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Coefficients of the Chromatic Polynomials of Connected Graphs

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Abstract: In this article, we provide some new results on the divisibility of certain coefficients of the chromatic polynomials of connected graphs, and characterize connected non-planar graphs of order n that minimize the absolute values of coefficients of the chromatic polynomials. Moreover, we also present a sufficient condition that a graph is planar in term of certain coefficients of its chromatic polynomial.

Key Words: Chromatic number, independent set, chromatic polynomial, uniquely k-colorable graph, a_i-minimum.

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§1. Introduction

In this article, all graphs considered are finite, simple and connected. Throughout this article, n and m will always denote, respectively, the number of vertices and the number of edges in a graph G. Terminologies and notations not explained here can be found in [1,2].

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the sets of vertices and edges of G, respectively. A λ-coloring of G is a function $\varphi : V(G) \to \{1, 2, \cdots, \lambda\}$ such that $\varphi(u) \neq \varphi(v)$ for any $uv \in E(G)$. In fact, a λ-coloring of G is also a partition of the vertex set $V(G)$ into λ classes where each class is exactly an independent set of vertices. The chromatic number $\chi(G)$ is the smallest λ for which G has a λ-coloring.

Let $P(G, \lambda)$ denote the chromatic polynomial of G. Whitney [10] showed that

$$P(G, \lambda) = \sum_{i=1}^{n} (-1)^{n-i} a_i(G) \lambda^i, \tag{1}$$

where $a_i(G)$ counts the number of spanning subgraphs of G that has exactly $n - i$ edges and that contain no broken cycles. The coefficient $a_1(G)$ in (1) is interesting in its own right. Read [6] first observed that G is connected iff $a_1(G) \geq 1$. Eisenberg [4] noted that G is a tree iff $a_1(G) = 1$. More than ten years later, Hong [5] proved that

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(1) if G is connected, then $a_1(G)$ is divisible by $(\chi(G)-1)!$ and

(2) G is connected and bipartite iff $a_1(G)$ is odd.

Let $\lfloor x\rfloor$ be the largest integer not exceeding the real x. For any integers k and i with $k \geq 0$ and $i > 0$, define

$$\phi(k,i) = \begin{cases} 1 & if\ k < 2i, \\ \lfloor k/i\rfloor!\phi(k-\lfloor k/i\rfloor, i) & if\ k \geq 2i. \end{cases} \qquad (2)$$

Recently, Dong obtained some results on the divisibility of certain $a_i(G)$ as follows:

Lemma 1.1([3]) *For any graph G, $a_i(G)$ is divisible by $\phi(\chi(G)-1, i)$ for each $i = 1, 2, \cdots, \chi(G)-1$.*

Lemma 1.2([3]) *Let G be a uniquely k-colorable graph. If $k = ip$ for some prime p and positive integer i, then $a_i(G)$ is not divisible by p.*

Lemma 1.3([3]) *For any uniquely k-colorable graph G, where $k \geq 2$, $a_1(G)$ is not divisible by $k!$.*

Obviously, they are an extension of Hong's results mentioned above. And by Lemma 1.1, if one may find the value of $\phi(\chi(G)-1, i)$, the divisibility of $a_i(G)$ will be a better understand. But it is very difficult for us.

Let $\Omega(n,m)$ be the family of all connected graphs with n vertices and m edges. For a given i, we say that a graph $G \in \Omega(n,m)$ is a_i-minimum if $a_i(G) \leq a_i(G')$ for all $G' \in \Omega(n,m)$. Since $a_n(G) = 1$ and $a_{n-1}(G) = m$ for each $G \in \Omega(n,m)$, all graphs in $\Omega(n,m)$ are both a_n-minimum and a_{n-1}-minimum. For any a_i-minimum $(1 \leq i \leq n-2)$, results on this subject can be found in [7–9].

Here we focus on the divisibility of certain $a_i(G)$ and a_i-minimum $(1 \leq i \leq n-2)$.

In section 2 we prove that for any graph G with $\chi(G) = k \geq 2$, $a_1(G)$ is divisible by $k!$ iff b_k is divisible by k, where b_k is the number of ways of partitioning $V(G)$ into k non-empty color classes (or independent sets); for any two positive integers k and i, if $\chi(G) \geq ki+1$, then $a_i(G)$ is divisible by $k!$.

In section 3 we prove that for the family $\mathcal{Q}_1$ (or $\mathcal{Q}_2$) of all non-planar connected graphs of order n containing K_5 (*or* $K_{3,3}$), if G is a graph obtained from K_5 (*or* $K_{3,3}$) by recursively attaching $n-5$ (or $n-6$) leaves, then G is a_i-minimum for each $1 \leq i \leq n-1$. And it is also shown that if G is a connected graph with $a_1(G) < 24$ or $a_2(G) < 50$, then G is a planar graph.

§2. Divisibility of Certain Coefficients

Based on Dong's results we now deal with the divisibility of certain coefficients of the chromatic polynomials.

Although it is difficult from (2) to determine the value of $\phi(k,i)$, we have the following.

Theorem 2.1 *For any graph G, if $\chi(G) = 2i+1$, then $a_i(G)$ is divisible by 2.*

Proof Since $\chi(G) = 2i + 1$, $\lfloor \frac{\chi(G)-1}{i} \rfloor ! = 2$ and $\phi(2i, i) = 2\phi(2i-2, i)$. By (2) we have

$$\phi(2i-2, i) = 1.$$

Thus, $\phi(2i, i) = 2$. By Lemma 1.1, the theorem follows. □

In what follows we present a more general result.

Theorem 2.2 *For any graph G and any two positive integers k and i, if $\chi(G) \geq ki + 1$, then $a_i(G)$ is divisible by $k!$.*

Proof Since $\chi(G) \geq ki + 1$, $\frac{\chi(G)-1}{i} \geq k$. It is clear that $\lfloor \frac{\chi(G)-1}{i} \rfloor !$ is divisible by $k!$. By Lemma 1.1 and (2), the theorem holds. □

If $i = 1$ in Theorem 2.2, then we have

Corollary 2.1. *For any graph G with $\chi(G) \geq k$, $a_1(G)$ is divisible by $(k-1)!$.*

It is easy to see that Hong's result (1) follows immediately from Corollary 2.1.

In addition, if $i = k = 2$ in Theorem 2.2, then we get

Corollary 2.2. *For any graph G with $\chi(G) \geq 5$, $a_2(G)$ is even.*

Since $\chi(K_5) = 5$, we yield the following result.

Corollary 2.3. *If G is a non-planar graph containing K_5, then $a_2(G)$ is even.*

It is well known that the chromatic polynomial $P(G, \lambda)$ can also be expressed in factorial form as follows:

$$P(G, \lambda) = \sum_{j=\chi(G)}^{n} b_j (\lambda)_j, \tag{3}$$

where n is the order of G,

$$(\lambda)_j = \lambda(\lambda-1)\cdots(\lambda-j+1) = P(K_j, \lambda)$$

and b_j is the number of ways of partitioning $V(G)$ into j non-empty color classes (or independent sets).

Theorem 2.3 *For any graph G with $\chi(G) = 4$, $a_2(G) \equiv b_4$ (mod 2).*

Proof Let G be a graph of order n with $\chi(G) = 4$. Combining (1) with (3), we have

$$a_i(G) = \sum_{j=\chi(G)}^{n} (-1)^{n-j} b_j a_i(K_j). \tag{4}$$

Thus,

$$a_2(G) = \sum_{j=4}^{n} (-1)^{n-j} b_j a_2(K_j).$$

Observe that $\chi(K_j) = j$, by Corollary 2.2, $a_2(K_j)$ is divisible by 2 for $j \geq 5$. Therefore $a_2(G)$ and $b_4 a_2(K_4)$ have the same parity.

One may find that $P(K_4, \lambda) = \lambda^4 - 6\lambda^3 + 11\lambda^2 - 6\lambda$. Hence $a_2(K_4) = 11$. It means that $b_4 a_2(K_4)$ and b_4 have the same parity. So have $a_2(G)$ and b_4. The theorem follows. □

A graph G is called uniquely k-colorable if $\chi(G) = k$ and there exists a unique way of partitioning $V(G)$ into k color class. We now have the following.

Corollary 2.4 *For any uniquely 4-colorable graph G, $a_2(G)$ is odd.*

Proof Since G is uniquely 4-colorable, we yield that $b_4 = 1$. It is clear that b_4 is odd. Combining this with Theorem 2.3, the corollary holds. □

Theorem 2.4 *For any graph G with $\chi(G) = k \geq 2$, $a_1(G)$ is divisible by $k!$ iff b_k is divisible by k.*

Proof By (4) we have

$$a_1(G) = \sum_{j=k}^{n} (-1)^{n-j} b_j a_1(K_j).$$

Consider that $P(K_j, \lambda) = (\lambda)_j$ and $a_1(K_j) = (j-1)!$. Thus, $a_1(K_j)$ is divisible by $k!$ if and only if $j > k$.

If b_k is divisible by k, then $(k-1)! b_k$ is divisible by $k!$. It is equivalent to $b_k a_1(K_k)$ divisible by $k!$. Consider that $a_1(K_j)$ is divisible by $k!$ for $j > k$, this results in $a_1(G)$ being divisible by $k!$. And it is clear that the process is reversible. Hence the theorem holds. □

For any uniquely k-colorable graph G, obviously $b_k = 1$ and $\chi(G) = k$. Thus, we have the following.

Corollary 2.5([3]) *For any uniquely k-colorable graph G, where $k \geq 2$, $a_1(G)$ is not divisible by $k!$.*

By Corollaries 2.1 and 2.5, we get

Corollary 2.6([3]) *If G is a uniquely k-colorable graph, where $k \geq 2$, then $a_1(G)$ is divisible by $(k-1)!$ but not by $k!$.*

If $k \geq 3$ in Corollary 2.6, then $a_1(G)$ is divisible by 2. Hence we have the following.

Corollary 2.7 *For any uniquely k-colorable graph G, where $k \geq 3$, $a_1(G)$ is even.*

Remark. One may construct a graph G with $\chi(G) = k$ such that $a_1(G)$ is even, but G is not uniquely k-colorable. As shown in Fig.1, we have $\chi(G) = 3$ and $a_1(G) = 2$. Obviously it is not uniquely 3-colorable.

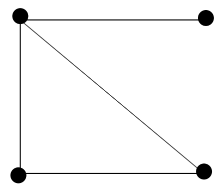

Fig. 1 $\quad (P(G, \lambda) = \lambda^4 - 4\lambda^3 + 5\lambda^2 - 2\lambda)$

§3. Least coefficients

We now concentrate on non-planar graph with least coefficients. For convenience, here we agree that

$$\sum_{i\in\emptyset} x_i = 0.$$

Let H be a graph of order r and $\mathscr{A}$ denote the set of graphs obtained from H by recursively attaching $n-r$ leaves. Then we have

Theorem 3.1 *For any graph $G \in \mathscr{A}$, $a_i(G) = \sum_{t=0}^{i-1} \binom{n-r}{t} a_{i-t}(H)$ $(i = 1, 2, \cdots, n)$.*

Proof It is clear that $P(G, \lambda) = (\lambda - 1)^{n-r} P(H, \lambda)$. Combining this with (1), one may find that

$$\begin{aligned}\sum_{i=1}^{n}(-1)^{n-i}a_i(G)\lambda^i =&(\lambda-1)^{n-r}\sum_{j=1}^{r}(-1)^{r-j}a_j(H)\lambda^j\\ =&\sum_{j=1}^{r}\sum_{t=0}^{n-r}(-1)^{n-(t+j)}\binom{n-r}{t}a_j(H)\lambda^{t+j}\\ =&\sum_{i=1}^{n}\sum_{t=0}^{i-1}(-1)^{n-i}\binom{n-r}{t}a_{i-t}(H)\lambda^i.\end{aligned}$$

Thus, we obtain that

$$a_i(G) = \sum_{t=0}^{i-1}\binom{n-r}{t}a_{i-t}(H)$$

for $i = 1, 2, \cdots, n$, as required. □

It follows immediately from Theorem 3.1 that

Corollary 3.1 *If G is a graph obtained from H by recursively attaching $n-r$ leaves, then $a_i(G) \geq a_i(H)$ for $1 \leq i \leq r$.*

In addition, it is easy to see that $a_1(G) = a_1(H)$ and $a_2(G) = a_2(H) + (n - r)a_1(H)$.

For any graph G and $e = uv \in E(G)$, let $G - e$ denote the graph obtained from G by deleting the edge e and $G \cdot e$ denote the graph obtained from G by identifying u and v and replacing all multi-edges by single ones. The following fundamental reduction formula is well known.

Lemma 3.1([6]) *Let G be a graph and $e \in E(G)$. Then*

$$P(G, \lambda) = P(G - e, \lambda) - P(G \cdot e, \lambda). \tag{5}$$

It follows from (5) that

$$a_i(G) = a_i(G - e) + a_i(G \cdot e) \tag{6}$$

for any integer $i \geq 1$.

It is clear that $a_i(G) \geq a_i(G - e)$ and $a_i(G) \geq a_i(G \cdot e)$ for each $i = 1, 2, \cdots, n$. Thus, we have the following.

Corollary 3.2 *Let G' be a connected subgraph of G. Then $a_i(G') \leq a_i(G)$ for $i = 1, 2, \cdots, |V(G')|$, where $|V(G')|$ is the number of vertices of G'.*

Proof Let n' be the order of G'. Assume that H is the spanning subgraph of G obtained from G' by recursively attaching $n - n'$ leaves. It is clear that H can be obtained from G by recursively deleting all edges $e \in E(G) - E(H)$. By repeatedly applying (6), we obtain that $a_i(G) \geq a_i(H)$ for all integer $i \geq 1$. Again by Corollary 3.1, we yield that $a_i(H) \geq a_i(G')$ for $1 \leq i \leq |V(G')|$. Therefore $a_i(G) \geq a_i(G')$ for $i = 1, 2, \cdots, |V(G')|$, as desired. □

The following two results can be obtained immediately from (6).

Corollary 3.3 *Let H be a graph obtained from G by subdividing an edge e of G. Then $a_i(H) \geq a_i(G)$ for $i = 1, 2, \cdots, n$. Specially, $a_1(H) \geq a_1(G)$, and equality holds iff $G - e$ is disconnected.*

Corollary 3.4 *If H be a subdivision of G, then $a_i(H) \geq a_i(G)$ for $i = 1, 2, \cdots, n$.*

Corollary 3.5 *Let G be a non-planar graph. Then*

(1) *If G has a subgraph which is a subdivision of K_5, $a_1(G) \geq 24$ and $a_2(G) \geq 50$;*

(2) *If G has a subgraph which is a subdivision of $K_{3,3}$, $a_1(G) \geq 31$ and $a_2(G) \geq 78$.*

Proof Let H be the subgraph of G which is a subdivision of K_5. Obviously, H is connected. By Corollary 3.4, we yield that $a_i(H) \geq a_i(K_5)$ for $i = 1, 2$. Again by Corollary 3.2, we obtained $a_i(G) \geq a_i(H)$ for $i = 1, 2$. Thus, $a_i(G) \geq a_i(K_5)$ for $i = 1, 2$. Observe that $P(K_5, \lambda) = \lambda^5 - 10\lambda^4 + 35\lambda^3 - 50\lambda^2 + 24\lambda$. Hence $a_1(G) \geq 24$ and $a_2(G) \geq 50$, as required.

Similarly, one can prove that (2) is true. This completes the proof of the corollary. □

Let $\mathscr{B}$ be the family of all connected graph with n vertices. For a given i, a graph $G \in \mathscr{B}$

is called a_i-minimum if $a_i(G) \le a_i(G')$ for all $G' \in \mathscr{B}$. It is clear that any tree of order n is a_i-minimum.

Let $\mathcal{Q}_1$ (or $\mathcal{Q}_2$) be the family of all non-planar graphs of order n containing K_5 (or $K_{3,3}$), and let $\mathcal{R}_1$ (or $\mathcal{R}_2$) be the family of graphs obtained from K_5 (or $K_{3,3}$) by recursively attaching $n-5$ (or $n-6$) leaves. By the proof of Theorem 3.1, it is easy to see that the graphs from $\mathcal{R}_1$ (or $\mathcal{R}_2$) are χ-equivalence.

Theorem 3.2 *Let G be a graph in $\mathcal{R}_k$. Then G is a_i-minimum in $\mathcal{Q}_k$ (k=1,2).*

Proof Assume that graph $G \in \mathcal{R}_1$. For any $H \in \mathcal{Q}_1$, since H is connected and contains K_5, there must be a connected generating subgraph $H' \in \mathcal{R}_1$ of H. By Corollary 3.2, we have $a_i(H') \le a_i(H)$. Observe that the graphs from $\mathcal{R}_1$ are χ-equivalence and $G, H' \in \mathcal{R}_1$. Thus, $a_i(G) = a_i(H') \le a_i(H)$. It means that G is a_i-minimum in $\mathcal{Q}_1$.

Similarly, one may prove that if $G \in \mathcal{R}_2$, G is a_i-minimum in $\mathcal{Q}_2$. □

Corollary 3.6 *Let G be a non-planar graph. Then*

(1) *if G contains K_5, $a_1(G) \ge 24$ and $a_2(G) \ge 50$;*

(2) *if G contains $K_{3,3}$, $a_1(G) \ge 31$ and $a_2(G) \ge 78$.*

Proof Let G be a non-planar graph of order n. If G contains $K_{3,3}$, then $G \in \mathcal{Q}_2$. By Theorem 3.2, there is a graph $H \in \mathcal{R}_2$ such that $a_i(H) \le a_i(G)$ for $i = 1, 2$. By Corollary 3.1, one may find that $a_i(H) \ge a_i(K_{3,3})$ for $i = 1, 2$. It is well known that $P(K_{3,3,}, \lambda) = \lambda^6 - 9\lambda^5 + 36\lambda^4 - 75\lambda^3 + 78\lambda^2 - 31\lambda$. Thus, $a_1(G) \ge 31$ and $a_2(G) \ge 78$.

A similar argument can be used to establish (1). □

By Corollary 3.6, one may obtain a sufficient condition that a graph is a planar graph in terms of coefficients of its chromatic polynomial.

Corollary 3.7 *If G is a connected graph with $a_1(G) < 24$ or $a_2(G) < 50$, then G is a planar graph.*

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