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ρ-Yamabe Solitons on 3-Dimensional Hyperbolic Kenmotsu Manifolds

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Abstract: In this paper, we examine the scalar curvature of a 3-dimensional hyperbolic Kenmotsu manifold, admitting a ρ-Yamabe soliton is constant, and the manifold reduces to a ρ-Einstein manifold. We have also examined if a 3-dimensional hyperbolic Kenmotsu manifold admits a ρ-Yamabe soliton, then the manifold reduces to an Einstein under some conditions. Here, we have also studied some different types of curvature properties under certain conditions. Finally, we construct an example of 3-dimensional hyperbolic Kenmotsu manifold admitting ρ-Yamabe soliton.

Key Words: Yamabe soliton, ρ-Yamabe soliton, ρ-Einstein manifold, hyperbolic Kenmotsu manifold.

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§1. Introduction

The study of self-similar solutions of the Ricci flow, the Yamabe flow and the curvature flow has an important place at the frontier between mathematics and physics. One of the most famous among these which is introduced by Richard Hamilton [10] is the concept of Yamabe solitons [18, 21].

The Yamabe flow is an evolution equation for metrics on a Riemannian manifold (M^n, g) defined as follows:

$$\frac{\partial}{\partial t}(g(t)) = -\kappa g(t), \qquad g(0) = g_0, \tag{1.1}$$

where κ is the scalar curvature of the metric $g(t)$.

The Ricci flow is an evolution equation for metrics on a Riemannian manifold (M^n, g) defined as follows:

$$\frac{\partial}{\partial t}(g(t)) = -2S, \tag{1.2}$$

where S denotes the Ricci tensor. In 2-dimension, the Yamabe flow is equivalent to the Ricci

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flow. However, in dimension stricly greater than two, the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow do not preserve in general.

A self-similar solution of the Yamabe flow is called Yamabe soliton on a (semi-)Riemannian manifold M^n [1, 5, 19] if there exists a smooth vector field V on M^n and a real number λ such that

$$\mathcal{L}_V g = (\kappa - \lambda)g, \tag{1.3}$$

where $\mathcal{L}_V g$ indicates the Lie derivative of g along the vector field V on M^n. Here, V is termed the soliton field of the Yamabe soliton. A Yamabe soliton is denoted by a triplet (g, V, λ). The beauty of Yamabe soliton depends on the soliton scalar λ. A soliton is named as shrinking, steady and expanding according to λ is negative, zero and positive, respectively. Yamabe solitons have been studied by several authors such as [3, 6, 7, 8, 16] and many others.

A Yamabe soliton becomes gradient Yamabe soliton if the soliton field V is gradient of some smooth function γ on M^n. In this case, the equation (1.3) reduces to

$$2Hess\gamma = (\kappa - \lambda)g, \tag{1.4}$$

where $Hess\gamma$ is the Hessian of the smooth function γ on M^n.

A ρ-Yamabe soliton is a generalization of Yamabe soliton [4], defined by the following equation:

$$\mathcal{L}_\zeta g = (\kappa - \lambda)g - \sigma\rho \otimes \rho, \tag{1.5}$$

where λ and σ are constants and ρ is a 1-form defined by $\rho(X) = g(X, \zeta)$, for any smooth vector field X on M^n and $\otimes$ represents the tensor product. If both λ and σ are smooth functions on M^n, then (1.5) is named as almost ρ-Yamabe soliton or a quasi-Yamabe soliton [4]. Moreover if $\sigma = 0$, then equation (1.5) reduces to (1.3) and hence the ρ-Yamabe soliton reduces to a Yamabe soliton whereas an almost ρ-Yamabe soliton reduces to an almost Yamabe soliton if in (1.5), λ is a smooth function on M^n and $\sigma = 0$.

The concept of almost contact hyperbolic (f, g, ρ, ζ)-structure was introduced by Upadhyay and Dube in [20]. Further, it was stadied by several geometers such as R. B. Pal [15], B. B. Sinha and R. N. Singh [17] and many others. A non-zero smooth vector field $U \in T_p(M)$ is called timelike, null, space-like and non-space-like according as $g_p(U, U) < 0$, $g_p(U, U) = 0$, $g_p(U, U) > 0$ and $g_p(U, U) \leq 0$, respectively, where $T_p(M)$ is the tangent space of M at $p \in M$ [14].

Let $\{e_1, e_2, \cdots, e_n = \zeta\}$ be a local orthonormal basis of vector fields in $n(= 2m + 1)$-dimensional semi-Riemannian manifold M^n. Then the Ricci tensor S and the scalar curvature κ of a $n(= 2m+1)$-dimensional almost hyperbolic contact metric manifold M^n endowed with a semi-Riemannian metric g are defined as follows:

$$S(X, Y) = \sum_{r=1}^{n} g(e_r, e_r)g(R(e_r, X)Y, e_r) \tag{1.6}$$

and

$$\kappa = \sum_{r=1}^{n} g(e_r, e_r) S(e_r, , e_r) \tag{1.7}$$

for all smooth vector fields X, Y on M^n, where ζ is the unit timelike vector field (i.e., $g(\zeta, \zeta) = -1$) and R is the curvature tensor of M^n [14].

In this paper, we start to investigate the ρ-Yamabe solitons on 3-dimensional hyperbolic Kenmotsu manifolds. Here, we establish that the scalar curvature of a 3-dimensional hyperbolic Kenmotsu manifold admitting a ρ-Yamabe soliton is constant, and the manifold reduces to a ρ-Einstein manifold. Next, we have proved that a 3-dimensional hyperbolic Kenmotsu manifold whose metric is ρ-Yamabe soliton reduces to an Einstein manifold under some conditions. Here, we also examine the nature of the manifold in terms of ρ-Yamabe soliton on hyperbolic Kenmotsu manifolds. Next, we have characterized the nature of ρ-Yamabe soliton when the Ricci tensor is parallel along ζ, Ricci operator is parallel along ζ. Next, we have shown that if a 3-dimensional hyperbolic Kenmotsu manifold with a cyclic parallel Ricci tensor admits a ρ-Yamabe soliton, then the manifold reduces to an Einstein manifold. Next, we have studied some different types of curvature properties under certain conditions. Finally, we construct an example of 3-dimensional hyperbolic Kenmotsu manifold admitting ρ-Yamabe soliton.

§2. **Preliminaries**

An odd-dimensional smooth manifold M^{2m+1} is named to be an almost hyperbolic contact metric manifold if it admits a timelike vector field ζ, a 1-form ρ, a fundamental tensor field φ of type $(1, 1)$, and a semi-Riemannian metric g satisfying [20]:

$$\varphi^2(X) = X + \rho(X)\zeta, \tag{2.1}$$

$$\rho(\zeta) = -1 \implies \varphi(\zeta) = 0, \tag{2.2}$$

$$rank(\varphi) = 2m, \tag{2.3}$$

$$\rho \circ \varphi = 0, \tag{2.4}$$

$$g(\varphi X, \varphi Y) = -g(X, Y) - \rho(X)\rho(Y), \tag{2.5}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \tag{2.6}$$

$$g(X, \zeta) = \rho(X) \tag{2.7}$$

for all smooth vector fields X, Y on M^{2m+1}. Then the structure $(\varphi, \zeta, \rho, g)$ on manifold M^{2m+1} named as almost hyperbolic contact metric structure.

If an almost hyperbolic contact metric manifold M^{2m+1} is fulfilled, the following condition [12]:

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\zeta - \rho(Y)\varphi X, \tag{2.8}$$

then, M^{2m+1} is called a hyperbolic Kenmotsu manifold [2], where ∇ denotes the Levi-Civita connection of g. From the antecedent equation, it is clear that

$$\nabla_X \zeta = -X - \rho(X)\zeta, \tag{2.9}$$

and

$$(\nabla_X \rho)Y = g(\varphi X, \varphi Y) = -g(X,Y) - \rho(X)\rho(Y). \tag{2.10}$$

Also in this manifold M^{2m+1} the following relations are satisfied for all smooth vector fields X and X on M^{2m+1} [2]:

$$R(X,Y)\zeta = \rho(Y)X - \rho(X)Y, \tag{2.11}$$

$$R(X,\zeta)\zeta = -X - \rho(X)\zeta, \tag{2.12}$$

$$R(\zeta,X)Y = g(X,Y)\zeta - \rho(Y)X, \tag{2.13}$$

$$S(X,\zeta) = 2m\rho(X), \tag{2.14}$$

$$S(\zeta,\zeta) = -2m, \tag{2.15}$$

$$Q\zeta = 2m\zeta, \tag{2.16}$$

where R, S are the Riemannian curvature tensor, Ricci tensor of the manifold M^{2m+1}, respectively, and Q the Ricci operator defined by $S(X,Y) = g(QX,Y)$.

It is well known that the tensor R on any 3-dimensional Riemannian manifold (M^3, g) always satisfies

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{\kappa}{2}[g(Y,Z)X - g(X,Z)Y] \tag{2.17}$$

for all smooth vector fields X, Y, Z on M^3.

Replacing X and Z by ζ in the above equation (2.17) and using the identities (2.2), (2.7) and (2.11) – (2.16) entails that

$$QX = (\frac{\kappa}{2} - 1)X + (\frac{\kappa}{2} - 3)\rho(X)\zeta. \tag{2.18}$$

Taking inner product of (2.18) with Y and using the relation $g(QX,Y) = S(X,Y)$ yields

$$S(X,Y) = (\frac{\kappa}{2} - 1)g(X,Y) + (\frac{\kappa}{2} - 3)\rho(X)\rho(Y), \tag{2.19}$$

which implies that the three-dimensional hyperbolic Kenmotsu manifolds are ρ Einstein manifolds.

In addition, we know that

$$(\mathcal{L}_\zeta g)(X,Y) = g(\nabla_X \zeta, Y) + g(X, \nabla_Y \zeta). \tag{2.20}$$

Utilizing (2.9) and (2.7) in the above equation (2.20), we get

$$(\mathcal{L}_\zeta g)(X,Y) = -2[g(X,Y) + \rho(X)\rho(Y)] \tag{2.21}$$

for all smooth vector fields X, Y on M^3.

§3. 3-Dimensional Hyperbolic Kenmotsu Manifolds Admitting ρ-Yamabe Solitons

Theorem 3.1 *Let the metric g of a 3-dimensional hyperbolic Kenmotsu manifold M^3 satisfy the ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$, ζ being the timelike vector field of M^3. Then the scalar curvature of the manifold is constant.*

Proof Let M^3 be a 3-dimensional hyperbolic Kenmotsu manifold which admit ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$. Then we have

$$(\mathcal{L}_\zeta g)(X, Y) = (\kappa - \lambda)g(X, Y) - \sigma\rho(X)\rho(Y) \tag{3.1}$$

for all smooth vector fields X, Y on M^3.

Substituting the value of $(\mathcal{L}_\zeta g)(X, Y)$ from (2.21) in the above equation (3.1) yields

$$(\kappa - \lambda + 2)g(X, Y) + (2 - \sigma)\rho(X)\rho(Y) = 0. \tag{3.2}$$

Replacing Y by ζ in equation (3.2) and using the relation (2.7) gives

$$(\kappa - \lambda + \sigma)\rho(X) = 0. \tag{3.3}$$

Since $\rho(X) \neq 0$, we get,

$$\kappa = \lambda - \sigma. \tag{3.4}$$

Since λ and σ both are constants, κ is also constant. □

Theorem 3.2 *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits Yamabe soliton (g, ζ, λ), then the timelike vector field ζ is a Killing vector field.*

Proof Now putting $\sigma = 0$ in equation (3.4) gives $\kappa = \lambda$, so, equation (3.1) reduces to

$$\mathcal{L}_\zeta g = 0. \tag{3.5}$$

This shows that the timelike vector field ζ is a Killing vector field. □

Corollary 3.3 *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits a ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$, ζ being the timelike vector field of M^3, then the manifold M^3 reduces to a ρ-Einstein manifold.*

Proof Now, from (2.19) and (3.4), we get

$$S(X, Y) = (\frac{\lambda - \sigma}{2} - 1)g(X, Y) + (\frac{\lambda - \sigma}{2} - 3)\rho(X)\rho(Y). \tag{3.6}$$

This completes the proof. □

§4. ρ-Yamabe Solitons on 3-Dimensional Hyperbolic Kenmotsu Manifold with Ricci Symmetric

Definition 4.1 *A hyperbolic Kenmotsu manifold of dimension three is said to be Ricci symmetric if*

$$(\nabla_Z S)(X,Y)=0 \tag{4.1}$$

for all smooth vector fields X, Y, Z on M^3 and ∇ is the Riemannian connection.

Theorem 4.1 *If a 3-dimensional Ricci symmetric hyperbolic Kenmotsu manifold M^3 admits ρ-Yamabe soliton (g,ζ,λ,σ), ζ being the timelike vector field of M^3, then $\lambda=\sigma+6$ and the manifold reduces to an Einstein manifold.*

Proof Now, taking covariant differentiation of (3.6) with respect to Z, we obtain

$$(\nabla_Z S)(X,Y)=(\frac{\lambda-\sigma}{2}-3)[\rho(X)(\nabla_Z\rho)(Y)+\rho(Y)(\nabla_Z\rho)(X)]. \tag{4.2}$$

In view of (2.10), we have

$$(\nabla_Z S)(X,Y)=(\frac{\lambda-\sigma}{2}-3)[\rho(X)g(\varphi Z,\varphi Y)+\rho(Y)g(\varphi Z,\varphi X)]. \tag{4.3}$$

Let us assume that the manifold is Ricci symmetric. Then, from (4.3), we get

$$(\frac{\lambda-\sigma}{2}-3)[\rho(X)g(\varphi Z,\varphi Y)+\rho(Y)g(\varphi Z,\varphi X)]=0. \tag{4.4}$$

Setting $Y=\zeta$ in the above equation (4.3)and using (2.2) gives

$$(\frac{\lambda-\sigma}{2}-3)g(\varphi Z,\varphi X)=0 \tag{4.5}$$

for all smooth vector fields Z, Y on M^3. It follows that

$$\lambda=\sigma+6.$$

Now, putting $\lambda=\sigma+6$ in equation (3.6), we have

$$S(X,Y)=2g(X,Y) \tag{4.6}$$

for all smooth vector fields X, Y on M^3. □

§5. ρ-Recurrent Ricci Tensor of a Hyperbolic Kenmotsu Manifolds of Dimension Three Admitting Rho-Yamabe Solitons

Definition 5.1 *The Ricci tensor S of a hyperbolic Kenmotsu manifold of dimension three is* *called ρ-recurrent if it satisfies*

$$(\nabla_Z S)(X,Y) = \rho(Z)S(X,Y) \tag{5.1}$$

for all smooth vector fields X, Y, Z on M^3.

Theorem 5.1 *There does not exist ρ-recurrent Ricci tensor of a hyperbolic Kenmotsu manifold M^3 of dimension three admitting a ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$, ζ being the timelike vector field of M^3*

Proof Using the relation (4.3) in the equation (5.1), we have

$$(\frac{\lambda-\sigma}{2} - 3)[\rho(X)g(\varphi Z, \varphi Y) + \rho(Y)g(\varphi Z, \varphi X)] = \rho(Z)S(X,Y). \tag{5.2}$$

Substituting $X = \zeta = Y$ in the above equation (5.2) and using the equations (2.2) and (2.15) yields

$$\rho(Z) = 0,$$

which is a contradiction. Hence, the proof. □

§6. The Ricci Tensor S and Ricci Operator Q Are Parallel Along ζ on 3-Dimensional Hyperbolic Kenmotsu Manifold Admitting ρ-Yamabe Soliton

Definition 6.1 *The Ricci tensor S of a hyperbolic Kenmotsu manifold of dimension three is parallel along the smooth vector field X on M^3 if it satisfies*

$$(\nabla_X S)(Y,Z) = 0 \tag{6.1}$$

for all smooth vector fields X, Y, Z on M^3.

Definition 6.2 *The Ricci operator Q of a hyperbolic Kenmotsu manifold of dimension three is parallel along the smooth vector field X on M^3 if it satisfies*

$$(\nabla_X Q)Y = 0 \tag{6.2}$$

for all smooth vector fields Y on M^3.

Theorem 6.3 *The Ricci tensor S and Ricci operator Q of a 3-dimensional hyperbolic Kenmotsu manifold M^3 admitting an ρ-Yamabe soliton $(g, \zeta, \lambda, \sigma)$ are parallel along the timelike vector field ζ of M^3.*

Proof From (4.3), we obtain,

$$(\nabla_\zeta S)(X,Y) = (\frac{\lambda-\sigma}{2} - 3)[\rho(X)g(\varphi\zeta, \varphi Y) + \rho(Y)g(\varphi\zeta, \varphi X)] \tag{6.3}$$

for all smooth vector fields X, Y on M^3.

Using the relation (2.2) in the above equation (6.3), we get,

$$(\nabla_\zeta S)(X,Y) = 0 \tag{6.4}$$

for all smooth vector fields X, Y on M^3,

which implies that S is parallel along the timelike vector field ζ.

Also, from (3.6), we obtain,

$$QX = (\frac{\lambda-\sigma}{2} - 1)X + (\frac{\lambda-\sigma}{2} - 3)\rho(X)\zeta. \tag{6.5}$$

Replace the expression of Q from (6.5) in

$$(\nabla_\zeta Q)X = \nabla_\zeta QX - Q(\nabla_\zeta X), \tag{6.6}$$

we obtain,

$$(\nabla_\zeta Q)X = (\frac{\lambda-\sigma}{2} - 3)((\nabla_\zeta \rho)X)\zeta. \tag{6.7}$$

Now, using the identities (2.10) and (2.2) in the above equation (6.7), we get

$$(\nabla_\zeta Q)X = 0 \tag{6.8}$$

for all smooth vector fields X on M^3.

Hence, Q is parallel to the timelike vector field ζ. □

§7. ρ-Yamabe Solitons on 3-Dimensional Hyperbolic Kenmotsu Manifolds with Cyclic Parallel Ricci Tensor

A. Gray [9] revealed the idea of cyclic parallel Ricci tensor and Ricci tensor of Codazzi type. Codazzi type of Ricci tensor means that the Levi-Civita connection ∇ of such metric is a Yang-Mills connection while keeping the metric of the manifold fixed. A Riemannian manifold (M^n, g) is said to have a cyclic parallel Ricci tensor if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the following condition:

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0 \tag{7.1}$$

for all smooth vector fields X, Y, Z on M^3. Ki et al. [13] proved that Carten hypersurfaces are manifolds with non-parallel Ricci tensor that satisfy cyclic parallel Ricci tensor.

Now, from equation (4.3),

$$(\nabla_Z S)(X,Y) = (\frac{\lambda-\sigma}{2} - 3)[\rho(X)g(\varphi Z, \varphi Y) + \rho(Y)g(\varphi Z, \varphi X)]. \tag{7.2}$$

In view of (7.2), it follows that

$$(\nabla_Y S)(Z,X) = (\frac{\lambda-\sigma}{2} - 3)[\rho(Z)g(\varphi Y,\varphi X) + \rho(X)g(\varphi Y,\varphi Z)] \tag{7.3}$$

and

$$(\nabla_X S)(Y,Z) = (\frac{\lambda-\sigma}{2} - 3)[\rho(Y)g(\varphi X,\varphi Z) + \rho(Z)g(\varphi X,\varphi Y)]. \tag{7.4}$$

Let us assume that the Ricci tensor S of type (0,2) is cyclic parallel. Then, using the relations (7.2), (7.3) and $(7,4)$ in equation (7.1), we get

$$(\frac{\lambda-\sigma}{2} - 3)[\rho(X)g(\varphi Z,\varphi Y) + \rho(Y)g(\varphi Z,\varphi X) + \rho(Z)g(\varphi Y,\varphi X)] = 0. \tag{7.5}$$

Putting $X = \zeta$ in the equation (7.5) and using the relation (2.2), we lead,

$$(\frac{\lambda-\sigma}{2} - 3)g(\varphi Z,\varphi Y) = 0 \tag{7.6}$$

for all smooth vector fields Z, Y on M^3,

which gives

$$\lambda = \sigma + 6. \tag{7.7}$$

Now substituting $\lambda = \sigma + 6$ in the equation (3.6), we get

$$S(X,Y) = 2g(X,Y). \tag{7.8}$$

Thus, we can state the following.

Theorem 7.1 *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 with cyclic parallel Ricci tensor admits a ρ-Yamabe soliton (g,ζ,λ,σ), ζ being the timelike vector field of M^3, then $\lambda = \sigma + 6$ and the manifold reduces to an Einstein manifold.*

§8. Curvature Properties on 3-Dimensional Hyperbolic Kenmotsu Manifold Admitting ρ-Yamabe Soliton

Let us assume that 3-dimensional hyperbolic Kenmotsu manifolds with ρ-Yamabe solitons satisfy the condition

$$R(\zeta,X)\cdot S = 0. \tag{8.1}$$

Then we have

$$S(R(\zeta,X)Y,Z) + S(Y,R(\zeta,X)Z) = 0 \tag{8.2}$$

for all smooth vector fields X, Y, Z on M^3.

Then, from (2.13), the equation (8.2) takes the form:

$$S(g(X,Y)\zeta - \rho(Y)X, Z) + S(Y, g(X,Z)\zeta - \rho(Z)X) = 0. \tag{8.3}$$

In view of (3.6), we obtain:

$$(\frac{\lambda-\sigma}{2}-3)[\{-g(X,Y)-\rho(X)\rho(Y)\}\rho(Z)+\{-g(X,Z)-\rho(X)\rho(Z)\}\rho(Y)]=0. \tag{8.4}$$

Using the relation (2.10) in the above equation (8.4), we get

$$(\frac{\lambda-\sigma}{2}-3)[g(\varphi X,\varphi Y)\rho(Z)+g(\varphi X,\varphi Z)\rho(Y)]=0. \tag{8.5}$$

Now putting $Z=\zeta$ in the equation (8.5) and using the relation (2.2) gives

$$(\frac{\lambda-\sigma}{2}-3)g(\varphi X,\varphi Y)=0 \tag{8.6}$$

for all smooth vector fields X, Y on M^3.

It follows that

$$\lambda=\sigma+6. \tag{8.7}$$

Also, we obtain from (3.6),

$$S(X,Y)=2g(X,Y). \tag{8.8}$$

Thus, we are in position to state the following.

Theorem 8.1 *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admitting a ρ-Yamabe soliton (g,ζ,λ,σ), ζ being the timelike vector field of M^3, satisfies the condition $R(\zeta,X)\cdot S=0$, then the soliton scalar $\lambda=\sigma+6$ and the manifold reduces to an Einstein manifold.*

Again, we assume that 3-dimensional hyperbolic Kenmotsu manifolds with ρ-Yamabe solitons satisfy the condition

$$S(\zeta,X)\cdot R=0, \tag{8.9}$$

which implies that

$$\begin{aligned}&S(X,R(Y,Z)W)\zeta-S(\zeta,R(Y,Z)W)X+S(X,Y)R(\zeta,Z)W\\&-S(\zeta,Y)R(X,Z)W+S(X,Z)R(Y,\zeta)W-S(\zeta,Z)R(Y,X)W\\&+S(X,W)R(Y,Z)\zeta-S(\zeta,W)R(Y,Z)X=0\end{aligned} \tag{8.10}$$

for all smooth vector fields X, Y, Z, W on M^3.

Now taking inner product of (8.1) with ζ, we obtain

$$\begin{aligned}&-S(X,R(Y,Z)W)-S(\zeta,R(Y,Z)W)\rho(X)+S(X,Y)\rho(R(\zeta,Z)W)\\&-S(\zeta,Y)\rho(R(X,Z)W)+S(X,Z)\rho(R(Y,\zeta)W)-S(\zeta,Z)\rho(R(Y,X)W)\\&+S(X,W)\rho(R(Y,Z)\zeta)-S(\zeta,W)\rho(R(Y,Z)X)=0.\end{aligned} \tag{8.11}$$

Now using the equations (3.6), (2.2), (2.11) and (2.12) and putting $Z=\zeta, W=\zeta$ in the above equation (8.11) entails that

$$(\frac{\lambda-\sigma}{2}+1)[g(X,Y)+\rho(X)\rho(Y)]=0 \tag{8.12}$$

for all smooth vector fields X, Y on M^3, which implies that

$$\lambda=\sigma-2. \tag{8.13}$$

Thus, we can conclude the following.

Theorem 8.2 *If a 3-dimensional hyperbolic Kenmotsu manifold M^3 admits an ρ-Yamabe soliton (g,ζ,λ,σ), ζ being the timelike vector field of M^3 satisfies the condition $S(\zeta,X)\cdot R=0$, then the soliton scalar $\lambda=\sigma-2$.*

§9. **Example**

We consider the three-dimensional manifold $M^3=\{(x,y,z)\in\mathbb{R}^3 : z\neq 0\}$, where the standard coordinates are in $\mathbb{R}^3$. Then the vector fields

$$e_1=e^z\frac{\partial}{\partial x}, e_2=e^z\frac{\partial}{\partial y}, e_3=\frac{\partial}{\partial z}=\zeta$$

are linearly independent at each point of M^3, and so they form a basis of the tangent space at each point of M^3. Let g be a semi-Riemannian metric defined by

$$g(e_i,e_j)=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}.$$

Let ρ be the 1-form defined by $g(V,e_3)=\rho(V)$ for any vector field V on M^3 and φ is a $(1,1)$ tensor field defined by $\varphi(e_1)=e_2, \varphi(e_2)=e_1, \varphi(e_3)=0$. Then, using the linearity property of φ and g we obtain

$$\rho(e_3)=-1, \varphi^2V=V+\rho(V)e_3, g(\varphi V,\varphi W)=-g(V,W)-\rho(V)\rho(W)$$

for any vector fields V, W on M^3.

Thus, for $e_3=\zeta$, the structure (φ,ζ,ρ,g) defines an almost hyperbolic contact metric structure on M^3. All possible Lie brackets for the example are as follows:

$$[e_1,e_1]=[e_2,e_2]=[e_3,e_3]=[e_1,e_2]=[e_2,e_1]=0, [e_1,e_3]=-e_1,$$

$$[e_3,e_1]=e_1, [e_3,e_2]=e_2, [e_2,e_3]=-e_2$$

.

Let ∇ be a Riemannian connection with respect to the semi-Riemannian metric g. Now using Koszul's formula

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]),$$

we can obtain

$$\nabla_{e_1}e_2 = \nabla_{e_2}e_1 = \nabla_{e_3}e_1 = \nabla_{e_3}e_2 = \nabla_{e_3}e_3 = 0$$

$$\nabla_{e_1}e_1 = -e_3, \nabla_{e_1}e_3 = -e_1, \nabla_{e_2}e_2 = e_3, \nabla_{e_2}e_3 = -e_2$$

From the above relations, we get

$$\nabla_X \zeta = -X - \rho(X)\zeta$$

fulfilled for any vector field X on M^3. Hence, the structure $(\varphi, \zeta, \rho, g)$ is a hyperbolic Kenmotsu structure on M^3. Consequently, $M^3(\varphi, \zeta, \rho, g)$ is a 3-dimensional hyperbolic Kenmotsu manifold. The non-zero components of the curvature tensor R as follows:

$R(e_1,e_2)e_1 = -e_2, R(e_2,e_1)e_1 = e_2, R(e_1,e_3)e_1 = -e_3, R(e_3,e_1)e_1 = e_3,$

$R(e_1,e_2)e_2 = -e_1, R(e_2,e_1)e_2 = e_1, R(e_2,e_3)e_2 = e_3, R(e_3,e_2)e_2 = -e_3,$

$R(e_1,e_3)e_3 = -e_1, R(e_1,e_3)e_3 = -e_1, R(e_2,e_3)e_3 = -e_2, R(e_3,e_2)e_3 = e_2.$

The Ricci tensor R is given by

$$S(X,Y) = \sum_{r=1}^{3} g(e_r,e_r)g(R(e_r,X)Y,e_r). \tag{9.1}$$

So, we have

$$S(e_1,e_1) = 2, S(e_2,e_2) = -2, S(e_3,e_3) = -2. \tag{9.2}$$

Again, the scalar curvature of the given hyperbolic Kenmotsu manifold can be calculated as follows:

$$\kappa = \sum_{r=1}^{3} g(e_r,e_r)S(e_r,,e_r) = S(e_1,e_1) - S(e_2,e_2) - S(e_3,e_3) = 6. \tag{9.3}$$

Now, from (3.6) we have

$$S(e_1,e_1) = (\frac{\lambda-\sigma}{2} - 1), S(e_2,e_2) = -(\frac{\lambda-\sigma}{2} - 1), S(e_3,e_3) = -(\frac{\lambda-\sigma}{2} - 1). \tag{9.4}$$

Now, from (9.1) and (9.3) we get

$$\lambda - \sigma = 6. \tag{9.5}$$

So, the values of κ and $\lambda - \sigma$ are the same, and hence the equation (3.4) is satisfied. Therefore, the structure $(g, \zeta, \lambda, \sigma)$ is a ρ-Yamabe soliton on 3-dimensional hyperbolic Kenmotsu manifolds $M^3(\varphi, \zeta, \rho, g)$, ζ being timelike vector field on M^3.

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