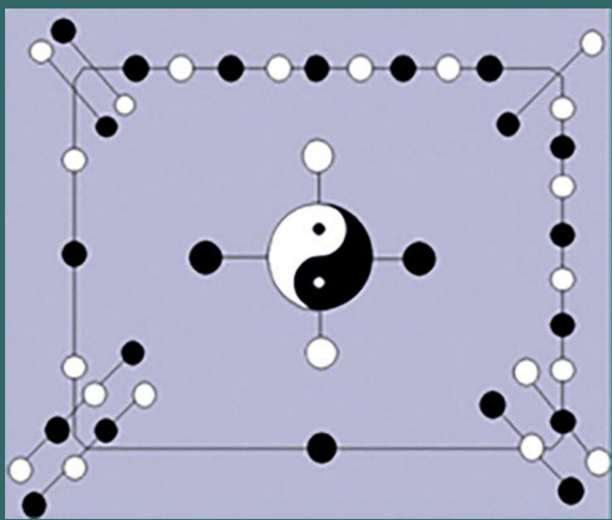
ISSN 1937 - 1055

VOLUME 4, 2022

INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

December, 2022

Vol.4, 2022 ISSN 1937-1055

International Journal of

Mathematical Combinatorics

(www.mathcombin.com)

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

December, 2022

Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.Linfan MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

Smarandache multi-spaces and Smarandache geometries with applications to other sciences;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Mathematical theory on gravitational fields and parallel universes;

Applications of Combinatorics to mathematics and theoretical physics.

Generally, papers on applications of combinatorics to other mathematics and other sciences are welcome by this journal.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews (USA), Zentralblatt Math (Germany), Index EuroPub (UK), Referativnyi Zhurnal (Russia), Mathematika (Russia), EBSCO (USA), Google Scholar, Baidu Scholar, Directory of Open Access (DoAJ), International Scientific Indexing (ISI, impact factor 2.012), Institute for Scientific Information (PA, USA), Library of Congress Subject Headings (USA), CNKI(China).

Subscription A subscription can be ordered by an email directly to

Linfan Mao

The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science Beijing, 100190, P.R.China, and also the President of Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA

Email: maolinfan@163.com

Price: US$48.00

Famous Words:

Try not to become a man of success but rather try to become a man of value.

By Albert Einstein, an American theoretical physicist.

International J.Math. Combin. Vol.4(2022), 1-10

New Condition
Related with Higher Triple Centralizers in Semiprime Rings

Salah Mehdi Salih and Afrah Mohammad Ibraheem

(Depertment of Mathematics, College of Education, Al- Mustansiriyah University, Baghdad, Iraq)

E-mail: dr.salahms2014@uomustansiriyah.edu.iq, afrah.wisam@uomustansiriyah.edu.iq

Abstract: The main objective of this work is shown that: if R is a 2-torsion free semiprime ring, consider that $T=(t_i)_{i\in\mathbb{N}}$ a family of additive mappings of R onto R, such that $2t_n(xyz)=\sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(x)+t_{i-1}(x)t_{i-1}(y)t_i(x)$, holds for all $x,y\in R$ and $n\in\mathbb{N}$, then T is Jordan higher centralizer of R.

Key Words: Semiprime rings, higher centralizer, higher triple centralizer, Jordan higher triple centralizer.

AMS(2010): 16N60, 39B05.

§1. Introduction

Let R be a ring. An additive mapping $T: R\rightarrow R$ is called a left (right) centralizer of R if $T(xy)=T(x)y\,(T(xy)=xT(y))$,is called a left (right) Jordan centralizer of R in case $T\left(x^2\right)=T(x)x(T\left(x^2\right)=xT(x))$, holds for all $x,y\in R$. We follow Zalar [7] and call T a centralizer in case T is both a left and a right centralizer. Vukman [5] proved that if $T: R\rightarrow R$ is an additive mapping such that $2T\left(x^2\right)=T(x)x+xT(x)$ holds for all $x\in R$ then T is a centralizer. In [1] and [2] A.M. Ibraheem and S.M. Salih presented and studied the concepts of higher triple left (resp. right) centralizer and Jordan higher triple left (resp. right) centralizer of rings and Γ-ring, and in [3] they provide that: if M is a 2-torsion free semiprime Γ-ring satisfying the assumption $p\alpha r\beta q=p\beta r\alpha q$, for all $p,r,q\in M$ and $\alpha,\beta\in\Gamma$, $F\in(f_i)_{i\in\mathbb{N}}$ is a family of additive mapping associated with a Jordan higher triple centralizer $T=(t_i)_{i\in\mathbb{N}}$ of M such that

$$2f_n(p\alpha r\beta q)=\sum_{i=1}^{n} f_i(p)\alpha t_{i-1}(r)\beta t_{i-1}(p)+t_{i-1}(p)\alpha t_{i-1}(pr)f_i(p)$$

hold for all $p,r\in M$ and $\alpha,\beta\in\Gamma$, and $n\in\mathbb{N}$ of M, then F is Jordan generalized higher triple centralizer of M. In this paper we prove that: if R is a 2-torsion free semiprime ring and $T\in(t_i)_{i\in\mathbb{N}}$ a family of additive mappings of R onto R, such that $2t_n(xyz)=\sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(x)+t_{i-1}(x)t_{i-1}(y)t_i(x)$, holds for all $x,y\in R$ and $n\in\mathbb{N}$, then T is Jordan higher centralizer of R.

[1]Received September 22, 2022, Accepted December 2, 2022.

§2. Main Results

For proving our main results, we need the following Lemmas:

Lemma 2.1([6]) *Let R be a 2-torsion free semiprime ring. Suppose that the identity $axb+bxc$ holds for all $x \in R$ and for some $a,b,c \in R$, then $(a+c)xb = 0$ for all $x \in R$.*

Lemma 2.2 *Let R be a semiprime ring, and $T = (t_i)_{i\in\mathbb{N}}$ be a family of an additive mappings of R onto R, such that*

$$2t_n(xyx) = \sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(y)t_i(x)$$

for all $x, y \in R$ and $n \in \mathbb{N}$, then

(*i*) $2t_n(xyz + zyx) = \sum\limits_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(z) + t_i(z)t_{i-1}(y)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(y)t_i(z) + t_{i-1}(z)t_{i-1}(y)t_i(x)$;

(*ii*) $\left(t_n\left(x^2-\right) \sum\limits_{i=1}^{n} t_i(x)t_{i-1}(x)\right) t_{i-1}(y)t_{i-1}(x)+t_{i-1}(x)t_{i-1}(y)\left(t_n\left(x^2\right) \sum\limits_{i=1}^{n} t_{i-1}(x)t_i(x)\right) = 0$;

(*iii*) *If R is a 2-torsion free ring, then*

$$2t_n\left(x^2\right) = \sum_{i=1}^{n} t_i(x)t_{i-1}(x) + t_{i-1}(x)t_i(x);$$

(*iv*) *In particular if R is a 2-torsion free commutative ring, then*

$$2t_n(xyz) = \sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(z) + t_{i-1}(x)t_{i-1}(y)t_i(z).$$

Proof (*i*) Linearizing the hypothesis on x, we get

$$\begin{aligned} 2t_n\left((x+z)y(x+z)\right) &= \sum_{i=1}^{n} t_i(x+z)t_{i-1}(y)t_{i-1}(x+z) + t_{i-1}(x+z)t_{i-1}(y)t_i(x+z), \\ &= \sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(x) + t_i(x)t_{i-1}(y)t_{i-1}(z) + t_i(z)t_{i-1}(y)t_{i-1}(x) \\ &\quad + t_i(z)t_{i-1}(y)t_{i-1}(z) + t_{i-1}(x)t_{i-1}(y)t_i(x) + t_{i-1}(x)t_{i-1}(y)t_i(z) \\ &\quad + t_{i-1}(z)t_{i-1}(y)t_i(x) + t_{i-1}(z)t_{i-1}(y)t_i(z) \cdots \end{aligned} \tag{1}$$

On the other hand

$$\begin{aligned} 2t_n\left((x+z)y(x+z)\right) &= 2t_n\left(xyx + xyz + zyx + zyz\right) \\ &= 2t_n(xyx) + 2t_n(zyz) + 2t_n(xyz + zyx). \end{aligned} \tag{2}$$

Comparing (1), (2) and with the hypothesis, we get

$$\begin{aligned} 2t_n\left((x+z)y(x+z)\right) &= \sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(z) \\ &\quad + t_i(z)t_{i-1}(y)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(y)t_i(z) + t_{i-1}(z)t_{i-1}(y)t_i(x). \end{aligned}$$

(ii) Putting x^2 for z in part (i), we get

$$\begin{aligned} 2t_n\left(x^2yx+xyx^2\right) &= \sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}\left(x^2\right) + t_i\left(x^2\right)t_{i-1}(y)t_{i-1}\left(x\right) \\ &\quad + t_{i-1}(x)t_{i-1}(y)t_{i-1}\left(x^2\right) + t_{i-1}\left(x^2\right)t_{i-1}(y)t_{i-1}\left(x\right)\ldots \end{aligned} \tag{3}$$

Replacing $t_{i-1}\left(x^2\right)$ by $t_{i-1}(x)t_{i-1}(x)$ in (3), we get

$$\begin{aligned} 2t_n\left(x^2yx+xyx^2\right) &= \sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}t_{i-1}(x)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(x)t_{i-1}(y)t_{i-1}\left(x\right) \\ &\quad + t_{i-1}(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(x)t_{i-1}(y)t_{i-1}\left(x\right)\cdots \end{aligned} \tag{4}$$

Replacing y by $yx+xy$ in the hypothesis, we get

$$\begin{aligned} 2t_n\left(x(yx+xy)x\right) &= \sum_{i=1}^{n} t_i(x)t_{i-1}(yx)t_{i-1}(x) + t_i(x)t_{i-1}(xy)t_{i-1}(x) \\ &\quad + t_{i-1}(x)t_{i-1}\left(yx\right)t_i(x) + t_{i-1}(x)t_{i-1}(yx)t_i(x)\ldots \end{aligned} \tag{5}$$

Replacing $t_{i-1}(yx)$ in (5) by $t_{i-1}(x)t_{i-1}(y)$, and $t_{i-1}(xy)$ by $t_{i-1}(y)t_{i-1}(x)$, we get

$$\begin{aligned} 2t_n\left(x^2yx+xyx^2\right) &= \sum_{i=1}^{n} t_i(x)t_{i-1}(x)t_{i-1}(y)t_{i-1}(x) + t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) \\ &\quad + t_{i-1}(x)t_{i-1}(y)t_{i-1}(x)t_i(x) + t_{i-1}(x)t_{i-1}(x)t_{i-1}(y)t_i(x)\ldots \end{aligned} \tag{6}$$

Comparing (4) and (6), we get

$$\left(t_n\left(x^2\right) - \sum_{i=1}^{n} t_i(x)t_{i-1}(x)\right)t_{i-1}(y)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(y)\left(t_n\left(x^2\right)\sum_{i=1}^{n} t_{i-1}(x)t_i(x)\right) = 0.$$

(iii) Taking $a = t_n\left(x^2\right) - \sum_{i=1}^{n} t_i(x)t_{i-1}(x)$, $b = t_n\left(x^2\right) - \sum_{i=1}^{n} t_{i-1}(x)t_i(x)$, $c = t_{i-1}(y)$ and $d = t_{i-1}(x)$ in (ii) we find that $acd + dcb = 0$ for all $c \in R$. Then, by Lemma 2.1, we have

$$\left(t_n\left(x^2\right) - \sum_{i=1}^{n} t_i(x)t_{i-1}(x) + t_n\left(x^2\right) - \sum_{i=1}^{n} t_{i-1}(x)t_i(x)\right)t_{i-1}(y)t_{i-1}(x) = 0,$$

$$\left(2t_n\left(x^2\right) - \sum_{i=1}^{n} t_i(x)t_{i-1}(x) - \sum_{i=1}^{n} t_{i-1}(x)t_i(x)\right)t_{i-1}(y)t_{i-1}(x) = 0 \tag{7}$$

for all $x, y \in R$ and $n \in \mathbb{N}$. Let

$$A_n(x) = \left(t_n\left(x^2\right) - \sum_{i=1}^{n} t_i(x)t_{i-1}(x) + t_n\left(x^2\right) - \sum_{i=1}^{n} t_{i-1}(x)t_i(x) \right),$$

then (7) becomes

$$A_n(x)t_{n-1}(y)t_{n-1}(x) = 0. \tag{8}$$

Replacing $t_{n-1}(y)$ by $t_{n-1}(x)t_{n-1}(y)A_n(x)$ in (8), we get

$$A_n(x)t_{n-1}(x)t_{n-1}(y)A_n(x)t_{n-1}(x) = 0$$

Since $t_{n-1}(y)$ is onto and R is semiprime, we have

$$A_n(x)t_{n-1}(x) = 0, \text{ for all } x \in R. \tag{9}$$

Left multiplying (8) by $t_{n-1}(x)$ and right multiplying by $A_n(x)$, we get

$$t_{n-1}(x)A_n(x)t_{n-1}(y)t_{n-1}(x)A_n(x) = 0.$$

Since $t_{n-1}(y)$ is onto and R is semiprime, we have

$$t_{n-1}(x)A_n(x) = 0, \text{ for all } x \in R. \tag{10}$$

Linearizing (9), we get

$$A_n(x+y)t_{n-1}(x+y) = 0, \text{ for all } x \in R$$

that leads to

$$\left(A_n(x) + A_n(y) + B_n(x,y)\right)\left(t_{n-1}(x) + t_{n-1}(y)\right),$$

where

$$\begin{aligned} B_n(x,y) &= 2t_n(xy+yx) - \sum_{i=1}^{n} t_i(x)t_{i-1}(y) \\ &\quad - \sum_{i=1}^{n} t_i(y)t_{i-1}(x) - \sum_{i=1}^{n} t_{i-1}(x)t_i(y) - \sum_{i=1}^{n} t_{i-1}(y)t_i(x) = 0. \end{aligned}$$

This yields that

$$\begin{aligned} &A_n(x)t_{n-1}(x) + A_n(y)t_{n-1}(x) + A_n(x)t_{n-1}(x) + B_n(x,y)t_{n-1}(x) \\ &\quad + A_n(x)t_{n-1}(y) + A_n(y)t_{n-1}(y) + B_n(x,y)t_{n-1}(y) = 0. \end{aligned} \tag{11}$$

Using (9) in (11), we get

$$A_n(y)t_{n-1}(x) + A_n(x)t_{n-1}(y) + B_n(x,y)t_{n-1}(x) + B_n(x,y)t_{n-1}(y) = 0. \tag{12}$$

Replacing x by $-x$ in (12), we get

$$A_n(x)t_{n-1}(y) - A_n(y)t_{n-1}(x) + B_n(x,y)t_{n-1}(x) - B_n(x,y)t_{n-1}(y) = 0. \tag{13}$$

Comparing (13) with (12), and since R is 2-torsion free, we arrive at

$$A_n(x)t_{n-1}(y) + B_n(x,y)t_{n-1}(x) = 0. \tag{14}$$

Right multiplying (14) by $A_n(x)$ and using (10), we get

$$A_n(x)t_{n-1}(y)A_n(x) = 0.$$

Since $t_{n-1}(y)$ is onto and R is semiprime, we have $A_n(x) = 0$ for all $x \in R$. That is,

$$2t_n\left(x^2\right) = \sum_{i=1}^{n} t_i(x)t_{i-1}(x) + t_{i-1}(x)t_i(x).$$

(iv) By using (i), and since R is commutative, we have

$$2t_n(xyz + xyz) = 2(2t_n(xyz)) = 2\left(\sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(z) + t_{i-1}(x)t_{i-1}(y)t_i(z)\right)$$

for all $x, y \in R$ and $n \in \mathbb{N}$. Since R is a 2-torsion free we get the require result. □

Theorem 2.3 *Let R be a 2-torsion free semiprime ring, and $T = (t_i)_{i\in\mathbb{N}}$ be a family of additive mappings of R onto R such that*

$$2t_n(xyz) = \sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(y)t_i(x)$$

holds for all $x, y \in R$ and $n \in \mathbb{N}$, then T is Jordan higher centralizer of R.

Proof Taking (iii) of Lemma 2.2 and linearizing it, we have

$$\begin{aligned} 2t_n((x+y)(x+y)) &= 2t_n\left(x^2\right) + 2t_n\left(xy+yx\right) + 2t_n\left(y^2\right) \\ &= \sum_{i=1}^{n} t_i(x)t_{i-1}(x) + t_i(y)t_{i-1}(y) + t_i(x)t_{i-1}(y) + t_i(y)t_{i-1}(x) \\ &\quad + t_{i-1}(x)t_i(x) + t_{i-1}(y)t_i(y) + t_{i-1}(x)t_i(y) + t_{i-1}(y)t_i(x) \end{aligned}$$

By applying (iii), we get

$$2t_n\left(xy+yx\right) = \sum_{i=1}^{n} t_i(x)t_{i-1}(y) + t_i(y)t_{i-1}(x) + t_{i-1}(x)t_i(y) + t_{i-1}(y)t_i(x) \tag{15}$$

for all $x, y \in R$. Replacing y by $2xyx$ in (15), we get

$$\begin{aligned} 2(2t_n(x(xyx)+xyx)x)) &= \sum_{i=1}^{n} t_i(x)t_{i-1}(xyx) \\ &\quad + t_i(xyx)t_{i-1}(x) + t_{i-1}(x)t_i(xyx) + t_{i-1}(xyx)t_i(x). \qquad (16) \end{aligned}$$

Now, replacing $t_{i-1}(xyx)$ by $t_{i-1}(x)t_{i-1}(y)t_{i-1}(x)$ in (16) and applying the condition of the theorem, we ge

$$\begin{aligned} 4t_n\left(x^2yx+xyx^2\right) = 2\sum_{i=1}^{n} & t_i(x)t_{i-1}(x)t_{i-1}(y)t_{i-1}(x) \\ & + t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(y)t_{i-1}(x)t_i(x) \\ & + t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(x)t_{i-1}(y)t_{i-1}(x) \\ & + 2t_{i-1}(x)t_{i-1}(y)t_{i-1}(x)t_i(x). \qquad (17) \end{aligned}$$

Comparing (17) with equation (6), we have

$$\begin{aligned} \sum_{i=1}^{n} & t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) + t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x) \\ & - t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x) - t_{i-1}(x)t_i(x)t_{i-1}(y)t_i(x) = 0. \qquad (18) \end{aligned}$$

Replacing $t_{i-1}(y)$ by $t_{i-1}(y)t_{i-1}(x)$ in (18), we get

$$\begin{aligned} \sum_{i=1}^{n} & t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x)t_{i-1}(x) + t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) \\ & - t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) - t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x)t_i(x) = 0. \qquad (19) \end{aligned}$$

Right multiplying (18) by $t_{i-1}(x)$, we get

$$\begin{aligned} \sum_{i=1}^{n} & t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x)t_{i-1}(x) + t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) \\ & - t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) - t_{i-1}(x)t_i(x)t_{i-1}(y)t_i(x)t_{i-1}(x) = 0. \qquad (20) \end{aligned}$$

Subtracting (19) from (20), we get

$$\sum_{i=1}^{n} t_{i-1}(x)t_{i-1}(x)t_{i-1}(y)\left[t_i(x),t_{i-1}(x)\right] - t_{i-1}(x)t_{i-1}(y)\left[t_i(x),t_{i-1}(x)\right]t_{i-1}(x) = 0 \qquad (21)$$

for all $x, y \in R$. Replacing $t_{i-1}(y)$ by $t_i(y)t_{i-1}(y)$ in (21), we get

$$\sum_{i=1}^{n} t_{i-1}(x)t_{i-1}(x)t_i(y)t_{i-1}(y)\left[t_i(x),t_{i-1}(x)\right] - t_{i-1}(x)t_i(y)t_{i-1}(y)\left[t_i(x),t_{i-1}(x)\right]t_{i-1}(x) = 0. \qquad (22)$$

Left multiplying (21) by t_i, we get

$$\sum_{i=1}^{n} t_i(x)t_{i-1}(x)t_{i-1}(x)t_{i-1}(y)\left[t_i(x),t_{i-1}(x)\right] - t_i(x)t_{i-1}(x)t_{i-1}(y)\left[t_i(x),t_{i-1}(x)\right]t_{i-1}(x) = 0. \quad (23)$$

Subtracting (22) from (23), we get

$$\sum_{i=1}^{n} \left[t_i(x), t_{i-1}(x)t_{i-1}(x)\right] t_{i-1}(y)\left[t_i(x),t_{i-1}(x)\right] - \left[t_i(x),t_{i-1}(x)\right]t_{i-1}(y)\left[t_i(x),t_{i-1}(x)\right]t_{i-1}(x) = 0. \quad (24)$$

Now, in (24) if we take

$$\begin{aligned} a &= \left[t_i(x), t_{i-1}(x)t_{i-1}(x)\right], \\ b &= \left[t_i(x), t_{i-1}(x)\right], \\ c &= \left[t_i(x), t_{i-1}(x)\right]t_{i-1}(x) \end{aligned}$$

and $d = t_{i-1}(x)$. Hence, we have $adb + bdc = 0$ implies that $(a + c)db = 0$, by Lemma 2.1 and so we get

$$\sum_{i=1}^{n} \left(\left[t_i(x), t_{i-1}(x)t_{i-1}(x)\right] - \left[t_i(x), t_{i-1}(x)\right]t_{i-1}(x)\right)t_{i-1}(y)\left[t_i(x), t_{i-1}(x)\right] = 0$$

This implies that,

$$\sum_{i=1}^{n} t_{i-1}(x)\left[t_i(x), t_{i-1}(x)\right]t_{i-1}(y)\left[t_i(x), t_{i-1}(x)\right] = 0 \quad (25)$$

for all $x, y \in R$. Replacing $t_{i-1}(y)$ by $t_{i-1}(y)t_{i-1}(x)$ in (25), we get

$$\sum_{i=1}^{n} t_{i-1}(x)\left[t_i(x), t_{i-1}(x)\right]t_{i-1}(y)t_{i-1}(x)\left[t_i(x), t_{i-1}(x)\right] = 0$$

for all $x, y \in R$. Since $t_{i-1}(y)$ is onto, and R is semiprime ring, we have

$$\sum_{i=1}^{n} t_{i-1}(x)\left[t_i(x), t_{i-1}(x)\right] = 0 \quad (26)$$

for all $x, y \in R$. Replacing $t_{i-1}(y)$ by $t_{i-1}(x)t_{i-1}(y)$ in (18), we get

$$\sum_{i=1}^{n} t_i(x)t_{i-1}(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) + t_{i-1}(x)t_i(x)t_{i-1}(x)t_{i-1}(y)t_{i-1}(x) - t_{i-1}(x)t_i(x)t_{i-1}(x)t_{i-1}(y)t_{i-1}(x) - t_{i-1}(x)t_i(x)t_{i-1}(x)t_{i-1}(y)t_i(x) = 0. \quad (27)$$

Left multiplying (18) by $t_{i-1}(x)$, we get

$$\sum_{i=1}^{n} t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x)$$
$$- t_{i-1}(x)t_{i-1}(x)t_i(x)t_{i-1}(y)t_{i-1}(x) - t_{i-1}(x)t_{i-1}(x)t_i(x)t_{i-1}(y)t_i(x) = 0. \tag{28}$$

Subtracting (27) from (28), we get

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)]\, t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) - t_{i-1}(x)\, [t_i(x), t_{i-1}(x)]\, t_{i-1}(y)t_{i-1}(x) = 0, \tag{29}$$

for all $x, y \in R$. Using (26) in (29), we get

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)]\, t_{i-1}(y)t_{i-1}(x)t_{i-1}(x) = 0. \tag{30}$$

Replacing $t_{i-1}(y)$ by $t_{i-1}(y)t_i(x)$ in (30), we get

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)]\, t_{i-1}(y)t_i(x)t_{i-1}(x)t_{i-1}(x) = 0. \tag{31}$$

Right multiplying (30) by $t_i(x)$, we get

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)]\, t_{i-1}(y)t_{i-1}(x)t_{i-1}(x)t_i(x) = 0. \tag{32}$$

Subtracting (31) from (32), we get

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)]\, t_{i-1}(y)\, [t_i(x), t_{i-1}(x)t_{i-1}(x)] = 0. \tag{33}$$

Now, we can rewritten (33) and using the relation (26), we get

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)]\, t_{i-1}(y)\, [t_i(x), t_{i-1}(x)]\, t_{i-1}(x) = 0. \tag{34}$$

Replacing $t_{i-1}(y)$ by $t_{i-1}(x)t_{i-1}(y)$ in (34), we get

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)]\, t_{i-1}(x)t_{i-1}(y)\, [t_i(x), t_{i-1}(x)]\, t_{i-1}(x) = 0.$$

Since $t_{i-1}(y)$ is onto, and R is semiprime ring, we have

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)]\, t_{i-1}(x) = 0, \text{ for all } x, y \in R. \tag{35}$$

Linearizing (26) and using (26) again, we have

$$\sum_{i=1}^{n} t_{i-1}(y)\,[t_i(x), t_{i-1}(x)] + t_{i-1}(x)\,[t_i(y), t_{i-1}(x)] + t_{i-1}(y)\,[t_i(y), t_{i-1}(x)]$$
$$+\, t_{i-1}(x)\,[t_i(x), t_{i-1}(y)] + t_{i-1}(y)\,[t_i(x), t_{i-1}(y)] + t_{i-1}(x)\,[t_i(y), t_{i-1}(y)] = 0. \tag{36}$$

Replacing x by $-x$ in (36), we get

$$\sum_{i=1}^{n} t_{i-1}(y)\,[t_i(x), t_{i-1}(x)] + t_{i-1}(x)\,[t_i(y), t_{i-1}(x)] + t_{i-1}(x)\,[t_i(x), t_{i-1}(y)]$$
$$= t_{i-1}(y)\,[t_i(y), t_{i-1}(x)] + t_{i-1}(y)\,[t_i(x), t_{i-1}(y)] + t_{i-1}(x)\,[t_i(y), t_{i-1}(y)] = 0. \tag{37}$$

Putting (37) in (36) and since R is 2-torsion free, we have

$$\sum_{i=1}^{n} t_{i-1}(y)\,[t_i(x), t_{i-1}(x)] + t_{i-1}(x)\,[t_i(y), t_{i-1}(x)] + t_{i-1}(x)\,[t_i(x), t_{i-1}(y)] = 0. \tag{38}$$

Left multiplying (38) by $[t_i(x), t_{i-1}(x)]$, and using (35), we get

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)]\, t_{i-1}(y)\,[t_i(x), t_{i-1}(x)] = 0, \text{ for all } \; x, y \in R.$$

Since $t_{i-1}(y)$ is onto, and R is semiprime ring, we have

$$\sum_{i=1}^{n} [t_i(x), t_{i-1}(x)] = 0. \tag{39}$$

Comparing (39) with part (iii) of Lemma 2.2, we get

$$t_n(x^2) = \sum_{i=1}^{n} t_i(x)t_{i-1}(x) \;\text{ and }\; t_n(x^2) = \sum_{i=1}^{n} t_{i-1}(x)t_i(x)$$

That is, T is a Jordan higher left and right centralizer of R, hold for all $x \in R$, and $n \in \mathbb{N}$. Therefore, T is Jordan higher centralizer of R. □

Proposition 2.4 *Let R be a 2-torsion free semiprime ring, and $T = (t_i)_i \in \mathbb{N}$ be a family of additive mappings of R onto R, such that*

$$2t_n(xyx) = \sum_{i=1}^{n} t_i(x)t_{i-1}(y)t_{i-1}(x) + t_{i-1}(x)t_{i-1}(y)t_i(x)$$

for all $x, y \in R$ and $n \in \mathbb{N}$,then T is Jordan higher triple centralizer of R.

Proof From [2] we have T is Jordan higher left and right centralizer of R, and by [4] we get, T is a Jordan higher triple left centralizer of R.

Similarly, we can prove that T is a Jordan higher triple right centralizer of R, Therefore, we have T is Jordan higher triple centralizer of R.

Acknowledgement

The authors would like to thank Mustansiriyah University (www.uomustansiriyah.edu.iq) Baghdad-Iraq for its support in the present work, and the referees for their valuable comments and suggestions.

References

[1] A.M. Ibraheem and S.M.Salih, Jordan higher triple left resp. right centralizers of prime G-Rings, *Diyala Journal for Pure Science*, International Conference for Pure Sciences (ICPS-2021),17(3), (2021), 25–36.

[2] S.M.Salih, and A.M.Ibraheem, On generalized higher triple centralizers of prime rings, *Academic J. for Engineering and Science*, 3(3), (2021), 41–46.

[3] S.M. Salih and A.M. Ibraheem, Jordan generalized higher triple centralizers of 2-torsion semiprime gamma rings, to appear.

[4] S.M. Salih, Generalized higher centralizer of prime rings, *International Journal of Current Research*, Vol. 8, 11, (2016), 40966–40975.

[5] J. Vukman, An identity related to centralizers in semiprime rings, *Comment. Math. Univ. Carolinae*, 40, 3,(1999), 447–456.

[6] J.Vukman, Centralizers on semiprime rings, *Comment. Math. Univ. Carolinae*, 42,(2001), 237–245.

[7] B. Zalar, On centralizers of semiprime rings, *Comment. Math. Univ. Carolinae*, 32, 4,(1991), 609–614.

International J.Math. Combin. Vol.4(2022), 11-20

Support Edge Regular Neutrosophic Fuzzy Graphs

N.R. Santhi Maheswari

PG and Research Department of Mathematics

G.Venkataswamy Naidu College, Kovilpatti - 628 502, Tamil Nadu, India

S. Sivabala

Research Scholar, PG and Research Department of Mathematics

G.Venkataswamy Naidu College, Kovilpatti - 628 502, Tamil Nadu, India)

E-mail: nrsmaths@yahoo.com, sivamaths13@gmail.com

Abstract: In this paper, we introduce support edge regular neutrosophic fuzzy graph, support totally edge regular neutrosophic fuzzy graphs and investigate some theorems and results of these graphs. Also the comparative study between edge regular neutrosophic fuzzy graph and support edge regular neutrosophic fuzzy graph are done here.

Key Words: Support edge regular neutrosophic fuzzy graph, support totally edge regular neutrosophic fuzzy graph.

AMS(2010): 03E72, 05C72

§1. Introduction

Prof.Smarandache [20] introduced notion of neutrosophic set which is useful for dealing real life problems having imprecise, indeterminacy and inconsistent data. They are generalization of the theory of fuzzy sets, intuitionistics fuzzy set, interval valued fuzzy set, and interval valued intuitionistic fuzzy sets. Nagoor Gani and Radha [4] introduced regular fuzzy graphs, total degree and totally regular fuzzy graphs. Irregular fuzzy graphs are introduced by Nagoor Gani, S. R. Latha [3]. Azriel Rosenfeld introduced fuzzy graphs in 1975 [5]. N. Shah [19] introduced the notion of neutrosophic graphs and different operations like union, intersection and complement in his work. A neutrosophic set is characterized by a truth membership degree (t), an indeterminacy membership degree(i), falsity membership degree(f) independently, which are with in the real standard or non standard unit interval $]^{-}0, 1^{+}[$. Divya and Dr. J. Malarvizhi introduced the notion of neutrosophic fuzzy graph and few fundamental operation on neutrosophic fuzzy graph [1]. N. R. Santhi Maheswari and C. Sekar introduced Neighbourly irregular graphs and semi neighbourly irregular graphs, m-neighbourly irregular Fuzzy graphs, m-neighbourly irregular intuitionistic Fuzzy Graph [7, 8, 9].

N. R. Santhi Maheswari and C. Sekar introduced edge irregular fuzzy graphs, neighbourly edge irregular fuzzy graphs, strongly edge irregular fuzzy graphs [10, 11, 12]. N. R. Santhi Ma-

[1]Received August 21, 2021, Accepted December 3, 2022.

heswari and C. Sekar introduced edge irregular bipolar fuzzy graphs, neighbourly edge irregular bipolar fuzzy graphs, strongly edge irregular bipolar fuzzy graphs [13, 14, 15]. N. R. Santhi Maheswari, M. Sudha and Durga introduced edge irregularity intuitionistic fuzzy graph [16]. R. Muneeswari and N. R. Santhi Maheswari introduced support edge regular fuzzy graph [6].

N. R. Santhi Maheswari and K. Amutha introduced support neighbourly edge irregular graphs and 1-neighbourly edge irregular graphs [17, 18]. S. Sivabala and N. R. Santhi Maheswari introduced support edge irregular neutrosophic fuzzy graphs, neighbourly and highly irregular neutrosophic fuzzy graphs and highly edge irregular neutrosophic fuzzy graphs [21, 22, 23].

These ideas motivate us to introduce support edge regular and support totally edge regular neutrosophic fuzzy graphs.

§2. Preliminaries

In this section, we mainly recall the notions related to neutrosophic set, fuzzy graph, neutrosophic fuzzy set and neutrosophic fuzzy graph.

Definition 2.1([20]) *Let X be a space of points with generic elements in X denoted by x. A neutrosophic set $A(NSA)$ is an object having the form*

$$A = \{< x : T_A(x), I_A(x), F_A(x) >, x \in X\},$$

where the functions $T, I, F \to]^{-}0, 1^{+}[$ define respectively a truth membership function, an indeterminacy membership function and a falsity membership function of the element $x \in X$ to the set A with the condition

$$^{-}0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$$

The functions $T_A(x), I_A(x), F_A(x)$ are real standard or non standard subsets of $]^{-}0, 1^{+}[$.

Definition 2.2([3]) *A fuzzy graph is a pair of functions $G = (\sigma, \mu)$, where σ is a fuzzy subset of a non-empty set V and is a symmetric fuzzy relation of σ i.e $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ such that $\mu(uv) \leq \sigma(u)\Lambda\sigma(v), \forall u, v \in V$ where uv denote the edge between u and v, and $\sigma(u)\Lambda\sigma(v)$ denotes the minimum of $\sigma(u)$ and $\sigma(v)$, σ is called the fuzzy vertex set of V and μ is called the fuzzy edge set of E.*

Definition 2.3([1]) *Let X be a space of points with generic elements in X denoted by x. A neutrosophic fuzzy set$A(NFSA)$ is characterized by truth membership function $T_A(x)$, an indeterminacy membership functions $I_A(x)$ and a falsity membership function $F_A(x)$.*

For each point $x \in X$, $T_A(x), I_A(x), F_A(x) \in [0, 1]$. A neutrosophic fuzzy set A can be written as

$$A = \{< x : T_A(x), I_A(x), F_A(x) >, x \in X\}.$$

Definition 2.4([1]) *Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be neutrosophic fuzzy sets on a set X. If $A = (T_A, I_A, F_A)$ is a neutrosophic fuzzy relation on a set X, then $A = (T_A, I_A, F_A)$*

is called a neutrosophic fuzzy relation on $B = (T_B, I_B, F_B)$ *if*

$$T_B(x,y) \leq T_A(x).T_A(y),$$
$$I_B(x,y) \leq I_A(x).I_A(y),$$
$$F_B(x,y) \leq F_A(x).F_A(y),$$

for all $x, y \in X$, *where . means the ordinary multiplication.*

Definition 2.5([1]) *A neutrosophic fuzzy graph* ($NFgraph$) *with underlying set* V *is defined to be a pair* $N_G = (A, B)$, *where*

(*i*) *The functions* $T_A, I_A, F_A : V \to [0,1]$ *denote the degree of truth membership, degree of indeterminacy membership and the degree of falsity membership of the element* $v_i \in V$ *respectively and* $0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$;

(*ii*) $E \subseteq V \times V$ *where the functions* $T_B, I_B, F_B : V \times V \to [0,1]$ *are defined by*

$$T_B(v_i, v_j) \leq T_A(v_i).T_A(v_j)$$
$$I_B(v_i, v_j) \leq I_A(v_i).I_A(v_j)$$
$$F_B(v_i, v_j) \leq F_A(v_i).F_A(v_j)$$

for all $v_i, v_j \in V$, *where . means ordinary multiplication denotes the degrees of truth membership, indeterminacy membership and falsity membership of the edge* $(v_i, v_j) \in E$ *respectively, where* $0 \leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$ *for all* $(v_i, v_j) \in E$ $(i, j = 1, 2, \cdots, n)$.

Definition 2.6([23]) *Let* $N_G = (A, B)$ *be a neutrosophic fuzzy graph. The degree of an edge* uv *in* N_G *defined by* $d_{N_G}(uv) = d_{N_G}(u) + d_{N_G}(v) - 2(T_B(uv), I_B(uv), F_B(uv))$, $d_{N_G}(u) = (deg_T(u), deg_I(u), deg_F(u))$, $d_{N_G}(v) = (deg_T(v), deg_I(v), deg_F(v))$, *where*

$$\begin{aligned} deg_T(u) &= \sum_{uv \in E} T_B(uv), \quad deg_I(u) = \sum_{uv \in E} I_B(uv), \\ deg_F(u) &= \sum_{uv \in E} F_B(uv), \quad deg_T(v) = \sum_{uv \in E} T_B(uv), \\ deg_I(v) &= \sum_{uv \in E} I_B(uv), \quad deg_F(v) = \sum_{uv \in E} F_B(uv). \end{aligned}$$

The minimum degree of an edge is $\delta_E(N_G) = \wedge\{d_{N_G}(uv)/uv \in E\}$. *The maximum degree of an edge is* $\Delta_E(N_G) = \vee\{d_{N_G}(uv)/uv \in E\}$.

Definition 2.7([23]) *Let* $N_G = (A, B)$ *be a neutrosophic fuzzy graph. The total degree of an edge* uv *in* N_G *defined by* $td_{N_G}(uv) = d_{N_G}(uv) + (T_B(uv), I_B(uv), F_B(uv))$. *The minimum total degree of an edge is* $\delta_{tE}(N_G) = \wedge\{td_{N_G}(uv)/uv \in E\}$. *The maximum total degree of an edge is* $\Delta_{tE}(N_G) = \vee\{td_{N_G}(uv)/uv \in E\}$. z

Definition 2.8([23]) *Let* N_G *be a neutrosophic fuzzy graph on* $G(V, E)$. *Then* N_G *is said to be an edge regular neutrosophic fuzzy graph if all edges having same edge degree.*

Definition 2.9([23]) *Let* N_G *be a neutrosophic fuzzy graph on* $G(V, E)$. *Then* N_G *is said to be an edge totally regular neutrosophic fuzzy graph if all edges having same edge total degree.*

Definition 2.10([21]) *Let $N_G = (A, B)$ be a neutrosophic fuzzy graph. The support of an edge e in N_G is the sum of edge degree of its neighbour edges. That is*

$$s_{N_G}(e) = \sum_{e \in N(e_i)} d_{N_G}(e_i), \quad d_{N_G}(e_i)$$

is the degree of an edge e_i, where

$$\begin{aligned} d_{N_G}(e) &= d_{N_G}(u) + d_{N_G}(v) - 2(T_B(e), I_B(e), F_B(e)), \ e = uv \in E, \\ d_{N_G}(u) &= (deg_T(u), deg_I(u), deg_F(u)), \quad d_{N_G}(v) = (deg_T(v), deg_I(v), deg_F(v)) \end{aligned}$$

with

$$\begin{aligned} deg_T(u) &= \sum_{uv \in E} T_B(uv), \quad deg_I(u) = \sum_{uv \in E} I_B(uv), \\ deg_F(u) &= \sum_{uv \in E} F_B(uv), \quad deg_T(v) = \sum_{uv \in E} T_B(uv), \\ deg_I(v) &= \sum_{uv \in E} I_B(uv), \quad deg_F(v) = \sum_{uv \in E} F_B(uv). \end{aligned}$$

Definition 2.11([21]) *Let $N_G = (A, B)$ be a neutrosophic fuzzy graph. The total support of an edge e in N_G defined by $ts_{N_G}(e) = s_{N_G}(e) + (T_B, I_B, F_B)(e)$.*

§3. Support Edge Regular Neutrosophic Fuzzy Graphs

In this section, we have defined support edge regular and support totally edge regular neutrosophic fuzzy graphs and discuss some properties.

Definition 3.1 *Let N_G be a neutrosophic fuzzy graph on $G(V, E)$. Then N_G is said to be an support edge regular if all the edges having same edge support or $s_{N_G}(e) = (k_1, k_2, k_3)$ for all edges in N_G.*

Definition 3.2 *Let N_G be a neutrosophic fuzzy graph on $G(V, E)$. Then N_G is said to be an support totally edge regular if all the edges having same total edge support or $ts_{N_G}(e) = (tk_1, tk_2, tk_3)$ for all edges in N_G.*

Example 3.3 Let N_G be a neutrosophic fuzzy graph. Consider the neutrosophic fuzzy graph given in Figure 1. We calculated support of an edge and total support of an edge are as follows:

$d_{N_G}(e_1) = d_{N_G}(v_1v_2) = d_{N_G}(v_1) + d_{N_G}(v_2) - 2(T_B(v_1v_2), I_B(v_1v_2), F_B(v_1v_2))$.

$d_{N_G}(v_1v_2) = (0.04, 0.08, 0.12) + (0.04, 0.08, 0.12) - 2(0.02, 0.04, 0.06)$.

$d_{N_G}(v_1v_2) = (0.04, 0.08, 0.12) = d_{N_G}(e_1)$.

Similarly, $d_{N_G}(e_2) = (0.04, 0.08, 0.12)$; $d_{N_G}(e_3) = (0.04, 0.08, 0.12)$; $s_{N_G}(e_1) = d_{N_G}(e_2) + d_{N_G}(e_3) = (0.08, 0.16, 0.24)$ and $s_{N_G}(e_2) = s_{N_G}(e_3)$.

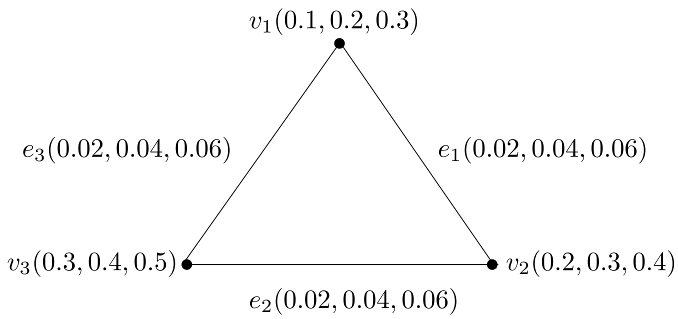


Figure 1

and $ts_{N_G}(e_1) = s_{N_G}(e_1) + (T_B, I_B, F_B)(e_1) = (0.10, 0.20, 0.30)$.

Similarly, $ts_{N_G}(e_2) = (0.10, 0.20, 0.30) = ts_{N_G}(e_3)$. Here every edges having same edge support and same total edge support. Hence this graph is support edge regular neutrosophic fuzzy graph and support totally edge regular neutrosophic fuzzy graph.

Remark 3.4 Every support edge regular neutrosophic fuzzy graph need not be support totally edge regular neutrosophic fuzzy graph.

Example 3.5 Let N_G be a neutrosophic fuzzy graph. Consider the neutrosophic fuzzy graph given in Figure.2. We calculated support of an edge and total support of an edge are as follows:

$s_{N_G}(e_1) = (0.06, 0.10, 0.14) = s_{N_G}(e_2) = s_{N_G}(e_3) = s_{N_G}(e_4)$.

$ts_{N_G}(e_1) = (0.07, 0.12, 0.17)$, $ts_{N_G}(e_2) = (0.08, 0.13, 0.18)$.

$ts_{N_G}(e_3) = (0.07, 0.12, 0.17)$, $ts_{N_G}(e_4) = (0.08, 0.13, 0.18)$.

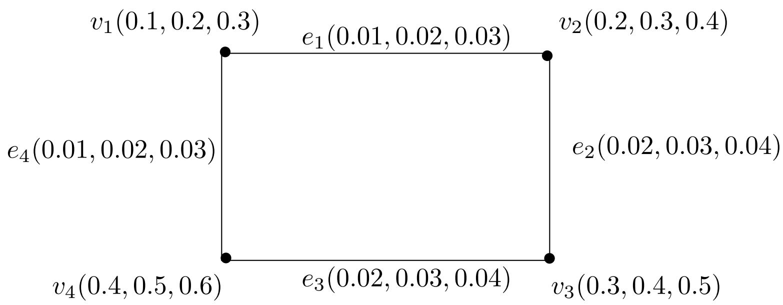


Figure.2

Here every edge has same edge support. Therefore this graph is support edge regular neutrosophic fuzzy graph. But total edge support of all edges are not same. Therefore this graph is not a support totally edge regular neutrosophic fuzzy graph.

Theorem 3.6 *Let N_G be neutrosophic fuzzy graph on $G(V, E)$ and (T_B, I_B, F_B) is constant function. Then, the following conditions are equivalent:*

(*i*) *N_G is support edge regular neutrosophic fuzzy graph;*

(*ii*) *N_G is support totally edge regular neutrosophic fuzzy graph.*

Proof Let N_G be neutrosophic fuzzy graph with (T_B, I_B, F_B) is constant function . i.e. $(T_B, I_B, F_B)(e) = (x, y, z)$, where (x, y, z) is constant for all $e \in E$. Suppose N_G is support edge regular neutrosophic fuzzy graph. Then all the edges have same edge support. Let e_1 and e_2 be two edges having same edge support. i.e. $s_{N_G}(e_1) = s_{N_G}(e_2)$.

Since $(T_B, I_B, F_B)(e) = (x, y, z)$, where (x, y, z) is constant for all $e \in E$, we have $s_{N_G}(e_1) + (x, y, z) = s_{N_G}(e_2) + (x, y, z)$. This implies that $s_{N_G}(e_1) + (T_B, I_B, F_B)(e_1) = s_{N_G}(e_2) + (T_B, I_B, F_B)(e_2)$, which implies that $ts_{N_G}(e_1) = ts_{N_G}(e_2)$, where e_1 and e_2 be two edges. Therefore all the edges having same total edge support.

Hence N_G is support totally edge regular neutrosophic fuzzy graph and $(i) \implies (ii)$ hold.

Suppose N_G is support totally edge regular neutrosophic fuzzy graph. Then all the edges having same total edge support. Without loss of generality, let e_1 and e_2 be two edges having same total edge supports, i.e. $ts_{N_G}(e_1) = ts_{N_G}(e_2)$. This implies that $s_{N_G}(e_1) + (x, y, z) = s_{N_G}(e_2) + (x, y, z)$.

Since $(T_B, I_B, F_B)(e) = (x, y, z)$, where (x, y, z) is constant for all $e \in E$, we have $s_{N_G}(e_1) = s_{N_G}(e_2)$, where e_1 and e_2 be two edges. Therefore all the edges having same edge support. Hence, N_G is support edge regular neutrosophic fuzzy graph and $(ii) \implies (i)$ hold. □

Theorem 3.7 *Let N_G be neutrosophic fuzzy graph on $G(V, E)$ and (T_B, I_B, F_B) is constant function. Then the following conditions are equivalent:*

(*i*) *N_G is edge regular neutrosophic fuzzy graph;*

(*ii*) *N_G is support edge regular neutrosophic fuzzy graph.*

Proof Let N_G be neutrosophic fuzzy graph on $G(V, E)$ and (T_B, I_B, F_B) is constant function. Suppose we take N_G is edge regular neutrosophic fuzzy graph. Then, $d_{N_G}(e_i) = (x_1, x_2, x_3)$, where (x_1, x_2, x_3) is constant for all e_i in N_G. We know that

$$s_{N_G}(e_i) = \sum_{e_j \in N(e_i)} d_{N_G}(e_j) = \sum_{e_k \in N(e_j)} d_{N_G}(e_k) = s_{N_G}(e_j)$$

since $d_{N_G}(e_i) = (x_1, x_2, x_3)$, is constant for all e_i in N_G. Therefore edge support of all edges are same. Hence this graph is support edge regular neutrosophic fuzzy graph. Hence $(i) \implies (ii)$ hold.

Suppose we take N_G is support edge regular neutrosophic fuzzy graph. Then the edge support of all edges are same. Now, To prove that the graph is edge regular neutrosophic fuzzy graph. Suppose we take the graph is not edge regular neutrosophic fuzzy graph. That is, $d_{N_G}(e_i) \neq d_{N_G}(e_j)$ for some integers i, j. This implies that $s_{N_G}(e_s) \neq s_{N_G}(e_t)$ for some s, t, where $d_{N_G}(e_i) \in s_{N_G}(e_s)$ and $d_{N_G}(e_j) \in s_{N_G}(e_t)$. This is contradiction. Therefore $d_{N_G}(e_i) = d_{N_G}(e_j)$. Hence this graph is edge regular neutrosophic fuzzy graph and $(i) \implies (ii)$ hold. □

Theorem 3.8 *Let N_G be neutrosophic fuzzy graph, where N_G is cycle with (T_B, I_B, F_B) is a constant function. Then N_G is support edge regular and support totally edge regular neutrosophic* *fuzzy graph.*

Proof Let N_G be neutrosophic fuzzy graph, a cycle of length n. Suppose $(T_B, I_B, F_B)(e_i) = (x, y, z)$, where (x, y, z) is constant for all $e \in E$. Then we know that

$$s_{N_G}(e_i) = \sum_{e_j \in N(e_i)} d_{N_G}(e_j).$$

Now,

$$s_{N_G}(e_i) = \begin{cases} d_{N_G}(e_2) + d_{N_G}(e_n) \text{if } i = 1 \\ d_{N_G}(e_{i-1}) + d_{N_G}(e_{i+1}) \text{ if } i = 2, 3, \cdots, n-1 \\ d_{N_G}(e_{i-1}) + d_{N_G}(e_1) \text{ if } i = n, \end{cases}$$

where

$$d_{N_G}(e_i) = \begin{cases} (T_B, I_B, F_B)(e_2) + (T_B, I_B, F_B)(e_n) \text{ if } i = 1 \\ (T_B, I_B, F_B)(e_{i-1}) + (T_B, I_B, F_B)(e_{i+1}) \text{ if } i = 2, 3, \cdots, n-1 \\ (T_B, I_B, F_B)(e_{i-1}) + (T_B, I_B, F_B)(e_1) \text{ if } i = n. \end{cases}$$

Since $(T_B, I_B, F_B)(e_i) = (x, y, z)$, where (x, y, z) is constant for all $e_i \in E$, we have all the edges having same edge support. Hence this graph is support edge regular neutrosophic fuzzy graph.

Since the graph is support edge regular neutrosophic fuzzy graph with (T_B, I_B, F_B) is a constant function, then by Theorem 3.6, the graph is suppor totally edge regular neutrosophic fuzzy graph. □

Theorem 3.9 *Let N_G be neutrosophic fuzzy graph, a star $K_{1,n}$ with (T_B, I_B, F_B) is a constant function. Then N_G is support edge regular and support edge totally regular neutrosophic fuzzy graph.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of N_G adjacent to the vertex x. Let $e_1, e_2, \cdots, e_n$ be the edges of a star N_G with (T_B, I_B, F_B) constant and

$$d_{N_G}(e_i) = ((x_1, y_1, z_1) + \cdots + (x_n, y_n, z_n)) - (x_i, y_i, z_i), \quad (1 \leq i \leq n).$$

Therefore all the edges e_i,$1 \leq i \leq n$ having same edge degrees. Therefore the graph is edge regular neutrosophic fuzzy graph. By theorem 3.7, the graph is support edge regular neutrosophic fuzzy graph. By theorem 3.6, the graph is support edge totally regular neutrosophic fuzzy graph. □

§4. Comparative Study Between Support Edge Regular N_G and Edge Regular N_G

Remark 4.1 Every support edge regular neutrosophic fuzzy graph need not be an edge regular neutrosophic fuzzy graph.

Example 4.2 Let N_G be neutrosophic fuzzy graph.

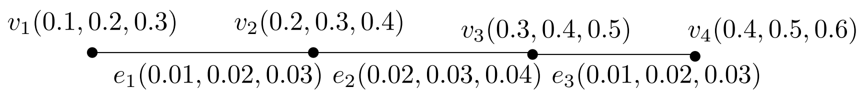


Figure 3

Consider the neutrosophic fuzzy graph given in Figure 3. We calculated degree of an edge and support of an edge are as follows:

$d_{N_G}(e_1) = (0.02, 0.03, 0.04)$, $d_{N_G}(e_2) = (0.04, 0.06, 0.08)$,

$d_{N_G}(e_3) = (0.02, 0.03, 0.04)$, $s_{N_G}(e_1) = (0.04, 0.06, 0.08)$,

$s_{N_G}(e_2) = (0.04, 0.06, 0.08)$, $s_{N_G}(e_3) = (0.04, 0.06, 0.08)$.

Here all the edges having same edge support but the edge degree of an edge e_2 which is distinct from the edges e_1 and e_3. Therefore this graph is support edge regular neutrosophic fuzzy graph but not a edge regular neutrosophic fuzzy graph.

Remark 4.3 Every edge totally regular neturosophic fuzzy graph need not be an support totally edge regular neutrosophic fuzzy graph.

Example 4.4 Let N_G be a neutrosophic fuzzy graph.

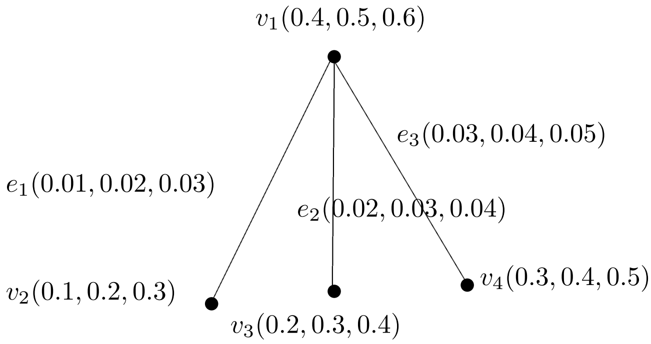


Figure 4

Consider the neutrosophic fuzzy graph given in Figure 4, we calculated degree of an edge, total degree of an edge, support of an edge and total support of an edge are as follows:

$d_{N_G}(e_1) = (0.05, 0.07, 0.09)$, $d_{N_G}(e_2) = (0.04, 0.06, 0.08)$, $d_{N_G}(e_3) = (0.03, 0.05, 0.07)$;

$td_{N_G}(e_1) = (0.06, 0.09, 0.12) = td_{N_G}(e_2) = td_{N_G}(e_3)$;

$s_{N_G}(e_1) = (0.07, 0.11, 0.15)$, $s_{N_G}(e_2) = (0.08, 0.12, 0.16)$, $s_{N_G}(e_3) = (0.09, 0.13, 0.17)$.;

$ts_{N_G}(e_1) = (0.08, 0.13, 0.18)$, $ts_{N_G}(e_2) = (0.10, 0.15, 0.20)$, $ts_{N_G}(e_3) = (0.12, 0.17, 0.22)$.

Here every edges having same edge total degree but all the edges having distinct total edge support. Therefore, this graph is edge totally regular neutrosophic fuzzy graph but not a support totally edge regular neutrosophic fuzzy graph.

References

[1] G. Divya and J. Malarvizhi, Some operations on neutrosophic fuzzy graph, *International Journal of Mathematical Archieve*, 8(9), 2017, 120-125.

[2] Florentin Smarandache, Surapati Pramavik, *New Trends in Neutrosophic Theory and its Applications*, Pons Edition, Brussels, Belgium, EU, 2016.

[3] A. Nagoor Gani and S. R. Latha, On irregular fuzzy graphs, *Applied Mathematical Sciences*, Vol.6, 11(2012), 517-523.

[4] A. Nagoor Gani and K. Radha, On regular fuzzy graphs, *Journal of Physical Sciences*, Vol.12, 2008, 33 - 40.

[5] A. Rosenfeld, *Fuzzy Graphs, Fuzzy Sets And Their Applications to Cognitive and Decision Process*, M.Eds. Academic Press, New York, 77-95, 1975.

[6] R. Muneeswari and N. R. Santhi Maheswari, Support edge regular fuzzy graphs, *International Journal of Advanced Research in Engineering and Technology*(IJARET), Volume 11, Issue 1, January 2020, 184-193.

[7] N. R. Santhi Maheswari and C. Sekar, Neighbourly irregular graphs and semi neighbourly irregular graphs, *Acta Ciencia Indica*, Vol. XLM, No.1 (2014), 71 - 77.

[8] N. R. Santhi Maheswari and C. Sekar, On m-neighbourly irregular fuzzy graphs, *International Journal of Mathematics and Soft Computing*, Vol.5, 2(2015), 145 - 153.

[9] N. R. Santhi Maheswari and C. Sekar, On m-neighboulry irregular intuitionistic fuzzy graph, *International Journal of Mathematical Combinatorics*, Vol.3(2016), 107 - 114.

[10] N. R. Santhi Maheswari and C. Sekar, On edge irregular fuzzy graphs, *International Journal of Mathematics and Soft Computing*, Vol.6, No.2, 2016, 131-143.

[11] N. R. Santhi Maheswari and C. Sekar, On neighbourly edge irregular fuzzy graphs, *International Journal of Mathematical Archieve*, Vol. 6, 10(2015), 224 - 231.

[12] N. R. Santhi Maheswari and C. Sekar, On strongly edge irregular fuzzy graphs, *Kragujevac Journal of Mathematics*, Vol. 40(1), 2016, 125 - 135.

[13] N. R. Santhi Maheswari and C. Sekar, On edge irregular bipolar fuzzy graphs, *International Journal of Modern Science and Engineering Technology*, Vol.3, 8(2016), 19-25.

[14] N. R. Santhi Maheswari and C. Sekar, On neighbourly edge irregular bipolar fuzzy graphs, *Annals of Pure and Applied Mathematics*, Vol.11, No.1, 2016, 1-8.

[15] N. R. Santhi Maheswari and C. Sekar, Strongly edge irregular bipolar fuzzy graphs, *International Journal of Mathematical Archieve*, 8(7), 2017, 1-7.

[16] N. R. Santhi Maheswari, M. Sudha and S. Durga, On edge irregularity intuitionistic fuzzy graphs, *International Journal of Innovatie Research in Sciences, Engineering and Technology*, Vol.6, 4(2017), 5770 - 5780.

[17] N. R. Santhi Maheswari and K. Amutha, Support neighbourly edge irregular graphs, *International Journal of Recent Technology and Engineering*, Vol 8, 3(2019), 5329-5332.

[18] N. R. Santhi Maheswari and K. Amutha, 1-Neighbourly edge irregular graphs, *Advances in Mathematics: Scientific Journal*, Special issue No.3, 2019, 200 - 207.

[19] N. Shah, Some studies in neutrosophic graphs, *Neutrosophic Sets and Systems*, Vol 12, 2016, 54- 64.

[20] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, *Inter. J.Pure Appl. Math.*, 24 (2005), 287-297.

[21] S. Sivabala and N. R. Santhi Maheswari, Support edge irregular neutrosophic fuzzy graphs, *International Journal of Advanced Research in Engineering and Technology (IJARET)*, Volume 11, 1(2020), 172 - 183.

[22] S. Sivabala and N. R. Santhi Maheswari, Neighbourly and highly irregular neutrosophic fuzzy graphs, *International Journal of Mathematical Combinatorics*, Volume 1, 2021, 30 - 46.

[23] S. Sivabala and N. R. Santhi Maheswari, Highly edge irregular neutrosophic fuzzy graphs, *Malaya Journal of Mathematik* (Communicated).

International J.Math. Combin. Vol.4(2022), 21-31

Some Hermite-Hadamard Type Inequalities For Trigonometrically ρ-Convex Functions via by an Identity

Huriye Kadakal

Bayburt University, Faculty of Education

Department of Primary Education, Baberti Campus, 69000 Bayburt-Türkiye

Mahir Kadakal

Bayburt University, Faculty of Applied Sciences

Department of Customs Management, Baberti Campus, 69000 Bayburt-Turkey)

E-mail: huriyekadakal@hotmail.com, mahirkadakal@gmail.com

Abstract: In this paper, using an identity with integral inequalities Hölder, power-mean, Hölder-İşcan and improved power-mean integral inequalities, we get some refinements of the Hermite-Hadamard type integral inequalites for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically ρ-convex.

Key Words: Convex function, trigonometrically convex function, trigonometrically ρ-convex function, Hermite-Hadamard integral inequality.

AMS(2010): 26A51, 26D10, 26D15.

§1. Preliminaries and Fundamentals

Throughout the paper I is a non-empty interval in $\mathbb{R}$. A function $f : I \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)y\right) \le tf\left(x\right) + \left(1-t\right)f\left(y\right)$$

is valid for all $x, y \in I$ and $t \in [0, 1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \emptyset$.

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. One of the most famous inequality for the class of convex functions is so called Hermite-Hadamard inequality, which states that: For a convex mapping $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f\left(a\right)+f(b)}{2}$$

[1]Received November 4, 2022, Accepted December 4, 2022.

is known as the Hermite-Hadamard integral inequality (for more information, see [5]). Since then, some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [2, 3, 4, 11].

Definition 1.1([10]) *Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. We say that $f : I \to \mathbb{R}$ is an h-convex function, or that f belongs to the class $SX(h, I)$, if f is non-negative and for all $x, y \in I$, $\alpha \in (0, 1)$ we have*

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha)f(x) + h(1-\alpha)f(y).$$

If this inequality is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$.

Definition 1.2([7]) *A non-negative function $f : I \to \mathbb{R}$ is called trigonometrically convex function on interval $[a, b]$, if for each $x, y \in [a, b]$ and $t \in [0, 1]$,*

$$f(tx + (1-t)y) \leq \left(\sin\frac{\pi t}{2}\right) f(x) + \left(\cos\frac{\pi t}{2}\right) f(y).$$

We will denote by $TC(I)$ the class of all trigonometrically convex functions on interval I. For $h(t) = \sin\frac{\pi t}{2}$, every trigonometrically convex function is also h-convex function.

Theorem 1.1(Hölder Inequality for Integrals, [9]) *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|^q$, $|g|^q$ are integrable functions on interval $[a, b]$, $q > 1$ then*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}}$$

with equality holding if and only if $A|f(x)|^p = B|g(x)|^q$, almost everywhere, where A and B are constants.

Theorem 1.2(Power-mean Integral Inequality) *Let $q \geq 1$. If f and g are real functions defined on interval $[a, b]$ and if $|f|$, $|f||g|^q$ are integrable functions on interval $[a, b]$, then*

$$\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|\, dx\right)^{1-\frac{1}{q}} \left(\int_a^b |f(x)|\,|g(x)|^q\, dx\right)^{\frac{1}{q}}.$$

In [6], İşcan achieved the following integral inequality which gives better approach than the classical Hölder integral inequality:

Theorem 1.3(Hölder-İşcan Inequality for Integrals) *Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on interval $[a, b]$ and $|f|^q$ and $|g|^q$ are integrable functions on interval*

$[a, b]$, *then*

$$\int_a^b |f(x)g(x)|\,dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_a^b (x-a)\,|f(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.$$

In [8], a different representation of the Hölder-İşcan inequality is given as follows:

Theorem 1.4(Improved power-Mean Integral Inequality) *Let* $p \ge 1$ *and* $\frac{1}{p}+\frac{1}{q}=1$. *If* f *and* g *are real functions defined on interval* $[a,b]$ *and* $|f|$ *and* $|f|\,|g|^q$ *are integrable functions on interval* $[a,b]$, *then*

$$\int_a^b |f(x)g(x)|\,dx \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_a^b (x-a)\,|f(x)|\,dx\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,dx\right)^{\frac{1}{q}}\right\}.$$

§2. Some Integral Inequalities for Trigonometrically ρ-Convexity

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard type integral inequalities for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is trigonometrically ρ-convex. Alomari and Darus [1] used the following lemma.

Lemma 2.1([1]) *Let* $f : I \subset \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I° *where* $a, b \in I$ *with* $a < b$. *If* $f' \in L[a,b]$, *then the following equality holds:*

$$f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx$$
$$=\frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2}+(1-t)a\right)dt+\int_0^1 (t-1)f'\left(tb+(1-t)\frac{a+b}{2}\right)dt\right].$$

Note that we will use the followings in this section:

$$\int_0^1 |t|\sin\frac{\pi t}{2}dt = \int_0^1 |t-1|\cos\frac{\pi t}{2}dt=\frac{4}{\pi^2},$$
$$\int_0^1 |t-1|\sin\frac{\pi t}{2}dt = \int_0^1 |t|\cos\frac{\pi t}{2}dt=\frac{2\pi-4}{\pi^2},$$
$$\int_0^1 |t|^p\,dt = \int_0^1 |t-1|^p\,dt=\frac{1}{p+1},$$

$$\begin{aligned}\int_0^1 \sin\frac{\pi t}{2}dt &= \int_0^1 \cos\frac{\pi t}{2}dt = \frac{2}{\pi} \\ \int_0^1 |t|\,dt &= \int_0^1 |t-1|\,dt = \frac{1}{2} \\ A &= A(u,v) = \frac{u+v}{2} \text{ arithmetic mean}\end{aligned}$$

Theorem 2.1 *Let $f : I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I and assume that $f' \in L[a,b]$. If $|f'|$ is trigonometrically ρ-convex function on interval $[a,b]$, then the following inequality*

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \frac{b-a}{2}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\frac{2\pi-4}{\pi^2}\right)A\left(|f'(a)|, |f'(b)|\right)\right)\end{aligned}$$

holds for $t \in [0,1]$.

Proof Using Lemma 2.1 and the inequalities

$$\left|f'(t\frac{a+b}{2} + (1-t)a\right| \leq \sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right| + \cos\frac{\pi t}{2}\left|f'(a)\right|$$

and

$$\left|f'(tb + (1-t)\left(\frac{a+b}{2}\right)\right| \leq \sin\frac{\pi t}{2}\left|f'(b)\right| + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|$$

we have

$$\begin{aligned}&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\ &\leq \left|\frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2} + (1-t)a\right)dt + \int_0^1 (t-1)f'\left(tb + (1-t)\frac{a+b}{2}\right)dt\right]\right| \\ &\leq \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt + \int_0^1 |t-1|\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|dt\right] \\ &\leq \frac{b-a}{4}\int_0^1 |t|\left[\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right| + \cos\frac{\pi t}{2}\left|f'(a)\right|\right]dt \\ &\quad + \frac{b-a}{4}\int_0^1 |t-1|\left[\sin\frac{\pi t}{2}\left|f'(b)\right| + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|\right]dt \\ &= \frac{b-a}{4}\left[\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1 |t|\sin\frac{\pi t}{2}dt + \left|f'(a)\right|\int_0^1 |t|\cos\frac{\pi t}{2}dt\right] \\ &\quad + \frac{b-a}{4}\left[\left|f'(b)\right|\int_0^1 |t-1|\sin\frac{\pi t}{2}dt + \left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1 |t-1|\cos\frac{\pi t}{2}dt\right] \\ &= \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|\frac{4}{\pi^2} + \left|f'(a)\right|\left(\frac{2\pi-4}{\pi^2}\right) + \left|f'(b)\right|\left(\frac{2\pi-4}{\pi^2}\right) + \left|f'\left(\frac{a+b}{2}\right)\right|\frac{4}{\pi^2}\end{aligned}$$

$$= \frac{b-a}{2}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\frac{2\pi-4}{\pi^2}\right)\frac{|f'(a)|+|f'(b)|}{2}\right)$$
$$= \frac{b-a}{2}\left[\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\frac{2\pi-4}{\pi^2}\right)A\left(|f'(a)|,|f'(b)|\right)\right].$$

This completes the proof of theorem. □

Theorem 2.2 *Let $f : I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I assume that $q > 1$. If the mapping $|f'|^q$ is trigonometrically ρ-convex function on interval $[a, b]$, then the following inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq 2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}\left[\left(A\left|f'\left(\frac{a+b}{2}\right)\right|^q, |f'(a)|^q\right)\right]^{\frac{1}{q}}$$
$$+2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}\left[A\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{\frac{1}{q}},$$

holds for $t \in [0, 1]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 2.1, Hölder's integral inequality and inequalities

$$\left|f'(t\frac{a+b}{2} + (1-t)a\right|^q \leq \sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \cos\frac{\pi t}{2}|f'(a)|^q$$

and

$$\left|f'(tb + (1-t)\left(\frac{a+b}{2}\right)\right|^q \leq \sin\frac{\pi t}{2}|f'(b)|^q + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q$$

which is the trigonometrically ρ-convexity of $|f'|^q$, we get

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \left|\frac{b-a}{4}\left[\int_0^1 tf'\left(t\frac{a+b}{2} + (1-t)a\right)dt + \int_0^1 (t-1)f'\left(tb + (1-t)\frac{a+b}{2}\right)dt\right]\right|$$
$$\leq \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|dt + \int_0^1 |t-1|\left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|dt\right]$$
$$\leq \frac{b-a}{4}\left(\int_0^1 |t|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(t\frac{a+b}{2} + (1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{4}\left(\int_0^1 |t-1|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \cos\frac{\pi t}{2}|f'(a)|^q\right)dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left(\sin\frac{\pi t}{2}|f'(b)|^q + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$
\begin{aligned}
&= \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\int_{0}^{1}\sin\frac{\pi t}{2}dt+|f'(a)|^{q}\int_{0}^{1}\cos\frac{\pi t}{2}dt\right]^{\frac{1}{q}} \\
&\quad+\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[|f'(b)|^{q}\int_{0}^{1}\sin\frac{\pi t}{2}dt+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\int_{0}^{1}\cos\frac{\pi t}{2}dt\right]^{\frac{1}{q}} \\
&= 2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}\left[\left(A\left|f'\left(\frac{a+b}{2}\right)\right|^{q},|f'(a)|^{q}\right)\right]^{\frac{1}{q}} \\
&\quad+2^{\frac{1}{q}}\frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2}{\pi}\right)^{\frac{1}{q}}\left[A\left(|f'(b)|^{q},\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)\right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof of theorem. □

Theorem 2.3 *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I such that $0 < \rho(b-a) < \pi$ and assume that $q \geq 1$. If the mapping $|f'|^q$ is trigonometrically ρ-convex function on interval $[a, b]$, then the following inequality*

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right| \\
&\leq \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\frac{4}{\pi^{2}}+|f'(a)|^{q}\frac{2\pi-4}{\pi^{2}}\right)^{\frac{1}{q}} \\
&\quad+\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(|f'(b)|^{q}\frac{2\pi-4}{\pi^{2}}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\frac{4}{\pi^{2}}\right)^{\frac{1}{q}},
\end{aligned}
$$

holds for $t \in [0, 1]$.

Proof Firstly, let $q > 1$. From Lemma 2.1, well known Power-mean integral inequality and the following inequalities

$$
\begin{aligned}
\left|f'(t\frac{a+b}{2}+(1-t)a\right|^{q} &\leq \sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\cos\frac{\pi t}{2}|f'(a)|^{q} \\
\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^{q} &\leq \sin\frac{\pi t}{2}|f'(b)|^{q}+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^{q},
\end{aligned}
$$

we have

$$
\begin{aligned}
&\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right| \\
&\leq \frac{b-a}{4}\left[\int_{0}^{1}|t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_{0}^{1}|t-1|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right] \\
&\leq \frac{b-a}{4}\left(\int_{0}^{1}|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}|t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
&\quad+\frac{b-a}{4}\left(\int_{0}^{1}|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}|t-1|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^{q}dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$= \frac{b-a}{4}\left(\int_0^1 |t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t|\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \cos\frac{\pi t}{2}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{4}\left(\int_0^1 |t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t-1|\left(\sin\frac{\pi t}{2}\left|f'(b)\right|^q + \cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{4}\left(\int_0^1 |t|\,dt\right)^{1-\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q \int_0^1 |t|\sin\frac{\pi t}{2}dt + \left|f'(a)\right|^q \int_0^1 |t|\cos\frac{\pi t}{2}dt\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{4}\left(\int_0^1 |t-1|\,dt\right)^{1-\frac{1}{q}}\left(\left|f'(b)\right|^q \int_0^1 |t-1|\sin\frac{\pi t}{2}dt + \left|f'\left(\frac{a+b}{2}\right)\right|^q \int_0^1 |t-1|\cos\frac{\pi t}{2}dt\right)^{\frac{1}{q}}$$
$$= \frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q \frac{4}{\pi^2} + \left|f'(a)\right|^q \frac{2\pi-4}{\pi^2}\right)^{\frac{1}{q}}$$
$$+\frac{b-a}{4}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\left|f'(b)\right|^q \frac{2\pi-4}{\pi^2} + \left|f'\left(\frac{a+b}{2}\right)\right|^q \frac{4}{\pi^2}\right)^{\frac{1}{q}}.$$

For $q = 1$ we use the estimates from the proof of Theorem 2.1, which also follow step by step the above estimates. This completes the proof of theorem. □

Corollary 2.1 *Under the assumption of Theorem* 2.3 *with* $q = 1$, *we have the following inequality:*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{4}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{2\pi-4}{\pi^2}A\left(\left|f'(a)\right|, \left|f'(ab|\right)\right)\right)$$

This inequality coincides with inequality in Theorem 2.1.

Theorem 2.4 *Let* $f : I \to \mathbb{R}$ *be a continuously differentiable function, let* $a < b$ *in* I *assume that* $q > 1$. *If the mapping* $|f'|^q$ *is trigonometrically* ρ*-convex function on interval* $[a, b]$, *then the following inequality*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\le \frac{b-a}{4}\left\{\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2\pi-4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{4}{\pi^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q + \frac{2\pi-4}{\pi^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{2\pi-4}{\pi^2}\left|f'(b)\right|^q + \frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|f'(b)\right|^q + \frac{2\pi-4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\},$$

holds for $t \in [0, 1]$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 2.1, Hölder-İşcan integral inequality and inequalities

$$\begin{aligned} \left|f'(t\frac{a+b}{2}+(1-t)a\right|^q &\le \sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\left|f'(a)\right|^q \\ \left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q &\le \sin\frac{\pi t}{2}\left|f'(b)\right|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q \end{aligned}$$

which is the trigonometrically ρ-convexity of $|f'|^q$, we obtain

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \\ &\le \frac{b-a}{4}\left[\int_0^1 |t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|dt+\int_0^1 |t-1|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt\right] \\ &\le \frac{b-a}{4}\left\{\left(\int_0^1(1-t)\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\ &\quad+\left(\int_0^1 t\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\quad+\left(\int_0^1(1-t)\left|t-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\ &\quad\left.+\left(\int_0^1 t\left|t-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\ &\le \frac{b-a}{4}\left\{\left(\int_0^1(1-t)\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right. \\ &\quad+\left(\int_0^1 t\left|t\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}} \\ &\quad+\left(\int_0^1(1-t)\left|t-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)\left(\sin\frac{\pi t}{2}\left|f'(b)\right|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}} \\ &\quad\left.+\left(\int_0^1 t\left|t-1\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 t\left(\sin\frac{\pi t}{2}\left|f'(b)\right|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right\} \\ &= \frac{b-a}{4}\left\{\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{2\pi-4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{4}{\pi^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right. \\ &\quad+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{2\pi-4}{\pi^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\ &\quad+\left(\frac{1}{p+2}\right)^{\frac{1}{p}}\left(\frac{2\pi-4}{\pi^2}\left|f'(b)\right|^q+\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\ &\quad\left.+\left(\frac{1}{(p+1)(p+2)}\right)^{\frac{1}{p}}\left(\frac{4}{\pi^2}\left|f'(b)\right|^q+\frac{2\pi-4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\}, \end{aligned}$$

where

$$\begin{aligned}
\int_0^1 (1-t)\,|t|^p\,dt &= \int_0^1 t\,|t-1|^p\,dt = \frac{1}{(p+1)(p+2)}, \\
\int_0^1 t\,|t|^p\,dt &= \int_0^1 (1-t)\,|t-1|^p\,dt = \frac{1}{p+2}, \\
\int_0^1 (1-t)\sin\frac{\pi t}{2}dt &= \int_0^1 t\cos\frac{\pi t}{2}dt = \frac{2\pi-4}{\pi^2}, \\
\int_0^1 (1-t)\cos\frac{\pi t}{2}dt &= \int_0^1 t\sin\frac{\pi t}{2}dt = \frac{4}{\pi^2}.
\end{aligned}$$

This completes the proof. □

Theorem 2.5 *Let $f : I \to \mathbb{R}$ be a continuously differentiable function, let $a < b$ in I assume that $q \geq 1$. If the mapping $|f'|^q$ is trigonometrically ρ-convex function on interval $[a,b]$, then the following inequality*

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
&\leq \frac{b-a}{4}\left\{ \left(\frac{1}{6}\right)^{1-\frac{1}{q}} \left(\frac{16-4\pi}{\pi^3}\left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{16-4\pi}{\pi^3}|f'(a)|^q \right)^{\frac{1}{q}} \right. \\
&\quad + \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left(\frac{8\pi-16}{\pi^3}\left| f'\left(\frac{a+b}{2}\right)\right|^q + \frac{2\pi^2-16}{\pi^3}|f'(a)|^q \right)^{\frac{1}{q}} \\
&\quad + \left(\frac{1}{3}\right)^{1-\frac{1}{q}} \left(\frac{2\pi^2-16}{\pi^3}|f'(b)|^q + \frac{8\pi-16}{\pi^3}\left| f'\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}} \\
&\quad \left. + \left(\frac{1}{6}\right)^{1-\frac{1}{q}} \left(\frac{16-4\pi}{\pi^3}|f'(b)|^q + \frac{16-4\pi}{\pi^3}\left| f'\left(\frac{a+b}{2}\right)\right|^q \right)^{\frac{1}{q}} \right\}
\end{aligned}$$

holds for $t \in [0,1]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 2.1, Improved power-mean integral inequality and inequalities

$$\begin{aligned}
\left| f'(t\frac{a+b}{2} + (1-t)a \right|^q &\leq \sin\frac{\pi t}{2}\left| f'\left(\frac{a+b}{2}\right)\right|^q + \cos\frac{\pi t}{2}|f'(a)|^q \\
\left| f'(tb + (1-t)\left(\frac{a+b}{2}\right) \right|^q &\leq \sin\frac{\pi t}{2}|f'(b)|^q + \cos\frac{\pi t}{2}\left| f'\left(\frac{a+b}{2}\right)\right|^q
\end{aligned}$$

which is the trigonometrically ρ-convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \\
&\leq \frac{b-a}{4}\left[\int_0^1 |t|\left| f'\left(t\frac{a+b}{2} + (1-t)a\right)\right| dt + \int_0^1 |t-1|\left| f'\left(tb + (1-t)\frac{a+b}{2}\right)\right| dt \right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\,|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1 t\,|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|t|\left|f'\left(t\frac{a+b}{2}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 (1-t)\,|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|t-1|\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1 t\,|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|t-1|\left|f'(tb+(1-t)\left(\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&\leq \frac{b-a}{4}\left\{\left(\int_0^1 (1-t)\,|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|t|\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\,|f'(a)|^q\right)dt\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1 t\,|t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|t|\left(\sin\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\cos\frac{\pi t}{2}\,|f'(a)|^q\right)dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^1 (1-t)\,|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-t)\,|t-1|\left(\sin\frac{\pi t}{2}\,|f'(b)|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1 t\,|t-1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\,|t-1|\left(\sin\frac{\pi t}{2}\,|f'(b)|^q+\cos\frac{\pi t}{2}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}\right\}\\
&= \frac{b-a}{4}\left\{\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{16-4\pi}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{16-4\pi}{\pi^3}\,|f'(a)|^q\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{8\pi-16}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{2\pi^2-16}{\pi^3}\,|f'(a)|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{3}\right)^{1-\frac{1}{q}}\left(\frac{2\pi^2-16}{\pi^3}\,|f'(b)|^q+\frac{8\pi-16}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\\
&\left.+\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\frac{16-4\pi}{\pi^3}\,|f'(b)|^q+\frac{16-4\pi}{\pi^3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right\},
\end{aligned}
$$

where

$$
\begin{aligned}
\int_0^1 (1-t)\,|t|\,dt &= \int_0^1 t\,|t-1|\,dt=\frac{1}{6}\\
\int_0^1 t\,|t|\,dt &= \int_0^1 (1-t)\,|t-1|\,dt=\frac{1}{3}\\
\int_0^1 (1-t)\,|t|\sin\frac{\pi t}{2}dt &= \int_0^1 (1-t)\,|t|\cos\frac{\pi t}{2}dt=\frac{16-4\pi}{\pi^3}\\
\int_0^1 t\,|t-1|\sin\frac{\pi t}{2}dt &= \int_0^1 t\,|t-1|\cos\frac{\pi t}{2}dt=\frac{16-4\pi}{\pi^3}\\
\int_0^1 t\,|t|\sin\frac{\pi t}{2}dt &= \int_0^1 (1-t)\,|t-1|\cos\frac{\pi t}{2}dt=\frac{8\pi-16}{\pi^3},\\
\int_0^1 t\,|t|\cos\frac{\pi t}{2}dt &= \int_0^1 (1-t)\,|t-1|\sin\frac{\pi t}{2}dt=\frac{2\pi^2-16}{\pi^3}.
\end{aligned}
$$

This completes the proof. □

Corollary 2.2 *If we choose $q = 1$ in Theorem* 2.5*, we get the following inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx \right| \leq \frac{b-a}{2}\left(\frac{4}{\pi^2}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\frac{2\pi-4}{\pi^2}\right) A\left(\left|f'(a)\right|, \left|f'(b)\right|\right)\right).$$

This inequality coincides with the inequality in Theorem 2.1*.*

References

[1] M. Alomari, M. Darus, Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense, *Appl Math Lett*, 2010, (23): 1071-1076

[2] SS. Dragomir, Refinements of the Hermite-Hadamard integral inequality for log-convex functions, *Aust. Math. Soc. Gaz.* 28(3) (2001), 129-134.

[3] SS. Dragomir and CEM. Pearce, Selected topics on Hermite-Hadamard inequalities and its applications, *RGMIA Monograph,* 2002.

[4] SS. Dragomir, J. Pečarić and LE.Persson, Some inequalities of Hadamard type, *Soochow Journal of Mathematics*, 21 (3)(2001), pp. 335-341.

[5] J. Hadamard, Etude sur les proprietes des fonctions entieres en particulier d'une fonction consideree par Riemann, *J. Math. Pures Appl.* 58(1893), 171-215.

[6] İ. İşcan, New refinements for integral and sum forms of Hölder inequality, *Journal of Inequalities and Applications*, 304 (2019) 1-11.

[7] H. Kadakal, Hermite-Hadamard type inequalities for trigonometrically convex functions, *Scientific Studies and Research. Series Mathematics and Informatics*, 2018, 28(2), 19-28.

[8] Kadakal M., İscan, I,İ. Kadakal H. and Bekar K., On improvements of some integral inequalities. *Honam Mathematical Journal*, 43(3) (2021), 441-452.

[9] D.S. Mitrinovic, J.E. Pecaric and A.M. Fink, *Classical and New Inequalities in Analysis*, The Netherlands: Kluwer Academic Publishers, 1993.

[10] S. Varošanec, On *h*-convexity, *J. Math. Anal. Appl.* 326 (2007) 303-311.

[11] G. Zabandan, A new refinement of the Hermite-Hadamard inequality for convex functions, *J. Inequal. Pure Appl. Math.* 10 2(2009), Article ID 45.

International J.Math. Combin. Vol.4(2022), 32-40

The Number of Chains of Subgroups in the Lattice of Subgroups of Group $Z_m \times A_n, n \leq 6, m \leq 3$

Mike Ogiugo[1], Amit Seghal[2], S. A. Adebisi[3] and M. EniOluwafe[4]

1. Department of Mathematics,School of Science, Yaba College of Technology, Nigeria
2. Department of Mathematics, Pt. Neki Ram Sharma Government College, Rohtak, India
3. Department of Mathematics, Faculty of Science, University of Lagos, Nigeria
4. Department of Mathematics, Faculty of Science, University of Ibadan, Nigeria

E-mail: ekpenogiugo@gmail.com, amit_sehgal_iit@yahoo.com
adesinasunday@yahoo.com, michael.enioluwafe@gmail.com

Abstract: In this paper, we established the number of chains of subgroups in the subgroup lattice of the Cartesian product of the alternating group and cyclic group using computational technique induced by the set of representatives of isomorphism classes of subgroups.

Key Words: Subgroups, alternating group, chains of subgroups and fuzzy subgroups.

AMS(2010): 20D06, 20D30, 20E15.

§1. Introduction

All groups are finite throughout this paper.For a given group G, the lattice of subgroups is $(L(G), \leq)$ here by $L(G)$ mean all subgroups of G and the partial order $\leq$ works as set inclusion. If chain of subgroups of G contains G then it is called rooted (more precisely G-rooted) otherwise it is known as unrooted.

In this paper the study of chains of subgroups describes the set containing all chains of subgroups of G, which ends in with G.A formula of lattice of a finite cyclic group, for number of chains of subgroups was given by Tărnăuceanu and Bentea [1] by giving its one variable generating function. J.M. Oh in his paper [2] determined the number of subgroups of a finite cyclic group of 4n by giving its multi variables generating function. The problem of counting chains of subgroups in the lattice of subgroups of for any given group G got attention of researchers specially classifying fuzzy subgroups of G under a specific type of equivalence relation (see [3], [4]).

§2. Preliminaries

A set of subgroups of G fully ordered by set inclusion is a chain of subgroups of group G. In this paper, the chain of subgroups of G which end in G.

[1]Received November 15, 2022, Accepted December 6, 2022.

Let us suppose a finite group G and $\mu\colon G \mapsto [0,1]$ be its fuzzy subgroup of G. By Putting $\mu(G) = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ where $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. Then, μ determines the chain of subgroups of G which ends in group G:

$$\mu G\alpha_1 \subset \mu G\alpha_2 \subset \cdots \subset \mu G\alpha_m = G$$

Also for any element x of group G and $i = \overline{1,r}$, we get

$$\mu(x) = \alpha_i \Leftrightarrow i = max\{j | x \in \mu G\alpha_j\} \Leftrightarrow x \in \mu G_{\alpha_i} \setminus \mu G\alpha_{1-r}$$

The authors of [7] identified the necessary and sufficient condition for equivalence of two fuzzy subgroups μ, η of G with respect to $\sim$, which is $\mu \sim \eta$ if and only if set of level subgroups of μ and η are same.For a fuzzy subgroup of group G, the corresponding equivalence classes are closely connected to the chains of subgroups in group G in this case.To determine these classes, we calculate the number of all chains of subgroups of G that terminate in G.

Let G be a finite group and $\delta(G)$ be the number of chains of subgroups of G that terminate in G. We have two kind of subgroup chains following:

(1) $G_1 \subset G_2 \subset \cdots \subset G_k = G$ with $G_1 \neq \{e\}$;

(2) $\{e\} \subset G_2 \subset \cdots \subset G_k = G$.

It is clear that the numbers of chains of types (1) and (2) are equal. So,

$$\delta(G) = 2x_k.$$

Obviously, this problem of counting all chains of subgroups of G depends entirely on the lattice of subgroups of G and not on the group itself. This leads to a more general problem.

Theorem 2.1 *Let $\delta(G)$ be the number of subgroup chains of group G that terminates in G. Then*

$$\delta(G) = \sum_{distictH \in Iso(G)} \delta(H) \times n(H),$$

where $Iso(G)$ is the set of representatives of isomorphism classes of subgroups of G, $n(H)$ denotes the size of the isomorphism class with representative H.

Proof Let fixes $\delta(H_1) = \delta(H_\alpha) = 1$, for which H_1 is the trivial group of and H_α is the improper subgroup of G for any $H_i \in Iso(G)$ and $i = \overline{1,\alpha}$. Then

$$\begin{aligned}\delta(G) &= n(H_1) * \delta(H_1) + n(H_1) * \delta(H_2) + n(H_3)\delta(H_3) + \cdots + n(H_\alpha)\delta(H_\alpha)\\ &= \sum_{H_i \in Iso(G)} \delta(H_i) \times n(H_i)\\ &= 2 + \sum_{distictH_i \in Iso(G)} \delta(H_i) \times n(H_i)\end{aligned}$$

This completes the proof. □

In this work, Theorem 2.1 is used to obtain the number of chains of subgroups of G that terminates in G. It also follows that

(*i*) $\delta(Z_p) = 2$, where p is prime;

(*ii*) $\delta(Z_{pq}) = 6$, where p and q are distinct prime;

(*iii*) $\delta(Z_{p^2}) = 4$, where p is any prime;

(*iv*) $\delta(Z_p \times Z_p) = 2p + 4$, where p is any prime;

(*v*) $\delta(Z_p \times Z_p \times Z_p) = 2p^3 + 8p^2 + 8p + 8$, where p is any prime;

(*vi*) $\delta(Z_{pq} \times Z_p) = 16 + 10p$, where p and q are distinct primes.

Above result are special cases of Theorems or Corollaries $5.1, 5.2, 5.4, 5.5, 5.6$ of [6] and Section 3 of [7].

Lemma 2.1([3]) *For groups S_3, A_4 and S_4, we have $\delta(S_3) = 10$, $\delta(A_4) = 24$ and $\delta(S_4) = 232$ respectively.*

Lemma 2.2([8]) *Let G be Dihedral group of order 2p, where p is any prime then , $\delta(G) = 4+2p$*

§3. The Number of Chains of Subgroups of $Z_2 \times A_n, n \leq 6$

Theorem 3.1 *The number of chains of subgroups of $Z_2 \times A_3$ is 6*

Proof The direct product A_3 and Z_2 is isomorphic to a cyclic group of order 6. Then, $\delta(Z_2 \times A_3) = 6$. □

Theorem 3.2 *The number of chains of subgroups of $Z_2 \times A_4$ is 200.*

Proof Notice that $Z_2 \times A_4$ is a non-Abelian group of order 24, it has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 7]$, $[Z_3, 4]$, $[Z_2 \times Z_2, 7]$, $[Z_6, 4]$, $[(Z_2 \times Z_2 \times Z_2), 1]$, $[A_4, 1]$ and $[(Z_2 \times A_4), 1]$.

So,

$$\begin{aligned}\delta(Z_2 \times A_4) &= 1 + \delta(H_e) + 7 \times \delta(Z_2) + 4 \times \delta(Z_3) + 7 \times \delta(Z_2 \times Z_2)\\ &\quad +4 \times \delta(Z_6) + \delta(Z_2 \times Z_2 \times Z_2) + \delta(A_4) = 200.\end{aligned}$$

This completes the proof. □

Theorem 3.3 *The number of chains of subgroups of group $Z_2 \times A_5$ is 3292.*

Proof Notice that $Z_2 \times A_5$ is non-Abelian of order 120, it has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 31]$, $[Z_3, 10]$, $[Z_2 \times Z_2, 35]$, $[Z_5, 6]$, $[Z_6, 10]$, $[(Z_2 \times Z_2 \times Z_2), 5]$, $[Z_{10}, 6]$, $[S_3, 20]$, $[D_5, 12]$, $[D_6, 10]$, $[D_{10}, 6]$, $[Z_2 \times A_4, 5]$, $[A_4, 5]$, $[A_5, 1]$ and $[(Z_2 \times A_5), 1]$.

So,

$$\begin{aligned}\delta(Z_2 \times A_5) &= 1 + \delta(H_e) + 31 \times \delta(Z_2) + 10 \times \delta(Z_3) + 35 \times \delta(Z_2 \times Z_2) + 6 \times \delta(Z_5) \\ &+10 \times \delta(Z_6) + 5 \times \delta(Z_2 \times Z_2 \times Z_2) + 6 \times \delta(Z_{10}) + 20 \times \delta(S_3) \\ &+12 \times \delta(D_5) + 10 \times \delta(D_6) + 6 \times \delta(D_{10}) + 5 \times \delta(A_4) + \delta(A_5) = 3292.\end{aligned}$$

This completes the proof. □

Proposition 3.1(By Proposition 3 of [8]) *Let the wreath product of the cyclic groups :* $(Z_3 \times Z_3) \ltimes Z_2$ *and* $\delta(Z_3 \times Z_3) \ltimes Z_4)$. *Then,* $\delta(Z_3 \times Z_3) \ltimes Z_2) = 158$ *and* $\delta(Z_3 \times Z_3) \ltimes Z_2) = 352$.

Theorem 3.4 *Suppose that* G *be the Cartesian product of group* $D_p \times Z_q$, *where* p, q *are distinct prime numbers, then*

$$\delta(G) = \begin{cases} 16p + 20 & \text{if } q = 2 \\ 14p + 12 & \text{if } p = q \\ 10p + 16 & \text{if } p \neq q, q \neq 2 \end{cases}$$

Proof Our proof is divided into 3 cases following:

Case 1. $q = 2$.

Notice that $D_p \times Z_q$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$H_e,\ Z_2(2p+1 \text{ times}),\ Z_p, Z_{pq}, D_p \ (2 \text{ times}),\ Z_2 \times Z_2 \text{ and } D_p \times Z_q.$$

Then,

$$\begin{aligned}\delta(D_p \times Z_q) &= 1 + \delta(H_e) + (2p+1) \times \delta(Z_2) + \delta(Z_p) \\ &+\delta(Z_{pq}) + p \times \delta(Z_2 \times Z_2) + 2 \times \delta(D_p = 16p + 20.\end{aligned}$$

Case 2. $p = q$.

Notice that $D_p \times Z_q$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$H_e,\ Z_2 \ (p \text{ times}),\ Z_p \ (p+1 \text{ times}), Z_{pq} \ (p \text{ times}),\ D_p, Z_p \times Z_p \text{ and } D_p \times Z_q.$$

Then,

$$\begin{aligned}\delta(D_p \times Z_q) &= 1 + \delta(H_e) + p \times \delta(Z_2) + (p+1) \times \delta(Z_p) \\ &+p \times \delta(Z_{pq}) + \delta(Z_p \times Z_p) + \delta(D_p = 14p + 12.\end{aligned}$$

Case 3. $p \neq q$, $q \neq 2$.

Notice that $D_p \times Z_q$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$H_e,\ Z_2\ (p \text{ times}),\ Z_p, Z_q, Z_{pq}, Z_{2p}\ (p \text{ times}),\ D_p \text{ and } D_p \times Z_q.$$

Then,

$$\begin{aligned}\delta(D_p \times Z_q) &= 1 + \delta(H_e) + p \times \delta(Z_2) + \delta(Z_p) \\ &+\delta(Z_q) + \delta(Z_{pq}) + p \times \delta(Z_{2p}) + \delta(D_p) = 10p + 16.\end{aligned}$$

This completes the proof. □

Theorem 3.5 *Let G be the direct product of Z_2 and $Z_3^2 \ltimes Z_2$, then, $\delta(G) = 1572$.*

Proof Notice that $Z_2 \times (Z_3 \times Z_3) \ltimes Z_2$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$[e, 1],\ [Z_2, 19],\ [Z_2 \times Z_2, 9],\ [Z_3 \times Z_3, 1],\ [Z_3, 4],\ [Z_6, 4],\ [Z_6 \times Z_3, 1],$$
$$[S_3, 24],\ [D_6, 12],\ [(Z_3 \times Z_3) \ltimes Z_2, 2] \text{ and } [Z_2 \times (Z_3 \times Z_3) \ltimes Z_2, 1].$$

So,

$$\begin{aligned}\delta(G) &= 1 + \delta(H_e) + 19 \times \delta(Z_2) + 4 \times \delta(Z_3) + 9 \times \delta(Z_2 \times Z_2) \\ &+\delta(Z_3 \times Z_3) + 4 \times \delta(Z_6) + \delta(Z_6 \times Z_3) + 24 \times \delta(S_3) \\ &+12 \times \delta(D_6) + 2 \times \delta((Z_3 \times Z_3) \ltimes Z_2) = 1572.\end{aligned}$$

This completes the proof. □

Theorem 3.6 *Let G be the direct product of Z_2 and $Z_3^2 \ltimes Z_4$,then, $\delta(G) = 4136$.*

Proof Notice that $Z_2 \times (Z_3 \times Z_3) \ltimes Z_4$ is a non-Abelian, A-group of order 72. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$[e, 1],\quad [Z_2, 19],\quad [Z_2 \times Z_2, 9],\quad [Z_3 \times Z_3, 1],\quad [Z_3, 4],\quad [Z_4, 18],\quad [Z_6, 4],$$
$$[Z_4 \times Z_2, 9],\quad [Z_6 \times Z_3, 1],\quad [S_3, 24],\quad [D_6, 12],\quad [(Z_3 \times Z_3) \ltimes Z_2, 2],$$
$$[(Z_3 \times Z_3) \ltimes Z_4, 2],\ [Z_2 \times (Z_3 \times Z_3) \ltimes Z_2, 1] \text{ and } [Z_2 \times (Z_3 \times Z_3) \ltimes Z_4, 1].$$

So,

$$\begin{aligned}\delta(G) &= 1 + \delta(H_e) + 19 \times \delta(Z_2) + 4 \times \delta(Z_3) + 18 \times \delta(Z_4) + 9 \times \delta(Z_2 \times Z_2) \\ &+\delta(Z_3 \times Z_3) + 9 \times \delta(Z_4 \times Z_2) + 4 \times \delta(Z_6) + \delta(Z_6 \times Z_3) + 24 \times \delta(S_3) \\ &+12 \times \delta(D_6) + 2 \times \delta((Z_3 \times Z_3) \ltimes Z_2) + 2 \times \delta((Z_3 \times Z_3) \ltimes Z_4) \\ &+\delta(Z_2 \times (Z_3 \times Z_3) \ltimes Z_2) = 4136.\end{aligned}$$

This completes the proof. □

Theorem 3.7 *The number of chains of subgroups of $Z_2 \times A_6$ is 301320.*

Proof Notice that $Z_2 \times A_6$ is non-Abelian of order 720. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e,1]$, $[Z_2,91]$, $[Z_2\times Z_2,165]$, $[Z_3,40]$, $[Z_4,90]$, $[Z_5,36]$, $[Z_6,40]$, $[Z_{10},36]$, $[Z_2\times Z_2\times Z_2,30]$, $[Z_3\times Z_3,10]$, $[Z_2\times D_4,45]$, $[Z_2\times S_4,30]$, $[Z_4\times Z_2,45]$, $[(Z_6\times Z_3),10]$, $[Z_2\times(Z_3\times Z_3)\ltimes Z_4,10]$, $[(Z_3\times Z_3)\ltimes Z_2,20]$, $[Z_3\times Z_3\ltimes Z_4,20]$, $[Z_2\times(Z_3\times Z_3)\ltimes Z_2,10]$, $[Z_{10},6]$, $[S_3,240]$, $[S_4,60]$, $[D_4,180]$, $[D_5,72]$, $[D_6,120]$, $[D_{10},36]$, $[Z_2\times A_4,30]$, $[A_4,30]$, $[A_5,12]$, $[A_6,1]$, $[(Z_2\times A_5),12]$ and $[(Z_2\times A_6),1]$.

So,

$$\begin{aligned}\delta(Z_2\times A_6) \;=\;& 1+\delta(H_e)+91\times\delta(Z_2)+40\times\delta(Z_3)+90\times\delta(Z_4)\\ &+36\times\delta(Z_5)+40\times\delta(Z_6)+36\times\delta(Z_{10})+165\times\delta(Z_2\times Z_2)\\ &+20\times\delta(Z_2\times Z_2\times Z_2)+30\times\delta(Z_2\times A_4)+10\times\delta(Z_2\times(Z_3\times Z_3)\ltimes Z_2)\\ &+10\times\delta(Z_2\times(Z_3\times Z_3)\ltimes Z_4)+12\times\delta(Z_2\times A_5)+45\times\delta(Z_2\times D_4)\\ &+30\times\delta(Z_2\times S_4)+10\times\delta(Z_3\times Z_3)+45\times\delta(Z_4\times Z_2)+10\times\delta(Z_6\times Z_3)\\ &+20\times\delta((Z_3\times Z_3)\ltimes Z_2)+20\times\delta((Z_3\times Z_3)\ltimes Z_4)+30\times\delta(A_4)\\ &+12\times\delta(A_5)+\delta(A_6)+240\times\delta(S_3)+60\times\delta(S_4)+180\times\delta(D_4)\\ &+72\times\delta(D_5)+120\times\delta(D_6)+36\times\delta(D_{10})=301320.\end{aligned}$$

This completes the proof. □

Then, we get the following theorem.

Theorem 3.8 *The number of chains of subgroups $Z_2\times A_n, n\le 6$, then*

$$\delta(Z_2\times A_n)=\begin{cases}6 & n=3\\ 200 & n=4\\ 3292 & n=5\\ 301320 & n=6\end{cases}$$

§4. The Number of Chains of Subgroups of $Z_3\times A_n, n\le 6$

Theorem 4.1 *The number of chains of subgroups of $Z_3\times A_3$ is* 10.

Proof The direct product A_3 and Z_3 is isomorphic to $Z_3\times Z_3$. So, $\delta(Z_3\times A_3)=10$. □

Theorem 4.2 *The number of chains of subgroups of $Z_3\times A_4$ is 208.*

Proof Notice that $Z_3\times A_4$ is a non-Abelian group of order 36. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e,1], [Z_2,3], [Z_2\times Z_2,1], [Z_3\times Z_3,4], [Z_3,13], [Z_6,3], [Z_6\times Z_2,1], [A_4,3]$ and $[Z_3\times A_4,1]$.

So,

$$\begin{aligned}\delta(Z_3 \times A_4) &= 1+\delta(H_e)+3\times\delta(Z_2)+13\times\delta(Z_3)+3\times\delta(Z_6)\\ &+\delta(Z_2\times Z_2)+4\times\delta(Z_3\times Z_3)+\delta(Z_6\times Z_2)+3\times\delta(A_4)=208.\end{aligned}$$

This completes the proof. □

Theorem 4.3 *The number of Chains of Subgroups of $Z_3 \times A_5$ is* 3440.

Proof Notice that $Z_3 \times A_5$ is non-Abelian group of order 180. It is isomorphic to the general linear group (2,4). The group has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 15]$, $[Z_2 \times Z_2, 5]$, $[Z_3 \times Z_3, 10]$, $[Z_3, 11]$, $[Z_3 \times D_5, 10]$, $[Z_3 \times S_3, 10]$, $[Z_3 \times A_4, 5]$, $[Z_5, 6]$, $[Z_6, 15]$, $[Z_{15}, 6]$, $[S_3, 10]$, $[A_4, 15]$, $[A_5, 1]$ and $[(Z_3 \times A_5), 1]$.

So,

$$\begin{aligned}\delta(Z_3 \times A_5) &= 1+\delta(H_e)+15\times\delta(Z_2)+11\times\delta(Z_3)+6\times\delta(Z_5)\\ &+15\times\delta(Z_6)+5\times\delta(Z_2\times Z_2)+10\times\delta(Z_3\times Z_3)\\ &+10\times\delta(Z_3\times D_5)+10\times\delta(Z_3\times S_3)+5\times\delta(Z_3\times A_4)\\ &+10\times\delta(S_3)+6\times\delta(Z_{15})+15\times\delta(A_4)+\delta(A_5)=3440.\end{aligned}$$

This completes the proof. □

Theorem 4.4 *Let G be the direct product of Z_3 and $Z_3^2 \ltimes Z_2$,then,* $\delta(G) = 1314$.

Proof Notice that $Z_3 \times (Z_3 \times Z_3) \ltimes Z_2$ has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 9]$, $[Z_3 \times Z_3, 13]$, $[Z_3, 13]$, $[Z_6, 9]$, $[Z_3 \times S_3, 12]$, $[S_3, 12]$, $[Z_3 \times Z_3 \times Z_3, 1]$, $[(Z_3 \times Z_3) \ltimes Z_2, 1]$ and $[Z_3 \times (Z_3 \times Z_3) \ltimes Z_2, 1]$.

So,

$$\begin{aligned}\delta(G) &= 1+\delta(H_e)+9\times\delta(Z_2)+13\times\delta(Z_3)+\delta(Z_3\times Z_3\times Z_3)\\ &+13\times\delta(Z_3\times Z_3)+9\times\delta(Z_6)+12\times\delta(Z_3\times S_3)\\ &+12\times\delta(S_3)+\delta((Z_3\times Z_3)\ltimes Z_2)=1314.\end{aligned}$$

This completes the proof. □

Theorem 4.5 *Let G be the direct product of Z_3 and $Z_3^2 \ltimes Z_4$,then,* $\delta(G) = 3160$.

Proof Notice that $Z_3 \times (Z_3 \times Z_3) \ltimes Z_4$ is a non-Abelian group of order 108. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$[e, 1]$, $[Z_2, 9]$, $[Z_3 \times Z_3, 13]$, $[Z_3, 13]$, $[Z_4, 9]$, $[Z_6, 9]$, $[Z_{12}, 9]$,

$$[Z_3 \times S_3, 12], \quad [S_3, 12], \quad [Z_3 \times Z_3 \times Z_3, 1], \quad [(Z_3 \times Z_3) \ltimes Z_2, 1],$$
$$[(Z_3 \times Z_3) \ltimes Z_4, 1], [Z_3 \times (Z_3 \times Z_3) \ltimes Z_2, 1] \text{ and } [Z_3 \times (Z_3 \times Z_3) \ltimes Z_4, 1].$$

So,

$$\begin{aligned}\delta(G) &= 1 + \delta(H_e) + 9 \times \delta(Z_2) + 13 \times \delta(Z_3) + 9 \times \delta(Z_4) \\ &+9 \times \delta(Z_6) + 9 \times \delta(Z_{12}) + \delta(Z_3 \times Z_3 \times Z_3) \\ &+13 \times \delta(Z_3 \times Z_3) + 12 \times \delta(Z_3 \times S_3) + 12 \times \delta(S_3) \\ &+\delta((Z_3 \times Z_3) \ltimes Z_2) + \delta((Z_3 \times Z_3) \ltimes Z_4) + \delta(Z_3 \times (Z_3 \times Z_3) \ltimes Z_2) = 3160.\end{aligned}$$

This completes the proof. □

Theorem 4.6 *The number of Chains of Subgroups of $Z_3 \times A_6$ is 212848.*

Proof Notice that $Z_3 \times A_6$ is non-Abelian of order 1080. It has the following set of representatives of isomorphism classes of subgroups with their sizes:

$$[e, 1], [Z_2, 45], [Z_2 \times Z_2, 30], [Z_3, 121], [Z_4, 45], [Z_5, 36], [Z_6, 45], [Z_{12}, 45], [Z_{15}, 36],$$
$$[Z_3 \times Z_3 \times Z_3, 10], [Z_3 \times Z_3, 130], [Z_3 \times D_4, 45], [Z_3 \times D_5, 36], [Z_3 \times S_3, 120],$$
$$[Z_3 \times S_4, 30], \quad [(Z_3 \times A_4), 30], \quad [(Z_3 \times Z_3) \ltimes Z_4, 10], \quad [(Z_3 \times Z_3) \ltimes Z_2, 10],$$
$$[Z_3 \times (Z_3 \times Z_3) \ltimes Z_4, 10], \quad [Z_3 \times (Z_3 \times Z_3) \ltimes Z_2, 10], \quad [Z_6 \times Z_2, 30], \quad [S_3, 120],$$
$$[S_4, 30], [D_4, 45] , [D_5, 36], [A_4, 90], [A_5, 12], [A_6, 1], [GL(2, 4), 12] \text{ and } [(Z_3 \times A_6), 1]$$

from the isomorphism class. So,

$$\begin{aligned}\delta(Z_3 \times A_6) &= 1 + \delta(H_e) + 45 \times \delta(Z_2) + 121 \times \delta(Z_3) + 45 \times \delta(Z_4) \\ &+36 \times \delta(Z_5) + 45 \times \delta(Z_6) + 45 \times \delta(Z_{12}) + 36 \times \delta(Z_{15}) \\ &+30 \times \delta(Z_2 \times Z_2) + 10 \times \delta(Z_3 \times Z_3 \times Z_3) + 130 \times \delta(Z_3 \times Z_3) \\ &+30 \times \delta(Z_3 \times A_4) + 10 \times \delta(Z_2 \times (Z_3 \times Z_3) \ltimes Z_2) \\ &+10 \times \delta(Z_2 \times (Z_3 \times Z_3) \ltimes Z_4) + 12 \times \delta(Z_3 \times A_5) \\ &+45 \times \delta(Z_3 \times D_8) + 120 \times \delta(Z_3 \times S_3) + 30 \times \delta(Z_3 \times S_4) \\ &+30 \times \delta(Z_6 \times Z_2) + 10 \times \delta((Z_3 \times Z_3) \ltimes Z_2) + 10 \times \delta((Z_3 \times Z_3) \ltimes Z_4) \\ &+10 \times \delta(Z_3 \times (Z_3 \times Z_3) \ltimes Z_2) + 10 \times \delta(Z_3 \times (Z_3 \times Z_3) \ltimes Z_4) \\ &+90 \times \delta(A_4) + 12 \times \delta(A_5) + \delta(A_6) + 120 \times \delta(S_3) + 30 \times \delta(S_4) \\ &+45 \times \delta(D_8) + 36 \times \delta(D_{10}) = 212848.\end{aligned}$$

This completes the proof. □

Then, we get the following theorem.

Theorem 4.7 *The number of chains of subgroups* $Z_3 \times A_n, n \leq 6$ *, then*

$$\delta(Z_2 \times A_n) = \begin{cases} 10 & n = 3 \\ 208 & n = 4 \\ 3440 & n = 5 \\ 212848 & n = 6 \end{cases}$$

§5. Conclusion

The study of the number of chains of subgroups in the lattice of subgroups for larger groups are interesting and give rise to potential applications to quantum computing and coding. In this paper, it is clearly observed from the results that the number of chains of subgroups of G do not depend on the order of G but the lattice of subgroups of G as in Theorems 3.8 and 4.7.

References

[1] M. Tărnăuceanu and L. Bentea, On the number of fuzzy subgroups of finite abelian groups, *Fuzzy Sets and Systems*, Vol.159 (2008), 1084-1096,

[2] J.M. Oh, The number of chains of subgroups of a finite cycle group, *European Journal of Combinatorics*, Vol. 33, No. 2(2012), 259-266,

[3] M.Ogiugo and M.EniOluwafe, Classifying a class of the fuzzy subgroups of the alternating group A_n, *Imhotep Mathematical Proceedings* Vol.4(2017), 27-33.

[4] A.C. Volf, Counting fuzzy subgroups and chains of subgroups, *Fuzzy Systems & Artificial Intelligence*, Vol. 10(2004), 191-200.

[5] M. Ogiugo,Amit Seghal and M.EniOluwafe, The number of chains of subgroups for certain finite symmetric groups (Submitted).

[6] Amit Sehgal, Sarita Sehgal, P. K. Sharma and Manjeet Jakhar, Fuzzy subgroups of a finite abelian group, *Advances in Fuzzy Sets and Systems*, Vol. 21(4)(2016), 291-302.

[7] Amit Sehgal and Manjeet Jakhar, The number of fuzzy subgroups for finite Abelian p-group of rank three, *Advances in Fuzzy Mathematics*, Vol.12(4)(2017), 1035-1045.

[8] Ogiugo M.and Amit Seghal, The number of chains of subgroups for certain finite alternating groups, *Annals of Pure and Applied Mathematics*, Vol 22(1)(2020), 65-70.

[9] Leili Kamali Ardekania, and Bijan Davvaz. Classifying fuzzy (normal) subgroups of the group $D_{2p} \times Z_q$ and finite groups of order $n \leq 20$, *Journal of Intelligent & Fuzzy Systems,* Vol. 33(2017), 3615-3627, DOI:10.3233/JIFS-17301.

[10] *GAP – Groups, Algorithms and Programming*, Version 4.8.7; (https://www.gap-system.org)

International J.Math. Combin. Vol.4(2022), 41-49

Support Strongly Edge Irregular FSGs

N.R. Santhi Maheswari

PG and Research Department of Mathematics
G.Venkataswamy Naidu College, Kovilpatti - 628 502, Tamil Nadu, India

L.Subha Lakshmi

Research Scholar (Reg. No.21212052092001), PG and Research Department of Mathematics
G.Venkataswamy Naidu College, Kovilpatti
Affiliated to Manonmaniam Sundaranar University, Abishekapatti-627 012, Tirunelveli, Tamil Nadu, India

E-mail: nrsmaths@yahoo.com, lakshmansubha2@gmail.com

Abstract: The discussions on support strongly edge irregular fuzzy soft graph(FSG), support strongly totally edge irregular FSG is made. Necessary condition for a FSG to be both support strongly and support strongly totally edge irregular is given. We also derive the conditions, the FSG satisfies, if it is support strongly and support strongly totally edge irregular.

Key Words: Support edge of FSG, support edge strongly irregular FSG, support edge irregular FSG, support edge totally strongly irregular FSG.

AMS(2010): 05C12, 03E72, 05C72.

§1. Introduction

The soft set theory expanded by Molodtstov dealt with uncertainity and unclear objects, and the notion of soft set theory was given by him in 1999 [8]. Maji.et al [7] evolved a new set with soft sets and fuzzy which proved much effective and defined basic operations on them with some applications to it. Akram and Nawaz looked into properties of fuzzy soft graphs [9]. Santhi and Sekar discussed edge irregular fuzzy graphs in [6]. They also worked on strongly edge irregular fuzzy graphs in [5]. Akilandeswari introduced and discussed properties of edge degree in a fuzzy soft graph and edge regular FSG [11]. Somasundari introduced support of a vertex in FSG [13] and discussed support neighbourly irregular FSG. Subha Lakshmi and Santhi Maheswari dealt with support strongly irregular FSG [17]. The support and total support of edge in FSG was introduced by Subha Lakshmi and NRS [18] and some properties of neighbourly edge irregular FSG was also discussed. These promoted the idea to develop support-SEI and support- STEI FSGs.

§2. Preliminaries

Definition 2.1 *A fuzzy graph G is a nonempty set V with functions $\sigma : V \to [0,1]$ and*

[1]Received October 26, 2022, Accepted December 8, 2022.

$\mu : V \times V \to [0,1]$: $\forall\ u, v \in V$, $\mu(uv) \le \sigma(u) \wedge \sigma(v)$, *where* σ *is fuzzy vertex set and* μ *is fuzzy edge set in* G.

Definition 2.2 *A pair* (F, A) *is soft set over the universal set* U, *where* $A \subseteq E$ *and* $F : a \to \mathscr{P}(U)$. *That is a soft set over* U *is parameterized collection of subsets of* U.

Definition 2.3 *An FSG* $\widetilde{G} = (\mathscr{G}^*, \mathscr{F}, \mathscr{K}, \mathscr{A})$ *is a 4-tuple:*

(1) $\mathscr{G}^*$ *is crisp graph;*
(2) $\mathscr{A}$ *is the parameter set;*
(3) $\widetilde{\mathscr{F}}, \mathscr{A}$ *is fuzzy soft set over vertex set V;*
(4) $\widetilde{\mathscr{K}}, \mathscr{A}$ *is fuzzy soft set over edge set E.*
The pair $\left(\widetilde{\mathscr{F}}(a), \widetilde{\mathscr{K}}(a)\right)$ *is fuzzy (sub)graph of* $\mathscr{G}^*, \forall\ a \in \mathscr{A}$.

The membership value of the edge in an FSG is given as

$$\widetilde{\mathscr{K}}(a)(xy) \le \min\left\{\widetilde{\mathscr{F}}(a)(x), \widetilde{\mathscr{F}}(a)(y)\right\}$$

and a FSG $\left(\widetilde{\mathscr{F}}(a), \widetilde{\mathscr{K}}(a)\right)$ is denoted as $\widetilde{\mathscr{H}}(a)$.

Definition 2.4 *If* $\widetilde{G}$ *is an FSG, then the degree of a vertex u is given as*

$$d_{\widetilde{G}}(u) = \sum_{a_i \in A} (\sum_{u \neq v} \widetilde{\mathscr{K}}(a_i)(uv)).$$

Definition 2.5 *If* $\widetilde{G}$ *is an FSG, then degree of edge uv is given as*

$$d_{\widetilde{G}}(uv) = d_{\widetilde{G}}(u) + d_{\widetilde{G}}(v) - 2(\sum_{a_i \in A} \widetilde{\mathscr{K}_{a_i}}(uv)).$$

Definition 2.6 *If* $\widetilde{G}$ *is an FSG, then total degree of edge uv is given as*

$$td_{\widetilde{G}}(uv) = d_{\widetilde{G}}(uv) + \sum_{a_i \in A} \widetilde{\mathscr{K}_{a_i}}(uv).$$

Certainly, it can be also expressed as

$$td_{\widetilde{G}}(uv) = d_{\widetilde{G}}(u) + d_{\widetilde{G}}(v) - \left(\sum_{a_i \in A} \widetilde{\mathscr{K}_{a_i}}(uv)\right).$$

Definition 2.7 *An FSG* $\widetilde{G}$ *is irregular if* $\widetilde{\mathscr{H}}(e)$ *is irregular for all* $a \in \mathscr{A}$.

Definition 2.8 *The support* $(2 - degree)$ *of a vertex* u *in an FSG* $\widetilde{G}$ *is the addition of degrees of its adjacent vertices and denoted as* $s_{\widetilde{G}}(u)$, *whereas its total support is given as*

$$ts_{\widetilde{G}}(u) = s_{\widetilde{G}}(u) + \sum_{a_i \in A} \widetilde{\mathscr{F}}(a_i)(u).$$

Definition 2.9 *The support* $(2-degree)$ *of an edge in a FSG is the sum of edge degrees of edges which are adjacent to given edge and can be defined as*

$$s_{\widetilde{G}}(uv) = \sum_{e_i \in N(uv), a_i \in A} \widetilde{\mathscr{K}}(a_i)(uv).$$

Definition 2.10 *The total support of an edge in an FSG is found as*

$$ts_{\widetilde{G}}(uv) = s_{\widetilde{G}}(uv) + \sum_{a_i \in A} \mathscr{K}(a_i)(uv).$$

§3. Support Strongly EI (s-SEI) and Support Strongly Totally EI (s-STEI) FSG

Definition 3.1 *A FSG is support-SEI if all the edges in the graph are adjacent to edges with distinct edge supports,i.e., all the edges in the FSG have different edge supports.*

Example 3.2 Consider the below example $\widetilde{G}$, where $V = \{u, v, w, x, y, z\}$ and $E = \{uv, vw, vx, xy, xz\}$, the parameter set $\mathscr{A} = a_1, a_2$, where $a_1 = \{uv, vw, vx, xy\}$ and $a_2 = \{vw, vx, xy, xz\}$, $\widetilde{\mathscr{H}}(a) = (\widetilde{\mathscr{F}}(a), \widetilde{\mathscr{K}}(a))$ be the fuzzy soft subgraph of $\widetilde{G}$.

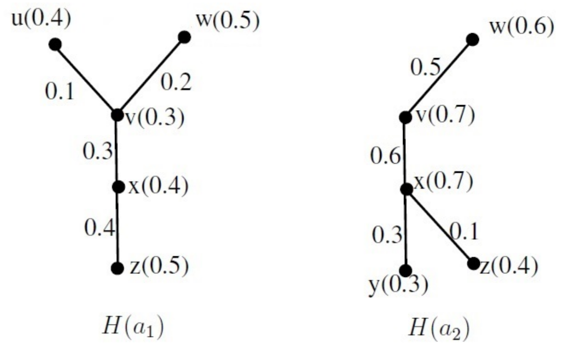


Figure 3.1

$\widetilde{F}$	**u**	**v**	**w**	**x**	**y**	**z**	$\widetilde{K}$	**uv**	**vw**	**vx**	**xy**	**xz**
(a_1)	0.4	0.3	0.5	0.4	0	0.5	(a_1)	0.1	0.2	0.3	0	0.4
(a_2)	0	0.7	0.6	0.7	0.3	0.4	(a_2)	0	0.5	0.6	0.3	0.1

Table 3.1

The degree of vertices are respectively given as $d_{\widetilde{G}}(u) = 0.1$, $d_{\widetilde{G}}(v) = 1.7$, $d_{\widetilde{G}}(w) = 0.7$, $d_{\widetilde{G}}(x) = 1.7$, $d_{\widetilde{G}}(y) = 0.3$, $d_{\widetilde{G}}(z) = 0.5$, the degree of the edges are respectively $d_{\widetilde{G}}(uv) = 0.1 + 1.7 - 2(0.1) = 1.6$, $d_{\widetilde{G}}(vw) = 1.0$, $d_{\widetilde{G}}(vx) = 1.6$, $d_{\widetilde{G}}(xy) = 1.4$, $d_{\widetilde{G}}(xz) = 1.2$ and the support of the edges are given as $s_{\widetilde{G}}(uv) = 2.6$, $s_{\widetilde{G}}(vw) = 3.2$, $s_{\widetilde{G}}(vx) = 5.2$, $s_{\widetilde{G}}(xy) = 2.8$, $s_{\widetilde{G}}(xz) = 3.0$. The support of all the edges in the FSG are different, so it is support SEI.

Definition 3.3 *A FSG is support-STEI when no edge in the graph is adjacent to edges with same total edge support. In other words, all the edges in the FSG have different total edge support.*

Example 3.4 Consider the example, with parameters $\mathscr{A} = \{a_1, a_2\}$, where $a_1 = \{uv, vw, wx\}$, $a_2 = \{uv, vw, wx\}$.

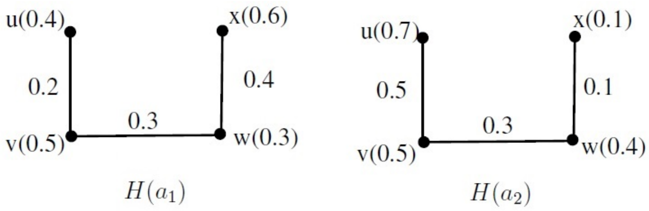


Figure 3.2

$\widetilde{F}$	**u**	**v**	**w**	**x**	$\widetilde{K}$	**uv**	**vw**	**wx**
(a_1)	0.4	0.5	0.3	0.6	(a_1)	0.2	0.3	0.4
(a_2)	0.7	0.5	0.4	0.1	(a_2)	0.5	0.3	0.1

Table 3.2

The degree of vertices and edges are found respectively to be $d_{\widetilde{G}}(u) = 0.7$, $d_{\widetilde{G}}(v) = 1.3$, $d_{\widetilde{G}}(w) = 1.1$, $d_{\widetilde{G}}(x) = 0.5$, $d_{\widetilde{G}}(uv) = 0.6$, $d_{\widetilde{G}}(vw) = 1.2$, $d_{\widetilde{G}}(wx) = 0.6$. And the support of the above edges are $s_{\widetilde{G}}(uv) = 1.2$, $s_{\widetilde{G}}(vw) = 1.2$, $s_{\widetilde{G}}(wx) = 1.2$. Thus, the total support of the above edges $ts_{\widetilde{G}}(uv) = 1.9$, $ts_{\widetilde{G}}(vw) = 1.8$, $ts_{\widetilde{G}}(wx) = 1.7$ are all not same, so it is support-STEI.

Remark 3.5 A support-SEI FSG need not be support-STEI, and vice versa.

Example 3.6 Consider below example, with parameters $\mathscr{A} = \{a_1, a_2, a_3\}$, where $a_1 = \{uv, uw\}$, $a_2 = \{uv, uw\}$, $a_3 = \{uv, uw, vw\}$.

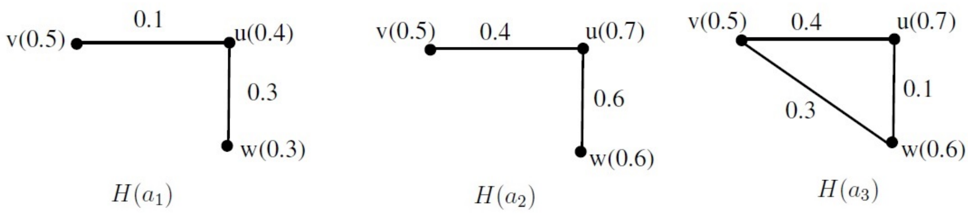


Figure 3.3

$\widetilde{\mathscr{F}}$	**u**	**v**	**w**	$\widetilde{\mathscr{K}}$	**uv**	**vw**	**wu**
(a_1)	0.4	0.5	0.3	(a_1)	0.1	0	0.3
(a_2)	0.7	0.5	0.6	(a_2)	0.4	0	0.6
(a_3)	0.7	0.5	0.6	(a_2)	0.4	0.3	0.1

Table 3.3

The support of the edges are distinct here, and consider the total support of the edges given as $ts_{\widetilde{G}}(uv) = 4.0$, $ts_{\widetilde{G}}(vw) = 4.5$, $ts_{\widetilde{G}}(wu) = 4.5$, where the edges vw and uw have same total edge support, so it is not support-STEI, but support-SEI.

Example 3.7 Consider the example 3.4, which is support-STEI, but not support-SEI.

Example 3.8 Consider the below example which is both support-SEI and support-STEI. The FSG is with the parameters $A = \{a_1, a_2\}$, where $a_1 = \{uv, vw, wu\}$, $a_2 = \{uv, vw, wu\}$.

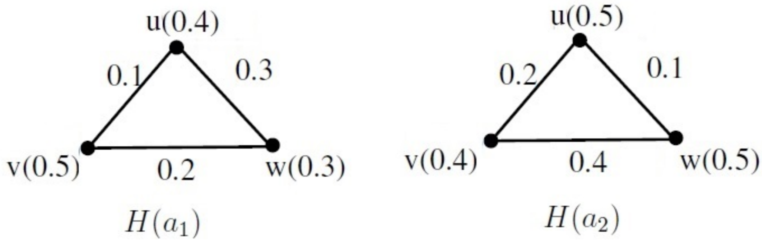


Figure 3.4

$\widetilde{F}$	**u**	**v**	**w**	$\widetilde{K}$	**uv**	**vw**	**wu**
(a_1)	0.4	0.5	0.3	(a_1)	0.1	0.3	0.3
(a_2)	0.5	0.4	0.5	(a_2)	0.2	0.4	0.1

Table 3.4

The degree of vertices are given as $d_{\widetilde{G}}(u) = 0.3 + 0.4 = 0.7$, $d_{\widetilde{G}}(v) = 0.9$, $d_{\widetilde{G}}(w) = 1.0$ and the edge degrees are given as $d_{\widetilde{G}}(uv) = 0.7 + 0.9 - 0.6 = 1.0$, $d_{\widetilde{G}}(vw) = 0.7$, $d_{\widetilde{G}}(uw) = 0.9$. The support of the edges are $s_{\widetilde{G}}(uv) = 1.6$, $s_{\widetilde{G}}(vw) = 1.9$, $s_{\widetilde{G}}(wu) = 1.7$.

The total support of the edges are $ts_{\widetilde{G}}(uv) = 1.9$, $ts_{\widetilde{G}}(vw) = 2.5$, $ts_{\widetilde{G}}(wu) = 2.1$ and they are all different so it is support-STEI.

§4. Properties of Support-SEI and Support-STEI FSGs

Theorem 4.1 *A FSG $\widetilde{G}$, is support-SEI, if its fuzzy soft (sub)graph $\widetilde{H}(a) = (\widetilde{F}(a), \widetilde{K}(a))$ is support-SEI, for all $a \in A$.*

Remark 4.2 The converse of Theorem 4.1, namely if G, is support-SEI, then its fuzzy subgraphs need not be s-SEI.

Example 4.3 Examine Example 3.2, which is support-SEI, while taking $\widetilde{H}(a_1)$, the support of the edges are $s_{\widetilde{H}(a_1)}(uv) = 2.2$, $s_{\widetilde{H}(a_1)}(vw) = 2.8$, $s_{\widetilde{H}(a_1)}(vx) = 4.8$, $s_{\widetilde{H}(a_1)}(xy) = 2.4$, $s_{\widetilde{H}(a_1)}(xz) = 2.2$ are all not distinct, so $H(a_1)$ is not support-SEI.

Theorem 4.4 *A FSG $\widetilde{G}$, is support-STEI, if its fuzzy soft (sub)graphs $\widetilde{H}(a) = (\widetilde{F}(a), \widetilde{K}(a))$ is support-STEI, for all $a \in A$.*

Remark 4.5 The converse need not be true.

Example 4.6 Given a FSG as below.

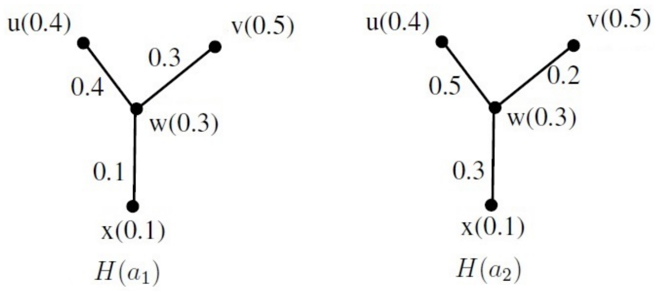


Figure 4.1

This FSG is support-STEI, but while considering $\widetilde{H}(a_1)$, the total support of the edges vw and vx are same, so it is not support-STEI.

Theorem 4.7 *If $\widetilde{G}$ is support-SEI FSG, then it is support-EI.*

Proof Given the FSG $\widetilde{G}$ is support-SEI, $\Rightarrow$ the support of all the edges in the graph are distinct, $\Rightarrow$ there exist at least one edge with edge support different from others, thus it is support-EI FSG. □

Theorem 4.8 *If $\widetilde{G}$ is support-STEI FSG, then it is support-TEI.*

Proof Given the FSG $\widetilde{G}$ is support-STEI, $\Rightarrow$ the total support of all the edges in the graph are distinct, $\Rightarrow$ there exist atleast one edge with total edge support different from others, thus it is support-TEI FSG. □

Remark 4.9 It is not necessary for a FSG, which is support-EI, to be support-SEI.

Remark 4.10 It is not necessary for a FSG, which is support-TEI, to be support-STEI.

Theorem 4.11 *If $\widetilde{G}$ is support-SEI FSG, then it is support edge neighbourly irregular.*

Proof Consider the given FSG, $\widetilde{G}$ is support-SEI. Then the support of all the edges are different, $\Rightarrow$ no two adjacent edges in $\widetilde{G}$ have same edge support. Therefore, it is support NEI FSG. □

Theorem 4.12 *If $\widetilde{G}$ is support-STEI FSG, then it is support edge totally neighbourly irregular.*

Proof Consider the given FSG, $\widetilde{G}$ is support-STEI. Then the total support of all the edges are distinct, $\Rightarrow$ no two adjacent edges in $\widetilde{G}$ have same total edge support. Therefore, it is support edge totally neighbourly irregular FSG. □

Remark 4.13 A support edge neighbourly EI FSG, not necesaarily be support-SEI.

Remark 4.14 A support edge totally neighbourly irregular FSG, not necesaarily be support-STEI.

Example 4.15 Consider the below example with parameters $A = \{a_1, a_2\}$, where $a_1 =$

$\{uv, vw, vx\}$, $a_2 = \{vw, vx, xy, xz\}$.

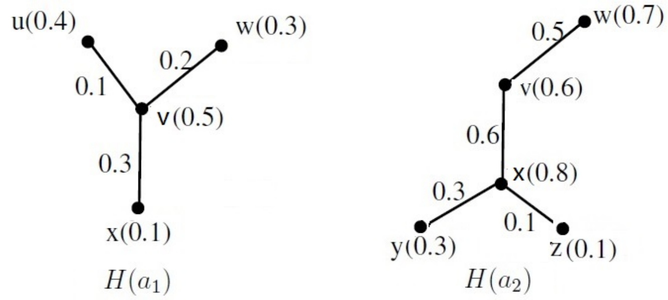


Figure 4.2

The support of the edges are given as $s_G(uv) = 2.2$, $s_G(vw) = 2.8$, $s_G(vx) = 4.8$, $s_G(xy) = 2.4$, $s_G(xz) = 2.2$. Whereas the total support of the edges are $ts_G(uv) = 2.3$, $ts_G(vw) = 3.5$, $ts_G(vx) = 5.7$, $ts_G(xy) = 2.7$, $ts_G(xz) = 2.3$. The given example is support edge NI, and support edge totally NI, but not support-SEI and support-STEI.

Theorem 4.16 *A FSG, whose edges have same support, then it is support-STEI iff* $\sum \widetilde{\mathscr{K}}(a)(e_i)$ *are all different,* $\forall a_i \in A$ *and* $e_i \in E$.

Proof Suppose $\widetilde{G}$, a FSG in which the support of all the edges are same, let that be $s_{\widetilde{G}}(e_i) = m$, for all $e_i \in E$. Given $\widetilde{G}$, is support-STEI, $\Rightarrow$ all the edges have different total edge support. $\Rightarrow$ no edges which are adjacent have same total edge support. Let e_j and e_{j+1} be adjacent edges, $\Rightarrow ts_{\widetilde{G}}(e_j) \neq ts_{\widetilde{G}}(e_{j+1})$.

And so,

$$\Rightarrow s_{\widetilde{G}}(e_j) + \sum \widetilde{K}(a_i)(e_j) \neq s_{\widetilde{G}}(e_{j+1}) + \sum \widetilde{K}(a_i)(e_{j+1})$$

$$\Rightarrow m + \sum \widetilde{K}(a_i)(e_j) \neq m + \sum \widetilde{K}(a_i)(e_{j+1}).$$

This implies

$$\sum \widetilde{K}(a_i)(e_j) \neq \sum \widetilde{K}(a_i)(e_{j+1}),$$

this holds for all the edges in $\widetilde{G}$.

Conversely assume $\sum \widetilde{K}(a_i)(e_i)$ are all distinct $\forall e_i \in E$. Suppose $\widetilde{G}$ is not support-STEI, $\Rightarrow$ there is at least a pair of edges with same total edge support. Let e_n and e_{n+1} be such edges, then

$$ts_{\widetilde{G}}(e_n) = ts_{\widetilde{G}}(e_{n+1}) \Rightarrow s_{\widetilde{G}}(e_n) + \sum \widetilde{K}(a_i)(e_n) = s_{\widetilde{G}}(e_n) + \sum \widetilde{K}(a_i)(e_{n+1}),$$

implies that

$$m + \sum \widetilde{K}(a_i)(e_n) = m + \sum \widetilde{K}(a_i)(e_{n+1}) \Rightarrow \sum \widetilde{K}(a_i)(e_n) = \sum \widetilde{K}(a_i)(e_{n+1}),$$

which $\Leftrightarrow$, thus $\widetilde{G}$ is support-STEI. □

Theorem 4.17 *A FSG,* $\widetilde{G}$*, in which* $\sum \widetilde{K}(a_i)(e_i)$ *are same for all* $e_i \in E$*, then* $\widetilde{G}$ *is not* *support-EI and support-edge totally irregular FSG.*

Proof Let $\widetilde{G}$ be a FSG, where

$$\sum \widetilde{K}(a_i)(e_i) = k, \quad \forall e_i \in E.$$

Let us consider a path in the graph $uvwx$, such that we have edges uv, vw, wx, where

$$\sum \widetilde{K}(a_i)(uv) = \sum \widetilde{K}(a_i)(vw) = \sum \widetilde{K}(a_i)(wx) = k.$$

The degree of the edges are $d_{\widetilde{G}}(uv) = d_{\widetilde{G}}(vw) = d_{\widetilde{G}}(wx) = k$. Then the support of these edges are $s_{\widetilde{G}}(uv) = s_{\widetilde{G}}(vw) = s_{\widetilde{G}}(wx) = k$, hence it is not support EI, also the total support of the edges are also same, hence not support edge totally irregular also. □

Theorem 4.18 *If a FSG, $\widetilde{G}$ is support-SEI and $\sum \widetilde{K}(a_i)(e_i)$ are different for all edges in E, then the given FSG is support-STEI.*

Proof Given that $\widetilde{G}$ is support-SEI, then no edge in the FSG has same support, i.e., $s_{\widetilde{G}}(e_i)$ are distinct for all $e_i \in E$. Let us consider the edges e_k and e_{k+1} having different edge supports. The support total of these edges are

$$s_{\widetilde{G}} e_k + \sum \widetilde{K}(a_i)(e_k) \quad \text{and} \quad s_{\widetilde{G}} e_{k+1} + \sum \widetilde{K}(a_i)(e_{k+1}),$$

respectively. It is given that $\sum \widetilde{K}(a_i)(e_i)$ are different for all the edges, so $ts_{\widetilde{G}} e_k \neq ts_{\widetilde{G}} e_{k+1}$, this holds for all pairs of edges in the given FSG. Thus we conclude that $\widetilde{G}$ is support-STEI.□

Theorem 4.19 *A FSG $\widetilde{G}$ is suppport-STEI, and $\sum \widetilde{K}(a_i)(e_i)$ is same for all the edges, then it is support-SEI, when $s_{\widetilde{G}}(e_i)$ are all distinct.*

Proof Given a FSG, $\widetilde{G}$ which is support-STEI. $\Rightarrow ts_{\widetilde{G}}(e_i)$ are different $\forall e_i \in E$. Given that

$$\sum \widetilde{K}(a_i)(e_i) = k, \quad \forall e_i \in E,$$

consider any pair of edges (say) e_n and e_m, then

$$ts_{\widetilde{G}}(e_n) \neq ts_{\widetilde{G}}(e_m) \;\Rightarrow\; s_{\widetilde{G}}(e_n) + \sum \widetilde{K}(a_i)(e_n) \neq s_{\widetilde{G}}(e_m) + \sum \widetilde{K}(a_i)(e_m),$$

$$s_{\widetilde{G}}(e_n) + k \neq s_{\widetilde{G}}(e_m) + k \;\Rightarrow\; s_{\widetilde{G}}(e_n) \neq s_{\widetilde{G}}(e_m).$$

This implies support of any pair of edges are distinct, thus $\widetilde{G}$ is support-SEI. □

References

[1] Amei Yu, Mei Lu and Feng Tian, On the spectral radius of graphs, *Linear Algebra and its Applications*, 387(2004), 41-49.

[2] Dasong Cao, Bounds on eigen values and chromatic numbers, *Linear algebra and its Ap-*

plications, 270(1998),1-13.

[3] Gary Chartrand, Paul Erdos, Ortrud R. Oellermann, How to define an irregular graphs, *College Math. Journal*, 19(1988).

[4] N.R. Santhi Maheswari, K. Amutha, Support neighbourly edge irregular graphs, *International Journal of Recent Technology and Engineering*, 8(3)(2019), September.

[5] N.R.Santhi Maheswari and C.Sekar, On strongly edge irregular fuzzy graphs, *Kragujevac Journal of Mathematics*, Vol.40, No.1(2016), 125-135.

[6] N.R.Santhi Maheswari and C.Sekar, On edge irregular fuzzy graphs, *International Journal of Mathematics and Soft Computing*, Vol.6, No.2(2016), 131-143.

[7] P. K. Maji, A. R. Roy, R. Biswas, An application of soft sets in a decision making problem, *Computers and Mathematics with Applications*, 44(89)(2002), 1077-1083.

[8] D. Molodtsov, Soft set theory-first results, *Computers and Mathematics with Applications*, 37(4-5)(1999), 19-31.

[9] Muhammad Akram and Saira Nawaz, On fuzzy soft graphs, *Italian Journal of Pure and Applied Mathematics -N*, 34(2015), 497-514.

[10] S.P.Nandhini and E.Nandhini, Strongly irregular fuzzy graphs, *International Journal of Mathematical Archieve*, Vol.5, No. 5(2016), 110-114.

[11] B.Akilandeswari, On edge regular fuzzy soft graph, *International journal for Research in Applied Science & Engineering Technology*, 2019, 1479-88.

[12] J. Krishnaveni and N.R. Santhi Maheswari, On support neighbourly irregular fuzzy graphs, *International Journal of Advanced Research in Engineering and Technology*, 11(1)(2020), 147-154.

[13] B.Somasundari and S.Mariammal, On support neighbourly irregular fuzzy soft graphs, *National Seminar Proceedings*, 2022, 144-161.

[14] A.Nagoor Gani and S.R.Latha, On irregular fuzzy graphs,(2012),vol.6, pp.517-523.

[15] L.Subha Lakshmi and N.R.Santhi Maheswari, On pseudo highly and pseudo strongly edge irregular graphs, *Proceedings of International Conference- ICRRMCSA'22*, 2022, 61-69.

[16] N.R.SanthiMaheswari and K.Amutha, Stepwise edge irregular graphs, *Advances in Mathematics: Scientific Journal*(Special issue), No.3(2019), 766-771.

[17] L.Subha Lakshmi and N.R.Santhi Maheswari, On support strongly irregular fuzzy soft graphs, *Third International Conference on Applied Mathematics and Intellectual Property Rights*, 2022.

[18] L.Subha Lakshmi and N.R.Santhi Maheswari, Support edge irregularity of FSG, *3rd International Conference on Mathematical Modelling, Analysis and Computing*, 2022.

[19] L.Subha Lakshmi and N.R.Santhi Maheswari, Support highly edge irregular FSGs, *NeuroQuantology*, Vol.20, No.v(2022), 897-900.

International J.Math. Combin. Vol.4(2022), 50-59

Optimality of the Generalization New Class of Caputo-Katugampola Fractional Optimal Control Problems

Moataz Abbas Holel and Sameer Qasim Hasan

(Department of Mathematics, College of Education, Mustansiriyah University, Baghdad, Iraq)

E-mail: moataz.abbas92@gmail.com

Abstract: In this work, a generalization of a new class of fractional order optimal control problems with Caputo-Katugampola derivative of order $\alpha, \beta \in (0, 1)$ and $\rho > 0$ was studied and considers the final time t_f is free. The necessary optimality conditions with Lagrange multipliers $\lambda(t) \in \mathbb{R}$ of fractional order optimal control problems were derived in case $t \in [A, t_f]$ and $a < A$. The formula for the integral by parts has been proven for the left Caputo-Katugampola fractional derivative that contributes to the finding and deriving the necessary optimality conditions.

Key Words: Fractional calculus, Caputo-Katugampola derivative, necessary optimality conditions, hamiltonian system.

AMS(2010): 49C20.

§1. Introduction

Fractional calculus contributes to many important aspects and fields of life such as science, engineering and physical applications, since the fractional order models gives an accurate description of non-linear and complex natural systems than integer order models , which prompted researchers to interest in studying these systems in advanced methods and in more than one way.

We mention some of its applications with optimal fractional control (FOCP) that are subject to dynamic constraints with the objective function problems, chaotic systems [1], bioscience [2], conformable to the FOCP [3], aerospace [4], economic [5] and so on [6-8]. The reader can refer to the books [9-12] for more details on fractional calculus.

Agrawal O. P. [13] is using Riemann-Liouville (R-LFD) to provide a general formulation and find an approximate solution for a class of (FOCPs).

Study of the necessary and sufficient optimality conditions for fractional optimal control problems for one-dimensional with Caputo fractional derivative by Pooseh S., et al. [14] and for system with JiHuan He's fractional derivative by Sayevand K., et al. [15] and for composition(FOCPs) by QasimHasan S., and Abbas Holel M. [16].

New types of fractional operators were introduced by U. Katugampola [17], These are done by generalizing the (R-L) and Hadamard fractional integrals, same author, introduce a

[1]Received November 15, 2022, Accepted December 9, 2022.

new fractional derivative [18], and see [19]. Almeida R., et al. [20] proves the existence and uniqueness theorem by using the Caputo-Katugampola derivative for a fractional Cauchy type problem.A definition has been given with some properties of generalized fractional derivatives by Jarad F., et al. [21].

This paper aims to study a generalization of a new class of (FOCPs) with left (C-KD) of order $\alpha, \beta \in (0,1)$ and $\rho > 0$,let $\mathcal{F}$, g are two differentiable functions with domain $[a, +\infty) \times \mathbb{R}^2$ and $\Psi : [a, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a differentiable function. And considering $(\mathcal{R}_1, \mathcal{R}_2) \neq (0,0)$, x_A is a fixed real number $t \in [A, t_f]$ and $a < A$, as follows:

$$minimize \;\; \mathbf{J}(x, u, t_f) = \int_A^{t_f} \mathcal{F}\left(x(t), u(t), t\right) dt + \Psi\left(t_f, x(t_f)\right), \tag{1}$$

subject to dynamic control system

$$\mathcal{R}_1 {}^{CK}_{\;a}D_t^{\alpha,\rho} x(t) + \mathcal{R}_2 {}^{CK}_{\;a}D_t^{\beta,\rho} x(t) = g\left(x(t), u(t), t\right), \tag{2}$$

$$x(A) = x_A. \tag{3}$$

This paper contains four sections: in Section 2, preliminaries and prove integration by parts formula for (C-KFD). The necessary optimality conditions are studied for a generalization class of (C-K FOCPs) in details in Section 3. The conclusions are introduced in Section 4.

§2. Preliminaries

The basic definitions of (FDs) and (FIs) are presented with proof of theorems are used in work.

Definition 2.1([18, 20]) *Let $\alpha > 0$, $\rho > 0$ and an interval $[a,b]$ of $\mathbb{R}$, where $0 < a < b$ The left and right (R-KFI) and (R-KFD) of a function $f \in L^1([a,b])$ are defined by*

$$ {}^{RK}_{\;a}D_t^{-\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{4}$$

$$ {}^{RK}_{\;t}D_b^{-\alpha,\rho} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\frac{\tau^\rho - t^\rho}{\rho}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau^{1-\rho}}, \tag{5}$$

$$ {}^{RK}_{\;a}D_t^{\alpha,\rho} f(t) = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right) \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} f(\tau) d\tau, \tag{6}$$

$$ {}^{RK}_{\;t}D_b^{\alpha,\rho} f(t) = \frac{-\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right) \int_t^b \frac{\tau^{\rho-1}}{(\tau^\rho - t^\rho)^\alpha} f(\tau) d\tau, \tag{7}$$

Definition 2.2([18, 20]) *Let $\alpha \in (0,1)$, $\rho > 0$ and an interval $[a,b]$ of $\mathbb{R}$, where $0 < a < b$. The left and right (C-KFD) are defined by*

$$ {}^{CK}_{\;a}D_t^{\alpha,\rho} f(t) = {}^{RK}_{\;a}D_t^{\alpha,\rho}[f(t) - f(a)] = \frac{\rho^\alpha}{\Gamma(1-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right) \int_a^t \frac{\tau^{\rho-1}}{(t^\rho - \tau^\rho)^\alpha} [f(\tau) - f(a)] d\tau, \tag{8}$$

$$ {}^{CK}_{\ t}D^{\alpha,\rho}_{b}f(t) = {}^{RK}_{\ t}D^{\alpha,\rho}_{b}[f(t)-f(b)] = \frac{-\rho^{\alpha}}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_t^b \frac{\tau^{\rho-1}}{(\tau^{\rho}-t^{\rho})^{\alpha}}[f(\tau)-f(b)]d\tau. \quad (9) $$

Theorem 2.1 *Let $\alpha \in (0,1)$, $\rho > 0$, then left and right (C-KFD) of a function $f \in \mathbb{C}^1[a,b]$ is given by*

$$ {}^{CK}_{\ a}D^{\alpha,\rho}_{t}f(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_a^t (t^{\rho}-\tau^{\rho})^{-\alpha} f'(\tau)d\tau, \quad (10) $$

$$ {}^{CK}_{\ t}D^{\alpha,\rho}_{b}f(t) = \frac{-\rho^{\alpha}}{\Gamma(1-\alpha)}\int_t^b (\tau^{\rho}-t^{\rho})^{-\alpha} f'(\tau)d\tau. \quad (11) $$

Proof We are proving the left (C-KFD) in (10) using Definition 2.2 in Eq. (8) and let

$$ {}^{CK}_{\ a}D^{\alpha,\rho}_{t}f(t) = \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right)\int_a^t u dv. \quad (12) $$

Now, using integration by part, derive relative to and substitute the result of Eq. (12), to get

$$ \begin{aligned} {}^{CK}_{\ a}D^{\alpha,\rho}_{t}f(t) &= \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}t^{1-\rho}\left[(1-\alpha)(\rho t^{\rho-1})\int_a^t \frac{1}{\rho(1-\alpha)}(t^{\rho}-\tau^{\rho})^{-\alpha} f'(\tau)d\tau\right] \\ &= \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_a^t (t^{\rho}-\tau^{\rho})^{-\alpha} f'(\tau)d\tau. \end{aligned} $$

This completes the proof. □

Theorem 2.2 *Let $f(t) \in C[a,b]$ and $g(t) \in C^1[a,b]$ be two functions and $\alpha \in (0,1)$, $\rho > 0$. Then*

$$ \int_a^b f(t)\cdot {}^{CK}_{\ a}D^{\alpha,\rho}_{t}g(t)dt = \int_a^b \left(g(t)t^{\rho-1}\right) {}^{RK}_{\ t}D^{\alpha,\rho}_{b}(t^{1-\rho}f(t))dt + \left[g(t)\,{}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{b}(t^{1-\rho}f(t))\right]_{t=a}^{t=b}. $$

Proof By using Theorem 2.1 to obtain:

$$ \int_a^b f(t)\cdot {}^{CK}_{\ a}D^{\alpha,\rho}_{t}g(t)dt = \int_a^b f(t)\left[\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_a^t (t^{\rho}-\tau^{\rho})^{-\alpha}\frac{d}{d\tau}g(\tau)d\tau\right]dt. \quad (13) $$

By using the Dirichlet's formula for Eq. (13), to get

$$ \int_a^b f(t)\cdot {}^{CK}_{\ a}D^{\alpha,\rho}_{t}g(t)dt = \int_a^b \frac{d}{dt}g(t)\left[\frac{\rho^{\alpha}}{\Gamma(1-\alpha)}\int_t^b (\tau^{\rho}-t^{\rho})^{-\alpha}\frac{\tau^{1-\rho}f(\tau)}{\tau^{1-\rho}}d\tau\right]dt. \quad (14) $$

Using definition of the right (R-KFI) of $\left(t^{1-\rho}f(t)\right)$ with order $(1-\alpha,\rho)$ in Eq. (14) to get

$$ \int_a^b f(t)\cdot {}^{CK}_{\ a}D^{-(1-\alpha,\rho)}_{t}g(t)dt = \int_a^b \frac{d}{dt}g(t)\,{}^{RK}_{\ t}D^{\alpha,\rho}_{b}(t^{1-\rho}f(t))dt = \int_a^b \frac{d}{dt}g(t)h(t)dt, $$

where $h(t) = {}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{b}\left(t^{1-\rho}f(t)\right)$.

Now, using integration by parts of above equation to obtain:

$$= \left[\begin{array}{c} g(t). \\ {}^{RK}_{t}D_b^{-(1-\alpha,\rho)}\left(t^{1-\rho}f(t)\right) \end{array} \right]_{t=a}^{t=b} - \int_a^b \left(g(t)t^{1-\rho}\right)(-1) \left[\begin{array}{c} \frac{-\rho^{\alpha}}{\Gamma(1-\alpha)}\left(t^{1-\rho}\frac{d}{dt}\right). \\ \int_t^b \left(\tau^{\rho}-t^{\rho}\right)^{-\alpha}\frac{\tau^{1-\rho}f(\tau)}{\tau^{1-\rho}}d\tau \end{array} \right] dt. \tag{15}$$

Using definition of right (R-KFD) of $\left(t^{1-\rho}f(t)\right)$ with order $(1-\alpha,\rho)$ in Eq. (7) to get

$$\int_a^b f(t)\cdot {}^{CK}_{a}D_t^{\alpha,\rho}g(t)dt = \int_a^b \left(g(t)t^{1-\rho}\right)\cdot {}^{RK}_{t}D_b^{\alpha,\rho}\left(t^{1-\rho}f(t)\right)dt + \left[g(t)\,{}^{RK}_{t}D_b^{-(1-\alpha,\rho)}(t^{1-\rho}f(t))\right]_{t=a}^{t=b}.$$

§3. Studying the Necessary Optimality Conditions of the Generalization a Class of (C-K)FOCPs

We study and derive the necessary optimality conditions the left (C-KFD) of order $\alpha,\beta\in(0,1)$, $\rho>0$ in the following theorem.

Theorem 3.1 *If (x,u,t_f) is a minimizer of (1) under the sum of two (C-K) FD of dynamic constraint (2), and the condition (3), then a function $\lambda(t)\in\mathbb{R}$, for (x,u,λ) satisfies:*

1.Hamiltonian system

$$\begin{cases} t^{\rho-1}\left[\mathcal{R}_1\,{}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)+\mathcal{R}_2\,{}^{RK}_{t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]=\frac{\partial H}{\partial x}\left(x(t),u(t),\lambda(t),t\right), \\ \mathcal{R}_1\,{}^{RK}_{a}D_t^{\alpha,\rho}x(t)+\mathcal{R}_2\,{}^{RK}_{a}D_t^{\beta,\rho}x(t)=\frac{\partial H}{\partial \lambda}\left(x(t),u(t),\lambda(t),t\right), \quad for all t\in[A,t_f], \end{cases} \tag{16}$$

and

$$\begin{cases} {}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{t}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)=0, \\ {}^{RK}_{t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{t}D_{A}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)=0, \end{cases} \quad for\ all\ t\in[a,A]$$

2.The stationary condition

$$\frac{\partial\mathcal{H}}{\partial u}\left(x(t),u(t),\lambda(t),t\right)=0, \quad for\ all\ t\in[A,t_f]. \tag{17}$$

3.The transversality conditions

$$\left[\begin{array}{c} \mathcal{H}\left(x(t),u(t),\lambda(t),t\right)+\mathcal{R}_1x'(t)\,{}^{RK}_{t}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t) \\ -\mathcal{R}_2x'(t)\,{}^{RK}_{t}D_{t_f}^{-(1-\beta,\rho)}t^{1-\rho}\lambda(t)-\mathcal{R}_1\lambda(t)\,{}^{CK}_{a}D_t^{\alpha,\rho}x(t) \\ -\mathcal{R}_2\lambda(t)\,{}^{CK}_{a}D_t^{\beta,\rho}x(t)+\frac{\partial\Psi}{\partial t}\left(t,x(t)\right) \end{array} \right]_{t=t_f}=0. \tag{18}$$

$$\left[\mathcal{R}_1\,{}^{RK}_{t}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)+\mathcal{R}_2\,{}^{RK}_{t}D_{t_f}^{-(1-\beta,\rho)}t^{1-\rho}\lambda(t)-\frac{\partial\Psi}{\partial x}\left(t,x(t)\right)\right]_{t=t_f}=0, \tag{19}$$

$$\begin{cases} {}^{RK}_{a}D_{t_f}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)-{}^{RK}_{a}D_{A}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)=0, \\ {}^{RK}_{a}D_{t_f}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)-{}^{RK}_{a}D_{A}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)=0. \end{cases}$$

Proof Constructing the problem as minimizing and by define the Hamiltonian function $\mathcal{H}(x(t), u(t), \lambda(t))$, as follows:

$$\mathcal{H}(x(t), u(t), \lambda(t), t) = \mathcal{F}(x(t), u(t)t) + \lambda(t) g(x(t), u(t), t), \tag{20}$$

$$\mathbf{J}^*(x, u, t_f, \lambda) = \int_A^{t_f} \begin{bmatrix} \mathcal{H}(x(t), u(t), \lambda(t), t) \\ -\lambda(t)\left\{\mathcal{R}_1 {}_a^{CK}D_t^{\alpha,\rho}x(t) + \mathcal{R}_2 {}_a^{CK}D_t^{\beta,\rho}x(t)\right\} \end{bmatrix} dt + \Psi(t_f, x(t_f)). \tag{21}$$

Using variations $x + \delta x$, $\lambda + \delta\ \lambda u + \delta u$, $, t_f + \delta t_f$ and $(\delta \mathbf{J}^* = 0)$, we conclude that

$$\begin{aligned} 0 \ = \ & \int_A^{t_f} \begin{bmatrix} \delta\mathcal{H}(x(t), u(t), \lambda(t), t) \\ -\delta\lambda(t) \left\{ \begin{matrix} \mathcal{R}_1 {}_a^{CK}D_t^{\alpha,\rho}x(t) \\ +\mathcal{R}_2 {}_a^{CK}D_t^{\beta,\rho}x(t) \end{matrix} \right\} \end{bmatrix} dt \\ & +\delta t_f \begin{bmatrix} \mathcal{H}(x(t), u(t), \lambda(t), t) \\ -\lambda(t) \left\{ \begin{matrix} \mathcal{R}_1 {}_a^{CK}D_t^{\alpha,\rho}x(t) \\ +\mathcal{R}_2 {}_a^{CK}D_t^{\beta,\rho}x(t) \end{matrix} \right\} \end{bmatrix}_{t=t_f} + \delta\Psi(t_f, x(t_f)). \end{aligned} \tag{22}$$

Thus,

$$\begin{aligned} = & \int_A^{t_f} \begin{bmatrix} \frac{\partial\mathcal{H}}{\partial x(t)}\delta x(t) + \frac{\partial\mathcal{H}}{\partial u(t)}\delta u(t) + \frac{\partial\mathcal{H}}{\partial\lambda(t)}\delta\lambda(t) \\ -\delta\lambda(t)\left\{\mathcal{R}_1 {}_a^{CK}D_t^{\alpha,\rho}x(t) + \mathcal{R}_2 {}_a^{CK}D_t^{\beta,\rho}x(t)\right\} \\ -\mathcal{R}_1\lambda(t) {}_a^{CK}D_t^{\alpha,\rho}\delta x(t) - \mathcal{R}_2\lambda(t) {}_a^{CK}D_t^{\beta,\rho}\delta x(t) \end{bmatrix} dt \\ & + \delta t_f \begin{bmatrix} \mathcal{H}(x(t), u(t), \lambda(t), t) \\ -\lambda(t)\left\{\mathcal{R}_1 {}_a^{CK}D_t^{\alpha,\rho}x(t) + \mathcal{R}_2 {}_a^{CK}D_t^{\beta,\rho}x(t)\right\} \end{bmatrix}_{t=t_f} \\ & + \frac{\partial\Psi}{\partial t}(t_f, x(t_f))\delta t_f + \frac{\partial\Psi}{\partial x}(t_f, x(t_f))(x'(t_f)\delta t_f + \delta x(t_f)). \end{aligned} \tag{23}$$

Now, integration by part the relation in Eq. (23) by using Theorem 2.2 in the form

$$\begin{aligned} & = \int_A^{t_f} \lambda(t) {}_a^{CK}D_t^{\alpha,\rho}\delta x(t)dt \\ & = \int_a^{t_f} \lambda(t) {}_a^{CK}D_t^{\alpha,\rho}\delta x(t)dt - \int_a^{A} \lambda(t) {}_a^{CK}D_t^{\alpha,\rho}\delta x(t)dt \\ & = \int_a^{t_f} \delta x(t) t^{\rho-1} {}_t^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt + \left[\delta x(t) {}_t^{RK}D_{t_f}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=a}^{t=t_f} \\ & \quad - \int_a^A \delta x(t) t^{\rho-1} {}_t^{RK}D_A^{\alpha,\rho}\left(t^{1-\rho}\lambda t\right)dt - \left[\delta x(t) {}_t^{RK}D_A^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=a}^{t=A} \\ & = \int_a^A \delta x(t) t^{\rho-1} {}_t^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt + \int_A^{t_f} \delta x(t) t^{\rho-1} {}_t^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt \end{aligned}$$

$$
\begin{aligned}
&+\left[\delta x(t) {}^{RK}_{t}D_{t_f}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=a}^{t=t_f}\\
&-\int_a^A \delta x(t) t^{\rho-1} {}^{RK}_{t}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt-\left[\delta x(t) {}^{RK}_{t}D_{A}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=a}^{t=A}\\
=&\int_a^A \delta x(t) t^{\rho-1}\left[{}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{t}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&+\int_A^{t_f} \delta x(t) t^{\rho-1} {}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt\\
&+\left[\delta x(t) {}^{RK}_{t}D_{t_f}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=a}^{t=t_f}-\left[\delta x(t) {}^{RK}_{t}D_{A}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=a}^{t=A}\\
=&\int_a^A \delta x(t) t^{\rho-1}\left[{}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{t}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&+\int_A^{t_f} \delta x(t) t^{\rho-1} {}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt\\
&+\left[\delta x(t) {}^{RK}_{t}D_{t_f}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=t_f}-\left[\delta x(t) {}^{RK}_{t}D_{t_f}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=a}\\
&-\left[\delta x(t) {}^{RK}_{t}D_{A}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=A}-\left[\delta x(t) {}^{RK}_{t}D_{A}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=a}.
\end{aligned}
$$

Since ($\delta x(A)=0$), we get

$$
\begin{aligned}
=&\int_a^A \delta x(t) t^{\rho-1}\left[{}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{t}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&+\int_A^{t_f} \delta x(t) t^{\rho-1} {}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)dt+\delta x(t_f)\left[{}^{RK}_{t}D_{t_f}^{-(1-\alpha,\rho)} t^{1-\rho}\lambda(t)\right]_{t=t_f}\\
&-\delta x(a)\left[{}^{RK}_{a}D_{t_f}^{-(1-\alpha,\rho)} a^{1-\rho}\lambda(a)-{}^{RK}_{a}D_{A}^{-(1-\alpha,\rho)} a^{1-\rho}\lambda(a)\right].
\end{aligned}
\tag{24}
$$

Also, in the same way we have

$$
\begin{aligned}
=&\int_a^A \delta x(t) t^{\rho-1}\left[{}^{RK}_{t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{t}D_{A}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&+\int_A^{t_f} \delta x(t) t^{\rho-1} {}^{RK}_{t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)dt+\delta x(t_f)\left[{}^{RK}_{t}D_{t_f}^{-(1-\beta,\rho)} t^{1-\rho}\lambda(t)\right]_{t=t_f}\\
&-\delta x(a)\left[{}^{RK}_{a}D_{t_f}^{-(1-\beta,\rho)} a^{1-\rho}\lambda(a)-{}^{RK}_{a}D_{A}^{-(1-\beta,\rho)} a^{1-\rho}\lambda(a)\right].
\end{aligned}
\tag{25}
$$

Substitute the results of Eq. (24) and Eq. (25), into Eq. (23) as follows

$$
\begin{aligned}
=&\int_A^{t_f}\left[\begin{array}{c}
\frac{\partial\mathcal{H}}{\partial x(t)}\delta x(t)+\frac{\partial\mathcal{H}}{\partial u(t)}\delta u(t)+\frac{\partial\mathcal{H}}{\partial\lambda(t)}\delta\lambda(t)\\
-\delta\lambda(t)\left\{\mathcal{R}_1 {}^{CK}_{a}D_{t}^{\alpha,\rho}x(t)+\mathcal{R}_2 {}^{CK}_{a}D_{t}^{\beta,\rho}x(t)\right\}\\
-\delta x(t)t^{\rho-1}\left[\mathcal{R}_1 {}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)+\mathcal{R}_2 {}^{RK}_{t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]
\end{array}\right]dt\\
&-\mathcal{R}_1\int_a^A \delta x(t) t^{\rho-1}\left[{}^{RK}_{t}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{t}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&-\mathcal{R}_2\int_a^A \delta x(t) t^{\rho-1}\left[{}^{RK}_{t}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}^{RK}_{t}D_{A}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt
\end{aligned}
$$

$$
\begin{aligned}
&-\delta x\left(t_{f}\right)\left[\mathcal{R}_{1}\left[{}_{t}^{RK}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)\right]_{t=t_f}+\mathcal{R}_{2}\left[{}_{t}^{RK}D_{t_f}^{-(1-\beta,\rho)}t^{1-\rho}\lambda(t)\right]_{t=t_f}\right]\\
&+\mathcal{R}_{1}\delta x(a)\left[{}_{a}^{RK}D_{t_f}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)-{}_{a}^{RK}D_{A}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)\right]\\
&+\mathcal{R}_{2}\delta x(a)\left[{}_{a}^{RK}D_{t_f}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)-{}_{a}^{RK}D_{A}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)\right]\\
&+\delta t_{f}\left[\begin{array}{c}\mathcal{H}\left(x(t),u(t),\lambda(t),t\right)\\ -\lambda(t)\left\{\mathcal{R}_{1}{}_{a}^{CK}D_{t}^{\alpha,\rho}x(t)+\mathcal{R}_{2}{}_{a}^{CK}D_{t}^{\beta,\rho}x(t)\right\}\end{array}\right]_{t=t_f}\\
&+\frac{\partial\Psi}{\partial t}\left(t_{f},x\left(t_{f}\right)\right)\delta t_{f}+\frac{\partial\Psi}{\partial x}\left(t_{f},x\left(t_{f}\right)\right)\left(x'\left(t_{f}\right)\delta t_{f}+\delta x\left(t_{f}\right)\right),
\end{aligned}
$$

Thus,

$$
\begin{aligned}
0=&\int_{A}^{t_f}\left[\begin{array}{c}\delta x(t)\left(\frac{\partial\mathcal{H}}{\partial x(t)}-t^{\rho-1}\left[\mathcal{R}_{1}{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)+\mathcal{R}_{2}{}_{t}^{RK}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]\right)\\ +\delta u(t)\frac{\partial\mathcal{H}}{\partial u(t)}+\delta\lambda(t)\left(\frac{\partial\mathcal{H}}{\partial\lambda(t)}-\mathcal{R}_{1}{}_{a}^{CK}D_{t}^{\alpha,\rho}x(t)-\mathcal{R}_{2}{}_{a}^{CK}D_{t}^{\beta,\rho}x(t)\right)\end{array}\right]dt\\
&-\mathcal{R}_{1}\int_{a}^{A}\delta x(t)t^{\rho-1}\left[{}_{t}^{RK}D_{t_f}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}_{t}^{RK}D_{A}^{\alpha,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&-\mathcal{R}_{2}\int_{a}^{A}\delta x(t)t^{\rho-1}\left[{}_{t}^{RK}D_{t_f}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)-{}_{t}^{RK}D_{A}^{\beta,\rho}\left(t^{1-\rho}\lambda(t)\right)\right]dt\\
&-\delta x\left(t_{f}\right)\left[\mathcal{R}_{1}{}_{t}^{RK}D_{t_f}^{-(1-\alpha,\rho)}t^{1-\rho}\lambda(t)+\mathcal{R}_{2}{}_{t}^{RK}D_{t_f}^{-(1-\beta,\rho)}t^{1-\rho}\lambda(t)-\frac{\partial\Psi}{\partial x}\left(t,x(t)\right)\right]_{t=t_f}\\
&+\delta t_{f}\left[\begin{array}{c}\mathcal{H}\left(x(t),u(t),\lambda(t),t\right)-\lambda(t)\left\{\mathcal{R}_{1}{}_{a}^{CK}D_{t}^{\alpha,\rho}x(t)+\mathcal{R}_{2}{}_{a}^{CK}D_{t}^{\beta,\rho}x(t)\right\}\\ +\frac{\partial\Psi}{\partial t}\left(t,x(t)\right)+\frac{\partial\Psi}{\partial x}\left(t,x(t)\right)x'(t)\end{array}\right]_{t=t_f}\\
&+\mathcal{R}_{1}\delta x(a)\left[{}_{a}^{RK}D_{t_f}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)-{}_{a}^{RK}D_{A}^{-(1-\alpha,\rho)}a^{1-\rho}\lambda(a)\right]\\
&+\mathcal{R}_{2}\delta x(a)\left[{}_{a}^{RK}D_{t_f}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)-{}_{a}^{RK}D_{A}^{-(1-\beta,\rho)}a^{1-\rho}\lambda(a)\right].
\end{aligned}
\tag{26}
$$

Now, we rewrite the transversality conditions in Eq. (26) by using the Taylor series for $f=(x+\delta x)$ about the point $x=t_f$ can be given as

$$
\begin{aligned}
&(x+\delta x)\left(t_{f}+\delta t_{f}\right)=x+\delta x)\left(t_{f}\right)+x'\left(t_{f}\right)\delta t_{f}+\mathbf{O}\left(\delta t_{f}^{2}\right),\\
&\underbrace{(x+\delta x)\left(t_{f}+\delta t_{f}\right)-x\left(t_{f}\right)}-\delta x\left(t_{f}\right)=x'\left(t_{f}\right)\delta t_{f}+\mathbf{O}\left(\delta t_{f}^{2}\right)\\
&x\left(t_{f}\right)-\delta x\left(t_{f}\right)=x'\left(t_{f}\right)\delta t_{f}+\mathbf{O}\left(\delta t_{f}^{2}\right),
\end{aligned}
$$

$$
\delta x\left(t_{f}\right)-\delta x_{t_f}=-x'\left(t_{f}\right)\delta t_{f}+\mathbf{O}\left(\delta t_{f}^{2}\right),\tag{27}
$$

$$
\delta x\left(t_{f}\right)=\delta x_{t_f}-x'\left(t_{f}\right)\delta t_{f}+\mathbf{O}\left(\delta t_{f}^{2}\right),\tag{28}
$$

where $\delta x_{t_f}=(x+\delta x)\left(t_f+\delta t_f\right)-x\left(t_f\right)$ and $\lim_{\gamma\to\infty}\frac{\mathbf{O}(\gamma)}{\gamma}$ is finite.

Substitute the Eq. (27) and Eq. (28), into Eq. (26), we have

$$
\begin{aligned}
&= \int_A^{t_f} \left[\begin{array}{c} \delta x(t) \left(\frac{\partial \mathcal{H}}{\partial x(t)} - t^{\rho-1} \left[\mathcal{R}_1 {}^{RK}_{\ t}D^{\alpha,\rho}_{t_f} \left(t^{1-\rho}\lambda(t)\right) + \mathcal{R}_2 {}^{RK}_{\ t}D^{\beta,\rho}_{t_f} \left(t^{1-\rho}\lambda(t)\right)\right]\right) \\ + \delta u(t) \frac{\partial \mathcal{H}}{\partial u(t)} + \delta\lambda(t) \left(\frac{\partial \mathcal{H}}{\partial \lambda(t)} - \mathcal{R}_1 {}^{CK}_{\ a}D^{\alpha,\rho}_{t} x(t) - \mathcal{R}_2 {}^{CK}_{\ a}D^{\beta,\rho}_{t} x(t) \right) \end{array} \right] dt \\
&- \mathcal{R}_1 \int_a^A \delta x(t) t^{\rho-1} \left[{}^{RK}_{\ t}D^{\alpha,\rho}_{t_f} \left(t^{1-\rho}\lambda(t)\right) - {}^{RK}_{\ t}D^{\alpha,\rho}_{A} \left(t^{1-\rho}\lambda(t)\right) \right] dt \\
&- \mathcal{R}_2 \int_a^A \delta x(t) t^{\rho-1} \left[{}^{RK}_{\ t}D^{\beta,\rho}_{t_f} \left(t^{1-\rho}\lambda(t)\right) - {}^{RK}_{\ t}D^{\beta,\rho}_{A} \left(t^{1-\rho}\lambda(t)\right) \right] dt \\
&\left[\begin{array}{c} -\mathcal{R}_1 \left(\delta x_t - x'(t)\delta t\right) {}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{t_f} t^{1-\rho}\lambda(t) \\ -\mathcal{R}_2 \left(\delta x_t - x'(t)\delta t\right) {}^{RK}_{\ t}D^{-(1-\beta,\rho)}_{b} t^{1-\rho}\lambda(t) \\ +\delta x(t) \frac{\partial \Psi}{\partial x}(t, x(t)) + \delta t \mathcal{H}(x(t), u(t), \lambda(t), t) \\ -\mathcal{R}_1 \delta t \lambda(t) {}^{CK}_{\ a}D^{\alpha,\rho}_{t} x(t) - \mathcal{R}_2 \delta t \lambda(t) {}^{CK}_{\ a}D^{\beta,\rho}_{t} x(t) \\ +\delta t \frac{\partial \Psi}{\partial x}(t, x(t)) + \frac{\partial \Psi}{\partial x}(t, x(t)) \left(\delta x_t - \delta x(t)\right) \end{array} \right]_{t=t_f} \\
&+ \mathcal{R}_1 \delta x(a) \left[{}^{RK}_{\ a}D^{-(1-\alpha,\rho)}_{t_f} a^{1-\rho}\lambda(a) - {}^{RK}_{\ a}D^{-(1-\alpha,\rho)}_{A} a^{1-\rho}\lambda(a) \right] \\
&+ \mathcal{R}_2 \delta x(a) \left[{}^{RK}_{\ a}D^{-(1-\beta,\rho)}_{t_f} a^{1-\rho}\lambda(a) - {}^{RK}_{\ a}D^{-(1-\beta,\rho)}_{A} a^{1-\rho}\lambda(a) \right] + \mathbf{O}\left(\delta t_f^2\right) = 0,
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&= \int_A^{t_f} \left[\begin{array}{c} \delta x(t) \left(\frac{\partial \mathcal{H}}{\partial x(t)} - t^{\rho-1} \left[\mathcal{R}_1 {}^{RK}_{\ t}D^{\alpha,\rho}_{t_f} \left(t^{1-\rho}\lambda(t)\right) + \mathcal{R}_2 {}^{RK}_{\ t}D^{\beta,\rho}_{t_f} \left(t^{1-\rho}\lambda(t)\right)\right]\right) \\ + \delta u(t) \frac{\partial \mathcal{H}}{\partial u(t)} + \delta\lambda(t) \left(\frac{\partial \mathcal{H}}{\partial \lambda(t)} - \mathcal{R}_1 {}^{CK}_{\ a}D^{\alpha,\rho}_{t} x(t) - \mathcal{R}_2 {}^{CK}_{\ a}D^{\beta,\rho}_{t} x(t) \right) \end{array} \right] dt \\
&-\mathcal{R}_1 \int_a^A \delta x(t) t^{\rho-1} \left[{}^{RK}_{\ t}D^{\alpha,\rho}_{t_f} \left(t^{1-\rho}\lambda(t)\right) - {}^{RK}_{\ t}D^{\alpha,\rho}_{A} \left(t^{1-\rho}\lambda(t)\right) \right] dt \\
&-\mathcal{R}_2 \int_a^A \delta x(t) t^{\rho-1} \left[{}^{RK}_{\ t}D^{\beta,\rho}_{t_f} \left(t^{1-\rho}\lambda(t)\right) - {}^{RK}_{\ t}D^{\beta,\rho}_{A} \left(t^{1-\rho}\lambda(t)\right) \right] dt \\
&+\delta t_f \left[\begin{array}{c} \mathcal{H}(x(t), u(t), \lambda(t), t) + \mathcal{R}_1 x'(t) {}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{t_f} t^{1-\rho}\lambda(t) \\ +\mathcal{R}_2 x'(t) {}^{RK}_{\ t}D^{-(1-\beta,\rho)}_{b} t^{1-\rho}\lambda(t) - \mathcal{R}_1 \lambda(t) {}^{CK}_{\ a}D^{\alpha,\rho}_{t} x(t) \\ -\mathcal{R}_2 \lambda(t) {}^{CK}_{\ a}D^{\beta,\rho}_{t} x(t) + \frac{\partial \Psi}{\partial t}(t, x(t)) \end{array} \right]_{t=t_f} \\
&-\delta x_{t_f} \left[\mathcal{R}_1 {}^{RK}_{\ t}D^{-(1-\alpha,\rho)}_{t_f} t^{1-\rho}\lambda(t) + \mathcal{R}_2 {}^{RK}_{\ t}D^{-(1-\beta,\rho)}_{t_f} t^{1-\rho}\lambda(t) - \frac{\partial \Psi}{\partial x}(t, x(t)) \right]_{t=t_f} \\
&+\mathcal{R}_1 \delta x(a) \left[{}^{RK}_{\ a}D^{-(1-\alpha,\rho)}_{t_f} a^{1-\rho}\lambda(a) - {}^{RK}_{\ a}D^{-(1-\alpha,\rho)}_{A} a^{1-\rho}\lambda(a) \right] \\
&+\mathcal{R}_2 \delta x(a) \left[{}^{RK}_{\ a}D^{-(1-\beta,\rho)}_{t_f} a^{1-\rho}\lambda(a) - {}^{RK}_{\ a}D^{-(1-\beta,\rho)}_{A} a^{1-\rho}\lambda(a) \right] + \mathbf{O}\left(\delta t_f^2\right) = 0. \qquad (29)
\end{aligned}
$$

Since the variation functions were chosen arbitrarily, we get the necessary optimality conditions from Eq. (29) for sum two C-KFOCPs. □

§4. Conclusion

In this paper, a new system for the generalization a class of (FOCPs) with Caputo-Katugampola derivativesin the case where the lower bound of the integral of J is greater than of a of ${}^{CK}_{a}D^{\alpha,\rho}_{t}x(t) + {}^{CK}_{a}D^{\beta,\rho}_{T}x(t)$ has been studied and derived. We are assuming that the end time t_f free. The necessary optimality conditions for the system are obtained when $\alpha, \beta \in (0, 1)$ and $\rho > 0$ and $a \in \mathbb{R}$ and consist of a Hamiltonian system, stationary condition and transversality conditions,which contributes to solving non-linear dynamical control systems with FDs to obtain approximate solutions for state and control variables with the help of the proposed numerical methods.

References

[1] Ghasemi S., Nazemi A., Tajik R., & Mortezaee M. (2021), On fractional optimal control problems with an application in fractional chaotic systems using a Legendre collocation-optimization technique, *Transactions of the Institute of Measurement and Control*, 43(6), 1268-1285.

[2] Ameen I. G., & Ali H. M. (2020), Application of fractional optimal control problems on some mathematical bioscience, In *Advanced Applications of Fractional Differential Operators to Science and Technology* (pp. 41-56). IGI Global.

[3] Chiranjeevi T., & Biswas R. K. (2022), Application of conformable fractional differential transform method for fractional optimal control problems. *IFAC-PapersOnLine*, 55(1), 643-648.

[4] Zhu, J., Trlat, E., & Cerf, M. (2017), Geometric optimal control and applications to aerospace, *Pacific Journal of Mathematics for Industry*, 9(1), 1-41.

[5] Lin Z, Wang H., Modeling and application of fractional-order economic growth model with time delay, *Fractal Fract.*, 5(3)(2021), 1-18. Congress on Applied Sciences, (pp. 101-120).

[6] Heydari MH, Razzaghi M., A new class of orthonormal basis functions: application for fractional optimal control problems, *Int J. Syst. Sci.*, 53(2)(2022), 240-252.

[7] Luo D, Wang JR, Fekan M., Applying fractional calculus to analyze economic growth modelling, *JAMSI*, 14(1)(2018), 25-36.

[8] Heydari MH, Razzaghi M., Piecewise Chebyshev cardinal functions: Application for constrained fractional optimal control problems, *Chaos Solitons Fractals*, 150(2021), 1-11.

[9] K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, Inc., New York, 1993.

[10] I. Podlubny, *Fractional Differential Equations*, Academic Press, 1999.

[11] J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado, *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, Springer, 2007.

[12] G. Samko A.A. Kilbas, O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[13] Agrawal OP., A general formulation and solution scheme for fractional optimal control problems, *Nonlinear Dyn.*, 38(1)(2004), 323-337.

[14] Pooseh S., Almeida R., & Torres D. F. (2013), Fractional order optimal control problems with free terminal time, *arXiv:* 1302.1717.

[15] Sayevand K., & Rostami M. (2018), Fractional optimal control problems: optimality conditions and numerical solution, *IMA Journal of Mathematical Control and Information*, 35(1), 123-148.

[16] QasimHasan S., & Abbas Holel M.(2018), Solution of some types for composition fractional order differential equations corresponding to optimal control problems, *Journal of Control Science and Engineering*, 2018.

[17] Katugampola UN., New approach to a generalized fractional integral, *Appl. Math. Comput.*, 218(3)(2011), 860-865.

[18] Katugampola UN., A new approach to generalized fractional derivatives, *Bull. Math. Anal. Appl.*, 6(4)(2014), 1-15.

[19] Almutairi O, Kiliman A., New generalized Hermite-Hadamard inequality and related integral inequalities involving Katugampola type fractional integrals, *Symmetry*, 12(4)(2020), 1-14.

[20] Almeida R., Malinowska A. B., & Odzijewicz T.(2016), Fractional differential equations with dependence on the Caputo-Katugampola derivative, *Journal of Computational and Nonlinear Dynamics*, 11(6).

[21] Jarad F., Abdeljawad T., & Baleanu D.(2017), On the generalized fractional derivatives and their Caputo modification, *J.Nonlinear Sciences and Applications*, Vol.10, No.5, 2607–2619.

International J.Math. Combin. Vol.4(2022), 60-65

A QSPR Analysis for Stress-Sum Index

C. N. Harshavardhana[1], R. Rajendra[2], P. Siva Kota Reddy[3] and Khaled A. A. Alloush[4]

1.Department of Mathematics, Government First Grade College for Women, Holenarasipur-573 211, India

2.Department of Mathematics, Field Marshal K.M. Cariappa College (A Constituent College of Mangalore University), Madikeri - 571201, India

3.Department of Mathematics, Sri Jayachamarajendra College of Engineering, JSS Science and Technology University, Mysuru-570 006, India

4.Department of Mathematics, Dar Al-Uloom University, Riyadh-13314, Saudi Arabia

E-mail: cnhmaths@gmail.com, rrajendrar@gmail.com,
pskreddy@jssstuniv.in; pskreddy@sjce.ac.in, khaledindia@gmail.com

Abstract: In this brief paper, the stress-sum index of molecular graphs and the physical characteristics of lower alkanes are analysed using QSPR, and linear regression models for boiling points, molar volumes, molar refractions, heats of vaporisation and critical temperatures are presented.

Key Words: Molecular graph, topological index, Stress sum index.

AMS(2010): 05C09, 05C92.

§1. Introduction

Let $G = (V, E)$ be a graph (finite, simple, connected and undirected). A geodesic between two vertices u and v in G is a shortest path between u and v. The molecular graph of a chemical compound is a simple connected graph considering atoms of chemical compounds as vertices and the chemical bonds between them as edges. For definitions in graph theory, the textbook of Harary [2] has been cited. As and when necessary, the non-standard notions will be provided in this article.

The topological indices are graph invariants (theoretical molecular descriptors) that play an important role in chemistry (See [3-14]). There are many important degree/distance based topological indices defined for graphs having numerous applications in chemistry [13] like Zagreb index, Wiener index, Harary index etc.

The concept of stress of a vertex in a network (graph) has been introduced by Shimbel [15] as a centrality measure in 1953. The concepts of stress number of a graph and stress regular graphs have been studied by K. Bhargava, N. N. Dattatreya, and R. Rajendra in [1]. The stress of a vertex v in a graph G, denoted by $\mathrm{str}_G(v)$ or briefly by $\mathrm{str}(v)$, is the number of geodesics

[1]Received November 18, 2022, Accepted December 10, 2022.
[2]Corresponding author: Dr.P.Siva Kota Reddy, Email: pskreddy@jssstuniv.in
[3]This joint work is dedicated to the memory of Shree K. Jagan Mohan Reddy, father-in-law of third author.

passing through it. The stress-sum index $\mathcal{SS}(G)$ of a simple graph G is defined (see [6]) by

$$\mathcal{SS}(G) = \sum_{uv \in E(G)} \mathrm{str}(u) + \mathrm{str}(v). \quad (1)$$

the quantitative structure-property relationship (QSPR) studies translate quantitative physical properties of chemical compounds into numerical data which helps to study correlation between properties of chemical compounds, their structure and simultaneously develop regression models. QSPR analysis for many topological indices can be found in literature. In this short paper, by a QSPR analysis for physical properties of lower alkanes involving stress-sum index of molecular graphs, we present best linear regression models for boiling points, molar volumes, molar refractions, heats of vaporization and critical temperatures of low alkanes.

§2. A QSPR Analysis

We carry a QSPR analysis for the physical properties - boiling points, molar volumes, molar refractions, heats of vaporization, critical temperatures, critical pressures and surface tensions of lower alkanes with stress-sum index of molecular graphs.

Table 1 gives the stress-sum index $\mathcal{SS}(G)$ of molecular graphs and the experimental values for the physical properties - Boiling points (bp) $°C$, molar volumes (mv) cm^3, molar refractions (mr) cm^3, heats of vaporization (hv) kJ, critical temperatures (ct) $°C$, critical pressures (cp) atm, and surface tensions(st) $dyne\ cm^{-1}$ of considered alkanes. The values given in the columns 3 to 9 in the Table 1 are taken from Needham et al. [4] (the same values can be found in [14]).

Table 1. Stress sum index, boiling points, molar volumes, molar refractions, heats of vaporization, critical temperatures, critical pressures and surface tensions of low alkanes

Alkane	$\mathcal{SS}(G)$	$\frac{bp}{°C}$	$\frac{mv}{cm^3}$	$\frac{mr}{cm^3}$	$\frac{hv}{kJ}$	$\frac{ct}{°C}$	$\frac{cp}{atm}$	$\frac{st}{dyne\ cm^{-1}}$
Pentane	20	36.1	115.2	25.27	26.4	196.6	33.3	16
2-Methylbutane	21	27.9	116.4	25.29	24.6	187.8	32.9	15
2,2-Dimethylpropane	24	9.5	122.1	25.72	21.8	160.6	31.6	
Hexane	40	68.7	130.7	29.91	31.6	234.7	29.9	18.42
2-Methylpentane	41	60.3	131.9	29.95	29.9	224.9	30	17.38
3-Methylpentane	40	63.3	129.7	29.8	30.3	231.2	30.8	18.12
2,2-Dimethylbutane	44	49.7	132.7	29.93	27.7	216.2	30.7	16.3
2,3-Dimethylbutane	42	58	130.2	29.81	29.1	227.1	31	17.37
Heptane	70	98.4	146.5	34.55	36.6	267	27	20.26
2-Methylhexane	71	90.1	147.7	34.59	34.8	257.9	27.2	19.29
3-Methylhexane	69	91.9	145.8	34.46	35.1	262.4	28.1	19.79
3-Ethylhexane	104	93.5	143.5	34.28	35.2	267.6	28.6	20.44

(Table continues)

Alkane	$\mathcal{SS}(G)$	$\frac{bp}{°C}$	$\frac{mv}{cm^3}$	$\frac{mr}{cm^3}$	$\frac{hv}{kJ}$	$\frac{ct}{°C}$	$\frac{cp}{atm}$	$\frac{st}{dyne\ cm^{-1}}$
2,2-Dimethylpentane	74	79.2	148.7	34.62	32.4	247.7	28.4	18.02
2,3-Dimethylpentane	70	89.8	144.2	34.32	34.2	264.6	29.2	19.96
2,4-Dimethylpentane	72	80.5	148.9	34.62	32.9	247.1	27.4	18.15
3,3-Dimethylpentane	59	86.1	144.5	34.33	33	263	30	19.59
2,3,3-Trimethylbutane	75	80.9	145.2	34.37	32	258.3	29.8	18.76
Octane	112	125.7	162.6	39.19	41.5	296.2	24.64	21.76
2-Methylheptane	113	117.6	163.7	39.23	39.7	288	24.8	20.6
3-Methylheptane	110	118.9	161.8	39.1	39.8	292	25.6	21.17
4-Methylheptane	109	117.7	162.1	39.12	39.7	290	25.6	21
3-Ethylhexane	104	118.5	160.1	38.94	39.4	292	25.74	21.51
2,2-Dimethylhexane	116	106.8	164.3	39.25	37.3	279	25.6	19.6
2,3-Dimethylhexane	110	115.6	160.4	38.98	38.8	293	26.6	20.99
2,4-Dimethylhexane	111	109.4	163.1	39.13	37.8	282	25.8	20.05
2,5-Dimethylhexane	114	109.1	164.7	39.26	37.9	279	25	19.73
3,3-Dimethylhexane	112	112	160.9	39.01	37.9	290.8	27.2	20.63
3,4-Dimethylhexane	108	117.7	158.8	38.85	39	298	27.4	21.62
3-Ethyl-2-methylpentane	105	115.7	158.8	38.84	38.5	295	27.4	21.52
3-Ethyl-3-methylpentane	108	118.3	157	38.72	38	305	28.9	21.99
2,2,3-Trimethylpentane	105	109.8	159.5	38.92	36.9	294	28.2	20.67
2,2,4-Trimethylpentane	117	99.2	165.1	39.26	36.1	271.2	25.5	18.77
2,3,3-Trimethylpentane	129	114.8	157.3	38.76	37.2	303	29	21.56
2,3,4-Trimethylpentane	111	113.5	158.9	38.87	37.6	295	27.6	21.14
Nonane	168	150.8	178.7	43.84	46.4	322	22.74	22.92
2-Methyloctane	169	143.3	179.8	43.88	44.7	315	23.6	21.88
3-Methyloctane	175	144.2	178	43.73	44.8	318	23.7	22.34
4-Methyloctane	163	142.5	178.2	43.77	44.8	318.3	23.06	22.34
3-Ethylheptane	156	143	176.4	43.64	44.8	318	23.98	22.81
4-Ethylheptane	138	141.2	175.7	43.49	44.8	318.3	23.98	22.81
2,2-Dimethylheptane	172	132.7	180.5	43.91	42.3	302	22.8	20.8
2,3-Dimethylheptane	164	140.5	176.7	43.63	43.8	315	23.79	22.34
2,4-Dimethylheptane	164	133.5	179.1	43.74	42.9	306	22.7	21.3
2,5-Dimethylheptane	166	136	179.4	43.85	42.9	307.8	22.7	21.3
2,6-Dimethylheptane	170	135.2	180.9	43.93	42.8	306	23.7	20.83

(Table continues)

Alkane	$\mathcal{SS}(G)$	$\frac{bp}{^\circ C}$	$\frac{mv}{cm^3}$	$\frac{mr}{cm^3}$	$\frac{hv}{kJ}$	$\frac{ct}{^\circ C}$	$\frac{cp}{atm}$	$\frac{st}{dyne\ cm^{-1}}$
3,3-Dimethylheptane	166	137.3	176.9	43.69	42.7	314	24.19	22.01
3,4-Dimethylheptane	160	140.6	175.3	43.55	43.8	322.7	24.77	22.8
3,5-Dimethylheptane	162	136	177.4	43.64	43	312.3	23.59	21.77
4,4-Dimethylheptane	164	135.2	176.9	43.6	42.7	317.8	24.18	22.01
3-Ethyl-2-methylhexane	154	138	175.4	43.66	43.8	322.7	24.77	22.8
4-Ethyl-2-methylhexane	157	133.8	177.4	43.65	43	330.3	25.56	21.77
3-Ethyl-3-methylhexane	158	140.6	173.1	43.27	43	327.2	25.66	23.22
3-Ethyl-4-methylhexane	153	140.46	172.8	43.37	44	312.3	23.59	23.27
2,2,3-Trimethylhexane	167	133.6	175.9	43.62	41.9	318.1	25.07	21.86
2,2,4-Trimethylhexane	169	126.5	179.2	43.76	40.6	301	23.39	20.51
2,2,5-Trimethylhexane	173	124.1	181.3	43.94	40.2	296.6	22.41	20.04
2,3,3-Trimethylhexane	165	137.7	173.8	43.43	42.2	326.1	25.56	22.41
2,3,4-Trimethylhexane	161	139	173.5	43.39	42.9	324.2	25.46	22.8
2,3,5-Trimethylpentane	165	131.3	177.7	43.65	41.4	309.4	23.49	21.27
2,4,4-Trimethylhexane	167	130.6	177.2	43.66	40.8	309.1	23.79	21.17
3,3,4-Trimethylhexane	163	140.5	172.1	43.34	42.3	330.6	26.45	23.27
3,3-Diethylpentane	152	146.2	170.2	43.11	43.4	342.8	26.94	23.75
2,2-Dimethyl-3-ethylpentane	160	133.8	174.5	43.46	42	338.6	25.96	22.38
2,3-Dimethyl-3-ethylpentane	159	142	170.1	42.95	42.6	322.6	26.94	23.87
2,4-Dimethyl-3-ethylpentane	155	136.7	173.8	43.4	42.9	324.2	25.46	22.8
2,2,3,3-Tetramethylpentane	170	140.3	169.5	43.21	41	334.5	27.04	23.38
2,2,3,4-Tetramethylpentane	156	133	173.6	43.44	41	319.6	25.66	21.98
2,2,4,4-Tetramethylpentane	176	122.3	178.3	43.87	38.1	301.6	24.58	20.37
2,3,3,4-Tetramethylpentane	166	141.6	169.9	43.2	41.8	334.5	26.85	23.31

§3. **Regression Models**

Using Table 1, a study was carried out with a linear regression model

$$P = A + B \cdot \mathcal{SS}(G)$$

where P = Physical property and $\mathcal{SS}(G)$ = stress-sum index. The correlation coefficient r, its square r^2, standard error (se), t-value and p-value are computed and tabulated in Table 2 followed by linear regression models.

Table 2. r,r^2, se, t and p for the physical properties (P) and stress-sum index

P	r	r^2	se	t	p
bp	0.9351	0.8744	(3.8323) (0.0290)	(9.6718) (21.6046)	$(2.4333E-14)$ $(6.6517E-32)$
mv	0.9716	0.9440	(1.4125) (0.0106)	(83.6459) (33.6272)	$(1.6608E-69)$ $(1.1145E-43)$
mr	0.9774	0.9553	(0.3838) (0.0029)	(67.3975) (37.8821)	$(2.702E-63)$ $(5.6412E-47)$
hv	0.9210	0.8484	(0.7220) (0.0054)	(35.5945) (19.3642)	$(3.0145E-45)$ $(3.7544E-29)$
ct	0.9155	0.8381	(5.2891) (0.0400)	(37.6222) (18.6270)	$(8.7678E-47)$ $(3.3851E-28)$
cp	−0.8672	0.7520	(0.4510) (0.0034)	(71.6614) (−14.257)	$(4.686E-65)$ $(5.6973E-22)$
st	0.7855	0.6170	(0.4216) (0.00312)	(40.3860) (10.1551)	$(2.9465E-47)$ $(5.7316E-15)$

The linear regression models for boiling points, molar volumes, molar refractions, heats of vaporization, critical temperatures, critical pressures and surface tensions of low alkanes are as follows:

$$bp = 37.0657 + 0.6270 \cdot \mathcal{SS}(G) \quad (2)$$

$$mv = 118.1525 + 0.3597 \cdot \mathcal{SS}(G) \quad (3)$$

$$mr = 25.8679 + 0.1101 \cdot \mathcal{SS}(G) \quad (4)$$

$$hv = 25.6994 + 0.1058 \cdot \mathcal{SS}(G) \quad (5)$$

$$ct = 198.9881 + 0.7461 \cdot \mathcal{SS}(G) \quad (6)$$

$$cp = 32.32099 - 0.04870 \cdot \mathcal{SS}(G) \quad (7)$$

$$st = 17.0289 + 0.0317 \cdot \mathcal{SS}(G) \quad (8)$$

The values of r,r^2, se, t and p in Table 2 for the physical properties are good except for critical pressures and surface tensions. As a result the linear regression models (2)-(6) can be employed as predictive tools.

§4. Conclusion

The physical properties of low alkanes - boiling points, molar volumes, molar refractions, heats of vaporisation, and critical temperatures, can be predicted using the linear regression models (2)-(6), as shown in Table 2. It demonstrates that stress-sum index can be used as predictive means in QSPR researches.

Acknowledgement

The authors thank the referees for their several helpful comments and suggestions.

References

[1] K. Bhargava, N. N. Dattatreya, and R. Rajendra, On stress of a vertex in a graph, *Palestine Journal of Mathematics*, accepted for publication.

[2] F. Harary, *Graph Theory*, Addison Wesley, Reading, Mass, 1972.

[3] K. B. Mahesh, R. Rajendra and P. S. K. Reddy, Square root stress sum index for graphs, *Proyecciones*, 40(4) (2021), 927-937.

[4] D. E. Needham, I. C. Wei and P. G. Seybold, Molecular modeling of the physical properties of alkanes, *J. Am. Chem. Soc.*, 110 (1988), 4186C4194.

[5] R. M. Pinto, R. Rajendra, P. S. K. Reddy and Ismail Naci CANGÜL, A QSPR analysis for physical properties of lower alkanes involving peripheral Wiener index, *Montes Taurus J. Pure Appl. Math.*, 4(2) (2022), 81C85.

[6] R. Rajendra, P. S. K. Reddy and C. N. Harshavardhana, Stress sum index for graphs, stress-sum index for graphs, *Sci. Magna*, 15(1) (2020), 94-103.

[7] R. Rajendra, K. B. Mahesh and P. S. K. Reddy, Mahesh inverse tension index for graphs, *Adv. Math., Sci. J.*, 9(12) (2020), 10163–10170.

[8] R. Rajendra, P. S. K. Reddy and C. N. Harshavardhana, Tosha index for graphs, *Proceedings of the Jangjeon Math. Soc.*, 24(1) (2021), 141-147.

[9] R. Rajendra, P. S. K. Reddy and Ismail Naci CANGÜL, Stress indices of graphs, *Advn. Stud. Contemp. Math.*, 31(2) (2021), 163-173.

[10] R. Rajendra, K. Bhargava, D. Shubhalakshmi and P. S. K. Reddy, Peripheral Harary index of graphs, *Palest. J. Math.*, 11(3) (2022), 323-336.

[11] R. Rajendra, P. S. K. Reddy, Smitha G Kini and M. Smitha, Peripheral geodesic index for graphs, *Adv. Appl. Math. Sci.*, 22(1) (2022), 13-24.

[12] P. S. K. Reddy, K. N Prakasha and Ismail Naci CANGÜL, Randić type Hadi index of graphs, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics*, 40(4) (2020), 175-181.

[13] H. Wiener, Structural determination of paraffin boiling points, *J. Amer. Chem. Soc.*, 69 (1) (1947), 17-20.

[14] K. Xu, K. C. Das and N. Trinajstic, *The Harary Index of a Graph*, Springer-Verlag, Berlin, Heidelberg, 2015.

[15] A. Shimbel, Structural parameters of communication networks, *Bulletin of Mathematical Biophysics*, 15 (1953), 501-507.

International J.Math. Combin. Vol.4(2022), 66-73

On the Bounds of the Radio Numbers of Stacked-Book Graph

T.C. Adefokun

(Department of Computer and Mathematical Sciences, Crawford University, Nigeria)

D.O.A Ajayi

(Department of Mathematics, University of Ibadan, Nigeria)

E-mail: tayoadefokun@crawforduniversity.edu.ng, olayide.ajayi@mail.ui.edu.ng

Abstract: A Stacked-book graph $G_{m,n}$ results from the Cartesian product of a star graph S_m and path P_n, where m and n are the orders of S_m and P_n respectively. A radio labeling problem of a simple and connected graph, G, involves a non-negative integer function $f : V(G) \to \mathbb{Z}^+$ on the vertex set $V(G)$ of G, such that for all $u, v \in V(G)$, $|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$, where $\text{diam}(G)$ is the diameter of G and $d(u, v)$ is the shortest distance between u and v. Suppose that f_{min} and f_{max} are the respective least and largest values of f on $V(G)$, then, span$f$, the absolute difference of f_{min} and f_{max}, is the span of f while the radio number $rn(G)$ of G is the least value of span$f$ over all the possible radio labels on $V(G)$. In this paper, we obtain the radio number for the stacked-book graph $G_{m,n}$ where $m \geq 4$ and n is even, and obtain bounds for $m = 3$ which improves existing upper and lower bounds for $G_{m,n}$ where $m = 3$.

Key Words: Radio number, Cartesian product of graphs, stacked-book graph.

AMS(2010): 05C15, 05C12.

§1. Introduction

The graph G considered in this paper is simple and undirected. The vertex and edge sets of G are $V(G)$ and $E(G)$. For $e = uv \in E(G)$, e connects two vertices u and v while $d(u, v)$ is the distance between u, v and $\text{diam}(G)$ is the diameter of G. Radio number labeling problem, which is mostly applied in frequency assignment for signal transmission, where it mitigates the problems of signal interference. It was first suggested in 1980 by Hale [6].

Let f be a non negative integer function on $V(G)$ such that the radio labeling condition, $|f(u) - f(v)| \geq \text{diam}G + 1 - d(u, v)$ is satisfied for every pair $u, v \in V(G)$. The span of f, span$f$, is the difference between f_{min} and f_{max}, the minimum and the maximum radio label on G respectively. Thus the smallest possible value of span$f$ is the radio number, $rn(G)$, of G. The radio labeling condition guarantees that every vertex on G has unique radio label. Therefore, $rn(G) \geq |V(G)| - 1$ is trivially true. However, establishing the radio number of graphs has proved to be quite tedious. Even so, such numbers have been completely determined for some

[1]Received November 2, 2022, Accepted December 12, 2022.

graphs. Liu and Zhu [10] showed that for path, P_n, $n \geq 3$,

$$rn(P_n) = \begin{cases} 2k(k-1)+1 & \text{if } n = 2k; \\ 2k^2+2 & \text{if } n = 2k+1. \end{cases}$$

This improves results in [5] and [4] by Chatrand, et. al. where the upper and lower bounds for the same class of graph are obtained. Furthermore, Liu and Xie [9], found the radio number for the square of a path, P_n^2 as:

$$rn(P_n^2) = \begin{cases} k^2+2 & \text{if } n \equiv 1(\text{mod } 4), \text{n} \geq 9; \\ k^2+1 & \text{if otherwise.} \end{cases}$$

Similar results are obtained in [8] for square of cycles. Jiang [7] completely solved the radio number problem for the grid graph ($P_m \Box P_n$), where for $m, n > 2$, it is noted that $rn(P_m \Box P_n) = \frac{mn^2+nm^2-n}{2} - mn - m + 2$, for m-odd and n even. Saha and Panigrahi [12] and Ajayi and Adefokun [1] obtained results on the radio numbers of Cartesian products of two cycles (toroidal grid) and of path and star graph (stacked-book graph) respectively. In the case of stacked-book graph $G = S_n \Box P_m$, $rn(G) \leq n^2m+1$, which the authors noted is not tight. Recent results on radio number include those on middle graph of path [2], trees, [3] and edge-joint graphs [11].

In this paper, for even positive integer n, we consider the stacked-book graph $G_{m,n}$ and derive the $rn(G_{m,n})$ for the case $m \geq 4$. Furthermore, new lower and upper bounds of the number are obtained for $m = 3$, which improve similar results in [1].

§2. Preliminaries

Let S_m be a star of order $m \geq 3$ and for each vertex $v_i \in V(S_m)$, $2 \leq i \leq m$, v_i is adjacent to v_1, the center vertex of S_m. Also, let P_n be a path such that $|V(P_n)| = n$. The Graph $G_{m,n} = S_m \Box P_n$, is obtained by the Cartesian product of S_m and P_n. The vertex set $V(G_{m,n})$ is the Cartesian product $V(S_m) \times V(P_n)$, such that for any $u_iv_j \in V(G_{m,n})$, then, $u_i \in V(S_m)$ and $v_j \in E(P_n)$. For $E(G_{m,n})$, u_iv_j u_kv_l is contained in $E(G_{m,n})$ for u_iv_j, $u_kv_l \in V(G_{m,n})$, then either $u_i = u_k$ and $v_jv_l \in E(P_m)$ or $u_iu_k \in E(S_m)$ and $v_j = v_l$. Geometrically, $V(G_{m,n})$ contains n number of S_m stars, namely $S_{m(1)}, S_{m(2)}, \cdots, S_{m(n)}$, such that for every pair $v_i \in S_{m(i)}$ and $v_{i+1} \in S_{m(i+1)}$, $v_iv_{i+1} \in E(G_{m,n})$. These are, in fact, the only type of edges on $G_{m,n}$ apart from those on its S_m stars. This geometry fetched $G_{m,n}$ the name *stacked-book* graph.

Remark 2.1 It is easy to see that $diam(G_{m,n}) = n+1$, being the number of edges from $u_iv_1 \to u_1v_1 \to u_1v_2 \to \cdots \to u_1v_n \to u_jv_n$, where $i \neq j$.

Remark 2.2 For convenience, we write u_iv_j as $u_{i,j}$ in certain cases and $u_{i,j}u_{k,l}$ is the edge induced by u_iv_j and u_kv_l.

Definition 2.1 *Let $G_{m,n} = S_m \Box P_n$. The vertex set $V_{(i)} \subset V(G_{m,n})$ is the set of vertices on* *star $S_{m(i)}$, defined by the set $\{u_1v_i, u_2v_i, \cdots, u_mv_i\}$.*

We introduce the following definition:

Definition 2.2 *Let $G_{m,n} = S_m\Box P_n$. Then, the pair $\left\{S_{m(i)}, S_{m(i+\frac{n}{2})}\right\}$ is a subgraph $G(i) \subseteq G_{m,n}$ induced by $V_{(i)}$ and $V_{(i+\frac{n}{2})}$.*

Remark 2.3 The maximum number of $G(i)$ subgraph in a $G_{m,n}$ graph, n even, is $\frac{n}{2}$ and the $diam(G(i)) = \frac{n}{2} + 2$.

Remark 2.4 Let $\left\{V_{(i)}, V_{(i+\frac{n}{2})}\right\}$ induce $G(i)$, such that $V_{(i)} = \{u_{1,i}, u_{2,i}, \cdots, u_{m,i}\}$ and $V_{(i+\frac{n}{2})} = \left\{u_{1,\frac{i+n}{2}}, u_{2,\frac{i+n}{2}}, \cdots, u_{m,\frac{i+n}{2}}\right\}$. Then, for $u \in V_{(i)}$, $v \in V_{(i+\frac{n}{2})}$ and $d(u,v) = p$, where $p \in \left\{\frac{n}{2}, \frac{n}{2}+1, \frac{n}{2}+2\right\}$ and for $u_{k,i}, v_{t,i+\frac{n}{2}}$,

$$p = \begin{cases} \frac{n}{2} & \text{if } k = t; \\ \frac{n}{2}+1 & \text{if } t = 1, k \neq t; \\ \frac{n}{2}+2 & \text{if } t \neq 1, k \neq 1, k \neq t. \end{cases}$$

§3. Results

In this section, we investigate the radio number of stacked-book graphs and obtain the exact radio number for $G_{m,n}$, for $m \geq 4$, n even.

Lemma 3.1 *Let S_m be a star on $G_{m,n}$ and f, a radio label function on $G_{m,n}$. Then spanf on S_m is $n(m-1)+1$.*

Proof Let the center vertex of S_m be v_1 and let $f(v_1)$ be the radio label on v_1. There exists some $v_2 \in V(S_m)$ such that $d(v_1, v_2) = 1$. Therefore, by the definition, $f(v_2) \geq f(v_1) + n + 1$. Suppose that $k \notin \{1, 2\}$, then $d(v_2, v_k) = 2$, for all $v_k \in V(S_m)$. Thus, without loss of generality, suppose that v_m is the last vertex on $V(S_m)$, then $f(v_m) \geq f(v_0) + (n+1) + n(m-2)$ and the claim follows. □

Remark 3.1 It is easy to confirm that given a star S_m with center vertex v_1, if for a positive integer α, $rn(S_m) = \alpha$, then either $f(v_1)$ is f_{min} or f_{max}.

Now we establish lower bound for $G(i)$.

Lemma 3.2 *Let $G(i)$ be a subgraph of $G_{m,n}$ and let f be the radio label on $V(G_{m,n})$. Then, $rn(G(i)) \geq f(v_1) + mn - \frac{n}{2} + 2$, where v_1 is the center vertex of $S_{m(i+\frac{n}{2})}$.*

Proof Let $S_{m(i)}$ and $S_{m(j)}$ be the stars on $G(i) \subset G_{m,n}$, where $j = i + \frac{n}{2}$. By Lemma 3.1, $f(v_m) = f(v_1) + mn - n + 1$, with $f(v_m) = \max\{f(v_t) : v_t \in V(S_m(j))\}$, and v_1 the center of $S_{m(j)}$. Now, let u_1 be the center vertex of $S_{m(i)}$. It is clear that $d(u_1, v_1) = \frac{n+2}{2}$. Thus, $f(u_1) \geq f(v_1) + mn - n + 1 + \frac{n+2}{2} = f(v_1) + mn - \frac{n}{2} + 2$.

Claim. For optimal radio labeling of $G(i)$, maximum label on $S_{m(i)}$ is at least $f(u_1)$.

Consider some $u_m \in V(S_m)$, such that $m \neq 1$ and $d(u_m, v_m) = \frac{n}{2} + 2$. Then $f(u_m) = f(v_1) + mn - \frac{n}{2} + 1$. By Lemma 3.1, the span$f$ of f for a star S_m is $mn - n + 1$. Now, $f(u_m) - (mn - n + 1) = f(v_i) + \frac{n}{2}$. Thus, by Remark 3.1, $f(u_1) = f(v_1 + \frac{n}{2})$. This is a contradiction, considering that $d(u_1, v_1) = \frac{n}{2}$. □

Lemma 3.3 *Let $G^+(i) \subset G_{m,n}$ be $G(i) \cup w_1$, where w_1 is the center vertex of $S_{m(j+1)}$ and let f be a radio labeling on $G_{m,n}$, where n is even. Then, the span$f$ of f on $G^+(i) \geq mn + 3$.*

Proof Let u_1 be the center vertex of $S_{m(i)}$. It can be verified that $d(u_1, w_1) = \frac{n}{2}$. By the proof of Lemma 3.1, $f(u_1) \geq f(v_1) + mn - \frac{n}{2} + 2$, where v_1 is the center vertex of $S_{m(j)}$. Thus by definition, $f(w_1) \geq f(v_1) + mn - \frac{n}{2} + 2 + \frac{n+2}{2} = f(v_1) + mn + 3$. Since $f(v_1)$ is the minimum label on $G(i)$, the result follows. □

Now we present the lower bound for stacked-book graph $G_{m,n}$, where n is an even integer and $m \geq 3$.

Theorem 3.1 *Let $G = G_{m,n}$ be a stacked-book graph with $m \geq 3$ and n an even integer. Furthermore, let f be the radio labeling on G. Then, $rn(G) \geq \frac{mn^2}{2} + n - 1$.*

Proof From the definition of $G(i)$, graph $G_{m,n}$ contains $\frac{n}{2}$ number of $G(i)$ subgraphs. Likewise, it can be seen that $G_{m,n}$ contains $\frac{n-1}{2}$ number of $G^+(i)$ subgraphs. Now, let $G(\frac{n}{2})$, induced by $S_{m(\frac{n}{2})}$ and $S_{m(n)}$ be the last $G(i)$ subgraphs on $G_{m,n}$ and $G^+(1), G^+(2), \cdots, G^+(\frac{n-1}{2})$ be the $\frac{n-1}{2}$ number of $G^+(i)$ graphs. By the earlier result, if $f(v_1) = 0$, then $rn(G_{m,n}) \geq \left(\frac{n-1}{2}\right)(mn + 3) + mn - \frac{n}{2} + 2 = \frac{mn^2}{2} + n - 1$. □

In what follows, we examine the upper bound of the stacked-book graph $G_{m,n}$.

Lemma 3.4 *Let $G(i)$ be a subgraph of $G_{m,n}$ induced by $\left\{V_{(i)}, V_{(i+\frac{n}{2})}\right\}$. Then for any pair $v \in V_{(i)}$ and $u \in V_{(i+\frac{n}{2})}$, such that $d(u, v) \geq \frac{n}{2} + 1$, $|f(v) - f(u)| \geq \frac{n}{2}$.*

Proof Let $u = u_{k,i} \in V_{(i)}$ and $v = u_{t,i+\frac{n}{2}} \in V_{(i+\frac{n}{2})}$. Since $d(u, v) > \frac{n}{2}$, then by Remark 2.4, $k \neq t$. Suppose that neither u nor v is the center vertex of their respective stars $S_{m(i)}$ and $S_{m(i+\frac{n}{2})}$. Then, $d(u, v) = diam(G(i))$. Now, let the radio label on u and v be $f(u)$ and $f(v)$ respectively. Suppose, without loss of generality, that $f(v) > f(u)$. Then $f(v) \geq f(u) + diam(G_{m,n}) + 1 - diam(G(i))$, which implies that

$$f(v) \quad \geq \quad f(u) + \frac{n}{2}.$$

This implies that $f(v) - f(u) \geq \frac{n}{2}$. Similarly, if $f(u) \geq f(v)$, then $f(u) - f(v) \geq \frac{n}{2}$ and thus, the claim follows. □

The following remarks can be confirmed by applying similar methods as in the proof of Lemma 3.4.

Remark 3.2 Suppose that either of u, v in Lemma 3.4, say u, is such that for any $u' \in V_{(i)}$,

$uu' \in E(S_{m(i)})$. Then $d(u,u') = \frac{n}{2}+1$ and $|f(u)-f(v)| \geq \frac{n}{2}+1$.

Remark 3.3 Let $u, u' \in V_{(i)}$. If $d(u,u') = 1$, then $|f(u)-f(u')| \geq n+1$ and $|f(u)-f(u')| \geq n$ for $d(u,u') = 2$.

Theorem 3.2 *Let $m > 3$ be odd and $G(i) \subseteq G_{m,n}$, be induced by $\left\{V_{(i)}, V_{(i+\frac{n}{2})}\right\}$. then, $rn(G(i)) \leq f(v_1) + mn - \frac{n}{2} + 2$, where v_1 is the center star $S_{m(1+\frac{n}{2})}$.*

Proof Let $V_{(i)} = \{u_{1,i}, u_{2,i}, \cdots, u_{m,i}\}$ and $V_{(t)} = \{u_{1,t}, u_{2,t}, \cdots, u_{m,t}\}$, where $t = i + \frac{n}{2}$. For $r \in [1,m]$, set $u_{r,i} \in V_{(i)}$ as α_r and $u_{r,t} \in V_{(t)}$ as β_r. From earlier remark, $d(\beta_r, \alpha_r) \in \left\{\frac{n}{2}+1, \frac{n}{2}+2\right\}$ for $r \neq s$. Now, for every pair α_s, β_r, where $\alpha_s \in V_{(i)}$, and $\beta_r \in V_{(t)}$, let $r \neq s$ except otherwise stated. Let α_1 and β_1 be the respective centers of the stars $S_{m(i)}$ and $S_{m(t)}$ induced by $V_{(i)}$ and $V_{(t)}$ and let the radio label on β_1 be $f(\beta_1)$ such that $f(\beta_1) = \min\{f(\beta_i) : 1 \leq i \leq m\}$. Since β_1 is the center of $S_{m(t)}$, then given $\alpha_2 \in V_{(i)}$, $d(\beta_1, \alpha_2) = \frac{n}{2}+1$. Now set $p = diam(G_{m,n}) + 1 - d(\beta_1, \alpha_r)$, $r \neq 1$. Hence, $p = n + 2 - (\frac{n}{2}+1) = \frac{n}{2}+1$. Suppose that $\alpha_j \in V_{(i)}$ and $\beta_k \in V_{(t)}$, such that $1 \neq j \neq k \neq 1$. Then, $d(\alpha_j, \beta_k) = \frac{n}{2}+2$. So we set $q = diam(G_{m,n}) + 1 - d(\alpha_j, \beta_k) = \frac{n}{2}$. For $f(\beta_1)$ and some $\alpha_2 \in V_{(i)}$, $f(\alpha_2) = f(\beta_1) + p$. Also, for $\beta_3 \in V_{(t)}$, $f(\beta_3) = f(\alpha_2) + q = f(\beta_1) + p + q$ and $f(\alpha_4) = f(\beta_1) + 2q + p$. We continue to label the vertices on both $V_{(i)}$ and $V_{(t)}$ alternatively based on the last value attained. Therefore, for m odd,

$$\begin{aligned} f(\beta_m) &= f(\alpha_{m-1}) + \frac{n}{2} \\ &= f(\beta_1) + (m-2)q + p. \end{aligned}$$

It can be seen that there does not exist $\alpha_d \in V_{(i)}$, such that $d > m$. So, we reverse the order of labeling, such that for β_m, α_3, $f(\alpha_3) = f(\beta_m) + q = f(\beta_1) + (m-2)q + 2p$. Also, for the pair α_3, β_2, $f(\beta_2) = f(\beta_1) + (m-2)q + 2q + p$. This continues until we reach the pair α_m, β_{m-1}, and obtain

$$f(\alpha_{m-1}) = f(\beta_1) + (2m-3)q + p.$$

Finally, we consider the pair β_{m-1} and α_1. Since α_1 is the center of $S_{(i)}$, then $d(\alpha_1, \beta_{m-1}) = \frac{n}{2}+1$ and hence,

$$\begin{aligned} f(\alpha_1) &= f(\alpha_{m-1}) + p \\ &= f(\beta_1) + (2m-3)q + 2p \\ &= f(\beta_1) + mn - \frac{n}{2} + 2. \end{aligned}$$

Hence, $rn(G(i)) \leq f(v_1) + mn - \frac{n}{2} + 2$, where m is odd and n even. □

Next, we directly apply Theorem 3.2 to get results following.

Lemma 3.5 *Let $\bar{G}(i)$ be induced by $\left\{S_{m(i)}, S_{m(i+\frac{n}{2})}, \gamma_1\right\}$, where γ_1 is the center of star $S_{m(i+\frac{n}{2}+1)}$, induced by $V_{(i+\frac{n}{2}+1)}$. Then, $f(\gamma_1) \leq f(\beta_1) + mn + 3$.*

Proof For α_1 and β_1 centers of stars $S_{(i)}$ and $S_{(i+\frac{n}{2})}$ respectively, let $f(\alpha_1) = f(\beta_1) + mn - \frac{n}{2} + 2$, as established in Theorem 3.2. Then, $d(\alpha_1, \gamma_1) = \frac{n}{2} + 1$. Therefore,

$$\begin{aligned} f(\gamma_1) &= f(\alpha_1) + p \\ &= f(\beta_1) + mn + 3. \end{aligned}$$

This completes the proof. □

Now, for β_1, the center of $S_{m(1+\frac{n}{2})}$, induced by $V_{(1+\frac{n}{2})}$. By setting $f(\beta_1) = 0$, we establish an upper bound for the radio number of a stacked-book graph $G_{m,n}$ in the next results.

Theorem 3.3 *For a graph $G_{m,n}$ with m odd and n even, $rn(G_{m,n}) \leq \frac{mn^2}{2} + n - 1$.*

Proof Let $\left\{v_{1(1)}, v_{2(1)}, v_{3(1)}, \cdots, v_{n(1)}\right\}$ be the set of the respective centers of stars $S_{m(1)}$, $S_{m(2)}$, $S_{m(3)}$, $\cdots$, $S_{m(n)}$ in $G_{m,n}$. Also, suppose that $f(v_{\frac{n}{2}+1(1)}) = 0$. From the Lemma 3.5, $f(v_{\frac{n}{2}+2(1)}) = mn+3$; $f(v_{\frac{n}{2}+3(1)}) = 2(mn+3)$ and so on. In the end, $f(v_{n(1)}) = (\frac{n}{2}-1)(mn+3)$. Also, let $v_{n-\frac{n}{2}(1)} = v_{\frac{n}{2}(1)}$ be the center of $S_{m(\frac{n}{2})} \subset G_{m,n}$ and let $S_{m(\frac{n}{2})}, S_{m(n)}$ induce the graph $G(\frac{n}{2}) \subset G_{m,n}$. By Theorem 3.2,

$$\begin{aligned} rn\left(G\left(\frac{n}{2}\right)\right) &\leq f(v_{n(1)}) + mn - \frac{n}{2} + 2 \\ &\leq \frac{mn^2}{2} + n - 1. \end{aligned}$$

This completes the proof. □

Theorem 3.4 *Let m, n be even. Then $rn(G_{m,n}) \leq \frac{mn^2}{2} + n - 1$.*

Proof The proof follows similar argument and technique as in Theorems 3.2, 3.3 and Lemma 3.5. □

Notice that Theorems 3.1, 3.3 and 3.4 establish the radio number of $G_{m,n}$, where $m \geq 4$ and n is even, as recapped in the next theorem.

Theorem 3.5 *Let $G_{m,n}$ be a stacked-book graph with $m \geq 4$ and n even, then, $rn(G_{m,n}) = \frac{mn^2}{2} + n - 1$.*

Next we consider the case where $m = 3$. First we present a result that is equivalent to Theorem 3.2 with respect to $m = 3$.

Theorem 3.6 *Let $G_{3,n}$ be a stacked-book graph, where n is even. Suppose that the pair $\left\{S_{3(i)}, S_{3(i+\frac{n}{2})}\right\}$ form a subgraph $G(i)$ of $G_{3,n}$. Then, $rn(G(i)) \leq f(u_1) + \frac{5n}{2} + 3$, where u_1 is the center vertex of $S_{3(i+\frac{n}{2})}$.*

Proof Let $V_{(i)} = \{v_1, v_2, v_3\}$ and $V_{(i+\frac{n}{2})} = \{u_1, u_2, u_3\}$ where $V_{(i)}$ and $V_{(i+\frac{n}{2})}$ are vertex sets of stars $S_{3(i)}$ and $S_{3(i+\frac{n}{2})}$ in $G_{3,n}$ respectively. Also, let v_1 and u_i be the respective center vertices of $S_{3(i)}$ and $S_{3(i+\frac{n}{2})}$. From earlier remark, $d(v_1, u_1) = \frac{n}{2} + 1$. Suppose that $f(u_1)$, the radio label of u_1 is the smallest possible radio label on $G(i)$, then,

$$\begin{aligned} f(v_2) &= f(v_1) + diam(G_{3,n}) + 1 - d(v_1, u_1) \\ &= f(u_i) + \frac{n}{2} + 1. \end{aligned}$$

For v_2, u_3, $d(v_2, u_3) = \frac{n}{2} + 2$, $f(u_3) = f(u_1) + n + 1$; For u_3, v_1, $d(u_3, v_1) = \frac{n}{2} + 1$, $f(v_1) = f(u_1) + \frac{3n}{2} + 2$; For v_1, u_2, $d(v_1, u_2) = \frac{n}{2} + 1$, $f(u_2) = f(u_1) + 2n + 3$ and finally, for the pair v_3, u_2, $d(v_3, u_3) = \frac{n}{2} + 2$ and $f(u_3) = f(u_1) + \frac{5n}{2} + 3$. Hence, $rn(G(i)) \leq f(u_1) + \frac{5n}{2} + 3$.□

Next, we obtain the following result.

Lemma 3.6 *Let κ_1 be the center of star $S_{3(i+\frac{n}{2})+1} \subseteq G_{3,n}$ and let $\bar{H}(1)$ be a subgraph of $G_{(3,m)}$ induced by $\left\{S_{3(i)}, S_{3(i+\frac{n}{2}),\kappa_1}\right\}$. Then $f(\kappa_1) \leq 3n + 1$.*

Proof The vertex with the maximum value of radio label in Theorem 3.6 is u_3. Let us adopt this, with $f(u_3) = f(u_1) + \frac{5n}{2} + 3$. Now, $d(u_3, \kappa_1) = \frac{n}{2} + 2$. Therefore, $f(\kappa_1) = f(u_1) + 3n + 3$. This completes the proof. □

In the final result here, we set $f(u_1) = 0$, for u_i, the center of star $S_{3(1+\frac{n}{2})}$.

Theorem 3.7 *Let n be an even positive integer. Then, $rn(G_{3,n}) \leq \frac{3n^2}{2} + n$.*

Proof The proof follows similar technique adopted in Theorem 3.4. □

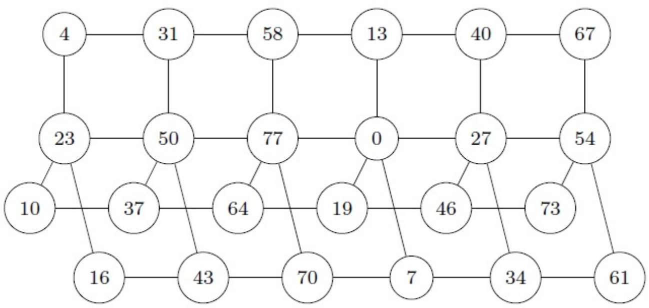


Figure 1. A $G_{4,6}$ graph with $rn(G_{4,6}) \leq 77$.

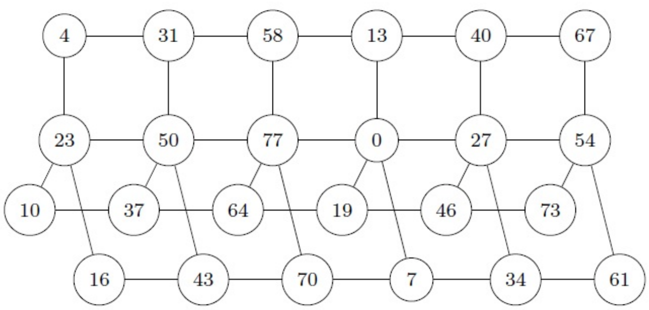


Figure 1. A $G_{3,6}$ graph with $rn(G_{3,6}) \leq 60$.

Notice that the radio numberings for $G_{4,6}$ and $G_{3,6}$ are shown in Figures 1 and 2. They demonstrate that that the radio numbers of the graphs can not be more than 77 and 60 respectively.

§4. Conclusion

It is noteworthy to look at some of the results in [7]. A $G_{3,n}$ is a $3 \times n$ grid. By [7], it is seen that $rn(G_{3,6}) = 59$, which is better than the result in Figure 2 above by 1. But this is still a considerable improvement compared with a upper bound of 109 suggested in [1]. In establishing the upper bound for $G_{3,n}$, it is observed that the number of the pair $u, v \in V(G_{3,n})$ for which $d(u,v) = \frac{diam(G_{3,n})+1}{2}$ is more than the case where $d(u,v) = \frac{n}{2}$ in each of the segments of radio labeling of the stacked-graph. However, the reverse proves to be the case in $G_{m,n}$, $m \geq 4$.

References

[1] Ajayi D.O. and Adefokun T.C., On bounds of radio number of certain product graphs, *J. Nigerian Math. Soc.*, 48, 2, (2018), 71–8.

[2] Bantva D., Vaiya S. and Zhou S., Radio numbers of trees, *Electron. Notes in Discrete Math.*, 48, (2015), 135–141.

[3] D.Bantva, Radio numbers of middle graph of paths, *Electron. Notes in Discrete Math.*, 63, (2017), 93–100.

[4] Chartrand G., Erwin D., and Zhang P., A graph labeling problem suggested by FM channel restrictions, *Bull. Inst. Combin. Appl.*, 43, (2005), 43–57.

[5] Chartrand G., Erwin D., Harary F. and Zhang P., Radio labelings of graphs, *Bull. Inst. Combin. Appl.* 33, (2001), 77-85.

[6] Hale W.K., Frequency assignment theory and applications, *Proc. IEEE*, 68, 12, (1980), 1497–1514.

[7] Jiang T.-S., The radio number of grid graphs, *arXiv*: 1401.658v1, (2014).

[8] Liu D.D.-F. and M. Xie, Radio number for square cycles, *Cong. Numer.*, 169, (2004), 101–125.

[9] Daphne Der-Fen Liu and M. Xie Radio number for square paths, *ARS. Combin.*, 90, (2009), 307–319.

[10] Daphne Der-Fen Liu and Zhu X., Multilevel distance labelings for paths and cycles, *SIAM J. Discrete Math.*, 19, 3, (2005), 610–621.

[11] Naseem A., Shabbir K. and Shaker H., The radio number of edge-joint graphs, *ARS Combin.*, 139, (2018), 337–351.

[12] Saha L. and Panigrahi P., On the radio numbers of toroidal grid, *Aust. Jour. Combin.*, 55, (2013), 273–288.

International J.Math. Combin. Vol.4(2022), 74-81

Pair Difference Cordial Labeling of Franklin Graph, Heawood Graph, Tietze Graph and Durer Graph

R. Ponraj

Department of Mathematics, Sri Paramakalyani College, Alwarkurichi–627 412, India

A. Gayathri

Research Scholor, Department of Mathematics

Manonmaniam Sundaranar University, Abhishekapati,Tirunelveli–627 012, India)

S. Somasundaram

Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli – 627 012, India

E-mail: ponrajmaths@gmail.com, gayugayathria555@gmail.com, somutvl@gmail.com

Abstract: Let $G = (V, E)$ be a (p, q) graph. Define

$$\rho = \begin{cases} \frac{p}{2}, & \text{if } p \text{ is even} \\ \frac{p-1}{2}, & \text{if } p \text{ is odd} \end{cases}$$

and $L = \{\pm 1, \pm 2, \pm 3, \cdots, \pm\rho\}$ called the set of labels. Consider a mapping $f : V \longrightarrow L$ by assigning different labels in L to the different elements of V when p is even and different labels in L to p-1 elements of V and repeating a label for the remaining one vertex when p is odd.The labeling as defined above is said to be a pair difference cordial labeling if for each edge uv of G there exists a labeling $|f(u) - f(v)|$ such that $\left|\Delta_{f_1} - \Delta_{f_1^c}\right| \leq 1$, where Δ_{f_1} and $\Delta_{f_1^c}$ respectively denote the number of edges labeled with 1 and number of edges not labeled with 1. A graph G for which there exists a pair difference cordial labeling is called a pair difference cordial graph. In this paper we investigate the pair difference cordial labeling behavior of Pair Difference Cordial Labeling of Franklin graph, Heawood graph, Tietze graph and Durer graph.

Key Words: Path, cycle, wheel, gear graph, ladder.

AMS(2010): 05C78.

§1. Introduction

In this paper we consider only finite, undirected and simple graphs. Cordial labeling was introduced in [2] by Cahit. Also cordial related labeling was studied in [15,16]. In [7] the notion of pair difference cordial labeling of a graph was introduced and also the pair difference cordial labeling behaviour of path, cycle, star, ladder have been studied. In [8,9,10,11,12,13,14], the

[1]Received November 2, 2022, Accepted December 15, 2022.

pair difference cordial labeling behaviour of snake related graph and butterfly graph have been investigated. In this paper we have study about the pair difference cordiality of some named graphs like Franklin graph, Heawood graph, Tietze graph and Durer graph.

§2. Preliminaries

Definition 2.1([3]) *The corona graph $G_1 \odot G_2$ is the graph obtained by taking one copy of G_1 and n copies of G_2 and joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy G_2, where G_1 is graph of order n.*

Definition 2.2([4]) *Let C_n be the cycle $a_1a_2a_3\cdots a_na_1$ where $n \equiv 0(mod 12)$. Then FG_n is the graph with the vertex set $V(FG_n) = V(C_n)$ and $E(FG_n) = E(C_n) \cup \{a_{4i+1}a_{4i+4} : 0 \le i \le \frac{n-4}{4}\}$ $\cup\{a_{4i+3}a_{4i+10} : 0 \le i \le \frac{n-8}{4}\}\cup\{a_2a_{n-5}, a_{n-1}a_6\}$. FG_n has n vertices and $\frac{3n}{2}$ edges. Note that FG_{12} is called the Franklin graph.*

Definition 2.3([3]) *Let C_n be the cycle $a_1a_2a_3\cdots a_na_1$ where $n \equiv 0(mod 14)$. Then HG_n is the graph with the vertex set $V(HG_n) = V(C_n)$ and $E(HG_n) = E(C_n) \cup \{a_{2i+1}a_{2i+6} : 0 \le i \le \frac{n-4}{2}\}$ $\cup\{a_2a_{n-3}, a_{n-1}a_4\}$. HG_n has n vertices and $\frac{3n}{2}$ edges. The graph HG_{14} is called the Heawood graph which is given in Figure 1.*

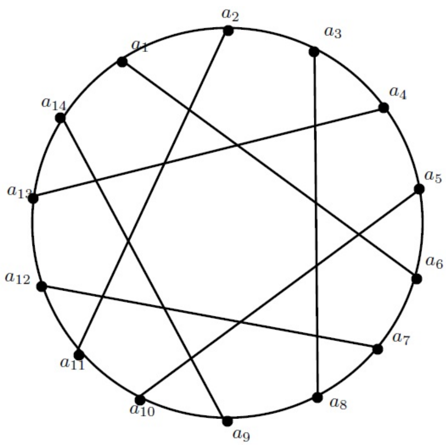


Figure 1

Definition 2.4([3]) *Let C_n be the cycle $v_1v_2v_3\cdots v_nv_1$ where $n \equiv 0(mod 9)$. Then TG_n is the graph with the vertex set $V(TG_n) = V(C_n) \cup\{u_{3i-2} : 1 \le i \le \frac{n}{3}\}$ and $E(TG_n) = E(C_n)$ $\cup\{u_{3i-2}v_{3i-2} : 1 \le i \le \frac{n}{3}\}\cup\{u_{3i-1}u_{3i+3} : 1 \le i \le \frac{n-3}{3}\} \cup\{u_{n-1}u_3\}$. TG_n has $\frac{4n}{3}$ vertices and $2n$ edges. Note that TG_9 is called the Tietze graph.*

Definition 2.5([3]) *Let C_n be the cycle $v_1v_2v_3\cdots v_nv_1$ where $n \equiv 0(mod 6)$. Then DG_n is the graph with the vertex set $V(DG_n) = V(C_n) \cup\{g_i : 1 \le i \le n\}$ and $E(DG_n) = E(C_n)$ $\cup\{g_iv_i : 1 \le i \le n\}\cup\{g_ig_{i+2} : 1 \le i \le n-2\} \cup\{g_ng_2, g_{n-1}g_1\}$. DG_n has $2n$ vertices and $3n$ edges. The graph DG_6 is called the Durer graph.*

§3. Main Results

Theorem 3.1 *FG_n is pair difference cordial for all values of $n \equiv 0(mod 12)$.*

Proof Take the vertex set and edge set from definition 2.2.

Assign the labels $1, 2, 3, \cdots, \frac{n}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ respectively and assign the labels $-1, -2, -3, \cdots, -\frac{n}{2}$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}$. Next assign the labels $-4, -6, -5, -7$ respectively to the vertices $a_{\frac{n+8}{2}}, a_{\frac{n+10}{2}}, a_{\frac{n+12}{2}}, a_{\frac{n+14}{2}}$ and assign the labels $-8, -10, -9, -11$ respectively to the vertices $a_{\frac{n+16}{2}}, a_{\frac{n+18}{2}}, a_{\frac{n+20}{2}}, a_{\frac{n+22}{2}}$. Proceeding like this until we reach the vertex a_n. Here $\Delta_{f_1^c} = \Delta_{f_1} = \frac{3n}{4}$. □

Theorem 3.2 *HG_n is pair difference cordial for all values of $n \equiv 0(mod 14)$.*

Proof Take the vertex set and edge set from Definition 2.3.

Assign the labels $1, 2, 3, \cdots, \frac{n}{2}$ to the vertices $a_1, a_2, a_3, \cdots, a_{\frac{n}{2}}$ respectively and assign the labels $-1, -2, -3, \cdots, -\frac{n}{2}$ respectively to the vertices $a_{\frac{n+2}{2}}, a_{\frac{n+4}{2}}, a_{\frac{n+6}{2}}$. Next assign the labels $-4, -6, -5, -7$ respectively to the vertices $a_{\frac{n+8}{2}}, a_{\frac{n+10}{2}}, a_{\frac{n+12}{2}}, a_{\frac{n+14}{2}}$ and assign the labels $-8, -10, -9, -11$ respectively to the vertices $a_{\frac{n+16}{2}}, a_{\frac{n+18}{2}}, a_{\frac{n+20}{2}}, a_{\frac{n+22}{2}}$. Proceeding like this until we reach the vertex a_n.

The Table 1 given below establish that this vertex labeling f is a pair difference cordial of HG_n for $n \equiv 0(mod 14)$.

Nature of $n = 14k$	$\Delta_{f_1^c}$	Δ_{f_1}
$n = 14k$, k is odd	$\frac{3n-2}{4}$	$\frac{3n+2}{4}$
$n = 14k$, k is even	$\frac{3n}{4}$	$\frac{3n}{4}$

Table 1

This completes the proof. □

Theorem 3.3 *TG_n is pair difference cordial for all values of $n \equiv 0(mod 9)$.*

Proof Take the vertex set and edge set from Definition 2.4.

Assign the labels $1, 2, 3, \cdots, \frac{2n}{3}$ respectively to the vertices $v_1, v_2, v_3, \cdots, v_{\frac{2n}{3}}$ and assign the labels $-1, -2, -3, \cdots, -\frac{n}{3}$ to the vertices $v_{\frac{2n+3}{3}}$, $v_{\frac{2n+6}{3}}$, $v_{\frac{2n+9}{3}}$, $\cdots, v_n$ respectively. Next assign the labels to the vertices $u_i, 1 \leq i \leq n$. There are two cases arises.

Case 1. n is even.

Assign the labels $-\frac{n+3}{3}, -\frac{n+9}{3}, -\frac{n+15}{3}, \cdots, -(\frac{2n}{3} - 3)$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n-3}{3}}$ and assign the labels $-\frac{n+6}{3}, -\frac{n+12}{3}, -\frac{n+18}{3}, \cdots, -(\frac{2n}{3} - 2)$ to the vertices $u_{\frac{n}{3}}$, $u_{\frac{n+3}{3}}$, $u_{\frac{n+6}{3}}$, $\cdots, u_{n-2}$ respectively. Finally assign the labels $-(\frac{2n}{3} - 1), -(\frac{2n}{3})$ respectively to the vertices u_{n-1}, u_n.

Case 2. n is odd.

Assign the labels $-\frac{n+6}{3}, -\frac{n+12}{3}, -\frac{n+18}{3}, \cdots, -(\frac{2n}{3} - 3)$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n-3}{3}}$ and assign the labels $-\frac{n+3}{3}, -\frac{n+9}{3}, -\frac{n+15}{3}, \cdots, -(\frac{2n}{3} - 2)$ to the vertices $u_{\frac{n+3}{3}}$, $u_{\frac{n+3}{3}}$,

$u_{\frac{n+6}{3}}, \cdots, u_{n-2}$ respectively. Lastly assign the labels $-(\frac{2n}{3}-1)$,$-(\frac{2n}{3})$ respectively to the vertices u_{n-1}, u_n. In both cases, we get $\Delta_{f_1^c} = \Delta_{f_1} = n$. □

Theorem 3.4 *DG_n is pair difference cordial for all values of $n \equiv 0(mod 6)$.*

Proof Take the vertex set and edge set from Definition 2.5.

Assign the labels $1, 2, 3, \cdots, n$ to the vertices $v_1, v_2, v_3, \cdots, v_n$ respectively and assign the labels $-1, -2, -3, , \cdots, -\frac{n}{2}$ respectively to the vertices g_1,$g_3, g_5, \cdots$,g_{n-1}. Now assign the labels $-\frac{n+4}{2}$,$-\frac{n+8}{2}$,$-\frac{n+12}{2}$,$\cdots$,$-(n-3)$ respectively to the vertices g_2,$g_4, g_6, \cdots$,$g_{\frac{n-6}{2}}$ and assign the labels $-\frac{n+2}{2}$,$-\frac{n+6}{2}$,$-\frac{n+10}{2}$,$\cdots$,$-(n-2)$ respectively to the vertices $g_{\frac{n-2}{2}}$,$g_{\frac{n+2}{2}}$,$g_{\frac{n+6}{2}}$,$\cdots, g_{n-2}$. Finally assign the labels $-(n-1), -n$ respectively to the vertices g_{n-1}, g_n.

Clearly $\Delta_{f_1^c} = \Delta_{f_1} = \frac{3n}{2}$. A pair difference cordial labeling of DG_{18} is given in Figure 2.

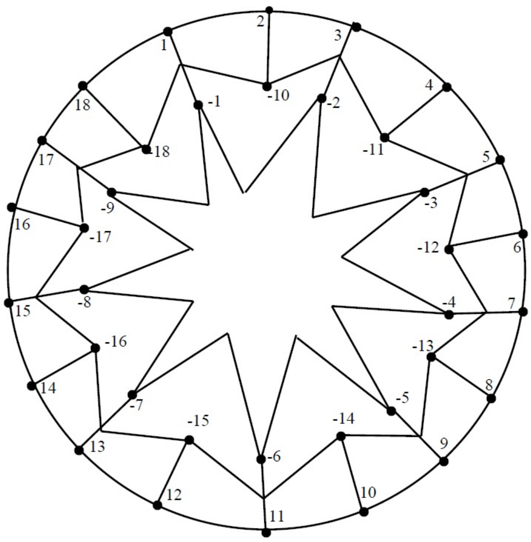


Figure 2

Theorem 3.5 *$FG_n \odot K_1$ is pair difference cordial for all values of $n \equiv 0(mod 12)$.*

Proof Let $V(FG_n \odot K_1) = V(FG_n) \cup \{x_i : 1 \leq i \leq n\}$ and $E(FG_n \odot K_1) = E(FG_n) \cup \{a_i x_i : 1 \leq i \leq n\}$. Note that $FG_n \odot K_1$ has $2n$ vertices and $\frac{5n}{2}$ edges.

Assign the labels $1, 5, 9, \cdots, n-3$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_{n-1}$ and assign the labels $4, 8, 12, \cdots, n$ to the vertices $x_2, x_4, x_6, \cdots, x_n$ respectively. Next assign the labels $2, 6, 10, \cdots, n-2$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{n-1}$ and assign the labels $3, 7, 11, \cdots, n-1$ to the vertices $a_2, a_4, a_6, \cdots, a_n$ respectively.

Now assign the labels $-1, -3, -5, \cdots, -(n-1)$ respectively to the vertices $x_1, x_2, x_3, \cdots, x_n$ and assign the labels $-2, -4, -6, \cdots, -(n)$ to the vertices $a_1, a_2, a_3, \cdots, a_n$ respectively.

Clearly $\Delta_{f_1^c} = \Delta_{f_1} = \frac{5n}{4}$. □

Theorem 3.6 *$HG_n \odot K_1$ is pair difference cordial for all values of $n \equiv 0(mod 14)$.*

Proof Let $V(HG_n \odot K_1) = V(HG_n) \cup \{x_i : 1 \leq i \leq n\}$ and $E(HG_n \odot K_1) = E(HG_n) \cup$

$\{a_ix_i : 1 \leq i \leq n\}$. Note that $HG_n \odot K_1$ has $2n$ vertices and $\frac{5n}{2}$ edges.

There are two cases arises.

Case 1. $n = 14k$ where k is odd.

Assign the labels $1, 5, 9, \cdots, n-5$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_{\frac{n-4}{2}}$ and assign the labels $4, 8, 12, \cdots, n-2$ to the vertices $x_2, x_4, x_6, \cdots, x_{\frac{n-2}{2}}$ respectively. Next assign the labels $-1, -3, -5, \cdots, -(n-3)$ respectively to the vertices $x_{\frac{n}{2}}$, $x_{\frac{n+2}{2}}$,$x_{\frac{n+4}{2}}$,$\cdots, x_{n-2}$ and assign the labels $-2, -4, -6, \cdots, -(n-2)$ to the vertices $a_{\frac{n}{2}}$, $a_{\frac{n+2}{2}}$,$a_{\frac{n+4}{2}}$,$\cdots, a_{n-2}$ respectively.

Now assign the labels $2, 6, 10, \cdots, n-4$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-4}{2}}$ and assign the labels $3, 7, 11, \cdots, n-3$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n-2}{2}}$ respectively. Lastly assign the labels $n-1, n, -(n-1), -n$ respectively to the vertices $a_{n-1}, a_n, x_{n-1}, x_n$.

Case 2. $n = 14k$ where k is even.

Assign the labels $1, 5, 9, \cdots, n-3$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_{\frac{n-2}{2}}$ and assign the labels $4, 8, 12, \cdots, n$ to the vertices $x_2, x_4, x_6, \cdots, x_{\frac{n}{2}}$ respectively. Next assign the labels $-1, -3, -5, \cdots, -(n-1)$ respectively to the vertices $x_{\frac{n+2}{2}}$, $x_{\frac{n+4}{2}}$,$x_{\frac{n+6}{2}}$,$\cdots, x_n$ and assign the labels $-2, -4, -6, \cdots, -(n)$ to the vertices $a_{\frac{n+2}{2}}$, $a_{\frac{n+4}{2}}$,$a_{\frac{n+6}{2}}$,$\cdots, a_n$ respectively.

Now assign the labels $2, 6, 10, \cdots, n-2$ respectively to the vertices $a_1, a_3, a_5, \cdots, a_{\frac{n-2}{2}}$ and assign the labels $3, 7, 11, \cdots, n-1$ to the vertices $a_2, a_4, a_6, \cdots, a_{\frac{n}{2}}$ respectively. Lastly assign the labels $n-1, n, -(n-1), -n$ respectively to the vertices $a_{n-1}, a_n, x_{n-1}, x_n$.

The Table 2 given below establish that this vertex labeling f is a pair difference cordial of $HG_n \odot K_1$, $n \equiv 0(mod14)$.

Nature of n	$\Delta_{f_1^c}$	Δ_{f_1}
$n = 14k$, k is odd	$\frac{5n+2}{4}$	$\frac{5n-2}{4}$
$n = 14k$, k is even	$\frac{5n}{4}$	$\frac{5n}{4}$

Table 2

This completes the proof. □

Theorem 3.7 *$DG_n \odot K_1$ is pair difference cordial for all values of $n \equiv 0(mod6)$.*

Proof Let $V(DG_n \odot K_1) = V(DG_n) \cup \{x_i, y_i : 1 \leq i \leq n\}$ and $E(DG_n \odot K_1) = E(DG_n) \cup \{g_ix_i, v_iy_i : 1 \leq i \leq n\}$. Note that $HG_n \odot K_1$ has $4n$ vertices and $5n$ edges.

Assign the labels $1, 5, 9, \cdots, 2n-3$ respectively to the vertices $y_1, y_3, y_5, \cdots, y_{n-1}$ and assign the labels $4, 8, 12, \cdots, 2n$ to the vertices $y_2, y_4, y_6, \cdots, y_n$ respectively. Next assign the labels $2, 6, 10, \cdots, 2n-2$ respectively to the vertices $v_1, v_3, v_5, \cdots, v_{n-1}$ and assign the labels $3, 7, 11, \cdots, 2n-1$ to the vertices $v_2, v_4, v_6, \cdots, v_n$ respectively.

Now assign the labels $-1, -3, -5, \cdots, -(2n-1)$ respectively to the vertices $x_1, x_2, x_3, \cdots, x_n$ and assign the labels $-2, -4, -6, \cdots, -(2n)$ to the vertices $g_1, g_2, g_3,$ $\cdots, g_n$ respectively.

Obviously $\Delta_{f_1^c} = \Delta_{f_1} = \frac{5n}{2}$. □

Theorem 3.8 *$TG_n \odot K_1$ is pair difference cordial for all values of $n \equiv 0(mod9)$.*

Proof Let $V(TG_n \odot K_1) = V(TG_n) \cup \{x_i, y_j : 1 \le i \le n, 1 \le j \le \frac{n}{3}\}$ and $E(TG_n \odot K_1) = E(TG_n) \cup \{v_i x_i, u_j y_j : 1 \le i \le n, 1 \le j \le \frac{n}{3}\}$. Note that $TG_n \odot K_1$ has $\frac{8n}{3}$ vertices and $\frac{10n}{3}$ edges. Our proof is divided into three cases.

Case 1. $n \equiv 0(mod 36)$.

Assign the labels $1, 5, 9, \cdots, n-3$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_{\frac{n-2}{2}}$ and assign the labels $4, 8, 12, \cdots, n$ to the vertices $x_2, x_4, x_6, \cdots, x_{\frac{n}{2}}$ respectively. Next assign the labels $2, 6, 10, \cdots, n-2$ respectively to the vertices $v_1, v_3, v_5, \cdots, v_{\frac{n-2}{2}}$ and assign the labels $3, 7, 11, \cdots, n-1$ to the vertices $v_2, v_4, v_6, \cdots, v_{\frac{n}{2}}$ respectively.

Assign the labels $-1, -5, -9, \cdots, -(n-3)$ respectively to the vertices $x_{\frac{n+2}{2}}, x_{\frac{n+6}{2}}, x_{\frac{n+10}{2}}, \cdots, x_{n-1}$ and assign the labels $-4, -8, -12, \cdots, -n$ to the vertices $x_{\frac{n+4}{2}}, x_{\frac{n+8}{2}}, x_{\frac{n+12}{2}}, \cdots, x_n$ respectively. Next assign the labels $-2, -6, -10, \cdots, -(n-2)$ respectively to the vertices $v_{\frac{n+2}{2}}, v_{\frac{n+6}{2}}, v_{\frac{n+10}{2}}, \cdots, v_{n-1}$ and assign the labels $-3, -7, -11, \cdots, -(n-1)$ to the vertices $v_{\frac{n+4}{2}}, v_{\frac{n+8}{2}}, v_{\frac{n+12}{2}}, \cdots, v_n$ respectively.

Now we assign the labels $(n+1), (n+3), (n+5), \cdots, \frac{4n-3}{3}$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n}{6}}$ and assign the labels $(n+2), (n+4), (n+6), \cdots, \frac{4n}{3}$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n}{6}}$ respectively. Next assign the labels $-(n+1), -(n+5), -(n+9), \cdots, -(\frac{4n-9}{3})$ respectively to the vertices $u_{\frac{n+6}{6}}, u_{\frac{n+12}{6}}, u_{\frac{n+18}{6}}, \cdots, u_{\frac{3n}{12}}$ and assign the labels $-(n+3), -(n+7), -(n+11), \cdots, -(\frac{4n-3}{3})$ respectively to the vertices $y_{\frac{n+6}{6}}, y_{\frac{n+12}{6}}, y_{\frac{n+18}{6}}, \cdots, y_{\frac{3n}{12}}$. Now assign the labels $-(n+2), -(n+6), -(n+10), \cdots, -(\frac{4n-6}{3})$ respectively to the vertices $u_{\frac{3n+12}{12}}, u_{\frac{3n+24}{12}}, u_{\frac{3n+36}{12}}, \cdots, u_n$ and assign the labels $-(n+4), -(n+8), -(n+12), \cdots, -(\frac{4n}{3})$ respectively to the vertices $y_{\frac{3n+12}{12}}, y_{\frac{3n+24}{12}}, y_{\frac{3n+36}{12}}, \cdots, y_n$.

Case 2. $n \equiv 9(mod 36)$.

Assign the labels $1, 5, 9, \cdots, n-4$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_{\frac{n-3}{2}}$ and assign the labels $4, 8, 12, \cdots, n-1$ to the vertices $x_2, x_4, x_6, \cdots, x_{\frac{n-1}{2}}$ respectively. Next assign the labels $2, 6, 10, \cdots, n-3$ respectively to the vertices $v_1, v_3, v_5, \cdots, v_{\frac{n-3}{2}}$ and assign the labels $3, 7, 11, \cdots, n-2$ to the vertices $v_2, v_4, v_6, \cdots, v_{\frac{n-1}{2}}$ respectively.

Assign the labels $-1, -5, -9, \cdots, -(n-4)$ respectively to the vertices $x_{\frac{n+1}{2}}, x_{\frac{n+5}{2}}, x_{\frac{n+9}{2}}, \cdots, x_{n-2}$ and assign the labels $-4, -8, -12, \cdots, -(n-1)$ to the vertices $x_{\frac{n+3}{2}}, x_{\frac{n+7}{2}}, x_{\frac{n+11}{2}}, \cdots, x_{n-1}$ respectively. Next assign the labels $-2, -6, -10, \cdots, -(n-3)$ respectively to the vertices $v_{\frac{n+1}{2}}, v_{\frac{n+5}{2}}, v_{\frac{n+9}{2}}, \cdots, v_{n-2}$ and assign the labels $-3, -7, -11, \cdots, -(n-2)$ to the vertices $v_{\frac{n+3}{2}}, v_{\frac{n+7}{2}}, v_{\frac{n+11}{2}}, \cdots, v_{n-1}$ respectively. Assign the labels $n, -n$ respectively to the vertices v_n, x_n.

Now we assign the labels $(n+1), (n+3), (n+5), \cdots, \frac{4n-6}{3}$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n-3}{6}}$ and assign the labels $(n+2), (n+4), (n+6), \cdots, \frac{4n-3}{3}$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n-3}{6}}$ respectively. Next assign the labels $-(n+1), -(n+5), -(n+9), \cdots, -(\frac{4n-18}{3})$ respectively to the vertices $u_{\frac{n+3}{6}}, u_{\frac{n+9}{6}}, u_{\frac{n+15}{6}}, \cdots, u_{\frac{n-5}{4}}$ and assign the labels $-(n+3), -(n+7), -(n+11), \cdots, -(\frac{4n-12}{3})$ respectively to the vertices $y_{\frac{n+3}{6}}, y_{\frac{n+9}{6}}, y_{\frac{n+15}{6}}, \cdots, y_{\frac{n-5}{4}}$. Now assign the labels $-(n+2), -(n+6), -(n+10), \cdots, -(\frac{4n-15}{3})$ respectively to the vertices $u_{\frac{n-1}{4}}, u_{\frac{n+3}{4}}, u_{\frac{n+7}{4}}, \cdots, u_{\frac{n-6}{3}}$ and assign the labels $-(n+4), -(n+8), -(n+12), \cdots, -(\frac{4n-9}{3})$ respectively to the vertices $y_{\frac{n-1}{4}}, y_{\frac{n+3}{4}}, y_{\frac{n+7}{4}}, \cdots, y_{\frac{n-6}{3}}$. Now assign the labels $-(\frac{4n-3}{3}), -(\frac{4n-6}{3}), -(\frac{4n}{3}), (\frac{4n}{3})$

respectively to the vertices $u_{\frac{n-3}{3}}, y_{\frac{n-3}{3}}, u_{\frac{n}{3}}, y_{\frac{n}{3}}$.

Case 3. $n \equiv 18(mod 36)$.

Assign the labels $1, 5, 9, \cdots, n-5$ respectively to the vertices $x_1, x_3, x_5, \cdots, x_{\frac{n42}{2}}$ and assign the labels $4, 8, 12, \cdots, n-3$ to the vertices $x_2, x_4, x_6, \cdots, x_{\frac{n-2}{2}}$ respectively. Next assign the labels $2, 6, 10, \cdots, n-4$ respectively to the vertices $v_1, v_3, v_5, \cdots, v_{\frac{n-4}{2}}$ and assign the labels $3, 7, 11, \cdots, n-2$ to the vertices $v_2, v_4, v_6, \cdots,$
$v_{\frac{n-2}{2}}$ respectively.

Assign the labels $-1, -5, -9, \cdots, -(n-5)$ respectively to the vertices $x_{\frac{n}{2}}, x_{\frac{n+4}{2}}, x_{\frac{n+8}{2}}, \cdots, x_{n-3}$ and assign the labels $-4, -8, -12, \cdots, -(n-3)$ to the vertices $x_{\frac{n+2}{2}}, x_{\frac{n+6}{2}}, x_{\frac{n+10}{2}}, \cdots, x_{n-2}$ respectively. Next assign the labels $-2, -6, -10, \cdots, -(n-4)$ respectively to the vertices $v_{\frac{n}{2}}, v_{\frac{n+4}{2}}, v_{\frac{n+8}{2}}, \cdots, v_{n-3}$ and assign the labels $-3, -7, -11, \cdots, -(n-2)$ to the vertices $v_{\frac{n+2}{2}}, v_{\frac{n+6}{2}}, v_{\frac{n+10}{2}}, \cdots, v_{n-2}$ respectively. Next we assign the labels $n-1, n, -(n-1), -n$ respectively to the vertices $x_{n-1}, v_{n-1}, x_n, v_n$.

Now we assign the labels $(n+1), (n+3), (n+5), \cdots, \frac{4n-3}{3}$ respectively to the vertices $u_1, u_2, u_3, \cdots, u_{\frac{n}{6}}$ and assign the labels $(n+2), (n+4), (n+6), \cdots, \frac{4n}{3}$ to the vertices $y_1, y_2, y_3, \cdots, y_{\frac{n}{6}}$ respectively. Next assign the labels $-(n+1), -(n+5), -(n+9), \cdots, -(\frac{4n-12}{3})$ respectively to the vertices $u_{\frac{n+6}{6}}, u_{\frac{n+12}{6}}, u_{\frac{n+18}{6}}, \cdots, u_{\frac{n-2}{4}}$ and assign the labels $-(n+3), -(n+7), -(n+11), \cdots, -(\frac{4n-6}{3})$ respectively to the vertices $y_{\frac{n+6}{6}}, y_{\frac{n+12}{6}}, y_{\frac{n+18}{6}}, \cdots, y_{\frac{n-2}{4}}$. Now assign the labels $-(n+2), -(n+6), -(n+10), \cdots, -(\frac{4n-15}{3})$ respectively to the vertices $u_{\frac{n+2}{4}}, u_{\frac{n+6}{4}}, u_{\frac{n+10}{4}}, \cdots, u_{\frac{n-3}{3}}$ and assign the labels $-(n+4), -(n+8), -(n+12), \cdots, -(\frac{4n-9}{3})$ respectively to the vertices $y_{\frac{n+2}{4}}, y_{\frac{n+6}{4}}, y_{\frac{n+10}{4}}, \cdots, y_{\frac{n-3}{3}}$. Finally assign the labels $-(\frac{4n-3}{3}), -(\frac{4n}{3})$ to the vertices $u_{\frac{n}{3}}, v_{\frac{n}{3}}$.

In all the three cases, $\Delta_{f_1^c} = \Delta_{f_1} = \frac{10n}{6}$. □

References

[1] N.Akgunes, and Y.Nacaroglu, On the sigma index of the corona products of monogenic semigroup graphs, *Journal of Universal Mathematics*, Vol.2(1) (2019), 68–74.

[2] I. Cahit, Cordial Graphs, A weaker version of graceful and harmonious graphs, *Ars combin.*, 23 (1987), 201–207.

[3] J.A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 19(2016).

[4] F. Harary, *Graph Theory*, Addision wesley, New Delhi, 1969.

[5] Y.Nacaroglu, and A. D. Maden, The multiplicative Zagreb coincides of graph operations, *Util.Math.,* Vol.102(2017), 19–38.

[6] Y.Nacaroglu, On the corona products of monogenic semigroup graphs, *Advances and Applications in Discrete Mathematics,* Vol.19(4) (2018), 409–420.

[7] R. Ponraj, A. Gayathri and S.Somasundaram, Pair difference cordial labeling of graphs, *J.Math. Comp.Sci.*, Vol.11(3)(2021), 2551–2567.

[8] R. Ponraj, A. Gayathri and S.Somasundaram, Pair difference cordiality of some snake and butterfly graphs, *Journal of Algorithms and Computation*, Vol.53(1)(2021), 149–163.

[9] R. Ponraj, A. Gayathri and S.Somasundaram, Pair difference cordial graphs obtained from the wheels and the paths, *J. Appl. and Pure Math.*, Vol.3 No.3-4(2021), pp. 97–114.

[10] R. Ponraj, A. Gayathri and S.Somasundaram, Pair difference cordiality of some graphs derived from ladder graph, *J.Math. Comp.Sci.*, Vol.11, No 5(2021), 6105–6124.

[11] R. Ponraj, A. Gayathri and S.Somasundaram, Some pair difference cordial graphs, *Ikonion Journal of Mathematics*, Vol.3(2)(2021), 17–26.

[12] R. Ponraj, A. Gayathri and S.Somasundaram, Pair difference cordial labeling of planar grid and mangolian tent, *Journal of Algorithms and Computation*, Vol.53(2)(2021), 47–56.

[13] R. Ponraj, A. Gayathri and S.Somasundaram, Pair difference cordiality of some special graphs, *J. Appl. and Pure Math.*, Vol.3 No. 5-6(2021), 263–274.

[14] R. Ponraj, A. Gayathri and S.Somasundaram, Pair difference cordiality of mirror graph, shadow graph and splitting graph of certain graphs, *Maltepe Journal of Mathematics*, Vol.4, No.1(2022), 24–32.

[15] U.M. Prajapati and S.J. Gajjar, Cordial labeling for complement graphs, *Mathematics Today*, Vol.30(2015), 99–118.

[16] U.M. Prajapati and S.J. Gajjar, Some results on prime cordial labeling of generalized prism graph $Y_{m,n}$, *Ultra Scientist*, Vol.27(3)A, (2015), 189–204.

International J.Math. Combin. Vol.4(2022), 82-86

The Computation for the Fuzzy Subgroups of the Algebraic Structure $D_{2^4} \times C_{2^4}$

S. A. Adebisi[1], M. Ogiugo [2] and M. EniOluwafe[3]

1. Department of Mathematics , Faculty of Science, University of Lagos, Nigeria

2. Department of mathematics,School of Science,Yaba College of Technology, Lagos, Nigeria

3. Department of mathematics,Faculty of Science, university of Ibadan, Nigeria

E-mail: adesinasunday@yahoo.com

Abstract: Any finite nilpotent group can be uniquely written as a direct product of p-groups In this paper, an attempt for the computation of $D_{2^4} \times C_{2^4}$ was made. This happens to be the computation of the number of distinct fuzzy subgroups of the cartesian product of the dihedral group of order 2^4 with a cyclic group of order sixteen.

Key Words: Finite p-groups, nilpotent group, fuzzy subgroups, dihedral group, inclusion-exclusion principle, maximal subgroups.

AMS(2010): 08A72, 20D15, 60A86, 20N25.

§1. Introduction

The classification of the fuzzy subgroups, most especially the finite p-groups cannot be underestimated. This aspect of pure mathematics has undergone a dynamic developments over the years. For instance, many researchers have treated cases of finite Abelian groups. Since then, the study has been extended to some other important classes of finite Abelian and non-Abelian groups such as the dihedral, quaternion, semidihedral, and hamiltonian groups.

§2. Methodology

The method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G. By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1<r_2\le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right). \qquad (2.1)$$

[1]Received August 31, 2022, Accepted December 16, 2022.

In [2], (2.1) was used to obtain the explicit formulas for some positive integers n.

Theorem 2.1(Marius) *The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is*

(*i*) $h(\mathbb{Z}_{p^n}) = 2^n$;

(*ii*) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

§3. The number of Fuzzy Subgroups for $\mathbb{Z}_8 \times \mathbb{Z}_8$

Lemma 3.1 *Let G be abelian such that $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then, $h(G) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 48$.*

Proof By the use of GAP (Group Algorithms and Programming), G has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Hence, we have that: $\frac{1}{2}h(G) = 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = h(\mathbb{Z}_2 \times \mathbb{Z}_4)$. And by Theorem 2.1, $h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 24 \Rightarrow h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 48$. □

Corollary 3.2 *Following the last lemma, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ and $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^8})$ = 1536, 4096, 10496 and 26112 respectively.*

Theorem 3.3 *Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_8$. Then, $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$.*

Proof Notice that there are three maximal subgroups of G, i.e., one is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}$), while two are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^n}$). We have

$$\begin{aligned}\frac{1}{2}h(G) &= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) \\ &= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) \\ &= h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})\end{aligned}$$

Hence,

$$\begin{aligned}h(G) &= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) \\ &= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) - 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) \\ &\quad +32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) - 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) + 16h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-4}}) \\ &= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) \\ &\quad +32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) - 64h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-5}}) + 32h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-5}}) + \cdots - 2^{j+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-j}}) \\ &\quad +2^j h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-j}}).\end{aligned}$$

For $n-j=3$

$$\begin{aligned}
&= 4h(\mathbb{Z}_4\times\mathbb{Z}_{2^n}) + 2^{n-3}h(\mathbb{Z}_8\times\mathbb{Z}_{2^3}) - 2^{n-1}h(\mathbb{Z}_4\times\mathbb{Z}_{2^3}) + \sum_{k=1}^{n-3}[2^{k+1}h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-k}})\\
&= 2^{n+2}[n^2+5n+3] + \sum_{k=1}^{n-3}h(\mathbb{Z}_4\times\mathbb{Z}_{2^{n-k}})\\
&= 2^{n+2}((n^2+5n+3) + \frac{1}{6}(n-3)(n^2+9n+14))\\
&= \frac{1}{3}(2^{n+1})(n^3+12n^2+17n-24)
\end{aligned}$$

for $n>2$. □

Theorem 3.4 *Suppose that $G = D_{2^3}\times\mathbb{C}_8$. Then, $h(G) = 5376$.*

Proof Notice that

$$\begin{aligned}
\frac{1}{2}h(G) &= h(D_{2^3}\times Z_4) + 2h(Z_{2^3}\times Z_2\times Z_2) - 4h(Z_{2^2}\times Z_2\times Z_2)\\
&\quad + h(Z_8\times Z_4) - 6h(Z_8\times Z_2) - 2h(Z_4\times Z_4) + 8h(Z_4\times Z_2) + h(Z_{2^3}) = 2688.
\end{aligned}$$

Therefore $h(G) = 2\times 2688 = 5376$. □

Theorem 3.5 *Let $G = D_{2^5}\times\mathbb{Z}_8$. Then, $h(G) = 111136$.*

Proof Notice that

$$\begin{aligned}
\frac{1}{2}h(G) &= h(D_{2^5}\times Z_{2^2}) + 2h(D_{2^4}\times Z_{2^3}) - 4h(D_{2^4}\times Z_{2^2}) + h(\mathbb{Z}_{2^4}\times Z_{2^3})\\
&\quad -2h(\mathbb{Z}_{2^4}\times Z_{2^2}) - 2h(\mathbb{Z}_{2^3}\times Z_{2^3}) + 8h(\mathbb{Z}_{2^3}\times Z_{2^2}) + h(\mathbb{Z}_{2^4}) - 4h(\mathbb{Z}_{2^3}) = 55568.
\end{aligned}$$

Therefore, $h(G) = 2\times 55568 = 111136$. □

Theorem 3.6 *Suppose that $G = D_{2^6}\times\mathbb{Z}_8$. Then, $h(G) = 492864$.*

Proof Certainly,

$$\begin{aligned}
\frac{1}{2}h(G) &= h(D_{2^6}\times Z_4) + 2h(D_{2^5}\times Z_{2^3}) - 4h(D_{2^5}\times Z_4) + h(\mathbb{Z}_{2^5}\times Z_{2^3})\\
&\quad -2h(\mathbb{Z}_{2^5}\times Z_{2^2}) - 2h(\mathbb{Z}_{2^4}\times Z_{2^3}) + 8h(\mathbb{Z}_{2^4}\times Z_{2^2}) + h(\mathbb{Z}_{2^5}) - 4h(\mathbb{Z}_{2^4}) = 246432.
\end{aligned}$$

Therefore, $h(G) = 2\times 246432 = 492864$. □

Theorem 3.7 *Let $G = D_{2^n}\times\mathbb{C}_2$, the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is given by : $h(G) = 2^{2n}(2n+1) - 2^{n+1}, n>3$.*

§4. The Number of Fuzzy Subgroups for $D_{2^n} \times \mathbb{C}_8$

Proposition 4.1 *Suppose that $G = D_{2^n} \times \mathbb{C}_8$. Then, the number of distinct fuzzy subgroups of G is given by*

$$2^{2(n-1)}(6n+113) + 2^n \left[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j)\right]$$
$$+ \frac{1}{3}(2^{n+2})\Bigg[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30$$
$$+ \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]\Bigg].$$

Proof Notice that

$$\begin{aligned} h(D_{2^n} \times C_8) &= 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times Z_4) + 2h(D_{2^{n-1}} \times C_8) \\ &\quad +4h(Z_{2^{n-2}} \times C_8) + 2^4 h(Z_{2^{n-3}} \times C_8) + 2^6 h(\mathbb{Z}_{2^{n-4}} \times C_8) - 2^8 h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3}) \\ &\quad -4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10} h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9 h(\mathbb{Z}_{2^{n-5}}) - 2^9 h(D_{2^{n-4}} \times C_{2^2}) \\ &\quad +2^8 h(D_{2^{n-4}} \times C_{2^3}) \\ &= 2^n + 2h(D_{2^n} \times C_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2 h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3}) \\ &\quad -2^{2(n-3)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2 h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 2^{2n-5} h(\mathbb{Z}_{2^2}) \\ &\quad -2^{2n-5} h(D_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)} h(D_{2^3} \times \mathbb{Z}_{2^3}) + 3\sum_{i=1}^{n-5} 2^{2ij} h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3}) \end{aligned}$$

as required. □

Proposition 4.2 (see [16]) *Suppose that $G = D_{2^n} \times \mathbb{C}_8$. Then, the number of distinct fuzzy subgroups of G is given by*

$$2^{2(n-1)}(6n+113) + 2^n \left[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j)\right]$$
$$+\frac{1}{3}(2^{n+2})\big[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30$$
$$+ \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]\Bigg].$$

Proof Calculation shows that

$$\begin{aligned} h(D_{2^n} \times C_8) &= 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times Z_4) + 2h(D_{2^{n-1}} \times C_8) \\ &\quad +4h(Z_{2^{n-2}} \times C_8) + 2^4 h(Z_{2^{n-3}} \times C_8) + 2^6 h(\mathbb{Z}_{2^{n-4}} \times C_8) - 2^8 h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3}) \end{aligned}$$

$$\begin{aligned}
&-4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10}h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9 h(\mathbb{Z}_{2^{n-5}}) - 2^9 h(D_{2^{n-4}} \times C_{2^2}) \\
&+2^8 h(D_{2^{n-4}} \times C_{2^3}) \\
= \;& 2^n + 2h(D_{2^n} \times C_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2 h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3}) \\
&-2^{2(n-3)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2 h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 2^{2n-5}h(\mathbb{Z}_{2^2}) \\
&-2^{2n-5}h(D_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)}h(D_{2^3} \times \mathbb{Z}_{2^3}) + 3\sum_{i=1}^{n-5} 2^{2ij}h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3})
\end{aligned}$$

as required. □

Theorem 4.3 *Let* $G = D_{2^4} \times \mathbb{C}_{2^4}$. *Then* , $h(G) = 61384$.

Proof There exist seven maximal subgroups. Among them, two isomorphic to $D_{2^4} \times \mathbb{C}_{2^3}$, two isomorphic to $D_{2^3} \times \mathbb{C}_{2^4}$, two isomorphic to $D_{2^4} \times \mathbb{C}_{2^2}$ while the seventh is isomorphic to $\mathbb{Z}_{2^4}$. Hence , we have that

$$\begin{aligned}
\frac{1}{2}h(G) \;=\;& 2h(D_{2^4} \times Z_{2^2}) + 2h(D_{2^4} \times Z_{2^3}) + 2h(D_{2^3} \times Z_{2^4}) \\
&-6h(D_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4}) \\
&+2h(D_{2^3} \times \mathbb{Z}_{2^3}) + 28h(\mathbb{Z}_{2^3} \times Z_{2^2}) + 2h(Z_{2^4} \times Z_{2^2}) + 2h(\mathbb{Z}_{2^4}) \\
&+h(Z_{2^3} \times \mathbb{Z}_{2^3}) - 35h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) + 21h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) - 7h(\mathbb{Z}_{2^3}\mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) \\
=\;& 2[h(D_{2^4} \times Z_{2^2}) + h(D_{2^4} \times Z_{2^3}) + h(D_{2^3} \times Z_{2^4}) - 2h(D_{2^3} \times \mathbb{Z}_{2^3}) - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) \\
&-h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) + 4h(D_{2^3} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^4}) + \frac{1}{2}h(Z_{2^4})].
\end{aligned}$$

And therefore,

$$h(G) = 4[700 + 8416 + 10744 - 10752C1088 + 162 + 704C40] = 4 \times 15346 = 61384. \quad \square$$

References

[1] Adebisi S. A, Ogiugo M. and EniOluwafe M. (2020), Computing the Number of Distinct Fuzzy Subgroups for the Nilpotent p-Group of $D_{2^n} \times C_4$, *International J.Math. Combin.*(www.mathcombin.com), Vol.1(2020), 86-89.

[2] Tarnauceanu, M. (2009), The number of fuzzy subgroups of finite cyclic groups and Delannoy numbers, *European J. Combin.*, (30), 283-287.

International J.Math. Combin. Vol.4(2022), 87-92

Some Generalized Result on Fixed Point Theorem in Complex Valued Intuitionistic Fuzzy Metric Space

Ram Milan Singh

(Department of Mathematics, Government Post graduate College, Tikamgarh, 472001, India)

E-mail: rammilansinghlig@gmail.com

Abstract: In this paper, we will present a number of common fixed point theorems for contraction conditions that satisfy specific requirements in complex valued intuitionistic fuzzy metric spaces.

Key Words: Common fixed point, intuitionistic fuzzy set, complex valued, continuous t-norm.

AMS(2010): 47H10.

§1. Introduction

In 1965, Zadeh [12] proposed the concept of fuzzy sets. Fuzzy set theory is a useful tool for describing situations involving imprecise or ambiguous data. Fuzzy sets deal with situations like these by assigning a degree of belonging to a set to each object. Since then, it has become a burgeoning field of study in engineering, medicine, social science, graph theory, metric space theory, and complex analysis, among other fields. Kramosil and Michalek [6] introduced fuzzy metric spaces in a variety of ways in 1975. With the help of continuous t-norms, George and Veermani [4] improved the concept of fuzzy metric spaces in 1994.

Buckley [3] was the one who originally established the concept of fuzzy complex numbers and fuzzy complex analysis. 1987. Some authors were influenced by Buckley's work. Re-examination of fuzzy complex numbers continues. The year was 2002, and Fuzzy sets were extended to complicated fuzzy sets by Ramot et al. [8]. as though it were a blanket statement Ramot et al. claim that a membership function defines a sophisticated fuzzy set. function with a range that extends beyond [0, 1] the complicated plane's unit circle Singh was born in the year 2016. The concept of complex valued fuzzy was introduced by et al.[10]. Using complex valued continuous to create metric spaces t -norm as well as the concept of convergent convergence. In a complex valued fuzzy sequence, Cauchy sequence in complex valued fuzzy metric spaces. By introducing the concept of non-membership grade to fuzzy set theory, Atanassov [1] created a stir in 1983.

In the complex valued intuitionistic fuzzy metric spaces, this work gives some common fixed point theorems for pairs of occasionally weakly compatible mappings satisfying various requirements.

[1]Received December 8, 2022, Accepted December 18, 2022.

§2. Preliminaries

Definition 2.1 *A binary operation* $* : r_s(\cos\theta + i\sin\theta) \times r_s(\cos\theta + i\sin\theta) \to r_s(\cos\theta + i\sin\theta)$, *where* $r_s \in [0,1]$ *and a fix* $\theta \in \left[0, \frac{\pi}{2}\right]$, *is called complex valued continuous t-norm if it satisfies the followings:*

(1) $*$ *is associative and commutative;*
(2) $*$ *is continuous;*
(3) $a * e^{i\theta} = a, \forall a \in r_s(\cos\theta + i\sin\theta)$;
(4) $\mathrm{a} *\ \mathrm{b} \le c * d$ *whenever* $a \le c$ *and* $b \le d, \forall \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \in r_s(\cos\theta + i\sin\theta)$.

Definition 2.2 *A binary operation* $: r_s(\cos\theta + i\sin\theta) \times r_s(\cos\theta + i\sin\theta) \to r_s(\cos\theta + i\sin\theta)$, *where* $r_s \in [0,1]$ *and a fix* $\theta \in \left[0, \frac{\pi}{2}\right]$ *is called complex valued continuous t-co norm if it satisfies the followings:*

(1) *is associative and commutative;*
(2) *is continuous;*
(3) $\mathrm{a} \diamond 0 = a, \forall a \in r_s(\cos\theta + i\sin\theta)$;
(4) $\mathrm{a} \diamond \mathrm{b} \le \mathrm{c} \diamond \mathrm{d}$ *whenever* $a \le c$ *and* $b \le d, \forall \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{d} \in r_s(\cos\theta + i\sin\theta\ )$.

Definition 2.3 *The following are examples for complex valued continuous t-norm:*

(*i*) $a * b = \min\{a, b\}, \forall a, b \in r_s(\cos\theta + i\sin\theta)$ *and a fix* $\theta \in \left[0, \frac{\pi}{2}\right]$;
(*ii*) $a * b = \max(a + b - (\cos\theta + i\sin\theta), 0)$, *for all* $a, b \in r_s(\cos\theta + i\sin\theta)$ *and a fix* $\theta \in \left[0, \frac{\pi}{2}\right]$.

Definition 2.4 *The following are examples for complex valued continuous t-conorm:*

(*i*) $\mathrm{a} \diamond b = \max\{a, b\}, \forall a, b \in r_s(\cos\theta + i\sin\theta)$ *and a fix* $\theta \in \left[0, \frac{\pi}{2}\right]$;
(*ii*) $a \diamond b = \min(a + b, 1)$, *for all* $a, b \in r_s(\cos\theta + i\sin\theta)$ *and a fix* $\theta \in \left[0, \frac{\pi}{2}\right]$.

Definition 2.5 *The 5-triplet* $(\ X, \mathrm{M}, \mathrm{N}, *, \diamond)$ *is said to be complex valued intuitionistic fuzzy metric space if* X *is an arbitrary non empty set,* $*$ *is a complex valued continuous* t-*norm,* $\diamond$ *is a complex valued continuous t-conorm and* $M, N : X \times X \times (0, \infty) \to r_s(\cos\theta + i\sin\theta)$ *are complex valued fuzzy sets, where* $r_s \in [0,1], r_s(\cos\theta + i\sin\theta)$ *are complex valued fuzzy sets, where* $r_s \in$ *and* $\theta \in \left[0, \frac{\pi}{2}\right]$, *satisfying the following conditions:*

for all $\mathrm{x}, \mathrm{y}, \mathrm{z} \in X, \mathrm{t}, \mathrm{s} \in (0, \infty), r_s \in [0,1]$ *and* $\theta \in \left[0, \frac{\pi}{2}\right]$,

($cf1$) $M(a, b, p) + M(a, b, p) \le (\cos\theta + i\sin\theta)$;

($cf2$) $M(a, b, p) > 0$;

($cf3$) $M(a, b, p) = (\cos\theta + i\sin\theta)$, *for all* $p \in (0, \infty)$ *if and only if* $\mathrm{a} = \mathrm{b}$;

($cf4$) $M(a, b, p) = M(b, a, p)$;

($cf5$) $M(a, b, p + s) \ge M(a, c, p) * M(c, b, s)$;

($cf6$) $M(a, b, p) : (0, \infty) \to r_s(\cos\theta + i\sin\theta)$ *is continuous;*

($cf7$) $N(a, b, p) < (\cos\theta + i\sin\theta)$;

($cf8$) $N(a, b, p) = 0$ *for all* $p \in (0, \infty)$ *if and only if* $\mathrm{a} = \mathrm{b}$;

$(cf9)$ *$N(a,b,p) = N(b,a,p)$;*

$(cf10)$ *$N(a,b,p+s) \leq N(a,c,p) \diamond N(c,b,s)$;*

$(cf11)$ *$N(a,b,p) : (0,\infty) \to r_s(\cos\theta + i\sin\theta)$ is continuous.*

The pair (M,N) is called a complex valued intuitionistic fuzzy metric space. The functions $M(a,b,p)$ and $N(a,b,p)$ denotes the degree of nearness and non-nearness between a and b with respect to t. It is noted that if we take $\theta = 0$, then complex valued intuitionistic fuzzy metric simply goes to real valued intuitionistic fuzzy metric.

§3. Main Results

Theorem 3.1 *Let $(X,M,N,*,\diamond)$ be a complex valued intuitionistic fuzzy metric space with*

$$\lim_{p\to\infty} M(a,b,p) = (\cos\theta + i\sin\theta) \quad and \quad \lim_{p\to\infty} N(a,b,p) = 0$$

for all $a,b \in X$ and let P,Q,A and B be self - mappings on X. Let the pairs $\{P,A\}$ and $\{Q,B\}$ be occasionally weakly compatible. If there exists $d \in (0,1)$ such that

$$M(Pa,Qb,dp) \geq \left\{ \begin{array}{c} M(Aa,Bb,p) * M(Pa,Ab,p)* \\ M(Qb,Bb,p) * M(Ps,Bb,p) \end{array} \right\}, \tag{3.1}$$

$$N(Px,By,kt) \leq \left\{ \begin{array}{c} N(Aa,Bb,p) \diamond N(Pa,Ab,p) \\ N(Qb,Bb,p) \diamond N(Ps,Bb,p) \end{array} \right\} \tag{3.2}$$

for all $a,b \in X$ and for all $p > 0$. Then P,Q,A and B have a unique common fixed point in X.

Proof The pairs $\{P,A\}$ and $\{Q,B\}$ be occasionally compatible, so there are points $a,b \in X$ such that Aa and $Qb = Bb$. Now, from (3.1) and (3.2) we have

$$\begin{aligned} M(Pa,Qb,dp) &\geq \left\{ \begin{array}{c} M(Aa,Bb,p) * M(Pa,Ab,p)* \\ M(Qb,Bb,p) * M(Ps,Bb,p) \end{array} \right\} \\ N(Px,By,kt) &\leq \left\{ \begin{array}{c} N(Aa,Bb,p) \diamond N(Pa,Ab,p)\diamond \\ N(Qb,Bb,p) \diamond N(Ps,Bb,p) \end{array} \right\} \\ M(Pa,Qb,dp) &= \left\{ \begin{array}{c} M(Pa,Qb,p) * M(Pa,Pa,p)* \\ M(Qb,Qb,p) * M(Pa,Qb,p) \end{array} \right\} \\ N(Px,By,kt) &= \left\{ \begin{array}{c} N(Pa,Qb,p) \diamond N(Pa,Pa,p)\diamond \\ N(Qb,Qb,p) \diamond N(Pa,Qb,p) \end{array} \right\} \end{aligned}$$

$$M(Pa,Qb,dp) = \left\{ \begin{array}{c} M(Pa,Qb,p) * (\cos\theta + i\sin\theta) * \\ (\cos\theta + i\sin\theta) * M(Pa,Qb,p) \end{array} \right\}$$

$$N(Px,By,kt) = \left\{ \begin{array}{c} N(Pa,Qb,p) \diamond 0 \diamond \\ 0 \circ N(Pa,Qb,p) \end{array} \right\}$$

$$M(Pa,Qb,dp) = M(Pa,Qb,p)$$

$$N(Px,By,kt) = N(Pa,Qb,p)$$

but $\{a_n\}$ be a sequence in a complex valued intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ with $\lim_{p\to\infty} M(a,b,p) = \cos\theta + \mathrm{i}\sin\theta$ and $\lim_{p\to\infty} N(a,b,p) = 0, \forall a, b \in X$.

If $\lim_{p\to 0} N(a,b,p) = 0$, there exists $d \in (0,1)$ such that $M\left(a_{n+1}, a_{n+2}, dp\right) \geqslant M\left(a_n, a_{n+1}, p\right)$ and $N\left(a_{n+1}, a_{n+2}, dp\right) \leqslant \left(a_n, a_{n+1} p\right)$, for all $p > 0$, then $\{a_n\}$ is a cauchy sequence in X. Then $Pc = Qb$ and consequently

$$Pa = Aa = Qb = Bb. \tag{3.3}$$

Now suppose that the pair P, A have another coincidence point $Pc = Ac$.

$$M(Pc,Qb,dp) \geq \left\{ \begin{array}{c} M(Ac,Bb,p) * M(Pc,Ac,p) * \\ M(Qb,Bb,p) * M(Pc,Bb,p) \end{array} \right\}$$

$$N(Pc,Qb,dp) \leq \left\{ \begin{array}{c} N(Ac,Bb,p) \diamond N(Pc,Ac,p) \diamond \\ N(Qb,Bb,p) \diamond N(Pc,Bb,p) \end{array} \right\}$$

$$M(Pc,Qb,dp) = \left\{ \begin{array}{c} M(Pc,Qb,p) * M(Pc,Pc,p) * \\ M(Qb,Qb,p) * M(Pc,Qb,p) \end{array} \right\}$$

$$N(Pc,Qb,dp) = \left\{ \begin{array}{c} N(Pc,Qb,p) \diamond N(Pc,Pc,p) \diamond \\ N(Qb,Qb,p) \diamond N(Pc,Qb,p) \end{array} \right\}$$

$$M(Pc,Qb,dp) = \left\{ \begin{array}{c} M(Pc,Qb,p) * (\cos\theta + i\sin\theta) * \\ (\cos\theta + i\sin\theta) * M(Pc,Qb,p) \end{array} \right\}$$

$$N(Pc,Qb,dp) = \left\{ \begin{array}{c} N(Pc,Qb,p) \diamond 0 \diamond \\ 0 \vee N(Pc,Qb,p) \end{array} \right\}$$

$$M(Pc,Qb,dp) = M(Pc,Qb,p)$$

$$N(Pc,Qb,dp) = N(Pc,Qb,p)$$

but $\{a_n\}$ be a sequence in a complex valued intuitionistic fuzzy metric space $(X, \mathrm{M}, \mathrm{N}, *, \diamond)$ with $\lim_{p\to\infty} M(a,b,p) = \cos\theta + \mathrm{i}\sin\theta$ and $\lim_{p\to\infty} N(a,b,p) = 0, \forall a, b \in X$.

If $\lim_{p\to 0} N(a,b,p) = 0$, there exists $d \in (0,1)$ such that $M\left(a_{n+1}, a_{n+2}, dp\right) \geqslant M\left(a_n, a_{n+1}, p\right)$ and $N\left(a_{n+1}, a_{n+2}, dp\right) \leqslant \left(a_n, a_{n+1}, p\right)$, for all $p > 0$, then $\{a_n\}$ is a cauchy sequence in X. Then $Pc = Qb$ and consequently

$$Pc = Ac = Qb = Bb. \tag{3.4}$$

From (3.3) and (3.4) we have $\mathrm{Pa} = \mathrm{Pc}$ and therefore the pair $\{\mathrm{P}, \mathrm{A}\}$ have a unique point of coincidence $\mathrm{v} = \mathrm{Pa} = \mathrm{Aa}, \mathrm{v}$ is the common fixed point of $\{\mathrm{P}, \mathrm{A}\}$.

But, $\{a_n\}$ be a sequence in a complex valued intuitionistic fuzzy metric space $(X, \mathrm{M}, \mathrm{N}, *, \diamond)$ with $\lim_{p\to\infty} M(a,b,p) = \cos\theta + \mathrm{i}\sin\theta$ and $\lim_{p\to\infty} N(a,b,p) = 0, \forall a, b \in X$.

If $\lim_{p\to 0} N(a,b,p) = 0$, there exists $d \in (0,1)$ such that $M\left(a_{n+1}, a_{n+2}, dp\right) \geqslant M\left(a_n, a_{n+1}, p\right)$ and $N\left(a_{n+1}, a_{n+2}, dp\right) \leqslant \left(a_n, a_{n+1}, p\right)$, for all $p > 0$, then $\{a_n\}$ is a cauchy sequence in X.Then we have $\mathrm{v} = \mathrm{w}$ and v is the common fixed point of P, Q, A and B. For uniqueness, let u is an another common fixed point of P, Q, A and B.

Therefore,

$$\begin{aligned}
M(v,w,dp) &= M(Pv,Qu,dp)\\
&\geq \left\{\begin{array}{c} M(Av,Bu,p) * M(Pv,Av,p) * \\ M(Qu,Bu,p) * M(Pv,Bu,p) \end{array}\right\}\\
&= \left\{\begin{array}{c} M(v,u,p) * M(v,v,p) * \\ M(v,v,p) * M(v,u,p) \end{array}\right\}\\
&= \left\{\begin{array}{c} M(v,u,p) * (\cos\theta + i\sin\theta) \\ *(\cos\theta + i\sin\theta) * M(v,u,p) \end{array}\right\}\\
&= M(v,u,p)\\
N(v,w,dp) &= N(Pv,Qu,dp)\\
&\leq \left\{\begin{array}{c} N(Av,Bu,p) \diamond N(Pv,Av,p) \diamond \\ N(Qu,Bu,p) \diamond N(Pv,Bu,p) \end{array}\right\}\\
&= \left\{\begin{array}{c} N(v,u,p) \diamond N(v,v,p) \diamond \\ N(v,v,p) \diamond N(v,u,p) \end{array}\right\}\\
&= \left\{\begin{array}{c} N(v,u,p) \diamond 0 \diamond 0 \diamond \\ N(v,u,p) \end{array}\right\} = N(v,u,p)\\
M(v,w,dp) &= M(Pv,Qw,dp)\\
&\geq \left\{\begin{array}{c} M(Av,Bw,p) * M(Pv,Av,p) * \\ M(Qw,Bw,p) * M(Pv,Bw,p) \end{array}\right\}\\
&= \{M(Pv,Qw,p) * M(Pv,Pv,p) *\}
\end{aligned}$$

$$
\begin{aligned}
&= \{M(Qw,Qw,p) * M(Pv,Bw,p)\} \\
&= \left\{ \begin{array}{c} M(Pv,Qw,p) * (\cos\theta + i\sin\theta) \\ *(\cos\theta + i\sin\theta) * M(Pv,Qw,p) \end{array} \right\} \\
&= M(Pv,Qw,p) = M(v,w,p) \\
N(v,w,dp) &= N(Pv,Qw,dp) \\
&\leq \left\{ \begin{array}{c} N(Av,Bw,p) \diamond N(Pv,Av,p) \diamond \\ N(Qw,Bw,p) \diamond N(Pv,Bw,p) \end{array} \right\} \\
&= \{N(Pv,Qw,p) \diamond N(Pv,Pv,p) \diamond \\
&= \{N(Pv,Qw,p) \diamond 0 \diamond 0\} \\
&= \left\{ \begin{array}{c} N(Pv,Qw,p) \diamond 0 \diamond 0 \diamond \\ N(Pv,Qw,p) \end{array} \right\} \\
&= N(Pv,Qw,p) = N(v,w,p)
\end{aligned}
$$

Consequently, $v = u$ and P, Q, A and B have a unique common fixed point. □

References

[1] K. Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20(1986), 87-96.

[2] A. Azam, B. Fisher and M. Khan, Common fixed point theorems in complex valued metric space, *Numerical Functional Analysis and Optimization*, 32(2011), 243-253.

[3] J. J. Buckley, Fuzzy complex numbers, *Fuzzy Sets and Systems*, 33(1989), 333-345.

[4] A. George and P. Veeramani, On some results in fuzzy metric spaces, *Fuzzy Sets and Systems*, 64(1994), 395-399.

[5] G. Jungck, Compatible mappings and common fixed points, *Internat. Math. J. Maths. Sci.*, 9(1986), 771-779.

[6] I. Kramosil and J. Michalek, Fuzzy metric and statistical spaces, *Kybernetica*, 11(1975), 336-344.

[7] J. H. Park, Intuitionstic fuzzy metric spaces, *Chaos, Solitons Fractals*, 22(2004), 1039-1046.

[8] D. Ramot, R. Milo, M. Friedman, and A. Kandel, Complex fuzzy sets, *IEEE Transactions of Fuzzy System*, 10(2002).

[9] R. Saadati, J. H. Park, On the intuitionstic fuzzy topological spaces, *Chaos Solitons Fractals*, 27(2009), 331-344.

[10] D. Singh, V. Joshi, M. Imdad, P. Kumar, A novel framework of complex valued fuzzy metric spaces fixed point theorems, *Journal of Intelligent and Fuzzy Systems*, 30(6)(2016), 3227 - 3238.

[11] Uday Dolas, A common fixed point theorem in fuzzy metric spaces using common E.A. Like property, *Journal of Applied Mathematics and Computation*, 2(6)(2018), 245-250.

[12] L. A. Zadeh, Fuzzy sets, *Inform and Control*, 8(1965) 338 - 353.

Papers Published in IJMC, 2022

Vol.1,2022

Vol.2,2022

Vol.3,2022

Vol.4,2022

Famous Words

Nothing else but Smarandache multispacesthe multilateral property of a particle P and the mathematical consistence determine that such an understanding is only local, not the whole reality on P, which leads to a central thesis for knowing the nature.

By Linfan MAO, a Chinese mathematician, philosophical critic.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats latex, eps, pdf may be submitted electronically to one member of the Editorial Board for consideration in the **International Journal of Mathematical Combinatorics** (*ISSN 1937-1055*). An effort is made to publish a paper duly recommended by a referee within a period of $2-4$ months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.
[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as JPG, EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

December 2022

Contents

An International Journal on Mathematical Combinatorics

ISSN 1937-1055