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On Right Distributive Torian Algebras

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Abstract: Torian algebras were introduced in [7]. In this paper, torian algebras $(X;*,0)$ which satisfy the condition $(y*z)*x=(y*x)*(z*x)$ for all $x,y,z\in X$ (called right distributive torian algebras) are studied. Their properties are investigated. It is shown that every right distributive torian algebra fixes its zero element. Moreover, necessary and sufficient conditions for a torian algebra to be right distributive are also presented.

Key Words: Torian algebras, right distributivity, Smarandachely torian algebra.

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§1. Introduction

In recent times, the study of algebras of type (2,0) has generated interest among mathematicians. Kim and Kim, in [1] introduced the notion of BE-algebras. In [2] and [3], Ahn and So introduced the notions of ideals and upper sets in BE-algebras and investigated related properties. In [6] and [7], Ilojide introduced the notions of obic algebras and torian algebras. The notion of ideals in torian algebras was also introduced and studied in [8]. In this paper, torian algebras $(X;*,0)$ which satisfy the condition $(y*z)*x=(y*x)*(z*x)$ for all $x,y,z\in X$ (called right distributive torian algebras) are studied. Their properties are investigated. It is shown that every right distributive torian algebra fixes its zero element. Moreover, necessary and sufficient conditions for a torian algebra to be right distributive are also presented.

§2. Preliminaries

Definition 2.1([6]) *A triple $(X;*,0)$; where X is a non-empty set, $*$ a binary operation on X, and 0 a constant element of X is called an obic algebra if the following axioms hold for all $x,y,z\in X$:*

(1) $x*0=x$;
(2) $[x*(y*z)]*x=x*[y*(z*x)]$;
(3) $x*x=0$.

Example 2.1([6]) Consider the multiplicative group $G=\{1,-1,i,-i\}$. Define a binary operation $*$ on G by $a*b=ab^{-1}$. Then $(G;*,1)$ is an obic algebra.

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Lemma 2.1([6]) *Let X be an obic algebra. Then for all $x, y \in X$, the following hold:*

$$x * y = [x * (y * x)] * x.$$

Definition 2.2([7]) *An obic algebra X is called torian if $[(x * y) * (x * z)] * (z * y) = 0$ for all $x, y, z \in X$. Otherwise, if there are $x, y, z \in X$, such that $[(x * y) * (x * z)] * (z * y) \neq 0$, such an obic algebra X is called Smarandachely torian.*

Lemma 2.2([7]) *Let X be a torian algebra. Then the following hold for all $x, y, z \in X$:*

$$(x * y) * z = (x * z) * y.$$

Definition 2.3([7]) *Let X be a torian algebra. An element $x \in X$ is said to fix 0 if $0 * x = 0$. If every element in X fixes 0, then X is said to fix 0.*

Lemma 2.3([7]) *Let X be a torian algebra. Define the relation $\sim$ on X by $x \sim y \Leftrightarrow x * y = 0$ for all $x, y \in X$. Then $(X; \sim)$ is a partially ordered set.*

Lemma 2.4([8]) *Let X be a torian algebra with the partial ordering $\sim$. Then, $[(x*y)*(z*y)] \sim (x * z)$ for all $x, y, z \in X$.*

Definition 2.4([7]) *A torian algebra X is called a weak property torian algebra (WPTA) if $x * y = 0$ and $y * x = 0$ imply that $x = y$ for all $x, y \in X$.*

Proposition 2.1([7]) *Let X be a WPTA. Then for all $x, y, z \in X$, the following hold:*

$$x * [x * (x * y)] = x * y.$$

Lemma 2.5 *Let X be a torian algebra with partial ordering $\sim$. Then $(x * y) \sim z \Leftrightarrow (x * z) \sim y$ for all $x, y, z \in X$.*

From now on, X will denote a weak property torian algebra.

§3. Main Results

Definition 3.1 *Let X be a torian algebra. An element $x \in X$ is said to be right distributive in X if $(y * z) * x = (y * x) * (z * x)$ for all $y, z \in X$.*

Example 3.1 For any torian algebra X, 0 is right distributive in X.

Remark 3.1 If every element in a torian algebra X is right distributive in X, then X is said to be a right distributive torian algebra.

The following Lemma follows from definition.

Lemma 3.1 *Let X be a right distributive toran algebra. Then the following hold for all*

$x, y, z \in X$:

(1) $(0 * z) * x = (0 * x) * (z * x)$;
(2) $y * x = (y * x) * (0 * x)$;
(3) $0 * x = 0$;
(4) $(x * z) * x = 0 * (z * x)$;
(5) $0 * z = 0 * (z * x)$;
(6) $(y * z) * z = y * z$;
(7) $(y * x) * z = (y * x) * (z * x)$;
(8) $[(0 * x) * z] * x = [(0 * x) * x] * (z * x)$;
(9) $(y * x) = (y * x) * [(0 * x) * x]$;
(10) $(x * z) * x = (0 * x) * (z * x)$;
(11) $(0 * x) * z = (0 * x) * (z * x)$;
(12) $(x * z) * x = 0$.

Proposition 3.1 *Let X be a right distributive torian algebra. Then the following hold for all $x, y, z \in X$:*

(1) $(0 * x) * [[z * (x * z)] * z] = (0 * z) * x$;
(2) $[y * (x * y)] * y = [[y * (x * y)] * y] * (0 * x)$;
(3) $[[x * (z * x)] * x] * x = 0 * [[z * (x * z)] * z]$;
(4) $0 * z = 0 * [[z * (x * z)] * z]$;
(5) $[[y * (z * y)] * y] * z = [y * (z * y)] * y$;
(6) $[[y * (x * y)] * y] * z = [[y * (x * y)] * y] * [z * (x * z)] * z$;
(7) $[(0 * x) * x] * [[z * (x * z)] * z] = [(0 * x) * z] * x$;
(8) $[y * (x * y)] * y = [[y * (x * y)] * y] * [(0 * x) * x]$;
(9) $[[x * (z * x)] * x] * x = (0 * x) * [[z * (x * z)] * z]$;
(10) $(0 * x) * z = (0 * x) * [[z * (x * z)] * z]$;
(11) $[[x * (z * x)] * x] * x = 0$.

Proof The proof follows from Lemmas 2.1 and 3.1. □

Proposition 3.2 *Let X be a right distributive torian algebra. Then the following hold for all $x, y, z \in X$:*

(1) $(0 * x) * [z * [z * (z * x)]] = (0 * z) * x$;
(2) $y * [y * (y * x)] = [y * [y * (y * x)]] * (0 * x)$;
(3) $[x * [x * (x * z)]] * x = 0 * [z * [z * (z * x)]]$;
(4) $0 * z = 0 * [z * [z * (z * x)]]$;
(5) $[y * [y * (y * z)]] * z = y * [y * (y * z)]$;
(6) $[y * [y * (y * x)]] * z = [y * [y * (y * x)]] * [z * [z * (z * x)]]$;
(7) $[(0 * x) *] * [z * [z * (z * x)]] = [(0 * x) * z] * x$;
(8) $y * [y * (y * x)] = [y * [y * (y * x)]] * [(0 * x) * x]$;
(9) $[x * [x * (x * z)]] * x = (0 * x) * [z * [z * (z * x)]]$;
(10) $(0 * x) * [z * [z * (z * x)]] = (0 * x) * z$;

(11) $[x * [x * (x * z)]] * x = 0$.

Proof The proof follows from Proposition 2.1 and Lemma 3.1. □

The following proposition follows from Lemma 3.1.

Proposition 3.3 *Every right distributive torian algebra fixes* 0.

Example 3.2 Consider the set $\mathbb{R}$ of real numbers. Define a binary operation $*$ on $\mathbb{R}$ by

$$x * y = \begin{cases} 0, & x \leq y \\ x, & x > y \end{cases}$$

Then, $(\mathbb{R}; *, 0)$ is a right distributive torian algebra.

Theorem 3.1 *Let X be a torian algebra such that $[(x * z) * y] * [(x * z) * (y * z)] = 0$ for all $x, y, z \in X$. Then X is right distributive if and only if $(x * y) * y = x * y$ for all $x, y \in X$.*

Proof Suppose $(x*y)*y = x*y$. Notice that $(x*z)*(y*z) = [(x*z)*z]*(y*z) \sim (x*z)*y$(by Lemma 2.4). So, $[(x * z) * (y * z)] * [(x * z) * y] = 0$. Now, by the hypothesis, we have $(x * z) * y = (x * z) * (y * z)$; giving us $(x * y) * z = (x * z) * (y * z)$ as required.

The converse is obvious from Lemma 3.1(6). The proof is complete. □

Corollary 3.1 *Let X be a torian algebra such that $[[x*(z*x)]*y]*[[[x*[(z*x)]*x]]*[[y*[(z*y)]*y]]] = 0$ for all $x, y, z \in X$. Then X is right distributive if and only if $[x*[(y*x)]*x]*y = [x*(y*x)]*x$ for all $x, y \in X$.*

Proof The proof follows from Theorem 3.1 and Lemma 2.1. □

Corollary 3.2 *Let X be a torian algebra such that $[[x * [x * (x * z)]] * y] * [[x * [x * (x * z)]] * [y * [y * (y * z)]]]] = 0$ for all $x, y, z \in X$. Then X is right distributive if and only if $[x * [*(x * y)]] * y = x * [x * (x * y)]$ for all $x, y \in X$.*

Proof The proof follows from Theorem 3.1 and Proposition 2.1. □

Theorem 3.2 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z, p, v \in X$:*

(1) $[x * (y * z)] * [x * (y * p)] \sim (z * p)$;
(2) $x \sim y \Rightarrow (z * y) \sim (z * x)$;
(3) $(x * y) \sim v \Rightarrow (x * v) \sim [x * (x * y)]$;
(4) $[(x * z) * y] * [(x * z) * (y * z)] = 0$.

Then, $[x * [x * [y * (y * x)]]] = [x * (x * y)] * (y * x)$ for all $x, y \in X$.

Proof Notice that $[x * (x * y)] * [x * [x * [y * (y * x)]]] \sim [y * [y * (y * x)]] = y * x$. Hence, $[x * (x * y)] * (y * x) \sim [x * [x * [y * (y * x)]]]$. Now let $[x * [y * (y * x)] = v$. Then we have $(x*v) \sim [y*(y*x)]$. Notice that $[y*(y*x)] \sim y$. So, $(x*y) \sim [x*[y*(y*x)]]$; giving us $(x*y) \sim v$;

so that $(x*v) \sim [x*(x*y)]$. Now notice also that $[y*(y*x)] = [y*(y*x)]*(y*x) \sim [x*(y*x)]$. Since $(x*v) \sim [y*(y*x)]$ and $[y*(y*x)] \sim [x*(y*x)]$, we have $(x*v) \sim [x*(y*x)]$.

Now, multiply both sides of the last relation on the right by v to get $[(x*v)*v] \sim [x*(y*x)]*v$. That is, $[(x*v)*v] \sim (x*v)*(y*x)$; giving us $(x*v) \sim [(x*v)*(y*x)]$; leading to $(x*v) \sim [[x*(x*y)]*(y*x)]$. Substituting back for v, we have $[x*[x*[y*(y*x)]]] \sim [x*(x*y)]*(y*x)$. Since $[x*(x*y)]*(y*x) \sim [x*[x*[y*(y*x)]]]$ and $[x*[x*[y*(y*x)]]] \sim [x*(x*y)]*(y*x)$, we conclude that $[x*[x*[y*(y*x)]]] = [x*(x*y)]*(y*x)$ as required. □

Corollary 3.3 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z, p, v \in X$:*

(1) $[x*[[y*(z*y)]*y]]*[[x*[y*(p*y)]*y]] \sim [[z*(p*z)]*z]$;

(2) $x \sim y \Rightarrow [[z*(y*z)]*z] \sim [[z*(x*z)]*z]$;

(3) $[[x*(y*x)]*x] \sim v \Rightarrow [[x*(v*x)]*x] \sim [x*[[x*(y*x)]*x]]$;

(4) $[[[x*(z*x)]*x]*y]*[[[x*[(z*x)]*x]]*[[y*[(z*y)]*y]]] = 0$.

Then, $[x*[x*[y*[y*(x*y)]*y]]] = [[x*[x*(y*x)]*x]]*[[y*[(x*y)]*x]]$ *for all* $x, y \in X$.

Proof The proof follows from Theorem 3.2 and lemma 2.1. □

Corollary 3.4 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z, p, v \in X$:*

(1) $[x*[y*[y*(y*z)]]]*[x*[y*[y*(y*p)]]] \sim [z*[z*(z*p)]]$;

(2) $x \sim y \Rightarrow [z*[z*(z*y)]] \sim [z*[z*(z*x)]]$;

(3) $[x*[x*(x*y)]] \sim v \Rightarrow [x*[x*(x*v)]] \sim [x*[x*[x*(x*y)]]]$;

(4) $[[x*[x*(x*z)]]*y]*[[x*[x*(x*z)]]*[y*[y*(y*z)]]] = 0$.

Then, $[x*[x*[y*[y*[y*(y*x)]]]]] = [x*[x*[x*(x*y)]]]*[y*[y*(y*x)]]$ *for all* $x, y \in X$.

Proof The Proof follows from Theorem 3.2 and Proposition 2.1. □

Theorem 3.3 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z, p, v \in X$:*

(1) $[x*(y*z)]*[x*(y*p)] \sim (z*p)$;

(2) $x \sim y \Rightarrow (z*y) \sim (z*x)$;

(3) $(x*y) \sim v \Rightarrow (x*v) \sim [x*(x*y)]$;

(4) $[(x*z)*y]*[(x*z)*(y*z)] = 0$.

Then $(x*y)*[x*(x*y)] = x*y$ *for all* $x, y \in X$.

Proof From Theorem 3.2, for all $x, y \in X$, we have

$$[x*(x*y)]*(y*x) = [x*[x*[y*(y*x)]]] \tag{1}$$

Put $x * y$ for x, and put x for y in expression (1). Then, the left hand side becomes

$$\begin{aligned}[(x * y) * [(x * y) * x]] * [x * (x * y) &= [(x * y) * [(x * x) * y] * [x * (x * y) \\ &= [(x * y) * (0 * y)] * [x * (x * y) \\ &= (x * y) * [x * (x * y)].\end{aligned}$$

Also, the right hand side becomes

$$(x * y) * [(x * y) * [x * [x * (x * y)]]] = (x * y) * [(x * y) * (x * y)] = x * y.$$

Hence, equating the left and right hand sides, we have $(x * y) * [x * (x * y)] = x * y$ as required. The proof is complete. □

Corollary 3.5 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z, p, v \in X$:*

(1) $[x * [[y * (z * y)] * y]] * [x * [[y * (p * y)] * y]] \sim [[z * (p * z)] * z]$;
(2) $x \sim y \Rightarrow [[z * (y * z)] * z] \sim [[[z * (x * z)] * z]$;
(3) $[[x * (y * x)] * x] \sim v \Rightarrow [[x * (v * x)] * x] \sim [x * [x * (y * x)] * x]$;
(4) $[[[x * (z * x)] * x] * y] * [[[x * (z * x)] * x] * [[y * (z * y)] * y]] = 0$.

Then, $[[x * (y * x)] * x] * [[x * [[x * (y * x)] * x] = [[x * (y * x)] * x]$ *for all* $x, y \in X$.

Proof The proof follows from Theorem 3.3 and Lemma 2.1. □

Corollary 3.6 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z, p, v \in X$:*

(1) $[x * [y * [y * (y * z)]]] * [x * [y * [y * (y * p)]]] \sim [z * [z * (z * p)]]$;
(2) $x \sim y \Rightarrow [z * [z * (z * y)]] \sim [z * [z * (z * x)]]$;
(3) $[x * [x * (x * y)]] \sim v \Rightarrow [x * [x * (x * v)]] \sim [[x * [x * [x * (x * y)]]]]$;
(4) $[[x * [x * (x * z)]] * y] * [[x * [x * (x * z)]] * [y * [y * (y * z)]]] = 0$.

Then, $[x * [x * (x * y)]] * [x * [x * (x * y)]]$ *for all* $x, y \in X$.

Proof The proof follows from Theorem 3.3 and Proposition 2.1. □

Remark 3.2 Let X be a torian algebra. We define $x * y^k = [(x * y) * y] * \cdots] * y$ (k times); where k is a natural number.

Theorem 3.4 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z \in X$:*

(1) $x \sim y \Rightarrow (x * z) \sim (y * z)$;
(2) $x * y^k = x * y^{k+1}$, *where* $k \in \mathbb{N}$; *the set of natural numbers;*
(3) $x * y^k = x * y^l$ *for all* $l \geq k \in \mathbb{N}$;
(4) $(x * z^k) * (y * z^k \sim (x * y)$.

Then, $(x * y) * z^k = (x * z^k) = (x * z^k) * (y * z^k)$ *for all* $x, y, z \in X$.

Proof By hypothesis, we have $x * z^k = x * z^{2k}$. Since, $(x * z^k) * (y * z^k) \sim (x * y)$, we have $[(x * z^k) * (y * z^k)] * z^k \sim (x * y) * z^k$; which gives $[(x * z^k) * z^k] * (y * z^k) \sim (x * y) * z^k$; which results to $(x * z^{2k}) * (y * z^k) \sim (x * y) * z^k$. Since $x * z^k = x * z^{2k}$, we now have

$$(x * z^k) * (y * z^k) \sim (x * y) * z^k \tag{1}$$

Notice that $(y * z^k) * y = 0$. So, $(y * z^k) \sim y$. We therefore have $[(x * z^k) * y] \sim [(x * z^k) * (y * z^k)]$; which gives

$$[(x * y) * z^k] \sim [(x * z^k) * (y * z^k)] \tag{2}$$

By expressions (1) and (2), we have $(x * y) * z^k = (x * z^k) * (y * z^k)$ as required. The proof is complete. □

Proposition 3.4 *Let X be a right distributive torian algebra. If $(x * y) * z^k = (x * z^k) * (y * z^k)$, then $x * z^k = x * z^{k+1}$ for all $x, y, z \in X; k \in \mathbb{N}$.*

Proof By hypothesis, we have $(x * z) * z^k = (x * z^k) * (z * z^k)$, which gives $x * z^{k+1} = x * z^k$ as required. The proof is complete. □

Theorem 3.5 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z \in X$:*

(1) $x \sim y \Rightarrow (x * z) \sim (y * z)$;

(2) $x * y^k = x * y^{k+1}$; *where $k \in \mathbb{N}$, the set of natural numbers;*

(3) $x * y^k = x * y^l$ *for all $l \geq k \in \mathbb{N}$.*

Then, $[y * (y * x)^k] * (x * y)^k = [x * (x * y)^k] * (y * x)^k$ for all $x, y \in X$.

Proof By hypothesis, we have

$$x * (x * y)^{k_1} = x * (x * y)^{k_1} \tag{3}$$

and

$$y * (y * x)^{k_2} = y * (y * x)^{k_2} \tag{4}$$

Let k be the maximum of k_1 and k_2. Then

$$x * (x * y)^k = x * (x * y)^{k+1} \tag{5}$$

and

$$y * (y * x)^k = y * (y * x)^{k+1} \tag{6}$$

Notice that $[x * (x * y)] * y = 0$. So, $x * (x * y) \sim y$ and from expression (5), we have

$$x * [(x * y)^k \sim y * (x * y)^k \tag{7}$$

Now, multiply expression (7) on both sides on the right by $y * x$ (k times) to get

$$[x * (x * y)^k] * (y * x)^k \sim [y * (x * y)^k] * (y * x)^k \tag{8}$$

Now apply Lemma 2.2 to expression (8) to get

$$[x * (x * y)^k] * (y * x)^k \sim [y * (y *)^k] * (x * y)^k \tag{9}$$

Also notice that $[y * (y * x)] * x = 0$. So, $[y * (y * x)] \sim x$; and so from expression (6), we have

$$[y * (y * x)^k] \sim [x * (y * x)^k] \tag{10}$$

Multiply both sides of expression (10) on the right by $x * y$ (k times) to get

$$[y * (y * x)^k] * (x * y)^k \sim [x * (y * x)^k] * (x * y)^k \tag{11}$$

Now apply Lemma 2.2 to expression (11) to get

$$[y * (y * x)^k] * (x * y)^k \sim [x * (x * y)^k] * (y * x)^k \tag{12}$$

From expressions (9) and (12), we have $[y * (y * x)^k] * (x * y)^k$
$= [x * (x * y)^k] * (y * x)^k$ as required. The proof is complete. □

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