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K-Banhatti Indices for Special Graphs and Vertex Gluing Graphs

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Abstract: The K-Banhatti indices was introduced by Kulli in 2016, defined as

$$B_1(G) = \sum_{ue}[d_G(u) + d_G(e)] \quad \text{and} \quad B_2(G) = \sum_{ue} d_G(u).d_G(e),$$

where ue means that the vertex u and edge e are incident and $d_G(e)$ denotes the degree of the edge e in G. In this paper, we formulate general formula for certain graphs.

Key Words: Indices, homeomorphism, graphs, bridge.

AMS(2010): 05C10, 97K30.

§1. Introduction

Topological indices is an useful tool to model physical and chemical properties of molecules to design pharmacologically active compounds, to recognize environmentally hazardous materials [1]. Applications see [7, 9, 10, 4].

Let $G(V, E)$ be a connected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u. The edge connecting the vertices u and v will be denoted by uv. Let $d_G(e)$ denote the degree of an edge $e = uv$ in G, which is defined by $d_G(e) = d_G(u) + d_G(v) - 2$. The vertices and edges of a graph are said to be its elements [3].

The first and second Banhatti index were introduced by Kulli [2, 5] and are defined as below

$$B_1(G) = \sum_{ue}[d_G(u) + d_G(e)] \quad \text{and} \quad B_2(G) = \sum_{ue} d_G(u).d_G(e).$$

where ue means that the vertex u and edge e are incident in G.

In this paper, we studied the K-Banhatti indices of some special graphs as well as a vertex gluing of graphs by establishing general formula.

§2. Basic Definitions

A K_4-homeomorphic graph/K_4-homeomorph as $K_4(e_1, e_2, e_3, e_4, e_5, e_6)$ is the graph obtained

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when the six edges of a complete graph with four vertices of (K_4) are subdivided edge is called a path and its length is the number of resulting segments (see Fig.1 for details).

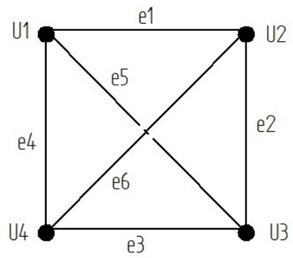

Fig.1 K_4-homeomorphic graph

A complete bipartite graph is a simple bipartite graph with partite sets U_1 and U_2, where every vertex in U_1 is adjacent with all the vertices in U_2. If $|U_1| = m$ and $|U_2| = n$, then such complete bipartite graph is denoted by $K_{m,n}$ [or $K(m, n)$]. So $K_{m,n}$ has order $m + n$ and size mn (see Fig.2 for details).

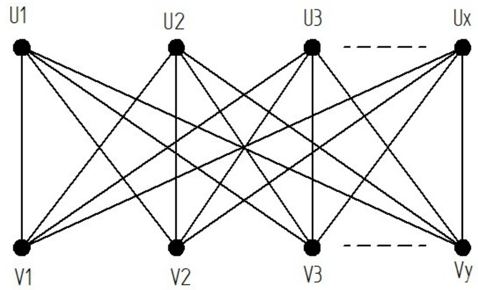

Fig.2 A complete bipartite $K_{x,y}$

A graph consisting of r paths joining two vertices is called an r-bridge graph, which is denoted by $T(e_1, e_2, \cdots, e_r)$, where $e_1, e_2, \cdots, e_r$ are the lengths of r paths. Clearly, an r-bridge graph is a generalized polygon tree (see Fig.3 following).

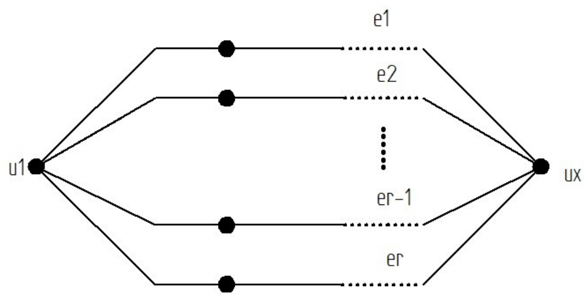

Fig.3 An r-bridge graph

A web graph $Web(r, s)$ is the graph obtained from the Cartesian product of the cycle C_r and the path P_s (see Fig.4).

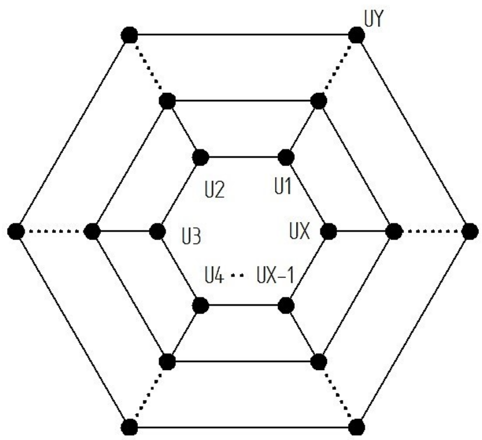

Fig.4 A web graph $Web(x, y)$

§3. K-Banhatti Indices of Some Special Graphs

This section demonstrates general formulas obtained for some special graphs.

Theorem 3.1 *Let $e_1, e_2, e_3, e_4, e_5, e_6$ be positive integers, then the K-Banhatti indices of a K_4-homeomorphism graph denoted by $K_4(e_1, e_2, e_3, e_4, e_5, e_6)$ will be as follows:*

(i) If e_1 or/and e_2 or/and e_3 or/and e_4 or/and e_5 or/and $e_6 = 1$, then the first and second Banhatti index to any one of them is, 14 and 24 respectively;

(ii) If e_1 or/and e_2 or/and e_3 or/and e_4 or/and e_5 or/and $e_6 \neq 1$ then the first and second Banhatti index to any one of them is, (number of edges) 11 and (number of edges) 15 respectively.

Proof (i) If e_1 or/and e_2 or/and e_3 or/and e_4 or/and e_5 or/and $e_6 = 1$, then any one of them will have one edge and two vertices with the same degree three. Thus the first and second Banhatti index to any one of them is 14 and 24 respectively.

(ii) If e_1 or/and e_2 or/and e_3 or/and e_4 or/and e_5 or/and e_6, then any one of them will have two or more edges and each of them will have two vertices in which at least one of the vertices is of degree two. Thus, the first and second Banhatti index to any one of them is (number of edges) 11 and (number of edges) 15 respectively. □

Example 3.2 Let $e_1, e_2, e_3, e_4, e_5, e_6$ be positive integers, the K-Banhatti indices of a K_4-homeomorphism graph denoted by $K_4(e_1, e_2, e_3, e_4, e_5, e_6)$ is,

$$B_1[K_4(e_1, e_2, e_3, e_4, e_5, e_6)] = \begin{cases} 11 \sum_{i=1}^{6} e_i & \text{if } e_i \neq 1, 1 \leq i \leq 6 \\ 84 & \text{if } \sum_{i=1}^{6} e_i = 1 \\ 52 + (e_4 + e_5 + e_6)11 & \text{if } e_1 = e_2 = e_3 = 1, e_4 = e_5 = e_6 \neq 1. \end{cases} \quad (3.1)$$

$$B_2[K_4(e_1,e_2,e_3,e_4,e_5,e_6)] = \begin{cases} 15\sum_{i=1}^{6} e_i & \text{if } e_i \neq 1, 1 \leq i \leq 6 \\ 144 & \text{if } e_i \neq 1, 1 \leq i \leq 6 = 1 \\ 72 + (e_4+e_5+e_6)15 & \text{if } e_1 = e_2 = e_3 = 1, e_4 = e_5 = e_6 \neq 1. \end{cases} \quad (3.2)$$

Theorem 3.3 *Let m, n be positive integers. The first and second Banhatti index of a complete bipartite graph denoted by $K_{m,n}$ is,*

$$B_1[K_{m,n}] = mn[3m+3n-4], \quad B_2[K_{m,n}] = mn(m+n)(m+n-2).$$

Proof In complete bipartite graph having mn number of edges each one of them has two vertices that have same degree which has the first vertex of degree m and the second vertex of degree n. Hence by the definitions of first and second Banhatti index, we get that

$$B_1[K_{m,n}] = mn[3m+3n-4], \quad B_2[K_{m,n}] = mn(m+n)(m+n-2).$$

This completes the proof. □

Theorem 3.4 *Let k be a positive integer, The first and second Banhatti index of a k-bridge graph denoted by $T(e_1, e_2, \cdots, e_k)$ is,*

$$B_1[T(e_1,e_2,\cdots,e_k)] = (e_1+e_2+\cdots+e_k)8, \quad B_2[T(e_1,e_2,\cdots,e_k)] = (e_1+e_2+\cdots+e_k)8.$$

Proof This result is proving by mathematical induction. Let $K = 2$, then $G = T(e_1, e_2)$, whose graph is shown in Fig.5.

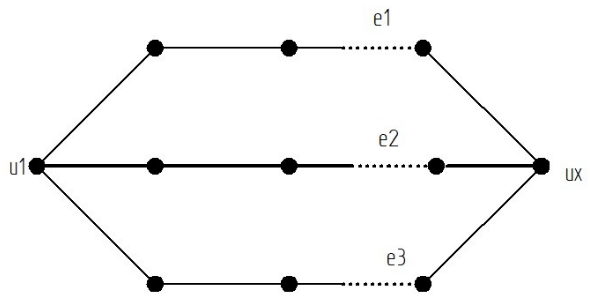

Fig.5

Thus,

$$B_1[T(e_1,e_2)] = (e_1+e_2)[3(2)+3(2)-4] = (e_1+e_2)8,$$
$$B_2[T(e_1,e_2)] = (e_1+e_2)[(2+2)^2 - 2(2+2)] = (e_1+e_2)8.$$

Hence, it is true for $k = 2$.

Let us assume that the result is true for $k = r$.

$$B_1[T(e_1, e_2, \cdots, e_r)] = 8(e_1 + e_2 + \cdots + e_r),$$
$$B_2[T(e_1, e_2 + \cdots + e_r)] = 8(e_1 + e_2 + \cdots + e_r).$$

Now, to prove that the result is true for $k = r + 1$. Let us consider a graph with $r + 1$ bridges such as those shown in Fig.6

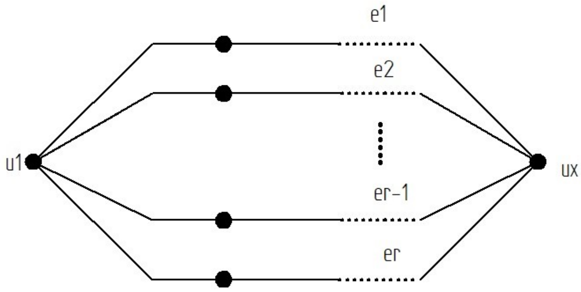

Fig.6

where e_i denotes the position of the edges of graph $T(e_1, e_2, \cdots, e_r)$ at the i^{th} position. The graph H is the path which contains endings V_1 and V_2 and e_{r+1} is the number of edges in H as follows (see Fig.7 for details).

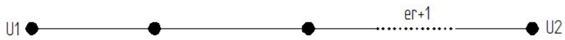

Fig.7

Connect the graph $T(e_1, e_2, \cdots, , e_r)$ with the graph H such that $V_1 = U_1$ and $V_2 = U_2$. the vertices $V_1 = U_1$ and $V_2 = U_2$ are of degree $r + 1$, as shown in Fig.8 following.

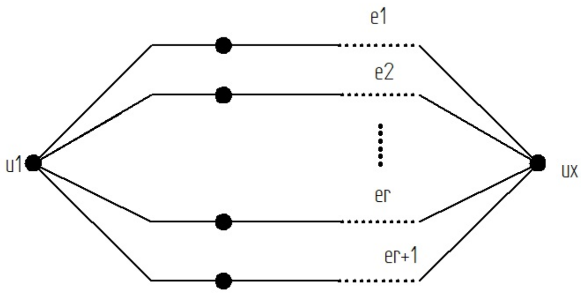

Fig.8

Thus,

$$\begin{aligned} B_1(T_{r+1}) &= B_1(T_r) + B_1(H) \\ &= 8(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) + 8e_{r+1} = (e_1 + e_2 + \cdots + e_{r+1})8. \\ B_2(T_{r+1}) &= B_2(T_r) + B_2(H) \\ &= 8(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) + 8e_{r+1} = (e_1 + e_2 + \cdots + 8e_{r+1}). \end{aligned}$$

Therefore, the result is also true for $k = r + 1$.

Hence, the result is true for all k by the induction principle.

$$B_1(T_{r+1}) = 8(e_1 + e + 2 + \cdots + e_r) = B_2(T_{r+1}). \qquad \square$$

§4. K.Banhatti Indices of Certain Vertex Gluing Graphs

This section contains the general formulas for first and second Banhatti index of certain vertex gluing graphs. Let K_4^2- homeomorphism be a graph obtained from two different K_4- homeomorphism graphs $K_4(e_1, e_2, e_3, e_4, e_5, e_6)$ and $K_4(e_7, e_8, e_9, e_{10}, e_{11}, e_{12})$ with one common vertex U_1 (vertex gluing of graph) (see Fig.9 for details).

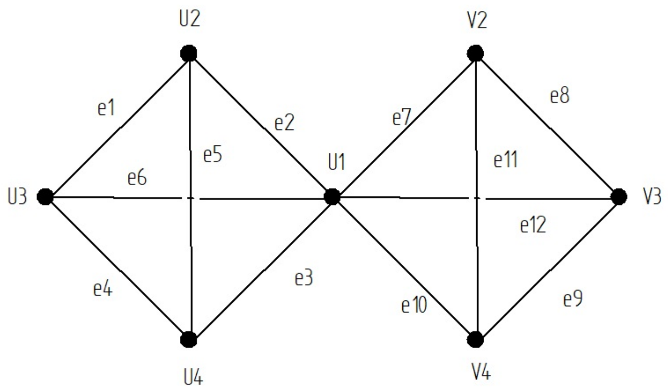

Fig.9 A graph K_4^2 - homeomorphism

Theorem 4.1 *If e_i is a positive integer for integers $1 \leq i \leq 12$, then the first and second Banhatti index of K_4^2-homeomorphism graph are respectively*

(1) *If $e_i = 1$ then $B_1(e_i) = 14$, $B_2(e_i) = 24$ for $i = 1, 4, 5, 8, 9, 11$;*

(2) *If $e_i \geq 2$ then $B_1(e_i) = 11$, $B_2(e_i) = 15$ for $1 \leq i \leq 12$;*

(3) *If $e_i = 1$ then $B_1(e_i) = 23, B_2(e_i) = 63$ for $i = 2, 3, 6, 7, 10, 12$,*

then,

$$B_1(K_4^2 - homeomorphism) = \sum_{i=1}^{12} B_1(e_i)$$
$$B_2(K_4^2 - homeomorphism) = \sum_{i=1}^{12} B_2(e_i)$$

Proof The proof is divided into three cases following.

Case 1. If $e_i = 1$ then $B_1(e_i) = 14$ and $B_2(e_i) = 24$, $i = 1, 4, 5, 8, 9, 11$ and any edge e_i has two vertices having the same degree three, then

$$B_1(e_i) = 3(3) + 3(3) - 4 = 14,$$
$$B_2(e_i) = (3+3)^2 - 2(3+3) = 24.$$

Case 2. If $e_i \geq 2$, then

$$B_1(e_i) = 11 \text{ and } B_2(e_i) = 15, 1 \leq i \leq 12$$

and all edges in this case has at least one vertex of degree two, then

$$B_1(e_i) = 3(3) + 3(2) - 4 = 1$$
$$B_2(e_i) = (3+2)^2 - 2(3+2) = 15.$$

Case 3. If $e_i = 1$ then $B_1(e_i) = 23$ and $B_2(e_i) = 63$, $i = 2, 3, 6, 7, 10, 12$, and all edges in this case have two vertices, the first one of degree three and second one of degree six. Then

$$B_1(e_i) = 3(3) + 3(6) - 4 = 23$$
$$B_2(e_i) = (3+6)^2 - 2(3+6) = 63.$$

This completes the proof. □

Let u_1-gluing of complete bipartite graph be a graph obtained from two different complete bipartite graphs $K_{x,y}$ and $K_{p,q}$ with common one vertex u_1 denoted by $K^{p;q}_{x,y}(u_1)$, a vertex gluing of graph (see Fig.10 for details).

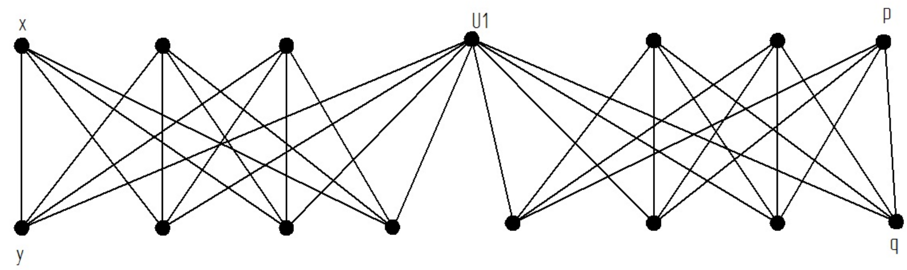

Fig.10 A u_1 - gluing of complete bipartite graph $K^{y;q}_{x,p}(u_1)$

Theorem 4.2 *Let x, y, p and q be positive integers. The first and second Banhatti index of the u_1- gluing of complete bipartite graph $K^{p;q}_{x,y}(u_1)$ is*

(*i*) $B_1[K^{p;q}_{x,y}(u_1)] = y(x-1)(3x+3y-4) + y(3x+3y+3q-4) + q(3p+3y+3q-4) + q(p-1)(3p+3q-4)$;

(*ii*) $B_2[K^{p;q}_{x,y}(u_1)] = y(x-1)(x+y)(x+y-2) + y(x+y+q)(x+y+q-2) + q(y+p+q)(y+p+q-2) + q(p-1)(p+q)(p+q-2)$.

Proof We consider two cases following.

Case 1. In complete bipartite graph $K_{x,y}$ there are xy edges. $y(x-1)$ of them are incident on two vertices of degree x and y. The remaining y will incidents on two vertices of degree x and $(y+q)$.

Case 2. In complete bipartite graph $K_{p,q}$ there are pq edges. $q(p-1)$ of them will incidents on two vertices of degree p and q. The remaining q will incidents on two vertices of degree p and $(y+q)$.

From Cases 1 and 2 we get that

$$\begin{aligned} B_1[K_{x,y}^{p,q}(u_1)] &= y(x-1)(3x+3y-4) \\ &\quad +y(3x+3y+3q-4)+q(3p+3y+3q-4)+q(p-1)(3p+3q-4), \\ B_2[K_{x,y}^{p,q}(u_1)] &= y(x-1)(x+y)(x+y-2)+y(x+y+q)(x+y+q-2) \\ &\quad +q(y+p+q)(y+p+q-2)+q(p-1)(p+q)(p+q-2). \end{aligned}$$

This completes the proof. □

Let u_1-gluing of $x,y-$ bridge graph be a graph obtained from two different $k-$bridge graphs T_1 and T_2 with common one vertex u_1 denoted by $K_x^y(u_1)$, a vertex gluing of graph (see Fig.11 for details).

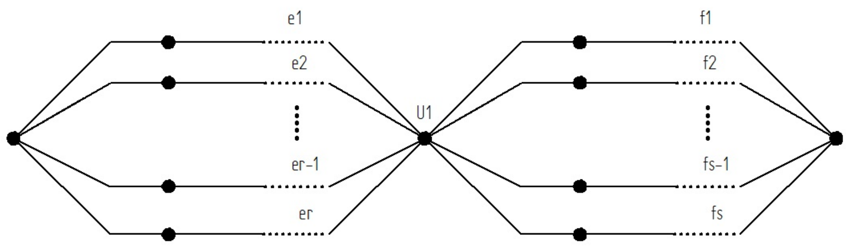

Fig.11 A u_1- gluing of $x,y-$bridge graph $T_x^y(u_1)$

Theorem 4.3 *Let x and y be positive integers. The first and second Banhatti index of the u_1- bridge graph $T_x^y(u_1)$ is*

$$B_1[T_x^y(u_1)] = \sum_{i=1}^{x} e_i(8) + \sum_{j=1}^{y} f_j(8) = B_2[T_x^y(u_1)].$$

Proof We have e_i, $i=1,2,3\cdots x$ and f_j, $j=1,2,3\cdots y$, the numbers of edges, all of them have atleast one vertex of degree two, then

$$B_1[T_x^y(u_1)] = \sum_{i=1}^{x} e_i(8) + \sum_{j=1}^{y} f_j(8) = B_2[T_x^y(u_1)].$$ □

Let u_1-gluing of web graph be a graph obtained from two different web graphs. Web(x,p) and web(y,q) with one common vertex u_1 denoted by $W_{x,p}^{y,q}(u_1)$, a vertex gluing of graph (see Fig.12 for details).

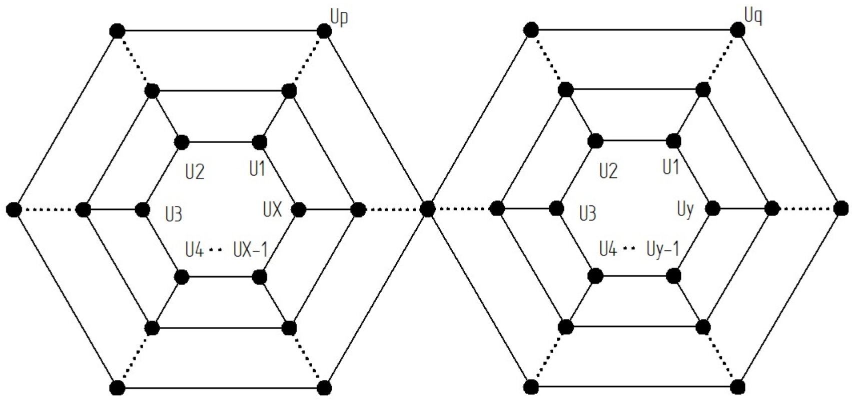

Fig.12 A U_1- gluing of web graph $W_{x,p}^{y;q}(u_1)$

Theorem 4.4 *Let x, p, y and q be positive integers. Then the first and second Banhatti index of the u_1- gluing of Web graph $W_{x,p}^{y;q}(u_1)$ is*

$$B_1[W_{x,p}^{y;q}(u_1)] = \begin{cases} a & if\ p, q = 2 \\ b & if\ p = 2 \\ c & if\ p, q \neq 2. \end{cases} \tag{4.1}$$

where $a = 52(x + y - 2) + 138$, $b = 14(3x + 2y - 5) + 17(2y - 1) + 20y(2q - 5) + 141$, $c = 28(x + y - 2) + 34(x + y - 1) + 20[x(2p - 5) + y(2q - 5)] + 144$ *and*

$$B_2[W_{x,p}^{y;q}(u_1)] = \begin{cases} d & if\ p, q = 2 \\ e & if\ p = 2 \\ f & if\ p, q \neq 2. \end{cases} \tag{4.2}$$

where $d = 72(x + y - 2) + 378$, $e = 24(3x + 2y - 5) + 35(2y - 1) + 48y(2q - 5) + 395$ *and* $f = 48(x + y - 2) + 70(x + y - 1) + 48[x(2p - 5) + y(2q - 5)] + 412$.

Proof We consider three cases and their edge and vertex partition of above web graph as follow.

Case 1. If

(3,3)	(3,6)
3(x+y-2)	6

Then, by definitions of K.Banhatti indices, we get

$$\begin{aligned} B_1[W_{x,p}^{y;q}(u_1)] &= 52(x + y - 2) + 138 \quad \text{and} \\ B_2[W_{x,p}^{y;q}(u_1)] &= 72(x + y - 2) + 378. \end{aligned}$$

Case 2. If

(3,3)	(3,4)	(3,6)	(4,4)	(4,6)
(3x+2y-5)	(2y-1)	5	y(2q-5)	1

Then, by definitions of K.Banhatti indices, we get

$$B_1[W_{x,p}^{y,q}(u_1)] = 14(3x+2y-5)+17(2y-1)+20y(2q-5)+141,$$
$$B_2[W_{x,p}^{y,q}(u_1)] = 24(3x+2y-5)+35(2y-1)+48y(2q-5)+395.$$

Case 3. If

(3,3)	(3,4)	(3,6)	(4,4)	(4,6)
2(x+y-2)	2(x+y-1)	4	x(2p-5)+y(2q-5)	2

Then, by definitions of K.Banhatti indices, we get

$$B_1[W_{x,p}^{y,q}(u_1)] = 28(x+y-2)+34(x+y-1)+20[x(2p-5)+y(2q-5)]+144,$$
$$B_2[W_{x,p}^{y,q}(u_1)] = 48(x+y-2)+70(x+y-1)+48[x(2p-5)+y(2q-5)]+412.$$

Hence, by combining all the three cases we get

$$B_1[W_{x,p}^{y,q}(u_1)] = \begin{cases} a & \text{if } p,q=2 \\ b & \text{if } p=2 \\ c & \text{if } p,q\neq 2. \end{cases}$$

where $a = 52(x+y-2)+138$, $b = 14(3x+2y-5)+17(2y-1)+20y(2q-5)+141$, $c = 28(x+y-2)+34(x+y-1)+20[x(2p-5)+y(2q-5)]+144$ and

$$B_2[W_{x,p}^{y,q}(u_1)] = \begin{cases} d & \text{if } p,q=2 \\ e & \text{if } p=2 \\ f & \text{if } p,q\neq 2. \end{cases}$$

where $d = 72(x+y-2)+378$, $e = 24(3x+2y-5)+35(2y-1)+48y(2q-5)+395$ and $f = 48(x+y-2)+70(x+y-1)+48[x(2p-5)+y(2q-5)]+412$. □

§5. Conclusions

Here, the general formula for K.Banhatti indices of certain graphs namely K_4- homeomorphism, complete bipartite, k-bridge graphs and vertex gluing of graphs are established.

Acknowledgments

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