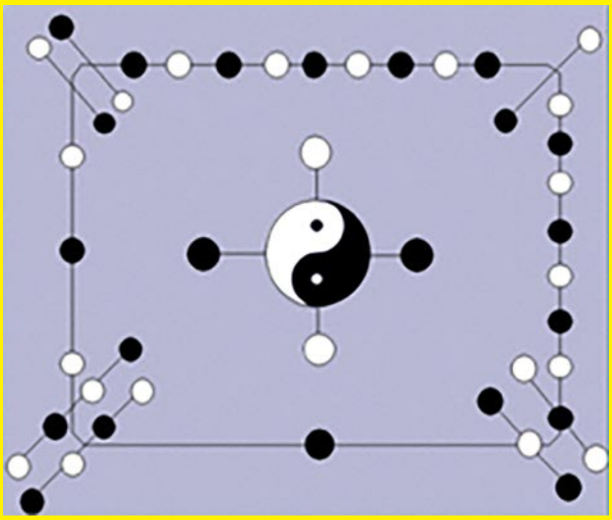
ISSN 1937 - 1055

VOLUME 4, 2021

INTERNATIONAL JOURNAL OF

MATHEMATICAL COMBINATORICS



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

December, 2021

Vol.4, 2021 **ISSN 1937-1055**

International Journal of

Mathematical Combinatorics

(http://fs.unm.edu/IJMC.htm, www.mathcombin.com/IJMC.htm)

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

December, 2021

Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.Linfan MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

Smarandache multi-spaces and Smarandache geometries with applications to other sciences;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

Mathematical theory on gravitational fields and parallel universes;

Applications of Combinatorics to mathematics and theoretical physics.

Generally, papers on applications of combinatorics to other mathematics and other sciences are welcome by this journal.

It is also available from the below international databases:

Serials Group/Editorial Department of EBSCO Publishing
10 Estes St. Ipswich, MA 01938-2106, USA
Tel.: (978) 356-6500, Ext. 2262 Fax: (978) 356-9371
http://www.ebsco.com/home/printsubs/priceproj.asp

and

Gale Directory of Publications and Broadcast Media, Gale, a part of Cengage Learning
27500 Drake Rd. Farmington Hills, MI 48331-3535, USA
Tel.: (248) 699-4253, ext. 1326; 1-800-347-GALE Fax: (248) 699-8075
http://www.gale.com

Indexing and Reviews: Mathematical Reviews (USA), Zentralblatt Math (Germany), Index EuroPub (UK), Referativnyi Zhurnal (Russia), Mathematika (Russia), EBSCO (USA), Google Scholar, Baidu Scholar, Directory of Open Access (DoAJ), International Scientific Indexing (ISI, impact factor 2.012), Scope(India), CNKI(China).

Subscription A subscription can be ordered by an email directly to

Linfan Mao
The Editor-in-Chief of *International Journal of Mathematical Combinatorics*
Chinese Academy of Mathematics and System Science Beijing, 100190, P.R.China, and also the President of Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA
Email: maolinfan@163.com

Price: US$48.00

Famous Words:

People can't create an opportunity but he can catch the has emerged.

By Percy Bysshe Shelley, a poet, also a writer of England.

International J.Math. Combin. Vol.4(2021), 1-10

On Parallel Ruled Surfaces in Minkowski 3-Space and Their Retractions

A.E. El-Ahmady

(Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt)

Malak E. Raslan and A.T. M-Zidan

(Mathematics Department, Faculty of Science, Damietta University, Damietta, Egypt)

E-mail: a.e_elahmady@yahoo.com, atm_zidan@yahoo.com

Abstract: In this paper, new types of retractions on ruled surfaces and parallel ruled surfaces as directrix retraction and ruling retraction in Minkowski 3-space has been presented. The topological folding of surfaces is deduced. The limit of these retractions and foldings have been obtained. The relations between the fundamental forms, Gaussian and mean curvatures before and after folding are discussed.

Key Words: Minkowski 3-space, parallel surfaces ruled surfaces, retractions, directrix and ruling retractions, folding, Frenet equations.

AMS(2010): 53A35, 53A05, 51B20, 58C05.

§1. Introduction

Parallel surface something like that a surface M^r, whose points are at a constant distance along the normal from another surface M is said to be parallel to M. In theory of surfaces. There are some special surfaces such as ruled surfaces, minimal surfaces, and surfaces of constant curvatures. So, there are infinite numbers of surfaces because we choose the constant distance along the normal arbitrarily. Parallel surface can be regarded as the locus of a point which are on the normals to M at a non-zero constant distance r from M. A surface M is ruled if through every point of M there is a straight line that lies on M. The most familiar examples are the plane and the curved surface of a cylinder or cone. A ruled surface can always be described (at least locally) as the set of points swept by a moving straight line. For example, a cone is formed by keeping one point of a line fixed whilst moving another point along a circle. A surface that can be (locally) unrolled onto a flat plane without tearing or stretching it is called a developable surface [8-16].

§2. Preliminary Notes

The Minkowski 3-space E_1^3 is the Euclidean space E^3 provided with the Lorentzian inner product

[1]Received September 13, 2021, Accepted November 26, 2021.

$$\langle u, v\rangle = u_1v_1 + u_2v_2 - u_3v_3,$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3) \in E_1^3$.

We say that a vector v in E_1^3 is space-like if $\langle v, v\rangle > 0$, time like if $\langle v, v\rangle < 0$ and light-like (null) if $\langle v, v\rangle = 0$ and $v \neq \mathbf{0}$. The norm of the vector $v \in E_1^3$ is defined [12,17] by

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

A subset$A$ of a topological space X is called retract of X if there exists a continuous map $r : X \to A$ called a retraction [5, 6] such that $r(a) = a$ for any $a \in A$.

Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ if f does not preserve the length it is called topological folding [1-4].

Definition 2.1([12,17]) *A surface M in the Minkowski 3-space E_1^3 is said to be space-like, time-like surface if, respectively the induced metric on the surface is a positive definite Riemannian metric, Lorentz metric. In other words, the normal vector on the space like (time like) surface is a time- like (space like) vector.*

Definition 2.2 *Let M be a surface. An immersion $x : M \to E_1^3$ is called space-like (resp. timelike, light-like) if all tangent planes $\left(T_pM, x^*\left(\langle\ ,\ \rangle_p\right)\right)$ are space-like (resp. time-like, light-like). A non-degenerate surface is a space like or time-like surface.*

An immersion $X : M \to L^3$ from a connected surface M into L^3 is said to be a time-like surface If the induced metric via ψ is a Lorentzian metric on M, which will be also denoted by $\langle\ ,\ \rangle$ or by I.

Definition 2.3 *A ruled surface is a surface that can be swept out by moving a line in space if therefor has a parameterization of the form*

$$c(u, v) = b(u) + v\delta(u),$$

where $b(u)$ is called the ruled surface directrix or the base curve and $\delta(u)$ is the director curve.

Alternatively, the surface can be represented a ruling joining corresponding points on two space curves. This is represented by

$$c(u, v) = (1 - v)c_A(u) + vc_B(u),\ \ 0 \le u, v \le 1,$$

where $c_A(u)$ and $c_B(u)$ are directrices. The two representations are identical if

$$b(u) = c_A(u) \ \text{ and } \ \delta(u) = c_B(u) - c_A(u).$$

The straight lines themselves are called rulings.

Definition 2.4 *A surface M in E_1^3 is constant mean curvature if and only if $g(H, H) = c$,* *where* $c \in \mathbb{R}$, g *is the standard metric in and* H *the mean curvature vector field. If* $c = 0$ *then* $H = 0$*, which means that* M *is a minimal surface.*

Let S *be an orientable surface and let* n *be the unit normal vector field of* S*, the surface* $\overline{S}$ *is parallel to* S *at distance* δ *if the point* $\overline{p}(u,v)$ *are defined by*

$$\overline{p}(u,v) = p(u,v) + \delta n(u,v),$$

where δ *is constant positive real number and* n *is the unit normal vector field on* M*.*

§3. Folding of Parallel Surfaces in Minkowski 3-Space E_1^3

Theorem 3.1 *Let* M *and* M^δ *are space-like surfaces in Minkowski 3-space and let* h *be a topological folding of* M *and* M^δ*, then,* $h(M)$ *and* $h\left(M^\delta\right)$ *are space-like surfaces in* E_1^3 *and, the folded surface* $h\left(M^\delta\right)$ *is parallel to the surface* $h(M)$ *if and only if* M^δ *is a parallel surface of* M*.*

Proof Let h be a topological folding of the space-like surfaces M and M^δ, where M^δ is parallel to M in E_1^3 and

$$h(M) = h(M(u,v))$$

for any space-like curve ψ on the space-like surface M. We have a curve $h(\psi)$ on the folded surface $h(M)$, where $h'(\psi)$ satisfies

$$\langle h'(\psi), h'(\psi)\rangle = h^{'2}(M(u,v))$$

since M is a space-like surface $\langle \psi', \psi'\rangle > 0$.

Also, this hold for any curve psi^δ on M^δ then $h(M)$ and $h\left(M^\delta\right)$ are space-like surfaces in E_1^3. Since

$$N = \frac{M_u \wedge M_v}{|M_u \wedge M_v|},$$

then

$$N_f = \frac{h(M)_u \wedge h(M)_v}{|h(M)_u \wedge h(M)_v|} = \frac{h^{'2}(M)(M_u \wedge h(M)_v}{|h^{'2}(M)M_u \wedge h(M)_v|} = \frac{h^{'2}(M_u \wedge h(M)_v}{|h^{'2}M_u \wedge h(M)_v|} = N$$

and

$$h\left(M^\delta\right) = h(M) + \delta_f N_f = h(M) + \delta_f N.$$

We can choose $\delta_f \leq \delta$. Then, the folded surface $h\left(M^\delta\right)$ is parallel to the surface $h(M)$. □

Corollary 3.1 *Let* M *be a space-like surface in* E_1^3 *and* $h(M)$ *be a topological folding of* M*. Then, the fundamental forms and the Gaussian and mean curvatures of* $h(M)$ *can be formed by the fundamental forms and the Gaussian and mean curvatures of* M*.*

Proof Let M be space-like surface in Minkowski 3-space and $h(M)$ be a topological folding of M. Then, by definition

$$\begin{aligned} h(M) &= h(M(u,v)), \\ I_h &= \langle dh(M), dh(M)\rangle , \\ I_h &= \left\langle \frac{\partial h(M)}{\partial u}, \frac{\partial h(M)}{\partial u}\right\rangle (du)^2 + 2\left\langle \frac{\partial h(M)}{\partial u}, \frac{\partial h(M)}{\partial v}\right\rangle dudv + \left\langle \frac{\partial h(M)}{\partial v}, \frac{\partial h(M)}{\partial v}\right\rangle (dv)^2. \end{aligned}$$

Hence, we get the first fundamental form of the folded surface to be

$$I_h = E^h(du)^2 + 2F^h dudv + G^h(dv)^2 = h^{'2} I.$$

By definition,

$$\begin{aligned} II_h &= \left\langle -dh(M), N^h\right\rangle , \\ II_h &= -\left\langle \frac{\partial h(M)}{\partial u}, \frac{\partial n^h}{\partial u}\right\rangle - 2\left\langle \frac{\partial h(M)}{\partial u}, \frac{\partial n^h}{\partial v}\right\rangle dudv - \left\langle \frac{\partial h(M)}{\partial v}, \frac{\partial n^h}{\partial v}\right\rangle (dv)^2. \end{aligned}$$

So, we get that the second fundamental form

$$II_h = e^h(du)^2 + 2f^h dudv + g^h(dv)^2 = h^{'2} II.$$

Similarly, by definition

$$\begin{aligned} III_h &= \left\langle dN^h, dN^h\right\rangle , \\ III_h &= \left\langle \frac{\partial N^h}{\partial u}, \frac{\partial N^h}{\partial u}\right\rangle + 2\left\langle \frac{\partial N^h}{\partial u}, \frac{\partial N^h}{\partial v}\right\rangle dudv + \left\langle \frac{\partial N^h}{\partial v}, \frac{\partial N^h}{\partial v}\right\rangle (dv)^2. \end{aligned}$$

Notice that $N^h = N$, we get that

$$\begin{aligned} III_h &= \left\langle \frac{\partial N^h}{\partial u}, \frac{\partial N^h}{\partial u}\right\rangle + 2\left\langle \frac{\partial N^h}{\partial u}, \frac{\partial N^h}{\partial v}\right\rangle dudv + \left\langle \frac{\partial N^h}{\partial v}, \frac{\partial N^h}{\partial v}\right\rangle (dv)^2 \\ &= \langle dN, dN\rangle = III \end{aligned}$$

and also, we have that

$$\begin{aligned} E_h &= h^{'2}(M)\langle M_u, M_u\rangle = h^{'2}(M)E, \\ G_h &= h^{'2}(M)\langle M_v, M_v\rangle = h^{'2}(M)G, \\ F_h &= h^{'2}(M)\langle M_u, M_v\rangle = h^{'2}(M)F \end{aligned}$$

and

$$\begin{aligned} e_h &= -\langle N_u, M_u\rangle = -h^{'2}(M)\langle N_u, M_u\rangle = h^{'2}(M)e; \\ f_h &= -\langle N_u, M_v\rangle = -h^{'2}(M)\langle N_u, M_v\rangle = h^{'2}(M)f; \\ g_h &= -\langle N_v, M_v\rangle = -h^{'2}(M)\langle N_v, M_v\rangle = h^{'2}(M)g, \end{aligned}$$

where the Gaussian curvature of the folded surface can be calculated by

$$K_h = \frac{e_h g_h - f_h^2}{E_h G_h - F_h^2} = \frac{h^{'4}\left(eg - f^2\right)}{h'^4\left(EG - F^2\right)} = \frac{eg - f^2}{EG - F^2} = K$$

and the mean curvature of the folded surface

$$H_h(p) = \frac{G_h e_h + E_h g_h - 2F_h f_h}{2\left(EG - F^2\right)} = \frac{h^{'4}\left(Ge + Eg - 2Ff\right)}{2h'^4\left(EG - F^2\right)} = H(p)$$

at a point $p \in h(M)$. □

Example 3.1 Consider $M = M(u,v)$ be a surface in Minkowski 3-space and $h(M(u,v))$ be a topological folding of m defined as $h(M) : M(u,v) \to \frac{M(u,v)}{m}, m \in \mathbb{N}$. Then, we get

$$\begin{aligned} h(M)_u &= h'(M)M_u, \quad h(M)_v = h'(M)M_v, \\ \langle h_u, h_u \rangle &= h'(M)\langle M_u, M_u \rangle, \quad \langle h_v, h_v \rangle = h'(M)\langle M_v, M_v \rangle \end{aligned}$$

and the normal vector of the folded surface is

$$N_h(p) = \frac{h_u(M) \wedge h_v(M)}{\| h_u(M) \wedge h_v(M) \|} = N(p).$$

We get that

$$\begin{aligned} I_h &= E_h(du)^2 + 2F_h dudv + G_h(dv)^2 \\ &= \frac{1}{m^2}\left(E^h(du)^2 + 2F^h dudv + G^h(dv)^2\right) = \frac{1}{m^2}I, \\ II_h &= e^h(du)^2 + 2f^h dudv + g^h(dv)^2 \\ &= e(du)^2 + 2f dudv + g(dv)^2 = \frac{1}{m^2}II, \\ III_h &= III. \end{aligned}$$

The Gaussian and mean curvatures of the folded surface can be calculated by

$$K_h = \frac{e_h g_h - f_h^2}{E_h G_h - F_h^2} = \frac{eg - f^2}{EG - F^2} = K \text{ and } H_h = H.$$

Theorem 3.2 *Let M and M^δ be two parallel surfaces in E_1^3 and $h(u,v)$ be a topological folding on M and M^δ. Then, the fundamental forms of $h(M^\delta)$ can be formed by the fundamental forms of $h(M)$ and we have $K_h = K_h^\delta$ and $H_h = H_h^\delta$, where K_h, K_h^δ, H_h and H_h^δ are the Gaussian and mean curvatures of the folded surfaces $h(M)$ and $h(M^\delta)$.*

Definition 3.1([15]) *A surface is called Weingarten surface or w-surface in E_1^3 if there is a nontrivial relation $\Phi(K,H) = 0$ or equivalently if the gradients of K and H are linearly dependent. In terms of the partial derivatives concerning u and v this is the equation $K_u H_v - K_v H_u = 0$, where K and H are Gaussian and mean curvatures of surface, respectively.*

Corollary 3.2 *Let M be a ruled surface in E_1^3 and $h(u,v)$ be an isometric folding of M. Then, $h(M)$ be a ruled surface and $h(M)$ can not be a developable surface.*

§4. Retraction of Ruled Surfaces in Minkowski 3-Space

Definition 4.1 *Let $X(u,v) = b(u) + v\delta(u)$ be a ruled surface in Minkowski 3-space with a retraction r defined by $r : X(u,v) \to X(u,v) - \{l(u)\}$, where $l(u)$ be the ruled surface directrix or the base curve which is called directrix retraction and the limit of this retraction be the base curve.*

Definition 4.2 *Let $X(u,v) = b(u) + v\delta(u)$, be a ruled surface in Minkowski 3-space with a retraction r defined by $r : X(u,v) \to X(u,v) - \{l(u)\}$, where $l(u)$ be a one of their rulings, which is called ruling retraction, and the limit of this retraction of X be a one of their rulings.*

Theorem 4.1 *The retraction of a ruled surface in E_1^3 is a ruled surface that can be swept out by moving a line in space if therefor has a parameterization of the form*

$$X(u,v) = b(u) + v\delta(u),$$

where $b(u)$ is called the ruled surface directrix or the base curve and $\delta(u)$ is the director curve.

Theorem 4.2 *The limit of retractions of a ruled surface in E_1^3 is a curve on the ruled surface.*

Proof Let $X(u,v) = b(u) + v\delta(u)$ be a ruled surface in E_1^3 and let $r = r(X(u,v))$ be a retraction of $X(u,v)$, where $r(u,v) : X(u,v) \to X(u,v) - \{l(u)\}$ with $l(u)$ being a curve on $X(u,v)$. Then, the limit of retractions of $X(u,v)$ is a curve on the surface $X(u,v)$ in E_1^3 and $r = r(X(u,v))$ satisfies

$$\begin{aligned}
r_1(u,v) : X(u,v) &\to X(u,v) - \{l_1(u)\},\\
r_2(u,v) : r_1(X) &\to r_1(X) - \{l_2(u)\},\\
r_3(u,v) : r_2(r_1(X)) &\to r_2(r_1(X)) - \{l_3(u)\},\\
\cdots\cdots\cdots\cdots &\cdots \cdots\cdots\cdots\cdots,\\
r_n : r_{n-1}(r_{n-2}(\cdots r_1(X))\cdots) &\to r_{n-1}(r_{n-2}(\cdots r_1(X))\cdots) - \{l_n(u)\}.
\end{aligned}$$

Then $\lim_{n\to\infty} r_n(X) = m(u)$, which is a curve on the surface $X(u,v)$. □

Corollary 4.1 *The limit of retractions of a non-developable ruled surface in E_1^3 is a curve on the ruled surface.*

Corollary 4.2 *The limit of retractions of the helicoid $X(u,v) = (u\cos v, u\sin v, v)$ in Minkowski 3-space be either a helix or a one of its rulings.*

Proof Let $X(u,v) = (u\cos v, u\sin v, v)$ be a helicoid which is a non-developable surface in Minkowski 3-space and $r(u,v) : X(u,v) \to X(u,v) - \{l(u)\}$ be a retraction of $X(u,v)$. We discuss two cases following.

Case 1. The directrex retraction defined as

$$r(u,v) : X(u,v) \to X(u,v) - \{l(u)\},$$

where the curve $\{l_1(u)\}$ is a helix on the surface. In this case, we get

$$\begin{aligned}
r_1(u,v) : X(u,v) &\to X(u,v) - \{l_1(u)\} = X_1(u,v),\\
r_2(u,v) : X_1(u,v) &\to X_1(u,v) - \{l_2(u)\} = X_2(u,v),\\
r_3(u,v) : X_2(u,v) &\to X_2(u,v) - \{l_3(u)\} = X_3(u,v),\\
\cdots\cdots\cdots &\cdots \cdots\cdots\cdots,\\
r_{n-1}(u,v) : X_{n-2}(u,v) &\to X_{n-2}(u,v) - \{l_{n-1}(u)\} = X_{n-1}(u,v),\\
\lim_{n\to\infty} r_n\left(X_n(u,v)\right) &= l(u)
\end{aligned}$$

where $l(u)$ is a helix in E_1^3.

Case 2. The ruling retraction defined as

$$r(u,v) : X(u,v) \to X(u,v) - \{l^*(u)\},$$

where the curve $\{l_1^*(u)\}$ is a one of its rulings of the surface by using a sequence of retractions then we have that the limit of ruling retractions of the helicoid is one of its rulings. □

Theorem 4.3 *The limit of retractions of the developable $X(u,v)$ in Minkowski 3-space be either a helix or a one of its rulings.*

Proof The proof is similar to the proof of Corollary 4.2. □

Corollary 4.3 *The limit of directrix retractions of the double cone $X(u,v)$ in Minkowski 3-space be a cone in E_1^3 and the limit of directrix retractions of this cone is a point which all rulings meeting.*

Proof Let $X(u,v)$ be a double cone in E_1^3 which is developable ruled surface with the property that all rulings meet in a point $p \in X(u,v)$, the directrix retraction of the upper nappe of $X(u,v)$ can be defined as

$$\begin{aligned}
r_1(X((u,v)) : X(u,v) &\to X(u,v) - \{s_1^1\},\\
r_2(X(u,v)) : X(u,v) &\to X(u,v) - \{s_2^1\},\\
r_3(X(u,v)) : X(u,v) &\to X(u,v) - \{s_3^1\},\\
\cdots\cdots\cdots &\cdots \cdots\cdots\cdots,\\
\lim_{n\to\infty} r_n\left(X_n(u,v)\right) &= X^*(u,v)
\end{aligned}$$

and we have the limit of directrix retractions of $X(u,v)$ to be a cone $X^*(u,v)$. We therefore get that the directrix retraction of a cone $X^*(u,v)$ can be defined as

$$\begin{aligned}
r_1\left(X^*(u,v)\right): X(u,v) &\rightarrow X^*(u,v)-\{s_1^{*1}\},\\
r_2\left(X^*(u,v)\right): X(u,v) &\rightarrow X^*(u,v)-\{s_2^{*1}\},\\
r_3\left(X^*(u,v)\right): X(u,v) &\rightarrow X^*(u,v)-\{s_3^{*1}\},\\
\cdots\cdots\cdots &\cdots \cdots\cdots\cdots,\\
\lim_{n\to\infty} r_n\left(X_n^*(u,v)\right) &= p.
\end{aligned}$$

Then the limit of directrix retractions of a cone $X^*(u,v)$ is meeting the ruling point. □

§5. Parallel Ruled Surfaces in E_1^3 and Their Retractions

Theorem 5.1([15]) *Let M be a time like surface and let M^δ be a parallel surface of M in E_1^3. Then, we get*

$$\overline{K}=\frac{K}{1-2\delta H+\delta^2 K}$$

and

$$\overline{H}=\frac{H-\delta K}{1-2\delta H+\delta^2 K},$$

where the Gaussian and mean curvatures of M and M^δ are denoted by K,H and $\overline{K},\overline{H}$, respectively.

Theorem 5.2 *Let $X(u,v)$ and $Y(u,v)=X(u,v)+\delta N(u,v)$ be parallel time like ruled surfaces with time-like rulings in Minkowski 3-space E_1^3 with Gaussian and mean curvatures K,H and $\overline{K},\overline{H}$, respectively. Then, $X(u,v)$ and $Y(u,v)$ are time-like developable surface with time-like rulings if*

$$\overline{K}=K=0 \quad \text{and} \quad \overline{H}=\frac{H}{1-2\delta H}.$$

Proof Let $X(u,v)$ and $Y(u,v)=X(u,v)+\delta N(u,v)$ be parallel ruled surfaces in Minkowski 3-space with Gaussian and mean curvatures K,H and $\overline{K},\overline{H}$, respectively. We find from the relation between the Gaussian and mean curvatures of the surfaces $X(u,v)$ and $Y(u,v)$ that

$$\overline{K}=\frac{K}{1-2\delta H+\delta^2 K}$$

and

$$\overline{H}=\frac{H+\delta K}{1-2\delta H+\delta^2 K}.$$

If $K=0$ we get that $\overline{K}=K=0$, and then $Y(u,v)$ are developable surfaces in E_1^3 and $1-2H\delta=c^*$, where c^* be constant. And then, we have

$$H=\frac{1-c^*}{2\delta}.$$

The relation between the mean curvatures of the surfaces $X(u,v)$ and $Y(u,v)$ is

$$\overline{H} = \frac{H}{1-2\delta H}.$$

Consequently, we know that

$$H = \frac{\overline{H}}{1+2\delta\overline{H}}.$$ □

Theorem 5.3([15]) *Let M be a space like surface and M^δ be a parallel surface of M in E_1^3. Then, the parallel surface of a space-like developable ruled surface is a space-like ruled surface.*

Theorem 5.4 *Let M be a developable space like ruled surface in E_1^3. Then, the ruling retraction $r(M) = M_r$ is a developable space like ruled surface, and the parallel surface M^δ of M is a space-like ruled Weingarten surface.*

Proof Since M_r is a ruled retraction of a developable space-like ruled surface M, M_r is a developable space-like ruled surface in E_1^3. From Theorem 3.1, the parallel surface M_r^δ of the developable space-like ruled surface M_r is a developable space-like ruled surface. Notice that M_r^δ and M_r both are developable surfaces. The Gaussian curvature of M_r^δ and M_r satisfy $K_r^\delta = K_r = 0$. We therefore get that $\Psi\left(K_r^\delta, H_r^\delta\right) = 0$. Thus, the parallel surface M_r^δ of M_r is a Weingarten surface. □

Corollary 5.1 *Let $r(M) = M_r$ be the directrix retraction of the space-like ruled surface M and let M_r^δ be a parallel ruled surface of a directrix retraction ruled surface M_r in E_1^3. Then the parallel surface M_r^δ is a Weingarten surface if M_r is a Weingarten surface.*

Theorem 5.5 *Let M and M^δ be two parallel surfaces in E_1^3 and $h(u,v)$ be a topological folding on M and M^δ if $h(M)$ is a minimal surface, then M^δ is a minimal surface also.*

Proof Let M and M^δ be two parallel surfaces in E_1^3. If M is a minimal surface, $H = 0$ and from

$$\overline{H} = \frac{H}{1-2\delta H},$$

we get $\overline{H} = 0$ and so, M^δ is a minimal surface. □

Corollary 5.2 *The directrix retraction $r(X(u,v))$ of the ruled surface $X(u,v)$ in Minkowski 3-space E_1^3 is a minimal surface if $k_1 = -k_2$, where k_1 and k_2 are the principal curvatures of the retracted ruled curve.*

Corollary 5.3 *Under the directrix retraction of the helicoid $X(u,v)$ in a Minkowski 3-space, the first fundamental forms, Gaussian and mean curvatures are invariant.*

References

[1] El-Ahmady A. E., The variation of the density on chaotic spheres in chaotic space-like Minkowski space-time, *Chaos, Solutions and Fractals*, Vol.31(2007), 1272-1278.

[2] El-Ahmady A. E. and A.T.M. Zidan, On helix in Minkowski 3-space and its retractions,

International J.Math. Combin., Vol.1(2020), 90-104.

[3] El-Ahmady A. E. and Zidan A.T.M., On hyperbola in Minkowski 3-space and its deformations, *International Journal of Applied Mathematics and Statistics*, 57(5)(2018), 115-127.

[4] El-Ahmady A. E., Malak E., Raslan and A.T.M. Zidan, On null curves in Minkowski 3-space and its fractal folding, *Modern Applied Science*, Vol. 14, 4(2020), 90-98).

[5] El-Ahmady A. E. and Zidan A.T.M., On the deformation retractions of Frenet curves in Minkowski 4Cspace, *Modern Applied Science*, Vol.14, 9(2020), 55-62.

[6] El-Ahmady A. E. and Zidan A.T.M., On the retraction of pseudo null space-like curves in Minkowski 3-space, *JP Journal of Geometry and Topology*, Vol.23, 2(2019), 89-106.

[7] Docarmo M. P., *Riemannian Geometry*, Boston: Birkhauser, 1992.

[8] H. S. Abdelaziz, M. Khalifa Saad and Sezai Kiziltug, Parallel surfaces of Weingarten type in Minkowski 3-space, *International Mathematical Forum*, Vol.7, 46(2012), 2293-2302.

[9] Kasap E., A method of the determination of a geodesic curve on ruled surface with time-like rulings, *Novi Sad J. Math.*, Vol.35, 2(2005), 103-110.

[10] Muradiye Çimdiker, Cumali Ekici, On the space like parallel ruled surfaces with Darboux frame, *International J.Math. Combin.*, Vol.2(2017), 60-69.

[11] Raving T. and S. Ku, Bertrand offsets of ruled surface and developable surface, *Computer - Aided Design*, 23(2)(1991), 145-152.

[12] R. Lopez, *Differential Geometry of Curves and Surfaces in Lorentz C Minkowski Space*, Instituto de Matematica Estatistica, University of Sao Paulo, Brazil, 2008.

[13] Robertson S. A., Isometric folding of Riemannian manifolds, *Proc. Roy. Soc Edinburgh*, 77(1977), 275- 284.

[14] Suk Ro J. and Won Yoon D., Tubes of Weingarten types in a Euclidean 3-space, *J. Chungcheong Mathematical Society*, 22(3)(2009), 359-366.

[15] Yasin Ünlütürk, Erdal Ozusaglam, On parallel surfaces in Minkowski 3-space, *TWMS J. App. Eng. Math.*, Vol.3, 2(2013), 214-222.

[16] Yasin Ünlütürk and Cumali Ekici, Parallel surfaces of space like ruled Weingarten surfaces in Minkowski 3-space, *New Trends in Mathematical Science*, V.1, 1(2013), 85-92.

[17] Walrave J.,*Curves and Surfaces in Minkowski Space*, Ph.D. thesis, ProQuest LLC, AnnArbor, Mich, USA, Katholieke Universiteit, Leuven, Belgium, 1995.

International J.Math. Combin. Vol.4(2021), 11-24

Backlund Transformations of Non-Null Curve Flows with Respect to Frenet Frame

Nevin Ertuğ Gürbüz

(Mathematics-Computer Department of Eskişehir Osmangazi University, Turkey)

E-mail: ngurbuz@ogu.edu.tr

Abstract: In this paper we present three classes of curve evolution connected with the generalized nonlinear Schrödinger equation and the generalized nonlinear heat system using visco-Da Rios equation in Minkowski 3-space. Later we obtain Backlund transformations of nonnul curve flows associated with visco-Da Rios equation in Minkowski 3-space.

Key Words: The visco Da Rios equation, Schrödinger equation, Backlund transformation

AMS(2010): 53A05, 53B25, 53B30, 53C80

§1. Introduction

The classic Backlund transformations are used to construct constant negative Gaussian curvature surfaces. It is used to obtain new pseudospherical surfaces using solution of an integrable partial differential equation.

In the last years Backlund transformations have been studied by many authors [1-9]. Nemeth studied Backlund transformations of constant torsion curves in 3-dimensional constant curvature spaces in Euclidean 3-space [5]. Palmer studied Backlund Transformations for surfaces in Minkowski Space [6]. Gürbüz extended Backlund transformations of constant torsion curves n dimensional Lorenzian space [7]. Abdel-Baky showed that the Minkowski versions of the Backlund's theorem and its application by using the method of moving frames [8]. Bracken studied Backlund transformations for several cases of a generalized KdV equation in Euclidean space [9]. Grbovic and Nesovic studied Backlund transformation and vortex filament equation for pseudo null curves in Minkowski 3-space [10]

The connection between moving curves and integrable systems has been studied by numerous authors [11-23]. If the position vector of a vortex filament is $\beta(\sigma, u)$, the following equation

$$\beta_u = \beta_\sigma \times \beta_{\sigma\sigma}$$

is called the Da Rios equation [11]. Hasimoto discovered the connection between the motion of vortex filament in an incompressible, inviscid three-dimensional fluid and solutions of the Nonlinear Schrödinger NLS equation in Euclidean 3-space [12]. Lakshamanan *et al.* presented motion of curves and surfaces and nonlinear evolution equations in (2+1) dimensions [13].

[1]Received October 25, 2021, Accepted December 5, 2021.

Barros et al. found out explicit solutions of the Betchov-Da Rios soliton equation in three-dimensional Lorentzian space forms [14].

Murugesh and Balakrishnan presented three classes of curve evolution associated with numerous soliton equations with respect to Frenet frame in Euclidean 3-space [24]. Gürbüz obtained three classes of curve evolution connected with the nonlinear Schrödinger equation according to Frenet frame in Minkowski 3-space [25].

The case when viscosity effects on a fluid dynamic, the following equation is

$$\beta_u = \beta_\sigma \times \beta_\sigma + \varsigma\beta_\sigma \tag{1}$$

is called the visco-Da Rios equation [26]. Here ς denotes viscosity and non-negatif constant. Gürbüz and Yoon presented the visco modified Heisenberg magnet model and physical applications with respect to the Frenet frame in Minkowski 3-space [27].

Qu and Kang studied Bäcklund transformations for integrable geometric curve flows in Euclidean 3-space [28]. In this paper, one give three classes of curve evolution connected with the visco Da Rios equation with respect to the Frenet frame in Minkowski 3-space are presented. Later Backlund transformations of three classes of the curve evolution connected with the generalized nonlinear heat flow and the repulsive type generalized nonlinear Schrödinger flow with respect to Frenet frame in Minkowski 3-space are studied.

§2. Backlund Transformations of the $GNLS^-$ and $GNLH$ Flows in $\mathbf{R}_1^3$

Let the β be a non-null curve with the arc length σ in 3 dimensional Minkowski space $\mathbf{R}_1^3$. Formulae of derivative of the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ have the following form [29]

$$\begin{aligned} \mathbf{T}_\sigma &= \varepsilon_2\kappa\mathbf{N}, \\ \mathbf{N}_\sigma &= -\varepsilon_1\kappa\mathbf{T}+\varepsilon_3\tau\mathbf{B}, \\ \mathbf{B}_\sigma &= -\varepsilon_2\tau\mathbf{N} \end{aligned}$$

satisfying

$$\begin{aligned} \langle\mathbf{T},\mathbf{T}\rangle_L &= \varepsilon_1, \langle\mathbf{N},\mathbf{N}\rangle_L = \varepsilon_2,\ \langle\mathbf{B},\mathbf{B}\rangle_L = \varepsilon_3, \\ \mathbf{T}\times_L\mathbf{N} &= \varepsilon_3,\ \ \mathbf{N}\times_L\mathbf{B} = \varepsilon_1\mathbf{T},\ \mathbf{B}\times_L\mathbf{T} = \varepsilon_2\mathbf{N},\ \varepsilon_i = \pm 1, i = 1,2,3 \end{aligned}$$

where $\times$ is cross product. The $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are tangent, normal and binormal vectors. The κ, τ are curvatures and torsion of the β.

In this section one present Backlund transformations of three classes of curve evolution connected with the generalized nonlinear heat GNLH flow and the repulsive type generalized nonlinear Schrödinger $GNLS^-$ flow in Minkowski 3-space $\mathbf{R}_1^3$.

Class I. Let β_1 be spacelike curve with timelike binormal according to the Frenet frame in $\mathbf{R}_1^3$. Assume that

$$\beta_{1\sigma} = \mathbf{T}. \tag{2}$$

The visco-Da Rios equation associated with the generalized nonlinear heat system is given by

$$\beta_{1u} = \beta_{1\sigma} \times \beta_{1\sigma\sigma} + \varsigma\beta_{1\sigma} \tag{3}$$

where ς is the viscosity and non-negatif constant.

The general time evolution according to the Frenet frame of the spacelike curve β_1 with timelike binormal for first class in $\mathbf{R}_1^3$ is given by

$$\begin{aligned} \mathbf{T}_u &= \eta_1\mathbf{N} + \eta_2\mathbf{B} && (4)\\ \mathbf{N}_u &= -\eta_\mathbf{1}\mathbf{T}+\gamma_\mathbf{1}\mathbf{B} && (5)\\ \mathbf{B}_u &= \eta_2\mathbf{T}+\gamma_\mathbf{1}\mathbf{N} && (6) \end{aligned}$$

where η_1, η_2 and γ_1 are smooth functions. Using Eqs.(2) and (3) one obtain

$$\mathbf{T}_u = \mathbf{T} \times \mathbf{T}_\sigma + \varsigma\mathbf{T} =\varsigma\mathbf{T}-\kappa\mathbf{B} \tag{7}$$

The compatibility conditions $\beta_{1\sigma u} = \beta_{1u\sigma}$ and $\mathbf{T}_{u\sigma} = \mathbf{T}_{\sigma u}$ the followings are obtained

$$\begin{aligned} \eta_1 &= \kappa(\tau+\varsigma),\ \eta_\mathbf{2} = -\kappa_\sigma && (8)\\ \gamma_1 &= -(\frac{\kappa_{\sigma\sigma}}{\kappa} + \tau(\tau+\varsigma)) && (9)\\ \kappa_u &= (\kappa(\tau+\varsigma))_\sigma + \kappa_\sigma\tau && (10) \end{aligned}$$

respectively, the second frame $\{\phi_1, \phi_2, \phi_3\}$ and Hasimoto-like transformation μ_1 using visco-Da Rios equation with respect to Frenet frame for the first class in Minkowski 3-space are given by

$$\begin{aligned} \phi_1 &= \mathbf{T}, && (11)\\ \phi_2 &= \frac{\mathbf{N}+\mathbf{B}}{\sqrt{2}}e^{\int\tau}\ , && (12)\\ \phi_3 &= \frac{\mathbf{N}-\mathbf{B}}{\sqrt{2}}e^{-\int\tau}, && (13)\\ \mu_1 &= \frac{\kappa}{\sqrt{2}}e^{\int\tau}. && (14) \end{aligned}$$

From Eqs.(11), (12) (13)and (14), the spatial variation of the frame $\{\phi_1, \phi_2, \phi_3\}$ associated with the generalized nonlinear heat system for first class is given by

$$\begin{aligned} \phi_{1\sigma} &= \mu_2\phi_2 + \mu_1\phi_3\\ \phi_{2\sigma} &= -\mu_1\phi_1\\ \phi_{3\sigma} &= -\mu_2\phi_1, \end{aligned}$$

where $\mu_2 = \frac{\kappa}{\sqrt{2}}e^{-\int\tau}$. The temporal variation of the frame $\{\phi_1, \phi_2, \phi_3\}$ associated with the generalized nonlinear heat system for first class is given by

$$\phi_{1u} = (\mu_{1\sigma} + \varsigma\mu_1)\phi_3 - (\mu_{2\sigma} - \varsigma\mu_2)\phi_2,$$

$$\begin{aligned}\phi_{2u} &= -(\mu_{1\sigma}+\varsigma\mu_1)\phi_1 + R_1\phi_2,\\ \phi_{3u} &= (\mu_{2\sigma}-\varsigma\mu_2)\phi_1 - R_1\phi_3.\end{aligned}$$

From $\phi_{2u\sigma} = \phi_{2\sigma u}$ one obtain the generalized nonlinear heat GNLH system for spacelike curve with timelike binormal with respect to Frenet frame for first class in Minkowski 3-space:

$$\begin{aligned}\mu_{1u} &= \mu_{1\sigma\sigma} + \varsigma\mu_{1\sigma} + R_1\mu_1\\ \mu_{2u} &= -\mu_{2\sigma\sigma} + \varsigma\mu_{2\sigma} - R_1\mu_2,\ R_1 = \mu_1\mu_2.\end{aligned}$$

3.1 Backlund Transformation of the Generalized Nonlinear Heat Flow with Respect to Frenet Frame for First Class in $\mathbf{R}_1^3$

Backlund transformation of the generalized nonlinear heat flow with respect to Frenet frame for first class is constructed: Let $\widetilde{\beta}_1$ be another curve associated with the spacelike curve β_1.

$$\widetilde{\beta}_1(\sigma,u) = \beta_1(\sigma,u) + f_1(\sigma,u)\mathbf{T} + f_2(\sigma,u)\mathbf{N} - f_3(\sigma,u)\mathbf{B}, \tag{15}$$

where f_1, f_2 and f_3 are functions of u and σ. Respectively, the derivative of according σ and u of Eq.(15) are obtained by

$$\widetilde{\beta}_{1\sigma} = (1+f_{1\sigma}-f_2\kappa)\mathbf{T} + (f_{2\sigma}+f_1\kappa+f_3\tau)\mathbf{N} - (f_{3\sigma}+f_2\tau)\mathbf{B}, \tag{16}$$

$$\begin{aligned}\widetilde{\beta}_{1u} \;=\; & (\varsigma + f_{1u} - f_2\kappa(\tau+\varsigma) + f_3\kappa_\sigma)\mathbf{T}\\ & +(f_{2u} + f_1\kappa(\tau+\varsigma) - f_3(\frac{\kappa_{\sigma\sigma}}{\kappa} + \tau(\tau+\varsigma))\mathbf{N}\\ & +(f_1\kappa_\sigma - f_2(\frac{\kappa_{\sigma\sigma}}{\kappa} + \tau(\tau+\varsigma)) + f_{3u} - \kappa)\mathbf{B}\end{aligned} \tag{17}$$

Let $\widetilde{\sigma}$ be the arclength parameter of the curve $\widetilde{\beta}_1$. Via Eq.(16), one derive

$$d\widetilde{\sigma} = \sqrt{(1+f_{1\sigma}-f_2\kappa)^2 + (f_{2\sigma}+f_1\kappa+f_3\tau)^2 - (f_{3\sigma}+f_2\tau)^2}\,d\sigma = \Theta d\sigma. \tag{18}$$

The tangent vector of $\widetilde{\beta}_1$ is derived by

$$\widetilde{\mathbf{T}} = \frac{d\widetilde{\beta}_1}{d\sigma}\frac{d\sigma}{d\widetilde{\sigma}} = \delta_1\mathbf{T} + \delta_2\mathbf{N} + \delta_3\mathbf{B}, \tag{19}$$

where

$$\delta_1 = \Theta^{-1}(1+f_{1\sigma}-f_2\kappa),\ \delta_2 = \Theta^{-1}(f_{2\sigma}+f_1\kappa+f_3\tau),\ \delta_3 = -\Theta^{-1}(f_{3\sigma}+f_2\tau)$$

From Eq.(19),

$$\frac{d^2\widetilde{\beta}_1}{d\widetilde{\sigma}^2} = \frac{\delta_{1\sigma}-\delta_2\kappa}{\Theta}\mathbf{T} + \frac{\delta_{2\sigma}+\delta_1\kappa-\delta_3\tau}{\Theta}\mathbf{N} + \frac{\delta_{3\sigma}-\delta_2\tau}{\Theta}\mathbf{B}. \tag{20}$$

From Eq.(20), the curvature $\widetilde{\kappa}$ of the curve $\widetilde{\beta}_1$ is derived by

$$\widetilde{\kappa}=\frac{\sqrt{(\delta_{1\sigma}-\delta_2\kappa)^2+(\delta_{2\sigma}+\delta_1\kappa-\delta_3\tau)^2-(\delta_{3\sigma}-\delta_2\tau)^2}}{\Theta}=\frac{\Lambda}{\Theta}. \tag{21}$$

The principal normal vector $\widetilde{\mathbf{N}}$ of the curve $\widetilde{\beta}_1$ is obtained by

$$\widetilde{\mathbf{N}}=\frac{\delta_{1\sigma}-\delta_2\kappa}{\Lambda}\mathbf{T}+\frac{\delta_{2\sigma}+\delta_1\kappa-\delta_3\tau}{\Lambda}\mathbf{N}+\frac{\delta_{3\sigma}-\delta_2\tau}{\Lambda}\mathbf{B}. \tag{22}$$

The binormal vector of $\widetilde{\mathbf{B}}$ of the curve $\widetilde{\beta}_1$ is found in the following

$$\widetilde{\mathbf{B}}=\omega_1\mathbf{T}+\omega_2\mathbf{N}+\omega_3\mathbf{B}, \tag{23}$$

where

$$\begin{aligned}
\omega_1 &= -\frac{\delta_3(\delta_{2\sigma}+\delta_1\kappa-\delta_3\tau)-\delta_2(\delta_{3\sigma}-\delta_2\tau)}{\Lambda}\\
\omega_2 &= \frac{\delta_3(\delta_{1\sigma}-\delta_2\kappa)-\delta_1(\delta_{3\sigma}-\delta_2\tau)}{\Lambda}\\
\omega_3 &= -\frac{\delta_1(\delta_1\kappa+\delta_{2\sigma}-\delta_3\tau)-\delta_2(\delta_{1\sigma}-\delta_2\kappa)}{\Lambda}
\end{aligned}$$

Since $\widetilde{\beta}_1$ and β_1 have with same integrable systems, the curve $\widetilde{\beta}_1$ satisfies the generalized nonlinear heat flow $\widetilde{\beta}_{1u}$ of the visco Da Rios equation for the first class in Minkowski 3-space following.

$$\widetilde{\beta}_{1u}=-\widetilde{\kappa}\widetilde{\mathbf{B}}+\varsigma\widetilde{\mathbf{T}}. \tag{24}$$

Theorem 3.1 *The generalized nonlinear heat flow Eq.(24) is invariant according to Backlund transformation Eq.(15) for second class in Minkowski 3-space if f_1, f_2 and f_3 satisfy the following system*

$$\varsigma+f_{1u}-f_2\kappa(\tau+\varsigma)+f_3\kappa_\sigma = -\frac{\Lambda}{\Theta}\omega_1+\varsigma\delta_1 \tag{25}$$

$$f_{2u}+f_1\kappa(\tau+\varsigma)+f_3\left(\frac{\kappa_{\sigma\sigma}}{\kappa}+\tau(\tau+\varsigma)\right) = -\frac{\Lambda}{\Theta}\omega_2+\varsigma\delta_2 \tag{26}$$

$$-f_1\kappa_\sigma-f_2\left(\frac{\kappa_{\sigma\sigma}}{\kappa}+\tau(\tau+\varsigma)\right)-f_{3u}-\kappa = -\frac{\Lambda}{\Theta}\omega_3+\varsigma\delta_3 \tag{27}$$

Proof Via Eqs.(17), (19), (21), (23), (24) one obtain Eqs.(25), (26) and (27). □

Class II. Let β_2 be a spacelike curve with timelike binormal according to Frenet frame in $\mathbf{R}^3_1$. Assume that

$$\beta_{2\sigma}=\mathbf{B}. \tag{28}$$

The modified visco-Da Rios equation associated with the repulsive type generalized nonlinear Schrödinger $GNLS^{-}$ equation is given by

$$\beta_{2u} = \beta_{2\sigma} \times \beta_{2\sigma\sigma} + \varsigma\beta_{2\sigma}, \quad (29)$$

where ς is the viscosity and non-negative constant.

The general time evolution of the Frenet frame for the second class in $\mathbf{R}_1^3$ is given by

$$\begin{aligned} \mathbf{T}_u &= \gamma_2\mathbf{N}+\zeta_1\mathbf{B}, & (30)\\ \mathbf{N}_u &= -\gamma_2\mathbf{T}+\zeta_2\mathbf{B}, & (31)\\ \mathbf{B}_u &= \zeta_1\mathbf{T}+\zeta_2\mathbf{N}. & (32) \end{aligned}$$

Via Eqs.(28) and (29) it is derived by

$$\mathbf{B}_{2u} = \mathbf{B}_{2\sigma} \times \mathbf{B}_{2\sigma\sigma} + \varsigma\mathbf{B}_{2\sigma} = \tau\mathbf{T}+\varsigma\mathbf{B} \quad (33)$$

From $\beta_{2\sigma u} = \beta_{2u\sigma}$ and $\mathbf{B}_{u\sigma} = \mathbf{B}_{\sigma u}$ via Eq.(29), Eq.(32) and Eq.(33) one can derive

$$\begin{aligned} \zeta_1 &= \tau_\sigma,\ \zeta_2 = \tau(\kappa-\varsigma), & (34)\\ \gamma_2 &= \frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa-\varsigma), & (35)\\ \tau_u &= -\kappa\tau_\sigma - (\tau(\kappa-\varsigma))_\sigma & (36) \end{aligned}$$

The second frame $\{\xi_1, \xi_2, \xi_2^*\}$ connected with the repulsive type $GNLS^{-}$ equation Eq.(28) and Hasimoto-like transformation ψ_2 are given by

$$\begin{aligned} \xi_1 &= \mathbf{B}, & (37)\\ \xi_2 &= \frac{\mathbf{T}+i\mathbf{N}}{\sqrt{2}} e^{i\int\kappa}, & (38)\\ \xi_2^* &= \frac{\mathbf{T}-i\mathbf{N}}{\sqrt{2}} e^{-i\int\kappa} & (39) \end{aligned}$$

and

$$\psi_2 = \frac{\tau}{\sqrt{2}} e^{i\int\kappa}, \quad (40)$$

where ξ_2^* is the complex conjugate of ξ_2. Using Eqs.(37), (38), (39) and (40), the spatial and time evolutions of the frame $\{\xi_1, \xi_2, \xi_2^*\}$ connected with the $GNLS^{-}$ equation for second class, respectively are given by

$$\begin{aligned} \xi_{1\sigma} &= -i\psi_2\xi_2^* + i\psi_2^*\xi_2,\\ \xi_{2\sigma} &= -i\psi_2\xi_1,\\ \xi_{2\sigma}^* &= i\psi_2^*\xi_1;\\ \xi_{1u} &= (\psi_{2\sigma}^* + i\varsigma\psi_2^*)\xi_2 + (\psi_{2\sigma} - i\varsigma\psi_2)\xi_2^*,\\ \xi_{2u} &= (\psi_{2\sigma} - i\varsigma\psi_2)\xi_1 + iR_2\xi_2, \end{aligned}$$

$$\xi^*_{2u} = (\psi^*_{2\sigma} + i\varsigma\psi^*_2)\xi_1 - iR_2\xi^*_2.$$

From $\xi_{2u\sigma} = \xi_{2\sigma u}$ one can obtain the repulsive type $GNLS^-$ equation for second class in in $\mathbf{R}^3_1$

$$\psi_{2u} = i\psi_{2\sigma\sigma} + iR_1\psi_2 + \varsigma\psi_{2\sigma},\ R_2 = -\psi_2\psi^*_2$$

3.2 Backlund Transformation of the $GNLS^-$ Flow with Respect to Frenet Frame for Second Class in $\mathbf{R}^3_1$

Backlund transformation of the $GNLS^-$ flow of spacelike curve β_2 with timelike binormal is constructed as the following.

Let $\widetilde{\beta}_2$ be another curve associated with the spacelike curve β_2 with timelike binormal $\mathbf{B}$ for second class in $\mathbf{R}^3_1$, it can be expressed

$$\widetilde{\beta}_2(\sigma,u) = \beta_2(\sigma,u) + g_1(\sigma,u)\mathbf{T} + g_2(\sigma,u)\mathbf{N} - g_3(\sigma,u)\mathbf{B}, \tag{41}$$

where g_1, g_2 and g_3 are the smooth functions of u and σ. By differentiating of according σ and u of Eq.(41), the following are obtained by

$$\frac{\partial\widetilde{\beta}_2}{\partial\sigma} = \widetilde{\beta}_{2\sigma} = (g_{1\sigma} - g_2\kappa)\mathbf{T} + (g_1\kappa + g_{2\sigma} + g_3\tau)\mathbf{N} + (1 - g_2\tau - g_{3\sigma})\mathbf{B}, \tag{42}$$

$$\begin{aligned} \widetilde{\beta}_{2u} &= (\tau + g_{1u} - g_2(\frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa-\varsigma)) - g_3\tau_\sigma)\mathbf{T} \\ &\quad + (g_1(\frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa-\varsigma)) + g_{2u} - g_3\tau(\kappa-\varsigma))\mathbf{N} \\ &\quad + (\varsigma + g_1\tau_\sigma + g_2\tau(\kappa-\varsigma) - g_{3u})\mathbf{B}. \end{aligned} \tag{43}$$

For the arclength parameter $\widetilde{\sigma}$ is of the curve $\widetilde{\beta}_2$ in $\mathbf{R}^3_1$ one can obtains

$$\begin{aligned} d\widetilde{\sigma} &= \sqrt{(g_{1\sigma} - g_2\kappa)^2 + (g_{2\sigma} + g_1\kappa + g_3\tau)^2 - (1 - g_2\tau - g_{3\sigma})^2}d\sigma \\ &= \Gamma d\sigma \end{aligned} \tag{44}$$

The tangent vector $\widetilde{\mathbf{T}}$ of the curve $\widetilde{\beta}_2$ for the second class in $\mathbf{R}^3_1$ is obtained by

$$\widetilde{\mathbf{T}} = \frac{d\widetilde{\beta}_2}{d\sigma}\frac{d\sigma}{d\widetilde{\sigma}} = \varphi_1\mathbf{T} + \varphi_2\mathbf{N} + \varphi_3\mathbf{B}, \tag{45}$$

where

$$\begin{aligned} \varphi_1 &= \Gamma^{-1}(g_{1\sigma} - g_2\kappa)\mathbf{T},\ \ \varphi_2 = \Gamma^{-1}(g_{2\sigma} + g_1\kappa + g_3\tau) \\ \varphi_3 &= -\Gamma^{-1}(1 - g_2\tau - g_{3\sigma}). \end{aligned}$$

Using Eq.(45), one can obtains

$$\frac{d^2\widetilde{\beta}}{d\widetilde{\sigma}^2} = \frac{\varphi_{1\sigma} - \varphi_2\kappa}{\Gamma}\mathbf{T} + \frac{\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau}{\Gamma}\mathbf{N} + \frac{\varphi_{3\sigma} - \varphi_2\tau}{\Gamma}\mathbf{B}. \tag{46}$$

Using Eqs.(46) the curvature $\widetilde{\kappa}$ of the curve $\widetilde{\beta}_2$ is obtained by

$$\widetilde{\kappa} = \frac{\sqrt{(\varphi_{1\sigma} - \varphi_2\kappa)^2 + (\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau)^2 - (\varphi_{3\sigma} - \varphi_2\tau)^2}}{\Gamma} = \frac{\Phi}{\Gamma}. \tag{47}$$

With aid of Eqs.(46) and (47), the principal normal vector $\widetilde{\mathbf{N}}$ of the curve $\widetilde{\beta}_2$ is obtained by

$$\widetilde{\mathbf{N}} = \frac{\varphi_{1\sigma} - \varphi_2\kappa}{\Phi}\mathbf{T} + \frac{\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau}{\Phi}\mathbf{N} + \frac{\varphi_{3\sigma} - \varphi_2\tau}{\Phi}\mathbf{B}. \tag{48}$$

From Eqs.(46), (47) and (48), the binormal vector of $\widetilde{\mathbf{B}}$ of the curve $\widetilde{\beta}_2$ can be obtained by

$$\widetilde{\mathbf{B}} = \rho_1\mathbf{T} + \rho_2\mathbf{N} + \rho_3\mathbf{B}, \tag{49}$$

where

$$\begin{aligned}
\rho_1 &= \frac{\varphi_2(\varphi_{3\sigma} - \varphi_2\tau) - \varphi_3(\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau)}{\Phi} \\
\rho_2 &= \frac{-\varphi_1(\varphi_{3\sigma} - \varphi_2\tau) + \varphi_3(\varphi_{1\sigma} - \varphi_2\kappa)}{\Phi} \\
\rho_3 &= \frac{-\varphi_1(\varphi_1\kappa + \varphi_{2\sigma} - \varphi_3\tau) + \varphi_2(\varphi_{1\sigma} - \varphi_2\kappa)}{\Phi}.
\end{aligned}$$

The torsion vector $\widetilde{\tau}$ of the curve $\widetilde{\beta}_2$ using Eqs.(48) and (49) in Minkowski 3-space is derived by

$$\widetilde{\tau} = -\frac{\Psi}{\Gamma\Phi}, \tag{50}$$

where

$$\begin{aligned}
\Psi &= (\rho_{1\sigma} - \rho_2\kappa)(\varphi_{1\sigma} - \varphi_2\kappa) + (\rho_{2\sigma} + \rho_1\kappa - \rho_3\tau)(\varphi_{2\sigma} + \varphi_1\kappa - \varphi_3\tau) \\
&\quad -(\varphi_{3\sigma} - \varphi_2\tau)(\rho_{3\sigma} - \rho_2\tau).
\end{aligned}$$

Since the curve $\widetilde{\beta}_2$ and the curve β_2 are expressed with same integrable systems, the curve $\widetilde{\beta}_2$ satisfies the following generalized nonlinear Schrödinger flow of the visco Da Rios equation connected with the second class in Minkowski 3-space:

$$\widetilde{\beta}_{2u} = \widetilde{\tau}\widetilde{\mathbf{T}} + \varsigma\widetilde{\mathbf{B}}. \tag{51}$$

Theorem 3.2 *The generalized Schrödinger flow Eq.(51) is invariant according to Backlund transformation Eq.(41) for second class in Minkowski 3-space if g_1, g_2 and g_3 satisfies the following system*

$$\begin{aligned}
\tau + g_{1u} - g_2\left(\frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa - \varsigma)\right) - g_3\tau_\sigma &= -\frac{\Psi}{\Gamma\Phi}\varphi_1 + \varsigma\rho_1 && (52) \\
g_1\left(\frac{\tau_{\sigma\sigma}}{\tau} - \kappa(\kappa - \varsigma)\right) + g_{2u} - g_3\tau(\kappa - \varsigma) &= -\frac{\Psi}{\Gamma\Phi}\varphi_2 + \varsigma\rho_2 && (53) \\
\varsigma + g_1\tau_\sigma + g_2\tau(\kappa - \varsigma) - g_{3u} &= -\frac{\Psi}{\Gamma\Phi}\varphi_3 + \varsigma\rho_3. && (54)
\end{aligned}$$

Proof Via Eqs.(43), (48), (49), (50), (51) one can obtain Eqs.(52), (53) and (54). □

Class 3 Let β_3 be spacelike curve with timelike binormal according to the Frenet frame in $\mathbf{R}_1^3$. Assume that

$$\beta_{3\sigma} = \mathbf{N}. \tag{55}$$

The modified visco-Da Rios equation associated with the repulsive type generalized nonlinear Schrödinger $GNLS^-$ equation is given by

$$\beta_{3u} = \beta_{3\sigma} \times \beta_{3\sigma\sigma} + \varsigma\beta_{3\sigma}, \tag{56}$$

where ς is the viscosity and non-negative constant. The general time evolution of the Frenet frame

$$\begin{aligned} \mathbf{T}_u &= -h_1\mathbf{N} + \gamma_3\mathbf{B} && (57)\\ \mathbf{N}_u &= h_1\mathbf{T}+h_2\mathbf{B} && (58)\\ \mathbf{B}_u &= \gamma_3\mathbf{T} - h_2\mathbf{N}. && (59) \end{aligned}$$

Via Eqs.(55) and (56) it is found

$$\beta_{3u} = -\tau\mathbf{T} + \varsigma\mathbf{N} - \kappa\mathbf{B}. \tag{60}$$

From $\beta_{3u\sigma} = \beta_{3\sigma u}$ and $\mathbf{N}_{3u\sigma} = \mathbf{N}_{3\sigma u}$ the following are obtained by

$$h_1 = -\tau_\sigma - \varsigma\kappa,\ h_2 = -(\kappa_\sigma + \varsigma\tau),\ \gamma_3 = \frac{1}{2}(\tau^2 - \kappa^2)\ , \tag{61}$$

$$\begin{aligned} \kappa_u &= \tau_{\sigma\sigma} + \varsigma\kappa_\sigma + \frac{1}{2}\tau(\kappa^2 - \tau^2),\\ \tau_u &= \kappa_{\sigma\sigma} + \varsigma\tau_\sigma + \frac{1}{2}\kappa(\kappa^2 - \tau^2). \end{aligned}$$

The third frame $\{\pi_1, \pi_2, \pi_3\}$ connected with the generalized nonlinear heat system and Hasimoto-like transformation Ω_1 for the spacelike curve with timelike binormal are given by:

$$\begin{aligned} \pi_1 &= \mathbf{N}, && (62)\\ \pi_2 &= \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{B}), && (63)\\ \pi_3 &= \frac{1}{\sqrt{2}}(\mathbf{B} - \mathbf{T}). && (64) \end{aligned}$$

and

$$\Omega_1 = \frac{1}{\sqrt{2}}(\kappa + \tau). \tag{65}$$

Using Eqs.(62), (63), (64) and (65), the spatial evolution of the frame $\{\pi_1, \pi_2, \pi_3\}$ connected with the generalized nonlinear heat equation of the spacelike curve with timelike binormal for third class, respectively are given by

$$\begin{aligned}\pi_{1\sigma} &= -\Omega_1\pi_2 + \Omega_2\pi_3\\ \pi_{2\sigma} &= \Omega_2\pi_1\\ \pi_{3\sigma} &= -\Omega_1\pi_1.\end{aligned}$$

where

$$\Omega_2 = \frac{1}{\sqrt{2}}(\kappa - \tau).$$

The time evolution of the frame $\{\pi_1, \pi_2, \pi_3\}$ connected with the generalized nonlinear heat equation of the spacelike curve with timelike binormal for the third class, respectively are given by

$$\begin{aligned}\pi_{1u} &= -(\Omega_{1\sigma} - \varsigma\Omega_1)\pi_2 - (\Omega_{2\sigma} + \varsigma\Omega_2)\pi_3\\ \pi_{2u} &= -(\Omega_{2\sigma} + \varsigma\Omega_2)\pi_1 + R_3\pi_2\\ \pi_{3u} &= -(\Omega_{1\sigma} - \varsigma\Omega_1)\pi_1 - R_3\pi_3\end{aligned}$$

From $\pi_{2u\sigma} = \pi_{2\sigma u}$ one obtains the generalized nonlinear heat system for the spacelike curve with timelike binormal with respect to Frenet frame for third class in Minkowski 3-space:

$$\begin{aligned}\Omega_{2u} &= -\Omega_{2\sigma\sigma} + \varsigma\Omega_{2\sigma} + R_3\Omega_2,\ R_3 = \Omega_1\Omega_2\\ \Omega_{1u} &= \Omega_{1\sigma\sigma} + \varsigma\Omega_{1\sigma} - R_3\Omega_1,\ R_3 = -\Omega_1\Omega_2\end{aligned}$$

3.3 Backlund Transformation Connected with visco-Da Rios Equation for Third Class in $\mathbf{R}_1^3$

Backlund transformation of the generalized Schrödinger flow for the spacelike curve β_3 with timelike binormal for the third class is constructed as the following:

Consider other nonnul curve $\widetilde{\beta}_3$ connected with β_3,

$$\widetilde{\beta}_3(\sigma, u) = \beta_3(\sigma, u) + h_1(\sigma, u)\mathbf{T} + h_2(\sigma, u)\mathbf{N} - h_3(\sigma, u)\mathbf{B}, \tag{66}$$

where h_1, h_2 and h_3 are functions of u and σ. If the derivatives of according σ and u of Eq.(66) are taken, one can obtain

$$\widetilde{\beta}_{3\sigma} = (h_{1\sigma} - h_2\kappa)\mathbf{T} + (1 + h_1\kappa + h_{2\sigma} + h_3\tau)\mathbf{N} - (h_{3\sigma} + h_2\tau)\mathbf{B}, \tag{67}$$

$$\begin{aligned}\widetilde{\beta}_{3u} &= (-\tau + h_{1u} - h_2(\tau_\sigma + \varsigma\kappa) - h_3(\frac{\tau^2 - \kappa^2}{2}))\mathbf{T}\\ &\quad +(\varsigma + h_1(\tau_\sigma + \varsigma\kappa) + h_{2u} - h_3(\kappa_\sigma + \varsigma\tau)\mathbf{N}\\ &\quad +(h_1(\frac{\tau^2 - \kappa^2}{2}) - h_2(\kappa_\sigma + \varsigma\tau) - h_{3u} - \kappa)\mathbf{B}\end{aligned} \tag{68}$$

Let $\widetilde{\sigma}$ be the arc length parameter of the curve $\widetilde{\beta}_3$, in this case

$$\begin{aligned} d\widetilde{\sigma} &= \sqrt{(h_{1\sigma}-h_2\kappa)^2+(1+h_1\kappa+h_{2\sigma}+h_3\tau)^2-(h_{3\sigma}+h_2\tau)^2}d\sigma \\ &= \chi d\sigma \end{aligned} \tag{69}$$

The tangent vector of the curve $\widetilde{\beta}_3$ is obtained by

$$\widetilde{\mathbf{T}} = \frac{d\widetilde{\beta}_3}{d\sigma}\frac{d\sigma}{d\widetilde{\sigma}} = m_1\mathbf{T}+m_2\mathbf{N}+m_3\mathbf{B}. \tag{70}$$

From Eq.(70),

$$\frac{d^2\widetilde{\beta}_3}{d\widetilde{\sigma}^2} = \frac{m_{1\sigma}-m_2\kappa}{\chi}\mathbf{T}+\frac{m_{2\sigma}-m_1\kappa+m_3\tau}{\chi}\mathbf{N}+\frac{m_2\tau+m_{3\sigma}}{\chi}\mathbf{B}. \tag{71}$$

From Eq.(70), the curvature $\widetilde{\kappa}$ of $\widetilde{\beta}_2$ is given by

$$\widetilde{\kappa} = \frac{\sqrt{(m_{1\sigma}-m_2\kappa)^2+(m_{2\sigma}+m_1\kappa-m_3\tau)^2-(m_{3\sigma}-m_2\tau)^2}}{\chi} = \frac{\Upsilon}{\chi}. \tag{72}$$

The normal vector $\widetilde{\mathbf{N}}$ of the curve $\widetilde{\beta}_3$ is given by

$$\widetilde{\mathbf{N}} = \frac{m_{1\sigma}-m_2\kappa}{\Upsilon}\mathbf{T}+\frac{m_{2\sigma}-m_1\kappa+m_3\tau}{\Upsilon}\mathbf{N}+\frac{m_2\tau+m_{3\sigma}}{\Upsilon}\mathbf{B}. \tag{73}$$

The vector of $\widetilde{\mathbf{B}}$ of the curve $\widetilde{\beta}_3$ is given by

$$\widetilde{\mathbf{B}} = n_1\mathbf{T}+n_2\mathbf{N}+n_3\mathbf{B}, \tag{74}$$

where

$$\begin{aligned} n_1 &= \frac{m_2(m_{3\sigma}+m_2\tau)-m_3(m_{2\sigma}-m_1\kappa+m_3\tau)}{\Upsilon} \\ n_2 &= \frac{m_3(m_{1\sigma}-m_2\kappa)-m_1(m_{3\sigma}+m_2\tau)}{\Upsilon} \\ n_3 &= \frac{m_2(m_{1\sigma}-m_2\kappa)-m_1(m_{2\sigma}-m_1\kappa+m_3\tau)}{\Upsilon} \end{aligned}$$

The torsion vector $\widetilde{\tau}$ of the curve $\widetilde{\beta}_3$ is obtained by

$$\widetilde{\tau} = \frac{\alpha}{\Upsilon\chi}, \tag{75}$$

where

$$\begin{aligned} \alpha &= -(n_{1\sigma}-n_2\kappa)(m_{1\sigma}-m_2\kappa)-(n_{2\sigma}+n_1\kappa-n_3\tau)(m_{2\sigma}-m_1\kappa+m_3\tau) \\ &\quad +(n_{3\sigma}-n_2\tau)(m_{3\sigma}+m_2\tau) \end{aligned}$$

Since the curve $\widetilde{\beta}_3$ and the curve β_3 have with same integrable systems, the curve $\widetilde{\beta}_3$ yields the following generalized nonlinear heat $GNLH$ flow of the visco Da Rios equation for the third class in Minkowski 3-space

$$\widetilde{\beta}_{3u} = -\widetilde{\tau}\widetilde{\mathbf{T}} - \widetilde{\kappa}\mathbf{B} + \varsigma\widetilde{\mathbf{N}}. \tag{76}$$

Theorem 3.3 *The generalized nonlinear heat flow Eq.(76) is invariant according to Backlund transformation Eq.(66) for third class in Minkowski 3-space if h_1, h_2 and h_3 satisfies the following system:*

$$-\tau + h_{1u} - h_2(\tau_\sigma + \varsigma\kappa) - h_3(\frac{\tau^2 - \kappa^2}{2}) = -\frac{\alpha}{\Upsilon\chi}m_1 - \frac{\Upsilon}{\chi}n_1 + \varsigma\frac{m_{1\sigma} - m_2\kappa}{\Upsilon} \tag{77}$$

$$\varsigma + h_1(\tau_\sigma + \varsigma\kappa) + h_{2u} - h_3(\kappa_\sigma + \varsigma\tau) = -\frac{\alpha}{\Upsilon\chi}m_2 - \frac{\Upsilon}{\chi}n_2 + \varsigma\frac{m_{2\sigma} - m_1\kappa + m_3\tau}{\Upsilon} \tag{78}$$

$$h_1(\frac{\tau^2 - \kappa^2}{2}) - h_2(\kappa_\sigma + \varsigma\tau) - h_{3u} - \kappa = -\frac{\alpha}{\Upsilon\chi}m_3 - \frac{\Upsilon}{\chi}n_3 + \varsigma\frac{m_2\tau + m_{3\sigma}}{\Upsilon} \tag{79}$$

Proof Via Eqs.(68), (70), (72), (73), (74), (76) one can obtain Eqs.(77), (78) and (79). □

§4. Conclusions

In this paper one obtained the first class connected with the generalized nonlinear heat $GNLH$ system to the spacelike curve evolution with timelike binormal according to Frenet frame in $\mathbf{R}^3_1$. Later, one presented Backlund transformation of $GNLH$ flow with the Frenet frame in Minkowski 3-space. We gave the second class connected with generalized nonlinear Schrödinger equation $GNLS^-$ of spacelike curve evolution with timelike binormal according to Frenet frame in $\mathbf{R}^3_1$ and obtained Backlund transformation of $GNLS$ flow with Frenet frame in Minkowski 3-space in $\mathbf{R}^3_1$. Finally, one presented the third class connected with the generalized nonlinear Schrödinger equation $GNLH$ of spacelike curve evolution with timelike binormal according to Frenet frame in $\mathbf{R}^3_1$ and obtained Backlund transformation of $GNLH$ flow with Frenet frame in Minkowski 3-space in $\mathbf{R}^3_1$.

References

[1] Bäcklund A.V., *Concerning Surfaces with Constant Negative Curvature*, Coddington E.M. Translator, New Era Printing Co.: Lancaster, PA, USA, 1905.

[2] S.S Chern, K. Tenenblat, Pseudospherical surfaces and evolution equations, *Stud. Appl. Math.*, 74(1986), 55–83.

[3] A. Calini and T. Ivey, Backlund transformations and knots of constant torsion, *J.Knot*

Theory Ramifications, 7 (1998), 719–746.

[4] C. Rogers and W. K. Schief, *Backlund and Darboux Transformations, Geometry of Modern Applications in Soliton Theory*, Cambridge University Press, Cambridge, 2002.

[5] S. Z. Nemeth, Backlund transformations of constant torsion curves in 3-dimensional constant curvature spaces, *Italian J. Pure Appl. Math.*, 7 (2000), 125–138.

[6] B. Palmer, Backlund transformations for surfaces in Minkowski space, *Journal of Mathematical Physics*, 31 (1990), 2872-2875.

[7] N.Gürbüz, Bäcklund transformations of constant torsion curves in R_1^n, *Hadronic Journal*, 29 (2006), 213-220.

[8] R.A Abdel-baky,The Bäcklund's theorem in Minkowski 3-space, *App. Math. and Comp.*, 160 (2005), 41-55.

[9] P. Bracken, Backlund transformations for several cases of a generalized KdV equation, *International Journal of Mathematics and Mathematical Sciences*, 63 (2004), 3369-3377.

[10] M. Grbovic and E. Nesovic, On Backlund transformation and vortex filament equation of pseudo null curve in Minkowski 3-space, *International Journal of Geometric Methods in Modern Physics*, 13 (2016), 1650077?.

[11] L. S. Da Rios, On the motion of an unbounded fluid with a vortex filament of an shape, *Rend. Circ. Mat. Palermo*, 22 (1906), 117.

[12] H. Hasimoto, Soliton on a vortex filament,*J. Fluid Mech.*, 51 (1972) 477–485.

[13] R. L. Ricca, The contributions of Da Rios and Levi-Civita to asymptotic potential theory and vortex filament dynamics, *Fluid Dyn. Res.*, 18 (1996), 245-268.

[14] M. Barros, A. Ferrandez, P. Lucas and M. Merono, Solutions of the Betchov–Da Rios soliton equations: A Lorentzian approach, *J. Geom. Phys.*, 33 (1999), 217–228.

[15] Q. Ding and J. Inoguchi, Schrodinger flows, binormal motion for curves and the second AKNS-hierarchies, *Chaos Solitons Fractals*, 21 (2004), 669–677.

[16] Z. W. Zhong, L. M. Li, Q. Y. Hai and W. Ke, Modified Heisenberg ferromagnet model and integrable equation, *Commun. Theory. Phys.*, 44 (2005), 415–418.

[17] N. Gürbüz, The differential formula of Hasimoto transformation in Minkowski 3-space, *International Journal of Mathematics and Mathematical Sciences*, 16 (2005), 2609–2616.

[18] S. C. Anco, R. Myrzakulov, Integrable generalizations of Schrodinger maps and Heisenberg spin models from Hamiltonian flows of curves and surfaces, *Journal of Geometry and Physics*, 60 (2010), 1576–1603.

[19] Marí Beffa G., Sanders J.A., Wang J.P., Integrable systems in three-dimensional Riemannian geometry, *J. Nonlinear Sci.*, 2002,12, 143–167.

[20] N Gürbüz, Intrinstic geometry of NLS equation and heat system in 3-dimensional Minkowski space, *Adv. Stud. Theory*, 4 (2010), 557-564.

[21] N.Gürbüz, The motion of timelike surfaces in timelike geodesic coordinates, *Int. J. Math. Anal*, 4 (2010), 349-356.

[22] N Gürbüz, DW Yoon, Hasimoto surfaces for two classes of curve evolution in Minkowski 3-space, *Demonstratio Mathematica*, 53 (2020), 277-284.

[23] Körpınar, T., A new version of Fermi walker derivative with constant energy for normal image of slant Helix in the Lie groups, *Differential Equations and Dynamical Systems*,

(2018), 1-9, https://doi.org/10.1007/s72591-018-0445-3.

[24] S. Murugesh, R. Balakrishnan, New connections between moving curves and soliton equations, *Physics Letters*, *A*, 290 (2001), 81-87.

[25] N Gürbüz, Anholonomy according to three formulations of non-null curve evolution, *International Journal of Geometric Methods in Modern Physics*, 14 (2017), 1750175.

[26] H.C. Pak, Motion of vortex filaments in 3-manifolds, *Bull.Korean Soc.*, 42 (2005), 75-85.

[27] N.E.Gürbüz, D. W. Yoon, The visco modified Heisenberg magnet equation and physical applications, *Optik, IJLEO*, https://doi.org/10.1016/j.ijleo.2021.167815 (2021).

[28] C. Qu, J. Han, J. Kang, Bäcklund transformations for integrable geometric curve flows, *Symmetry*, 7 (2015), 1376-1394.

[29] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

International J.Math. Combin. Vol.4(2021), 25-33

The Natural Lift Curve of the Spherical Indicatrix of a Curve According to Bishop Frame in Euclidean 3-Space

Evren ERGÜN

Çarşamba Chamber of Commerce Vocational School of Ondokuz
Mayıs University, Çarşamba 55500, Samsun, Turkey

E-mail: eergun@omu.edu.tr

Abstract: In this study, we dealt with the natural lift curves of the spherical indicatrices of a curve according to Bishop frame.Furthermore, some interesting results about the original curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2\right)$.

Key Words: Natural lift, bishop frame, geodesic spray.

AMS(2010): 53A04, 51N20, 51M05.

§1. Introduction

Classical differential geometry began with the emergence of derivative and integral calculus, which helped solve geometric problems such as determining the tangents and curvatures of a curve.

Richard L. Bishop [1] brought forward the best answer to this as "there are 3 more than one way to crack a curve". Bishop observed that parallel vector fields on a C^2 regular curve form a 3-dimensional vector space. He clarified the equations of the Bishop roof, which is named after him; hence it is sometimes referred to as the Relatively Parallel Adapted Frame, Bishop, [1]. Fenchel W. [7] stated that a point $\gamma(t)$ on a curve, when plotting the curve, the Frenet vectors $\{T, N, B\}$ change and thus spherical signs are formed.

Thorpe J.A. [4], together with the geodesic spray concepts, gave the theorem that "for a curve γ to be an integral curve for the geodesic spray X of the natural lift γ, and only if γ is a geodesic over "M.Çalışkan, Sivridağ and Hacısalihoğlu [5], using these concepts and theorem given by Thorpe [4] in E^3.

§2. Preliminaries

Let $\gamma : I \longrightarrow \mathbb{R}^3$ be a parameterized curve. We denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the curve γ, where T, N and B are the tangent, the principal normal and the binormal vector fields of the curve γ, respectively.

[1]Received November 5, 2021, Accepted December 10, 2021.

Let γ be a reguler curve in $\mathbb{R}^3$. Then ([6])

$$T = \frac{\gamma^{'}}{\|\gamma^{'}\|},\ N = B \times T,\ B = \frac{\gamma^{'} \times \gamma^{''}}{\|\gamma^{'} \times \gamma^{''}\|}$$

and if γ is a unit speed curve, then

$$T = \gamma^{'},\ N = \frac{\gamma^{''}}{\|\gamma^{''}\|},\ B = T \times N.$$

Let γ be a unit speed space curve with curvature κ and torsion τ. Let Frenet vector fields of γ be $\{T, N, B\}$. Then, Frenet formulas are given ([6]) by

$$T^{'} = \kappa N,\ N^{'} = -\kappa T + \tau B,\ B^{'} = -\tau N,$$

where $\kappa = \left\langle T^{'}, N \right\rangle$ and $\tau = \left\langle N^{'}, B \right\rangle$.

Definition 2.1 *Let $\gamma : I \to \mathbb{R}^3$ be a unit speed spacelike or timelik space curve. Let $T = \dot{\gamma}$ be the tangent vector defined at each point of the curve. In this case, M_1 and M_2 vectors are perpendicular to the tangent vector T at each point and any two vector fields in the normal plane, on the curve γ, $\{T, N, B\}$, there is always a frame $\{T, M_1, M_2\}$,as an alternative to the moving frame. $\{T, M_1, M_2\}$ is Bishop frame to this alternative frame*

Then, Frenet formulas are given by [1]

$$\begin{aligned}
\dot{T} &= k_1 M_1 + k_2 M_2, \\
\dot{M_1} &= k_1 T, \\
\dot{M_2} &= k_2 T, \\
\kappa(t) &= \sqrt{k_1^2 + k_2^2},\ \varphi(t) = \arctan\left(\frac{k_1}{k_2}\right),\ \tau(t) = \dot{\varphi}, \\
k_1 &= \kappa \cos\varphi,\ k_2 = \kappa \sin\varphi, \\
T &= T, \\
M_1 &= \cos\varphi N - \sin\varphi B, \\
M_2 &= \sin\varphi N + \cos\varphi B
\end{aligned}$$

where the differentiable functions k_1 and k_2 are the Bishop curvatures.

Definition 2.2([1]) *Let M be a hypersurface in $\mathbb{R}^3$ and let $\alpha : I \longrightarrow M$ be a parameterized curve. γ is called an integral curve of X if*

$$\frac{d}{ds}(\gamma(s)) = X(\gamma(s))\ \ (\textit{for all } t \in I),$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup_{P\in M} T_P M = \chi(M)$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of vector fields on M.

Definition 2.3([4]) *For any parameterized curve $\gamma : I \longrightarrow M$, $\overline{\gamma} : I \longrightarrow TM$ given by*

$$\overline{\gamma}(s) = \left(\gamma(s), \gamma^{'}(s)\right) = \gamma^{'}(s)\,|_{\gamma(s)}$$

is called the natural lift of γ on TM. Thus, we can write

$$\frac{d\overline{\gamma}}{ds} = \frac{d}{ds}\left(\gamma^{'}(s)\,|_{\gamma(s)}\right) = D_{\gamma^{'}(s)}\gamma^{'}(s),$$

where D is the Levi-Civita connection on $\mathbb{R}^3$.

Definition 2.4([4]) *A $X \in \chi(TM)$ is called a geodesic spray if for $V \in TM$*

$$X(V) = -\langle S(V), V\rangle N. \tag{1}$$

Theorem 2.5([4]) *The natural lift $\overline{\gamma}$ of the curve γ is an integral curve of geodesic spray X if and only if γ is a geodesic on M.*

§3 The Natural Lift Curve of the Spherical Indicatrix of a Curve According to Bishop Frame

Let ∇, $\bar{\nabla}$ be Levi-Civita connections on $\mathbb{R}^3$ and S^2 and ξ be a unit normal vector field of S^2. Then Gauss equations are given by the followings

$$\nabla_X Y = \bar{\nabla}_X Y + \langle S(X), Y\rangle \xi,$$

where S is the shape operator of S^2 and

$$S = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let γ_T be the spherical indicatrix of tangent vectors of γ and $\overline{\gamma}_T$ be the natural lift of the curve γ_T. In this case the equation for γ_T is $\gamma_T = T$. If $\overline{\gamma}_T$ is an integral curve of the geodesic spray, then from Theorem 2.5 we have

$$\bar{\nabla}_{\dot{\gamma}_T}\dot{\gamma}_T = 0,$$

that is,

$$\nabla_{\dot{\gamma}_T}\dot{\gamma}_T = \bar{\nabla}_{\dot{\gamma}_T}\dot{\gamma}_T - \left\langle S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right\rangle \xi$$

$$\nabla_{\dot{\gamma}_T}\dot{\gamma}_T = -\left\langle S\left(\dot{\gamma}_T\right), \dot{\gamma}_T\right\rangle T$$

$$\nabla_{\dot{\gamma}_T}\dot{\gamma}_T = -\left(k_1^2 + k_2^2\right) T$$

$$\frac{d}{ds_T}\left(k_1 M_1 + k_2 M_2\right) = -\left(k_1^2 + k_2^2\right) T$$

$$\frac{d}{ds}\left(k_1 M_1 + k_2 M_2\right)\frac{ds}{ds_T} + \left(k_1^2 + k_2^2\right) T = 0$$

$$\left(\dot{k}_1 M_1 - k_1^2 T + \dot{k}_2 M_2 - k_2^2 T\right)\frac{ds}{ds_T} + \left(k_1^2 + k_2^2\right) T = 0$$

$$\left(-\frac{k_1^2 + k_2^2}{\sqrt{k_1^2 + k_2^2}} + \left(k_1^2 + k_2^2\right)\right) T + \left(\frac{\dot{k}_1}{\sqrt{k_1^2 + k_2^2}}\right) M_1 + \left(\frac{\dot{k}_2}{\sqrt{k_1^2 + k_2^2}}\right) M_2 = 0$$

Since $\{T, M_1, M_2\}$ bishop frame is linearly independent system, we have

$$\begin{aligned}
\left(\frac{k_1^2 + k_2^2}{\sqrt{k_1^2 + k_2^2}} - \left(k_1^2 + k_2^2\right)\right) &= 0, \\
\left(\frac{\dot{k}_1}{\sqrt{k_1^2 + k_2^2}}\right) &= 0, \\
\left(\frac{\dot{k}_2}{\sqrt{k_1^2 + k_2^2}}\right) &= 0.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
k_1^2 + k_2^2 &= 1, \\
i)\ k_1 &= 1,\ k_2 = 0 \\
ii)\ k_1 &= 0,\ k_2 = 1
\end{aligned}$$

Proposition 3.1 *If the natural lift $\overline{\gamma}_T$ of γ_T is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,*

$$\begin{cases}
T = T,\ M_1 = N,\ \ M_2 = B,\ (k_1 = 1,\ k_2 = 0) \\
\\
T = T,\ M_1 = -B,\ \ M_2 = N, (\ k_1 = 0,\ k_2 = 1)
\end{cases}$$

Let γ_{M_1} be the spherical indicatrix of γ relative to M_1 and $\overline{\gamma}_{M_1}$ be the natural lift of the curve γ_{M_1}. In this case the equation for γ_{M_1} is $\gamma_{M_1} = M_1$. If $\overline{\gamma}_{M_1}$ is an integral curve of the geodesic spray, then from Theorem 2.5 we have

$$\bar{\nabla}_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = 0,$$

that is,

$$\nabla_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = \bar{\nabla}_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} - \left\langle S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1}\right\rangle \xi$$

$$\nabla_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = -\left\langle S\left(\dot{\gamma}_{M_1}\right), \dot{\gamma}_{M_1}\right\rangle M_1$$

$$\nabla_{\dot{\gamma}_{M_1}} \dot{\gamma}_{M_1} = -k_1^2 M_1$$

$$\frac{d}{ds_{M_1}}\left(-k_1 T\right) = -k_1^2 M_1$$

$$\frac{d}{ds}\left(-k_1 T\right)\frac{ds}{ds_{M_1}} + k_1^2 M_1 = 0$$

$$\left(-\dot{k}_1 T - k_1^2 M_1 - k_1 k_2 M_2\right)\frac{ds}{ds_{M_1}} + k_1^2 M_1 = 0$$

$$\left(-\frac{\dot{k}_1}{k_1^2}\right) T + \left(\frac{-k_1 + k_1^2}{k_1^2}\right) M_1 + \left(-\frac{k_2}{k_1}\right) M_2 = 0$$

Since $\{T, M_1, M_2\}$ bishop frame is linearly independent system, we have

$$\begin{aligned} \frac{\dot{k}_1}{k_1^2} &= 0, \\ \left(-k_1 + k_1^2\right) &= 0, \\ k_2 &= 0. \end{aligned}$$

Hence, we have

$$k_1 = constant \ \ (k_1 \neq 0), \ k_2 = 0$$

Proposition 3.2 *If the natural lift $\overline{\gamma}_{M_1}$ of γ_{M_1} is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows,*

$$T = T, \ M_1 = N, \ \ M_2 = B.$$

Let γ_{M_2} be the spherical indicatrix of γ relative to M_2 and $\overline{\gamma}_{M_2}$ be the natural lift of the curve γ_{M_2}. In this case the equation for γ_{M_2} is $\gamma_{M_2} = M_2$. If $\overline{\gamma}_{M_2}$ is an integral curve of the geodesic spray, then from Theorem 2.5 we have

$$\bar{\nabla}_{\dot{\gamma}_{M_2}} \dot{\gamma}_{M_2} = 0,$$

that is,

$$\nabla_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2} = \bar{\nabla}_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2} - \left\langle S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right\rangle \xi$$

$$\nabla_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2} = -\left\langle S\left(\dot{\gamma}_{M_2}\right), \dot{\gamma}_{M_2}\right\rangle M_2$$

$$\nabla_{\dot{\gamma}_{M_2}}\dot{\gamma}_{M_2} = -k_2^2 M_2$$

$$\frac{d}{ds_{M_2}}\left(-k_2 T\right) = -k_2^2 M_2$$

$$\frac{d}{ds}\left(-k_2 T\right)\frac{ds}{ds_{M_2}} + k_2^2 M_2 = 0$$

$$\left(-\dot{k}_2 T - k_1 k_2 M_1 - k_2^2 M_2\right)\frac{ds}{ds_{M_2}} + k_2^2 M_2 = 0$$

$$\left(-\frac{\dot{k}_2}{k_2}\right) T + \left(-\frac{k_1}{k_2}\right) M_1 + \left(-1 + k_2^2\right) M_2 = 0$$

Since $\{T, M_1, M_2\}$ Bishop frame is linearly independent system, we have

$$\frac{\dot{k}_2}{k_2} = 0, \quad \frac{k_1}{k_2} = 0, \quad -1 + k_2^2 = 0.$$

Hence, we have

$$k_1 = 0, \ k_2 = \pm 1$$

Proposition 3.3 *If the natural lift $\overline{\gamma}_{M_2}$ of γ_{M_2} is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows*

$$\begin{aligned} T &= T, \ M_1 = -B, \quad M_2 = N, \\ T &= T, \ M_1 = B, \quad M_2 = -N, \\ k_1 &= 0, \ k_2 = \pm 1. \end{aligned}$$

Example 3.4 Given the arclength timelike curve $\gamma(s) = \left(\frac{\sqrt{5}}{3}s, \frac{2}{9}\cos 3s, \frac{2}{9}\sin 3s\right)$.In this trihedron, it is easy to show that its Frenet apparatus are

$$\begin{aligned} T(s) &= \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right), \\ N(s) &= \left(0, -\cos 3s, -\sin 3s\right), \\ B(s) &= \left(\frac{2}{3}, \frac{\sqrt{5}}{3}\sin 3s, -\frac{\sqrt{5}}{3}\cos 3s\right), \end{aligned}$$

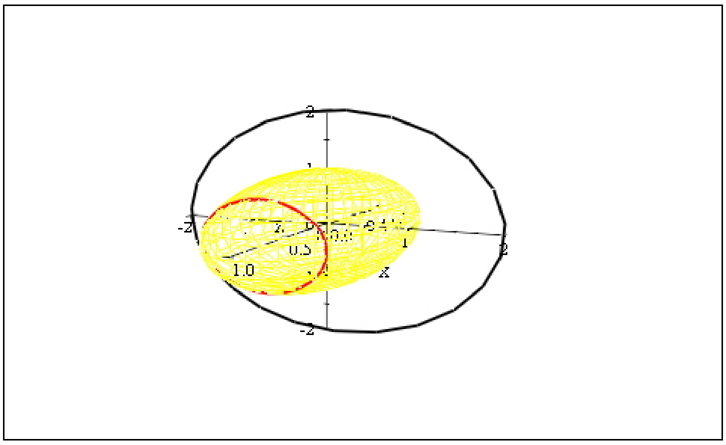
If the natural lift $\overline{\gamma}_T$ of γ_T is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows

$$\begin{cases} T = T,\ M_1 = N, \quad M_2 = B,\ (k_1 = 1,\ k_2 = 0) \\ T = T,\ M_1 = -B, \quad M_2 = N, (\ k_1 = 0,\ k_2 = 1) \end{cases}$$

with

$$\begin{aligned} T(s) &= \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right), \\ M_1(s) &= (0, -\cos 3s, -\sin 3s), \\ M_2(s) &= \left(\frac{2}{3}, \frac{\sqrt{5}}{3}\sin 3s, -\frac{\sqrt{5}}{3}\cos 3s\right), \\ & (k_1 = 1,\ k_2 = 0) \end{aligned}$$

$$P(s,t) = (\sin 3s \cos 3t, \sin 3s \sin 3t, \cos 3s)$$



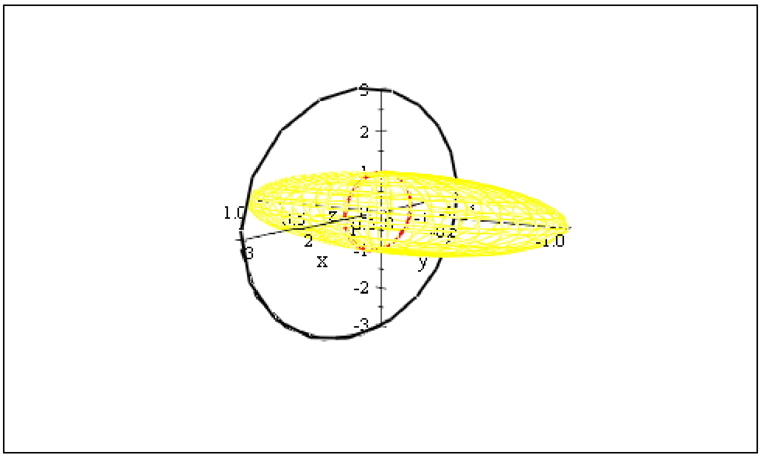
$\gamma_T = \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right)$ and $\overline{\gamma}_T = (0, -2\cos 3s, -2\sin 3s)$

If the natural lift $\overline{\gamma}_{M_1}$ of γ_{M_1} is an integral curve of the geodesic on the tangent bundle $T(S^2)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows

$$T = T,\ M_1 = N, \quad M_2 = B.$$

with

$$\begin{aligned} T(s) &= \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right), \\ M_1(s) &= (0, -\cos 3s, -\sin 3s), \\ M_2(s) &= \left(\frac{2}{3}, \frac{\sqrt{5}}{3}\sin 3s, -\frac{\sqrt{5}}{3}\cos 3s\right), \\ P(s,t) &= (\cos 3s \sin 3t, \cos 3s \cos 3t, \sin 3s). \end{aligned}$$

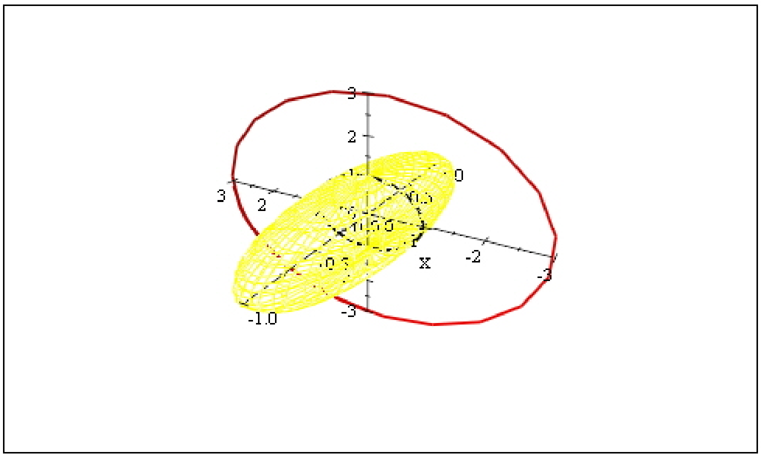


$\gamma_{M_1} = (0, -\cos 3s, -\sin 3s)$ and $\bar{\gamma}_{M_1} = (0, 3\sin 3s, -3\cos 3s)$

If the natural lift $\overline{\gamma}_{M_2}$ of γ_{M_2} is an integral curve of the geodesic on the tangent bundle $T\left(S^2\right)$, then there is a relationship between frames $\{T, N, B\}$ and $\{T, M_1, M_2\}$ as follows

$$\begin{aligned}
T &= T,\ M_1 = -B,\quad M_2 = N, \\
T &= T,\ M_1 = B,\quad M_2 = -N, \\
k_1 &= 0,\ k_2 = \pm 1, \\
T(s) &= \left(\frac{\sqrt{5}}{3}, -\frac{2}{3}\sin 3s, \frac{2}{3}\cos 3s\right), \\
M_1(s) &= \left(-\frac{2}{3}, -\frac{\sqrt{5}}{3}\sin 3s, \frac{\sqrt{5}}{3}\cos 3s\right), \\
M_2(s) &= (0, -\cos 3s, -\sin 3s).
\end{aligned}$$

$$P(s,t) = (\cos 3s \sin 3t, \cos 3s \cos 3t, \sin 3s)$$



$\gamma_{M_2} = (0, -\cos 3s, -\sin 3s)$ and $\bar{\gamma}_{M_2} = (0, 3\sin 3s, -3\cos 3s)$

References

[1] R. L. Bishop, There is more than one way to frame a curve, *The American Mathematical Monthly*, 82:3, 246-251, 1975.

[2] L. C. B.Silva, *Differential Geometry of Rotation Minimizing Frames, Spherical Curves and Quantum Mechanics of a Constrained Particle*, Doctoral Dissertation Universidade Federal De Pernambuco, 133, Brazil, 2017.

[3] G. Arreaga, R. Capovilla ve J. Guven, Frenet-Serret dynamics, *Classical Quantum Gravity*, 18:23, 5065-5083, 2001.

[4] J.A. Thorpe, *Elementary Topics Iin Differential Geometry*, Springer-Verlag, New York, Heidelberg-Berlin, 1979.

[5] M.Çalışkan, A.İ. Sivridağ and H.H. Hacısalihoğlu, Some characterizations for the natural lift curves and the geodesic spray, *Communications*, Fac. Sci.Univ. Ankara Ser. A Math. 33 , Num. 28,235-242, 1984.

[6] M.P. do Carmo, *Differential Geometry of Curves and Surfaces*, Englewood Cliffs, New Jersey 1976.

[7] Fenchel, W. On the differential geometry of closed space curves, *Bulletin of the American Mathematical Society*, 57, 44–54, (1951).

International J.Math. Combin. Vol.4(2021), 34-40

Almost Semi-Equivelar Maps on Torus and Klein Bottle

Anand Kumar Tiwari

(Department of Applied Science, Indian Institute of Information Technology, Allahabad 211 015, India)

Amit Tripathi

(Department of Applied Science & Humanities, Rajkiya Engineering College, Banda -210201, India)

E-mail: anand@iiita.ac.in, tiwari.iitp@gmail.com, amittripathi@recbanda.ac.in

Abstract: A map M is called an almost semi-equivelar if all the vertices of M have same type face-cycle except one. The maps on the surfaces of square pyramid and the pentagonal pyramid (2 out of 92 Johnson solids) provide almost semi-equivelar maps on the sphere. In this paper, for the first time, we study and classify an almost semi-equivelar map on close surfaces, other than sphere, particularly on torus and Klein bottle on at most 15 vertices.

Key Words: Semi-equivelar map, almost semi-equivelar map, torus, Klein bottle.

AMS(2010): 52B70, 57M20

§1. Introduction

For the topological graph theory related terms, we refer to [7]. A surface (closed) F is a connected, compact 2-dimensional manifold without boundary. An embedding of a connected, simple graph G into F is called a map M on F if the closure of each connected component of $F \setminus G$ is a p-gonal 2-disk D_p, also called face of the map, such that the non-empty intersection of any two faces is either a vertex or an edge, see [1]. The vertices and edges of the underlying graph G are called the vertices and edges of M. In this paper, we deal with the maps on torus and Klein bottle. Let M_1 and M_2 be two maps with vertex sets $V(M_1)$ and $V(M_2)$ respectively. Then M_1 is isomorphic to M_2, denoted as $M_1 \cong M_2$, if there is a bijective map $f : V(M_1) \to V(M_2)$ that preserves the incidence of edges and faces.

In a map M, a cycle of consecutive faces $(D_{p_1}, \cdots D_{p_k})$ around a vertex v such that $D_{p_1} \cap D_{p_2}, \ldots, D_{p_k} \cap D_{p_1}$ are edges, is called the face-cycle of v. A map is called semi-equivelar of type $(D_{p_1}. \cdots .D_{p_k})$ if the face-cycle of each vertex is same and of the type $(D_{p_1}, D_{p_2}, \cdots D_{p_k})$ up to a cyclic permutation. The well known five Platonic solids and thirteen Archimedean solids provide all possible types semi-equivelar maps on the sphere, *i.e.*, the surface of Euler characteristic 2. Such maps have been studied extensively by many researchers for the surfaces other than the sphere. For a recent progress on such maps for the surfaces of Euler characteristic 0, i.e., on torus and Klein bottle, see ([4], [5], [3], [13], [11]), for the surface of Euler characteristic -1, see ([13], [14]) and for the surfaces of Euler characteristic -2, see ([6], [10], [9]).

A map M is called an almost semi-equivelar map, briefly ASEM, if all the vertices have

[1]Received November 10, 2021, Accepted December 12, 2021.

same face-cycle except one vertex. If M has one vertex with face-cycle (f_1) and remaining vertices with face-cycle (f_2), we say that M is ASEM of the type $[(f_1)_1 : (f_2)]$. The surfaces of the square pyramid and pentagonal pyramid (2 of the 92 Johnson solids) provide ASEMs of types $[(3,3,3,3)_1 : (3,3,4)]$ and $[(3,3,3,3,3)_1 : (3,3,4)]$ respectively on sphere, see [8]. In this paper, we study and classify ASEMs on torus and Klein bottle.

The article is organized as follows:, we describe almost semi-equivelar map of the type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ for the surfaces of Euler characteristic $\chi = 0$ and give a methodology to enumerate this type maps in Section 2. In Section 3, we present the results obtained during the enumeration. In Section 4, we illustrate the methodology and prove the result. We conclude the article by presenting some observations related to such almost semi-equivelar maps.

§2. Methodology

Let M be an ASEM of the type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$. Let v be a vertex in M with the face-cycle $(D_{p_1}, D_{p_2}, \cdots, D_{p_k})$. Then the union of these disks, i.e., $\cup_{i=1}^{k} D_{p_i}$ is a 2-disk D_n with the boundary cycle C_n, where $n = (p_1 + p_2 + \ldots + p_k) - 2k$. Let us call this cycle C_n as the link of v and denote it as $\mathrm{lk}(v)$. Thus $\mathrm{lk}(v)$ is a six or seven length cycle if the face-cycle of v is $(3,3,3,3,3,3)$ or $(3,3,4,3,4)$ respectively. We use the following notations to represent these links.

The notation $\mathrm{lk}(v) = C_6(v_1, v_2, v_3, v_4, v_5, v_6)$ means that the face-cycle of v is $(3,3,3,3,3,3)$, i.e., triangular faces $[v, v_1, v_2]$, $[v, v_2, v_3]$, $[v, v_3, v_4]$, $[v, v_4, v_5]$, $[v, v_5, v_6]$ and $[v, v_6, v_1]$ are incident at v.

The notation $\mathrm{lk}(v) = C_7(v_1, v_2, v_3, \boldsymbol{v}_4, v_5, v_6, \boldsymbol{v}_7)$ means that the face-cycle of v is $(3,3,4,3,4)$, i.e., triangular faces $[v, v_1, v_2]$, $[v, v_2, v_3]$, $[v, v_5, v_6]$ and quadrangular faces $[v, v_3, v_4, v_5]$, $[v, v_6, v_7, v_1]$ are incident at v. Note that, here bold symbols $\boldsymbol{v}_i$ shows that $\boldsymbol{v}_i$ is not adjacent to v.

Let $V(M) = \{u, v_1, \ldots, v_n\}$ be the vertex set of M. Here the vertex u has the face-cycle $(3,3,3,3,3,3)$ and the remaining have $(3,3,4,3,4)$. Let $\mathrm{lk}(u) = C_6(v_1, v_2, v_3, v_4, v_5, v_6)$. Now, we use the following steps to enumerate the ASEMs.

Step 1. Without loss of generality, choose v_1 (one can choose any v_i for $1 \le i \le 6$) such that $\mathrm{lk}(v_1) = C_7(v_2, u, v_6, \boldsymbol{\alpha}_1, \alpha_2, \alpha_3, \boldsymbol{\alpha}_4)$, where α_i, for $1 \le i \le 4$, is some $v_j \in V$.

Step 2. For each choice of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ in $V(M) \times V(M) \times V(M) \times V(M)$, we construct $\mathrm{lk}(v_1)$.

Step 3. For each possibility of $\mathrm{lk}(v_1)$ obtained from Step 2, we repeat Step 1 and Step 2 until we do not get links of remaining vertices.

Step 4. Define an isomorphism (if possible) between the maps, which will lead to the enumeration of the maps.

Applying the above methodology for $|V| \le 15$, we obtain the following result.

§3. Result

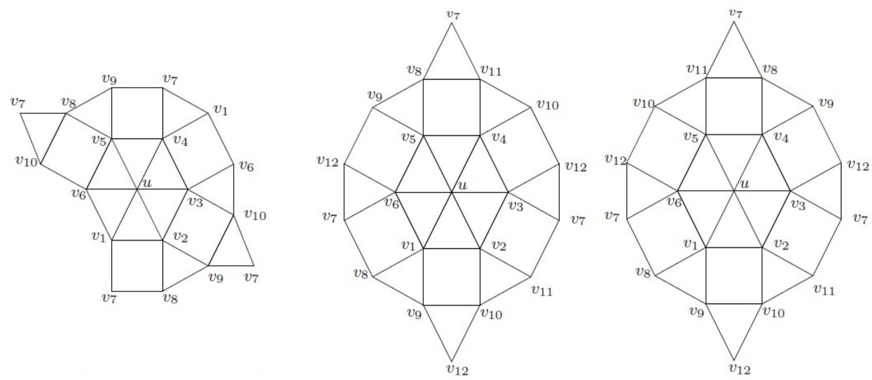
Theorem 3.1 *Let M be an ASEM of type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ with the vertex set*

$V(M)$. Then

(1) M does not exist for $|V(M)| \leq 9$;

(2) A unique M exists for $|V(M)| \leq 12$, this is $K_1[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ on Kelin bottle with 11 vertices;

(3) There exist exactly three such maps for $|V(M)| \leq 15$ on the surfaces of Euler characteristic $\chi = 0$. The other two maps are $K_2[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ and $T_1[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ on torus and Klein bottle, respectively, with $|V(M)| = 13$.



$K_1[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ $K_2[(3,3,3,3,3,3)_1 : (3,3,4,3,4))]$ $T_1[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$

Figure 3.1 ASEMs on Klein bottle

§4. Proof: Classification of ASEMs of Type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$

Let M be an ASEM of type $[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ with the vertex set $V(M) = V_{(3,3,3,3,3,3)} \cup V_{(3,3,4,3,4)}$, where $V_{(3,3,3,3,3,3)}$ and $V_{(3,3,4,3,4)}$ denote the sets of vertices with face-cycles $(3,3,3,3,3,3)$ and $(3,3,4,3,4)$ respectively. Thus for $|V| \leq 15$, we let $V_{(3,3,3,3,3,3)} = \{0\}$ and $V_{(3,3,4,3,4)} = \{1, 2, \cdots, n\}$, where $n \leq 14$.

Let $\mathrm{lk}(0) = C_6(1,2,3,4,5,6)$. This implies $\mathrm{lk}(1) = C_7(2,0,6,\boldsymbol{a},b,c,\boldsymbol{d})$, where $a,b,c,d \in V_{(3,3,4,3,4)}$. It is easy to see that $(a,b,c,d) \in \{(3,4,7,8),(4,3,7,8),(7,8,4,5),(7,8,5,4),(7,8,9,10)\}$. Observe that, $(3,4,7,8) \cong (7,8,4,5)$ and $(4,3,7,8) \cong (7,8,5,4)$ by the map $(2,6)(3,5)(7,8)$. Thus, we search for $(a,b,c,d) \in \{(4,3,7,8),(3,4,7,8),(7,8,9,10)\}$.

Case 1. In the case $(a,b,c,d) = (4,3,7,8)$, $\mathrm{lk}(1) = C_7(2,0,6,\mathbf{4},3,7,\mathbf{8})$. This implies $\mathrm{lk}(3) = C_7(2,0,4,\mathbf{6},1,7,\boldsymbol{e})$ and we see that two distinct quadrangular faces share more than one vertex. This is not allowed. Thus $(a,b,c,d) \neq (4,3,7,8)$

Case 2. In case $(a,b,c,d) = (3,4,7,8)$, we get $\mathrm{lk}(1) = C_7(2,0,6,\mathbf{3},4,7,\mathbf{8})$. Now considering $\mathrm{lk}(i)$, for $1 \leq i \leq 6$ and the fact that two distinct quadrangular faces share at most one vertex, we see easily that $\mathrm{lk}(4) = C_7(5,0,3,\mathbf{6},1,7,\mathbf{9})$. This implies $\mathrm{lk}(7) = C_7(8,10,9,\mathbf{5},4,1,\mathbf{2})$ and $\mathrm{lk}(2) = C_7(3,0,1,\mathbf{7},8,e,\boldsymbol{f})$. Observe that $(e,f) \in \{(9,10),(11,9),(11,10),(11,12)\}$.

Subcase 2.1. When $(e,f) = (11,i)$, for $i \in \{9,10\}$, then $\mathrm{lk}(2) = C_7(3,0,1,\mathbf{7},8,11,\boldsymbol{i})$ and $\mathrm{lk}(3) = C_7(2,0,4,\mathbf{1},6,i,\mathbf{11})$. This implies, $\deg(i) > 5$. Hence $(e,f) \neq (11,9)$ or $(11,10)$.

Subcase 2.2. When $(e,f)=(11,12)$, then $\mathrm{lk}(2)=C_7(3,0,1,\mathbf{7},8,11,\mathbf{12})$. This implies $\mathrm{lk}(3)=C_7(2,0,4,\mathbf{1},6,12,\mathbf{11})$. Then $\mathrm{lk}(6)=C_7(5,0,1,\mathbf{4},3,12,\mathbf{13})$, $\mathrm{lk}(5)=C_7(4,0,6,\mathbf{12},13,9,\mathbf{7})$ and $\mathrm{lk}(9)=C_7(14,13,5,\mathbf{4},7,10,\mathbf{15})$, $\mathrm{lk}(10)=C_7(8,7,9,\mathbf{14},15,j,\mathbf{k})$, now we see that j and k have no admissible values in $V_{(3,3,4,3,4)}$. Thus $(e,f)\neq(11,12)$

Subcase 2.3 When $(e,f)=(9,10)$ then successively we get $\mathrm{lk}(2)=C_7(3,0,1,\mathbf{7},8,9,\mathbf{10})$, $\mathrm{lk}(3)=C_7(2,0,4,\mathbf{1},6,10,\mathbf{9})$. This implies $\mathrm{lk}(6)=C_7(5,0,1,\mathbf{4},3,10,\boldsymbol{g})$, where $g\in\{8,11\}$. If $g=11$, $\deg(10)>5$, a contradiction. On the other hand when $g=8$, completing successively, we get $\mathrm{lk}(6)=C_7(5,0,1,\mathbf{4},3,10,\mathbf{8})$, $\mathrm{lk}(5)=C_7(4,0,6,\mathbf{10},8,9,\mathbf{7})$, $\mathrm{lk}(9)=C_7(2,8,5,\mathbf{4},7,10,\mathbf{3})$, $\mathrm{lk}(8)=C_7(2,8,5,\mathbf{6},10,7,\mathbf{1})$ and $\mathrm{lk}(10)=C_7(8,7,9,\mathbf{2},3,6,\mathbf{5})$. Then, $M\cong K_1[(3,3,3,3,3,3)_1:(3,3,4,3,4)]$ by the map $0\to u$, $i\to v_i$ for $1\le i\le 10$.

Case 3. In case $(a,b,c,d)=(7,8,9,10)$, $\mathrm{lk}(1)=C_7(2,0,6,\mathbf{7},8,9,\mathbf{10})$. This implies $\mathrm{lk}(2)=C_7(3,0,1,\mathbf{9},10,e,\boldsymbol{f})$. It is easy to see that $(e,f)\in\{(5,6),(7,11),(11,7),(11,8),(11,12)\}$. Observe that, $(7,11)\cong(11,8)$ by the map $(1,2)(3,6)(4,5)(7,8,11)(9,10)$. Therefore, we search for $(e,f)\in\{(5,6),(11,7),(11,8),(11,12)\}$.

Subcase 3.1. If $(e,f)=(5,6)$, $\mathrm{lk}(2)=C_7(3,0,1,\mathbf{9},10,5,\mathbf{6})$ and $\mathrm{lk}(6)=C_7(1,0,5,\mathbf{2},3,7,\mathbf{8})$. This implies $\mathrm{lk}(5)=C_7(4,0,6,\mathbf{3},2,10,\boldsymbol{g})$, where $g\in\{7,8,11\}$. If $g=7$ successively we get $\mathrm{lk}(5)=C_7(4,0,6,\mathbf{3},2,10,\mathbf{7})$, $\mathrm{lk}(10)=C_7(7,11,9,\mathbf{1},2,5,\mathbf{4})$, $\mathrm{lk}(7)=C_7(8,11,10,\mathbf{5},4,6,\mathbf{1})$. This implies $C_5(0,1,8,7,5)\subseteq\mathrm{lk}(6)$. If $g=11$, $\mathrm{lk}(5)=C_7(4,0,6,\mathbf{3},2,10,\mathbf{11})$, successively, we get $\mathrm{lk}(10)=C_7(9,12,11,\mathbf{4},5,2,\mathbf{1})$, $\mathrm{lk}(4)=C_7(3,0,5,\mathbf{10},11,13,\mathbf{7})$, $\mathrm{lk}(3)=C_7(2,0,4,\mathbf{13},7,6,\mathbf{5})$ and $\mathrm{lk}(7)=C_7(8,14,13,\mathbf{4},3,6,\mathbf{1})$. Now a small computation shows that lk(8) can not be completed for the given $V_{(3,3,4,3,4)}$. If $g=8$, $\mathrm{lk}(5)=C_7(4,0,6,\mathbf{3},2,10,\mathbf{8})$. This implies $\mathrm{lk}(10)=C_7(8,h,9,\mathbf{1},2,5,\mathbf{4})$. Clearly $h=7$, completing successively, we get $\mathrm{lk}(8)=C_7(1,9,4,\mathbf{5},10,7,\mathbf{6})$, $\mathrm{lk}(3)=C_7(2,0,4,\mathbf{9},7,6,\mathbf{5})$, $\mathrm{lk}(9)=C_7(1,8,4,\mathbf{3},7,10,\mathbf{2})$, $\mathrm{lk}(7)=C_7(8,10,9,\mathbf{4},3,6,\mathbf{1})$, $\mathrm{lk}(4)=C_7(3,0,5,\mathbf{10},8,9,\mathbf{7})$. This gives a map $M\cong K_1[(3,3,3,3,3,3)_1:(3,3,4,3,4)]$ by the map $0\mapsto u$, $1\mapsto v_2$, $2\mapsto v_1$, $3\mapsto v_6$, $4\mapsto v_5$, $5\mapsto v_4$, $6\mapsto v_3$, $7\mapsto v_{10}$, $8\mapsto v_9$, $9\mapsto v_8$, $10\mapsto v_7$.

Subcase 3.2. When $(e,f)=(11,7)$ then $\mathrm{lk}(2)=C_7(1,0,3,\mathbf{7},11,10,\mathbf{9})$. This implies $\mathrm{lk}(3)=C_7(4,0,2,\mathbf{11},7,g,\boldsymbol{h})$, where $(g,h)\in\{(8,12),(12,9),(12,10),(12,13)\}$.

If $(g,h)=(8,12)$, $\mathrm{lk}(8)=C_7(1,9,12,\mathbf{4},3,7,\mathbf{6})$. This implies $\mathrm{lk}(9)=C_7(12,8,1,\mathbf{2},10,i,\boldsymbol{j})$. Observe that, $(i,j)\in\{(5,6),(13,11),(13,14)\}$. If $(i,j)=(5,6)$, considering lk(6), we see that the degree of 7 is more than 5. If $(i,j)=(13,11)$ or $(13,14)$, we see lk(12) can not be completed. Thus $(g,h)\neq(8,12)$

If $(g,h)=(12,9)$, $\mathrm{lk}(4)=C_7(5,0,3,\mathbf{12},9,i,\boldsymbol{j})$, where $(i,j)\in\{(8,11),(10,13)\}$. In case $(i,j)=(10,13)$, $\mathrm{lk}(10)=C_7(2,11,13,\mathbf{5},4,9,\mathbf{1})$ and $\mathrm{lk}(11)=C_7(13,10,2,\mathbf{3},7,k,\boldsymbol{l})$, where $(k,l)\in\{(6,5),(8,12),(8,14)\}$. When $(k,l)=(6,5)$ then $C_5(0,4,10,13,11,6)\subseteq\mathrm{lk}(5)$. When $(k,l)=(8,12)$ then, successively, considering lk(8), lk(9), lk(12), we see that $\deg(7)>5$. When $(k,l)=(8,14)$ then considering lk(8), we see that $\deg(9)>5$. On the other hand when $(i,j)=(8,11)$, completing successively we get $\mathrm{lk}(8)=C_7(1,9,4,\mathbf{5},11,7,\mathbf{6})$, $\mathrm{lk}(9)=C_7(1,8,4,\mathbf{3},12,10,\mathbf{2})$, $\mathrm{lk}(11)=C_7(2,10,5,\mathbf{4},8,7,\mathbf{3})$, $\mathrm{lk}(10)=C_7(2,11,5,\mathbf{6},12,9,\mathbf{1})$, $\mathrm{lk}(5)=C_7(4,0,6,\mathbf{12},10,11,\mathbf{8})$, $\mathrm{lk}(6)=C_7(1,0,5,\mathbf{10},12,7,\mathbf{8})$, $\mathrm{lk}(7)=C_7(3,12,6,\mathbf{1},8,11,\mathbf{2})$. This gives $M\cong T_1[(3,3,3,3,3,3)_1:(3,3,4,3,4)]$ by the map $0\mapsto u$, $i\mapsto v_i$ for $1\le i\le 12$.

If $(g,h)=(12,10)$, $\mathrm{lk}(3)=C_7(2,0,4,\mathbf{10},12,7,\mathbf{11})$. This implies $\mathrm{lk}(10)=C_7(2,11,12,\mathbf{3},$

$4, 9, \mathbf{1})$ or $\text{lk}(10) = C_7(2, 11, 4, \mathbf{3}, 12, 9, \mathbf{1})$. In the first case, *i.e.*, when $\text{lk}(10) = C_7(2, 11, 12, \mathbf{3}, 4, 9, \mathbf{1})$ we get $\text{lk}(4) = C_7(3, 0, 5, \mathbf{13}, 9, 10, \mathbf{12})$, $\text{lk}(9) = C_7(1, 8, 13, \mathbf{5}, 4, 10, \mathbf{2})$, now observe that, $\text{lk}(7) = C_7(3, 12, 8, \mathbf{1}, 6, 11, \mathbf{2})$ or $\text{lk}(7) = C_7(3, 12, 6, \mathbf{1}, 8, 11, \mathbf{2})$. When $\text{lk}(7) = C_7(3, 12, 8, \mathbf{1}, 6, 11, \mathbf{2})$ then, successively considering lk(8) and lk(12), we see $\deg(10) > 5$, while for $\text{lk}(7) = C_7(3, 12, 6, \mathbf{1}, 8, 11, \mathbf{2})$, considering lk(8), we see three quadrangular faces incident at 11. Thus $\text{lk}(10) \neq C_7(2, 11, 12, \mathbf{3}, 4, 9, \mathbf{1})$. On the other hand when $\text{lk}(10) = C_7(2, 11, 4, \mathbf{3}, 12, 9, \mathbf{1})$ then $\text{lk}(4) = C_7(5, 0, 3, \mathbf{12}, 10, 11, \boldsymbol{i})$, clearly $i = 8$, now completing successively we get $\text{lk}(11) = C_7(2, 10, 4, \mathbf{5}, 8, 7, \mathbf{3})$, $\text{lk}(7) = C_7(3, 12, 6, \mathbf{1}, 8, 11, \mathbf{2})$, $\text{lk}(12) = C_7(3, 7, 6, \mathbf{5}, 9, 10, \mathbf{4})$, $\text{lk}(6) = C_7(1, 0, 5, \mathbf{9}, 12, 7, \mathbf{8})$, $\text{lk}(5) = C_7(4, 0, 6, \mathbf{12}, 9, 8, \mathbf{11})$, $\text{lk}(8) = C_7(1, 9, 5, \mathbf{4}, 11, 7, \mathbf{6})$, $\text{lk}(9) = C_7(1, 8, 5, \mathbf{6}, 12, 10, \mathbf{2})$. This gives $M \cong K_2[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ by the map $0 \mapsto u$, $i \mapsto v_i$ for $1 \leq i \leq 12$.

When $(g, h) = (12, 13)$, then $\text{lk}(7) = C_7(3, 12, 6, \mathbf{1}, 8, 11, \mathbf{2})$ or $\text{lk}(7) = C_7(3, 12, 8, \mathbf{1}, 6, 11, \mathbf{2})$. In the first case when $\text{lk}(7) = C_7(3, 12, 6, \mathbf{1}, 8, 11, \mathbf{2})$, we get $\text{lk}(6) = C_7(5, 0, 1, \mathbf{8}, 7, 12, \boldsymbol{i})$, where $i \in \{9, 10, 14\}$. If $i = 9$ and 10, considering lk(12), we see $\deg(9) > 5$ and $\text{lk}(10) > 5$ respectively. If $i = 14$, $\text{lk}(12) = C_7(3, 7, 6, \mathbf{5}, 14, 13, \mathbf{4})$. This implies $\text{lk}(5) = C_7(4, 0, 6, \mathbf{12}, 14, j, \boldsymbol{k})$ and $\text{lk}(4) = C_7(5, 0, 3, \mathbf{12}, 13, k, \boldsymbol{j})$. Observe that j and k have no admissible values from V. On the other hand when $\text{lk}(7) = C_7(3, 12, 8, \mathbf{1}, 6, 11, \mathbf{2})$ then $\text{lk}(12) = C_7(8, 7, 3, \mathbf{4}, 13, i, \boldsymbol{j})$, where $(i, j) \in \{(10, 11), (14, 9), (14, 10)\}$. If $(i, j) = (10, 11)$, successively considering lk(12), lk(10), lk(11), we see that $\deg(6) > 5$. If $(i, j) = (14, 9)$, $\text{lk}(9) = C_7(10, k, 14, \mathbf{12}, 8, 1, \mathbf{2})$ and we get no value for k from V. If $(i, j) = (14, 10)$, $\text{lk}(8) = C_7(1, 9, 10, \mathbf{14}, 12, 7, \mathbf{6})$, this implies $C_4(1, 2, 10, 8) \subseteq \text{lk}(9)$. So $(g, h) \neq (12, 13)$

Subcase 3.3. When $(e, f) = (11, 8)$ then $\text{lk}(8) = C_7(1, 9, 3, \mathbf{2}, 11, 7, \mathbf{6})$ or $\text{lk}(8) = C_7(1, 9, 11, \mathbf{2}, 3, 7, \mathbf{6})$. In case, $\text{lk}(8) = C_7(1, 9, 11, \mathbf{2}, 3, 7, \mathbf{6})$, we have $\text{lk}(9) = C_7(11, 8, 1, \mathbf{2}, 10, i, \boldsymbol{j})$, for $(i, j) \in \{(4, 5), (5, 4), (12, 13)\}$. If $(i, j) = (4, 5)$, $\text{lk}(11) = C_7(2, 10, 5, \mathbf{4}, 9, 8, \mathbf{3})$, now considering lk(10), we see that the set $\{4, 5\}$ is an edge and non-edge both. If $(i, j) = (5, 4)$, $\text{lk}(11) = C_7(2, 10, 4, \mathbf{5}, 9, 8, \mathbf{3})$, now considering lk(10) we see that the set $\{4, 5\}$ is an edge and non-edge both. If $(i, j) = (12, 13)$, $\text{lk}(11) = C_7(2, 10, 13, \mathbf{12}, 9, 8, \mathbf{3})$, now considering lk(10), we see that the set $\{12, 13\}$ is an edge and non-edge both. On the other hand when $\text{lk}(8) = C_7(1, 9, 3, \mathbf{2}, 11, 7, \mathbf{6})$ then $\text{lk}(9) = C_7(1, 8, 3, \mathbf{4}, 12, 10, \mathbf{2})$, $\text{lk}(3) = C_7(2, 0, 4, \mathbf{12}, 9, 8, \mathbf{11})$. This implies $\text{lk}(4) = C_7(5, 0, 3, \mathbf{9}, 12, g, \boldsymbol{h})$, where $(g, h) \in \{(7, 11), (13, 14)\}$. If $(g, h) = (13, 14)$, $\text{lk}(5) = C_7(6, 0, 4, \mathbf{13}, 14, i, \boldsymbol{j})$ and $\text{lk}(6) = C_7(5, 0, 1, \mathbf{8}, 7, j, \boldsymbol{i})$, but we see that i and j have no admissible value from V. If $(g, h) = (7, 11)$, $\text{lk}(7) = C_7(4, 12, 6, \mathbf{1}, 8, 11, \mathbf{5})$, this implies $\text{lk}(6) = C_7(5, 0, 1, \mathbf{8}, 7, 12, \boldsymbol{i})$. Observe that $i = 10$, now completing successively we get $\text{lk}(6) = C_7(1, 0, 5, \mathbf{10}, 12, 7, \mathbf{8})$, $\text{lk}(5) = C_7(4, 0, 6, \mathbf{12}, 10, 11, \mathbf{7})$, $\text{lk}(10) = C_7(2, 11, 5, \mathbf{6}, 12, 9, \mathbf{1})$, $\text{lk}(11) = C_7(2, 10, 5, \mathbf{4}, 7, 8, \mathbf{3})$, $\text{lk}(12) = C_7(4, 7, 6, \mathbf{5}, 10, 9, \mathbf{3})$. This gives $M \cong K_2[(3,3,3,3,3,3)_1 : (3,3,4,3,4)]$ via $0 \mapsto u$, $1 \mapsto v_4$, $2 \mapsto v_3$, $3 \mapsto v_2$, $4 \mapsto v_1$, $5 \mapsto v_6$, $6 \mapsto v_5$, $7 \mapsto v_8$, $8 \mapsto v_{11}$, $9 \mapsto v_{10}$, $10 \mapsto v_{12}$, $11 \mapsto v_7$, $12 \mapsto v_9$.

Subcase 3.4. When $(e, f) = (11, 12)$ then $\text{lk}(3) = C_7(4, 0, 2, \mathbf{11}, 12, g, \boldsymbol{h})$. It is easy to see that $(g, h) \in \{(7, 13), (8, 9), (9, 8), (13, 7), (13, 8), (13, 9), (13, 10), (13, 14)\}$.

If $(g, h) = (7, 13)$, then $\text{lk}(7) = C_7(6, 12, 3, \mathbf{4}, 13, 8, \mathbf{1})$ or $\text{lk}(7) = C_7(8, 12, 3, \mathbf{4}, 13, 6, \mathbf{1})$. In case $\text{lk}(7) = C_7(6, 12, 3, \mathbf{4}, 13, 8, \mathbf{1})$, $\text{lk}(6) = C_7(5, 0, 1, \mathbf{8}, 7, 12, \boldsymbol{i})$. Observe that $i \in \{9, 10, 13\}$.

If $i = 9$, $\text{lk}(9) = C_7(1, 8, 5, \mathbf{6}, 12, 10, \mathbf{2})$ or $\text{lk}(9) = C_7(1, 8, 12, \mathbf{6}, 5, 10, \mathbf{2})$, but for both the cases of lk(9), we see $\deg(12) > 5$. If $i = 10$ then $\text{lk}(12) = C_7(3, 7, 6, \mathbf{5}, 10, 11, \mathbf{2})$ and we get $C_4(2, 3, 12, 10) \subseteq \text{lk}(11)$. If $i = 13$, $\text{lk}(12) = C_7(3, 7, 6, \mathbf{5}, 13, 11, \mathbf{2})$ and we get $\deg(13) > 5$. Thus $\text{lk}(7) \neq C_7(6, 12, 3, \mathbf{4}, 13, 8, \mathbf{1})$ On the other hand when $\text{lk}(7) = C_7(8, 12, 3, \mathbf{4}, 13, 6, \mathbf{1})$ then $\text{lk}(6) = C_7(5, 0, 1, \mathbf{8}, 7, 13, \boldsymbol{i})$. Observe that $i \in \{9, 10, 11, 14\}$. If $i = 9$, considering successively lk(13), lk(10), lk(5), we get $\deg(9) > 5$. If $i = 10$, $\text{lk}(13) = C_7(4, j, 10, \mathbf{5}, 6, 7, \mathbf{3})$, where $j \in \{9, 11\}$. If $j = 9$, considering successively lk(10), lk(9) and lk(5), we get $\deg(10) > 5$ and if $j = 11$, we get three consecutive triangular faces incident at 11. When $i = 11$, three consecutive triangular faces at 10. When $i = 14$, $\text{lk}(13) = C_7(4, j, 14, \mathbf{5}, 6, 7, \mathbf{3})$ and we get no value for j in V. So $(g, h) \neq (7, 13)$

If $(g, h) = (8, 9)$, successively we get $\text{lk}(8) = C_7(3, 12, 7, \mathbf{6}, 1, 9, \mathbf{4})$, $\text{lk}(9) = C_7(4, 13, 10, \mathbf{2}, 1, 8, \mathbf{3})$. This implies $\text{lk}(4) = C_7(5, 0, 3, \mathbf{8}, 9, 13, \boldsymbol{j})$, where $i \in \{7, 11, 14\}$. When $i = 7$, $\text{lk}(7) = C_7(5, 12, 8, \mathbf{1}, 6, 13, \mathbf{4})$, now considering lk(4) and lk(6), we see that two distinct quadrangular faces share more than one vertex. When $i = 11$ then successively we get $\text{lk}(11) = C_7(5, 10, 2, \mathbf{3}, 12, 13, \mathbf{4})$, $\text{lk}(10) = C_7(2, 11, 5, \mathbf{6}, 13, 9, \mathbf{1})$. But $\deg(13) > 5$. If $i = 14$, successively, we get $\text{lk}(13) = C_7(4, 9, 10, \mathbf{12}, 7, 14, \mathbf{5})$, $\text{lk}(7) = C_7(6, 14, 13, \mathbf{10}, 12, 8, \mathbf{1})$ and $\text{lk}(6) = C_7(5, 0, 1, \mathbf{8}, 7, 14, \boldsymbol{j})$, now observe that the set $\{5, 14\}$ forms an edge and non-edge both. So, $(g, h) \neq (8, 9)$

Computing similarly for $(g, h) \in \{(8, 9), (9, 8), (13, 7), (13, 8), (13, 9), (13, 10), (13, 14)\}$, one can see easily that no map exists and therefore $(e, f) \neq (11, 12)$. This completes the exhaustive search and thus the proof. □

§5. Discussion

Note that ASEMs are a generalization of maps on Johnson solids to the close surfaces other than the 2-sphere. One can construct infinitely many types ASEMs on the sphere as follows: consider an n-gonal disk D_n, $n \geq 4$, with vertex set $V(D_n) = \{a_1, a_2, \cdots, a_n\}$ and a vertex a out side this disk, now join a to each a_i, $1 \leq i \leq n$, by an edge. This gives ASEMs of type $[(3.3.\ldots.3(n\text{-times}))_1 : (3.3.n)]$ for each $n \geq 4$. In the present work, existence of ASEMs is shown for the surfaces of Euler characteristic 0, that is, for the torus and Klein bottle. This study motivates us to explore other types of ASEMs on the torus and Klein bottle including other closed surfaces. As a consequence, the following natural question occurs.

Question 5.1 *Can we construct ASEMs of types, other than* $[(3, 3, 3, 3, 3, 3)_1 : (3, 3, 4, 3, 4)]$, *on the torus or Klein bottle?*

Acknowledgment. The first author expresses his thanks to IIIT Allahabad for providing the facility and resources to carry this research work.

Refernces

[1] U. Brehm, Polyhedral maps with few edges, *Topics in Combinatorics and Graph Theory*,

Physica-Verlag: Heidelberg, 1990.

[2] U. Brehm, B. Datta and N. Nilakantan, The edge-minimal polyhedral maps of Euler characteristic -8, *Beitr. Algebra Geom.*, 43(2001), 583-596.

[3] U. Brehm, W. Kuhnel, Equivelar maps on the torus, *European J. Combin.*, 29(2008), 1843-1861.

[4] B. Datta and N. Nilakantan, Equivelar Polyhedra with few vertices, *Discrete Comput. Geom.*, 26(2001), 429-461.

[5] B. Datta and A.K. Upadhyay, Degree-regular triangulations of torus and Klein bottle, *Proc. Indian Acad. Sci. Math. Sci.*, 115(2005), 279-307.

[6] B. Datta and A.K. Upadhyay, Degree-regular triangulations of the double torus, *Forum Math.*, 18(2006), 1011-1025.

[7] J.L. Gross and T.W. Tucker, *Topological Graph Theory*, John Wiley and Sons, New York, 1987.

[8] N.W. Johnson. Convex solids with regular faces, *Canadian J. Math.*, 18 (1966), 169-200.

[9] J. Karabas and R. Nedela, Archimedean solids of genus 2, *Electron. Notes Disc. Math.*, 28(2007), 331-339.

[10] J. Karabas and R. Nedela, Archimedean maps of higher genera, *Math. Comp.*, 81(2012), 569-583.

[11] A.K. Tiwari and A.K. Upadhyay, Semi-equivelar maps on the surfaces of Euler characteristic -1, *Note Mat.*, 37(2017), 91-102.

[12] A.K. Tiwari and A.K. Upadhyay, Semi-equivelar maps on the torus and the Klein bottle with few vertices, *Math. Slovaca*, 67(2017), 519-532.

[13] A.K. Upadhyay, A.K. Tiwari and D. Maity, Semi-equivelar maps. *Beitr. Algebra Geom.*, 55(2012), 229-242.

International J.Math. Combin. Vol.4(2021), 41-48

Coefficients of the Chromatic Polynomials of Connected Graphs

Shude Long

(Department of Mathematics, Chongqing University of Arts and Sciences, Chongqing 402160, P.R.China)

Junliang Cai

(School of Mathematical Sciences, Beijing Normal University, Beijing 100875, P.R.China)

E-mail: longshude@163.com, caijunliang@bnu.edu.cn

Abstract: In this article, we provide some new results on the divisibility of certain coefficients of the chromatic polynomials of connected graphs, and characterize connected non-planar graphs of order n that minimize the absolute values of coefficients of the chromatic polynomials. Moreover, we also present a sufficient condition that a graph is planar in term of certain coefficients of its chromatic polynomial.

Key Words: Chromatic number, independent set, chromatic polynomial, uniquely k-colorable graph, a_i-minimum.

AMS(2010): 05C15

§1. Introduction

In this article, all graphs considered are finite, simple and connected. Throughout this article, n and m will always denote, respectively, the number of vertices and the number of edges in a graph G. Terminologies and notations not explained here can be found in [1,2].

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the sets of vertices and edges of G, respectively. A λ-coloring of G is a function $\varphi : V(G) \to \{1, 2, \cdots, \lambda\}$ such that $\varphi(u) \neq \varphi(v)$ for any $uv \in E(G)$. In fact, a λ-coloring of G is also a partition of the vertex set $V(G)$ into λ classes where each class is exactly an independent set of vertices. The chromatic number $\chi(G)$ is the smallest λ for which G has a λ-coloring.

Let $P(G, \lambda)$ denote the chromatic polynomial of G. Whitney [10] showed that

$$P(G, \lambda) = \sum_{i=1}^{n} (-1)^{n-i} a_i(G) \lambda^i, \tag{1}$$

where $a_i(G)$ counts the number of spanning subgraphs of G that has exactly $n - i$ edges and that contain no broken cycles. The coefficient $a_1(G)$ in (1) is interesting in its own right. Read [6] first observed that G is connected iff $a_1(G) \geq 1$. Eisenberg [4] noted that G is a tree iff $a_1(G) = 1$. More than ten years later, Hong [5] proved that

[1]Supported by the the NNSFC under Grant No. 11371133 and the Natural Science Foundation Project of CQ under Grant No. cstc2019jcyj-msxmX0724.

[2]Received November 10, 2021, Accepted December 12, 2021.

(1) if G is connected, then $a_1(G)$ is divisible by $(\chi(G)-1)!$ and

(2) G is connected and bipartite iff $a_1(G)$ is odd.

Let $\lfloor x \rfloor$ be the largest integer not exceeding the real x. For any integers k and i with $k \geq 0$ and $i > 0$, define

$$\phi(k,i) = \begin{cases} 1 & if\ k < 2i, \\ \lfloor k/i \rfloor ! \phi(k - \lfloor k/i \rfloor, i) & if\ k \geq 2i. \end{cases} \tag{2}$$

Recently, Dong obtained some results on the divisibility of certain $a_i(G)$ as follows:

Lemma 1.1([3]) *For any graph G, $a_i(G)$ is divisible by $\phi(\chi(G)-1, i)$ for each $i = 1, 2, \cdots, \chi(G)-1$.*

Lemma 1.2([3]) *Let G be a uniquely k-colorable graph. If $k = ip$ for some prime p and positive integer i, then $a_i(G)$ is not divisible by p.*

Lemma 1.3([3]) *For any uniquely k-colorable graph G, where $k \geq 2$, $a_1(G)$ is not divisible by $k!$.*

Obviously, they are an extension of Hong's results mentioned above. And by Lemma 1.1, if one may find the value of $\phi(\chi(G)-1, i)$, the divisibility of $a_i(G)$ will be a better understand. But it is very difficult for us.

Let $\Omega(n,m)$ be the family of all connected graphs with n vertices and m edges. For a given i, we say that a graph $G \in \Omega(n,m)$ is a_i-minimum if $a_i(G) \leq a_i(G')$ for all $G' \in \Omega(n,m)$. Since $a_n(G) = 1$ and $a_{n-1}(G) = m$ for each $G \in \Omega(n,m)$, all graphs in $\Omega(n,m)$ are both a_n-minimum and a_{n-1}-minimum. For any a_i-minimum $(1 \leq i \leq n-2)$, results on this subject can be found in [7–9].

Here we focus on the divisibility of certain $a_i(G)$ and a_i-minimum $(1 \leq i \leq n-2)$.

In section 2 we prove that for any graph G with $\chi(G) = k \geq 2$, $a_1(G)$ is divisible by $k!$ iff b_k is divisible by k, where b_k is the number of ways of partitioning $V(G)$ into k non-empty color classes (or independent sets); for any two positive integers k and i, if $\chi(G) \geq ki+1$, then $a_i(G)$ is divisible by $k!$.

In section 3 we prove that for the family $\mathcal{Q}_1$ (or $\mathcal{Q}_2$) of all non-planar connected graphs of order n containing K_5 (*or* $K_{3,3}$), if G is a graph obtained from K_5 (*or* $K_{3,3}$) by recursively attaching $n-5$ (or $n-6$) leaves, then G is a_i-minimum for each $1 \leq i \leq n-1$. And it is also shown that if G is a connected graph with $a_1(G) < 24$ or $a_2(G) < 50$, then G is a planar graph.

§2. Divisibility of Certain Coefficients

Based on Dong's results we now deal with the divisibility of certain coefficients of the chromatic polynomials.

Although it is difficult from (2) to determine the value of $\phi(k,i)$, we have the following.

Theorem 2.1 *For any graph G, if $\chi(G) = 2i+1$, then $a_i(G)$ is divisible by 2.*

Proof Since $\chi(G) = 2i + 1$, $\lfloor \frac{\chi(G)-1}{i} \rfloor ! = 2$ and $\phi(2i, i) = 2\phi(2i-2, i)$. By (2) we have

$$\phi(2i-2, i) = 1.$$

Thus, $\phi(2i, i) = 2$. By Lemma 1.1, the theorem follows. □

In what follows we present a more general result.

Theorem 2.2 *For any graph G and any two positive integers k and i, if $\chi(G) \geq ki + 1$, then $a_i(G)$ is divisible by $k!$.*

Proof Since $\chi(G) \geq ki + 1$, $\frac{\chi(G)-1}{i} \geq k$. It is clear that $\lfloor \frac{\chi(G)-1}{i} \rfloor !$ is divisible by $k!$. By Lemma 1.1 and (2), the theorem holds. □

If $i = 1$ in Theorem 2.2, then we have

Corollary 2.1. *For any graph G with $\chi(G) \geq k$, $a_1(G)$ is divisible by $(k-1)!$.*

It is easy to see that Hong's result (1) follows immediately from Corollary 2.1.

In addition, if $i = k = 2$ in Theorem 2.2, then we get

Corollary 2.2. *For any graph G with $\chi(G) \geq 5$, $a_2(G)$ is even.*

Since $\chi(K_5) = 5$, we yield the following result.

Corollary 2.3. *If G is a non-planar graph containing K_5, then $a_2(G)$ is even.*

It is well known that the chromatic polynomial $P(G, \lambda)$ can also be expressed in factorial form as follows:

$$P(G, \lambda) = \sum_{j=\chi(G)}^{n} b_j (\lambda)_j, \qquad (3)$$

where n is the order of G,

$$(\lambda)_j = \lambda(\lambda - 1) \cdots (\lambda - j + 1) = P(K_j, \lambda)$$

and b_j is the number of ways of partitioning $V(G)$ into j non-empty color classes (or independent sets).

Theorem 2.3 *For any graph G with $\chi(G) = 4$, $a_2(G) \equiv b_4$ (mod 2).*

Proof Let G be a graph of order n with $\chi(G) = 4$. Combining (1) with (3), we have

$$a_i(G) = \sum_{j=\chi(G)}^{n} (-1)^{n-j} b_j a_i(K_j). \qquad (4)$$

Thus,

$$a_2(G) = \sum_{j=4}^{n}(-1)^{n-j}b_j a_2(K_j).$$

Observe that $\chi(K_j) = j$, by Corollary 2.2, $a_2(K_j)$ is divisible by 2 for $j \geq 5$. Therefore $a_2(G)$ and $b_4a_2(K_4)$ have the same parity.

One may find that $P(K_4, \lambda) = \lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda$. Hence $a_2(K_4) = 11$. It means that $b_4a_2(K_4)$ and b_4 have the same parity. So have $a_2(G)$ and b_4. The theorem follows. □

A graph G is called uniquely k-colorable if $\chi(G) = k$ and there exists a unique way of partitioning $V(G)$ into k color class. We now have the following.

Corollary 2.4 *For any uniquely 4-colorable graph G, $a_2(G)$ is odd.*

Proof Since G is uniquely 4-colorable, we yield that $b_4 = 1$. It is clear that b_4 is odd. Combining this with Theorem 2.3, the corollary holds. □

Theorem 2.4 *For any graph G with $\chi(G) = k \geq 2$, $a_1(G)$ is divisible by $k!$ iff b_k is divisible by k.*

Proof By (4) we have

$$a_1(G) = \sum_{j=k}^{n}(-1)^{n-j}b_j a_1(K_j).$$

Consider that $P(K_j, \lambda) = (\lambda)_j$ and $a_1(K_j) = (j-1)!$. Thus, $a_1(K_j)$ is divisible by $k!$ if and only if $j > k$.

If b_k is divisible by k, then $(k-1)!b_k$ is divisible by $k!$. It is equivalent to $b_ka_1(K_k)$ divisible by $k!$. Consider that $a_1(K_j)$ is divisible by $k!$ for $j > k$, this results in $a_1(G)$ being divisible by $k!$. And it is clear that the process is reversible. Hence the theorem holds. □

For any uniquely k-colorable graph G, obviously $b_k = 1$ and $\chi(G) = k$. Thus, we have the following.

Corollary 2.5([3]) *For any uniquely k-colorable graph G, where $k \geq 2$, $a_1(G)$ is not divisible by $k!$.*

By Corollaries 2.1 and 2.5, we get

Corollary 2.6([3]) *If G is a uniquely k-colorable graph, where $k \geq 2$, then $a_1(G)$ is divisible by $(k-1)!$ but not by $k!$.*

If $k \geq 3$ in Corollary 2.6, then $a_1(G)$ is divisible by 2. Hence we have the following.

Corollary 2.7 *For any uniquely k-colorable graph G, where $k \geq 3$, $a_1(G)$ is even.*

Remark. One may construct a graph G with $\chi(G) = k$ such that $a_1(G)$ is even, but G is not uniquely k-colorable. As shown in Fig.1, we have $\chi(G)=3$ and $a_1(G)=2$. Obviously it is not uniquely 3-colorable.

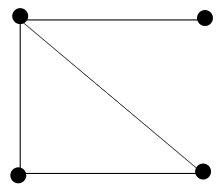


Fig. 1 $(P(G,\lambda)=\lambda^4-4\lambda^3+5\lambda^2-2\lambda)$

§3. **Least coefficients**

We now concentrate on non-planar graph with least coefficients. For convenience, here we agree that

$$\sum_{i\in\emptyset} x_i = 0.$$

Let H be a graph of order r and $\mathscr{A}$ denote the set of graphs obtained from H by recursively attaching $n-r$ leaves. Then we have

Theorem 3.1 *For any graph $G\in\mathscr{A}$, $a_i(G)=\sum_{t=0}^{i-1}\binom{n-r}{t}a_{i-t}(H)$ $(i=1,2,\cdots,n)$.*

Proof It is clear that $P(G,\lambda)=(\lambda-1)^{n-r}P(H,\lambda)$. Combining this with (1), one may find that

$$\begin{aligned}\sum_{i=1}^{n}(-1)^{n-i}a_i(G)\lambda^i =&(\lambda-1)^{n-r}\sum_{j=1}^{r}(-1)^{r-j}a_j(H)\lambda^j\\ =&\sum_{j=1}^{r}\sum_{t=0}^{n-r}(-1)^{n-(t+j)}\binom{n-r}{t}a_j(H)\lambda^{t+j}\\ =&\sum_{i=1}^{n}\sum_{t=0}^{i-1}(-1)^{n-i}\binom{n-r}{t}a_{i-t}(H)\lambda^i.\end{aligned}$$

Thus, we obtain that

$$a_i(G)=\sum_{t=0}^{i-1}\binom{n-r}{t}a_{i-t}(H)$$

for $i=1,2,\cdots,n$, as required. □

It follows immediately from Theorem 3.1 that

Corollary 3.1 *If G is a graph obtained from H by recursively attaching $n-r$ leaves, then $a_i(G)\geq a_i(H)$ for $1\leq i\leq r$.*

In addition, it is easy to see that $a_1(G) = a_1(H)$ and $a_2(G) = a_2(H) + (n - r)a_1(H)$.

For any graph G and $e = uv \in E(G)$, let $G - e$ denote the graph obtained from G by deleting the edge e and $G \cdot e$ denote the graph obtained from G by identifying u and v and replacing all multi-edges by single ones. The following fundamental reduction formula is well known.

Lemma 3.1([6]) *Let G be a graph and $e \in E(G)$. Then*

$$P(G,\lambda) = P(G - e,\lambda) - P(G \cdot e,\lambda). \tag{5}$$

It follows from (5) that

$$a_i(G) = a_i(G - e) + a_i(G \cdot e) \tag{6}$$

for any integer $i \geq 1$.

It is clear that $a_i(G) \geq a_i(G - e)$ and $a_i(G) \geq a_i(G \cdot e)$ for each $i = 1, 2, \cdots, n$. Thus, we have the following.

Corollary 3.2 *Let G' be a connected subgraph of G. Then $a_i(G') \leq a_i(G)$ for $i = 1, 2, \cdots, |V(G')|$, where $|V(G')|$ is the number of vertices of G'.*

Proof Let n' be the order of G'. Assume that H is the spanning subgraph of G obtained from G' by recursively attaching $n - n'$ leaves. It is clear that H can be obtained from G by recursively deleting all edges $e \in E(G) - E(H)$. By repeatedly applying (6), we obtain that $a_i(G) \geq a_i(H)$ for all integer $i \geq 1$. Again by Corollary 3.1, we yield that $a_i(H) \geq a_i(G')$ for $1 \leq i \leq |V(G')|$. Therefore $a_i(G) \geq a_i(G')$ for $i = 1, 2, \cdots, |V(G')|$, as desired. □

The following two results can be obtained immediately from (6).

Corollary 3.3 *Let H be a graph obtained from G by subdividing an edge e of G. Then $a_i(H) \geq a_i(G)$ for $i = 1, 2, \cdots, n$. Specially, $a_1(H) \geq a_1(G)$, and equality holds iff $G - e$ is disconnected.*

Corollary 3.4 *If H be a subdivision of G, then $a_i(H) \geq a_i(G)$ for $i = 1, 2, \cdots, n$.*

Corollary 3.5 *Let G be a non-planar graph. Then*

(1) *If G has a subgraph which is a subdivision of K_5, $a_1(G) \geq 24$ and $a_2(G) \geq 50$;*

(2) *If G has a subgraph which is a subdivision of $K_{3,3}$, $a_1(G) \geq 31$ and $a_2(G) \geq 78$.*

Proof Let H be the subgraph of G which is a subdivision of K_5. Obviously, H is connected. By Corollary 3.4, we yield that $a_i(H) \geq a_i(K_5)$ for $i = 1, 2$. Again by Corollary 3.2, we obtained $a_i(G) \geq a_i(H)$ for $i = 1, 2$. Thus, $a_i(G) \geq a_i(K_5)$ for $i = 1, 2$. Observe that $P(K_5, \lambda) = \lambda^5 - 10\lambda^4 + 35\lambda^3 - 50\lambda^2 + 24\lambda$. Hence $a_1(G) \geq 24$ and $a_2(G) \geq 50$, as required.

Similarly, one can prove that (2) is true. This completes the proof of the corollary. □

Let $\mathscr{B}$ be the family of all connected graph with n vertices. For a given i, a graph $G \in \mathscr{B}$

is called a_i-minimum if $a_i(G) \leq a_i(G')$ for all $G' \in \mathscr{B}$. It is clear that any tree of order n is a_i-minimum.

Let $\mathcal{Q}_1$ (or $\mathcal{Q}_2$) be the family of all non-planar graphs of order n containing K_5 (or $K_{3,3}$), and let $\mathcal{R}_1$ (or $\mathcal{R}_2$) be the family of graphs obtained from K_5 (or $K_{3,3}$) by recursively attaching $n-5$ (or $n-6$) leaves. By the proof of Theorem 3.1, it is easy to see that the graphs from $\mathcal{R}_1$ (or $\mathcal{R}_2$) are χ-equivalence.

Theorem 3.2 *Let G be a graph in $\mathcal{R}_k$. Then G is a_i-minimum in $\mathcal{Q}_k$ (k=1,2).*

Proof Assume that graph $G \in \mathcal{R}_1$. For any $H \in \mathcal{Q}_1$, since H is connected and contains K_5, there must be a connected generating subgraph $H' \in \mathcal{R}_1$ of H. By Corollary 3.2, we have $a_i(H') \leq a_i(H)$. Observe that the graphs from $\mathcal{R}_1$ are χ-equivalence and $G, H' \in \mathcal{R}_1$. Thus, $a_i(G) = a_i(H') \leq a_i(H)$. It means that G is a_i-minimum in $\mathcal{Q}_1$.

Similarly, one may prove that if $G \in \mathcal{R}_2$, G is a_i-minimum in $\mathcal{Q}_2$. □

Corollary 3.6 *Let G be a non-planar graph. Then*

(1) *if G contains K_5, $a_1(G) \geq 24$ and $a_2(G) \geq 50$;*
(2) *if G contains $K_{3,3}$, $a_1(G) \geq 31$ and $a_2(G) \geq 78$.*

Proof Let G be a non-planar graph of order n. If G contains $K_{3,3}$, then $G \in \mathcal{Q}_2$. By Theorem 3.2, there is a graph $H \in \mathcal{R}_2$ such that $a_i(H) \leq a_i(G)$ for $i = 1, 2$. By Corollary 3.1, one may find that $a_i(H) \geq a_i(K_{3,3})$ for $i = 1, 2$. It is well known that $P(K_{3,3,}, \lambda) = \lambda^6 - 9\lambda^5 + 36\lambda^4 - 75\lambda^3 + 78\lambda^2 - 31\lambda$. Thus, $a_1(G) \geq 31$ and $a_2(G) \geq 78$.

A similar argument can be used to establish (1). □

By Corollary 3.6, one may obtain a sufficient condition that a graph is a planar graph in terms of coefficients of its chromatic polynomial.

Corollary 3.7 *If G is a connected graph with $a_1(G) < 24$ or $a_2(G) < 50$, then G is a planar graph.*

References

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory With Applications*, Macmillan, 1976.

[2] F.M. Dong, K.M. Koh, K.L. Teo, *Chromatic Polynomial and Chromaticity of Graphs*, World Scientific Publishing Co. Pte. Ltd., Singapore, 2005.

[3] F.M. Dong, K.M. Koh, C.A. Soh, Divisibility of certain coefficients of the chromatic polynomials, *Discrete Math.*, 275 (2004), 311–317.

[4] B. Eisenberg, Characterization of a tree by means of coefficients of the chromatic polynomial, *Trans. New York Acad. Sci.*, 34 (1972), 146–153.

[5] Y. Hong, A note on the coefficients of chromatic polynomials, *J. East China Norm. Univ. Natur. Sci. Ed.*, (4) (1984), 33–35 (Chinese).

[6] R.C. Read, An introduction to chromatic polynomials, *J. Combin. Theory*, 4 (1968), 52–71.

[7] J. Rodriguez, A. Satyanarayana, Chromatic polynomials with least coefficients, *Discrete Math.*, 172 (1997), 115–119.

[8] A. Sakaloglu, A. Satyanarayana, Graphs with least number of colorings, *J. Graph Theory*, 19 (1995), 523–533.

[9] A. Sakaloglu, A. Satyanarayana, Planar graphs with least chromatic coefficients, *Discrete Math.*, 172 (1997), 121–130.

[10] H. Whitney, A logical expansion in mathematics, *Bull. Amer. Math. Soc.*, 38 (1932), 572–579.

International J.Math. Combin. Vol.4(2021), 49-56

Reverse Hyper-Zagreb Indices of the Cartesian Product of Graphs

Gowtham Kalkere Jayanna

(Department of Mathematics, University College of Science, Tumkur University, Karnataka State, India)

E-mail: gowtham_k_j@yahoo.com

Abstract: In this paper, found some exact expressions for the first and second reverse hyper Zagreb indices of Cartesian product of two simple connected graphs are determined. And repeated same for the forgotten topological index F and F_1 reverse index of a molecular graphs.

Key Words: Reverse Zagreb index, reverse hyper-Zagreb index, Cartesian product of graph.

AMS(2010): 05C90, 05C35, 05C12, 05C07.

§1. Introduction

Let G be a finite, undirected and simple graph with vertex set $V(G)$ and edge set $E(G)$. Let $\Delta(G)$ be the maximum degree of vertex among the set $V(G)$. The degree of the vertex v is denoted by d_v. In Chemical Science, a molecular graph is a graph in which atoms are taken as vertices and bonds as edges.Topological index is used to characterize some property of molecular graphs and those are used in theoretical Chemistry [1].

Forty years ago Gutman and Trinajestic [2], in which they defined first and second Zagreb indices as,

$$M_1 = M_1(G) = \sum_{v \in V(G)} d_v^2 \tag{1}$$

and

$$M_2 = M_2(G) = \sum_{uv \in E(G)} d_u d_v \tag{2}$$

respectively, where uv denotes an edge connecting the vetices u and v. Before we move to main results, first look to the know definitions which is in [3].

Definition 1.1 *Let G be a simple connected graph and v be a vertex of G. Then, the reverse vertex degree C_v of the vertex v is defined as $C_v = \Delta(G) - d_v + 1$.*

Definition 1.2 *Let G be a simple connected graph and v be a vertex of G.Then, the total reverse vertex degree $TR(G)$ of the graph G is the sum of the reverse vertex degree of the vertices of G.*

[1]Received November 10, 2021, Accepted December 14, 2021.

That is,

$$TR(G) = \sum_{u \in V(G)} C_u$$

Definition 1.3 *Let G be a simple connected graph and v be a vertex of G. Then, the first reverse Zagreb alpha index of G defined as*

$$CM_1^{\alpha}(G) = \sum_{v \in V(G)} C_v^2$$

Definition 1.4 *Let G be a simple connected graph and v be a vertex of G. Then, the first reverse Zagreb beta index of G defined as*

$$CM_1^{\beta}(G) = \sum_{uv \in E(G)} (C_u + C_v)$$

Definition 1.5 *Let G be a simple connected graph and v be a vertex of G. Then, the second reverse Zagreb index of G defined as*

$$CM_2(G) = \sum_{uv \in E(G)} (C_u C_v)$$

Definition 1.6([6]) *The first and second reverse hyper-Zagreb indices of a simple graph G is defined as*

$$HCM_1(G) = \sum_{uv \in E(G)} (C_u + C_v)^2 \quad and \quad HCM_2(G) = \sum_{uv \in E(G)} (C_u C_v)^2$$

Definition 1.7[7] *The F-reverse index $FC(G)$ and $F_1C(G)$ index of a graph G is defined as,*

$$FC(G) = \sum_{uv \in E(G)} (c_u^2 + c_v^2) \quad and \quad F_1C(G) = \sum_{u \in V(G)} c_u^3.$$

Definition 1.8 *The Cartesian product $G \times H$ of graph G and H is a graph such that,*

(1) *the vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$;*

(2) *$(a, b)(c, d)$ is an edge of $G \times H$ if $a = c$ and $bd \in E(G)$, or $b = d$ and $ac \in E(G)$.*

In [5] first and second Zagreb indices of the Cartesian product of graphs are computed and other topological indices of the product of graphs are found in [8], [9] and [10]. Further, in [3] S. Ediz found expressions for the reverse Zagreb indices of cartesian product of two simple connected graphs. By the motivation of [3] in this paper, Reverse hyper-Zagreb indices and forgotten reverse index of cartesian product of two simple connected graphs are determined.

§2. Preliminarily

The following are the preliminarily that have helped in our results and those are found in [3].

Lemma 2.1 *Let G and H be two simple connected graphs and $(a,b) \in E(G\times H)$. Then*

(a) $|V(G\times H)| = |V(G)||V(H)|$;
(b) $|E(G\times H)| = |V(G)||E(H)| + |E(G)||V(H)|$;
(c) $d_{G\times H}((a,b)) = d_a + d_b$.

Corollary 2.2 *Let G and H be two simple connected graphs. Then $\Delta_{G\times H} = \Delta_G + \Delta_H$.*

Proof Let $d_a = \Delta_G$ and $d_b = \Delta_H$. Then $V(G\times H)$. Clearly from above lemma,we have the result. □

Proposition 2.3 *Let $(a,b) \in V(G\times H)$. Then $C_{(a,b)} = C_a + C_b - 1$.*

§3. Reverse Hyper-Zagreb Indices of Cartesian Product of Graphs

In this section, we found reverse hyper-Zagreb indices of Cartesian product of two simply connected graph.

Theorem 3.1 *If G and H be two connected simple graphs then,*

$$\begin{aligned} HCM_1(G\times H) =& 4|E(H)|CM_1^{\alpha}(G) - 8|E(H)|TR(G) + 4|E(H)||V(G)| + |V(G)|HCM_1(H) \\ &+ 4[\ TR(G) - |V(G)|\ ]CM_1^{\beta}(H) + 4|E(G)|CM_1^{\alpha}(H) - 8|E(G)|TR(H) \\ &+ 4|E(G)||V(H)| + |V(H)|HCM_1(G) + 4[\ TR(H) - |V(H)|\ ]CM_1^{\beta}(G). \end{aligned}$$

Proof By Proposition 2.3, $C_{(a,b)} = C_a + C_b - 1$. So

$$\begin{aligned} HCM_1(G\times H) &= \sum_{(a,b)(c,d)\in E(G\times H)} [C_{(a,b)} + C_{(c,d)}]^2 \\ &= \sum_{u\in V(G)} \sum_{bd\in E(H)} [(C_u + C_b - 1) + (C_u + C_d - 1)]^2 \\ &\quad + \sum_{v\in V(H)} \sum_{ac\in E(G)} [(C_v + C_a - 1) + (C_v + C_c - 1)]^2 \\ &= \sum_{u\in V(G)} \sum_{bd\in E(H)} [2(C_u - 1) + (C_b + C_d)]^2 \\ &\quad + \sum_{v\in V(H)} \sum_{ac\in E(G)} [2(C_v - 1) + (C_a + C_c)]^2 \\ &=4 \sum_{u\in V(G)} \sum_{bd\in E(H)} (C_u - 1)^2 + \sum_{u\in V(G)} \sum_{bd\in E(H)} C_b^2 + \sum_{u\in V(G)} \sum_{bd\in E(H)} C_d^2 \end{aligned}$$

$$
\begin{aligned}
&+4\sum_{u\in V(G)}\sum_{bd\in E(H)}(C_u-1)C_b+4\sum_{u\in V(G)}\sum_{bd\in E(H)}(C_u-1)C_d\\
&+\sum_{u\in V(G)}\sum_{bd\in E(H)}C_bC_d+\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_v-1)^2\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}C_a^2+\sum_{v\in V(H)}\sum_{ac\in E(G)}C_c^2+4\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_v-1)C_a\\
&+4\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_v-1)C_c+\sum_{v\in V(H)}\sum_{ac\in E(G)}C_aC_c\\
=&4\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_u^2-2C_u+1]+\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_b^2+C_d^2+2C_bC_d]\\
&+4\sum_{u\in V(G)}\sum_{bd\in E(H)}(C_u-1)(C_b+C_d)+4\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_v^2-2C_v+1]\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_a^2+C_c^2+2C_aC_c]+4\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_v-1)(C_a+C_c)\\
=&4|E(H)|CM_1^{\alpha}(G)-8|E(H)|TR(G)+4|E(H)||V(G)|+|V(G)|HCM_1(H)\\
&+4[\ TR(G)-|V(G)|\ ]CM_1^{\beta}(H)+4|E(G)|CM_1^{\alpha}(H)-8|E(G)|TR(H)\\
&+4|E(G)||V(H)|+|V(H)|HCM_1(G)+4[\ TR(H)-|V(H)|\ ]CM_1^{\beta}(G)
\end{aligned}
$$

This completes the proof. □

Theorem 3.2 *If G and H be two connected simple graphs then,*

$$
\begin{aligned}
HCM_2(G\times H)=&|E(H)|[CM_1^2(G)]^2-4|E(H)|TR(G)CM_1^2(G)+6|E(H)|CM_1^2(G)\\
&-4TR(G)|E(H)|+|E(H)||V(G)|+\big(CM_1^2(G)-2TR(G)\\
&+|V(G)|\big)\left(HCM_1(H)+2CM_2(H)\right)+2CM_2(G)CM_1^{\beta}(H)\left(TR(G)-|V(G)|\right)\\
&-6CM_1^{\alpha}(G)CM_1^{\alpha}(H)+6TR(G)CM_1^{\beta}(G)-2|V(G)|CM_1^{\beta}(H)\\
&+HCM_2(G)|V(H)|+|E(G)|[CM_1^2(H)]^2-4|E(G)|TR(H)CM_1^2(H)\\
&+6|E(G)|CM_1^2(H)-4TR(H)|E(G)|+|E(G)||V(H)|+\big(CM_1^2(H)-2TR(H)\\
&+|V(H)|\big)\left(HCM_1(G)+2CM_2(G)\right)+2CM_2(H)CM_1^{\beta}(G)\left(TR(H)-|V(H)|\right)\\
&-6CM_1^{\alpha}(H)CM_1^{\alpha}(G)+6TR(H)CM_1^{\beta}(H)-2|V(H)|CM_1^{\beta}(G)+HCM_2(H)|V(G)|
\end{aligned}
$$

Proof By Proposition 2.3, $C_(a,b)=C_a+C_b-1$. So

$$
\begin{aligned}
HCM_2(G\times H)=&\sum_{(a,b)(c,d)\in E(G\times H)}(C_{(a,b)}C_{(c,d)})^2\\
=&\sum_{u\in V(G)}\sum_{bd\in E(H)}[(C_u+C_b-1)(C_u+C_d-1)]^2\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}[(C_v+C_a-1)(C_v+C_c-1)]^2\\
=&\sum_{u\in V(G)}\sum_{bd\in E(H)}[(C_u-1)^2+(C_b+C_d)(C_u-1)+C_bC_d]^2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}[(C_v-1)^2+(C_a+C_c)(C_u-1)+C_aC_c]^2\\
=&\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_u^4-4C_u^3+6C_u^2-4C_u+1]\\
&+[(C_b+C_d)^2+2C_bC_d](C_u^2-2C_u+1)\\
&+2C_bC_d(C_b+C_d)(C_u-1)-6C_u^2(C_b+C_d)+6C_u(C_b+C_d)\\
&-2(C_b+C_d)+C_b^2C_d^2+\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_v^4-4C_v^3+6C_v^2-4C_v+1]\\
&+[(C_+C_c)^2+2C_aC_c](C_v^2-2C_v+1)+2C_aC_c(C_a+C_c)(C_v-1)\\
&-6C_v^2(C_a+C_c)+6C_v(C_a+C_c)-2(C_a+C_c)+C_a^2C_c^2\\
=&\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_u^4-4C_u^3+6C_u^2-4C_u+1]\\
&+\sum_{u\in V(G)}\sum_{bd\in E(H)}[(C_b+C_d)^2+2C_bC_d](C_u^2-2C_u+1)\\
&+2\sum_{u\in V(G)}\sum_{bd\in E(H)}C_bC_d(C_b+C_d)(C_u-1)-6\sum_{u\in V(G)}\sum_{bd\in E(H)}C_u^2(C_b+C_d)\\
&+6\sum_{u\in V(G)}\sum_{bd\in E(H)}C_u(C_b+C_d)-2\sum_{u\in V(G)}\sum_{bd\in E(H)}(C_b+C_d)\\
&+\sum_{u\in V(G)}\sum_{bd\in E(H)}C_b^2C_d^2+\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_v^4-4C_v^3+6C_v^2-4C_v+1]\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}[(C_+C_c)^2+2C_aC_c](C_v^2-2C_v+1)\\
&+2\sum_{v\in V(H)}\sum_{ac\in E(G)}C_aC_c(C_a+C_c)(C_v-1)-6\sum_{v\in V(H)}\sum_{ac\in E(G)}C_v^2(C_a+C_c)\\
&+6\sum_{v\in V(H)}\sum_{ac\in E(G)}C_v(C_a+C_c)-2\sum_{v\in V(H)}\sum_{ac\in E(G)}(C_a+C_c)\\
&+\sum_{v\in V(H)}\sum_{ac\in E(G)}C_a^2C_c^2\\
=&|E(H)|[CM_1^2(G)]^2-4|E(H)|TR(G)CM_1^2(G)+6|E(H)|CM_1^2(G)-4TR(G)|E(H)|\\
&+|E(H)||V(G)|+\left(CM_1^2(G)-2TR(G)+|V(G)|\right)(HCM_1(H)+2CM_2(H))\\
&+2CM_2(G)CM_1^{\beta}(H)\,(TR(G)-|V(G)|)-6CM_1^{\alpha}(G)CM_1^{\alpha}(H)\\
&+6TR(G)CM_1^{\beta}(G)-2|V(G)|CM_1^{\beta}(H)+HCM_2(G)|V(H)|+|E(G)|[CM_1^2(H)]^2\\
&-4|E(G)|TR(H)CM_1^2(H)+6|E(G)|CM_1^2(H)-4TR(H)|E(G)|+|E(G)||V(H)|\\
&+\left(CM_1^2(H)-2TR(H)+|V(H)|\right)(HCM_1(G)+2CM_2(G))\\
&+2CM_2(H)CM_1^{\beta}(G)\,(TR(H)-|V(H)|)-6CM_1^{\alpha}(H)CM_1^{\alpha}(G)+6TR(H)CM_1^{\beta}(H)\\
&-2|V(H)|CM_1^{\beta}(G)+HCM_2(H)|V(G)|.
\end{aligned}
$$

This completes the proof. □

Theorem 3.3 *If G and H be two connected molecular graphs then,*

$$\begin{aligned}FC(G\times H)=&2|E(H)|CM_1^{\alpha}(G)+|V(G)|FC(H)+2TR(G)CM_1^{\beta}(H)-4|E(H)|TR(G)\\&-2|V(G)|CM_1^{\beta}(H)+2|V(G)||E(H)|+2|E(G)|CM_1^{\alpha}(H)+|V(H)|FC(G)\\&+2TR(H)CM_1^{\beta}(G)-4|E(G)|TR(H)-2|V(H)|CM_1^{\beta}(G)+2|V(H)||E(G)|\end{aligned}$$

Proof By Proposition 2.3, $C_(a,b)=C_a+C_b-1$. So

$$\begin{aligned}FC(G\times H)=&\sum_{(a,b)(c,d)\in E(G\times H)}(C_{(a,b)}^2+C_{(c,d)}^2)\\=&\sum_{u\in V(G)}\sum_{bd\in E(H)}[(C_u+C_b-1)^2+(C_u+C_d-1)^2]\\&+\sum_{v\in V(H)}\sum_{ac\in E(G)}[(C_v+C_a-1)^2+(C_v+C_c-1)^2]\\=&\sum_{u\in V(G)}\sum_{bd\in E(H)}2[C_u^2]+\sum_{u\in V(G)}\sum_{bd\in E(H)}[C_b^2+C_d^2]\\&+\sum_{u\in V(G)}\sum_{bd\in E(H)}2C_u(C_b+C_d)-\sum_{u\in V(G)}\sum_{bd\in E(H)}4C_u\\&-\sum_{u\in V(G)}\sum_{bd\in E(H)}2(C_b+C_d)+\sum_{u\in V(G)}\sum_{bd\in E(H)}2\\&+\sum_{v\in V(H)}\sum_{ac\in E(G)}2[C_v^2]+\sum_{v\in V(H)}\sum_{ac\in E(G)}[C_a^2+C_c^2]\\&+\sum_{v\in V(H)}\sum_{ac\in E(G)}2C_v(C_a+C_c)-\sum_{v\in V(H)}\sum_{ac\in E(G)}4C_v\\&-\sum_{v\in V(H)}\sum_{ac\in E(G)}2(C_a+C_c)+\sum_{v\in V(H)}\sum_{ac\in E(G)}2\\=&2|E(H)|CM_1^{\alpha}(G)+|V(G)|FC(H)+2TR(G)CM_1^{\beta}(H)\\&-4|E(H)|TR(G)-2|V(G)|CM_1^{\beta}(H)\\&+2|V(G)||E(H)|+2|E(G)|CM_1^{\alpha}(H)\\&+|V(H)|FC(G)+2TR(H)CM_1^{\beta}(G)\\&-4|E(G)|TR(H)-2|V(H)|CM_1^{\beta}(G)+2|V(H)||E(G)|\end{aligned}$$

This completes the proof. □

Theorem 3.4 *If G and H be two connected molecular graphs then,*

$$\begin{aligned}F_1C(G\times H)=&|V(H)|F_1C(G)+|V(G)|F_1C(H)-3TR(G)TR(H)[TR(G)+TR(H)]\\&-3|V(H)|CM_1^{\alpha}(G)-3|V(H)|CM_1^{\alpha}(G)-3|V(G)|CM_1^{\alpha}(H)-6TR(G)TR(H)\\&+3|V(H)|TR(G)+3|V(G)|TR(H)-|V(G)||V(H)|.\end{aligned}$$

Proof By Proposition 2.3, $C_(a,b) = C_a + C_b - 1$. So

$$\begin{aligned}
F_1C(G\times H) =& \sum_{a\in V(G)}\sum_{b\in V(H)} (C_a + C_b - 1)^3 \\
=& \sum_{a\in V(G)}\sum_{b\in V(H)} [(C_a + C_b)^3 - 3(C_a + C_b)^2 + 3(C_a + C_b) - 1] \\
=& \sum_{a\in V(G)}\sum_{b\in V(H)} C_a^3 + \sum_{a\in V(G)}\sum_{b\in V(H)} C_b^3 - \sum_{a\in V(G)}\sum_{b\in V(H)} 3C_aC_b(C_a + C_b) \\
& - 3\sum_{a\in V(G)}\sum_{b\in V(H)} (C_a^2 + C_b^2) - \sum_{a\in V(G)}\sum_{b\in V(H)} 6C_aC_b \\
& + \sum_{a\in V(G)}\sum_{b\in V(H)} (C_a + C_b) - \sum_{a\in V(G)}\sum_{b\in V(H)} 1 \\
=&|V(H)|F_1C(G) + |V(G)|F_1C(H) - 3TR(G)TR(H)[TR(G) + TR(H)] \\
& - 3|V(H)|CM_1^{\alpha}(G) - 3|V(G)|CM_1^{\alpha}(H) - 6TR(G)TR(H) \\
& + 3|V(H)|TR(G) + 3|V(G)|TR(H) - |V(G)||V(H)|
\end{aligned}$$

This completes the proof. □

§4. **Conclusions**

In this paper, we have obtained the exact values of the first and second reverse hyper Zagreb indices of Cartesian product of two simple connected graphs. And repeated same for the F and F_1 reverse index of a molecular graphs. It would be useful to study mathematical properties and formulas for some important graphs families. Same can be done with other operations (such as the composition, join, disjunction and symmetric difference of graphs, bridge graphs and Kronecker product of graphs) can be derived similarly.

References

[1] I.Gutman and O.E.Polansky, *Mathematical Concepts in Organic Chemistry*, Springer, Berlin, (1986).

[2] I. Gutman, N. Trinajstic, Graph theory and molecular orbitals, Total π-electron energy of alternant hydrocarbons, *Chem. Phy. Letters*, 17(1972),535-538.

[3] S. Ediz, Murat Cancan, Reverse Zagreb indices of cartesian product of graphs, *International Journal of Mathematics and Computer Science*, 11(1)(2016), 51-58.

[4] W. Imrich, S.Klavzar, *Product Graphs: Structure and Recognition*, John Wiley and Sons, New York, USA, 2000.

[5] M. H. Khalifeh , H. Y. Azari , A. R. Ashrafi, The first and second Zagreb indices of some graph operations, *Disc. Appl. Math.*, 157(2009), 804–811.

[6] V.R.Kulli, Reverse Zagreb and reverse hyper-Zagreb indices and their polynomials of

Rhombus silicate networks, *Annals of Pure and Applied Mathematics*, Vol. 16(1)(2018), 47-51; Zagreb indices of some graph operations, *Disc. Appl. Math.*, 157(2009), 804–811.

[7] V.R.Kulli, Computing f-reverse index and f-reverse polynomial of certain network, *International Journal of Mathematical Archive*, 9(8)(2018), 27-33.

[8] P. Paulraja, V. Sheeba Agnes, Gutman index of product graphs, *Discrete Math. Algorithms Appl.*, 6(4)(2014), 20.

[9] G. H. Fath-Tabar, S. Hossein-Zadeh, A. Hamzeh, On the first geometricarithmetic index of product graphs, *Util. Math.*, 86(2011), 279–287.

[10] H. Yousefi-Azari, B. Manoochehrian, A. R. Ashrafi, The PI index of product graphs, *Appl. Math. Letters*, 21 (6)2008(), 624–627.

International J.Math. Combin. Vol.4(2021), 57-64

A New Approach to the Nonhomogeneous Sturm-Liouville Fuzzy Problem with Fuzzy Forcing Function

Hülya GÜLTEKİN ÇİTİL

(Department of Mathematics, Faculty of Arts and Sciences, Giresun University, Giresun, Turkey)

E-mail: hulyagultekin55@hotmail.com, hulya.citil@giresun.edu.tr

Abstract: This paper is on the solution of a nonhomogeneous Sturm-Liouville fuzzy problem with fuzzy forcing function. The problem is studied by a different solution method. Example is solved.

Key Words: Sturm-Liouville theory, fuzzy differential equation, fuzzy boundary value problem, fuzzy eigenvalue.

AMS(2010): 03E72, 34A07, 34B24.

§1. Introduction

The Sturm-Liouville theory is used in mathematical physics [1,2]. Sturm-Liouville fuzzy problems with reel and fuzzy coefficients were studied by Gültekin Çitil and Altınışık [3,4,5]. Also, fuzzy eigenvalue problems were investigated under the approach of generalized differentiability [6,7] and the fuzzy problem with eigenvalue parameter in the boundary condition was studied [8,9]. On the other hand, Gültekin Çitil investigated the problem with fuzzy eigenvalue parameter and she studied the problem with fuzzy eigenvalue parameter in one of the boundary conditions [10,11].

The aim of this study is to investigate the solution of a nonhomogeneous Sturm-Liouville fuzzy problem with fuzzy forcing function by a different solution method.

§2. Preliminaries

Definition 2.1([12]) *A fuzzy number is a mapping* u:$\mathbb{R} \to [0,1]$ *with the following properties:*

(1) u *is normal;*

(2) u *is convex fuzzy set;*

(3) u *is upper semi-continuous on* $\mathbb{R}$*;*

(4) $cl\,\{x \in \mathbb{R} | u(x) > 0\}$ *is compact where* cl *denotes the closure of a subset.*

[1]Received November 16, 2021, Accepted December 15, 2021.

Definition 2.2([13]) *Let $U = R$ (where R is the set of real numbers). Let a, b and c real numbers such that $a \leq c \leq b$. A set $\tilde{u}$ with membership function*

$$\mu(x) = \begin{cases} \frac{x-a}{c-a}, & a < x < c \\ 1, & x = c \\ \frac{b-x}{b-c}, & c < x < b \\ 0, & otherwise \end{cases}$$

is a called a triangular fuzzy number and is denoted by $\tilde{u} = (a, c, b)$.

Definition 2.3([13]) *The classical set $A_\alpha = \{x \in \mathrm{U} | \mu_{\tilde{A}}(x) \geq \alpha\}, 0 < \alpha \leq 1$ is called the α-cut of $\tilde{A}$.*

Definition 2.4([13]) *For a triangular fuzzy number $\tilde{u} = (a, c, b)$ the α-cuts are intervals $u_\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, where*

$$\underline{u}_\alpha = a + \alpha(c - a), \overline{u}_\alpha = b + \alpha(c - b).$$

Definition 2.5([13]) *Let $F_a(.)$, $F_c(.)$, $F_b(.)$ be continuous functions on an interval I. The fuzzy set $\tilde{F}$ determined by the membership function*

$$\mu_{\tilde{F}}(y(.)) = \begin{cases} \alpha, & y = F_a + \alpha(F_c - F_a) \text{ and } 0 \leq \alpha \leq 1 \\ \alpha, & y = F_b + \alpha(F_c - F_b) \text{ and } 0 \leq \alpha \leq 1 \\ 0, & otherwise \end{cases}$$

is called a triangular fuzzy function and is denoted by $\tilde{F} = (F_a, F_c, F_b)$.

§3. A Nonhomogeneous Sturm-Liouville Fuzzy Problem

Consider the nonhomogeneous Sturm-Liouville fuzzy problem with fuzzy forcing function

$$\tau y := y'' + \mathrm{q}(x) y,$$

$$\tau y + \widetilde{\lambda} y = \widetilde{f}(x), \quad x \in (0, \ell) \tag{3.1}$$

$$\mathrm{A} y(0) + \mathrm{B} y'(0) = \widetilde{\beta} \tag{3.2}$$

$$\mathrm{C} y(\ell) + \mathrm{D} y'(\ell) = \widetilde{\delta}, \tag{3.3}$$

where $\mathrm{q}(x)$ is positive and continuous function, $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D} \geq 0$, $A^2 + B^2 \neq 0, C^2 + D^2 \neq 0$, $\widetilde{\lambda} = (\underline{\lambda}, \lambda, \overline{\lambda})$ is positive fuzzy eigenvalue, $\widetilde{f}(x) = (\underline{\mathrm{f}}(\mathrm{x}), \mathrm{f}(\mathrm{x}), \overline{\mathrm{f}}(\mathrm{x}))$ fuzzy forcing function, $\widetilde{\beta} = (\underline{\beta}, \beta, \overline{\beta})$, $\widetilde{\delta} = (\underline{\delta}, \delta, \overline{\delta})$ are symmetric triangular fuzzy numbers and $y(x)$ is positive fuzzy function.

3.1 Solution Method

Let's divide the problem (3.1)-(3.3) three problems following.

(i) The first problem is

$$\tau y + \lambda y = f(x), \tag{3.4}$$

$$Ay(0) + \mathrm{B}y'(0) = \beta, \tag{3.5}$$

$$Cy(\ell) + \mathrm{D}y'(\ell) = \delta, \tag{3.6}$$

(ii) The second problem is

$$\tau y = 0, \tag{3.7}$$

$$Ay(0) + By'(0) = \left(\underline{\beta} - \beta, 0, \overline{\beta} - \beta\right), \tag{3.8}$$

$$Cy(\ell) + \mathrm{D}y'(\ell) = \left(\underline{\delta} - \delta, 0, \overline{\delta} - \delta\right), \tag{3.9}$$

(iii) The third problem is

$$\tau y + \left(\underline{\lambda} - \lambda, 0, \overline{\lambda} - \lambda\right) y = \left(\underline{f}(x) - f(x), 0, \overline{f}(x) - f(x)\right), \tag{3.10}$$

$$Ay(0) + By'(0) = 0, \tag{3.11}$$

$$Cy(\ell) + \mathrm{D}y'(\ell) = 0. \tag{3.12}$$

We discuss the three problems respectively in the following cases.

Case 1. For the problem (i), consider the boundary value problem

$$\tau y + \lambda y = 0 \tag{3.13}$$

$$Ay(0) + \mathrm{B}y'(0) = 0, \tag{3.14}$$

$$Cy(\ell) + \mathrm{D}y'(\ell) = 0. \tag{3.15}$$

Let $\varphi_\lambda(x)$ be the solution of differential equation (3.13) satisfying the conditions $y(0) = \mathrm{B}$, $y'(0) = -\mathrm{A}$ and $\chi_\lambda(x)$ be the solution of differential equation (3.13) satisfying the conditions $y(\ell) = \mathrm{D}$, $y'(\ell) = -\mathrm{C}$. The eigenvalues of the boundary value problem (3.13) – (3.15) are the zeros of the Wronskian function

$$W(\lambda) = W(\varphi_\lambda, \chi_\lambda)(x) = \varphi_\lambda(x)\chi'_\lambda(x) - \chi_\lambda(x)\varphi'_\lambda(x).$$

Then, if λ is not the eigenvalue of the fuzzy boundary value problem (3.13) – (3.15), since $W(\lambda) \neq 0$, the solutions $\varphi_\lambda(x)$ and $\chi_\lambda(x)$ will be linear independent. According to this, the general solution of the fuzzy differential equation (3.13) is

$$y_\lambda(x) = c_{1\lambda}\varphi_\lambda(x) + c_{2\lambda}\chi_\lambda(x).$$

Using the method variation of parameters, we can search for the general solution of the differential equation (3.4) as

$$y_\lambda(x) = c_{1\lambda}(x)\,\varphi_\lambda(x) + c_{2\lambda}(x)\,\chi_\lambda(x). \tag{3.16}$$

Choosing the functions $\varphi_\lambda(x)$ and $\chi_\lambda(x)$ that

$$c'_{1\lambda}(x)\,\varphi_\lambda(x) + c'_{2\lambda}(x)\,\chi_\lambda(x) = 0,$$

$$c'_{1\lambda}(x)\,\varphi_{\lambda'}(x) + c'_{2\lambda}\chi_{\lambda'}(x) = f(x).$$

From this,

$$c'_{1\lambda}(x) = -\frac{1}{W(\lambda)}\chi_\lambda(x)\,f(x),\; c'_{2\lambda}(x) = \frac{1}{W(\lambda)}\varphi_\lambda(x)\;f(x)$$

are obtained. Thus, we have

$$c_{1\lambda}(x) = \frac{1}{W(\lambda)}\int_x^\ell \chi_\lambda(t)\,f(t)\,dt + c_{1\lambda},\; c_{2\lambda}(x) = \frac{1}{W(\lambda)}\int_0^x \varphi_\lambda(t)\;f(t)dt + c_{2\lambda}$$

Substituing these equations in (3.16), the general solution of the differential equation (3.4) is obtained as

$$y_\lambda(x) = \frac{1}{W(\lambda)}\left\{\varphi_\lambda(x)\int_x^\ell \chi_\lambda(t)\,f(t)\,dt + \chi_\lambda(x)\int_0^x \varphi_\lambda(t)\;f(t)dt\right\} + c_{1\lambda}\varphi_\lambda(x) + c_{2\lambda}\chi_\lambda(x)$$

Using the boundary condition (3.5), we have

$$c_{2\lambda}\left(A\chi_\lambda(0) + B\chi'_\lambda(0)\right) = \beta.$$

Also, $A\chi_\lambda(0) + B\chi'_\lambda(0) = W(\lambda)$ and since λ is not eigenvalue, $W(\lambda) \neq 0$. Thus,

$$c_{2\lambda} = \frac{\beta}{A\chi_\lambda(0) + B\chi'_\lambda(0)}.$$

Similarly,

$$c_{1\lambda} = \frac{\delta}{C\varphi_\lambda(\ell) + D\varphi'_\lambda(\ell)}$$

is obtained. Then, the solution of the boundary value problem (3.4) – (3.6) is

$$\begin{aligned} y_\lambda(x) &= \frac{1}{W(\lambda)}\left\{\varphi_\lambda(x)\int_x^\ell \chi_\lambda(t)\,f(t)\,dt + \chi_\lambda(x)\int_0^x \varphi_\lambda(t)\;f(t)dt\right\} \\ &+ \left(\frac{\delta}{C\varphi_\lambda(\ell) + D\varphi'_\lambda(\ell)}\right)\varphi_\lambda(x) + \left(\frac{\beta}{A\chi_\lambda(0) + B\chi'_\lambda(0)}\right)\chi_\lambda(x). \end{aligned} \tag{3.17}$$

Case 2. For the problem (ii), let $y_1(x)$ and $y_2(x)$ be linear independent solutions of the differential equation $\tau y = 0$. The solution of fuzzy boundary value problem (3.7) – (3.9) is

$$Y_1(x) = \left(\underline{\beta} - \beta, 0, \overline{\beta} - \beta\right) w_1(x) + \left(\underline{\delta} - \delta, 0, \overline{\delta} - \delta\right) w_2(x), \tag{3.18}$$

where

$$w_1(x) = \frac{y_2(\ell)\, y_1(x) - y_1(\ell)\, y_2(x)}{y_1(0)\, y_2(\ell) - y_1(\ell)\, y_2(0)}, \qquad w_2(x) = \frac{y_1(0)\, y_2(x) - y_2(0)\, y_1(x)}{y_1(0)\, y_2(\ell) - y_1(\ell)\, y_2(0)}.$$

Case 3. For the problem (iii), let $\tilde{y}(x)$ be the solution of differential equation $\tau y + (\underline{\lambda} - \lambda)\,\mathrm{y} = \underline{f}(x) - f(x)$ and $\tilde{\tilde{y}}(x)$ be the solution of differential equation $\tau y + \left(\overline{\lambda} - \lambda\right)\mathrm{y} = \overline{f}(x) - f(x)$. Then, the solution of fuzzy boundary value problem (3.10) – (3.12) is

$$Y_2(x) = \left(min\left\{\tilde{y}(x), 0, \tilde{\tilde{y}}(x)\right\}, 0, max\left\{\tilde{y}(x), 0, \tilde{\tilde{y}}(x)\right\}\right). \tag{3.19}$$

From (3.17), (3.18) and (3.19), fuzzy solution of fuzzy boundary value problem (3.1)-(3.3) is

$$\mathrm{y}(\mathrm{x}) = y_\lambda(x) + Y_1(x) + Y_2(x).$$

Example 3.1 Consider the fuzzy problem

$$y'' + \tilde{\lambda} y = \tilde{f}(\mathrm{x}), \quad x \in (0,1) \tag{3.20}$$

$$y(0) = \tilde{1}, \tag{3.21}$$

$$y(1) = \tilde{2}, \tag{3.22}$$

where $\tilde{\lambda} = (\lambda + 1, \lambda, \lambda + 3)$ is positive fuzzy eigenvalue, $\tilde{f}(\mathrm{x}) = (\mathrm{x} - 2, \mathrm{x}, \mathrm{x} + 2)$ fuzzy forcing function, $\tilde{1} = (0, 1{,}2)$, $\tilde{2} = (1, 2{,}3)$ are symmetric triangular fuzzy numbers and $y(x)$ is positive fuzzy function.

(i) The first problem is

$$y'' + \lambda y = x, \tag{3.23}$$

$$y(0) = 1, \tag{3.24}$$

$$y(1) = 2. \tag{3.25}$$

Consider the boundary value problem

$$y'' + \lambda y = 0 \tag{3.26}$$

$$y(0) = 0, \tag{3.27}$$

$$y(1) = 0. \tag{3.28}$$

Let

$$\varphi_\lambda(x) = \sin\left(\sqrt{\lambda} x\right)$$

be the solution of differential equation (3.26) satisfying the conditions $y(0)=0$ and

$$\chi_\lambda(x)=\sin\left(\sqrt{\lambda}\right)cos\left(\sqrt{\lambda}x\right)-\cos\left(\sqrt{\lambda}\right)sin\left(\sqrt{\lambda}x\right)$$

be the solution of differential equation (3.26)satisfying the conditions $y(1)=0$. From this, making the necessary operations, $W(\varphi_\lambda,\chi_\lambda)(x)=W(\lambda)$ is obtained as

$$W(\lambda)=-\sqrt{\lambda}sin\left(\sqrt{\lambda}\right).$$

From (3.17), the solution of the problem (3.23)-(3.25) is obtained as

$$\begin{aligned} y_\lambda(x) &= \frac{1}{-\sqrt{\lambda}sin\left(\sqrt{\lambda}\right)}\left\{\frac{\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}}\left(1-x\cos\left(\sqrt{\lambda}(1-x)\right)-\frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}(1-x)\right)\right)\right. \\ &+\left(\frac{cos\left(\sqrt{\lambda}x\right)\sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}}\right)\left(x\cos\left(\sqrt{\lambda}\right)+\frac{\sin\left(\sqrt{\lambda}\right)}{\sqrt{\lambda}}\right) \\ &\left.-\frac{1}{\sqrt{\lambda}}\left(x\sin\left(\sqrt{\lambda}\right)cos^2\left(\sqrt{\lambda}x\right)+\frac{\cos\left(\sqrt{\lambda}\right)sin\left(\sqrt{\lambda}x\right)}{\sqrt{\lambda}}\right)\right\} \\ &+\frac{\sin\left(\sqrt{\lambda}\right)cos\left(\sqrt{\lambda}x\right)+\left(2-\cos\left(\sqrt{\lambda}\right)\right)sin\left(\sqrt{\lambda}x\right)}{sin(\sqrt{\lambda})}. \end{aligned}$$

(ii) The second problem is

$$y^{''}=0, \tag{3.29}$$

$$y(0)=(-1,0,1), \tag{3.30}$$

$$y(1)=(-1,0,1). \tag{3.31}$$

Let $y_1(x)=x$ and $y_2(x)=1$ be linear independent solutions of the differential equation $y^{''}=0$. Then, from (3.18) the solution of fuzzy boundary value problem $(3.29)-(3.31)$ is

$$Y_1(x)=(-1,0,1)(1-x)+(-1,0,1)x \tag{3.32}$$

(iii) The third problem is

$$y^{''}+(1,0,3)y=(-2,0,2), \tag{3.33}$$

$$y(0)=0, \tag{3.34}$$

$$y(1)=0. \tag{3.35}$$

The solution of the boundary value problem $y^{''}$+y$=-2$, $y(0)=0$, $y(1)=0$ is

$$\tilde{y}(x)=2\cos(x)+\frac{2(1-\cos(1))}{\sin(1)}\sin(x)-2,$$

and the solution of the boundary value problem y''+3y= 2, $y\,(0)$=0, $y\,(1)$=0 is

$$\tilde{\tilde{y}}\,(x) = -2\cos\left(\sqrt{3}x\right) + \frac{2\left(\cos\left(\sqrt{3}\right)-1\right)}{\sin\left(\sqrt{3}\right)}\sin\left(\sqrt{3}x\right) + 2.$$

Then, the solution of fuzzy boundary value problem (3.33)-(3.35) is

$$Y_2\,(x) = \left(min\left\{\tilde{y}\,(x)\,, 0, \tilde{\tilde{y}}\,(x)\right\}, 0, max\left\{\tilde{y}\,(x)\,, 0, \tilde{\tilde{y}}\,(x)\right\}\right). \tag{3.36}$$

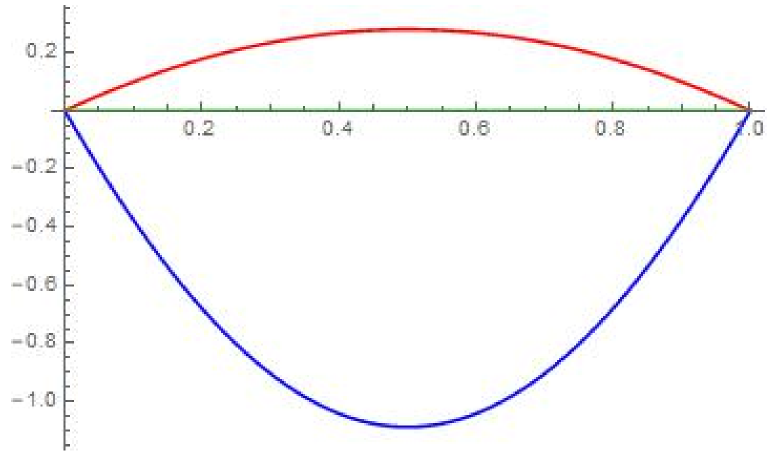


Figure 1 Graphic of (3.36) for $\alpha = 0$ cut (blue and red lines) and $\alpha = 1$ cut (green line)

Consequently, the fuzzy solution of fuzzy boundary value problem (3.20) − (3.22) is

$$\text{y (x)} = y_\lambda\,(x) + Y_1\,(x) + Y_2\,(x)\,.$$

§4. Conclusions

In this study, a nonhomogeneous Sturm-Liouville fuzzy problem with fuzzy forcing function is investigated. The problem is solved by a different solution method. Example is solved on studied problem. Examples can be multiplied.

This paper is a new approach to the studied problem.

References

[1] Fulton C. T., Two point boundary value problems with eigenvalue parameter contained in the boundary conditions, *Proc. Soc. Edinburg*, 77A(1977), 293-308.

[2] Binding P. A., Browne P. J., Watson B. A., Sturm-Liouville problems with boundary conditions rationally dependent on the eigenparameter, II, *Journal of Computational and Applied Mathematics*, 148(1)(2002), 147-168.

[3] Gültekin Çitil H., Altınışık N., On the eigenvalues and the eigenfunctions of the Sturm-Liouville fuzzy boundary value problem, *Journal of Mathematical and Computational Science*, 7(4)(2017), 786-805.

[4] Gültekin Çitil H., Altınışık N., The examination of eigenvalues and eigenfunctions of the Sturm-Liouville fuzzy problem according to boundary conditions, *International Journal of Mathematical Combinatorics*, 1(2018), 51-60.

[5] Gültekin Çitil H., Altınışık N., The eigenvalues and the eigenfunctions of the Sturm-Liouville fuzzy problem with fuzzy coefficient boundary conditions, *Journal of Science and Arts*, 4(45)(2018), 947-958.

[6] Gültekin Çitil H., The eigenvalues and the eigenfunctions of the Sturm-Liouville fuzzy boundary value problem according to the generalized differentiability. Scholars Journal of Physics, Mathematics and Statistics, 4(4)(2017), 185-195.

[7] Ceylan T., Altınışık N., Eigenvalue problem with fuzzy coefficients of boundary conditions, *Scholars Journal of Physics, Mathematics and Statistics*, 5(2)(2018), 187-193.

[8] Gültekin Çitil H., Important notes for a fuzzy boundary value problem, *Applied Mathematics and Nonlinear Sciences*, 4(2)(2019), 305–314.

[9] Ceylan T., Altınışık N., Fuzzy eigenvalue problem with eigenvalue parameter contained in the boundary condition, *Journal of Science and Arts*, 3(44)(2018), 589-602.

[10] Gültekin Çitil H.,Sturm-Liouville fuzzy problem with fuzzy eigenvalue parameter, *International Journal of Mathematical Modelling & Computations*, 9(3)(2019), 187- 195.

[11] Gültekin Çitil H., The problem with fuzzy eigenvalue parameter in one of the boundary conditions, *An International Journal of Optimization and Control: Theories & Applications*, 10(2)(2020), 159-165.

[12] Liu, H.-K., Comparison results of two-point fuzzy boundary value problems, *International Journal of Computational and Mathematical Sciences*, 5(1)(2011), 1-7.

[13] Gasilov N.A., Amrahov Ş.E, Fatullayev A.G., Hashimoğlu I.F., Solution method for a boundary value problem with fuzzy forcing function, *Information Sciences*, 317(2015), 349-368.

International J.Math. Combin. Vol.4(2021), 65-74

On Support Regular Interval-Valued Fuzzy Graphs

N.R. Santhi Maheswari and K. Priyadharshini

PG and Research Department of Mathematics, G.Venkataswamy Naidu College, Kovilpatti- 628502,
Affiliated to Manonmaniam Sundaranar University, Tirunelveli-627012, Tamil Nadu, India

E-mail: priyakan611@gmail.com, nrsmaths@yahoo.com

Abstract: In this paper, support regular interval-valued fuzzy graphs and support totally regular interval-valued fuzzy graphs are defined. Comparative study between support regular interval-valued fuzzy graph and support totally regular interval-valued fuzzy graph is done. A necessary and sufficient condition under which they are equivalent is provided. Characterization of support regular interval-valued fuzzy graph in which underlying crisp graph is a cycle is investigated. Also, whether the results hold for support totally regular interval-valued fuzzy graphs is examined.

Key Words: Support(2-degree) of a vertex in fuzzy graph, interval-valued fuzzy graph.

AMS(2010): 05C12, 03E72, 05C72.

§1. Introduction

In this paper, we consider only finite, simple, connected graphs. We denote the vertex set and the edge set of a graph G by V (G) and E(G) respectively. The degree of a vertex v is the number of edges incident at v, and it is denoted by d(v). A graph G is regular if all its vertices have the same degree. The notion of fuzzy sets was introduced by Zadeh as a way of representing uncertainity and vagueness [29]. The first definition of fuzzy graph was introduced by Haufmann in 1973. In 1975, A. Rosenfeld introduced the concept of fuzzy graphs [9]. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas. Irregular fuzzy graphs play a central role in combinatorics and theoretical computer science.In 1975, Zadeh introduced the notion of interval-valued fuzzy sets as an extension of fuzzy set [30] in which the values of the membership degree are intervals of numbers instead of the numbers. In 2011, Akram and Dudek [1] defined interval-valued fuzzy graphs and give some operations on them.

§2. Review of Literatures

Nagoorgani and Radha introduced the concept of degree, total degree, regular fuzzy graphs in 2008 [6]. Nagoorgani and Latha introduced the concept of irregular fuzzy graphs, neighbourly

[1]Received October 25, 2021, Accepted December 18, 2021.

irregular fuzzy graphs and highly irregular fuzzy graphs in 2012 [7]. N.R.Santhi Maheswari and C.Sekar introduced $(2, k)$-regular fuzzy graphs and totally $(2, k)$-regular fuzzy graphs, $(r, 2, k)$-regular fuzzy graphs, (m, k)-regular fuzzy graphs and (r, m, k)-regular fuzzy graphs [10, 14, 15, 16]. N.R.Santhi Maheswari and C. Sekar introduced 2-neighbourly irregular fuzzy graphs and m-neighbourly irregular fuzzy graphs [21, 13]. N.R.Santhi Maheswari and C.Sekar introduced an edge irregular fuzzy graphs, neighbourly edge irregular fuzzy graphs and strongly edge irregular fuzzy graph [17, 11, 18]. D.S.Cao, introduced 2-degree of vertex v is the the sum of the degrees of the vertices adjacent to v and it is denoted by t(v) [3]. A.Yu, M.Lu and F.Tian, introduced pseudo degree (average degree) of a vertex v is $(t(v))/d(v)$, where $d(v)$ is the number of edges incident at the vertex v [2]. N.R.Santhi Maheswari and C.Sekar introduced 2-degree of a vertex in fuzzy graphs , pseudo degree of a vertex in fuzzy graph and pseudo regular fuzzy graphs [12]. N.R Santhi Maheswari and M.Sutha introduced concept of pseudo irregular fuzzy graphs and highly pseudo irregular fuzzy graphs [19]. N.R.Santhi Maheswari and M.Rajeswari introduced the concept of strongly pseudo irregular fuzzy graphs [20]. N.R.Santhi Maheswari and V.Jeyapratha introduced the concept of neighbourly pseudo irregular fuzzy graphs [22]. N.R.Santhi Maheswari and K.Amutha introduced support neighbourly edge irregular graphs and 1-neighbourly edge irregular graphs, Pseudo Edge Regular and Pseudo Neighbourly edge irregular graphs [23, 24, 25]. J.Krishnaveni and N.R.Santhi Maheswari introduced support and total support of a vertex in fuzzy graphs, support neighbourly irregular fuzzy graphs and support neighbourly totally irregular fuzzy graphs []4. N.R.Santhi Maheswari and K.Priyadharshini introduced support highly irregular fuzzy graphs [26]. These motivate us to introduce support regular interval-valued fuzzy graphs and support totally regular interval-valued fuzzy graphs and discussed some of its properties.

§3. Preliminaries

We present some known definitions and results for ready reference to go through the work presented in this paper. By graph, we mean a pair $G^* = (V, E)$, where V is the set and E is a relation on V. The elements of V are vertices of G^* and the elements of E are edges of G^*.

Definition 3.1([3]) *The 2-degree (support) of v is defined as the sum of the degrees of the vertices adjacent to v and it is denoted by* $t(v)$*.*

Definition 3.2([2]) *The average (pseudo) degree of v is defined as* $t(v))/(d(v)$*, where* $t(v)$ *is the 2-degree of v and* $d(v)$ *is the degree of v and it is denoted by* $da(v)$*.*

Definition 3.3([2]) *A graph is called pseudo-regular if every vertex of G has equal (pseudo) average-degree.*

Definition 3.4([6]) *A fuzzy graph* $G : (\sigma, \mu)$ *is a pair of functions* (σ, μ)*, where* $\sigma : V \to [0, 1]$ *is a fuzzy subset of a non-empty set* V *and* $\mu : VXV \to [0, 1]$ *is a symmetric fuzzy relation on* σ *such that for all* u, v *in* V*, the relation* $\sigma(uv) \leq \sigma(u)\Lambda\sigma(v)$ *is satisfied. A fuzzy graph G is called complete fuzzy graph if the relation* $\sigma(uv) = \sigma(u)\Lambda\sigma(v)$ *is satisfied.*

Definition 3.5([6]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. The degree of a vertex u in G is denoted by $d(u)$ and is defined as $d(u) = \sum \mu(uv)$, for all $uv \in E$.*

Definition 3.6([6]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. The total degree of a vertex u in G is denoted by $td(u)$ and is defined as $td(u) = d(u) + \sigma(u)$, for all $u \in V$.*

Definition 3.7([7]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be an irregular fuzzy graph, if there is a vertex which is adjacent to the vertices with distinct degrees.*

Definition 3.8([7]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be a totally irregular fuzzy graph if there is a vertex which is adjacent to the vertices with distinct total degrees.*

Definition 3.9([7]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be a neighbourly irregular fuzzy graph if every two adjacent vertices of G have distinct degrees.*

Definition 3.10([7]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be a neighbourly totally irregular fuzzy graph if every two adjacent vertices have distinct total degrees.*

Definition 3.11([7]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be a highly irregular fuzzy graph if every vertex of G is adjacent to vertices with distinct degrees.*

Definition 3.12([7]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be a highly totally irregular fuzzy graph if every vertex of G is adjacent to vertices with distinct total degrees.*

Definition 3.13([6]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be a regular fuzzy graph if all the vertices of G have same degree.*

Definition 3.14([6]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be a totally regular fuzzy graph if all the vertices of G have same total degree.*

Definition 3.15([4]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. The support (2-degree) of a vertex v in G is defined as the sum of degrees of the vertices adjacent to v and is denoted by $s(v)$. That is, $s(v) = \sum dG(u)$, where $dG(u)$ is the degree of the vertex u which is adjacent with the vertex v.*

Definition 3.16([4]) *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. The total support of a vertex v in G is denoted by $ts(v)$ and is defined as $ts(v) = s(v) + \sigma(v)$, for all $v \in V$.*

Definition 3.17([4]) *A graph G is said to be a support neighbourly irregular fuzzy graph if every two adjacent vertices of G have distinct supports.*

Definition 3.18([4]) *A graph G is said to be a support neighbourly totally irregular graph if every two adjacent vertices of G have distinct total supports.*

Definition 3.19([4]) *A graph G is said to be a support highly irregular fuzzy graph if every vertex of G is adjacent to the vertices having distinct supports.*

Definition 3.20([26]) *A graph G is said to be a support highly totally irregular graph if every vertex of G is adjacent to the vertices having distinct total supports.*

Definition 3.21 *An interval-valued fuzzy graph with an underlying set V is defined to be the pair (A, B), where $A = (\mu_A^-, \mu_A^+)$ is an interval-valued fuzzy set on V such that $\mu_A^-(x) \leq \mu_A^+(x)$, for all $x \in V$ and $B = (\mu_B^-, \mu_B^+)$ is an interval-valued fuzzy set on E such that $\mu_B^-(x, y) \leq min((\mu_A^-(x), \mu_A^-(y)))$ and $\mu_B^+(x, y) \leq min((\mu_A^+(x), \mu_A^+(y)))$, for all edge $xy \in E$. Hence A is called an interval-valued fuzzy vertex set on V and B is called an interval-valued fuzzy edge set on E.*

Definition 3.22 *Let $G : (A, B)$ be an interval-valued fuzzy graph. The negative degree of a vertex $u \in G$ is defined as $d_G^-(u) = \sum \mu_B^-(u, v)$, for $uv \in E$. The positive degree of a vertex $u \in G$ is defined as $d_G^+(u) = \sum \mu_B^+(u, v)$, for $uv \in E$ and $\mu_B^+(uv) = \mu_B^-(uv) = 0$ if uv not in E. The degree of a vertex u is defined as $d_G(u) = (d_G^-(u), d_G^+(u))$.*

Definition 3.23 *Let $G : (A, B)$ be an interval-valued fuzzy graph on $G^*(V, E)$. The total degree of a vertex $u \in V$ is denoted by $td_G(u)$ and is defined as $td_G(u) = (td_G^-(u), td_G^+(u))$, where $td_G^-(u) = \sum \mu_B^-(u, v) + (\mu_A^-(u))$ and $td_G^+(u) = \sum \mu_B^+(u, v) + (\mu_A^+(u))$.*

Definition 3.24 *Let $G : (A, B)$ be an interval-valued fuzzy graph on $G^*(V, E)$, where $A = (\mu_A^-, \mu_A^+)$ and $B = (\mu_B^-, \mu_B^+)$ be two interval-valued fuzzy sets on a non-empty set V and $E \subseteq V \times V$ respectively. Then G is said to be regular interval-valued fuzzy graph if all the vertices of G has same degree (c_1, c_2).*

Definition 3.25 *Let $G : (A, B)$ be an interval-valued fuzzy graph on $G^*(V, E)$, then G is said to be totally regular interval-valued fuzzy graph if all the vertices of G has same total degree (c_1, c_2).*

§4. Support Regular Interval-Valued Fuzzy Graphs and the Totally Support Regular Interval-Valued Fuzzy Graphs

In this section, we define support regular interval-valued fuzzy graph and totally support regular interval-valued fuzzy graph and discussed about its properties.

Definition 4.1 *Let $G : (A, B)$ be an interval-valued fuzzy graph on $G^* : (V, E)$. The support of a vertex a in G is denoted by $s_G(a)$ and is defined as $s_G(a) = (s_G^-(a), s_G^+(a))$, where $s_G^-(a) = \sum_{u \in N(a)} d_G^-(a)$ and $s_G^+(a) = \sum_{u \in N(a)} d_G^+(a)$, for all $a \in V$.*

Definition 4.2 *Let $G : (A, B)$ be an interval-valued fuzzy graph on $G^* : (V, E)$. If $s_G(v) = (k_1, k_2)$, for all v in V; then G is called (k_1, k_2) - support regular interval-valued fuzzy graph.*

Example 4.3 Consider a fuzzy graph on graph on $G^*(V, E)$ following.

Figure 1

Here, $d_G(a) = (0.3, 0.6) = d_G(b) = d_G(c) = d_G(d)$. And, $s_G(a) = s_G(b) = s_G(c) = s_G(d) = (0.9, 1.8)$, for all $a \in V$. Hence G is $(0.9, 1.8)$- support regular interval-valued fuzzy graph.

Definition 4.3 *Let $G : (A, B)$ be an interval-valued fuzzy graph on G*(V,E). The total support of a vertex a in G is denoted by $ts_G(a)$ and is defined as $ts_G(a) = (ts_G^-(a), ts_G^+(a)) = (s_G^-(a) + \mu_A^-(a), s_G^+(a) + \mu_A^+(a))$ for all $a \in V$.*

Definition 4.4 *Let $G : (A, B)$ be an interval-valued fuzzy graph on G*(V,E). If all the vertices of G have the same total support (k_1, k_2), then G is said to be a (k_1, k_2)- support totally regular interval-valued fuzzy graph.*

Example 4.6 Consider an interval-valued fuzzy graph $G : (A, B)$ on graph on $G^*(V, E)$ following.

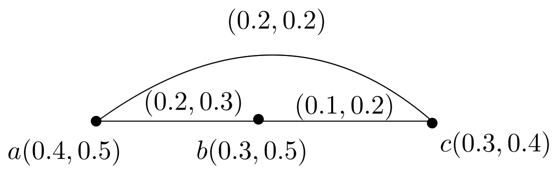


Figure 2

Here, $ts_G(a) = ts_G(b) = ts_G(c) = (1, 1.4)$. Hence G is a $(1, 1.4)$- totally support regular interval-valued fuzzy graph.

Remark 4.7 *A support regular interval-valued fuzzy graph need not be totally support regular interval-valued fuzzy graph.*

Example 4.8 Consider an interval-valued fuzzy graph $G : (A, B)$ on a crisp graph $G^*(V, E)$.

Notice that in Figure 1, $s_G(a) = s_G(b) = s_G(c) = s_G(d) = (0.9, 1.8)$. But $ts_G(a) = (1.3, 2.3), ts_G(b) = (1.1, 2.4), ts_G(c) = (1.2, 2.2), ts_G(d) = (1.2, 2.3)$. Hence support regular interval-valued fuzzy graph need not be totally support regular interval-valued fuzzy graph.

Remark 4.9 *A totally support regular interval-valued fuzzy need not be a support regular interval-valued fuzzy graph.*

Example 4.10 Consider an interval-valued fuzzy graph $G : (A, B)$ on graph $G^*(V, E)$.

Notice that in Figure 2, $s_G(a) = (0.6, 0.9), s_G(b) = (0.7, 0.9), s_G(c) = (0.7, 1)$. But $ts_G(a) = ts_G(b) = ts_G(c) = (1, 1.4)$. Hence totally support regular interval-valued fuzzy graph need not be support regular interval-valued fuzzy graph.

Example 4.11 A graph which is both support regular interval-valued fuzzy graph and support totally regular interval-valued fuzzy graph is given below.

Consider an interval-valued fuzzy graph $G : (A, B)$ on graph $G^*(V, E)$ following.

Figure 3

Here, the graph is $(0.6, 1.2)$- support regular interval-valued fuzzy graph and $(1, 1.7)$-totally support regular interval-valued fuzzy graph.

Theorem 4.12 *Let $G : (A, B)$ be an interval-valued fuzzy graph on $G * (V, E)$. Then $A(u) = (\mu_A^-(u), \mu_A^+(u))$, for all $u \in V$ is a constant function if and only if the following are equivalent:*

(i) G is a support regular interval-valued fuzzy graph;

(ii) G is a support totally regular interval-valued fuzzy graph.

Proof Assume that $A(u) = (\mu_A^-(u), \mu_A^+(u)) = (c_1, c_2)$, for all $u \in V$, where c_1 and c_2 are constant. Suppose G is a support regular interval-valued fuzzy graph. Then $s_G(u) = (k_1, k_2)$, for all $u \in V$. Now, $ts_G(u) = s_G(u) + (\mu_A^-(u), \mu_A^+(u)) = (k_1, k_2) + (c_1, c_2) = (k_1 + c_1, k_2 + c_2)$, for all $u \in V$. Hence G is a support totally regular fuzzy graph. Thus $(i) \Rightarrow (ii)$ is proved. Suppose G is a support totally regular fuzzy graph. Then $ts_G(u) = (k_1, k_2)$ for all $u \in V$ $\Rightarrow s_G(u) + (\mu_A^-(u), \mu_A^+(u)) = (k_1, k_2)$, for all $u \in V \Rightarrow s_G(u) + (c_1, c_2) = (k_1, k_2)$, for all $u \in V$ $\Rightarrow s_G(u) = (k_1 - c_1, k_2 - c_2)$, for all $u \in V$. Hence G is a support regular fuzzy graph. Thus $(ii) \Rightarrow (i)$ is proved. Hence (i) and (ii) are equivalent.

Conversely, suppose (i) and (ii) are equivalent. Let G be a support regular interval-valued fuzzy graph and a support totally regular interval-valued fuzzy graph. Then $s_G(u) = (k_1, k_2)$ and $ts_G(u) = (c_1, c_2)$, for all $u \in V$. Now $ts_G(u) = (c_1, c_2)$ for all $u \in V \Rightarrow s_G(u) + (\mu_A^-(u), \mu_A^+(u)) = (c_1, c_2)$, for all $u \in V \Rightarrow (k_1, k_2) + (\mu_A^-(u), \mu_A^+(u)) = (c_1, c_2)$, for all $u \in V \Rightarrow (\mu_A^-(u), \mu_A^+(u)) = (c_1, c_2) - (k_1, k_2) = (c_1 - k_1, c_2 - k_2)$, for all $u \in V$. Hence $A = (\mu_A^-(u), \mu_A^+(u))$ is a constant function. □

Theorem 4.13 *Let $G : (A, B)$ be an interval-valued fuzzy graph on $G^*(V, E)$. If G is both support regular and support totally regular interval-valued fuzzy graph then $A(u) = (\mu_A^-(u), \mu_A^+(u))$, for all $u \in V$ is a constant function.*

Proof Assume that G is both support regular and support totally regular interval-valued fuzzy graph. Then $s_G(u) = (c_1, c_2)$ and $ts_G(u) = (k_1, k_2)$, for all $u \in V$. Now,

$$\begin{aligned} ts_G(u) = (k_1, k_2) &\Rightarrow s_G(u) + (\mu_A^-(u), \mu_A^+(u)) = (k_1, k_2) \\ &\Rightarrow (c_1, c_2) + (\mu_A^-(u), \mu_A^+(u)) = (k_1, k_2) \\ &\Rightarrow (\mu_A^-(u), \mu_A^+(u)) = (k_1, k_2) - (c_1, c_2) = (k_1 - c_1, k_2 - c_2) = \text{constant}. \end{aligned}$$

Hence, $(\mu_A^-(u), \mu_A^+(u))$ is a constant function. □

Remark 4.14 The converse of the theorem 4.13 need not be true.

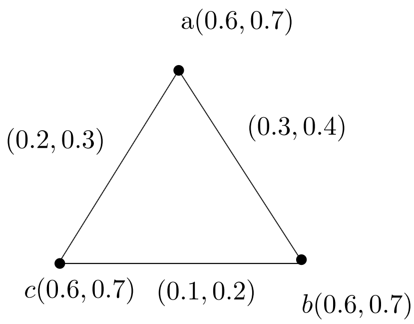


Figure 4

Here,$s_G(a) = (0.7, 1.1), s_G(b) = (0.8, 1.2), s_G(c) = (0.9, 1.3)$, $ts_G(a) = (1.3, 1.8), ts_G(b) = (1.4, 1.9), ts_G(c) = (1.5, 2)$ and $A = (\mu_A^-(u), \mu_A^+(u))$, for all $u \in V$ is a constant function. But G is neither support regular interval-valued fuzzy graph nor a support totally regular interval-valued fuzzy graph.

Theorem 4.15 *Let $G : (A, B)$ be an interval-valued fuzzy graph on $G^*(V, E)$, a cycle of length n. If $B = (\mu_B^-(uv), \mu_B^+(uv))$, for all $uv \in E$, is a constant function, then G is a support regular interval-valued fuzzy graph.*

Proof If $B = (\mu_B^-(uv), \mu_B^+(uv))$,for all $uv \in E$, is a constant function say $(\mu_B^-(uv), \mu_B^+(uv)) = (c_1, c_2)$; for all $uv \in E$. Then $s_G(u) = 4(c_1, c_2)$, for all $u \in V$: Hence G is $4(c_1, c_2)$- support regular interval-valued fuzzy graph. □

Remark 4.16 *The converse of Theorem* 4.15 *needs not be true.*

Example 4.17 Consider an interval-valued fuzzy graph $G : (A, B)$ on $G^* : (V, E)$ following.

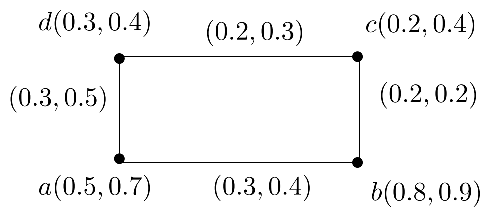


Figure 5

Here, B is not constant. But $s_G(a) = s_G(b) = s_G(c) = s_G(d) = (1, 1.4)$, for all $u \in V$. Therefore the graph is $(1, 1.4)$- support regular interval-valued fuzzy graph.

Theorem 4.18 *Let $G : (A, B)$ be an interval-valued fuzzy graph on $G^*(V, E)$, an even cycle of length n. If the alternate edges have same membership values, then G is a support regular interval-valued fuzzy graph.*

Proof If the alternate edges have the same membership values, then

$$\mu(ei) = \begin{cases} (c_1, c_2), & \text{if } i \text{ is odd;} \\ (k_1, k_2), & \text{if } i \text{ is even.} \end{cases}$$

Now, if $(c_1, c_2) = (k_1, k_2)$, then B is a constant function. So, by above theorem G is a support regular fuzzy graph. If $(c_1, c_2) \neq (k_1, k_2)$; then $dG(v) = (c_1, c_2)+(k_1, k_2) = (c_1+k_1, c_2+k_2)$ for all $v \in V$. So, $s(v) = 2(c_1, c_2) + 2(k_1, k_2)$. Hence, G is a $2(c_1 + k_1, c_2 + k_2)$-support regular interval-valued fuzzy graph. □

Remark 4.19 *The above theorem does not hold for a support totally regular interval-valued fuzzy graph.*

Example 4.20 Consider an interval-valued fuzzy graph on $G^*(V, E)$ following.

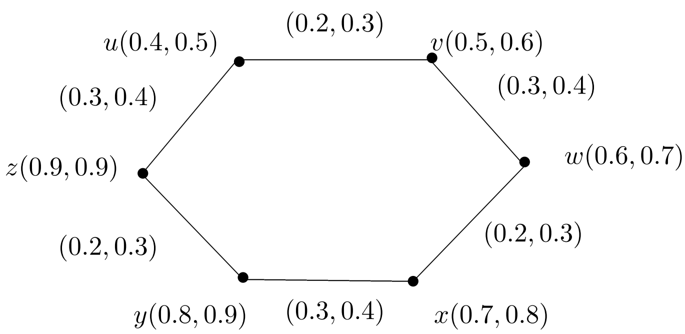


Figure 6

Here, $s_G(u) = (1, 1.4)$ for all $u \in V$, $ts_G(u) = (1.4, 1.9) \neq (1.5, 2) = ts_G(v)$ and the alternate edges have the same membership values, but G is not a support totally regular fuzzy graph.

Proposition 4.21 *If v is a pendant vertex, then support of a vertex v is the degree of the vertex which is adjacent with v. (or) If v is a pendant vertex then $s_G(v) = dG(u)$, where u is the vertex adjacent with v.*

Theorem 4.22 *If G is a (k_1, k_2)-regular interval-valued fuzzy graph on $G^*(V, E)$, an r-regular graph then $s_G(v) = rd_G(v)$, for all $v \in G$.*

Proof Let G be a (k_1, k_2) - regular interval valued fuzzy graph on $G^*(V, E)$, an r-regular graph. Then $d_G(v) = (k_1, k_2)$, for all $v \in G$ and $dG^*(v) = r$, for all $v \in G$. So, $s_G(v) =$

$\sum d_G(v_i)$, where each v_i (for $i = 1, 2, \cdots, r$) is adjacent with vertex $v \Rightarrow s_G(v) = \sum d_G(vi) = r(k_1, k_2) = rd_G(v)$. □

Theorem 4.23 *Let G be an interval-valued fuzzy graph on $G^*(V, E)$, an r-regular graph. Then G is support regular interval-valued fuzzy graph if G is a regular interval-valued fuzzy graph.*

Proof Let G be a (k_1, k_2) - regular interval-valued fuzzy graph on $G^*(V, E)$, an r-regular graph. Then,

$\Rightarrow s_G(v) = rdG(v)$, for all $v \in G$
$\Rightarrow s(v) = r(k_1, k_2)$, for all $v \in G$
$\Rightarrow$ all the vertices have same support $r(k_1, k_2)$.

Hence G is an $r(k_1, k_2)$ - support regular interval-valued fuzzy graph. □

References

[1] Akram, M. and W.A.Dudek, Interval-valued fuzzy graphs, *Computers Math Appl.,* 61(2011), 289-299.

[2] A.Yu, M.Lu and F.Tian, On the spectral radius of graphs, *Linear Algebra and Its Applications*, 387 (2004), 41-49.

[3] Dasong Cao, Bounds on eigen values and chromatic numbers, *Linear algebra and its Applications*,270(1998), 1-13.

[4] J.Krishnaveni and N.R.Santhi Maheswari, Support neighbourly Irregular fuzzy Graphs,*International Journal of Advanced Research in Engineering and Technology*, Vol.11, 1(2020), 147-154.

[5] R.Muneeswari and N.R.Santhi Maheswari,vSupport Edge Regular Fuzzy Graphs,*International Journal of Advanced Research in Engineering and Technology*, Vol.11, 1(2020), 184-193.

[6] A.Nagoor Gani and K.Radha, On regular fuzzy graphs, *Journal of Physical Sciences*, 12 (2008) 33-40.

[7] A.Nagoor Gani and S.R.Latha, On Irregular Fuzzy graphs, *Applied Mathematical Sciences*, 6 (2012) 517-523.

[8] K.Priyadharshini and N.R.Santhi Maheswari, A study on support regular graphs containing a given graph and its complement, *International Journal of Advanced Research in Engineering and Technology*, Vol.11, 2(2020), 369-378.

[9] A.Rosenfeld, Fuzzy graphs, In: L.A.Zadeh,K.S.Fu, M.Shimura, eds., *Fuzzy sets and Their Applications*, Academic Press (1975), 77-95.

[10] N.R.Santhi Maheswari and C.Sekar, On (2, k)- regular fuzzy graph and totally $(2, k)$-regular fuzzy graph, *International Journal of Mathematics and Soft Computing*, 4(2)(2014), 59-69.

[11] N.R.Santhi Maheswari and C.Sekar, On neighbourly edge irregular fuzzy graphs, *International Journal of Mathematical Archive*, 6(10)(2015), 224-231.

[12] N.R.Santhi Maheswari and C.Sekar, On pseudo regular fuzzy graphs, *Annals of Pure and Applied Mathematics*, 11(1)(2016), 105-113.

[13] N.R.Santhi Maheswari and C.Sekar, On m-neighbourly Irregular fuzzy graphs, *International Journal of Mathematics and Soft Computing*, 5(2)(2015), 145-153.

[14] N. R. Santhi Maheswari and C. Sekar, On $(r, 2, k)$- regular fuzzy graphs, *Journal of Combinatorial Mathematics and Combinatorial Computing*, 97(2016), 11-21.

[15] N. R. Santhi Maheswari and C. Sekar, On (m, k) -Regular fuzzy graphs, *International Journal of Mathematical Archieve*, 7(1)(2016), 1-7.

[16] N. R. Santhi Maheswari and C. Sekar, On (r, m, k) - Regular Fuzzy Graphs, *International J.Math.Combin.*, Vol.1(2016), 18-26.

[17] N. R. Santhi Maheswari and C. Sekar, On Edge Irregular Fuzzy Graphs ,*International Journal of Mathematics and Soft Computing*, Vol.6, 2(2016), 131 - 143.

[18] N. R. Santhi Maheswari and C. Sekar, On Strongly Edge Irregular fuzzy graphs, *Kragujevac Journal of Mathematics*, Vol.40, 1(2016), 125-135.

[19] N. R. Santhi Maheswari and M.Sudha, Pseudo irregular fuzzy graphs and highly pseudo irregular fuzzy graphs, *International Journal of Mathematical Archieve*, 7(4)(2016), 99-106.

[20] N.R.Santhi Maheswari and M.Rajeswari, On Strongly pseudo irregular fuzzy graphs, *International Journal of Mathemaical Archive*, 7(6)(2016), 145-151.

[21] N.R.Santhi Maheswari and C.Sekar, On 2-neighbourly irregular fuzzy graphs, *Utilitas Mathematica*, June 2018.

[22] N.R.Santhi Maheswari and V.Jeyapratha, On neighbourly pseudo irregular fuzzy graphs, *International Journal of Mathematical Combinatorics*, Vol.4(2018), 45-52.

[23] N. R. Santhi Maheswari and K. Amutha, Support neighbourly edge irregular graphs, *International Journal of Recent Technology and Engineering*, Vol 8, 3(2019), 5329-5332.

[24] N.R.Santhi Maheswari and K.Amutha, 1- Neighbourly edge irregular graphs, *Advances in Mathematics: Scientific Journal*, Special issue 3(2019), 200-207.

[25] N.R.Santhi Maheswari and K.Amutha, Pseudo edge regular and neighbourly pseudo edge irregular graphs, *Advances and Applications in Mathematical Sciences*(Communicated).

[26] N.R.Santhi Maheswari and K.Priyadharshini, Support highly irregular fuzzy graphs, *Advances and Applications in Mathematical Sciences* (Communicated).

[27] S.Sivabala and N.R.Santhi Maheswari, Support edge regular fuzzy graphs, *International Journal of Advanced Research in Engineering and Technology*, Vol.11, 1(2020), 172-183.

[28] D.Subburaj and N.R.Santhi Maheswari, Support edge regular fuzzy graphs,*International Journal of Advanced Research in Engineering and Technology*, Vol.11, 1(2020), 163-171.

[29] L.A. Zadeh, Fuzzy Sets, *Information and Control*, 8 (1965), 338-353.

[30] L.A. Zadeh, The concept of a linguistic and application to approximate reasoning, *Information Science*, 8 (1975), 199-249.

International J.Math. Combin. Vol.4(2021), 75-89

Economic Logic in the Carbon Neutrality with Reflection on Human Civilization

Linfan MAO

1. Chinese Academy of Mathematics and System Science, Beijing 100190, P.R.China
2. Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA

E-mail: maolinfan@163.com

Abstract: Clearly, the excess carbon emission is a natural accumulating of 300 years of humans' industrial activities that released carbon dioxide, also with other greenhouse gases. The main objective of carbon neutralization is to correct human's wanton interference to the nature, realize self-absorption of carbon emissions from human activities and then, only zero emissions to the nature so as to achieve the harmony and symbiosis of humans with the nature. In fact, this is to regard the humans and the nature consist of a binary system in biology and to re-understand the nature according to the principle of everything mutually reinforcing. But then, how to achieve carbon peak and then, carbon neutrality and *what is the underlying logic of its impact on economic development*? Taking carbon dioxide as a single substance flow, a particular case of continuity flow or mathematical combinatorics, this paper analyzes the way to achieve carbon neutrality and its impact on energy, industry and transportation industries according to the binary system, which points out that the Chinese civilization is the only possible way for humans to achieve carbon neutrality and return to the beautiful picture of green mountains with clear waters.

Key Words: Continuity flow, mathematical combinatorics, substance flow, carbon emission, carbon neutralization, natural circulation, conservation law on node, binary system, Chinese civilization.

AMS(2010): 05C22,91B76.

§1. Introduction

The notion of harmonious coexistence and symbiosis of humans with all living things is a philosophical proposition that regards the universe as a biological system and realizes the sustainable development of humans ourselves. Carbon neutralization refers to the balance between the total amount of carbon of human activities emitted to the nature and the amount of natural consumption. It is a reflection of the philosophical proposition of "*the harmony of humans with the nature*" in the Chinese culture. It is a reflection of scientific civilization and human industrialization activities as well as the adjustment of industry and industrial layout, to guide a civilized consumption direction in the mark. During the general debate of 75th Session of the

[1]Received October 5, 2021, Accepted December 20, 2021.

United Nations General Assembly, September 22, 2020, China pledged to improve the country's independent innovation capacity and strive to peak its carbon emissions by 2030 and achieve carbon neutrality by 2060, which is an arduous, also a complex systematic work. According to the statistics, China's carbon emissions accounted for 30.7% of the global carbon emissions in 2020, accounting for 9.899 billion tons, ranking first in the world statistics, with a gap of 5.442 billion tons between China and the United States, which ranked second with 4.457 billion tons of emissions. It should be noted that to close the gap on the second place the target of carbon emission should be reduced with 5.442 million tons, without mention the gap of carbon neutrality.

Then, *what is the carbon emission*? The carbon emission is a general term or abbreviation of greenhouse gas emission, refers to human activities or naturally formed greenhouse gases such as water vapor (H_2O), freon, carbon dioxide (CO_2), nitrous oxide (N_2O), methane (CH_4), ozone (O_3), organofluorines, perfluorocarbons, etc. However, the carbon dioxide is the main contributor, contributing about 60% to greenhouse gases, which is the governance priorities of abatement and control. Certainly, the economic development satisfies the people's ever-growing material and cultural needs but the energy conservation and emission reduction bring benefits also to the people. In this situation, as a whole for the contradictory relationship in the production with consumption, it is a very important thing of humans. Therefore, it is necessary to scientifically analyze the connotation of carbon neutrality, analyze the excess carbon emission of existing industrial production layout, reflect also on the human civilization formed over thousands years, and explore the coordinated development, namely the *humans in harmony with the nature* advocated by the traditional Chinese culture.

§2. Carbon Balance of Substance Flow

All things in the universe form a whole, an organic system that is complementary to each other, and led to a naturally series forms of combination mechanisms such as those of matter made up of molecules, molecules made up of atoms, and atoms made up of elementary particles. Among them, atoms cannot be born and cannot be perished also, can only be converted from one form to another, i.e., matter is immortal or conserved. This fact is reflected in carbon emissions, i.e., the total amount of six greenhouse gases discharged is equal to the total amount of natural consumption and floating in the air, which forms a natural conservation mechanism, namely the carbon emission equivalent to the natural consumption. What the emissions can't absorbed by the nature are floating in the air, form the greenhouse gases and cause the global temperatures to rise.

The substance flow analysis (SFA) is a method to describe the flow status of a specific substance (element or compound) in a specific system [11], and the conservation relationship among the inflow I, inventory R and outflow O on each node of the system follows a conserved equation

$$I = R + O. \tag{1}$$

Here, the carbon dioxide is taken as a substance flow [11] to analyze the circulating process of carbon dioxide in the nature, including the subjects of carbon emission and the way of absorption to maintain the balance mechanism. As we known, the carbon C and water H_2O are the main elements of livings in the earth. The carbon dioxide CO_2 is a molecule composed of carbon C and oxygen O and has three forms, namely the gaseous, liquid and solid. Here, its gaseous is commonly known as carbon dioxide, the liquid carbon dioxide is gaseous liquefaction under high pressure and low temperature and its solid carbon dioxide is liquid rapid coagulation under low pressure, also known as dry ice. The main origins of carbon emissions includes: ①Σ_1:breathing of living animals and plants; ②Σ_2:decomposing, fermenting, decaying and deteriorating of the carcasses of animals and plants; ③Σ_3:ordinary consumption of residents for survival needs; ④Σ_4:coal, petroleum, paraffin, natural gas, wood and other fuels burning; ⑤Σ_5:coal, petroleum and other production of chemical products; ⑥Σ_6:industrial production, especially energy consuming industrial production; ⑦Σ_7:transportation, especially energy-consuming transportation; ⑧Σ_8:residents consumption; ⑨Σ_9: waste disposal, especially waste disposal that consumes a certain amount of energy.

Then, *how are these carbon emissions absorbed*? There are four main types of consumption on carbon emissions that are known in the following.

Type 1. R_1**: photosynthesis.** The photosynthesis refers to the process in which green plants absorb light energy, synthesize carbon dioxide and water into energy-rich organic matter, and release oxygen at the same time including the absorption, transmission and conversion of light energy to electrical energy and electrical energy to chemical energy, as shown in Figure 1. The photosynthesis is the most common and largest natural reaction process on the earth. It plays a natural regulatory role in the synthesis of organic matter, solar energy storage and air purification and it is mainly realized by green plants, including green plants and algae in waters which convert carbon dioxide and water into oxygen and organic matter through photosynthesis. By the respiration, oxygen and organic matter decomposition into carbon dioxide, water in the nature which plays an irreplaceable role in maintaining the natural carbon-oxygen balance. At the same time, plants constantly release oxygen to enrich animal breathing, combustion and other consumption caused by oxygen deficiency in the process of photosynthesis, maintain the balance of oxygen in the nature for humans and other creatures to coexist peacefully to provide material security.

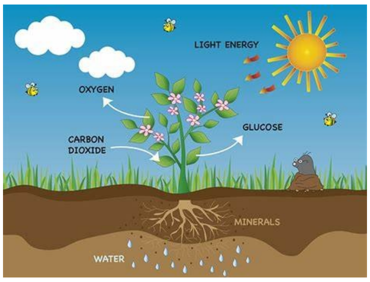


Figure 1. Photosynthesis

Type 2. R_2**: water absorption.** The carbon dioxide dissolved in water will react chemically to form carbonic acid, which is usually applied to make carbonated drinks. The chemical equation is $CO_2 + H_2O \to H_2CO_3$. Notice that the carbon dioxide dissolved in water forms carbonic acid is unstable, easy to decompose and reduce to carbon dioxide and water, i.e., $H_2CO_3 \to CO_2 + H_2O$ floating in water. Certainly, the water absorption of carbon dioxide does not means that carbon dioxide is dissolved in water but that the water, including seawater contains a large number of green unicellular or multicellular plants which absorb carbon dioxide through photosynthesis similar to that of terrestrial green plants.

Type 3. R_3**: soil absorption.** The respiration of microorganisms and plant roots in soil emits a certain amount of carbon dioxide, resulting in a higher concentration of carbon dioxide in soil than in air. The absorption of carbon dioxide in soil mainly occurs in arid and semi-arid regions. Most of these soils are alkaline, containing calcium ions, which react with carbon dioxide to form calcium carbonate. Notice that this consumption eventually forms calcium carbonate, which can lead to soil calcification or desertification, which is not conducive to humans survival.

Type 4. R_4**: humans disposal.** The carbon dioxide produced by industrial production is purposefully collected and sequestered to prevent it from being released into the air.

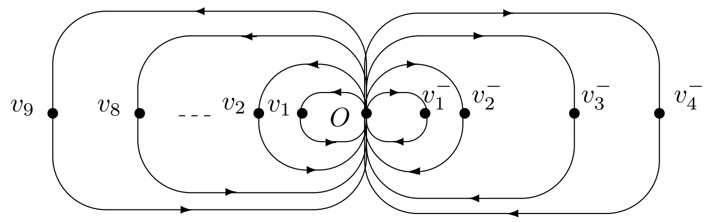


Figure 2. Carbon dioxide flow

The $\overrightarrow{G}-$flow of carbon dioxide is shown in Figure 3, which is in fact a particular case of continuity flow with all end-operators $1_{\mathscr{B}}$ of Banach space $\mathscr{B}$, a globally mathematical element on non-living body introduced in [8] and [9] or mathematical combinatorics. Here, the vertex v_1 refers to the breathing of living animals and plants; v_2 refers to the decomposition, fermentation, decay and deterioration of animals and plant corpses; v_3 refers to the ordinary consumption of residents for survival; v_4 refers to the combustion of coal, petroleum, paraffin, natural gas, wood and other fuels; V_5 refers to the production of chemical products of coal and petroleum, etc.; v_6 is the industrial production, especially energy consuming industrial production; v_7 is the transportation industry, especially energy consuming transportation; v_8 is the residential consumption; v_9 is the waste disposal, especially energy consuming waste disposal and the vertex v_1^- is the consumption by photosynthesis; v_2^- is the water consumption, v_3^- is the soil absorption, v_4^- is the human disposal on carbon dioxide. The input flow of the node O is $\Sigma_i, 1 \leq i \leq 9$, the output flow is $R_k, 1 \leq k \leq 4$ and the node inventory is R_{air}, the amount of carbon dioxide in the air.

Thus, the conservation equation of carbon dioxide flow at node O is ([8-9])

$$\sum_{i=1}^{9} \Sigma_i = \sum_{i=1}^{4} R_i + R_{air}. \tag{2}$$

In general, the amount of carbon dioxide in the air satisfies the condition of $R_{air} \leq R_{human}$, where R_{human} is the threshold of the allowable amount of carbon dioxide in air for human survival. When land, water, soil and human disposal are insufficient to absorb carbon dioxide emissions, resulting in $R_{air} \geq R_{human}$, i.e

$$\sum_{i=1}^{9} \Sigma_i > \sum_{i=1}^{4} R_i + R_{human} \tag{3}$$

and once the R_{air} exceeds R_{human} is no longer suitable for human survival. Even R_{air} is less than R_{human}, it can also induce the extreme natural disasters. For example, $R_{air} \geq 2\%$ is thought to cause the destruction of the atmospheric ozone layer, global temperature rise, ice sheet melting, sea level rise, extreme weather, drought and virus mutations such as SARS and novel Coronavirus (COVID-19) have caused attrition of humans worldwide. A few international public organizations estimate that 5 million people die every year from air pollution, famine and disease caused by climate change and excessive carbon emissions, and this number is expected to rise to 6 million on 2030 if the current patterns of fossil fuel consumption remain unchanged.

Figure 3. Negative consequences of carbon overshoot

§3. Carbon Imbalance Caused by Humans Industrialization

Then, *what is the carbon neutralization*? The carbon neutralization refers to the greenhouse gas emissions directly or indirectly generated by enterprises, organizations or individuals within a certain period of time are all offset to achieve *0 emissions* to nature through by the afforestation, soil organic carbon, carbon sequestration and other forms. That is,

$$\sum_{i=4}^{9} \Sigma_i \leq R_4, \tag{4}$$

which is a human initiative to return to the natural carbon equilibrium

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = R_1 + R_2 + R_3 + R_{air}, \tag{5}$$

at the vertex O in Figure 2.

First, the carbon emissions are balanced in the farming societies. At that time, carbon emissions consisted of animal and plant respiration, decomposition, fermentation, decay, deterioration of animal and plant carcasses, and common consumption for survival, namely Σ_1, Σ_2 and Σ_3. After 13.8 billion years of evolution, the earth became a cyclic ecosystem, meet the basic requirements of carbon and other harmful elements absorption balance.

And then, *when does the carbon balance start to get out of balance*? Due to the large-scale industrial production, carbon emissions increase sharply, which destructed the earth's green vegetation, reduce the ability of natural carbon absorption and damage to the ecological environment. This is expressed in the *Kyoto Protocol* and the *Paris Agreement* under the *United Nations Framework Convention on Climate Change*. For example, the Article 2 of the *Paris Agreement* explicitly requires all signatories to strengthen the global response to the threat of climate change by holding the increase in global average temperature below 2^oC above the pre-industrial levels and striving to limit the increase in temperature to 1.5^oC above pre-industrial levels, recognizing that this will significantly reduce the risks and impacts of climate change. Here, the *pre-industrial level* refers to the farming society (manual workshop) era, that is, to the carbon emission balance at that time as the standard.

The large-scale industrial production is the main *cause* of natural carbon imbalance, while excessive humans consumption is the *effect* of carbon imbalance. Industrial production comes from the development of science and technology, that is, science's cognition of nature is applied in the scope of humans activities, forming large-scale production and especially, the result of *money* directed by the logic of capital. In this production process, those beneficial to consumption and also beneficial to big money by a large number of mining, processing and manufacturing for people's consumption but the products that seem to have no value for consumption such as waste residue, waste gas, waste water and solid waste are consciously or unconsciously thrown to the nature in the belief that it can tolerates all the evil consequences of human actions. And *is the earth's environment still suitable for human life as a result of this natural regression to the carbon imbalance*? The answer is not necessarily! The humans ourselves do not know whether the period of this natural return will be past years, decades or hundreds of years, but it is certain that the return process will be accompanied with frequently natural disasters and the loss or destruction of humans and animals, because *the nature treats everything as mere nothing* without particularly generous to humans.

Here, the carbon emission sources of industrialization processes are analyzed as follows:

(1) Energy production. Energy is the basis of large-scale industrial production, including energy development and production, such as coal, oil, natural gas extraction, storage and electricity production. No matter the energy consumption is primary or secondary, the source is combustion. The chemical reaction equation such as $C + O_2 \to CO_2$, $2C + 2O \to 2CO$, $2CO + O_2 \to 2CO_2$ is accompanied by a large amount of carbon dioxide emissions, consumes lots of oxygen. At the same time, there is a lack of research on whether the massive development of coal, oil and natural gas will upset the earth's other ecological balance in addition to the carbon imbalance. As we know, water is the source of life, water shortage on earth is like human body water loss, lack of oil, lack of gas human body shrivel and even death. Similarly, the existence of things is the reason, the massive exploitation of the earth's coal, oil, natural gas and other fossil energy would result in the imbalance of materials that currently maybe unknown.

(2) Industrial production. Industrial processing, manufacturing and production based on energy consumption with carbon emissions in the process, also with harmful substances. The processes include: ①the produce cement, lime, molten steel, coke, clay brick and other furnace firing; ②the process steel rolling, alloy, metal surface coating; ③the machinery manufacturing; ④the produce chemical fertilizer, medicine, pesticide and paint, food additives refining or production, etc. Among these processes, unless the physical deformation of industrial products such as forging and pressing, a large number of industrial processes involve chemical reactions. The most basic question is whether these processes are reversible. The answer is that a large number of industrial processes are irreversible! Therefore, using industrial reverse processes to

deal with waste including carbon emissions is not feasible for humans. Even if a small amount of industrial process is reversible, according to the law of substance conservation the consumption of material is the same as that of production but the energy required is far greater than that of a production, and it is impossible to produce economic benefits. At this point, the economic benefits obtained by human industrialization are essentially at the cost of the earth's ecological environment.

(3) Construction. The production of building materials production of finished products or semi-finished products such as fire, precast concrete components made of cement, sand and water, the condensation and the chemical reaction, condensation, concrete, mortar, and energy consumption in the process of construction such as construction machinery, electric welding, spray gun, etc., there are carbon dioxide emissions. At the same time, a large number of high-rise buildings, residential pavement and road hardening as well as river hardening, crowded into the living space of green vegetation, carbon dioxide emissions to the upper air but artificially blocked the flow of atmosphere, water and stratum, broke the natural absorption of carbon dioxide process.

(4) Transportation. The fossil-based transportation such as cars, trains, ships and airplanes releases carbon dioxide into the air when coal, oil and natural gas are burned.

(5) Light industry and food processing industry. The light industry and food processing that are based on refining, burning or consuming energy release carbon dioxide in the process.

(6) Agricultural production and household consumption. The agriculture is the production process of green vegetation on earth, theoretically there should be no carbon emissions to the ecological environment pollution. However, accompanying with the industrialization of agriculture, large-scale mechanized operations and product processing, there are also emit pollutants including waste gas, waste water, animal excrement, dust and smoke, including carbon emissions while releasing productivity. In addition, in order to satisfy some people's pursuit of off-season fruits and vegetables, the widespread implementation of *plastic greenhouses* in agricultural production consumes water and electricity and other resources and at the same time, the man-made discarded plastic film is not degraded by the nature, forming the earth white pollution. On the other hand, the planting of green plants such as fruits and vegetables can absorb carbon dioxide from the atmosphere during photosynthesis nature, release oxygen required to human survival but the existence of greenhouses artificially cut off their photosynthesis and oxygen release, this fact reflects also the human's activity is causing the carbon imbalance in the nature.

Here, we take a few data from Chinese carbon emission database in 2019 as an example to show the proportion of carbon emissions from human industrialization activities in the total carbon emissions: ① Power and heat production accounted for 50%; ② Non-metallic mineral products, black metal smelting, processing and other heavy industry accounted for 33%; ③ Transportation, storage and postal service accounted for 8%; ④ Light industry production accounted for 1%; ⑤ Service industry accounted for 2%; ⑥ Agriculture, construction and residential carbon emissions accounted for 6%.

Even so, *stop the industrialization of human activities can return to the natural balance of carbon*? Up to now, the simple industrialization of humans cannot return to the natural carbon balance state because over the past 300 years, the population continues to increase, human's construction land continues to expand, forest and green vegetation reduction, land desertification, water and green plants reduction. In stark contrast, human industrialization has been disturbing the nature for more than 300 years. The continuous increase in the stock of supernaturally absorbed carbon emissions has been broken the carbon balance. The natural restoration needs to last for hundreds or thousands of years, and it is impossible to return to the carbon balance in a short period of time. At the same time, the cessation of industrial production will certainly affect the survival and development of humans and it is impossible to reverse back to the agricultural era. In this case, humans need to decide their own survival, *whether to let go, continue to destroy the earth, destroy humans themselves or humans and the nature coordinated development*? The answer is self-evident because we have only one earth. The hope of finding another planet to live on can only be an illusion because humans and the nature live in symbiosis. What is needed is the self-restraint of humans, including the change of consumption pattern and the realization of zero emissions of pollutants caused by human activities, and the carbon neutrality is humans initiative actions.

§4. **Economic Logic in Carbon neutrality**

The ultimate goal of carbon neutrality is to achieve carbon zero emission through the transition of energy conservation and emission reduction, which is a coordinated development between humans and the nature. The *Guidance of Accelerating the Development of Establishing and Perfecting the Green Low Carbon Circular Economy System* by the State Council of China explicitly proposes that, to implement the carbon amount to ensure peak carbon neutral goal under the premise of full the whole process of implementation, namely the all-directions, the whole process of planning, investment, design, construction, production, circulation, life, green consumption, and the development should be based on efficient use of resources, strict protection of the ecological environment and effective control of greenhouse gas emissions.

Certainly, the key to peaking carbon emissions by 2030 is to adjust the energy arrange and industrial distribution and to conserve energy, reduce emissions and consume green energy [3-4]. It plans to reduce carbon emissions every year from 2035 and become carbon neutral at 2060. The carbon neutral core lies in the fact that after 2035 to green energy as the main power supply, with carbon neutral technology development as the breakthrough, eliminate those industries and enterprises that can't carry out carbon zero emissions, and on the basis of organizational economic benefits to the people, this is a new concept of economic development. It is estimated that 86% of global CO_2 emissions come from fossil fuel use and 14% from land use. Of this, about 46% is retained in the atmosphere, 23% is absorbed by the ocean and about 31% is absorbed by the land. According to the commitment of the *Paris Agreement* signed by China, the carbon peak is to ensure that the rise of carbon emissions in global average temperature is limited to 1.5^oC above pre-industrial level and to control the growth of carbon emissions so as to achieve a steady decline in carbon emissions after 2035.

4.1. Energy Restructuring. Energy industry is the power base of industrial development and the energy distribution is related to industrial adjustment and industrial distribution, also the residents' life. According to official data released by the *National Bureau of Statistics of China*, energy output will be 3.9 billion tons of raw coal, 1.947.69 million tons of crude oil, 1919.25 billion cubic meters of natural gas, and 7.779.06 billion kilowatt-hours of electricity in 2020. For example, according to the statistics of *China Energy Big Data Report* (2021), the installed capacity of domestic thermal power will be 1.25 billion kW, accounting for 56.07%; Hydropower 370 million kw, accounting for 16.6%; Nuclear power 0.4989 billion kW, 2.24%; Grid-connected wind power 280 million kw, accounting for 12.56%; Grid-connected solar power 250 million kW, accounting for 11.21%; Biomass power generation is 0.2952 billion kW, accounting for 1.32 % until 2020. Among them, thermal power accounted for 56.07%.

Then, *how does the domestic energy distribution need to be adjusted before carbon peak 2030*? First, the existing thermal power projects should be maintained and no new projects should be added. We will improve and supervise the environmental protection of existing thermal power projects, urge them to upgrade their technologies, gradually replace coal with gas and reduce pollutant emission targets to meet the needs. At the same time, we will significantly increase the grid connection of hydropower, wind power and photovoltaic power generation and develop clean energy technologies such as hydropower, geothermal energy, marine energy, hydrogen energy, biomass energy and photovoltaic power generation in light on local conditions so as to increase their share in the energy distribution. As mentioned above, thermal power accounts for 56.07% of the installed power generation capacity alone. The replacement of fossil energy and emission reduction tasks are quite heavy.

After 2035, it needs to reduce the use of fossil energy every year. The clean energy technologies such as those of hydropower, geothermal energy, ocean energy, hydrogen, biomass energy and thermal power generation gradually as the main energy supplies in the market and the proportion of fossil energy such as coal, oil, natural gas and other fossil energy reduced year by year, gradually out of the energy market for using clean energy or carbon dioxide emissions of fossil fuels were completely disposed by humans until 2060. It is necessary to change the power configuration mode, adjust or dismantle some power generation and distribution facilities. This means that to achieve carbon neutrality by 2060, the first is to promote the replacement of fossil fuels with clean energy; the second is to develop carbon sequestration technology to collect carbon emissions from fossil energy and sequestration or utilization and the third is to promote the natural absorption of carbon emissions by photosynthesis, restore green plants in land and water including the sides and roofs of buildings and structures built by humans. If necessary, it needs to demolish redundant buildings and structures for planting green plants and open fruit and vegetable greenhouses to absorb carbon emissions from human activities.

4.2. Industrial Layout Adjustment. Energy is the power source of modern industry. Industrial production is realized by energy supply. The adjustment of energy structure will inevitably affect the industrial layout. Therefore, the core of industrial layout adjustment is energy conservation and emission reduction before carbon peak in 2030. On the basis of the existing industrial layout, limit and reduce the energy consumption of heavily polluting and energy-intensive industrial projects such as petrochemical, chemical, iron and steel, smelting, building materials and other industries is the key to achieve carbon peak, whose carbon emissions account for more than 70% of the country's total emissions. After a smooth transition period of carbon peak after 2030, the petrochemical, chemical, iron and steel, smelting and building materials industries will be urged to introduce environmental protection technologies or devices into their projects to carry out technological transformation, so as to realize artificial carbon absorption or storage. If necessary, the production of national defense and necessities shall be maintained while all other projects shall be shut down. At the same time, restore the green living space occupied by these industrial projects, remove or plant green plants on their sides and roofs to absorb carbon dioxide. The core of this period lies in the practical implementation of the *Circular Economy Promotion Law of China*, the implementation of e-cological economic transformation of industrial projects, that is the output of one project is the raw material or semi-finished product of the next or several projects, the establishment of waste disposal centers and the implementation of zero emission of harmful substances including carbon dioxide [5] as shown in Figure 4. At the same time, the closed buildings and structures still need to carry out environmental protection, the demolition of all kinds of items to clean up and resources integration, recycling, in order to build a domestic circular economy system.

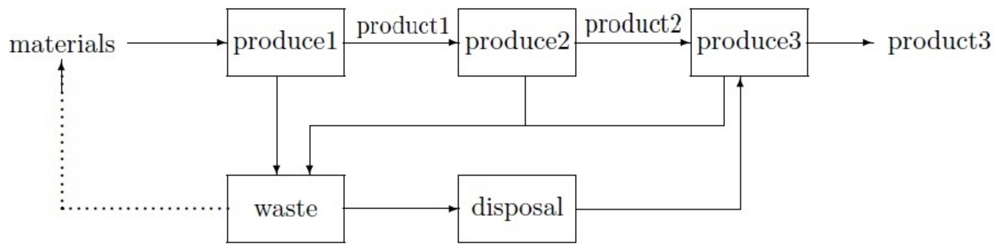


Figure 4. Circular model of economy

It should be noted that the traditional industry is based on fossil energy and a large amount of carbon is emitted in the industrial production, that is, the combustion process of fossil energy. After 2035, to achieve carbon neutrality by 2060, a fundamental question is whether clean energy can replace industrial production from fossil fuels. If not, it is necessary to propose new alternative energy schemes or research carbon absorption or carbon sequestration technologies to achieve zero emissions of carbon from industrial production. If not, the industrial production needs to be scaled down. Let's take the domestic power production as an example, clean energy such as hydropower, wind power and photovoltaic power only accounts for 43.93%, which is a large gap. Moreover, the wind power, photovoltaic power and tidal power are greatly affected by the nature and their power supply is unstable. So few of them can meet the requirements of grid-connection. If there are no new energy alternatives that rely on industrial projects of power supply in order to eliminate, ensure the livelihood of the people power or green electricity can't supply, it should construct the sort of small wind or solar power stations for replacing the gap. However, this way is bound to affect the industrial production, also affect the change of residents' consumption structure, namely driving less or not driving.

4.3. Transportation Restructuring. Under the existing road network it is necessary to develop new energy vehicles. Before the carbon peak in 2030, the electricity, gas or hydrogen can be used as the alternative energy for transportation. Although the electricity and gas still have certain carbon emissions, their emissions are small compared with oil. After 2035, the clean energy should be adopted to gradually replace petroleum-based or electricity-consuming vehicles. Among them, some breakthroughs have been made in using hydrogen energy to replace fuel oil on ships but it will take some time to complete. The core of using electricity to replace fuel lies in whether clean energy can meet the electricity demand of vehicles after a large number of thermal power plants shut down. Notice that the most need to pay attention to the first is the aircraft, the second is high-speed rail because the aircraft use so much fuel, there is no consensus on which energy source to replace them. In the experiment, people hope to use solar energy instead but whether it can meet aviation needs time to test. As we known, the high-speed trains consume electricity directly. It will also take time to test whether the electricity generation can meet the demand of high-speed trains after adopting clean energy.

4.4. Adjustment of Consumption Structure. In the dual relationship of production with consumption, the consumption decides the production and the production promotes consumption. The scale of production and construction content are determined by the basic necessities of life. At the same time, it guides the development of industries that produce for free consumption. According to Maslow's hierarchy of needs, a human first meets the physiological needs such as food and water, and then the safety needs such as personal safety, life stability, freedom from pain and disease were followed by social contact,namely the respect and self-fulfillment. So, *how does carbon peak and carbon neutrality affect consumption structure*?

First of all, in order to achieve the goal of peaking carbon emissions by 2030, consumers need to consciously comply with energy conservation and emission reduction and reduce the consumption of products with high energy consumption in production. For example, ① Dress in accordance with the norms of social behavior, natural and simple prevail, do not pursue too fancy, luxury or strange and let the light textile industry do not pursue too printing and dyeing colors or strange, do not artificially discharge harmful chemicals to nature; ② Take "*eat all off in the plate*" as the standard for food, avoid extravagance and waste or excessive consumption because more and more research makes clear that the rich disease of a person such as those of wait like fat of 3 high disease, coronary heart disease, diabetes, obesity, alcohol liver accompany by the person overeat, eat too good and too fine. Return to the diet to satisfy the stomach, can effectively reduce the rich disease, put an end to the phenomenon of excessive medical treatment in hospitals and return to the way of Chinese traditional medicine; ③ Home decoration and home appliances are both with green environmental protection products, do not pursue luxury and the building materials comply with green environmental protection from the procurement of raw materials to the processing and use of finished or semi-finished building materials and then guide home appliances to save energy, environmental protection, do not emit harmful substances to nature; ④ Advocate green travel, do not drive or buy green vehicles for travelling such as gas vehicles, electric vehicles, etc..

Secondly, after 2035, in order to achieve carbon neutrality by 2060, households need to promote the role of solar energy, biomass energy and other clean energy in using electricity, water, heating and cooking, consciously implement green consumption and carry out carbon recovery or storage so as to achieve zero carbon emissions. Among them, clean energy such as hydropower, solar energy and biomass energy can effectively guarantee family life but the problem of family car is similar to that of the above transportation structure adjustment, that is, whether clean energy can completely replace fossil energy. If not completely replaced, one is to solve the problem of carbon emission recovery consumption and storage in the use of household cars; another is to drive less, do not drive, which presents a challenge for energy storage technology such as the hydrogen cell with long duration.

4.5. Carbon Sequestration Technology. Return to the natural balance of carbon dioxide, namely the equation (2) to achieve carbon neutrality, i.e., the amount of carbon dioxide floating in the air is basically the same as before the industrial activities of humans, the premise is to carry out harmless sequestration and disposal of carbon dioxide emitted by industrial activities of humans. One is the energy conservation and emission reduction, led to change the layout of human production and consumption; another is the developing of carbon dioxide harmless sequestration and disposal technology, including geological sequestration, marine sequestration, mineral sequestration and ecological sequestration.

(1) Geological storage technology. The geological sequestration of carbon dioxide is to inject carbon dioxide as a substance flow into deep underground suitable places for storage through pipeline technology and make use of the gas tightness of geological structures such as oil fields, natural gas storage, salt formations and unrecoverable coal seams, abandoned mines and so on. In special cases, some of the carbon dioxide reacts with the surrounding material to form carbonate minerals. There have been some engineering examples of geological sequestration of carbon dioxide but it is not determined whether the formation structure will be unstable after injection. At the same time, when natural disasters such as earthquakes occur in the region, they can cause massive eruptions of carbon dioxide that can kill people also. For example, when a natural disaster occurred in the crater of a volcano in Cameroon in 1996, the release of carbon dioxide killed more than 1,500 persons, animals living within 14 kilometers are all killed.

(2) Marine storage. The idea of marine sequestration is to use the pipeline technology to transport liquid carbon dioxide to a depth of 1000m-3000m in the sea and use seawater to sequestration carbon dioxide. It is known that the carbon dioxide is normally denser than seawater, would react with seabed material to form a solid or liquid paste of carbon dioxide, slowing its decomposition into the above-ground environment. The problem of carbon dioxide sequestration in the sea is that there is no conclusion on the impact of such sequestration on the marine ecological environment. *Would humans destroy the marine ecological environment while they protect the ground ecological environment*? The answer is certainly not because this would be also a crisis of humans survival.

(3) Mineral storage. The mineral sequestration of carbon dioxide involves the acidification of various naturally occurring minerals with carbon dioxide to obtain stable carbonates, including alkaline and alkaline earth oxides. This is a permanent sequestration method that will not cause harm to humans. It is theoretically feasible but how to make the carbon dioxide emitted by humans react with such minerals has not been solved technically and if just by its natural reaction, the process is very slow and the hardening rate artificially needs to consume a lot of energy, which is not economical.

(4) Ecological conservation. The ecological sequestration refers to the use of plants, green

algae and other autotrophic microorganisms in land and water to photosynthesize with carbon dioxide to realize the conversion to organic carbon under certain conditions, the purpose of storing carbon dioxide, which is actually the nature's way of storing carbon dioxide, an ecological way. Correspondingly, it is necessary to increase the forests, vegetation, land microorganisms, grasslands, crops, tundra and swamps on the land and remove plastic greenhouses so that green crops can photosynthesize with the carbon dioxide, and then promote the natural mechanism of carbon reduction.

§5. Chinese Civilization for Achieving Carbon Neutrality

As the imbalance of carbon balance is caused by the large-scale industrialization of humans, it is necessary to further analyze its underlying reasons so as to realize the carbon neutralization.

5.1. Reflections on Science with Applications. Notice that the large-scale industrial production is based on scientific cognition of nature. However, *does science know things in the nature as a whole or as a part*? The answer is certainly as a partial cognition! *Why local cognition*? Because people's cognition of thing in the nature depends on the six of humans, namely the vision, hearing, smell, touch, taste and consciousness because humans can perceive things but their scope are limited, too large or too small cognition is not clear for humans. For example, we will never know whether the universe is spherical or circular in shape because we cannot fly out of it and whether or not parallel universes are useful in explaining some natural phenomena. Another example is why there is uncertainty principle in quantum mechanics. It is because humans cannot observe the behavior of particles in the microscopic world but can only discover the laws of particles in the range that humans can observe. That is why quantum mechanics uses probability statistics as a tool to study the wide range of possibilities of particles rather than certainty. This confirms from one side that human beings are only knowing the objective existence that can be observed, that is, humans are partially but not completely knowable to natural things ([6-10]).

5.2. Chinese Civilization to Carbon Neutrality. The humans and the nature constitute a binary system which is similar to the predator-prey model in biological world, i.e. they interact and influence each other. In this binary system, human activities on the natural too intrusive result in the nature adjusting the universe to follow the natural laws, including global warming, drought, severe weather, floods, virus and other natural phenomena, the essence of which is natural in the adjustment, adapt to the universe of conservation of mass, the natural reaction of the nature to human activities including cutting human to return to a new balance. In this case, what needs to be changed is for humans to correctly understand the relationship of humans with the nature and change themselves actively because humans can never compete with nature.

The view that human's cognition of the nature is a partial cognition has existed in the middle ages of Chinese culture. For example, Laozi said at the beginning of his *Tao Te Ching*: "*Tao that experienced is not the eternal Tao; Name named is not the eternal name* " is to indicate this point ([1-2]). A few of humans think that there are no science in ancient China.

This claim is sure if you think science is the local recognition on natural things and did not develop science in ancient China. However, it is a systemic recognition on the nature by Chinese because the Chinese know that our understanding of the universe is local, the application of science must include two aspects, i.e., one is beneficial to human society and meets the needs of human survival; another, it should not destroys the ecological environment of the earth and even the nature. In other words, science and application also constitute a binary system. They complement each other and are a kind of unity of opposites. That is, there are indeed sciences in ancient China, a system science in the eyes of today's human.

The carbon imbalance is the result of economic development without acknowledging the limitations of scientific cognition and unilaterally exaggerating the role of science. Because in the economic society, especially in the era that pursuit of profits by capital, most humans pursue *selfish interests*. In this situation, it is easy to exaggerate a scientific achievement that is beneficial to human society while ignoring its harmful side to the nature, which led lots of humans in the pursuit of economic benefits under the premise of unrestrained abuse.

Notice that the human science is a partial knowledge of nature, *how to use scientific results is the correct application*? Laozi pointed out "*the human follows the earth, the earth follows the universe, the universe follows the Tao and the Tao follows only itself* " in Chapter 25 of *Tao Te Ching*, as a way of survival for humans on the partial cognition, which requires human self-restraint. The word *follows* is generally interpreted as *imitation*. That is, to live on the earth, people need to follow the living rules of things on the earth, the earth needs to follow the rules of heaven, the heaven needs to follow the rules of Tao, and the Tao needs to follow the rules of the nature. In this way, the universal things can live altogether. However, according to the viewpoint that humans with the nature constitute a binary system, the *follow* besides *imitation*, also means *controlled or restricted* in fact, i.e., humans' behavior is restricted by the earth, and the movement of the earth is restricted by the heaven, the movement of heaven is limited by Tao, which derives from nature. Just imagine, the excess carbon emissions caused by human industrialization is due to the misunderstanding of the relationship of humans with the nature, believing that *science*, the partial knowledge of the nature can control the universe for humans' *selfish interests*. I think this is the top issue that needs to review for thousands years of human civilization in the carbon emissions!

Then, *how should humans consume in harmony with nature*? The answer is implied in the sentence of *Tao Te Ching*: "*the five colors blind the eye, the five sounds deafen the ear, the five flavors numb the taste, the maddening to ride the field and the desires wither the heart.*" That is to say, all things in the world are dazzling and easy to arouse people's private desires and go astray, and urge human to follow the "*saints* " because saint's eating is "*for the belly but not for the eyes* ". By contrast today's excess emissions of carbon is just the result of the pursuit of the *blind*, *deaf*, *taste* or *crazy* in the human civilization, stated in the *Tao Te Ching* [1].

Human civilization is not the resource plunder or war between humans or between humans and other creatures. Conservation is the premise of sustainable development of humans, and the nature will never allow any single creature to dominate the universe. For example, the unexplained destruction of dinosaurs. Chinese culture holds with the theory that "*humans are an integral part of the nature*" because the ancient Chinese realized that humans and the nature consist of a system. The standing of humans in the universe is insignificant, which requires humans' excellent virtue in living and hold with "*virtues the same as the heaven and the earth, appearing or showing the same as the sun and the moon, order the same as the four seasons and well or evil the same as the ghosts or the god*" [12]. The idea of carbon neutrality is nothing more than a retelling of the ancient Chinese notion of the harmony in humans with the nature. Consequently, to achieve the goal of carbon peak, the reduction of energy and carbon emission is needed. However, the fundamental way to achieve the carbon neutrality lies in the harmony of humans with the nature, advocated by Chinese civilization because the carbon neutrality can be only truly realized on the earth in this way.

References

[1] Peirong Fu, *Fu Peirong interprets Laozi* (in Chinese), Oriental Publishing House, 2012.

[2] Peirong Fu, *Fu Peirong interprets the mean in universities* (in Chinese), Oriental Publishing House, 2012.

[3] Linfan Mao, Make clear the implementation plan and practice green procurement (in Chinese), *Bulletin of the Government Procurement Information*, 2014-07-07.

[4] Linfan Mao, Industrial green procurement technology program and optimization model (in Chinese), *Tendering and Bidding*, Issue 10, 2014.

[5] Linfan Mao, A topological model for ecologically industrial systems, *International J.Math. Combin.*, Vol.1(2014), 109-116.

[6] Linfan Mao, Biological n-system with global stability, *Bull.Cal.Math.Soc.*, Vol.108, 6(2016), 403-430.

[7] Linfan Mao, Mathematical 4th Crisis: to Reality, *International J.Math. Combin.*, Vol.3(2018), 147-158.

[8] Linfan Mao, *Mathematical Reality - My Philosophy on Mathematics with Reality*, the Education Publisher Inc., USA, 2018.

[9] Linfan Mao, Science's dilemma – a review on science with applications, *Progress in Physics*, Vol.15, 2(2019), 78–85.

[10] Linfan Mao, Harmonic flow's dynamics on animals in microscopic level with balance recovery, *International J.Math.Combin.*, Vol.1(2019), 1-44.

[11] Q.Yang, *Animal Histology and Embryology*(2nd Edition), China Agricultural University Press, Beijing, 2018.

[12] Zengwei Yuan and Jun Bi, *Industrial Ecology*, Science Press, Beijing, 2010.

Papers Published in IJMC, 2021

Vol.1,2021

Vol.2,2021

Vol.3,2021

Vol.4,2021

Famous Words

There is nothing either good or bad but thinking makes it so.

By William Shakespeare, a writer, prominent dramatist and poet of England.

Author Information

Submission: Papers only in electronic form are considered for possible publication. Papers prepared in formats, viz., .tex, .dvi, .pdf, or.ps may be submitted electronically to one member of the Editorial Board for consideration in the **International Journal of Mathematical Combinatorics** (*ISSN 1937-1055*). An effort is made to publish a paper duly recommended by a referee within a period of $3-4$ months. Articles received are immediately put the referees/members of the Editorial Board for their opinion who generally pass on the same in six week's time or less. In case of clear recommendation for publication, the paper is accommodated in an issue to appear next. Each submitted paper is not returned, hence we advise the authors to keep a copy of their submitted papers for further processing.

Abstract: Authors are requested to provide an abstract of not more than 250 words, latest Mathematics Subject Classification of the American Mathematical Society, Keywords and phrases. Statements of Lemmas, Propositions and Theorems should be set in italics and references should be arranged in alphabetical order by the surname of the first author in the following style:

Books

[4]Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, InfoQuest Press, 2009.

[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

Figures: Figures should be drawn by TEXCAD in text directly, or as EPS file. In addition, all figures and tables should be numbered and the appropriate space reserved in the text, with the insertion point clearly indicated.

Copyright: It is assumed that the submitted manuscript has not been published and will not be simultaneously submitted or published elsewhere. By submitting a manuscript, the authors agree that the copyright for their articles is transferred to the publisher, if and when, the paper is accepted for publication. The publisher cannot take the responsibility of any loss of manuscript. Therefore, authors are requested to maintain a copy at their end.

Proofs: One set of galley proofs of a paper will be sent to the author submitting the paper, unless requested otherwise, without the original manuscript, for corrections after the paper is accepted for publication on the basis of the recommendation of referees. Corrections should be restricted to typesetting errors. Authors are advised to check their proofs very carefully before return.

December 2021

Contents

An International Journal on Mathematical Combinatorics

ISSN 1937-1055