\mathcal{G}$ is a neutrosophic α-homogeneous. In the case of type (R), we have a similar proof, where $R = 2,3, \dots ,8$.

Corollary 3.16: A nice NTG of type (R), where $R = 2,4,5$ is neutrosophic α-irresolute – homogeneous.

Proof: Let $\mathcal{G}$ be a nice NTG of type (2) and $a, b \in \mathcal{G}$. Then for any fixed element $k \in \mathcal{G}, r_k$ is a neutrosophic α-irresolute – homeomorphism, therefore, it is true when $k = a^{-1} \cdot b$. Thus, $r_{a^{-1}b}(g) = g \cdot a^{-1} \cdot b$ is a neutrosophic α-irresolute – homeomorphism. But $r_{a^{-1}b}(a) = b$, therefore $\mathcal{G}$ is a neutrosophic α-irresolute – homogeneous. In the case of type (R), we have a similar proof, where $R = 4,5$.

Definition 3.17: Let $\mathcal{G}$ be a NTG of type (2), (5), and $\mathcal{F}$ be a fundamental system of neutrosophic α-open nhds of the identity element e. Then for any fixed element $k \in \mathcal{G}, r_k$ is a neutrosophic α-irresolute – homeomorphism. So $\mathcal{F}(k) = \{r_k(\mathcal{A}) = \mathcal{A} \cdot k : \mathcal{A} \in \mathcal{F}\}$ is a fundamental system of neutrosophic α-open nhds of k.

Proposition 3.18: Let $\mathcal{G}$ be a NTG of type (2), (5). Any fundamental system $\mathcal{F}$ of neutrosophic α-open nhds of e in $\mathcal{G}$ has the below postulates:

(i) If $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, then $\exists \mathcal{C} \in \mathcal{F}$ such that $\mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}$.

(ii) If $g \in \mathcal{A} \in \mathcal{F}$, then $\exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B} \cdot g \subseteq \mathcal{A}$.

(iii) If $\mathcal{A} \in \mathcal{F}$, then $\exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B}^{-1} \cdot \mathcal{B} \subseteq \mathcal{A}$.

(iv) If $\mathcal{A} \in \mathcal{F}, k \in \mathcal{G}$, then $\exists \mathcal{B} \in \mathcal{F}$ such that $k^{-1} \cdot \mathcal{B} \cdot k \subseteq \mathcal{A}$.

(v) $\forall \mathcal{A} \in \mathcal{F}, \exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B}^{-1} \subseteq \mathcal{A}$.

(vi) $\forall \mathcal{A} \in \mathcal{F}, \exists \mathcal{C} \in \mathcal{F}$ such that $\mathcal{C}^2 \subseteq \mathcal{A}$.

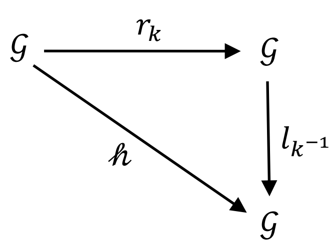
Proof:

(i) Let $\mathcal{A}, \mathcal{B} \in \mathcal{F}$, then $\mathcal{A} \cap \mathcal{B} \in \mathcal{F}$, so $\exists \mathcal{C} \in \mathcal{F}$ such that $\mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}$.

(ii) Let $\mathcal{A} \in \mathcal{F}$ and $g \in \mathcal{A}$ implies $\mathcal{A} \cdot g^{-1} \in \mathcal{F}$, then $\exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B} \subseteq \mathcal{A} \cdot g^{-1}$. Thus, $\mathcal{B} \cdot g \subseteq \mathcal{A}$.

(iii) The function $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, given by $\mu(g, h) = g^{-1} \cdot h$ is a Ne-α-irresolute because $\mathcal{G}$ is a NTG of type (2), (5). Thus $\mu^{-1}(\mathcal{A})$ is a neutrosophic α-open nhd in $\mathcal{G} \times \mathcal{G}$ contains (e, e) and hence includes a set of the from $\mathcal{U} \times \mathcal{V}$, where $\mathcal{U}, \mathcal{V}$ are neutrosophic α-open and provide e. But $\mathcal{U} \cap \mathcal{V}$ is a neutrosophic α-open contains e, so $\exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B} \subseteq \mathcal{U} \cap \mathcal{V}$ then $\mathcal{B} \subseteq \mathcal{U}$ and $\mathcal{B} \subseteq \mathcal{V}$. Thus $\mathcal{B} \times \mathcal{B} \subseteq \mathcal{U} \times \mathcal{V} \subseteq \mu^{-1}(\mathcal{A})$, then $\mu(\mathcal{B} \times \mathcal{B}) \subseteq \mathcal{A}$ but $\mu(\mathcal{B} \times \mathcal{B}) = \mathcal{B}^{-1} \cdot \mathcal{B} \subseteq \mathcal{A}$.

(iv) The function $\hbar: \mathcal{G} \to \mathcal{G}$ given by $\hbar(g) = k^{-1} \cdot g \cdot k$ is a Ne-α-irresolute. Since $l_{k^{-1}}, r_k$ is Ne-α-irresolute. So $l_{k^{-1}} \circ r_k$ is a Ne-α-irresolute from $\mathcal{G}$ to $\mathcal{G}$ put $\hbar = l_{k^{-1}} \circ r_k, \hbar(g) = (l_{k^{-1}} \circ r_k)(g) = l_{k^{-1}}(r_k(g)) = l_{k^{-1}}(g \cdot k) = k^{-1} \cdot g \cdot k$.



So, $\hbar^{-1}(\mathcal{A})$ is a neutrosophic α-open nhd and contains e, hence $\exists \mathcal{B} \in \mathcal{F}$, $\mathcal{B} \subseteq \hbar^{-1}(\mathcal{A})$ then $\hbar(\mathcal{B}) \subseteq \mathcal{A}$. Thus, $\hbar(\mathcal{B}) = k^{-1} \cdot \mathcal{B} \cdot k \subseteq \mathcal{A}$.

(v) Since I the inverse function in a NTG of type (2) is a Ne-α-irresolute, then $I^{-1}(\mathcal{A})$ is a neutrosophic α-open contains e so $\exists \mathcal{B} \in \mathcal{F}$ such that $\mathcal{B} \subseteq I^{-1}(\mathcal{A})$ then $I(\mathcal{B}) \subseteq \mathcal{A}$. Thus, $I(\mathcal{B}) = \mathcal{B}^{-1} \subseteq \mathcal{A}$.

(vi) Since μ in a NTG of type (5) is a Ne-α-irresolute. So $\mu^{-1}(\mathcal{A})$ is a neutrosophic α-open contains (e, e) and thus contains a neutrosophic set of the from $\mathcal{U} \times \mathcal{V}$, where $\mathcal{U}, \mathcal{V}$ are neutrosophic α-open and contain e then $\mathcal{U} \cap \mathcal{V}$ is a neutrosophic α-open and contain e $\exists \mathcal{C} \in \mathcal{F}$ such that $\mathcal{C} \subseteq \mathcal{U} \cap \mathcal{V}$, then $\mathcal{C} \times \mathcal{C} \subseteq \mathcal{U} \times \mathcal{V} \subseteq \mu^{-1}(\mathcal{A})$. Thus, $\mu(\mathcal{C} \times \mathcal{C}) = \mathcal{C} \cdot \mathcal{C} = \mathcal{C}^2 \subseteq \mathcal{A}$.

Definition 3.19: A neutrosophic α-open nhd $\mathcal{C}$ of g is called symmetric if $\mathcal{C}^{-1} = \mathcal{C}$.

Proposition 3.20: Let $\mathcal{G}$ be a NTG of type (R), where $R = 1,2,\ldots,8$, and let $\mathcal{B}$ be any neutrosophic α-open nhd of a point $g \in \mathcal{G}$. Then $\mathcal{B} \cup \mathcal{B}^{-1}$ is symmetric neutrosophic α-open nhd of g.

Proof: Let $\mathcal{B}$ is a neutrosophic α-open nhd of g, then $\mathcal{B} \cup \mathcal{B}^{-1}$ is a neutrosophic α-open nhd of g;

$\mathcal{B} \cup \mathcal{B}^{-1} = \{b: b \in \mathcal{B} \ or \ b \in \mathcal{B}^{-1}\} = \{b: b^{-1} \in \mathcal{B} \ or \ b^{-1} \in \mathcal{B}^{-1}\}$

$= \{b: b^{-1} \in \mathcal{B} \cup \mathcal{B}^{-1}\} = \{b: b \in (\mathcal{B} \cup \mathcal{B}^{-1})^{-1}\} = (\mathcal{B} \cup \mathcal{B}^{-1})^{-1}$.

That is, $\mathcal{B} \cup \mathcal{B}^{-1}$ is symmetric neutrosophic α-open nhd of g.

Proposition 3.21: Let $\mathcal{B}$ be any neutrosophic α-open nhd of e in a nice NTG of type (R), where $R = 1,2,\ldots\ldots,8$. Then $\mathcal{B} \cdot \mathcal{B}^{-1}$ is symmetric neutrosophic α-open nhd of e.

Proof: Let $\mathcal{B}$ be a neutrosophic α-open nhd of e and since $\mathcal{G}$ is a nice, then $\mathcal{B} \cdot \mathcal{B}^{-1}$ is neutrosophic α-open nhd of e;

$\mathcal{B} \cdot \mathcal{B}^{-1} = \{x \cdot y^{-1}: x, y \in \mathcal{B}\} = \{(x^{-1})^{-1} \cdot y^{-1}: x, y \in \mathcal{B}\} = (\mathcal{B}^{-1})^{-1} \cdot \mathcal{B}^{-1} = (\mathcal{B} \cdot \mathcal{B}^{-1})^{-1}$.

That is, $\mathcal{B} \cdot \mathcal{B}^{-1}$ is symmetric neutrosophic α-open nhd of e.

4. Conclusion

In this work, we examined the conceptions of eight different types of neutrosophic topological groups, each of which, depending on the notions of neutrosophic α-open sets and neutrosophic α-continuous function. In the future, we plan to rsearch the ideas of neutrosophic topological subgroups and the neutrosophic topological quotient groups as well as defining the perception of neutrosophic topological product groups with some results.

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A Contemporary Approach on Neutrosophic Nano Topological Spaces

[1] D. Sasikala and [2] K.C. Radhamani
[1]Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore, Tamilnadu, India, Email: dsasikala@psgrkcw.ac.in
[2]Department of Mathematics, Dr.N.G.P.Arts and Science College, Coimbatore, Tamilnadu, India, Email: radhamani@drngpasc.ac.in

Abstract: In this article, we implement a new notion of sets namely neutrosophic nano j-closed set, neutrosophic nano generalized closed set, neutrosophic nano generalized j-closed set and neutrosophic nano generalized j*-closed set in neutrosophic nano topological spaces. We also provide some appropriate examples to study the properties of these sets. The existing relations between some of these sets in neutrosophic nano topological space have been investigated.

Keywords: Neutrosophic nano j-closed set, neutrosophic nano generalized closed set, neutrosophic nano generalized j-closed set, neutrosophic nano generalized j*-closed set.

I. Introduction

In recent years, Topology plays a vast role in research area. In particular, the concept of neutrosophy is a trending tool in topology. We use fuzzy concept where we consider only the membership value. The intuitionistic fuzzy concept is used where the membership and the non-membership values are considered. But, more real life problems deal with indeterminacy. The suitable concept for the situation where the indeterminacy occurs is neutrosophy which is represented by the degree of membership (truth value), the degree of non-membership (falsity value) and the degree of indeterminacy.

The fuzzy concept was initially proposed by Zadeh [22] in 1965 and Chang [7] introduced Fuzzy topological spaces in 1968. Atanasov [6] defined intuitionistic fuzzy set and Coker [8] developed intuitionistic fuzzy topology. In 2005, Smarandache [17] introduced neutrosophic set and many researchers used this concept in engineering, medicine and many fields where the situation of indeterminacy arises. Abdel-Basset et.al, [1 - 5] working with many practical problems by using neutrosophy concept in the recent days. Salama et.al, [14] introduced the generalization of neutrosophic sets, neutrosophic closed sets and neutrosophic crisp sets in neutrosophic topological spaces.

The nano topology which has the maximum of five elements was introduced by Lellis Thivagar [9]. He applied nano topology for nutrition modelling [11] and medical diagnosis [12]. Zhang et.al [23], worked on neutrosophic rough sets over two universes. Lellis Thivagar initiated [10] neutrosophic nano topology and some closed sets on neutrosophic nano topological spaces were derived by recent researchers.

Sasikala and Arockiarani [15] introduced generalized j-closed set. Sasikala and Radhamani [16] introduced nano j-closed set in nano topological spaces. In this paper, we present a new set called neutrosophic nano j-closed set and work with some interesting examples. Also we investigate some of the properties of the introduced sets.

II. Preliminaries

Definition 2.1[9] Let U be a nonempty finite set of objects called the universe and R be an equivalence relation on U , called the indiscernibility relation. The pair (U,R) is said to be the approximation space. Let $X \subseteq U$.

(i) The lower approximation of X with respect to the relation R is the set of all objects, which can be for certain classified as X and it is denoted by $L_R(X)$. i.e., $L_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \subseteq U\}$, where $R(x)$ denotes the equivalence class determined by x.

(ii) The upper approximation of X with respect to the relation R is the set of all objects, which can be possibly classified as X and it is denoted by $U_R(X)$ .i.e., $U_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \cap X \neq \varphi\}$

(iii) The boundary region of X with respect to the relation R is the set of all objects, which can be classified neither as X nor as not X and it is denoted by $B_R(X)$ .i.e., $B_R(X) = U_R(X) - L_R(X)$

Remark 2.2[9] If (U,R) is an approximation space and $X,Y \subseteq U$, then

(i) $L_R(X) \subseteq X \subseteq U_R(X)$

(ii) $L_R(\varphi) = U_R(\varphi) = \varphi$ and $L_R(U) = U_R(U) = U$

(iii) $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$

(iv) $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$

(v) $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$

(vi) $L_R(X \cap Y) = L_R(X) \cap L_R(Y)$

(vii) $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$ whenever $X \subseteq Y$

(viii) $U_R(X^C) = [L_R(X)]^C$ and $L_R(X^C) = [U_R(X)]^C$

(ix) $U_R U_R(X) = L_R U_R(X) = U_R(X)$

(x) $L_R L_R(X) = U_R L_R(X) = L_R(X)$

Definition 2.3[9] Let U be an universe, R be an equivalence relation on U and $\tau_R(X) = \{U, \varphi, L_R(X),\ U_R(X), B_R(X)\}$ where $X \subseteq U$. Then by the properties mentioned in remark 2.2, $\tau_R(X)$ satisfies the following axioms:

(i) U and φ are in $\tau_R(X)$

(ii) The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$

(iii) The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$

Then $\tau_R(X)$ forms a topology on U called the nano topology with respect to X . We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called nano open sets. The complement of nano open sets are called nano closed sets.

Definition 2.4[9] Let $(U,\tau_R(X))$ be a nano topological space. A subset A is called nano generalized closed (briefly Ng-closed) set if $NCl(A)\subseteq V$ where $A\subseteq V$ and V is nano open in U.

Definition 2.5[16] A subset A of a nano topological space $(U,\tau_R(X))$ is called a nano j-open set if $A\subseteq N\,Int[N\,PCl(A)]$. The complement of nano j-open set is called a nano j-closed (briefly Nj-closed) set.

i.e., if A is Nj-closed, then $N\,Cl[N\,PInt(A)]\subseteq A$.

Definition 2.6[16] A subset A of a nano topological space $(U,\tau_R(X))$ is called a nano generalized j-closed (briefly Ngj-closed) set if $N\,JCl(A)\subseteq V$ where $A\subseteq V$ and V is nano open in U.

Definition 2.7[17] Let X be an universe of discourse with a general element x, the neutrosophic set is an object having the form $A=\{\prec x,\mu_A(X),\sigma_A(X),\gamma_A(X)\succ, x\in X\}$ where $\mu,\sigma,$ and γ each take the values from 0 to 1 and called as the degree of membership, degree of indeterminacy, and the degree of non-membership of the element $x\in X$ to the set A with the condition $0\leq \mu_A(x)+\sigma_A(x)+\sigma_A(x)\leq 3$.

Definition 2.8[10] Let U be a nonempty set and R be an equivalence relation on U. Let F be a neutrosophic set in U with the membership function μ_F, the indeterminacy function σ_F, and the non-membership function γ_F. The neutrosophic nano lower, neutrosophic nano upper approximations and neutrosophic nano boundary of F in the approximation (U,R), denoted by $\underline{N},\overline{N}$ and $BN(F)$ are respectively defined as follows:

(i) $\underline{N}(F)=\{\prec x,\mu_{\underline{R}(A)}(x),\sigma_{\underline{R}(A)}(x),\gamma_{\underline{R}(A)}(x)\succ / y\in [x]_R, x\in U\}$

(ii) $\overline{N}(F)=\{\prec x,\mu_{\overline{R}(A)}(x),\sigma_{\overline{R}(A)}(x),\gamma_{\overline{R}(A)}(x)\succ / y\in [x]_R, x\in U\}$

(iii) $BN(F)=\overline{N}-\underline{N}$

Where $\mu_{\underline{R}(A)}(x)=\wedge_{y\in[x]_R}\mu_A(y)$, $\sigma_{\underline{R}(A)}(x)=\wedge_{y\in[x]_R}\sigma_A(y)$, $\gamma_{\underline{R}(A)}(x)=\vee_{y\in[x]_R}\gamma_A(y)$,

$\mu_{\overline{R}(A)}(x)=\vee_{y\in[x]_R}\mu_A(y)$, $\sigma_{\overline{R}(A)}(x)=\vee_{y\in[x]_R}\sigma_A(y)$, $\gamma_{\overline{R}(A)}(x)=\wedge_{y\in[x]_R}\gamma_A(y)$

Definition 2.9[10] Let U be an universe, R be an equivalence relation on U and F be a neutrosophic set in U. If the collection $\tau_N(F)=\{0_N,1_N,\underline{N}(F),\overline{N}(F),BN(F)\}$ forms a topology, then it is said to be a neutrosophic nano topology. We call $(U,\tau_N(F))$ as the neutrosophic nano topological space. The elements of $\tau_N(F)$ are called neutrosophic nano open sets.

Definition 2.10[17] Let U be a nonempty set and the neutrosophic sets A and B are in the form $A=\{\prec x:\mu_A(x),\sigma_A(x),\gamma_A(x)\succ, x\in U\}$, $B=\{\prec x:\mu_B(x),\sigma_B(x),\gamma_B(x)\succ, x\in U\}$. Then the following statements hold:

(i) $0_N=\{\prec x,0,0,1\succ: x\in U\}$ and $1_N=\{\prec x,1,1,0\succ: x\in U\}$

(ii) $A\subseteq B\ iff\ \mu_A(x)\leq\mu_B(x),\sigma_A(x)\leq\sigma_B(x)\,or\,\sigma_A(x)\geq\sigma_B(x),\gamma_A(x)\geq\gamma_B(x)\ for all\ x\in U$

(iii) $A=B\ iff\ A\subseteq B\ and\ B\subseteq A$

(iv) $A^C=\{\prec x,\gamma_A(x),1-\sigma_A(x),\mu_A(x)\succ, x\in U\}$

(v) $A \cap B = \{x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\ for all\ x \in U\}$

(vi) $A \cup B = \{x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \wedge \gamma_B(x)\ for all\ x \in U\}$

(vii) $A - B = \{x, \mu_A(x) \wedge \gamma_B(x), \sigma_A(x) \wedge 1 - \sigma_B(x), \gamma_A(x) \vee \mu_B(x)\ for all\ x \in U\}$

Definition 2.11[10] $[\tau_N(F)]^C$ is called the dual neutrosophic nano topology of $\tau_N(F)$. The elements of $[\tau_N(F)]^C$ are called neutrosophic nano closed (N_N closed) sets. Thus, a neutrosophic set $N(G)$ of U is neutrosophic nano closed iff $U - N(G)$ is neutrosophic nano open in $\tau_N(F)$.

Definition 2.12[10] Let $(U, \tau_N(A))$ be a neutrosophic nano topological space and $A = \{\prec x, \mu_A(x), \sigma_A(x), \gamma_A(x) \succ : x \in U\}$ be a neutrosophic set in X. Then the neutrosophic closure and neutrosophic interior of A are defined by $NCl(A) =$ intersection of all closed sets which contains A and $NInt(A) =$ union of all open sets which is contained in A.

A is a neutrosophic open set iff $A = NInt(A)$ and A is a neutrosophic closed set iff $A = NCl(A)$

III. NEUTROSOPHIC NANO j-CLOSED SETS

Definition 3.1 Let $(U, \tau_N(A))$ be a neutrosophic nano topological space. Then a neutrosophic nano subset A in $(U, \tau_N(A))$ is said to be neutrosophic nano j-closed (briefly N_Nj-closed) set if $N_N Cl(N_N PInt(A)) \subseteq A$.

Theorem 3.2 Every neutrosophic nano closed set is a neutrosophic nano j-closed set.

Proof. Let A be a neutrosophic nano closed set. i.e., $N_N Cl(A) = A$. We know that $N_N Int(A) \subseteq N_N PInt(A) \subseteq A$ which implies $N_N Cl(N_N PInt(A)) \subseteq N_N Cl(A) = A$. Hence every neutrosophic nano closed set is neutrosophic nano j-closed.

Remark 3.3 The converse part of the above theorem need not be true as seen from the following example.

Example 3.4 Let $(U, \tau_N(A))$ be a neutrosophic nano topological space with $U = \{p1, p2, p3\}$, the universe of discourse and $R_U = \{\{p1, p2\}, \{p3\}\}$, the equivalence relation on U.

Let $A = \{\prec p1, (0.5, 0.4, 0.3) \succ, \prec p2, (0.5, 0.6, 0.4) \succ, \prec p3, (0.2, 0.5, 0.2) \succ\}$ be the neutrosophic nano subset of U.

Now, $N_N L_R(A) = \{\prec p1, (0.5, 0.4, 0.4) \succ, \prec p2, (0.5, 0.4, 0.4) \succ, \prec p3, (0.2, 0.5, 0.2) \succ\}$,
$N_N U_R(A) = \{\prec p1, (0.5, 0.6, 0.3) \succ, \prec p2, (0.5, 0.6, 0.3) \succ, \prec p3, (0.2, 0.5, 0.2) \succ\}$,
$N_N B_R(A) = \{\prec p1, (0.4, 0.6, 0.5) \succ, \prec p2, (0.4, 0.6, 0.5) \succ, \prec p3, (0.2, 0.5, 0.2) \succ\}$ and the neutrosophic nano topology formed by the subset A is $\tau_N(A) = \{0_N, 1_N, N_N L_R(A), N_N U_R(A), N_N B_R(A)\}$.

Here the subsets are called neutrosophic nano open sets and the neutrosophic nano closed sets are

$0_N, 1_N, [N_N L_R(A)]^C, [N_N U_R(A)]^C$ and $[N_N B_R(A)]^C$, where

$[N_N L_R(A)]^C = \{\prec p1, (0.4, 0.6, 0.5) \succ, \prec p2, (0.4, 0.6, 0.5) \succ, \prec p3, (0.2, 0.5, 0.2) \succ\}$,

$[N_N U_R(A)]^C = \{\prec p1, (0.3, 0.4, 0.5) \succ, \prec p2, (0.3, 0.4, 0.5) \succ, \prec p3, (0.2, 0.5, 0.2) \succ\}$, and

$[N_N B_R(A)]^C = \{\prec p1, (0.5, 0.4, 0.4) \succ, \prec p2, (0.5, 0.4, 0.4) \succ, \prec p3, (0.2, 0.5, 0.2) \succ\}$.

Now, $N_N Int(A) = \{\prec p1, (0.5, 0.6, 0.3) \succ, \prec p2, (0.5, 0.6, 0.3) \succ, \prec p3, (0.2, 0.5, 0.2) \succ\}$ and $N_N PInt(A) = \{\prec p1, (0.5, 0.6, 0.3) \succ, \prec p2, (0.5, 0.6, 0.3) \succ, \prec p3, (0.2, 0.5, 0.2) \succ\}$.

Let us take a closed set in $\tau_N(A)$ and let it be B.
i.e., $B=\{\prec p1,(0.5,0.4,0.4)\succ,\prec p2,(0.5,0.4,0.4)\succ,\prec p3,(0.2,0.5,0.2)\succ\}$.

Clearly $N_N Cl(N_N PInt(B))=B^C\subseteq B \Rightarrow B$ is NNj-closed.

Let us take an another NNj-closed set $C=\{\prec p1,(0.6,0.4,0.2)\succ,\prec p2,(0.6,0.5,0.3)\succ,\prec p3,(0.3,0.5,0.1)\succ\}$. But C is not a neutrosophic nano closed set. Hence a NNj-closed set need not be a NN closed set.

Theorem 3.5 The union (intersection) of two NNj-closed (open) sets need not be a NNj-closed (open) set as seen in the following example.

Example 3.6 Let $(U,\tau_N(A))$ be a neutrosophic nano topological space with $U=\{p1,p2,p3\}$, the universe of discourse and $R_U=\{\{p1,p2\},\{p3\}\}$, the equivalence relation on U.

Let $A=\{\prec p1,(0.5,0.4,0.3)\succ,\prec p2,(0.5,0.6,0.4)\succ,\prec p3,(0.2,0.5,0.2)\succ\}$ be the neutrosophic nano subset of U. The sets $\{\prec p1,(0.4,0.6,0.5)\succ,\prec p2,(0.4,0.6,0.5)\succ,\prec p3,(0.2,0.5,0.2)\succ\}$ and $\{\prec p1,(0.5,0.4,0.4)\succ,\prec p2,(0.5,0.4,0.4)\succ,\prec p3,(0.2,0.5,0.2)\succ\}$ are NNj-closed sets. But $\{\prec p1,(0.5,0.6,0.4)\succ,\prec p2,(0.5,0.6,0.4)\succ,\prec p3,(0.2,0.5,0.2)\succ\}$ which is the intersection of the above two sets is not a NNj-closed sets.

Theorem 3.7 Every neutrosophic nano j-closed set is a neutrosophic nano pre closed set.

Proof. Let A be a neutrosophic nano j-closed set. i.e., $N_N Cl(N_N PInt(A))\subseteq A$. We know that $N_N Int(A)\subseteq N_N PInt(A)$ which implies $N_N Cl(N_N Int(A))\subseteq N_N Cl(N_N PInt(A))\subseteq A$. Therefore A is a neutrosophic nano pre closed set. Hence every NNj-closed set is NN pre closed.

Remark 3.8 The converse part of the above theorem need not be true as seen from the following example.

Example 3.9 Let $U=\{p1,p2,p3\}$ be the universe with the equivalence relation $R_U=\{\{p1,p3\},\{p2\}\}$ and let the neutrosophic nano subset on U be $A=\{\prec p1,(0.3,0.4,0.2)\succ,\prec p2,(0.4,0.5,0.1)\succ,\prec p3,(0.5,0.2,0.3)\succ\}$. Here $N_N L_R(A)=\{\prec p1,(0.3,0.2,0.3)\succ,\prec p2,(0.4,0.5,0.1)\succ,\prec p3,(0.3,0.2,0.3)\succ\}$, $N_N U_R(A)=\{\prec p1,(0.5,0.4,0.2)\succ,\prec p2,(0.4,0.5,0.1)\succ,\prec p3,(0.5,0.4,0.2)\succ\}$, and $N_N B_R(A)=\{\prec p1,(0.3,0.4,0.3)\succ,\prec p2,(0.1,0.5,0.4)\succ,\prec p3,(0.3,0.4,0.3)\succ\}$. Then the neutrosophic nano topology formed by A is $\tau_N(A)=\{0_N,1_N,N_N L_R(A),N_N U_R(A),N_N B_R(A)\}$.

The subsets of $\tau_N(A)$ are called neutrosophic nano open sets and the neutrosophic nano closed sets are $0_N,1_N,[N_N L_R(A)]^C,[N_N U_R(A)]^C \text{ and } [N_N B_R(A)]^C$ where $[N_N L_R(A)]^C=\{\prec p1,(0.3,0.8,0.3)\succ,\prec p2,(0.1,0.5,0.4)\succ,\prec p3,(0.3,0.8,0.3)\succ\}$, $[N_N U_R(A)]^C=\{\prec p1,(0.2,0.6,0.5)\succ,\prec p2,(0.1,0.5,0.4)\succ,\prec p3,(0.2,0.6,0.5)\succ\}$, and $[N_N B_R(A)]^C=\{\prec p1,(0.3,0.6,0.3)\succ,\prec p2,(0.4,0.5,0.1)\succ,\prec p3,(0.3,0.6,0.3)\succ\}$. Then $N_N Int(A)=\{\prec p1,(0.3,0.4,0.3)\succ,\prec p2,(0.4,0.5,0.1)\succ,\prec p3,(0.3,0.4,0.3)\succ\}$, $N_N PInt(A)=\{\prec p1,(0.3,0.4,0.2)\succ,\prec p2,(0.4,0.5,0.1)\succ,\prec p3,(0.4,0.4,0.3)\succ\}$ and $Cl(A)=1_N$.

Clearly the set A itself is a neutrosophic nano pre closed set, but not a neutrosophic nano j-closed set, since $N_N Cl(N_N PInt(A))=1_N$, which is not contained in A.

Theorem: 3.10 Every neutrosophic nano regular closed set is a neutrosophic nano j-closed set.

Proof. We know that every N_N regular closed set is a N_N closed set and also every N_N closed set is a N_Nj-closed set. Hence every N_N regular closed set is a N_Nj-closed set.

Remark 3.11 The converse part of the above theorem need not be true as seen in the following example.

Example 3.12 Let $U=\{p1,p2,p3\}$ be the universe, $R_U=\{\{p1,p2\},\{p3\}\}$ be the equivalence relation on U, and $A=\{\prec p1,(0.1,0.4,0.2)\succ,\prec p2,(0.4,0.2,0.3)\succ,\prec p3,(0.5,0.3,0.3)\succ\}$ be the neutrosphic nano subset of U. Then $N_NL_R(A)=\{\prec p1,(0.1,0.2,0.3)\succ,\prec p2,(0.1,0.2,0.3)\succ,\prec p3,(0.5,0.3,0.3)\succ\}$, $N_NU_R(A)=\{\prec p1,(0.4,0.4,0.2)\succ,\prec p2,(0.4,0.4,0.2)\succ,\prec p3,(0.5,0.3,0.3)\succ\}$, $N_NB_R(A)=\{\prec p1,(0.3,0.4,0.2)\succ,\prec p2,(0.3,0.4,0.2)\succ,\prec p3,(0.3,0.3,0.5)\succ\}$, and the neutrosophic nano topology formed by A is $\tau_N(A)=\{0_N,1_N,N_NL_R(A),N_NU_R(A),N_NB_R(A)\}$.

Here the subsets are called neutrosophic nano open sets and the neutrosophic nano closed sets are $0_N,1_N,[N_NL_R(A)]^C,[N_NU_R(A)]^C, and\ [N_NB_R(A)]^C$ where

$[N_NL_R(A)]^C=\{\prec p1,(0.3,0.8,0.1)\succ,\prec p2,(0.3,0.8,0.1)\succ,\prec p3,(0.3,0.7,0.5)\succ\}$,

$[N_NU_R(A)]^C=\{\prec p1,(0.2,0.6,0.4)\succ,\prec p2,(0.2,0.6,0.4)\succ,\prec p3,(0.3,0.7,0.5)\succ\}$, and

$[N_NB_R(A)]^C=\{\prec p1,(0.2,0.6,0.3)\succ,\prec p2,(0.2,0.6,0.3)\succ,\prec p3,(0.5,0.7,0.3)\succ\}$. Then

$N_NInt(A)=\{\prec p1,(0.1,0.2,0.3)\succ,\prec p2,(0.1,0.2,0.3)\succ,\prec p3,(0.5,0.3,0.3)\succ\}$ and $Cl(A)=1_N$.

Let $B=\{\prec p1,(0.2,0.2,0.2)\succ,\prec p2,(0.3,0.4,0.2)\succ,\prec p3,(0.5,0.4,0.2)\succ\}$ be an another neutrosophic nano subset on U. Clearly $N_NCl(N_NPInt(B))=N_NCl(N_NL_R(A))=[N_NB_RA)]^C\subseteq B$.

But $N_NCl(N_NInt(B))\neq B$. Hence a N_Nj-closed set need not be a N_N regular closed.

Definition 3.13 Let $(U,\tau_N(A))$ be a neutrosophic nano topological space. Then a neutrosophic nano subset A in $(U,\tau_N(A))$ is said to be neutrosophic nano generalized closed (briefly N_Ng-closed) set if $N_NCl(A)\subseteq V$ whenever $A\subseteq V$ and V is neutrosophic nano open in U.

Theorem 3.14 Every neutrosophic nano closed set is a neutrosophic nano generalized closed set.

Proof. Let A be the neutrosophic nano closed set. Let $A\subseteq V$ and V is neutrosophic nano open set in U. Since A is N_N closed, $N_NCl(A)\subseteq A$. i.e., $N_NCl(A)\subseteq A\subseteq V$. Hence A is N_Ng-closed set. Hence every N_N closed set is N_Ng-closed.

Remark 3.15 The converse of the above theorem need not be true as seen in the following example.

Example 3.16 Let $(U,\tau_N(A))$ be a neutrosophic nano topological space with $U=\{p1,p2,p3\}$, the universe of discourse and $R_U=\{\{p1,p2\},\{p3\}\}$, the equivalence relation on U.

Let $A=\{\prec p1,(0.5,0.4,0.3)\succ,\prec p2,(0.5,0.6,0.4)\succ,\prec p3,(0.2,0.5,0.2)\succ\}$ be the neutrosophic nano subset of U.

Now, $N_NL_R(A)=\{\prec p1,(0.5,0.4,0.4)\succ,\prec p2,(0.5,0.4,0.4)\succ,\prec p3,(0.2,0.5,0.2)\succ\}$, $N_NU_R(A)=\{\prec p1,(0.5,0.6,0.3)\succ,\prec p2,(0.5,0.6,0.3)\succ,\prec p3,(0.2,0.5,0.2)\succ\}$, $N_NB_R(A)=\{\prec p1,(0.4,0.6,0.5)\succ,\prec p2,(0.4,0.6,0.5)\succ,\prec p3,(0.2,0.5,0.2)\succ\}$ and the neutrosophic nano topology formed by the subset A is $\tau_N(A)=\{0_N,1_N,N_NL_R(A),N_NU_R(A),N_NB_R(A)\}$.

Let $V=N_NU_R(A)$ and $B=\{\prec p1,(0.4,0.5,0.6)\succ,\prec p2,(0.3,0.3,0.5)\succ,\prec p3,(0.1,0.4,0.3)\succ\}$.

Clearly B is a N_Ng-closed set, since $Cl(B) \subseteq V$ whenever $B \subseteq V$. But it is not a N_N closed set.

Definition 3.17 Let $(U, \tau_N(F))$ be a neutrosophic nano topological space. Then a neutrosophic nano subset A in $(U, \tau_N(F))$ is said to be neutrosophic nano generalized j-closed (briefly $N_N$gj-closed) set if $N_N JCl \subseteq V$ whenever $A \subseteq V$ and V is neutrosophic nano open in U.

Definition 3.18 Let $(U, \tau_N(F))$ be a neutrosophic nano topological space. Then a neutrosophic nano subset A in $(U, \tau_N(F))$ is said to be neutrosophic nano generalized j*-closed (briefly $N_N$gj*-closed) set if $N_N JCl \subseteq V$ whenever $A \subseteq V$ and V is neutrosophic nano j-open in U.

Theorem 3.19 If A is a neutrosophic nano gj-closed set in $(U, \tau_R(X))$ and $A \subseteq B \subseteq N_N JCl(A)$, then B is neutrosophic nano generalized j-closed set in $(U, \tau_R(X))$.

Proof. Let $B \subseteq V$ where V is neutrosophic nano open in U. Then $A \subseteq B$ implies $A \subseteq V$. Since A is $N_N$gj-closed, $N_N JCl(A) \subseteq V$. Also $A \subseteq N_N JCl(B)$ implies $N_N JCl(B) \subseteq N_N JCl(A)$. Thus $N_N JCl(B) \subseteq V$ and therefore B is $N_N$gj-closed set in U.

Theorem 3.20 Every neutrosophic nano closed set is a neutrosophic nano generalized j-closed.

Proof. Let A be a neutrosophic nano closed set in U. Let $A \subseteq V$ and V is neutrosophic nano open in U. Since A is neutrosophic nano closed, $N_N Cl(A) = A \subseteq V$. Also $N_N JCl(A) \subseteq N_N Cl(A) \subseteq V$, where V is N_N open in U. Therefore A is a neutrosophic nano generalized j-closed set. Hence every N_N closed set is $N_N$gj-closed.

Remark 3.21 The converse part of the above theorem need not be true as seen in the following example.

Example 3.22 Let $U = \{p1, p2, p3\}$ be the universe, $R_U = \{\{p1, p2\}, \{p3\}\}$ be the equivalence relation on U, and $A = \{\prec p1,(0.1,0.4,0.2) \succ, \prec p2,(0.4,0.2,0.3) \succ, \prec p3,(0.5,0.3,0.3) \succ\}$ be the neutrosphic nano subset of U. Then $N_N L_R(A) = \{\prec p1,(0.1,0.2,0.3) \succ, \prec p2,(0.1,0.2,0.3) \succ, \prec p3,(0.5,0.3,0.3) \succ\}$, $N_N U_R(A) = \{\prec p1,(0.4,0.4,0.2) \succ, \prec p2,(0.4,0.4,0.2) \succ, \prec p3,(0.5,0.3,0.3) \succ\}$, $N_N B_R(A) = \{\prec p1,(0.3,0.4,0.2) \succ, \prec p2,(0.3,0.4,0.2) \succ, \prec p3,(0.3,0.3,0.5) \succ\}$, and the neutrosophic nano topology formed by A is $\tau_N(A) = \{0_N, 1_N, N_N L_R(A), N_N U_R(A), N_N B_R(A)\}$. Let the open set $V = \{\prec p1,(0.4,0.4,0.2) \succ, \prec p2,(0.4,0.4,0.2) \succ, \prec p3,(0.5,0.3,0.3) \succ\}$.

Let $B = \{\prec p1,(0.2,0.3,0.3) \succ, \prec p2,(0.2,0.3,0.4) \succ, \prec p3,(0.1,0.2,0.3) \succ\}$. Clearly $B \subseteq V$.

Also $N_N JCl(B) \subseteq V$. Hence B is a $N_N$gj-closed set, but not a N_N closed set.

Theorem 3.23 Every neutrosophic nano j-closed set is a neutrosophic nano generalized j-closed set.

Proof. Let A be a N_Nj-closed set. Let $A \subseteq V$ and V is neutrosophic nano open in U. Since A is N_Nj-closed, $N_N JCl(A) \subseteq A \subseteq V$. Therefore A is $N_N$gj-closed. Hence every N_Nj-closed set is $N_N$gj-closed.

Remark 3.24 The converse of the above theorem need not be true as seen in the following example.

Example 3.25 In example 3.22, B is a $N_N$gj-closed set. But $N_N Cl(N_N PInt(B))$ is not contained in V. i.e., B is not a N_Nj-closed set. Hence every $N_N$gj-closed set need not a N_Nj-closed set.

Theorem 3.26 Every N_Ng-closed set is a $N_N$gj-closed set.

Proof. Let A be a N_Ng-closed set. Then $N_N Cl(A) \subseteq V$ whenever $A \subseteq V$ and V is neutrosophic nano open in U. Since $N_N JCl(A) \subseteq N_N Cl(A) \subseteq V$, we have $N_N JCl(A) \subseteq V$ whenever $A \subseteq V$ and V is N_N open in U. Therefore A is $N_N$gj-closed. Hence every N_Ng-closed set is a $N_N$gj-closed set.

Theorem 3.27 Every N_Nj-closed set is a $N_N$gj*-closed set.

Proof. Let A be a NNj-closed set. Let $A \subseteq V$ and V is neutrosophic nano j-open in U. Since A is NNj-closed, $N_N JCl(A) = A \subseteq V$, V is NNj- open in U. Therefore A is NNgj*-closed. Hence every NNj-closed set is a NNgj*-closed set.

Theorem 3.28 Every NNgj*-closed set is a NNgj-closed set.

Proof. Let A be a NNgj*-closed set. Let $A \subseteq V$ and V is neutrosophic nano open in U. Since every NN open set is NNj-open, V is NNj-open in U. Since A is NNgj*-closed set, we have $N_N JCl(A) \subseteq V$. Therefore $N_N JCl(A) \subseteq V$ whenever $A \subseteq V$ and V is NNj-open in U. Therefore A is NNgj-closed. Hence every NNgj*-closed set is a NNgj-closed set.

IV. Conclusion

Neutrosophic nano j-closed set, neutrosophic nano generalized closed set, neutrosophic nano generalized j-closed set, neutrosophic nano generalized j*-closed set were introduced and some of their properties were discussed in this paper. The concept can be used for real life decision making problems where the situations of indeterminacy occurs. The practical problems may be solved by finding CORE values through the criterion reduction.

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Editors-in-Chief:

Prof. Dr. Florentin Smarandache
Department of Mathematics and Science
University of New Mexico
705 Gurley Avenue
Gallup, NM 87301, USA
E-mail: smarand@unm.edu

Dr. Mohamed Abdel-Basset
Department of Operations Research
Faculty of Computers and Informatics
Zagazig University
Zagazig, Ash Sharqia 44519, Egypt
E-mail:mohamed.abdelbasset@fci.zu.edu