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Minimal Structures and Grill in Neutrosophic Topological Spaces

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Abstract: The main objective of this article is to introduce the notion of minimal structure (in short M-structure) and grill in neutrosophic topological space (in short N-T-space). Besides, we establish its relation with some existing notions on different types of open sets in N-T-space, and investigate some basic properties of the class of M-structure and grill in NT-space. Further, we furnish some suitable examples of M-structure and grill via NT-space.

Keywords: *Neutrosophic Set*; *M-structure*; *Neutrosophic Topology*; *Neutrosophic Grill Topology*.

1. Introduction: The concept of fuzzy set (in short F-set) theory was grounded by Zadeh [29] in 1965. Uncertainty plays an important role in our everyday life problems. Zadeh [29] associated the membership value with the elements to control the uncertainty. It was not sufficient to control uncertainty, so Atanaosv [3] added non-membership value along with the membership value, and introduced the notion of intuitionistic fuzzy set (in short IF-set). Still it was difficult to handle some real world problems under uncertainty, in particular for problems on decision making. In order to overcome this difficulty, Smarandache [24] considered the elements with truth-membership, indeterminacy-membership and false-membership values, and grounded the idea of neutrosophic set (in short N-set) theory in 1998. Till now, the concept of N-set has been applied in many branches of science and technology.

The notions of N-T-space was first grounded by Salama and Alblowi [22], followed by Salama and Alblowi [23], Iswaraya and Bageerathi [15], who studied the concept of neutrosophic semi-open set (in short NSO-set) and neutrosophic semi-closed set (in short NSC-set). Arokiarani et al. [2] studied some new notations and mappings via N-T-spaces. Iswarya and Bageerathi [15] established the notion of neutrosophic semi-open sets via neutrosophic topological space. Afterwards, Rao and Srinivasa [21] introduced and studied neutrosophic pre-open set (in short NPO-set) and neutrosophic pre-closed set (in short NPC-set) via N-T-space. Das and Pramanik [7] grounded the notion of generalized neutrosophicx *b*-open set via neutrosophic topological space. Later on, Das and Pramanik [8] introduced the notion of neutrosophic ϕ-open set and neutrosophic ϕ-continuous function via neutrosophic topological space. Recently, Das and Tripathy [11] presented the notion of neutrosophic simply *b*-open set via neutrosophic topological space. Thereafter, Das and Tripathy [13] introduced the idea of pairwise neutrosophic *b*-open sets via N-T-space. Parimala et al. [19] introduced the notion of neutrosophic nano ideal topological space. Later on Parimala et al. [20] grounded the idea of neutrosophic $\alpha\psi$-homomorphism via neutrosophic topological spaces.

Makai et al. [16] introduced and studied the concept of minimal structure (in short M-structure) via topological space. It is found to have useful applications and the notion was investigated by Madok [17]. The notion of minimal structure in a fuzzy topological space was introduce by Alimohammady and Roohi [1] and further investigated by Tripathy and Debnath [27] and others.

In this article, we introduce the idea of minimal structure and grill via N-T-spaces. We establish its relation with some existing notions on several types of open sets via N-T-spaces. Besides, we investigate some basic properties of the class of minimal structures and grill via N-T-spaces. Further, we furnish some suitable examples on minimal structures and grill via N-T-spaces.

The rest of the paper is divided into following sections:

In section 2, we provide some definitions and results those are very useful for the preparation of the main results of this article. In section 3, we introduce the concept of M-structure and gril via NT-spaces, and proved some basic results on them. In section 4, we introduce an operator $(\ )^*$ on M-structure via NT-spaces, and established several interesting results based on it. Finally, in section 5, we conclude the work done in this article.

2. Preliminaries & Definitions:

In this section, we provide some existing results on neutrosophic set and neutrosophic topology those are relevant to main results of this article.

Definition 2.1.[24] Suppose that Ĝ be a fixed set. Then *W*, an N-set over Ĝ is a set contains triplet having truth, indeterminacy and false membership values that can be characterized independently, denoted by T_W, I_W, F_W in the unit interval [0, 1].

We denote the N-set *W* as follows:

$$W = \{(r, T_W(r), I_W(r), F_W(r)): r \in \hat{G}\}, \text{ where } T_W(r), I_W(r), F_W(r) \in [0, 1], \text{ for all } r \in \hat{G}.$$

Since, no restriction on the values of $T_W(r)$, $I_W(r)$ and $F_W(r)$ is imposed, so we have

$$0 \leq T_W(r) + I_W(r) + F_W(r) \leq 3, \text{ for all } r \in \hat{G}.$$

Example 2.1. Suppose that Ĝ={b, c} be a fixed set. Clearly, W={(b,0.4,0.8,0.7), (c,0.2,0.8,0.7)} is an N-set over Ĝ.

Definition 2.2.[24] Suppose that $W = \{(r, T_W(r), I_W(r), F_W(r)): r\in\hat{G}\}$ be an N-set over Ĝ. Then, W^c i.e., the complement of W is defined by $W^c = \{(r, 1\text{-}T_W(r), 1\text{-}I_W(r), 1\text{-}F_W(r)): r\in\hat{G}\}$.

Example 2.2. Suppose that Ĝ={b, c} be a fixed set. Let W={(b,0.5,0.5,0.7), (c,0.5,0.7,0.8)} be an N-set over Ĝ. Then, the complement of W is W^c={(b,0.5,0.5,0.3), (c,0.5,0.3,0.2)}.

Definition 2.3.[24] An N-set $W = \{(r, T_W(r), I_W(r), F_W(r)): r\in\hat{G}\}$ is called a subset of an N-set $L = \{(r, T_L(r), I_L(r), F_L(r)): r\in\hat{G}\}$ (i.e., $W\subseteq L$) if and only if $T_W(r) \leq T_L(r)$, $I_W(r) \geq I_L(r)$, $F_W(r) \geq F_L(r)$, for each $r\in\hat{G}$.

Example 2.3. Suppose that Ĝ={b, c} be a fixed set. Let M={(b,0.5,0.5,0.7), (c,0.5,0.7,0.8)} and W={(b,0.7,0.2,0.5), (c,0.9,0.5,0.3)} be two N-sets over Ĝ. Clearly, $M\subseteq W$.

Definition 2.4.[24] Assume that $W = \{(r, T_W(r), I_W(r), F_W(r)): r\in\hat{G}\}$ and $L=\{(r, T_L(r), I_L(r), F_L(r)): r\in\hat{G}\}$ be any two N-sets over a fixed set Ĝ. Then, their intersection and union are defined as follows:

$$W\cap L = \{(r, T_N(r)\wedge T_L(r), I_N(r)\vee I_L(r), F_N(r)\vee F_L(r)): r\in\hat{G}\},$$

$$W\cup L = \{(r, T_N(r)\vee T_L(r), I_N(r)\wedge I_L(r), F_N(r)\wedge F_L(r)): r\in\hat{G}\}.$$

Example 2.4. Suppose that Ĝ={b, c} be a fixed set. Let W={(b,0.5,0.5,0.7), (c,0.5,0.7,0.8)} and M= {(b,0.7,0.2,0.5), (c,0.9,0.5,0.3)} be two N-sets over Ĝ. Then, $W\cup M$={(b,0.7,0.2,0.5), (c,0.9,0.5,0.3)} and $W\cap M$={(b,0.5,0.5,0.7), (c,0.5,0.7,0.8)}.

Definition 2.5.[22] The null N-set (0_N) and absolute N-set (1_N) over Ĝ are represented as follows:

(i) $0_N =\{(r, 0, 0, 1) : r\in\hat{G}\}$;

(ii) $0_N =\{(r, 0, 1, 0) : r\in\hat{G}\}$;

(iii) $0_N =\{(r, 0, 1, 1) : r\in\hat{G}\}$;

(iv) $0_N =\{(r, 0, 0, 0) : r\in\hat{G}\}$;

(v) $1_N =\{(r, 1, 0, 1) : r\in\hat{G}\}$;

(vi) $1_N =\{(r, 1, 1, 0) : r\in\hat{G}\}$;

(vii) $1_N =\{(r, 1, 0, 0) : r\in\hat{G}\}$;

($viii$) $1_N =\{(r, 1, 1, 1) : r\in\hat{G}\}$.

Clearly, $0_N\subseteq 1_N$. We have, for any N-set W, $0_N \subseteq W \subseteq 1_N$.

Throughout the article, we will use $0_N =\{(r, 0, 1, 1) : r\in\hat{G}\}$ and $1_N =\{(r, 1, 0, 0) : r\in\hat{G}\}$.

Definition 2.6.[22] Assume that Ĝ be a fixed set. Then τ, a family of N-sets over Ĝ is called an N-T on Ĝ if the following condition holds:

(*i*) 0_N, $1_N \in \tau$;

(*ii*) W_1, $W_2 \in \tau \Rightarrow W_1 \cap W_2 \in \tau$;

(*iii*) $\{W_i : i \in \Delta\} \subseteq \tau \Rightarrow \cup_{i\in\Delta} W_i \in \tau$.

Then, the pair (Ĝ, τ) is called an N-T-space. If $W \in \tau$, then W is called an neutrosophic open set (in short N-O-set) in (Ĝ, τ), and the complement of W is called an neutrosophic closed set (in short N-C-set) in (Ĝ, τ).

Example 2.5. Suppose that W, E and Z be three N-sets over a fixed set Ĝ={p, q} such that:

W={(p,0.7,0.5,0.7), (q,0.5,0.1,0.5): p, $q \in$ Ĝ};

E={(p,0.6,0.9,0.8), (q,0.5,0.3,0.8): p, $q \in$ Ĝ};

Z={(p,0.5,1.0,0.8), (q,0.4,0.4,0.9): p, $q \in$ Ĝ}.

Then, the collection τ={0_N, 1_N, W, E, Z} forms a neutrosophic topology on Ĝ. Here, 0_N, 1_N, W, E, Z are NOSs in (Ĝ, τ), and their complements 1_N, 0_N, W^c={(p,0.3,0.5,0.3), (q,0.5,0.9,0.5): p, $q \in$ Ĝ}, E^c={(p,0.4,0.1, 0.2), (q,0.5,0.7,0.2): p, $q \in$ Ĝ} and Z={(p,0.5,0.0,0.2), (q,0.6,0.6,0.1): p, $q \in$ Ĝ} are NCSs in (Ĝ, τ).

The neutrosophic interior and neutrosophic closure of an N-set are defined as follows:

Definition 2.7.[22] Assume that τ be an N-T on Ĝ. Suppose that W be an N-set over Ĝ. Then,

(*i*) neutrosophic interior (in short N_{int}) of W is the union of all N-O-sets in (Ĝ, τ) those are contained in W, i.e., $N_{int}(W) = \cup\{E : E$ is an N-O-set in Ĝ such that $E \subseteq W\}$;

(*ii*) neutrosophic closure (in short N_{cl}) of W is the intersection of all N-C-sets in (Ĝ, τ) those containing W, i.e., $N_{cl}(W) = \cap\{F : F$ is an N-C-set in Ĝ such that $W \subseteq F\}$.

Clearly $N_{int}(W)$ is the largest N-O-set contained in W, and $N_{cl}(W)$ is the smallest N-C-set containing W.

Example 2.6. Suppose that (Ĝ, τ) be an NT-space as shown in **Example 2.5.** Suppose that U={(p,0.5,0.5,0.7), (q,0.5,0.7,0.8)} be an N-set over Ĝ. Then, $N_{int}(U)=0_N$ and $N_{cl}(U)$={(p,0.5,0.0,0.2), (q,0.6,0.6,0.1)}.

Proposition 2.1.[22] For any N-set B in (Ĝ, τ), we have the following:

(*i*) $N_{int}(B^c) = (N_{cl}(B))^c$;

(*ii*) $N_{cl}(B^c) = (N_{int}(B))^c$.

Definition 2.2.[21] Suppose that (Ĝ, τ) be an N-T-space, and W be an N-set over Ĝ. Then, W is called

(*i*) NSO-set if and only if $W \subseteq N_{cl}(N_{int}(W))$;

(*ii*) NPO-set if and only if $W \subseteq N_{int}(N_{cl}(W))$.

The collection of all NSO sets and NPO sets in (Ĝ, τ) are denoted by NSO(τ) and NPO(τ).

Example 2.7. Suppose that Ĝ={*a*, *b*} be a fixed set. Clearly, (Ĝ, τ) is an NT-space, where τ={0_N, 1_N, {(*a*, 0.3,0.3,0.4), (*b*,0.4,0.4,0.3): *a*, *b*∈Ĝ}, {(*a*,0.4,0.1,0.4), (*b*,0.5,0.3,0.1): *a*, *b*∈Ĝ}}. Then, the N-set *Q*={(*a*,0.6, 0.1,0.4), (*b*,0.9,0.2,0.1): *a*, *b*∈Ĝ} is an NSO set, and *P*={(*a*,0.3,0.2,0.9), (*b*,0.3,0.3,0.4): *a*, *b*∈Ĝ} is an NPO set in (Ĝ, τ).

Definition 2.8.[2] Assume that (Ĝ, τ) be an N-T-space. Then *W*, an N-set over Ĝ is called an neutrosophic *b*-open set (in short N-*b*-O-set) in (Ĝ, τ) if and only if $W \subseteq N_{int}(N_{cl}(W)) \cup N_{cl}(N_{int}(W))$.

An N-set *G* is called an neutrosophic *b*-closed set (in short N-*b*-C-set) in (Ĝ, τ) if and only if its complement is an N-*b*-O-set in (Ĝ, τ). The collection of all neutrosophic *b*-open sets in (Ĝ, τ) is denoted by N-*b*-O(τ).

Example 2.8. Suppose that (Ĝ, τ) be an NT-space as shown in **Example 2.7**. Then, the neutrosophic set *P*={(*a*,0.3,0.2,0.9), (*b*,0.3,0.3,0.4): *a*, *b*∈Ĝ} is an neutrosophic *b*-open set in (Ĝ, τ).

3. Minimal Structure in Neutrosophic Topological Space

In this section, we procure the notions of M-structure and grill in N-T-space.

Definition 3.1. A family *M* of neutrosophic subsets of Ĝ i.e., M ⊂ P(Ĝ), where P(Ĝ) is the collection of all N-sets defined over Ĝ is said to be a M-structure on Ĝ if 0_N and 1_N belong to *M*. By (Ĝ, *M*), we denote the neutrosophic minimal space (in short N-M-space). The members of *M* are called neutrosophic minimal-open (in short N-m-O) subset of Ĝ.

Example 3.1. Let *W*, *E* and *Z* be three neutrosophic sets over a fixed set Ĝ={*b*, *c*} such that:

W={(*b*,0.7,0.5,0.7), (*c*,0.5,0.1,0.5): *b*, *c*∈Ĝ};

E={(*b*,0.6,0.9,0.8), (*c*,0.5,0.3,0.8): *b*, *c*∈Ĝ};

Z={(*b*,0.5,1.0,0.8), (*c*,0.4,0.4,0.9): *b*, *c*∈Ĝ}.

Clearly, the collection M={0_N, 1_N, *W*, *E*, *Z*} forms a neutrosophic minimal structure on Ĝ, and the pair (Ĝ, M) is a neutrosophic minimal structure space.

Definition 3.2. The complement of N-m-O set *W* is an neutrosophic m-closed set (in short N-m-C set) in (Ĝ, *M*).

Example 3.2. Let us consider a neutrosophic minimal structure space as shown in Example 3.1. Here, 0_N, 1_N, *W*, *E*, *Z* are neutrosophic minimal open sets in (Ĝ, M), and $(0_N)^c = 1_N$, $(1_N)^c = 0_N$, W^c =

{(b,0.3,0.5,0.3), (c,0.5,0.9,0.5)}, E^c={(b,0.4,0.1,0.2), (c, 0.5,0.7,0.2)}, Z^c = {(b,0.5,0.0,0.2), (c,0.6,0.6,0.1)} are neutrosophic minimal closed sets in (Ĝ, M).

Example 3.3. From the above definitions, it is clear that NPO-sets, NSO-sets, N-α-O-sets, N-b-O-sets are N-m-O sets.

Example 3.4. Let W, E and Z be three neutrosophic sets over a non-empty set Ĝ ={b, c} such that:
W={(b,0.7,0.5,0.7), (c,0.5,0.5,0.1) : b, $c \in X$};
E={(b,0.6,0.8,0.9), (c,0.5,0.8,0.3) : b, $c \in X$};
Z={(b,0.5,0.8,1.0), (c,0.4,0.9,0.4) : b, $c \in X$}.

Here, the family τ={0_N, 1_N, W, E, Z} forms a neutrosophic topology on X, and so (Ĝ, τ) is a neutrosophic topological space. Suppose M = $\tau \cup$NPO(τ)$\cup$NSO(τ)$\cup$N-b-O(τ), then (Ĝ, M) is a neutrosophic minimal structure. Now, from the above it is clear that, every neutrosophic pre-open sets, neutrosophic semi-open sets, neutrosophic b-open sets in (Ĝ, τ) are neutrosophic m-open sets in (Ĝ, M). Further, it is also seen that, every neutrosophic m-open set in (Ĝ, M) is also a neutrosophic pre-open set, neutrosophic semi-open set, neutrosophic b-open set in (Ĝ, τ).

Remark 3.1. We can define neutrosophic minimal interior (in short N_{mint}), neutrosophic minimal closure (in short N_{mcl}) etc. in an N-M-space as we have define in the previous section.

Example 3.5. Suppose that (Ĝ, M) be a neutrosophic minimal structure space as shown in Example 3.1. Then, the neutrosophic minimal interior of D={(b,0.2,0.6,0.4), (c,0.4,0.9,0.7)} is $N_{m\text{-}int}(D)$={(b,0,1,1), (c,0,1,1)}, and the neutrosophic minimal closure of D={(b,0.2,0.6,0.4), (c,0.4,0.9,0.7)} is $N_{m\text{-}cl}(D)$={(b,0.3, 0.5,0.3), (c,0.5,0.9,0.5)} respectively.

In view of the definitions given in this article, we state the following result without proof.

Theorem 3.1. Suppose that (Ĝ, M) be an N-M-space. Then, for any N-sets S and R over Ĝ, the following holds:

(i) $(N_{mcl}(S))^c = N_{mcl}(S^c)$ and $(N_{mint}(S))^c = N_{mcl}(S^c)$.

(ii) $N_{mcl}(S)=S$ if and only if S is an N-m-C set in (Ĝ, M).

(iii) $N_{mint}(S)=S$ if and only if S is an N-m-O set in (Ĝ, M).

(iv) $S \subseteq R \Rightarrow N_{mcl}(S) \subseteq N_{mcl}(R)$ and $N_{mint}(S) \subseteq N_{mint}(R)$.

(v) $S \subseteq N_{mcl}(S)$ and $N_{mint}(S) \subseteq S$.

(vi) $N_{mcl}(N_{mcl}(S))=N_{mcl}(S)$ and $N_{mint}(N_{mint}(S))=N_{mint}(S)$.

Theorem 3.2. Assume that (Ĝ, *M*) be an N-M-space. Suppose that *M* satisfies the property *B*. Then, for an neutrosophic subset *S* of Ĝ, the followings hold:

(*i*) $S \in M$ if and only if $N_{mint}(S)=A$.

(*ii*) *S* is N-m-C set if and only if $N_{mcl}(S)=S$.

(*iii*) $N_{mint}(S) \in M$ and $N_{mcl}(S)$ is an N-m-C set.

Definition 3.3. An N-M-structure *M* on a non-empty set Ĝ is said to be have property *B* if the union of only family of neutrosophic subsets belonging *M* belongs to *M*.

Example 3.6. Suppose that *R*, *E* and *Y* be three neutrosophic sets over a fixed set Ĝ={*b*, *c*} such that:

R={(*b*,0.8,0.5,0.8), (*c*,0.6,0.1,0.6): *b*, *c*∈Ĝ};

E={(*b*,0.7,0.9,0.9), (*c*,0.6,0.3,0.9): *b*, *c*∈Ĝ};

Y={(*b*,0.6,1.0,0.9), (*c*,0.5,0.4,1.0): *b*, *c*∈Ĝ}.

Here, the collection M={0_N, 1_N, *R*, *E*, *Y*} forms a N-M-structure on Ĝ, and so the pair (Ĝ, M) is a neutrosophic minimal structure space. Clearly, the N-M-structure on Ĝ satisfied the property *B* which was stated in Definition 3.3.

Definition 3.4. An N-M-structure (Ĝ, *M*) satisfies the property *J* if the finite intersection of N-m-O sets is an N-m-O set.

Example 3.7. Let us consider a N-M-structure M on Ĝ as shown in Example 3.6. Clearly, the N-M-structure M on Ĝ satisfied the property *j* which was stated in Definition 3.4.

Remark 3.2. If a N-M-structure M on Ĝ is a neutrosophic topology on Ĝ, then M satisfied the property *j* which was stated in Definition 3.4.

Definition 3.5. A family *G* ($0_N \notin G$) of N-sets over Ĝ is called a grill on Ĝ if *G* satisfies the following condition:

(*i*) $S \in G$ and $S \subseteq R \Rightarrow R \in G$;

(*ii*) *S*, $R \subseteq$ Ĝ and $S \cup R \in G \Rightarrow S \in G$ or $R \in G$.

Example 3.8. Suppose that Ĝ = {*b*, *c*} be a fixed set. Then, the collection M={1_N, {(*b*,0.9,0.0,0.0), (*c*,0.9,0.0,0.0)}, {(*b*,0.8,0.0,0.0), (*c*,0.8,0.0,0.0)}} forms a grill on Ĝ.

Remark 3.3. Since $0_N \notin G$, so *G* is not a N-M-structure on Ĝ. An N-M-structure with a grill is called as a neutrosophic grill minimal space (in short N-G-M-space), denoted by (Ĝ, *M*, *G*).

4. ( )*-Operator on Neutrosophic Minimal Structure:

Definition 4.1. Suppose that (Ĝ, M, G) be an N-G-M-space. A function $(\ )^{*m}$: P(Ĝ) → P(Ĝ) called as neutrosophic minimal local function (N-M-L-function) is defined by

$$(\ )^{*m} = \{x \in \hat{G} : S \cap U \in M, \text{ for all } U \in M(x)\}.$$

Now we discuss about some properties of the neutrosophic minimal local function $()^{*m}$ in (Ĝ, M, G).

Definition 4.2. Assume that (Ĝ, M, G) be a N-G-M-space. Then, the boundary of a N-set S over Ĝ is defined by $(\partial S)^{*m} = (S)^{*m} \cap (S^c)^{*m}$.

We state the following two results without prove in view of the above definitions.

Proposition 4.1. Suppose that (Ĝ, M, G) be a N-G-M-space. Then, the following holds:

(i) $(0_N)^{*m} = 0_N$;

(ii) $(S)^{*m} = 0_N$, if $S \notin G$;

(iii) $(S)^{*mP} \subseteq (S)^{*mQ}$, where P and Q are neutrosophic grill on Ĝ with $P \subseteq Q$.

Proposition 4.2. Suppose that P(Ĝ) be the collection of all neutrosophic sets defined over Ĝ. Assume that (Ĝ, M, G) be a N-G-M-space. Then, for $S \in$ P(Ĝ),

(i) $(S)^{*m} = S \cup (\partial S)^{*m}$;

(ii) $(S)^{*m} = \cap_{n \in \Delta} F_n$, where $\{F_n\}_{n \in \Delta}$ is the collection of all $(\ )^{*m}$-closed sets in (Ĝ, M, G).

Theorem 4.1. Assume that (Ĝ, M, G) be a N-G-M-space. Then, the following holds:

(i) $S, R \in$ P(Ĝ) and $S \subseteq R \Rightarrow (S)^{*m} \subseteq (R)^{*m}$;

(ii) For $S \subseteq \hat{G}$, $N_{mcl}(S)^{*m} \subseteq (S)^{*m}$;

(iii) For $S \subseteq \hat{G}$, $(S)^{*m}$ is a N-m-C set;

(iv) For $S \subseteq \hat{G}$, $(S)^{*m} \subseteq N_{mcl}(S)$;

(v) For $S \subseteq \hat{G}$, $[(S)^{*m}]^{*m} \subseteq (S)^{*m}$.

Proof. (i) Assume that $S \subseteq R$ and $x \in (S)^{*m}$. Then, for all $U \in M$, we have by definition, that $U \cap S \in G$. Thus by definition of neutrosophic grill we have $U \cap R \in G$. Hence, $x \in (R)^{*m}$. Therefore, $(S)^{*m} \subseteq (R)^{*m}$.

(ii) Assume that $x \notin N_{mcl}(S)$, for some $S \in \hat{G}$. Then by a known result there exists an $U_x \in M$ such that $U_x \cap S = 0_N \notin G$. Therefore, $x \notin (S)^{*m}$. Hence, we have $(S)^{*m} \subseteq N_{mcl}(S)$.

(iii) Assume that $x \in N_{mcl}(S)^{*m}$ and $U \in M(x)$, then $U \cap (S)^{*m} \neq 0_N$. Next, let $y \in U \cap (S)^{*m}$. Then, we have $y \in U$ and $y \in (S)^{*m}$. Therefore, $U \cap S \in G$, which implies $x \in (S)^{*m}$.

Thus, we have $N_{mcl}(S)^{*m} \subseteq (S)^{*m}$.

(iv) Suppose that $S \in$ P(Ĝ). Then, we have $(S)^{*m} \subseteq N_{mcl}(S)^{*m}$ and $N_{mcl}(S)^{*m} = (S)^{*m}$. Thus, we have for any $(S)^{*m} = N_{mcl}(S)^{*m}$, since $N_{mcl}(S)^{*m}$ is an N-m-C set, so $(S)^{*m}$ is an N-m-C set.

(*v*) Suppose that $S\in P(\hat{G})$. Then from (*iv*) we have $N_{mcl}(S)^{*m} \subseteq (S)^{*m}$. Further on considering $(S)^{*m}$ is place of S, from (vi) we have $((S)^{*m})^{*m} \subseteq N_{mcl}(S)^{*m}$. Hence, from these two inclusion we have $((S)^{*m})^{*m} \subseteq (S)^{*m}$.

Theorem 4.2. Assume that $(\hat{G}, M, G)$ be a N-G-M-space. Suppose that $(\hat{G}, M)$ satisfies the property J. Then, the following holds:

(*i*) $(S\cup R)^{*m} = (S)^{*m} \cup (R)^{*m}$, for $S, R \subseteq M$;

(*ii*) $(S\cap R)^{*m} \subseteq (S)^{*m} \cap (R)^{*m}$, for $S, R \subseteq M$.

Proof. (*i*) We have $S \subseteq S\cup R$, $R \subseteq S\cup R$. Thus, we have $(S)^{*m} \subseteq (S\cup R)^{*m}$ and $(R)^{*m} \subseteq (S\cup R)^{*m}$. This implies, $(S)^{*m} \cup (R)^{*m} \subseteq (S\cup R)^{*m}$ (1)

Suppose that $x \notin (S)^{*m}\cup(R)^{*m}$. Therefore there exists $U_1, U_2 \in M(x)$ such that $U_1\cap S\notin G$, $U_2\cap R\notin G$. This implies, $(U_1\cap S) \cup (U_2\cap R)\notin G$.

Now, $U_1, U_2 \in M(x) \Rightarrow U_1\cap U_2\in M(x)$ and $(S\cup R) \cap (U_1\cap U_2) \subseteq (U_1\cap S) \cup (U_2\cap R)\notin G$.

Therefore, $x\notin(S\cup R)^{*m}$. Thus, we have $(S\cup R)^{*m} \subseteq (S)^{*m} \cup (R)^{*m}$ (2)

From (1) and (2) we have, $(S\cup R)^{*m} = (S)^{*m} \cup (R)^{*m}$.

(*ii*) We have $S\cap R \subseteq S$ and $S\cap R \subseteq R$. This implies, $(S\cap R)^{*m} \subseteq (S)^{*m}$ and $(S\cap R)^{*m} \subseteq (R)^{*m}$. Therefore, $(S\cap R)^{*m} \subseteq (S)^{*m} \cap (R)^{*m}$.

Theorem 4.3. Suppose that $(\hat{G}, M, G)$ be a N-G-M-space. Assume that $(\hat{G}, M)$ satisfies the property J. Then, the following holds:

(*i*) For $W\subseteq M$ and $S\subseteq\hat{G}$, $W\cap(S)^{*m} = W\cap(W\cap S)^{*m}$;

(*ii*) For $S, R \subseteq \hat{G}$, $[(S)^{*m} \setminus (R)^{*m}] = [(S\setminus R)^{*m} \setminus (R)^{*m}]$;

(*iii*) For $S, R \subseteq \hat{G}$, with $R\notin G$. $(S\cup R)^{*m} = (S)^{*m} = (S\setminus R)^{*m}$.

Proof. (*i*) It is known that $(W\cap S) \subseteq S$.

Now, $(W\cap S) \subseteq S$

$\Rightarrow (W\cap S)^{*m} \subseteq (S)^{*m}$

$\Rightarrow W\cap(W\cap(S))^{*m} \subseteq W\cap(S))^{*m}$ (3)

Assume that $x\in W\cap(S)^{*m}$ and $V\in M(x)$.

Then, we have $W\cap V\in M(x)$ and $x\in(S)^{*m}$ implies $(W\cap V) \cap S\in G$.

Thus, $(W\cap S) \cap V\in G$. Thus, we have $x\in(W\cap S)^{*m}$, which implies $x\in W\cap(W\cap S)^{*m}$.

Hence, $(W\cap S)^{*m} \subseteq W\cap(W\cap S)^{*m}$ (4)

From (3) and (4), we have $W\cap(S)^{*m} = W\cap(W\cap S)^{*m}$.

(*ii*) We have,

$(S)^{*m} = [(S\setminus R) \cup (S\cap R)]^{*m}$

$= (S\setminus R)^{*m} \cup (S\cap R)^{*m}$ [by part (*i*)]

$\subseteq [(S\setminus R)^{*m} \cup (R)^{*m}]$.

Thus, we have $[(S)^{*m}\setminus(R)^{*m}] \subseteq [(S\setminus R)^{*m} \cup (R)^{*m}]$.

We have, $S\setminus R \subseteq S \Rightarrow (S\setminus R)^{*m} \subseteq (S)^{*m}$. This implies, $[(S\setminus R)^{*m}\setminus(R)^{*m}] \subseteq [(S)^{*m}\setminus(R)^{*m}]$.

Hence, we have $[(S)^{*m}\setminus(R)^{*m}] = [(S\setminus R)^{*m}\setminus(R)^{*m}]$.

(*iii*) By Theorem 4.4 (*i*), we have for S, $R \subseteq \hat{G}$, $(S\cup R)^{*m} = (S)^{*m} \cup (R)^{*m}$.

Further the earlier result we have $R\notin G$ implies $(R)^{*m} = 0_N$, so $(S)^{*m} \cup (R)^{*m} = (S)^{*m}$.

We have, $S\setminus R \subseteq S \Rightarrow (S\setminus R)^{*m} \subseteq (S)^{*m}$, by part (*iii*) we have,

$[(S)^{*m} \setminus (R)^{*m}] \subseteq (S\setminus R)^{*m}$, since $(R)^{*m} = 0_N$ implies $(S)^{*m} \subseteq (S\setminus R)^{*m}$.

Thus, we have $(S)^{*m} = (S\setminus R)^{*m}$.

Definition 4.3. Suppose that ($\hat{G}$, M, G) be an N-G-M-space. Then, the mapping N_{clmG}: $P(\hat{G}) \rightarrow P(\hat{G})$ is define by $N_{clmG}(S) = S\cup(S)^{*m}$, for all $S\in P(\hat{G})$.

Theorem 4.4. The mapping N_{clmG}: $P(\hat{G}) \rightarrow P(\hat{G})$ satisfies the Kuratowski closure axioms.

Proof. We have, $N_{clmG}(0_N) = 0_N \cup (0_N)^{*m} = 0_N \cup 0_N = 0_N$. By the definition of N_{clmG}, we have for all $S\in P(\hat{G})$, $N_{clmG}(S) = S \cup (S)^{*m} \supseteq S$.

Further, we have

$$
\begin{aligned}
N_{clmG}(S\cup R) &= (S\cup R) \cup (S\cup R)^{*m} \\
&= ((S\cup R) \cup ((S)^{*m} \cup (R)^{*m}) && \text{[by Theorem 4.3]} \\
&= (S\cup(S)^{*m}) \cup (R\cup(R)^{*m}) = N_{clmG}(S) \cup N_{clmG}(R).
\end{aligned}
$$

Suppose that $S\in P(\hat{G})$. Then, we have

$$
\begin{aligned}
N_{clmG}(N_{clmG}(S)) &= N_{clmG}(S\cup(S)^{*m}) \\
&= [(S\cup(S)^{*m})] \cup [(S\cup(S)^{*m}]^{*m} \\
&= [(S\cup(S)^{*m})] \cup [(S)^{*m} \cup ((S)^{*m})^{*m}] && \text{[by Theorem 4.3]} \\
&= S\cup(S)^{*m}
\end{aligned}
$$

Hence, the mapping N_{clmG} satisfies the Kuratowski closure axioms.

5. Conclusions: In this article, we have introduced the notion of minimal structure and grill via N-T-spaces. Besides, we have established its relation with some existing notions on several types of open sets via N-T-spaces, and investigated some basic properties of the class of minimal structures and grill via N-T-spaces. Further, we have furnished some suitable examples on minimal structures and grill via N-T-spaces. In the future, it is hoped that the notion of minimal structures and grill on N-T-spaces can also be applied in neutrosophic supra topological space [5], quadripartitioned neutrosophic topological space [6], pentapartitioned neutrosophic topological space [12], neutrosophic bitopological space [18], neutrosophic tri-topological space [10], neutrosophic soft topological space [9], neutrosophic multiset topological space [14], multiset mixed topological space [4], etc.

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