International J.Math. Combin. Vol.4(2020), 47-65

On Some Fixed Point Theorems for Generalized ψ-Weak Contraction Mappings in Partial Metric Spaces Using C-Class Function

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Abstract: The main goal of this paper is to establish some fixed point theorems for generalized ψ-weak contraction mappings in the setting of complete partial metric spaces using C-class function. Also we give some examples in support of our results. As applications of our results, we obtain some fixed point results for contractive mappings of integral type. Our results extend, generalize and modify several results from the current existing literature regarding partial metric spaces and contractive conditions.

Key Words: Fixed point, coincidence point, generalized ψ-weak contraction mapping, partial metric space.

AMS(2010): 47H10,54H25.

§1. Introduction and Preliminaries

Let (X, d) be a metric space and let $f\colon X \to X$ be a self-mapping. Then,

(i) A point $x \in X$ is called a fixed point of f if $x = fx$;

(ii) f is called contraction if there exists a fixed constant $0 \leq c < 1$ such that

$$d(f(x), f(y)) \leq c\, d(x, y) \tag{1.1}$$

for all $x, y \in X$. If X is complete, then every contraction has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point of X (the Banach contraction principle). Obviously, every contraction is a continuous function. The Banach contraction mappings principle is the opening and vital result in the direction of fixed point theory. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. This principle is constructive in nature and is one of the most useful techniques in the study of nonlinear equations. Subsequently, several authors have devoted their concentration to expanding and improving this theory (see, e.g., [3, 4, 9, 10, 12, 13, 19, 20]).

[1]Received August 12, 2020, Accepted November 19, 2020.

Partial metric spaces, introduced by Matthews ([16, 17]) are a generalizations of the notion of metric space in which, in definition of metric the condition $d(x,x) = 0$ is replaced by the condition $d(x,x) \leq d(x,y)$. In [17], Matthews discussed some properties of convergence of sequences and proved the fixed point theorem for contraction mapping on partial metric spaces: any mapping T of a complete partial metric space X onto itself that satisfies, where $0 \leq b < 1$, the inequality $p(T(x), T(y)) \leq b\, p(x,y)$ for all $x, y \in X$, has a unique fixed point. Also, the concept of PMS provides to study denotational semantics of dataflow networks [16, 17, 21, 23].

The definition of partial metric space is given by Matthews ([16]) as follows:

Definition 1.1([16]) *Let X be a nonempty set and let $p\colon X \times X \to \mathbb{R}^+$ be a function satisfy*

(pm1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(pm2) $p(x,x) \leq p(x,y)$;
(pm3) $p(x,y) = p(y,x)$;
(pm4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

for all $x, y, z \in X$. Then p is called partial metric on X and the pair (X,p) is called partial metric space (in short PMS).

It is clear that if $p(x,y) = 0$, then from $(pm1)$ and $(pm2)$ we obtain $x = y$. But if $x = y$, $p(x,y)$ may not be zero. Various applications of this space has been extensively investigated by many authors (see [15], [22] for details).

Remark 1.2([11]) Let (X,p) be a partial metric space.

(r1) The function $d_p\colon X \times X \to \mathbb{R}^+$ defined as $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a (usual) metric on X and (X, d_p) is a (usual) metric space;

(r2) The function $d_m\colon X \times X \to \mathbb{R}^+$ defined as $d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\}$ is a (usual) metric on X and (X, d_m) is a (usual) metric space.

It is clear that d_p and d_m are equivalent. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Example 1.3([6]) Let $X = \mathbb{R}^+$ and $p\colon X \times X \to \mathbb{R}^+$ given by $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.

Example 1.4([6]) Let $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$. Then $p\Big([a,b],[c,d]\Big) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on X.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows ([16]).

Definition 1.5([16]) *Let (X,p) be a partial metric space. Then,*

(a1) *A sequence $\{x_n\}$ in (X,p) is said to be convergent to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n, x)$;*

(a2) *A sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m, x_n)$ exists and finite;*

(a3) (X,p) *is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$ *with respect to* τ_p. *Furthermore,*

$$\lim_{m,n\to\infty} p(x_m,x_n) = \lim_{n\to\infty} p(x_n,x) = p(x,x);$$

(a4) *A mapping* $G\colon X \to X$ *is said to be continuous at* $x_0 \in X$ *if for every* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that* $G\Big(B_p(x_0,\delta)\Big) \subset B_p\Big(G(x_0),\varepsilon\Big)$.

Definition 1.6([18]) *Let* (X,p) *be a partial metric space. Then,*

(b1) *A sequence* $\{x_n\}$ *in* (X,p) *is called* 0*-Cauchy if* $\lim_{m,n\to\infty} p(x_m,x_n) = 0$;

(b2) (X,p) *is said to be* 0*-complete if every* 0*-Cauchy sequence* $\{x_n\}$ *in* X *converges to a point* $x \in X$, *such that* $p(x,x) = 0$.

Definition 1.7([1], Weak Contraction Mapping) *Let* (X,d) *be a complete metric space. A mapping* $f\colon X \to X$ *is said to be weakly contractive if*

$$d(f(x),f(y)) \le d(x,y) - \psi(d(x,y)), \tag{1.2}$$

where $x,y \in X$, $\psi\colon [0,\infty) \to [0,\infty)$ *is continuous and non-decreasing,* $\psi(x) = 0$ *if and only if* $x = 0$ *and* $\lim_{x\to\infty} \psi(x) = \infty$.

If we take $\psi(x) = cx$ where $0 < c < 1$ then (1.2) reduces to (1.1).

Definition 1.8 *Let* (X,p) *be a partial metric space. A point* $y \in X$ *is called point of coincidence of two self mappings* T *and* f *on* X *if there exists a point* $x \in X$ *such that* $y = Tx = fx$.

In 2014, Ansari [5] introduced and study C-class function and proved some fixed point theorems.

Definition 1.9([5]) *A mapping* $F\colon [0,\infty)\times[0,\infty) \to R$ *is called a* C*-class function if it is continuous and satisfies the following axioms:*

(*i*) $F(s,t) \le s$;

(*ii*) $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$, *for all* $s,t \in [0,\infty)$.

An extra condition on F is that $F(0,0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ denotes the set of all C-class functions. The following example shows that $\mathcal{C}$ is nonempty.

Example 1.10([5]) Define a function $F\colon [0,\infty)\times[0,\infty) \to R$ by

(*i*) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;

(*ii*) $F(s,t) = ms, 0 < m < 1$, $F(s,t) = s \Rightarrow s = 0$, ; (*iii*) $F(s,t) = \frac{s}{(1+t)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0 \,\text{or}\, t = 0$;

(*iv*) $F(s,t) = \frac{log(t+a^s)}{1+t}$, $a > 1$, $F(s,t) = s \Rightarrow s = 0 \,\text{or}\, t = 0$;

(*v*) $F(s,t) = \frac{ln(1+a^s)}{2}$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;

(vi) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;,

(vii) $F(s,t) = s log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

$(viii)$ $F(s,t) = s - \left(\frac{1+s}{2+s}\right)\left(\frac{t}{1+t}\right)$, $F(s,t) = s \Rightarrow t = 0$;

(ix) $F(s,t) = s\beta(s)$, where $\beta\colon [0,\infty) \to [0,\infty)$ and is continuous, $F(s,t) = s \Rightarrow s = 0$;,

(x) $F(s,t) = s - \left(\frac{t}{k+t}\right)$, $F(s,t) = s \Rightarrow t = 0$;

(xi) $F(s,t) = s - \varphi(s)$, $F(s,t) = s \Rightarrow s = 0$, here $\varphi\colon [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$;

(xii) $F(s,t) = sh(s,t)$, $F(s,t) = s \Rightarrow s = 0$, here $h\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ is a continuous function such that $h(s,t) < 1$ for all $t, s > 0$;

$(xiii)$ $F(s,t) = s - \left(\frac{2+t}{1+t}t\right)$, $F(s,t) = s \Rightarrow t = 0$;

(xiv) $F(s,t) = \sqrt[n]{ln(1+s^n)}$, $F(s,t) = s \Rightarrow s = 0$;

(xv) $F(s,t) = \phi(s)$, $F(s,t) = s \Rightarrow s = 0$, here $\phi\colon [0,\infty) \to [0,\infty)$ is a upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$;

(xvi) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$;

$(xvii)$ $F(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t} dx$, where Γ is the Euler gamma function.

Then F are elements of $\mathcal{C}$.

Definition 1.11([5]) *A function $\psi\colon [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied*

(1) *ψ is non-decreasing and continuous function*

(2) *$\psi(t) = 0$ if and only if $t = 0$.*

Remark 1.12([5]) We denote Ψ the class of all altering distance functions.

Lemma 1.13([16, 17]) *Let (X,p) be a partial metric space. Then,*

(c1) *A sequence $\{x_n\}$ in (X,p) is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X,d_p);*

(c2) *(X,p) is complete if and only if the metric space (X,d_p) is complete;*

(c3) *A subset E of a partial metric space (X,p) is closed if a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges to some $x \in X$, then $x \in E$.*

Lemma 1.14([2]) *Assume that $x_n \to z$ as $n \to \infty$ in a partial metric space (X,p) such that $p(z,z) = 0$. Then $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.*

The purpose of this paper is to prove a unique fixed point theorem and a coincidence point theorem under generalized ψ-weak contraction in the setting of partial metric spaces using C-class function. Our results extend, generalize and improve several results from the existing literatures.

§2. Main Results

In this section, we shall establish a unique fixed point theorem and a coincidence point theorem in a complete partial metric space. We begin with the following.

Let (X,p) be a partial metric space and $\mathcal{T}\colon X\to X$ be a mapping. We set

$$\theta_1(x,y)=\max\Big\{p(x,y),p(x,\mathcal{T}x),\frac{1}{4}[p(x,\mathcal{T}y)+p(y,\mathcal{T}x)]\Big\}, \tag{2.1}$$

$$\theta_2(x,y)=\min\Big\{p(x,\mathcal{T}x),p(y,\mathcal{T}y)\Big\}. \tag{2.2}$$

With the above setting, we introduce the following definition.

Definition 2.1 *Let (X,p) be a partial metric space. A mapping $\mathcal{T}\colon X\to X$ is called a generalized ψ-weak contraction if*

$$\psi\Big(p(\mathcal{T}x,\mathcal{T}y)\Big)\le F\Big(\psi(\theta_1(x,y)),\psi(\theta_2(x,y))\Big), \tag{2.3}$$

for all $x,y\in X$, where F is a C-class function, that is, $F\in\mathcal{C}$, $\psi\colon[0,\infty)\to[0,\infty)$ is nondecreasing and continuous function with $\psi(t)=0$ if and only if $t=0$.

Now, we are in a position to prove our main result.

Theorem 2.2 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X\to X$ be a generalized ψ-weak contraction mapping, that is, satisfying condition (2.3). Then $\mathcal{T}$ has a unique fixed point.*

Proof Let $x_0\in X$ and $\{x_n\}$ be a sequence defined as $x_{n+1}=\mathcal{T}x_n$ for any $n\in\mathbb{N}$. If for some $n\in\mathbb{N}$, $x_n=x_{n+1}=\mathcal{T}x_n$, then x_n is a fixed point of $\mathcal{T}$. So, we assume that $x_n\neq x_{n+1}$. It follows from (2.3) and $(pm4)$ that

$$\begin{aligned}\psi\Big(p(x_n,x_{n+1})\Big) &= \psi\Big(p(\mathcal{T}x_{n-1},\mathcal{T}x_n)\Big)\\ &\le F\Big(\psi\big(\theta_1(x_{n-1},x_n)\big),\psi\big(\theta_2(x_{n-1},x_n)\big)\Big),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\theta_1(x_{n-1},x_n) &= \max\Big\{p(x_{n-1},x_n),p(x_{n-1},\mathcal{T}x_{n-1}),\frac{1}{4}[p(x_{n-1},\mathcal{T}x_n)+p(x_n,\mathcal{T}x_{n-1})]\Big\}\\ &= \max\Big\{p(x_{n-1},x_n),p(x_{n-1},x_n),\frac{1}{4}[p(x_{n-1},x_{n+1})+p(x_n,x_n)]\Big\}\\ &= \max\Big\{p(x_{n-1},x_n),p(x_{n-1},x_n),\frac{1}{4}[p(x_{n-1},x_n)+p(x_n,x_{n+1})\\ &\quad -p(x_n,x_n)+p(x_n,x_n)]\Big\}=p(x_{n-1},x_n),\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\theta_2(x_{n-1},x_n) &= \min\Big\{p(x_{n-1},\mathcal{T}x_{n-1}),p(x_n,\mathcal{T}x_n)\Big\}\\ &= \min\Big\{p(x_{n-1},x_n),p(x_n,x_{n+1})\Big\}=p(x_{n-1},x_n).\end{aligned} \tag{2.6}$$

From equations (2.4)-(2.6), we obtain

$$\begin{aligned}\psi\Big(p(x_n,x_{n+1})\Big) &\leq F\Big(\psi\big(p(x_{n-1},x_n)\big),\psi\big(p(x_{n-1},x_n)\big)\Big)\\ &\leq \psi\big(p(x_{n-1},x_n)\big). \qquad (2.7)\end{aligned}$$

Hence, we have

$$p(x_n,x_{n+1}) \leq p(x_{n-1},x_n).$$

It follows that the sequence $\{p(x_n,x_{n+1})\}$ is monotonically decreasing. Hence

$$p(x_n,x_{n+1}) \to 0 \text{ as } n \to \infty. \tag{2.8}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in X. Suppose on the contrary that the sequence $\{x_n\}$ is not Cauchy. Then there exists $\varepsilon > 0$ and increasing sequences of integers $\{m(k)\}$ and $\{n(k)\}$ such that for all integers k,

$$n(k) > m(k) > k, \tag{2.9}$$

$$p(x_{m(k)},x_{n(k)}) \geq \varepsilon. \tag{2.10}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.10). Then

$$p(x_{m(k)},x_{n(k)-1}) < \varepsilon. \tag{2.11}$$

Now, we have

$$\begin{aligned}\varepsilon &\leq p(x_{m(k)},x_{n(k)})\\ &\leq p(x_{m(k)},x_{n(k)-1}) + p(x_{n(k)-1},x_{n(k)}) - p(x_{n(k)-1},x_{n(k)-1})\\ &\leq p(x_{m(k)},x_{n(k)-1}) + p(x_{n(k)-1},x_{n(k)})\\ &< \varepsilon + p(x_{n(k)-1},x_{n(k)}) \text{ (by (2.11))}.\end{aligned} \tag{2.12}$$

Letting $k \to +\infty$ in equation (2.12) and using (2.8), we get

$$\lim_{k\to\infty} p(x_{m(k)},x_{n(k)}) = \varepsilon. \tag{2.13}$$

Again

$$\begin{aligned}p(x_{n(k)},x_{m(k)}) &\leq p(x_{n(k)},x_{n(k)-1}) + p(x_{n(k)-1},x_{m(k)-1})\\ &\quad +p(x_{m(k)-1},x_{m(k)}) - p(x_{n(k)-1},x_{n(k)-1})\\ &\quad -p(x_{m(k)-1},x_{m(k)-1})\\ &\leq p(x_{n(k)},x_{n(k)-1}) + p(x_{n(k)-1},x_{m(k)-1})\\ &\quad +p(x_{m(k)-1},x_{m(k)}),\end{aligned} \tag{2.14}$$

whereas

$$\begin{aligned} p(x_{n(k)-1}, x_{m(k)-1}) &\le p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) \\ &\quad + p(x_{m(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{n(k)}) \\ &\quad - p(x_{m(k)}, x_{m(k)}) \\ &\le p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) \\ &\quad + p(x_{m(k)}, x_{m(k)-1}). \qquad (2.15) \end{aligned}$$

Now, on letting $k \to +\infty$ in (2.14), (2.15), using (2.8) and (2.13), we obtain

$$\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{2.16}$$

Now setting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in inequality (2.3) and using $(pm4)$, we obtain

$$\begin{aligned} \psi\Big(p(x_{m(k)}, x_{n(k)})\Big) &= \psi\Big(p(\mathcal{T}x_{m(k)-1}, \mathcal{T}x_{n(k)-1})\Big) \\ &\le F\Big(\psi\big(\theta_1(x_{m(k)-1}, x_{n(k)-1})\big), \psi\big(\theta_2(x_{m(k)-1}, x_{n(k)-1})\big)\Big), \qquad (2.17) \end{aligned}$$

where

$$\begin{aligned} \theta_1(x_{m(k)-1}, x_{n(k)-1}) &= \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, \mathcal{T}x_{m(k)-1}), \\ &\quad \frac{1}{4}\big[p(x_{m(k)-1}, \mathcal{T}x_{n(k)-1}) + p(x_{n(k)-1}, \mathcal{T}x_{m(k)-1})\big]\Big\} \\ &= \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad \frac{1}{4}\big[p(x_{m(k)-1}, x_{n(k)}) + p(x_{n(k)-1}, x_{m(k)})\big]\Big\} \\ &= \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad \frac{1}{4}\big[p(x_{m(k)-1}, x_{m(k)}) + p(x_{m(k)}, x_{n(k)}) \\ &\quad - p(x_{m(k)}, x_{m(k)}) + p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) \\ &\quad - p(x_{n(k)}, x_{n(k)})\big]\Big\} \\ &\le \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad \frac{1}{4}\big[p(x_{m(k)-1}, x_{m(k)}) + p(x_{m(k)}, x_{n(k)}) \\ &\quad + p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)})\big]\Big\}. \end{aligned}$$

On letting $k \to +\infty$ and using (2.8), (2.13) and (2.16), we get

$$\theta_1(x_{m(k)-1}, x_{n(k)-1}) \to \varepsilon, \tag{2.18}$$

and

$$\begin{aligned}\theta_2(x_{m(k)-1}, x_{n(k)-1}) &= \min\Big\{p(x_{m(k)-1}, \mathcal{T}x_{m(k)-1}), p(x_{n(k)-1}, \mathcal{T}x_{n(k)-1})\Big\} \\ &= \min\Big\{p(x_{m(k)-1}, x_{m(k)}), p(x_{n(k)-1}, x_{n(k)})\Big\}.\end{aligned}$$

On letting $k \to +\infty$ and using (2.8), we get

$$\theta_2(x_{m(k)-1}, x_{n(k)-1}) \to 0. \tag{2.19}$$

Thus, using equation (2.17), (2.18) and (2.19), we obtain

$$\psi(\varepsilon) \quad \leq \quad F\big(\psi(\varepsilon), \psi(0)\big) \leq \psi(\varepsilon),$$

which implies $\psi(\varepsilon) = 0$. That is $\varepsilon = 0$, which is a contradiction. Thus the sequence $\{x_n\}$ is a Cauchy sequence and hence convergent. Thus by Lemma 1.13 this sequence will also Cauchy in (X, d_p). In addition, since (X, p) is complete, (X, d_p) is also complete. Thus there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Moreover by Lemma 1.14,

$$p(z, z) = \lim_{n\to\infty} p(z, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0, \tag{2.20}$$

implies

$$\lim_{n\to\infty} d_p(z, x_n) = 0. \tag{2.21}$$

Now, we show that z is a fixed point of $\mathcal{T}$. Notice that due to (2.20), we have $p(z, z) = 0$. Putting $x = x_{n-1}$ and $y = z$ in equation (2.3), we obtain

$$\begin{aligned}\psi\Big(p(x_n, \mathcal{T}z)\Big) &= \psi\Big(p(\mathcal{T}x_{n-1}, \mathcal{T}z)\Big) \\ &\leq F\Big(\psi(\theta_1(x_{n-1}, z)), \psi(\theta_2(x_{n-1}, z))\Big) \\ &\leq \psi(\theta_1(x_{n-1}, z)),\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}\theta_1(x_{n-1}, z) &= \max\Big\{p(x_{n-1}, z), p(x_{n-1}, \mathcal{T}x_{n-1}), \frac{1}{4}[p(x_{n-1}, \mathcal{T}z) \\ &\quad +p(z, \mathcal{T}x_{n-1})]\Big\} \\ &= \max\Big\{p(x_{n-1}, z), p(x_{n-1}, x_n), \frac{1}{4}[p(x_{n-1}, \mathcal{T}z) \\ &\quad +p(z, x_n)]\Big\}.\end{aligned} \tag{2.23}$$

On letting $n \to +\infty$ in (2.23), using (2.20) and Lemma 1.14, we get

$$\theta_1(x_{n-1}, z) \to \frac{p(z, \mathcal{T}z)}{4}. \tag{2.24}$$

On letting $n \to +\infty$ in (2.22), using (2.24) and continuity of ψ, we get

$$\psi\Big(p(z, \mathcal{T}z)\Big) \leq \psi\Big(\frac{p(z, \mathcal{T}z)}{4}\Big).$$

The above inequality is possible only if $p(z, \mathcal{T}z) = 0$. Thus $z = \mathcal{T}z$. This shows that z is a fixed point of $\mathcal{T}$. Now to prove the uniqueness of the fixed point of $\mathcal{T}$. For this, assume that z' be another fixed point of $\mathcal{T}$ such that $z' = \mathcal{T}z'$ with $z' \neq z$. Now, using (2.3), (2.20) and condition ($pm3$), we have

$$\begin{aligned} \psi(p(z, z')) &= \psi\Big(p(\mathcal{T}z, \mathcal{T}z')\Big) \\ &\leq F\Big(\psi\big(\theta_1(z, z')\big), \psi\big(\theta_2(z, z')\big)\Big) \leq \psi\big(\theta_1(z, z')\big), \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} \theta_1(z, z') &= \max\Big\{p(z, z'), p(z, \mathcal{T}z), \frac{1}{4}[p(z, \mathcal{T}z') + p(z', \mathcal{T}z)]\Big\} \\ &= \max\Big\{p(z, z'), p(z, z), \frac{1}{4}[p(z, z') + p(z', z)]\Big\} \\ &= p(z, z'). \end{aligned} \tag{2.26}$$

From (2.25) and (2.26), we get

$$\psi(p(z, z')) \leq \psi(p(z, z')).$$

The above inequality is possible only if $p(z, z') = 0$. Thus $z = z'$. This shows the fixed point of $\mathcal{T}$ is unique. This completes the proof. □

If we take $\max\Big\{p(x, y), p(x, \mathcal{T}x), \frac{1}{4}\big[p(x, \mathcal{T}y] + p(y, \mathcal{T}x)\big]\Big\} = p(x, y)$, $F(s, t) = ks$, $0 < k < 1$ and $\psi(t) = t$ for all $t \geq 0$ in the Theorem 2.2, then we obtain the following result in the form of a Banach contraction principle ([7]).

Corollary 2.3 *Let (X, p) be a complete partial metric space. Let $\mathcal{T} \colon X \to X$ be a mapping satisfying the inequality*

$$p(\mathcal{T}x, \mathcal{T}y) \leq k\, p(x, y)$$

for all $x, y \in X$, where $0 < k < 1$ is a constant. Then $\mathcal{T}$ has a unique fixed point in X.

Remark 2.4 Corollary 2.3 extends Banach fixed point theorem from complete metric space to the setting of complete partial metric space.

If we take $F(s, t) = ks$, $0 < k < 1$ and $\psi(t) = t$ for all $t \geq 0$ in the Theorem 2.2, then we obtain the following result.

Corollary 2.5 *Let (X, p) be a complete partial metric space. Let $\mathcal{T} \colon X \to X$ be a mapping* *satisfying the inequality*

$$p(\mathcal{T}x, \mathcal{T}y) \le k \max\Big\{p(x,y), p(x,\mathcal{T}x), \frac{1}{4}\big[p(x,\mathcal{T}y) + p(y,\mathcal{T}x)\big]\Big\},$$

for all $x, y \in X$, where $0 < k < 1$ is a constant. Then $\mathcal{T}$ has a unique fixed point in X.

The following result is obtain from Corollary 2.5.

Corollary 2.6 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X \to X$ be a mapping satisfying the inequality*

$$p(\mathcal{T}x, \mathcal{T}y) \le a_1\, p(x,y) + a_2\, p(x,\mathcal{T}x) + \frac{a_3}{4}\big[p(x,\mathcal{T}y) + p(y,\mathcal{T}x)\big]$$

for all $x, y \in X$, where $a_1, a_2, a_3 \ge 0$ are constants such that $a_1 + a_2 + a_3 < 1$. Then $\mathcal{T}$ has a unique fixed point in X.

Proof Follows from Corollary 2.5, by noting that

$$\begin{aligned} & a_1\, p(x,y) + a_2\, p(x,\mathcal{T}x) + \frac{a_3}{4}\big[p(x,\mathcal{T}y) + p(y,\mathcal{T}x)\big] \\ & \le (a_1 + a_2 + a_3) \max\Big\{p(x,y), p(x,\mathcal{T}x), \frac{1}{4}\big[p(x,\mathcal{T}y) + p(y,\mathcal{T}x)\big]\Big\}. \end{aligned}$$ □

If we take $F(s,t) = s - t$ in the Theorem 2.2, then we obtain the following result.

Corollary 2.7 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X \to X$ be a mapping satisfying the inequality*

$$\psi\Big(p(\mathcal{T}x, \mathcal{T}y)\Big) \le \psi\Big(\theta_1(x,y)\Big) - \psi\Big(\theta_2(x,y)\Big),$$

for all $x, y \in X$, where $\theta_1(x,y)$, $\theta_2(x,y)$ and ψ are as in Theorem 2.2. Then $\mathcal{T}$ has a unique fixed point in X.

If we take $\max\Big\{p(x,y), p(x,\mathcal{T}x), \frac{1}{4}\big[p(x,\mathcal{T}y]+p(y,\mathcal{T}x)\big]\Big\} = p(x,y)$, $F(s,t) = ks$, $0 < k < 1$ and $\psi(t) = t$ for all $t \ge 0$ in the Theorem 2.2, then we obtain the following result due to Matthews [17].

Corollary 2.8([17], Theorem 5.3) *Let (X,p) be a complete partial metric space. Suppose that $\mathcal{T}\colon X \to X$ be a mapping satisfying the condition*

$$p(\mathcal{T}x, \mathcal{T}y) \le k\, p(x,y), \tag{2.27}$$

for all $x, y \in X$ and $0 < k < 1$ is a constant. Then $\mathcal{T}$ has a unique fixed point.

If we take $F(s,t) = s$ and

$$\max\Big\{p(x,y), p(x,\mathcal{T}x), \frac{1}{4}\big[p(x,\mathcal{T}y] + p(y,\mathcal{T}x)\big]\Big\} = p(x,y)$$

in the Theorem 2.2, then we obtain the following result.

Corollary 2.9 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X\to X$ be a mapping satisfying the inequality:*

$$\psi\Big(p(\mathcal{T}x,\mathcal{T}y)\Big)\le\psi\big(p(x,y)\big),$$

for all $x,y\in X$, where ψ is as in Theorem 2.2. Then $\mathcal{T}$ has a unique fixed point in X.

Remark 2.10 If we take $\psi(t)=t$ for all $t\ge 0$ in Corollary 2.9, then we obtain Theorem 5.3 of Matthews [17].

If we take $F(s,t)=\frac{s}{(1+s)^r}$ for $r>0$ in the Theorem 2.2, then we obtain the following result.

Corollary 2.11 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X\to X$ be a mapping satisfying the inequality*

$$\psi\Big(p(\mathcal{T}x,\mathcal{T}y)\Big)\le\frac{\theta_1(x,y)}{\Big(1+\theta_1(x,y)\Big)^r},$$

for all $x,y\in X$, where $r>0$ and $\theta_1(x,y)$ and ψ are as in Theorem 2.2. Then $\mathcal{T}$ has a unique fixed point in X.

Theorem 2.12 *Let $\mathcal{T}$ and f be two self-maps on a complete partial metric space X satisfying the inequality*

$$\psi\Big(p(\mathcal{T}x,\mathcal{T}y)\Big)\le F\Big(\psi\Big(M_1(x,y)\Big),\psi\Big(M_2(x,y)\Big)\Big),\tag{2.28}$$

where

$$M_1(x,y)=\max\Big\{p(fx,fy),p(fx,\mathcal{T}x),\frac{1}{4}[p(fx,\mathcal{T}y)+p(fy,\mathcal{T}x)]\Big\},$$

and

$$M_2(x,y)=\min\Big\{p(fx,\mathcal{T}x),p(fy,\mathcal{T}y)\Big\},$$

for all $x,y\in X$, where $F\in\mathcal{C}$ and $\psi\in\Psi$. If $\mathcal{T}(X)\subset f(X)$ and $f(X)$ is a complete subspace of X, then $\mathcal{T}$ and f have a coincidence fixed point.

Proof Let $x_0\in X$ and choose a point x_1 in X such that $\mathcal{T}x_0=fx_1,\ldots,\mathcal{T}x_n=fx_{n+1}$. Then, from (2.28) and $(pm4)$, we get

$$\begin{aligned}\psi\Big(p(fx_n,fx_{n+1})\Big)&=\psi\Big(p(\mathcal{T}x_{n-1},\mathcal{T}x_n)\Big)\\&\le F\Big(\psi\Big(M_1(x_{n-1},x_n)\Big),\psi\Big(M_2(x_{n-1},x_n)\Big)\Big),\end{aligned}\tag{2.29}$$

where

$$\begin{aligned} M_1(x_{n-1},x_n) &= \max\Big\{p(fx_{n-1},fx_n),p(fx_{n-1},\mathcal{T}x_{n-1}),\frac{1}{4}[p(fx_{n-1},\mathcal{T}x_n) \\ &\quad +p(fx_n,\mathcal{T}x_{n-1})]\Big\} \\ &= \max\Big\{p(fx_{n-1},fx_n),p(fx_{n-1},fx_n),\frac{1}{4}[p(fx_{n-1},fx_{n+1}) \\ &\quad +p(fx_n,fx_n)]\Big\} \\ &= \max\Big\{p(fx_{n-1},fx_n),p(fx_{n-1},fx_n),\frac{1}{4}[p(fx_{n-1},fx_n) \\ &\quad +p(fx_n,fx_{n+1})-p(fx_n,fx_n)+p(fx_n,fx_n)]\Big\} \\ &= p(fx_{n-1},fx_n), \end{aligned} \tag{2.30}$$

and

$$\begin{aligned} M_2(x_{n-1},x_n) &= \min\Big\{p(fx_{n-1},\mathcal{T}x_{n-1}),p(fx_n,\mathcal{T}x_n)\Big\} \\ &= \min\Big\{p(fx_{n-1},fx_n),p(fx_n,fx_{n+1})\Big\} \\ &= p(fx_{n-1},fx_n). \end{aligned} \tag{2.31}$$

From equation (2.29)-(2.31), we get

$$\begin{aligned} \psi\Big(p(fx_n,fx_{n+1})\Big) &\leq F\Big(\psi\Big(p(fx_{n-1},fx_n)\Big),\psi\Big(p(fx_{n-1},fx_n)\Big)\Big) \\ &\leq \psi\Big(p(fx_{n-1},fx_n)\Big). \end{aligned} \tag{2.32}$$

Hence, we have

$$p(fx_n,fx_{n+1}) \leq p(fx_{n-1},fx_n).$$

It follows that the sequence $\{p(fx_n,fx_{n+1})\}$ is monotonically decreasing. Hence

$$p(fx_n,fx_{n+1}) \to 0 \text{ as } n\to\infty. \tag{2.33}$$

Now, we shall show that $\{fx_n\}$ is a Cauchy sequence in X. As in Theorem 2.2, we can easily show that $\{fx_n\}$ is a Cauchy sequence in X. Thus, by Lemma 1.13 this sequence will also Cauchy in (X,d_p). In addition, since (X,p) is complete, (X,d_p) is also complete. Thus there exists $u\in X$ such that $x_n\to u \Rightarrow fx_n\to fu$ as $n\to\infty$, since $f(X)$ is a complete subspace of X. Moreover, by Lemma 1.14

$$p(fu,fu)=\lim_{n\to\infty}p(fu,fx_n)=\lim_{n,m\to\infty}p(fx_n,fx_m)=0, \tag{2.34}$$

implies

$$\lim_{n\to\infty}d_p(fu,fx_n)=0. \tag{2.35}$$

Now, we show that u is a coincidence point of $\mathcal{T}$ and f. Notice that due to (2.34), we have $p(fu, fu) = 0$. Putting $x = x_{n-1}$ and $y = u$ in equation (2.28), we obtain

$$\begin{aligned} \psi\Big(p(fx_n, \mathcal{T}u)\Big) &= \psi\Big(p(\mathcal{T}x_{n-1}, \mathcal{T}u)\Big) \\ &\leq F\Big(\psi(M_1(x_{n-1}, u)), \psi(M_2(x_{n-1}, u))\Big) \leq \psi(M_1(x_{n-1}, u)), \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} M_1(x_{n-1}, u) &= \max\Big\{p(fx_{n-1}, fu), p(fx_{n-1}, \mathcal{T}x_{n-1}), \frac{1}{4}[p(fx_{n-1}, \mathcal{T}u) + p(fu, \mathcal{T}x_{n-1})]\Big\} \\ &= \max\Big\{p(fx_{n-1}, fu), p(fx_{n-1}, fx_n), \frac{1}{4}[p(fx_{n-1}, \mathcal{T}u) + p(fu, fx_n)]\Big\}. \end{aligned} \tag{2.37}$$

On letting $n \to +\infty$ in equation (2.37), using (2.34) and Lemma 1.14, we obtain

$$M_1(x_{n-1}, u) \to \frac{p(fu, \mathcal{T}u)}{4}. \tag{2.38}$$

On letting $n \to +\infty$ in equation (2.36), using (2.38) and Lemma 1.14, we obtain

$$\psi\Big(p(fu, \mathcal{T}u)\Big) \leq \psi\Big(\frac{p(fu, \mathcal{T}u)}{4}\Big). \tag{2.39}$$

The above inequality is possible only if $p(fu, \mathcal{T}u) = 0$. Thus $fu = \mathcal{T}u$. This shows that u is a coincidence point of $\mathcal{T}$ and f, that is, $fu = u = \mathcal{T}u$. This completes the proof. □

§3. Illustrations

Example 3.1 Let $X = \mathbb{R}$ and defined $p\colon X^2 \to \mathbb{R}^+$ by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then p is a partial metric on X and (X, p) is a partial metric space. Let $\mathcal{T}\colon X \to X$ be defined by $\mathcal{T}(x) = \frac{x}{7}$ and $\psi(t) = t$ for all $t \geq 0$, where $\psi\colon [0, \infty) \to [0, \infty)$ is continuous and non-decreasing function. Without loss of generality we assume that $x \geq y$. Then, choosing $x = 1$ and $y = \frac{1}{2}$, we have

$$\begin{aligned} p(x, y) = \max\{x, y\} &= x, \\ p(\mathcal{T}x, \mathcal{T}y) &= \max\Big\{\frac{x}{7}, \frac{y}{7}\Big\} = \frac{x}{7}, \\ p(x, \mathcal{T}x) &= \max\Big\{x, \frac{x}{7}\Big\} = x, \\ p(y, \mathcal{T}y) &= \max\Big\{y, \frac{y}{7}\Big\} = y, \\ p(x, \mathcal{T}y) &= \max\Big\{x, \frac{y}{7}\Big\} = x, \\ p(y, \mathcal{T}x) &= \max\Big\{y, \frac{x}{7}\Big\} = y, \\ \theta_1(x, y) &= \max\Big\{p(x, y), p(x, \mathcal{T}x), \frac{1}{4}[p(x, \mathcal{T}y) + p(y, \mathcal{T}x)]\Big\} \\ &= \max\{x, x, \frac{1}{4}(x + y)\} = x, \end{aligned}$$

$$\theta_2(x,y) = \min\big\{p(x,\mathcal{T}x), p(y,\mathcal{T}y)\big\} = \min\{x,y\} = y.$$

Result Analysis

(1) Now, consider the equation (2.27), we have

$$\begin{aligned}\psi\Big(p(\mathcal{T}(x),\mathcal{T}(y))\Big) &= \psi\Big(\frac{x}{7}\Big) = \frac{x}{7}\\ &\leq \psi(x) - \psi(y) = x - y,\end{aligned}$$

or

$$\frac{x}{7} \leq x - y.$$

Putting $x = 1$ and $y = \frac{1}{2}$, we have

$$\frac{1}{7} \leq 1 - \frac{1}{2} = \frac{1}{2},$$

which is true. Thus $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.7. Hence, by applying Corollary 2.7, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

(2) Consider the inequality (2.27), we have

$$\frac{x}{7} \leq k\,x$$

or

$$k \geq \frac{1}{7}.$$

If we take $0 < k < 1$, then $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.3 or Corollary 2.8. Hence, by applying Corollary 2.3, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

(3) Consider the inequality (2.27), we have

$$\frac{x}{7} \leq k\,x$$

or

$$k \geq \frac{1}{7}.$$

If we take $0 < k < 1$, then $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.5. Hence, by applying Corollary 2.5, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

(4) Consider the inequality (2.28), we have

$$\psi\Big(\frac{x}{7}\Big) = \frac{x}{7} \leq \psi(x) = x$$

or

$$\frac{1}{7} \leq 1,$$

which is true. Thus, $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.9. Hence, by applying Corollary 2.9, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

(5) Consider the inequality (2.28) and taking $r = 1$, we have

$$\psi\Big(\frac{x}{7}\Big) = \frac{x}{7} \leq \frac{x}{1+x}.$$

Putting $x = 1$, we get

$$\frac{1}{7} \leq \frac{1}{1+1} = \frac{1}{2},$$

which is true. Thus, $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.11. Hence, by applying Corollary 2.11, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

Example 3.2 Let $X = \{1, 2, 3, 4\}$ and $p\colon X \times X \to \mathbb{R}$ be defined by

$$p(x,y) = \begin{cases} |x-y| + \max\{x,y\}, & \text{if } x \neq y, \\ x, & \text{if } x = y \neq 1, \\ 0, & \text{if } x = y = 1, \end{cases}$$

for all $x, y \in X$. Then (X, p) is a complete partial metric space.

Define the mapping $\mathcal{T}\colon X \to X$ by

$$\mathcal{T}(1) = 1,\ \mathcal{T}(2) = 1,\ \mathcal{T}(3) = 2,\ \mathcal{T}(4) = 2.$$

Now, we have

$$p(\mathcal{T}(1), \mathcal{T}(2)) = p(1,1) = 0 \leq \frac{3}{4}.3 = \frac{3}{4}p(1,2),$$

$$p(\mathcal{T}(1), \mathcal{T}(3)) = p(1,2) = 3 \leq \frac{3}{4}.5 = \frac{3}{4}p(1,3),$$

$$p(\mathcal{T}(1), \mathcal{T}(4)) = p(1,2) = 3 \leq \frac{3}{4}.7 = \frac{3}{4}p(1,4),$$

$$p(\mathcal{T}(2), \mathcal{T}(3)) = p(1,2) = 3 \leq \frac{3}{4}.4 = \frac{3}{4}p(2,3),$$

$$p(\mathcal{T}(2), \mathcal{T}(4)) = p(1,2) = 3 \leq \frac{3}{4}.6 = \frac{3}{4}p(2,4),$$

$$p(\mathcal{T}(3), \mathcal{T}(4)) = p(2,2) = 2 \leq \frac{3}{4}.5 = \frac{3}{4}p(3,4).$$

Thus, $\mathcal{T}$ satisfies all the conditions of Corollary 2.3 and Corollary 2.8 with $k = \frac{3}{4} < 1$. Now, by applying Corollary 2.3, $\mathcal{T}$ has a unique fixed point, which in this case is 1.

Example 3.3 Let $X = \{0, 1, 2, 3, \dots\}$. Define $p\colon X \times X \to \mathbb{R}^+$ as $p(x,y) = \max\{x,y\}$ with

$\mathcal{T}, f\colon X \to X$ be defined respectively as follows: $f(x) = x$ for all $x \in X$ and

$$\mathcal{T}(x) = \begin{cases} x-1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly (X, p) is a partial metric space. Define the mapping $\psi\colon [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t$ for all $t \geq 0$ and taking $F(s,t) = s - t$. Now, let $x \leq y$. Then choose $x = \frac{1}{2}$ and $y = 1$, we have $p(\mathcal{T}x, \mathcal{T}y) = y - 1$, $p(fx, fy) = y$, $p(fx, \mathcal{T}x) = x$, $p(fy, \mathcal{T}y) = y$, $p(fx, \mathcal{T}y) = x$, $p(fy, \mathcal{T}x) = y$ and

$$\begin{aligned} M_1(x,y) &= \max\Big\{p(fx,fy), p(fx,\mathcal{T}x), \frac{1}{4}[p(fx,\mathcal{T}y) + p(fy,\mathcal{T}x)]\Big\} \\ &= \max\big\{y, x, \frac{1}{4}(x+y)\big\} = y, \end{aligned}$$

$$\begin{aligned} M_2(x,y) &= \min\Big\{p(fx,\mathcal{T}x), p(fy,\mathcal{T}y)\Big\} \\ &= \min\big\{x, y\big\} = x. \end{aligned}$$

Now, we have

$$p(\mathcal{T}x, \mathcal{T}y) = y - 1 \leq y - x.$$

Putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$0 \leq 1 - \frac{1}{2} = \frac{1}{2}.$$

The above inequality holds good. Thus $\mathcal{T}$ and f have the properties mentioned in Theorem 2.12. Hence the conditions of Theorem 2.12 are satisfied. Here it is seen that 0 is the point of coincidence of $\mathcal{T}$ and f, that is, $f(x) = 0 = T(x)$.

§4. Applications

As an application of our results, we introduce some fixed point theorems of integral type. Denote Φ the set of functions $\phi\colon [0, +\infty) \to [0, +\infty)$ satisfying the following hypothesis

($\mathcal{H}_1$) ϕ is a Lebesgue-integrable mapping on each compact subset of $[0, +\infty)$;

($\mathcal{H}_2$) for any $\varepsilon > 0$ we have $\int_0^\varepsilon \phi(s)ds > 0$.

Now, we have the following results.

Corollary 4.1 *Let (X, p) be a complete partial metric space. Let $\mathcal{T}\colon X \to X$ be a mapping.*

Suppose that there exists $0 < k < 1$ *such that for* $\phi \in \Phi$*, we have*

$$\int_0^{p(\mathcal{T}x,\mathcal{T}y)} \phi(s)ds \leq k \int_0^{p(x,y)} \phi(s)ds \tag{4.1}$$

for all $x, y \in X$*. Then* $\mathcal{T}$ *has a unique fixed point.*

Proof Follows from Corollary 2.3 or Corollary 2.8 by taking

$$t = \int_0^t \phi(s)ds. \tag{4.2}$$

This completes the proof. □

Remark 4.2 Corollary 4.1 extends Theorem 2.1 of Branciari [8] from complete metric space to the setting of complete partial metric space.

Corollary 4.3 *Let* (X, p) *be a complete partial metric space. Let* $\mathcal{T}\colon X \to X$ *be a mapping. Suppose that there exists* $0 < k < 1$ *such that for* $\phi \in \Phi$*, we have*

$$\int_0^{p(\mathcal{T}x,\mathcal{T}y)} \phi(s)ds \leq k \int_0^{\max\left\{p(x,y),p(x,\mathcal{T}x),\frac{1}{4}\left[p(x,\mathcal{T}y)+p(y,\mathcal{T}x)\right]\right\}} \phi(s)ds \tag{4.3}$$

for all $x, y \in X$*. Then* $\mathcal{T}$ *has a unique fixed point.*

Proof Follows from Corollary 2.5 by taking

$$t = \int_0^t \phi(s)ds. \tag{4.4}$$

This completes the proof. □

§5. Conclusion

In this article, we establish a unique fixed point theorem and a coincidence point theorem under generalized ψ-weak contractive mappings in the framework of complete partial metric spaces and give some examples in support of our results. As application of our results, we obtain some fixed point theorems for mappings satisfying contractive condition of integral type. Our results extend, generalize and modify several results from the existing literature.

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