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On a New Class of Harmonic p-Valent Functions Defined by Convolution Structure

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Abstract: In this paper, we investigate several properties for the harmonic classes $\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ and $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. We obtain coefficient bounds, distortion theorem, extreme points, convex compinations and integral operator for these classes.

Key Words: Multivalent harmonic function, Hadamard product, sense-preserving, distortion bounds, integral operator.

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§1. Introduction and Preliminaries

A real-valued function u is said to be harmonic in a domain $\mathfrak{D}\subset\mathbb{C}$ if it has continuous second order partial derivatives in $\mathfrak{D}$, which satisfy the Laplace equation

$$\Delta u:=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

We say that a complex-valued continuous function $f:\mathfrak{D}\to\mathbb{C}$ is harmonic in $\mathfrak{D}$ if both functions $u:=\mathrm{Re}f$ and $v:=\mathrm{Im}f$ are real-valued harmonic functions in $\mathfrak{D}$. We note that every complex-valued function f harmonic in $\mathfrak{D}$ with $0\in\mathfrak{D}$, can be uniquely represented as

$$f=h+\overline{g},$$

where $h,\ g$ are analytic functions in $\mathfrak{D}$ with $g(0)=0$. Then we call h the analytic part and g the co-analytic part of f (see [6]). The Jacobian of f is given by

$$J_f(z)=|h'(z)|^2-|g'(z)|^2\qquad(z\in\mathfrak{D}).$$

The mapping f is locally univalent if $J_f(z)\neq 0$ in $\mathfrak{D}$. A result of Lewy [16] shows that the converse is true for harmonic mappings. Therefore, f is locally univalent and sense-preserving if and only if

$$|h'(z)|>|g'(z)|\qquad(z\in\mathfrak{D}).$$

Duren [12] also Ahuja [1] and Ponnusamy and Rasila [24, 25].

For $p\geq 1$, denote by $\xi(p)$ the set of all multivalent harmonic functions $f=h+\overline{g}$ defined

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in the open unit disc $\mathcal{U}$, where h and g defined by

$$h(z) = z^p + \sum_{n=p+t}^{\infty} a_n z^n, \quad g(z) = \sum_{n=p+t-1}^{\infty} b_n z^n, \quad |b_{p+t-1}| < 1, \; t \in \mathbb{N} = \{1, 2, \cdots\} \tag{1.1}$$

are analytic functions in $\mathcal{U}$.

Let $\mathcal{F}(z) = \psi(z) + \overline{\varphi(z)}$ be a fixed multivalent harmonic function, where

$$\psi(z) = z^p + \sum_{n=p+t}^{\infty} |A_n| \, z^n, \; \varphi(z) = \sum_{n=p+t-1}^{\infty} |B_n| \, z^n, \; |B_{p+t-1}| < 1, \; t \in \mathbb{N} = \{1, 2, \cdots\}. \tag{1.2}$$

The Hadamard product (or convolution) of functions $f(z)$ and $\mathcal{F}(z)$ of the form

$$(f * \mathcal{F})(z) = z^p + \sum_{n=p+t}^{\infty} |a_n A_n| \, z^n + \sum_{n=p+t-1}^{\infty} |b_n B_n| \, \overline{z}^n. \tag{1.3}$$

Studies of convolution play an serious role in geometric function theory. It has a several researchers of this field. In 1975, Schild and Silverman [28] studied the diverse interesting results on the convolution of analytic functions. Later on, Choi et al. [5], Darwish [7], Darwish and Aouf [8], Domokos [11], Nishiwaki and Owa [20], Nishiwaki et al. [2], Owa [23] and Srivastava et al. [31] studied the generalized convolution for analytic functions only. For detailed study see the excellent text book by Ruscheweyh [27], see also [4], [9], [10], [13], [14], [15], [22], [26], [29], [30].

A function $f(z) \in \xi(p)$ is said to be in the class $\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ if

$$\Re\left\{ \frac{z(f * \mathcal{F})''(z) + (f * \mathcal{F})'(z)}{(f * \mathcal{F})'(z) + \tau z (f * \mathcal{F})''(z)} \right\} \geq \rho \left| \frac{z(f * \mathcal{F})''(z) + (f * \mathcal{F})'(z)}{(f * \mathcal{F})'(z) + \tau z (f * \mathcal{F})''(z)} - p \right| + p\varkappa \tag{1.4}$$

where $0 \leq \varkappa < 1, \; \rho \geq 0, \; 0 \leq \tau < 1$ and $z \in \mathcal{U}$.

Finally, denote by $\mathcal{T}\xi(p)$ the subclass of functions $f(z) = h(z) + \overline{g(z)}$ in $\xi(p)$ where

$$h(z) = z^p - \sum_{n=p+t}^{\infty} |a_n| \, z^n, \quad g(z) = -\sum_{n=p+t-1}^{\infty} |b_n| \, z^n, \quad b_{p+t-1} < 1. \tag{1.5}$$

Let $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau) = \mathcal{T}\xi(p) \cap \xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$.

We note that

(i) $\xi_1(1, \varkappa, \rho, \tau) = KM_H(\alpha, \beta, \gamma)$

$$\Re\left\{ \frac{zf''(z) + f'(z)}{f'(z) + \gamma z f''(z)} \right\} \geq \beta \left| \frac{zf''(z) + f'(z)}{f'(z) + \gamma z f''(z)} - 1 \right| + \alpha \quad \text{(see [2])}.$$

(ii) $\xi_1(1, \alpha, 0, \tau) = C(\lambda, \alpha)$

$$\Re\left\{ \frac{zf''(z) + f'(z)}{f'(z) + \lambda z f''(z)} \right\} > \alpha \qquad \text{(see [19])}.$$

In this paper, we obtained the coefficient bounds for the classes $\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ and $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. Further distortion theorem, extreme points, convex compinations and integral operator for the classe $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

§2. Coefficient Bounds

Now, we begin with a sufficient condition for functions in $\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Theorem 2.1 *Let $f = h + \overline{g}$ with h and g given by* (1.1). *If*

$$\sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho) - p(\tau(n-1)+1)(\rho+\varkappa)\right]}{p^2(1-\varkappa-\tau(p-1)(\rho+\varkappa))} |a_n A_n| + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho) - p(\tau(n-1)+1)(\rho+\varkappa)\right]}{p^2(1-\varkappa-\tau(p-1)(\rho+\varkappa))} |b_n B_n| \leq 1, \tag{2.1}$$

where $0 \leq \varkappa < 1,\ \rho \geq 0,\ 0 \leq \tau < 1$, then the harmonic function f is orientation preserving in $\mathcal{U}$ and $f \in \xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Proof To verify that f is orientation preserving, we show

$$\begin{aligned}
\left|(h(z)*\psi(z))'\right| &= \left| pz^{p-1} + \sum_{n=p+t}^{\infty} n\,|a_n A_n|\, z^{n-1} \right| \\
&\geq p\,|z|^{p-1} - \sum_{n=p+t}^{\infty} n\,|a_n A_n|\,|z|^{n-1} \\
&= p\,|z|^{p-1} \left(1 - \sum_{n=p+t}^{\infty} \frac{n}{p} |a_n A_n|\,|z|^{n-p} \right) \\
&\geq p\,|z|^{p-1} \left(1 - \sum_{n=p+t}^{\infty} \frac{n}{p} |a_n A_n| \right) \\
&\geq p\,|z|^{p-1} \left\{ 1 - \sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho) - (p(n-1)+1)(\rho+\varkappa)\right]}{p^2(1-\varkappa-\tau(p-1)(\rho+\varkappa))} |a_n A_n| \right\} \\
&\geq p\,|z|^{p-1} \left\{ \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho) - (p(n-1)+1)(\rho+\varkappa)\right]}{p^2(1-\varkappa-\tau(p-1)(\rho+\varkappa))} |b_n B_n| \right\} \\
&\geq p\,|z|^{p-1} \left\{ \sum_{n=p+t-1}^{\infty} \frac{n}{p} |b_n B_n| \right\} \\
&\geq \left| \sum_{n=p+t-1}^{\infty} \frac{n}{p} |b_n B_n|\, z^{n-1} \right| = \left|(g(z)*\varphi(z))'\right|.
\end{aligned}$$

Then, if $\psi(z) = 0$ and $\varphi(z) = 0$, we have $|h'(z)| = |g'(z)|$.

Next, we prove $f(z) \in \xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ by establishing condition (1.4). It is sufficient to show that

$$\Re\left\{\frac{z(f*\mathcal{F})''(z)+(f*\mathcal{F})'(z)}{(f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z)}(1+\rho e^{i\theta})-p\rho e^{i\theta}\right\}\geq p\varkappa\ (-\pi\leq\theta\leq\pi),$$

or equivalently

$$\Re\left\{\frac{(1+\rho e^{i\theta})\left(z(f*\mathcal{F})''(z)+(f*\mathcal{F})'(z)\right)-p\rho e^{i\theta}\left((f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z)\right)}{(f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z)}\right\}\geq p\varkappa. \tag{2.2}$$

If we put

$$A(z)=(1+\rho e^{i\theta})\left(z(f*\mathcal{F})''(z)+(f*\mathcal{F})'(z)\right)-p\rho e^{i\theta}\left((f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z)\right)$$

and

$$B(z)=(f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z).$$

Since, $\Re(w)\geq p\varkappa$ if and only if $|A(z)+p(1-\varkappa)B(z)|\geq|A(z)-p(1+\varkappa)B(z)|$, it suffices to show

$$|A(z)+p(1-\varkappa)B(z)|-|A(z)-p(1+\varkappa)B(z)|\geq 0.$$

But

$$\begin{aligned}
&|A(z)+p(1-\varkappa)B(z)|\\
&=\Bigg|(1+\rho e^{i\theta})\Bigg[z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2}\\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)+pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1}\\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}\Bigg]-p\rho e^{i\theta}\Bigg[pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1}\\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}+\tau z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2}\\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)\Bigg]+p(1-\varkappa)\Bigg[pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1}\\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}+\tau z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2}\\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)\Bigg]\Bigg|\\
&=\Bigg|p^2\left(2+\tau(p-1)(1-\varkappa-\rho e^{i\theta})-\varkappa\right)z^{p-1}\\
&\quad+\sum_{n=p+t}^{\infty}n\left[n(1+\rho e^{i\theta})+p(\tau(n-1)+1)(1-\varkappa-\rho e^{i\theta})\right]\left|a_nA_n\right|z^{n-1}\\
&\quad+\sum_{n=p+t-1}^{\infty}n\left[n(1+\rho e^{i\theta})+p(\tau(n-1)+1)(1-\varkappa-\rho e^{i\theta})\right]\left|b_nB_n\right|\overline{z}^{n-1}\Bigg|.
\end{aligned}$$

Also

$$
\begin{aligned}
&|A(z)-p(1+\varkappa)B(z)| \\
&=\Bigg|(1+\rho e^{i\theta})\Bigg[z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2} \\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)+pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}\Bigg]-p\rho e^{i\theta}\Bigg[pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}+\tau z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2} \\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)\Bigg]-p(1+\varkappa)\Bigg[pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}+\tau z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2} \\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)\Bigg]\Bigg| \\
&=\Big|-p^2\left[(\rho e^{i\theta}+\varkappa+1)\tau(p-1)+\varkappa\right]z^{p-1} \\
&\quad+\sum_{n=p+t}^{\infty}n\left[n(1+\rho e^{i\theta})-p(\tau(n-1)+1)(\rho e^{i\theta}+\varkappa+1)\right]\left|a_nA_n\right|z^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}n\left[n(1+\rho e^{i\theta})-p(\tau(n-1)+1)(\rho e^{i\theta}+\varkappa+1)\right]\left|b_nB_n\right|\overline{z}^{n-1}\Bigg|.
\end{aligned}
$$

Then

$$
\begin{aligned}
&|A(z)+p(1-\varkappa)B(z)|-|A(z)-p(1+\varkappa)B(z)|\geq 2p^2\left[1-\varkappa-\tau(p-1)(\rho e^{i\theta}+\varkappa)\right]|z|^{p-1} \\
&\quad+\sum_{n=p+t}^{\infty}2n\left[p(\tau(n-1)+1)(\rho e^{i\theta}+\varkappa)-n(1+\rho e^{i\theta})\right]\left|a_nA_n\right||z|^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}2n\left[p(\tau(n-1)+1)(\rho e^{i\theta}+\varkappa)-n(1+\rho e^{i\theta})\right]\left|b_nB_n\right||z|^{n-1} \\
&>2\Bigg\{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]-\sum_{n=p+t}^{\infty}n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right] \\
&\quad\left|a_nA_n\right|-\sum_{n=p+t-1}^{\infty}n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]\left|b_nB_n\right|\Bigg\}>0.
\end{aligned}
$$

The last expression is non-negative by (2.1), thus $f \in \xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$.

For $\sum\limits_{n=p+t}^{\infty} |x_n| + \sum\limits_{n=p+t-1}^{\infty} |y_n| = 1$, the function

$$\begin{aligned} f(z) \quad &= \quad z^p + \sum_{n=p+t}^{\infty} \frac{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]}{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]} x_n z^n \\ &\quad + \sum_{n=p+t-1}^{\infty} \frac{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]}{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]} \overline{y_n z^n}. \end{aligned}$$

This completes the proof. □

In the following theorem, it is shown that the condition (2.1) is also necessary for function $f = h + \overline{g}$, where h and g are of the form (1.5) and belongs to the class $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$.

Theorem 2.2 *Let the function $f = h + \overline{g}$ be so that h and g are given by (**??**). Then $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ if and only if*

$$\begin{aligned} &\sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]}{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]} |a_n A_n| \\ &\qquad + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]}{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]} |b_n B_n| \leq 1, \end{aligned} \tag{2.3}$$

where $0 \leq \varkappa < 1,\ \rho \geq 0,\ 0 \leq \tau < 1,\ z \in \mathcal{U}$.

Proof Since $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau) \subset \zeta_{\mathcal{F}}(p, \varkappa, \rho, \tau)$, we need only to prove the only if part of the theorem.

Assume that $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$. Then by (1.4), we have

$$\Re\left\{\frac{z(f*\mathcal{F})''(z)+(f*\mathcal{F})'(z)}{(f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z)}(1+\rho e^{i\theta})-p\rho e^{i\theta}\right\} \geq p\varkappa.$$

This is equivalent to

$$\Re\left\{\frac{\begin{array}{l} p^2\left[1-\tau(p-1)\rho e^{i\theta}\right]z^{p-1}-\sum\limits_{n=p+t}^{\infty} n\left[n+(n-p(\tau(n-1)+1))\rho e^{i\theta}\right] \\ |a_n A_n|\, z^{n-1}-\sum\limits_{n=p+t-1}^{\infty} n\left[n+(n-p(\tau(n-1)+1))\rho e^{i\theta}\right]|b_n B_n|\, \overline{z}^{n-1} \end{array}}{\begin{array}{l} p(1+\tau(p-1))z^{p-1}-\sum\limits_{n=p+t}^{\infty} n(1+\tau(n-1))\,|a_n A_n|\, z^{n-1} \\ \qquad -\sum\limits_{n=p+t-1}^{\infty} n(1+\tau(n-1))\,|b_n B_n|\, \overline{z}^{n-1} \end{array}}-\alpha p\right\} \geq 0. \tag{2.4}$$

This condition must hold for all values of z, such that $|z| = r < 1$. Choosing the values of z on the positive specific values, $0 \leq z = r < 1$ and noting that $\Re(-e^{i\theta}) \geq -\left|e^{i\theta}\right| = -1$, the above inequality reduces to

$$\begin{aligned} p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right] &- \sum_{n=p+t}^{\infty} n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]|a_nA_n| \\ &- \sum_{n=p+t-1}^{\infty} n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]|b_nB_n| \geq 0. \end{aligned}$$

This gives (2.3) and the proof is complete. □

§3. Distortion Bounds

Theorem 3.1 *Let $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. Then for $|z| = r < 1$, we have*

$$\begin{aligned} |f(z)| \leq\ & (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \left(\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}\right. \\ & \left. - \frac{(p+t-1)\left[(p+t-1)(1+\rho)-p(1+\tau(p+t-2))(\varkappa+\rho)\right]}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}|a_{p+t-1}A_{p+t-1}|\right) r^{p+t} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} |f(z)| \geq\ & (1-|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} - \left(\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}\right. \\ & \left. - \frac{(p+t-1)\left[(p+t-1)(1+\rho)-p(1+\tau(p+t-2))(\varkappa+\rho)\right]}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}|a_{p+t-1}A_{p+t-1}|\right) r^{p+t}. \end{aligned} \tag{3.2}$$

Proof Assume that $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. Then by (2.3), we get

$$\begin{aligned} |f(z)| &= \left| z^p - \sum_{n=p+t}^{\infty} |a_nA_n|\, z^n - \sum_{n=p+t-1}^{\infty} |b_nB_n|\, \overline{z}^n \right| \\ &\leq (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \sum_{n=p+t}^{\infty} (|a_nA_n|+|b_nB_n|)\, r^{p+t} \\ &\leq (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]} \\ &\quad \times \sum_{n=p+t}^{\infty} \frac{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} (|a_nA_n|+|b_nB_n|)\, r^{p+t} \\ &\leq (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]} \\ &\quad \times \sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} (|a_nA_n|+|b_nB_n|)\, r^{p+t} \\ &= (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]} \end{aligned}$$

$$
\begin{aligned}
&\times\left\{1-\frac{(p+t-1)\left[(p+t-1)(1+\rho)-p(1+\tau(p-t-2))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\left|a_{p+t-1}A_{p+t-1}\right|\right\}r^{p+t}\\
=&\ (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1}+\left\{\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}\right.\\
&\left.-\frac{(p+t-1)\left[(p+t-1)(1+\rho)-p(1+\tau(p-t-2))(\varkappa+\rho)\right]}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}\left|a_{p+t-1}A_{p+t-1}\right|\right\}r^{p+t}.
\end{aligned}
$$

The relation (3.2) can be proved by using similar statements. So the proof is complete. □

§4. Extreme Points

In this section we determine the extreme points of the closed convex hull of the class $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Theorem 4.1 *Let $f(z)$ given by (1.5). Then $f(z)\in\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ if and only if $f(z)$ can be expressed in the form*

$$
f(z)=\sum_{n=p+t-1}^{\infty}\left(\mu_n h_n(z)+\delta_n g_n(z)\right),\qquad z\in\mathcal{U}, \tag{4.1}
$$

where $h_p(z)=z^p$,

$$
h_n(z)=z^p-\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}z^n, n=p+t,p+t+1,\cdots
$$

and

$$
g_n(z)=z^p-\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\overline{z}^n, n=p+t-1,p+t\cdots,
$$

$$
\mu_{p+t-1}\equiv\mu_p=1-\left(\sum_{n=p+t}^{\infty}\mu_n+\sum_{n=p+t-1}^{\infty}\delta_n\right),\quad \mu_n,\delta_n\geq 0.
$$

Particularly, the extreme points of $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ are $\{h_n\}$ and $\{g_n\}$.

Proof Assume that $f(z)$ can be expressed by (4.1). Then, we have

$$
\begin{aligned}
f(z)\ =&\ \sum_{n=p+t-1}^{\infty}(\mu_n+\delta_n)z^p-\sum_{n=p+t}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\mu_n z^n\\
&-\sum_{n=p+t-1}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\delta_n\overline{z}^n
\end{aligned}
$$

$$
f(z)\ =\ z^p-\sum_{n=p+t}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\mu_n z^n
$$

$$-\sum_{n=p+t-1}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\delta_n\overline{z}^n.$$

Therefore

$$\begin{aligned}&\sum_{n=p+t}^{\infty}\frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\mu_n\\&\quad+\sum_{n=p+t-1}^{\infty}\frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\delta_n\\&=\sum_{n=p+t}^{\infty}\mu_n+\sum_{n=p+t-1}^{\infty}\delta_n=\sum_{n=p+t-1}^{\infty}(\mu_n+\delta_n)-\mu_{p+t-1}=1-\mu_p\leq 1.\end{aligned}$$

So $f(z)\in\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Conversely, let $f(z)\in\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$, by putting

$$\mu_n=\frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\left|a_nA_n\right|,n=p+t,p+t+1,\cdots$$

and

$$\delta_n=\frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\left|b_nB_n\right|,n=p+t-1,p+t,\cdots$$

We define $\mu_p\equiv\mu_{p+t-1}=\left(1-\sum\limits_{n=p+t}^{\infty}\mu_n-\sum\limits_{n=p+t-1}^{\infty}\delta_n\right)$.

Then, note that $0\leq\mu_n\leq 1$ $(n=p+t,p+t+1,\cdots)$, $0\leq\delta_n\leq 1$ $(n=p+t-1,p+t,\cdots)$. Hence

$$\begin{aligned}f(z)&=z^p-\sum_{n=p+t}^{\infty}\left|a_nA_n\right|z^n-\sum_{n=p+t-1}^{\infty}\left|b_nB_n\right|\overline{z}^n\\&=z^p-\sum_{n=p+t}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\mu_nz^n\\&\quad-\sum_{n=p+t-1}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\delta_n\overline{z}^n\\&=z^p-\sum_{n=p+t}^{\infty}\left(z^p-h_n(z)\right)\mu_n-\sum_{n=p+t-1}^{\infty}\left(z^p-g_n(z)\right)\delta_n\\&=\left(1-\sum_{n=p+t}^{\infty}\mu_n-\sum_{n=p+t-1}^{\infty}\delta_n\right)z^p+\sum_{n=p+t}^{\infty}\mu_nh_n(z)+\sum_{n=p+t}^{\infty}\delta_ng_n(z)\\&=\mu_ph_p(z)+\sum_{n=p+t}^{\infty}\mu_nh_n(z)+\sum_{n=p+t-1}^{\infty}\delta_ng_n(z)=\sum_{n=p+t-1}^{\infty}\left(\mu_nh_n(z)+\delta_ng_n(z)\right),\end{aligned}$$

that is the required representation. □

§5. Convolution and Convex Combination

In this section, we determine the convolution properties and convex combination.

For harmonic function

$$f_k(z) = z^p - \sum_{n=p+t}^{\infty} |a_{n,k}|\, z^n - \sum_{n=p+t-1}^{\infty} |b_{n,k}|\, \overline{z}^n \qquad (k = 1, 2), \tag{5.1}$$

are in the class $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$, we denote by $(f_1 * f_2)(z)$ the Hadamard product or (the convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = z^p - \sum_{n=p+t}^{\infty} |a_{n,1}|\, |a_{n,2}|\, z^n - \sum_{n=p+t-1}^{\infty} |b_{n,1}|\, |b_{n,2}|\, \overline{z}^n. \tag{5.2}$$

Using this definition, we show that the class $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ is closed under convolution.

Theorem 5.1 *For $0 \le \eta \le \varkappa < 1$, let the function $f_1 \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ and $f_2 \in \mathcal{T}\xi_{\mathcal{F}}(p, \eta, \rho, \tau)$. Then*

$$(f_1 * f_2)(z) \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau) \subset \mathcal{T}\xi_{\mathcal{F}}(p, \eta, \rho, \tau). \tag{5.3}$$

Proof Since f_1 be in the class $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ and f_2 be in the class $\mathcal{T}\xi_{\mathcal{F}}(p, \eta, \rho, \tau)$ and $|a_{n,2}| < 1$ and $|b_{n,2}| < 1$. We need to prove the coefficients of $(f_1 * f_2)(z)$ satisfy the condition given by (2.1), we obtain

$$\begin{aligned}
&\sum_{n=p+t}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\eta)]}{p^2\,[1-\eta-\tau(p-1)(\rho+\eta)]}\, |a_{n,1}|\, |a_{n,2}| \\
&\quad + \sum_{n=p+t-1}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\eta)]}{p^2\,[1-\eta-\tau(p-1)(\rho+\eta)]}\, |b_{n,1}|\, |b_{n,2}| \\
&\le \sum_{n=p+t}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\eta)]}{p^2\,[1-\eta-\tau(p-1)(\rho+\eta)]}\, |a_{n,1}| \\
&\quad + \sum_{n=p+t-1}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\eta)]}{p^2\,[1-\eta-\tau(p-1)(\rho+\eta)]}\, |b_{n,1}| \\
&\le \sum_{n=p+t}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\varkappa)]}{p^2\,[1-\varkappa-\tau(p-1)(\rho+\varkappa)]}\, |a_{n,1}| \\
&\quad + \sum_{n=p+t-1}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\varkappa)]}{p^2\,[1-\varkappa-\tau(p-1)(\rho+\varkappa)]}\, |b_{n,1}| \le 1.
\end{aligned}$$

Therefore $(f_1 * f_2)(z) \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau) \subset \mathcal{T}\xi_{\mathcal{F}}(p, \eta, \rho, \tau)$ for $0 \le \eta \le \varkappa < 1$. □

Next, we show that $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ is closed under convex combinations of its members.

Theorem 5.2 *The class $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ is closed under convex combinations.*

Proof For $j = 1, 2, 3, \cdots$, vLet $f_j \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$, where f_j is given by

$$f_j(z) = z^p - \sum_{n=p+t}^{\infty} |a_{n,j} A_{n,j}|\, z^n - \sum_{n=p+t-1}^{\infty} |b_{n,j} B_{n,j}|\, \overline{z}^n.$$

Then, by (2.3)

$$\sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |a_{n,j} A_{n,j}| + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |b_{n,j} B_{n,j}| \le 1. \tag{5.4}$$

For $\sum\limits_{j=1}^{\infty} t_j = 1, 0 \le t_j \le 1$, the convex combination of f_j's can be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z^p - \sum_{n=p+t}^{\infty} \left(\sum_{j=1}^{\infty} t_j\, |a_{n,j} A_{n,j}| \right) z^n - \sum_{n=p+t-1}^{\infty} \left(\sum_{j=1}^{\infty} t_j\, |b_{n,j} B_{n,j}| \right) \overline{z}^n.$$

Then by (5.4), we have

$$\begin{aligned}
&\sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} \left(\sum_{j=1}^{\infty} t_j\, |a_{n,j} A_{n,j}| \right) \\
&\quad + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} \left(\sum_{j=1}^{\infty} t_j\, |b_{n,j} B_{n,j}| \right) \\
&= \sum_{j=1}^{\infty} t_j \left\{ \sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |a_{n,j} A_{n,j}| \right. \\
&\quad \left. + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |b_{n,j} B_{n,j}| \right\} \\
&\le \sum_{j=1}^{\infty} t_j = 1.
\end{aligned}$$

Therefore $\sum\limits_{j=1}^{\infty} t_j f_j(z) \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$.

This complete the proof. □

§6. Integral Operator

Finally, we examine a closure property of the class $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ under the generalized Bernardi-Livingston integral operator (see [3, 17, 18]).

Definition 6.1 *The Bernardi operator is defined by*

$$L_{c,p}f(z) = \frac{c+p}{z^c}\int_0^\infty t^{c-1}f(t)dt, \quad c > -1. \tag{6.1}$$

If $f(z) = z^p + \sum\limits_{n=p+t}^{\infty} a_n z^n$, *then*

$$L_{c,p}f(z) = z^p + \sum_{n=p+t}^{\infty} \frac{c+p}{n+c} a_n z^n. \tag{6.2}$$

Remark 6.2 If $f = h + \overline{g}$, where

$$h(z) = z^p - \sum_{n=p+t}^{\infty} a_n z^n, \quad g(z) = -\sum_{n=p+t-1}^{\infty} b_n z^n, \quad (a_n, b_n \geq 0).$$

Then

$$L_{c,p}f(z) = L_{c,p}(h(z)) + \overline{L_{c,p}(g(z))}. \tag{6.3}$$

Theorem 6.3 *If* $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$, *then* $L_{c,p}f(z)$ $(c > -1)$ *is also in* $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Proof By (6.2) and (6.3), we get

$$\begin{aligned} L_{c,p}f(z) &= L_{c,p}\left(z^p - \sum_{n=p+t}^{\infty} |a_n A_n|\, z^n - \sum_{n=p+t-1}^{\infty} |b_n B_n|\, \overline{z}^n\right) \\ &= z^p - \sum_{n=p+t}^{\infty} \frac{c+p}{n+c} |a_n A_n|\, z^n - \sum_{n=p+t-1}^{\infty} \frac{c+p}{n+c} |b_n B_n|\, \overline{z}^n \\ &= z^p - \sum_{n=p+t}^{\infty} x_n\, |a_n A_n|\, z^n - \sum_{n=p+t-1}^{\infty} y_n\, |b_n B_n|\, \overline{z}^n. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} \left(\frac{c+p}{n+c}\, |a_n A_n|\right) \\ &\quad + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} \left(\frac{c+p}{n+c}\, |b_n B_n|\right) \\ &\leq \sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\, |a_n A_n| \\ &\quad + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\, |b_n B_n| \leq 1. \end{aligned}$$

Since $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$, by using Theorem 2.2, then $L_{c,p}f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. This complete the proof of Theorem 6.3. □

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