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Dynamic Network with E-Index Applications

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Abstract: Unlike particles in the classical dynamics, the dynamical behavior of a complex network maybe not synchronized but fragmented, even a heterogenous moving in the eyes of human beings, which finally results in characterizing a complex network by random method or probability with statistics sometimes. However, such a dynamics on complex network is quite different from dynamics on particles because all mathematics are established on compatible systems but none on a heterogenous one. Naturally, a heterogenous system produces a contradictory system in general which was abandoned in classical mathematics but exists everywhere, i.e., it is inevitable if we would like to understand the reality of things in the world. Thus, we should establish such a mathematics on those of elements that contradictions appear together peacefully but without loss of the individual characters. For this objective, the network or in general, the continuity flow is the best candidate of the element, i.e., mathematical elements over a topological graph $\overrightarrow{G}$ in space. The main purpose of this paper is to establish such a mathematical theory on networks, including algebraic operations, differential and integral operations on networks, G-isomorphic operators, i.e., network mappings remains the unchanged underlying graph $\overrightarrow{G}$ with a generalization of the fundamental theorem of calculus, algebraic or differential equations with flow solutions and also, the dynamical equations of network with applications to other sciences by e-indexes on network. All of these results show the importance, i.e., quantitatively characterizing the reality of things by mathematical combinatorics.

Key Words: Complex network, Smarandache system, Smarandache multispace, contradictory system, continuity flow, calculus on network, dynamic equation of network, e-index, mathematical combinatorics.

AMS(2010): 05C10, 05C21, 34A12, 34D06, 35A08, 46B25, 51D20, 68M10.

§1. Introduction

Usually, standing on different viewing points brings about different models for understanding the reality of things in the world, which causes the knowledge is local or partial, not the whole on things and results in the limitation of humans. For thousands of years, one would like to divide a matter into sub-matters, i.e. its composition such as those of molecular, atoms and electrons and further, elementary particles ([25]), and a living thing into cells and genes for

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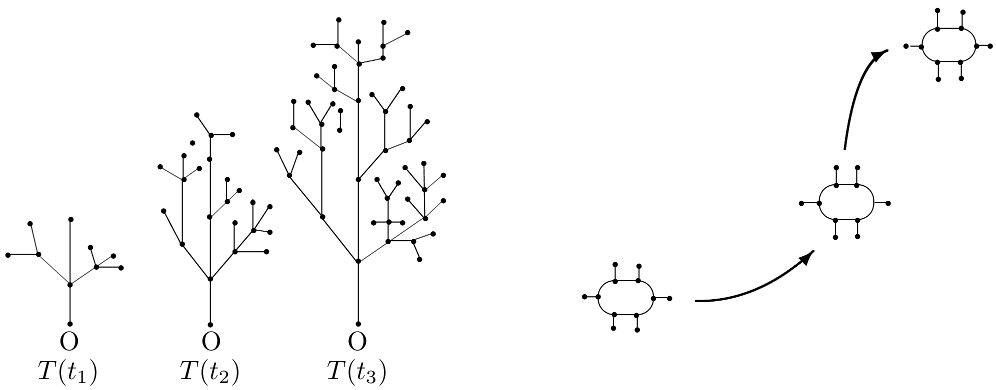
holding on its true face ([29]), which is essentially to equivalent a matter or thing to a complex network inherited by its fundamental units standing on a microscopic viewpoint. However, we are short of mathematics for characterizing the behavior of groups, particularly, a biological or adaptive system unless those of on the coordinated groups. Thus, we are more expected for establishing mathematics on groups, not only on those of isolated or ordered elements for holding on the reality of things.

According to the life cycle theory, there are series of stages for a living thing "*from birth to death*", i.e., birth, growth, maturity, decline and finally, death ([30]). Certainly, the *birth* is by chance but the *death* is inevitable, the growth, surviving and decline is the evolution or moving of a living thing such as those shown in Fig.1, where (a) is the evolution process of a tree and (b) is a mature horse runs on the earth.

(a) (b)

Fig.1

Then, *how do we characterize the evolution of the tree or moving of the horse appearing in Fig.*1? Usually, we characterize the pattern of a particle by differential equation in physics. Geometrically, we can depict the evolution of the tree or the running of the horse on the earth respectively by (a) or (b) in Fig.2.



(a) (b)

Fig.2

Certainly, the particle dynamics ignores the internal structure of the tree or the horse, abstracts them to points and characterizes their moving behavior by dynamic equations such as the Newtonian equations

$$\left(-\frac{\partial U}{\partial x_1}, -\frac{\partial U}{\partial x_2}, \cdots, -\frac{\partial U}{\partial x_n}\right) = \left(m\frac{d^2x_1}{dt^2}, m\frac{d^2x_2}{dt^2}, \cdots, m\frac{d^2x_m}{dt^2}\right), \tag{1.1}$$

where $U(\mathbf{x})$ is the potential energy of the field and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. However, we can not apply such an equation (1.1) for establishing the evolution equations of the tree, and also the running horse in Fig.1 because they are not particles but complex networks, unless their components are all in synchronized or ignored by us. Then, *what is the right dynamic equations on the tree or the horse in Fig.*1 *by the microscopic viewing?* They must be dynamic equations on complex networks or networks, i.e, graph dynamics different from that of a particle or a rigid body ([5]). Such a dynamics is essentially on group of elements, maybe not all synchronized but with internal relationship, i.e., non-harmonious groups defined by mathematics following:

Non-Harmonious Group. *A non-harmonious group is such a group $\mathscr{T}$ consisting of elements P_i, $1 \leq i \leq p, p \geq 2$ with internal relations that P_i is constrained on equation $\mathscr{F}_i = 0$ in space on time t.*

Such a non-harmonious group is in fact a *Smarandache system* or *Smarandache multispace* because it posses *Smarandache denied axioms* (See [7], [8], [9] and [26] for details), also the parallel universe ([28]) in physical terminology. Notice that there is an inherited graph $\overrightarrow{G}_T$ defined by ([10])

$$\begin{aligned} V\left(\overrightarrow{G}_T\right) &= \{P_i \mid 1 \leq i \leq p\}, \\ E\left(\overrightarrow{G}_T\right) &= \{(P_i, P_j) \mid \text{if } P_i \text{ is interrelated with } P_j \text{ for } 1 \leq i, j \leq p\} \end{aligned}$$

However, there is naturally also a topological line graph $\overrightarrow{G}_{LT}$ inherited in a non-harmonious group $\mathscr{T}$ with respective edge and vertex sets following

$$\begin{aligned} E\left(\overrightarrow{G}_{LT}\right) &= \{P_i \mid 1 \leq i \leq p\}, \\ V\left(\overrightarrow{G}_{LT}\right) &= \{\text{maximal subsets } \{P_{i_1}, P_{i_2}, \cdots, P_{i_s}\}, 1 \leq i_1, i_2, \cdots, i_s \leq p, \\ &\quad\ \text{where } P_{i_1}, P_{i_2}, \cdots, P_{i_s} \text{ have interrelation}\}, \end{aligned}$$

which is more useful for holding on the reality of mattes because nearly all living, non-living matters are non-harmonious groups with inherited line graph structures in the eyes of humans standing on a microscopic viewpoint.

Notice that such an inherited graph $\overrightarrow{G}_{LT}$ maybe more larger than the graph shown in Fig.1. For instance, we have known that a human body consists of $5 \times 10^{14} - 6 \times 10^{14}$ cells, i.e., the inherited graphs $\overrightarrow{G}_T, \overrightarrow{G}_{LT}$ of a human body by cells have respectively $5 \times 10^{14} - 6 \times 10^{14}$ vertices or edges. They are too larger graphs that nearly impossible to deal with them just by hands of humans. This fact implies that we should establish a mathematics on such non-harmonious groups for holding on the truth of matters, not only on its isolated elements but view the non-harmonious group $\mathscr{T}$ as a mathematical element entirely, i.e., mathematics over graphs or networks.

Notice that the evolution of the tree in Fig.1(a) is inclusive, i.e., the later includes the former $T(t_3) \supset T(t_2) \supset T(t_1)$ or the later develops from the former

$$T(t_3) = T(t_2) \bigcup (T(t_3) \setminus T(t_2)) = T(t_1) \bigcup (T(t_2) \setminus T(t_1)) \bigcup (T(t_3) \setminus T(t_2)) \qquad (1.2)$$

and also, all real networks such as those of internet, social relationship network, trading network, power and traffic network, $\cdots$, etc., are with the same advanced model. However, the running horse in Fig.1(b) is inclusive but unchanged, i.e., its inherited topological structure $\overrightarrow{G}_{LT}$ is invariable in running. Thus, the dynamics of the tree or the horse in Fig.1 and generally, a matter $\mathscr{T}$ can be always characterized by the motion of its inherited graph $\overrightarrow{G}_{LT}$ evolved at time t in space.

However, *can we conclude that a matter* $\mathscr{T} = \overrightarrow{G}_{LT}$*, the inherited graph of* $\mathscr{T}$*?* Certainly not because if we let $\mathscr{T}$ consisting of parts P_i, characterized by $\nu_j(P_i)$ with $i, j \geq 1$, we have

$$\mathscr{T} = \bigcup_{i \geq 1} P_i = \bigcup_{i \geq 1} \left(\bigcup_{j \geq 1} \nu_j(P_i) \right) \qquad (1.3)$$

in logic but the graph $\overrightarrow{G}_{LT}$ describes only the inherited structure but overlooked other characters of $\mathscr{T}$, which implies that a real model on $\mathscr{T}$ should retrieves all those of neglected characters on matter $\mathscr{T}$ in $\overrightarrow{G}_{LT}$, i.e., a dynamics on matters $\mathscr{T}$ should establishes on labeled graphs $\overrightarrow{G}^L_{LT}$ with labelling

$$\begin{aligned} & L : P_i \to P_i, \quad 1 \leq i \leq p, \\ & L : \{P_{i_1}, P_{i_2}, \cdots, P_{i_s}\} \to \bigcap_{k=1}^{s} P_{i_k} \quad \text{or} \quad L : (v, u) \to \bigcap_{k=1}^{s} \left(\bigcap_{l=1}^{s_l} \nu_l(P_{i_k}) \right), \end{aligned} \qquad (1.4)$$

where $\{i_1, i_2, \cdots, i_s\} \subset \{1, 2, \cdots, p\}$, i.e., network or its generalization, continuity flow following defined on the microscopic viewpoint, not only on particles or inherited graphs.

Definition 1.1([22-24]) *A continuity flow* $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ *is an oriented embedded graph* $\overrightarrow{G}$ *in a topological space* $\mathscr{S}$ *associated with a mapping* $L : v \to L(v),\ (v, u) \to L(v, u)$, *2 end-operators* $A^+_{vu} : L(v, u) \to L^{A^+_{vu}}(v, u)$ *and* $A^+_{uv} : L(u, v) \to L^{A^+_{uv}}(u, v)$ *on a Banach space* $\mathscr{B}$ *over a field* $\mathscr{F}$ *such as those shown in Fig.*3

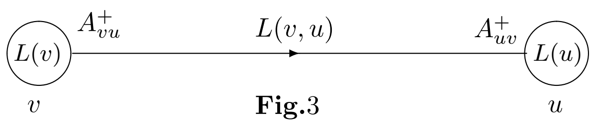


Fig.3

with $L(v, u) = -L(u, v)$, $A^+_{vu}(-L(v, u)) = -L^{A^+_{vu}}(v, u)$ *for* $\forall (v, u) \in E\left(\overrightarrow{G}\right)$ *holding with continuity equation*

$$\sum_{u \in N_G(v)} L^{A^+_{vu}}(v, u) = L(v) \quad for \quad \forall v \in V\left(\overrightarrow{G}\right) \qquad (1.5)$$

and all such continuity flows are denoted by $\mathscr{G}_{\mathscr{B}}$.

Notice that if we label edges by elements in a Banach space $\mathscr{B}$ and define the labels on vertices to be an induced labeling by

$$L(v) = \sum_{u \in N_G(v)} L^{A^+_{vu}}(v,u)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$, we can always get a continuity flow $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ on $\overrightarrow{G}$, and furthermore, if we let $\mathscr{B} = \mathbb{Z}$ and $\mathscr{A} = \{1_{\mathbb{Z}}\}$, a continuity flow $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ is nothing else but a network N.

In such induced continuity flows, the linear operator of $\mathscr{B}$, i.e., end-operators in $\mathscr{A}$ with criterion is in particular importance.

Definition 1.2([3]) *Let $\mathscr{B}$ be a Banach space over a field $\mathscr{F}$ and $\mathbf{T} : \mathscr{B} \to \mathscr{B}$ be an operator on Banach space $\mathscr{G}^{\pm}_{\mathscr{B}}$ over a field $\mathscr{F}$. Then, $\mathbf{T}$ is linear if*

$$\mathbf{T}(\lambda \cdot \mathbf{A} + \mu \cdot \mathbf{B}) = \lambda \cdot \mathbf{T}(\mathbf{A}) + \mu \cdot \mathbf{T}(\mathbf{B})$$

for $\forall \mathbf{A}, \mathbf{B} \in \mathscr{B}$ and $\lambda, \mu \in \mathscr{F}$.

Theorem 1.3([3]) *Let $\mathscr{B}_1, \mathscr{B}_2$ be Banach spaces over a field $\mathbb{F}$ with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then, a linear operator $\mathbf{T} : \mathscr{B}_1 \to \mathscr{B}_2$ is continuous if and only if it is bounded, or equivalently,*

$$\|\mathbf{T}\| := \sup_{\mathbf{0} \neq \mathbf{v} \in \mathscr{B}_1} \frac{\|\mathbf{T}(\mathbf{v})\|_2}{\|\mathbf{v}\|_1} < +\infty.$$

Now, *could we establish a mathematics on continuity flows $(\overrightarrow{G}; L, \mathscr{A})$ underlying a graph in family $\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_m\}$ by viewing each of them as a mathematical elements entirely for integer $m \geq 1$?* The answer is positive, particularly for linear operators $\mathscr{A}$. In fact, the papers [6], [10] established a geometrical theory on non-harmonious groups and [11] discussed non-mathematical systems by combinatorial method, which is the fundamental of mathematics on non-harmonious groups. The papers [13]-[25] establish mathematics on continuity flows by functionals with applications to physics and biology. All the discussions are viewing continuity flows to be purely elements of Banach flow space. The main purpose of this paper is to establish such a mathematics on continuity flows paying more attentions to structure of $\overrightarrow{G}$ such as those of algebraic operations, differential and integral operations on continuity flows, G-isomorphic operators, i.e., mappings on continuity flows remains the unchanged underlying graph $\overrightarrow{G}$ with a generalization of the fundamental theorem of calculus, algebraic or differential equations with flow solutions and particularly, the dynamical equations of networks with applications to other sciences by e-indexes, which implies the truth of results appearing in [13]-[25] holds also on G-isomorphic operators.

For terminologies and notations not mentioned here, we follow references [1] for mechanics, [3] for functionals, [4] for complex network, [7] for topology and graphs, [8], [26] for Smarandache systems and multispaces.

§2. Algebraic Operations

Notice that a continuity flow $\left(\overrightarrow{G};L,\mathscr{A}\right)$ is a labelled graph. An algebraic operation on continuity flows should posses both of the algebraic and graph properties. We define the operations of addition "+" and multiplication "·" as follows:

Definition 2.1 *Let $G^L, G'^L \in \mathscr{G}^t_{\mathscr{B}}$, $\lambda\in\mathscr{F}$. Define*

$$\overrightarrow{G}^L+\overrightarrow{G'}^{L'} = \left(\overrightarrow{G}\setminus\overrightarrow{G'}\right)^L\bigcup\left(\overrightarrow{G}\bigcap\overrightarrow{G'}\right)^{L+L'}\bigcup\left(\overrightarrow{G'}\setminus\overrightarrow{G}\right)^{L'}, \tag{2.1}$$

$$\overrightarrow{G}^L\cdot\overrightarrow{G'}^{L'} = \left(\overrightarrow{G}\setminus\overrightarrow{G'}\right)^L\bigcup\left(\overrightarrow{G}\bigcap\overrightarrow{G'}\right)^{L\cdot L'}\bigcup\left(\overrightarrow{G'}\setminus\overrightarrow{G}\right)^{L'}, \tag{2.2}$$

$$\lambda\cdot\overrightarrow{G}^L = \overrightarrow{G}^{\lambda\cdot L} \tag{2.3}$$

where, $L(v,u)$ and $L'(v,u)\in\mathscr{B}$, $L+L' : (v,u)\to L(v,u)+L'(v,u)$, $L\cdot L' : (v,u)\to L(v,u)\cdot L'(v,u)$ respectively with substituting end-operators $\mathcal{A}^{*+}_{vu}$, $\mathcal{A}^{*+}_{vu}$ and $\mathcal{A}^{**+}_{vu}$ action on $(v,u)\in E\left(\overrightarrow{G}\right)$ such that

$$\begin{aligned}
\mathcal{A}^{*+}_{vu} &: (L(v,u))+L'(v,u)\to L^{A^+_{vu}}(v,u)+{L'}^{{A'}^+_{vu}}(v,u),\\
\mathcal{A}^{*+}_{vu} &: L(v,u,)\cdot L'(v,u)\to L^{A^+_{vu}}(v,u)\cdot {L'}^{{A'}^+_{vu}}(v,u)),\\
\mathcal{A}^{**+}_{vu} &: \lambda\cdot L(v,u)\to \lambda\cdot L^{\mathscr{A}^+_{vu}}(v,u).
\end{aligned}$$

Let $\overrightarrow{G}^L,\overrightarrow{G'}^{L'}\in\mathscr{G}_{\mathscr{B}}$. A calculation shows that the labels on vertices of $\overrightarrow{G}$ are

$$L+L'(v)=\begin{cases} L(v) & \text{if } v\in\overrightarrow{G}\setminus\overrightarrow{G}',\\ L(v)+L'(v) & \text{if } v\in\overrightarrow{G}\bigcap\overrightarrow{G}',\\ L'(v) & \text{if } v\in\overrightarrow{G}'\setminus\overrightarrow{G}\end{cases}$$

and

$$L\cdot L'(v)=\begin{cases} L(v) & \text{if } v\in\overrightarrow{G}\setminus\overrightarrow{G}',\\ \sum\limits_{u\in N} L^{A^+_{vu}}(v,u)\cdot {L'}^{{A'}^+_{vu}}(v,u)) & \text{if } v\in\overrightarrow{G}\bigcap\overrightarrow{G}',\\ L'(v) & \text{if } v\in\overrightarrow{G}'\setminus\overrightarrow{G}\end{cases}$$

by definition. Particularly, if $\overrightarrow{G'}=\overrightarrow{G}$, we know that

$$L+L'(v)=L(v)+L'(v) \quad\text{and}\quad L\cdot L'(v)=\sum_{u\in N} L^{A^+_{vu}}(v,u)\cdot {L'}^{{A'}^+_{vu}}(v,u))$$

for $v\in\overrightarrow{G}$. The following convention are throughout in this paper.

Convention 2.2 *If $L(v,u)=\mathbf{0}$ for an edge $(v,u)\in E\left(\overrightarrow{G}^L\right)$, we always identify $\overrightarrow{G}^L$ with*

$\left(\overrightarrow{G}\setminus(v,u)\right)^L$, *i.e.*, $\overrightarrow{G}^L = \left(\overrightarrow{G}\setminus(v,u)\right)^L$.

Notice that the number of vertices of odd valency in a graph must be even. Thus, we can always transform a non-Eulerian graph to an Euleran graph by adding edges but with $\mathbf{0}$ flows between its odd vertices, which is essentially the same as the original continuity flows by Convention 2.2. We consider algebraic operations on continuity flows $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ following.

Definition 2.3 *Let* $a_1, a_2, \cdots, a_n \in \mathscr{B}$ *and* $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n} \in \mathscr{G}_{\mathscr{B}}$.

(1) Constant Elements. *Define* $a_i = \overrightarrow{G}^{I_{a_i}}$ *with* $I_{a_i} : (v,u) \to a_i$ *for* $\forall(v,u) \in E(G)$. *Particularly,* $0 = \overrightarrow{G}^{I_0} = \mathbf{O}$ *and* $1 = \overrightarrow{G}^{I_1} = \mathbf{I}$.

(2) Sum and Product. *Define*

$$
\begin{aligned}
a_1\overrightarrow{G}_1^{L_1} + a_2\overrightarrow{G}_2^{L_2} + \cdots + a_n\overrightarrow{G}_n^{L_n} &= \left(\bigcup_{i=1}^{n} G_i\right)^{a_1L_1+a_2L_2+\cdots+a_nL_n}, \\
\left(a_1\overrightarrow{G}_1^{L_1}\right)\cdot\left(a_2\overrightarrow{G}_2^{L_2}\right)\cdots\left(a_n\overrightarrow{G}_n^{L_n}\right) &= \left(\bigcup_{i=1}^{n} G_i\right)^{a_1L_1\cdot a_2L_2\cdot\cdots\cdot a_nL_n}.
\end{aligned}
$$

(3) Polynomial. *Define*

$$a_0 + a_1\overrightarrow{G}^L + a_2\overrightarrow{G}^{L^2} + \cdots + a_n\overrightarrow{G}^{L^n} = \overrightarrow{G}^{a_0+a_1L+a_2L^2+\cdots+a_nL^n}.$$

(4) Units. *Flows* $\mathbf{O}$ *and* $\mathbf{I}$ *are respectively the unit in* $(\mathscr{G}_{\mathscr{B}};+)$ *and* $(\mathscr{G}_{\mathscr{B}};\cdot)$ *because of*

$$
\begin{aligned}
\mathbf{O} + \overrightarrow{G}^L = \overrightarrow{G}^L + \mathbf{O} = \overrightarrow{G}^L, \\
\mathbf{I}\cdot\overrightarrow{G}^L = \overrightarrow{G}^L\cdot\mathbf{I} = \overrightarrow{G}^L.
\end{aligned}
$$

And we have operation properties of $\mathbf{O}$ *and* $\mathbf{I}$ *following:*

$$
\begin{aligned}
&\mathbf{O}+\mathbf{O}=\mathbf{O}, \quad \mathbf{O}+\mathbf{I}=\mathbf{I}+\mathbf{O}=\mathbf{I}, \\
&\mathbf{I}\cdot\mathbf{I}=\mathbf{I}, \quad \mathbf{O}\cdot\mathbf{O}=\mathbf{O}, \quad \mathbf{I}\cdot\mathbf{O}=\mathbf{O}\cdot\mathbf{I}=\mathbf{O}.
\end{aligned}
$$

(5) Inverse. *For* $\forall\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$, *if* $X + \overrightarrow{G}^L = \mathbf{O}$ *then* X *is defined to be the additive inverse of* $\overrightarrow{G}^L$. *Similarly, if* $Y\cdot\overrightarrow{G}^L = \mathbf{I}$ *then* Y *is defined to be the multiplication inverse of* $\overrightarrow{G}^L$.

Clearly,

$$X = -\overrightarrow{G}^L = \overrightarrow{G}^{-L} \quad \text{and} \quad Y = \frac{1}{\overrightarrow{G}^L} = \overrightarrow{G}^{\frac{1}{L}} = \overrightarrow{G}^{L^{-1}}.$$

We get the following equalities

$$-\overrightarrow{G}^L = \overrightarrow{G}^{-L} \quad \text{and} \quad \frac{1}{\overrightarrow{G}^L} = \overrightarrow{G}^{\frac{1}{L}}. \tag{2.4}$$

Applying formula (2.4), we immediately get the fraction, i.e.,

$$\frac{a_1\overrightarrow{G}_1^{L_1}+a_2\overrightarrow{G}_2^{L_2}+\cdots+a_n\overrightarrow{G}_n^{L_n}}{b_1\overrightarrow{G'}_1^{L'_1}+b_2\overrightarrow{G'}_2^{L'_2}+\cdots+b_n\overrightarrow{G'}_n^{L'_n}} = \frac{\left(\bigcup\limits_{i=1}^{n}G_i\right)^{a_1L_1+a_2L_2+\cdots+a_nL_n}}{\left(\bigcup\limits_{i=1}^{n}G'_i\right)^{b_1L'_1+b_2L'_2+\cdots+b_nL'_n}}$$

$$= \left(\bigcup_{i=1}^{n}G_i\right)^{a_1L_1+\cdots+a_nL_n}\cdot\frac{1}{\left(\bigcup\limits_{i=1}^{n}G'_i\right)^{b_1L'_1+\cdots+a_nL'_n}}$$

$$= \left(\left(\bigcup_{i=1}^{n}G_i\right)\bigcup\left(\bigcup_{i=1}^{n}G'_i\right)\right)^{\frac{a_1L_1+a_2L_2+\cdots+a_nL_n}{b_1L'_1+b_2L'_2+\cdots+b_nL'_n}}.$$

Notice that there are no the commutative laws

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$$

for $\forall\overrightarrow{G}^{L_1},\overrightarrow{G}^{L_2}\in\mathscr{G}_{\mathscr{B}}$ in general. However, we have

Theorem 2.4 *Let $\forall\overrightarrow{G}^{L_1},\overrightarrow{G}^{L_2}\in\mathscr{G}_{\mathscr{B}}$. Then,*

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$$

if and only if

$$L_1(v,u)\cdot L_2(v,u)=L_2(v,u)\cdot L_1(v,u)$$

and the same end-operators $\mathcal{A}_{vu}^{12+}=\mathcal{A}_{vu}^{21+}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, where $\mathcal{A}_{vu}^{12+}$ and $\mathcal{A}_{vu}^{21+}$ are end-operators on (v,u) in $\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}$ or $\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$, respectively.

Proof By (2.2), we know that

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_1\cdot L_2}\quad\text{and}\quad\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}=\overrightarrow{G}^{L_2\cdot L_1}.$$

Whence,

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$$

if and only if

$$L_1(v,u)\cdot L_2(v,u)=L_2(v,u)\cdot L_1(v,u)$$

and the same end-operators $\mathcal{A}_{vu}^{12+}=\mathcal{A}_{vu}^{21+}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. □

Corollary 2.5 *Let $\mathscr{B}$ be a commutative ring and let $\overrightarrow{G}^{L}\in\mathscr{G}_{\mathscr{B}}$ with $1_{\mathscr{B}}$ end-operator on $(v,u)\in E\left(\overrightarrow{G}\right)$. Then*

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$$

for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \mathscr{G}_{\mathscr{B}}$.

Proof It is obvious that

$$\mathcal{A}_{vu}^{12+} = \mathcal{A}_{vu}^{21+} = 1_{\mathscr{B}} \quad \text{and} \quad L_1(v,u) \cdot L_2(v,u) = L_2(v,u) \cdot L_1(v,u)$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ in this case if $\mathscr{B}$ is a commutative ring. □

Notice that if $(\mathscr{B};+,\cdot)$ is a division ring, i.e., $(\mathscr{B};+)$ and $(\mathscr{B};\cdot)$ are both of groups, then Corollary 2.5 implies the following conclusion.

Theorem 2.6 *If $(\mathscr{B};+,\cdot)$ is a division ring and every $\overrightarrow{G}^{L} \in \mathscr{G}_{\mathscr{B}}$ has $1_{\mathscr{B}}$ end-operator on $(v,u) \in E\left(\overrightarrow{G}\right)$, then $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ is a division ring. Furthermore, $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ is a field if $(\mathscr{B};+,\cdot)$ is a field.*

Proof Clearly, $(\mathscr{G}_{\mathscr{B}};+)$ and $(\mathscr{G}_{\mathscr{B}};\cdot)$ are both of Abelian groups with associative laws, i.e.,

$$\overrightarrow{G}^{L_1} \cdot \left(\overrightarrow{G}^{L_2} + \overrightarrow{G}^{L_3}\right) = \overrightarrow{G}^{L_1} \cdot \overrightarrow{G}^{L_2} + \overrightarrow{G}^{L_1} \cdot \overrightarrow{G}^{L_3}$$

and

$$\left(\overrightarrow{G}^{L_1} + \overrightarrow{G}^{L_2}\right) \cdot \overrightarrow{G}^{L_3} = \overrightarrow{G}^{L_1} \cdot \overrightarrow{G}^{L_3} + \overrightarrow{G}^{L_2} \cdot \overrightarrow{G}^{L_3}$$

for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2}, \overrightarrow{G}^{L_3} \in \mathscr{G}_{\mathscr{B}}$ because of

$$L_1 \cdot (L_2 + L_3) = L_1 \cdot L_2 + L_1 \circ L_3 \quad \text{and} \quad (L_1 + L_2) \cdot L_3 = L_1 \circ L_3 + L_2 \cdot L_3,$$

i.e., $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ is a division ring.

By Corollary 2.5, $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ is commutative if $(\mathscr{B};+,\cdot)$ is commutative, i.e., $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ is a field if $(\mathscr{B};+,\cdot)$ is a field. This completes the proof. □

Example 2.7 Let U and D be 2×2 matrixes over $\mathbb{R}$ determined by

$$U = \left\{ \left. \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right| a,b,c,d \in \mathbb{R} \right\}, \quad W = \left\{ \left. \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right| a,b \in \mathbb{R} \right\}$$

and $\overrightarrow{G}$ a digraph. For continuity flows $\overrightarrow{G}^{L}$ with all end-operators being the unit 1, and

$$L: \ (v,u) \to U, \quad (v,u) \in E\left(\overrightarrow{G}\right).$$

Then,

(1) $\left\{ \overrightarrow{G}^{L} \right| \left. L : (v,u) \to U \right\}$ maybe not commutative. For example, for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ let

$$L_1(v,u) = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad L_2(v,u) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$
\begin{aligned}
L_1(v,u)\cdot L_2(v,u) &= \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix},\\
L_2(v,u)\cdot L_1(v,u) &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 1 & 1 \end{pmatrix}.
\end{aligned}
$$

Thus,

$$L_1(v,u)\cdot L_2(v,u) \neq L_2(v,u)\cdot L_1(v,u),$$

i.e., $\left\{ \overrightarrow{G}^L \middle| \ L:(v,u)\to U \right\}$ is not commutative in this case by Theorem 3.1.

(2) Let

$$L_1(v,u) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad L_2(v,u) = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}$$

for $(v,u)\in E\left(\overrightarrow{G}\right)$. Then,

$$
\begin{aligned}
L_1(v,u)\cdot L_2(v,u) &= \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix},\\
L_2(v,u)\cdot L_1(v,u) &= \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix},
\end{aligned}
$$

i.e.,

$$L_1(v,u)\cdot L_2(v,u) = L_2(v,u)\cdot L_1(v,u)$$

for $(v,u)\in E\left(\overrightarrow{G}\right)$. We know that $\left\{ \overrightarrow{G}^L \middle| \ L:(v,u)\to W \right\}$ is commutative by Theorem 2.4.

§3. G-isomorphic Operators

By definition, the continuity flow is vectors associating with shapes, i.e., structures. Such a kind of operators that remains the topological structure $\overrightarrow{G}$ unchanged is particularly important.

3.1 G-isomorphic Operators on Continuity Flows

Definition 3.1 *An operator $f:\overrightarrow{G}_1^{L_1}\to\overrightarrow{G}_2^{L_2}$ on $\mathscr{G}_{\mathscr{B}}$ is G-isomorphic if it holds with conditions*

(i) there is an isomorphism $\varphi:\overrightarrow{G}_1\to\overrightarrow{G}_2$ of graph;
(ii) $L_2=f\circ\varphi\circ L_1$ for $\forall (v,u)\in E\left(\overrightarrow{G}_1\right)$.

We can therefore denote a G-isomorphic operator by $f:\overrightarrow{G}^{L_1}\to\overrightarrow{G}^{L_2}$. Particularly, let $\varphi=\mathrm{id}_{\overrightarrow{G}}$. Then, such an operator is determined by an equation

$$L_2 = f\circ L_1 \tag{3.1}$$

for $\forall(v,u)\in E(G)$ or in other words, a G-isomorphic operator is mapping on vectors with an invariant structure of graph.

Furthermore, if $\mathscr{B}$ is a function field on a variable t, i.e., $\mathscr{F}[t]$, we can therefore know such a G-isomorphic operator f holds with an equation

$$f\left(\overrightarrow{G}^{L[t]}\right)=\overrightarrow{G}^{f(L[t])}, \tag{3.2}$$

which enables us to get a few interesting equalities following.

(1) $a\left(\overrightarrow{G}^{L[t]}\right)^n=\overrightarrow{G}^{aL^n[t]}$ for $a\in\mathbb{R}$ and $n\in\mathbb{Z}^+$;

(2) $a^{\overrightarrow{G}^{L[t]}}=\overrightarrow{G}^{a^{L[t]}}$, $\log\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\log L[t]}$ for $0\neq a\in\mathbb{R}$,
$e^{\overrightarrow{G}^{L[t]}}=\overrightarrow{G}^{e^{L[t]}}$, $\ln\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\ln L[t]}$;

(3) $\sin\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\sin L[t]}$, $\cos\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\cos L[t]}$, $\tan\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\tan L[t]}$, $\cot\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\cot L[t]}$,
$\sinh\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\sinh L[t]}$, $\cosh\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\cosh L[t]}$,
$\coth\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\coth L[t]}$, $\tanh\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\tanh L[t]}$;

(4) $\left(\mathbf{I}+a\overrightarrow{G}^{L[t]}\right)^n=\overrightarrow{G}^{(1+aL[t])^n}$, $\left(\mathbf{I}+\frac{a\mathbf{I}}{\overrightarrow{G}^{L[t]}}\right)^n=\overrightarrow{G}^{\left(1+\frac{a}{L[t]}\right)^n}$ for $n\in\mathbb{Z}^+$, $a\in\mathbb{R}$;

(5) $\dfrac{\overrightarrow{G}^{nL[t]}}{\mathbf{I}-\overrightarrow{G}^{L[t]}}=\mathbf{I}+\overrightarrow{G}^{L[t]}+\overrightarrow{G}^{2L[t]}+\cdots+\overrightarrow{G}^{(n-1)L[t]}$ for $1\leq n\in\mathbb{Z}^+$.

Furthermore, we get the exponential map following.

Theorem 3.2 *Let $\overrightarrow{G}^{L[t]}\in\mathscr{G}_{\mathscr{B}}$, where $\mathscr{B}$ is a field. Then,*

$$e^{\overrightarrow{G}^{L[t]}}=\mathbf{I}+\frac{\overrightarrow{G}^{L[t]}}{1!}+\frac{\overrightarrow{G}^{2L[t]}}{2!}+\cdots+\frac{\overrightarrow{G}^{nL[t]}}{n!}+\cdots.$$

Proof Notice that

$$\begin{aligned}\mathbf{I}+\frac{\overrightarrow{G}^{L[t]}}{1!}+\frac{\overrightarrow{G}^{2L[t]}}{2!}+\cdots+\frac{\overrightarrow{G}^{nL[t]}}{n!}+\cdots &= \mathbf{I}+\overrightarrow{G}^{\frac{L[t]}{1!}}+\overrightarrow{G}^{\frac{2L[t]}{2!}}+\cdots+\overrightarrow{G}^{\frac{nL[t]}{n!}}+\cdots\\ &= \overrightarrow{G}^{1+\frac{L[t]}{1!}+\frac{2L[t]}{2!}+\cdots+\frac{nL[t]}{n!}+\cdots}=\overrightarrow{G}^{e^{L[t]}}.\end{aligned}$$

By equation (3.2), we know that $e^{\overrightarrow{G}^{L[t]}}=\overrightarrow{G}^{e^{L[t]}}$. Thus,

$$e^{\overrightarrow{G}^{L[t]}}=\mathbf{I}+\frac{\overrightarrow{G}^{L[t]}}{1!}+\frac{\overrightarrow{G}^{2L[t]}}{2!}+\cdots+\frac{\overrightarrow{G}^{nL[t]}}{n!}+\cdots. \qquad \square$$

By equation (3.2), it is clear that

$$\begin{aligned}e^{\overrightarrow{G}^{L[t]}}\cdot e^{\overrightarrow{G}^{L'[t]}} &= \overrightarrow{G}^{e^{L[t]}}\cdot\overrightarrow{G}^{e^{L'[t]}}=\overrightarrow{G}^{e^{L[t]}\cdot e^{L'[t]}}\\ &= \overrightarrow{G}^{e^{L[t]+L'[t]}}=e^{\overrightarrow{G}^{L[t]+L'[t]}},\end{aligned}$$

which is similar to that of $e^x\cdot e^y=e^{x+y}$ as the usual.

3.2 Extended Operators on Continuity Flows

Let $\overrightarrow{G}, \overrightarrow{H}$ be graphs with $\overrightarrow{G} \prec \overrightarrow{H}$. It is interesting to find an operator $f : \overrightarrow{G}^{L_1} \to \overrightarrow{H}^{L_2}$ for characterizing the trail from $\overrightarrow{G}^{L_1}$ to $\overrightarrow{H}^{L_2}$. By Convention 2.1, if $L(v,u) = \mathbf{0}$ for an edge $(v,u) \in E\left(G^L\right)$, we identify G^L with $(G\backslash(v,u))^L$ because there are no difference on flows between G^L with $(G\backslash(v,u))^L$.

Definition 3.3 *Let $\overrightarrow{G}, \overrightarrow{H}$ be graphs with $\overrightarrow{G} \prec \overrightarrow{H}$. An operator $f : \overrightarrow{G}^{L_1} \to \overrightarrow{H}^{L_2}$ on $\mathscr{G}_{\mathscr{B}}$ is extended if it holds with conditions*

(i) there is an isomorphism $\varphi : \overrightarrow{G} \to \overrightarrow{G}$ of graph;
(ii) $L_2 = f \circ \varphi \circ L_1$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ but $f : \mathbf{0} \to L_2(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{H} \setminus \overrightarrow{G}\right)$.

Certainly, such an extended operator maps a continuity flow to its extended flow. However, by Convention 2.2, we view such an extended operator f to be a H-isomorphic operator by the following ways

(1) Extend L_1 to L_1' by $L_1'(v,u) = L_1(v,u)$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ but $L_1'(v,u) = \mathbf{0}$ for $(v,u) \in E\left(\overrightarrow{H}^{L_2} \setminus \overrightarrow{G}^{L_1}\right)$;

(2) Extend $\varphi|_{\overrightarrow{G}}$ to $\varphi|_{\overrightarrow{H}}$ constraint by $\varphi|_{\overrightarrow{H}} = \varphi|_{\overrightarrow{G}}$ on graph $\overrightarrow{G}$.

By Definition 3.3, if an extended operator f exists, then its inverse f^{-1} must be existed because f is a $1-1$ mapping. Such a f^{-1} is called a contracted operator. For example, let $\overrightarrow{G}^{L_1}$, $\overrightarrow{G}^{L_2}$ be 2 continuity flows. An extended isomorphism $f(\mathbf{v_i}) = \mathbf{u_i}$ for $1 \le i \le 4$ but $f(0) = \mathbf{u_5}$ with its inverse f^{-1} is shown in Fig.4.

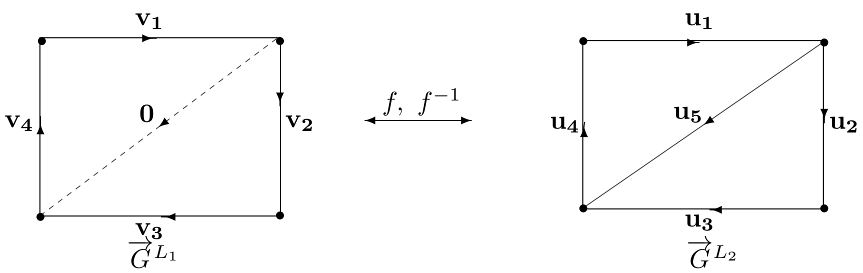


Fig.4

and if $f = ax + b$, $0 \ne a, b \in \mathbb{R}$, we can also get $\overrightarrow{G}^{L_2}$ by $\overrightarrow{G}^{L_1}$ shown in Fig.5.

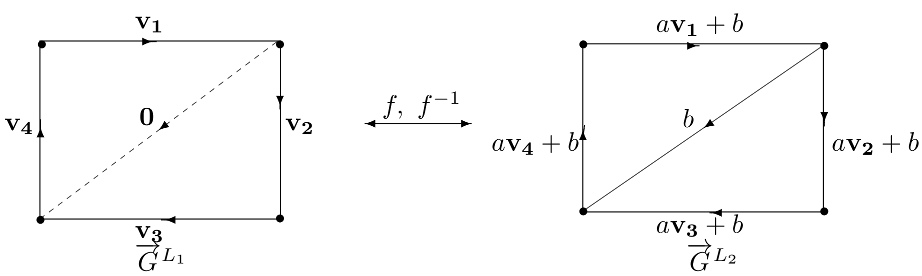


Fig.5

i.e., $\overrightarrow{G}^{L_2} = a\left(\overrightarrow{G}^{L_1}\right) + b = \overrightarrow{G}^{aL_1+b}$ with $f^{-1} = \dfrac{x}{a} - b$, both are linear G-isomorphic operators.

Generally, we get a result following.

Theorem 3.4 *Let $\emptyset \neq \overrightarrow{G}_1, \overrightarrow{G}_2 \in \mathscr{G}$, maybe with $\overrightarrow{G}_1 \simeq \overrightarrow{G}_2$ or not. There must be a G-isomorphic operator f such that*

$$f\left(\overrightarrow{G}_1^{L_1}\right) = \overrightarrow{G}_2^{L_2}$$

for $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$.

Proof Notice that $\overrightarrow{G}_1, \overrightarrow{G}_2 \neq \emptyset$. Let $G = \overrightarrow{G}_1 \bigcup \overrightarrow{G}_2$ and

$$L_1' = \begin{cases} L_1(v,u) & \text{if (v, u)} \in \mathrm{E}\left(\overrightarrow{G}_1\right), \\ \mathbf{0} & \text{otherwise;} \end{cases} \qquad L_2' = \begin{cases} L_2(v,u) & \text{if (v, u)} \in \mathrm{E}\left(\overrightarrow{G}_2\right), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then,

$$\overrightarrow{G}_1^{L_1} = \overrightarrow{G}^{L_1'} \quad \text{and} \quad \overrightarrow{G}_2^{L_2} = \overrightarrow{G}^{L_2'}$$

by Convention 2.2. Let φ be an automorphism of $\overrightarrow{G}$ and let $f : \overrightarrow{G}^{L_1'} \to \overrightarrow{G}^{L_2'}$ be an automorphism $f : \overrightarrow{G}^{L_1} \to \overrightarrow{G}^{L_2}$ with $L_2' = f \circ \varphi \circ L_1'$. Certainly, f is a G-isomorphic operator from $\overrightarrow{G}^{L_1'}$ to $\overrightarrow{G}^{L_2'}$, i.e.,

$$f\left(\overrightarrow{G}_1^{L_1}\right) = \overrightarrow{G}_2^{L_2}.$$

This completes the proof. □

3.3 Continuous Operators

Theorem 3.4 enables us to discuss the continuity behaviours of operators on $\mathscr{G}_{\mathscr{B}}$.

Definition 3.5 *Let $(\mathscr{B}; +, \cdot)$ be a normed space over field $\mathscr{F}$ with norm $\|\mathbf{v}\|$, $\mathbf{v} \in \mathscr{B}$ and $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$. The norm of $\overrightarrow{G}^L$ is defined by*

$$\left\|\overrightarrow{G}^L\right\| = \sum_{(v,u)\in E\left(\overrightarrow{G}\right)} \|L(v,u)\|,$$

i.e., the norm $\|\ \|$ is a mapping with $\|\ \| : \mathscr{G}_{\mathscr{B}}^t \to \mathbb{R}^+$.

For example, if $\mathbf{v_1} = (0,1), \mathbf{v_2} = (1,0), \mathbf{v_3} = (1,1), \mathbf{v_4} = (1,-1)$ with $\mathbf{v_5} = \mathbf{0}$ in Fig.4, then

$$\begin{aligned} \left\|\overrightarrow{G}^{L_1}\right\| &= \|\mathbf{v_1}\| + \|\mathbf{v_2}\| + \|\mathbf{v_3}\| + \|\mathbf{v_4}\| + \|\mathbf{0}\| \\ &= \sqrt{0^2+1^2} + \sqrt{1^2+0^2} + \sqrt{1^2+1^2} + \sqrt{1^2+(-1)^2} + 0 = 2\left(1+\sqrt{2}\right). \end{aligned}$$

Certainly, we are easily known that $\mathscr{G}_{\mathscr{B}}$ is a normed space by Definition 3.5, i.e., for $\forall \overrightarrow{G}^L, \overrightarrow{G}_1^{L_1}$ and $\overrightarrow{G}_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$,

(1) $\left\|\overrightarrow{G}^L\right\| \geq 0$ and $\left\|\overrightarrow{G}^L\right\| = 0$ if and only if $\overrightarrow{G}^L = \overrightarrow{G}^{\mathbf{0}} = \mathbf{O}$;

(2) $\left\|\overrightarrow{G}^{\xi L}\right\| = \xi\left\|\overrightarrow{G}^{L}\right\|$ for any scalar $\xi \in \mathscr{F}$;

(3) $\left\|\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right\| \leq \left\|\overrightarrow{G}_1^{L_1}\right\| + \left\|\overrightarrow{G}_2^{L_2}\right\|$.

Definition 3.6 *For $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$, the distance between $\overrightarrow{G}_1^{L_1}$ and $\overrightarrow{G}_2^{L_2}$ is defined by*

$$d\left(\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}\right) = \left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}_2^{L_2}\right\|.$$

By Definition 2.1, we know that

$$\overrightarrow{G}_1^{L_1} - \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1 - L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{L_2}.$$

Therefore,

$$\begin{aligned} d\left(\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}\right) &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)} \|L_1(e)\| \\ &+ \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)} \|L_1(e)\| + \sum_{e \in E\left(\overrightarrow{G}_2 \bigcap \overrightarrow{G}_1\right)} \|L_2(e)\|. \end{aligned}$$

For example, if $\mathbf{u_1} = (1,0), \mathbf{u_2} = (0,1), \mathbf{u_3} = (-1,-1), \mathbf{u_4} = (-1,1)$ and $\mathbf{u_5} = (-2,2)$ in Fig.4, then the distance of $\overrightarrow{G}^{L_1}$ and $\overrightarrow{G}^{L_2}$ is

$$\begin{aligned} d\left(\overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2}\right) &= \sum_{i=1}^{5} \|v_i - u_i\| = \sqrt{(-1)^2 + 1^2} + \sqrt{1^1 + (-1)^2} \\ &+ \sqrt{2^2 + 2^2} + \sqrt{2^2 + (-2)^2} + \sqrt{2^2 + (-2)^2} = 8\sqrt{2}. \end{aligned}$$

Definition 3.7 *Let f be a G-isomorphic operator on $\mathscr{G}_{\mathscr{B}}$, $\overrightarrow{G}^L$, $\overrightarrow{G}_0^{L_0} \in \mathscr{G}_{\mathscr{B}}$ dependent on a variable t. Then, f is G-continuous at $\overrightarrow{G}_0^{L_0}$, denoted by $\lim\limits_{L \to L_0} f(\overrightarrow{G}^L) = f(\overrightarrow{G}_0^{L_0})$ if for any number $\epsilon > 0$, there is always a number $\delta > 0$ such that*

$$d\left(f\left(\overrightarrow{G}^L[t]\right),\ f\left(\overrightarrow{G}_0^{L_0}[t_0]\right)\right) < \epsilon \tag{3.3}$$

if $d(\overrightarrow{G}^L[t], \overrightarrow{G}_0^{L_0}[t_0]) < \delta$. Furthermore, such an operator f is completely continuous, denoted by $\lim\limits_{t \to t_0} f(\overrightarrow{G}^L) = f(\overrightarrow{G}_0^{L_0})$ if the inequality (3.3) holds with $|t - t_0| < \delta$.

Clearly, a completely continuous operator does not depends on the structure of graph $\overrightarrow{G}$, i.e., it is G-free or in other words, it is G-continuous over any graph G.

Theorem 3.8 *Let f be a G-isomorphic operator on $\mathscr{G}_{\mathscr{B}}$, $\overrightarrow{G}^L$, $\overrightarrow{G}_0^{L_0} \in \mathscr{G}_{\mathscr{B}}$, where G is the union of all graphs in $\mathscr{G}$. Then,*

$$\lim_{L \to L_0} f\left(\overrightarrow{G}^L\right) = f\left(\overrightarrow{G}_0^{L_0}\right) \quad \textit{or} \quad \lim_{t \to t_0} f\left(\overrightarrow{G}^L\right) = f\left(\overrightarrow{G}_0^{L_0}\right)$$

if and only if f is continuous on L or $f \circ L$ is continuous on t for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

Proof Let $\overrightarrow{H} = \bigcup\limits_{\overrightarrow{G}_i \in \mathscr{G}} \overrightarrow{G}_i$. Without loss of generality, by Convention 2.2 we can let $\overrightarrow{G}^L = \overrightarrow{H}^L$ and $\overrightarrow{G}_0^{L_0} = \overrightarrow{H}^{L_0}$. By definition, f is G-continuous or completely continuous if for a number $\epsilon > 0$ there is always a number $\delta > 0$ such that if $d\left(\overrightarrow{H}^L[t], \overrightarrow{H}^{L_0}[t_0]\right) < \delta$ or $|t - t_0| < \delta$ then

$$d\left(f\left(\overrightarrow{H}^L[t]\right),\ f\left(\overrightarrow{H}^{L_0}[t_0]\right)\right) < \epsilon, \quad \text{i.e.,} \quad d\left(\overrightarrow{H}^{f(L)}[t],\ \overrightarrow{H}^{f(L_0)}[t_0]\right) < \epsilon,$$

which implies that

$$\left\|\overrightarrow{H}^{f(L)}[t] - \overrightarrow{H}^{f(L_0)}[t_0]\right\| < \epsilon, \quad \text{i.e.,} \quad \sum_{e \in E\left(\overrightarrow{H}\right)} \|(f(L[t]) - f(L_0[t_0]))(e)\| < \epsilon$$

by Definition 3.6. Notice that $\|e\| \geq 0$ for $e \in E\left(\overrightarrow{H}\right)$.

Conversely, for a number $\epsilon > 0$, if there is a number $\delta > 0$ such that

$$\|(f \circ L[t] - f \circ L_0[t_0])(e)\| < \frac{\epsilon}{\varepsilon\left(\overrightarrow{H}\right)}$$

for $\forall e \in E\left(\overrightarrow{H}\right)$ if $d\left(\overrightarrow{H}^L[t], \overrightarrow{H}^{L_0}[t_0]\right) < \delta$ or $|t - t_0| < \delta$, we get that

$$\begin{aligned} d\left(f\left(\overrightarrow{H}^L[t]\right),\ f\left(\overrightarrow{H}^{L_0}[t_0]\right)\right) &= \left\|\overrightarrow{H}^{f \circ L}[t] - \overrightarrow{H}^{f \circ L_0}[t_0]\right\| \\ &= \sum_{e \in E\left(\overrightarrow{H}\right)} \|(f \circ L[t] - f \circ L_0[t_0])(e)\| \\ &\leq \varepsilon\left(\overrightarrow{H}\right) \times \frac{\epsilon}{\varepsilon\left(\overrightarrow{H}\right)} = \epsilon \end{aligned}$$

where $\varepsilon\left(\overrightarrow{H}\right)$ is the size of $\overrightarrow{H}$. We therefore know that

$$d\left(f\left(\overrightarrow{H}^L[t]\right),\ f\left(\overrightarrow{H}^{L_0}[t_0]\right)\right) < \epsilon \ \Leftrightarrow\ \|(f(L[t]) - f(L_0[t_0]))(e)\| < \epsilon \tag{3.4}$$

for $\forall e \in E\left(\overrightarrow{H}\right)$.

Similarly, we can also know that

$$d\left(\overrightarrow{H}^L[t],\ \overrightarrow{H}^{L_0}[t_0]\right) < \epsilon \ \Leftrightarrow\ \|L[t] - L_0[t_0])(e)\| < \epsilon \tag{3.5}$$

for $\forall e \in E\left(\overrightarrow{G}\right)$.

By the equivalences (3.4) and (3.5), we are easily knowing that

$$\lim_{L[t] \to L[t_0]} f\left(\overrightarrow{H}^{L[t]}\right) = f\left(\overrightarrow{H}^{L_0[t_0]}\right), \quad \text{i.e.,} \quad \lim_{L \to L_0} f\left(\overrightarrow{G}^L\right) = f\left(\overrightarrow{G}_0^{L_0}\right) \tag{3.6}$$

if and only if f is continuous on L by definition, and

$$\lim_{t\to t_0} f\left(\overrightarrow{H}^{L[t]}\right) = f\left(\overrightarrow{H}^{L_0[t_0]}\right), \quad \text{i.e.,} \quad \lim_{t\to t_0} f\left(\overrightarrow{G}^{L}\right) = f\left(\overrightarrow{G}_0^{L_0}\right) \tag{3.7}$$

if and only if $f\circ L$ is continuous on t for $\forall (v,u)\in E\left(\overrightarrow{H}\right)$. This completes the proof. □

Notice that the composition of continuous functions is also continuous. We therefore know the conclusion following by Theorem 3.8.

Corollary 3.9 *If f respect to L and L respect to t both are continuous, then*

$$\lim_{t\to t_0} f\left(\overrightarrow{G}^{L[t]}\right) = f\left(\overrightarrow{G}_0^{L[t_0]}\right).$$

Example 3.10 Let $f = aL^2 + b$ with $0 \neq a, b \in \mathbb{R}$ and L shown in Fig.6.

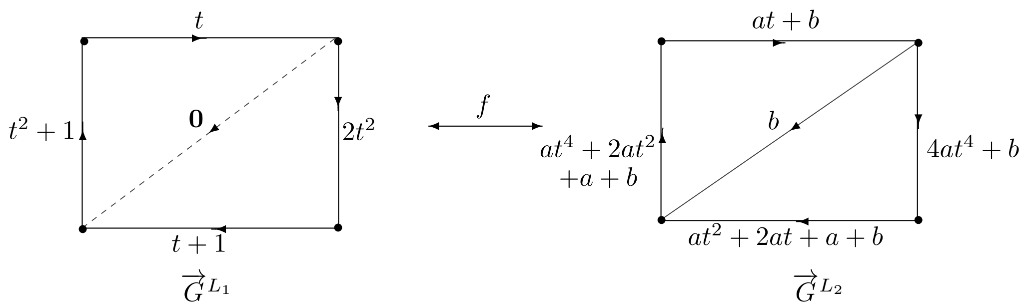


Fig.6

We know that f is G-isomorphic by Theorem 3.8 and furthermore, it is also complete.

§4. Calculus and E-index on Continuity Flows

4.1 Differential and Integral Operators

Definition 4.1 *Let $\overrightarrow{G}^{L}[t]\in\mathscr{G}_{\mathscr{B}}$ dependent on variable t and let f be a G-isomorphic operator on $\mathscr{G}_{\mathscr{B}}$ with $f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right)\to f\left(\overrightarrow{G}^{L}[t]\right)$ if $\Delta t\to 0$. Then, f is defined to be G-differential if*

$$\lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right) - f\left(\overrightarrow{G}^{L}[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t] - \overrightarrow{G}^{L}[t]} \in \mathscr{G}_{\mathscr{B}},$$

denoted by

$$\frac{df}{dt} = \lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right) - f\left(\overrightarrow{G}^{L}[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t] - \overrightarrow{G}^{L}[t]} \quad or \quad \dot{f} = \lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right) - f\left(\overrightarrow{G}^{L}[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t] - \overrightarrow{G}^{L}[t]}.$$

Let C be a continuity flow $\overrightarrow{G}^C$ with $C:(v,u)\rightarrow constant$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. Clearly,

$$\frac{df}{dt}\left(\overrightarrow{G}^L[t]\right)=F\left(\overrightarrow{G}^L[t]\right)\ \Rightarrow\ \frac{df}{dt}\left(\overrightarrow{G}^L[t]+C\right)=F\left(\overrightarrow{G}^L[t]\right)$$

and the integral operator on $\mathscr{G}^t_{\mathscr{B}}$ is defined by

$$\int F\left(\overrightarrow{G}^L[t]\right)dt\ =\ f\left(\overrightarrow{G}^L[t]\right)+C.$$

By definition, we know formulae on differential and integral operators following.

$$\int\left(\frac{df}{dt}\left(\overrightarrow{G}^L[t]\right)\right)dt\ =\ f\left(\overrightarrow{G}^L[t]\right)+C \tag{4.1}$$

and

$$\frac{df}{dt}\left(\int\left(f\left(\overrightarrow{G}^L[t]\right)\right)dt\right)\ =\ f\left(\overrightarrow{G}^L[t]\right). \tag{4.2}$$

The following conclusion is gotten immediately by definition.

Theorem 4.2 *The differential $\frac{d}{dt}$ and integral $\int$ both are linear on $\mathscr{G}_{\mathscr{B}}$.*

Now, let $\overrightarrow{H}=\bigcup\limits_{\overrightarrow{G}_i\in\mathscr{G}}\overrightarrow{G}_i$ and $\overrightarrow{H}^{L'}=\overrightarrow{G'}^{L'},\overrightarrow{H}^L=\overrightarrow{G}^L$ by Convention 2.2. By definition, we know that

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right)-f\left(\overrightarrow{G}^L[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t]-\overrightarrow{G}^L[t]} &= \lim_{\Delta t\to 0}\frac{\overrightarrow{H}^{f(L')[t+\Delta t]-f(L)[t]}}{\overrightarrow{H}^{L'[t+\Delta t]-L[t]}}\\ &= \lim_{\Delta t\to 0}\overrightarrow{H}^{\frac{f(L')[t+\Delta t]-f(L)[t]}{L'[t+\Delta t]-L[t]}}=\overrightarrow{H}^{\lim\limits_{\Delta t\to 0}\frac{f(L')[t+\Delta t]-f(L)[t]}{L'[t+\Delta t]-L[t]}}.\end{aligned}$$

Thus, f is G-differential if f itself is differential on L for $\forall e\in E\left(\overrightarrow{H}\right)$.

Conversely, if f is differential on L for $\forall e\in E\left(\overrightarrow{H}\right)$, then it is clear that

$$\begin{aligned}\overrightarrow{H}^{\dot{f}(L)} &= \overrightarrow{H}^{\lim\limits_{\Delta t\to 0}\frac{f(L')[t+\Delta t]-f(L)[t]}{L'[t+\Delta t]-L[t]}}\\ &= \lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right)-f\left(\overrightarrow{G}^L[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t]-\overrightarrow{G}^L[t]}\in\mathscr{G}_{\mathscr{B}},\end{aligned}$$

i.e., f is G-differential. We therefore get the conclusion following.

Theorem 4.3 A G-isomorphic operator $f:\ \mathscr{G}_{\mathscr{B}}\rightarrow\mathscr{G}_{\mathscr{B}}$ is differential if and only if $f(L)$ is differential on L.

A calculation by equations $(3.2),(4.1)$ and (4.2) shows that

1) $\frac{dC}{dt} = \mathbf{O}$, $\int \mathbf{O}dt = \mathbf{C}$;

2) $\frac{d}{dt}\left(\alpha\overrightarrow{G}^t\right) = \overrightarrow{G}^\alpha$, $\int \overrightarrow{G}^\alpha dt = \alpha\overrightarrow{G}^t + C$, where $t : (v,u) \to t$, $\alpha : (v,u) \to \alpha$ for $(v,u) \in E\left(\overrightarrow{G}\right)$, $t, \alpha \in \mathbb{R}$;

3) $\frac{d}{dt}\left(\overrightarrow{G}^{nL}\right) = n\overrightarrow{G}^{(n-1)L}$, $\int \overrightarrow{G}^{(n-1)L}dt = \frac{1}{n}\overrightarrow{G}^{nL}$, $n \in \mathbb{Z}^+$;

4) $\frac{d}{dt}\left(e^{\overrightarrow{G}^L}\right) = \overrightarrow{G}^{\frac{de^L}{dt}} = \overrightarrow{G}^{e^L} = e^{\overrightarrow{G}^L}$, $\int e^{\overrightarrow{G}^L}dt = e^{\overrightarrow{G}^L}$;

5) $\frac{d}{dt}\left(\ln\left|\overrightarrow{G}^L\right|\right) = \overrightarrow{G}^{\frac{d\ln|L|}{dt}} = \overrightarrow{G}^{\frac{1}{L}} = \frac{1}{\overrightarrow{G}^L}$, $\int \frac{dt}{\overrightarrow{G}^L} = \ln\left|\overrightarrow{G}^L\right|$, $L \neq \mathbf{0}$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$,

and similarly, we easily know

$$\begin{array}{llll}
\frac{d}{dt}\left(\sin\left(\overrightarrow{G}^L\right)\right), & \int \sin\left(\overrightarrow{G}^L\right)dt, & \frac{d}{dt}\left(\cos\left(\overrightarrow{G}^L\right)\right), & \int \cos\left(\overrightarrow{G}^L\right)dt, \\
\frac{d}{dt}\left(\tan\left(\overrightarrow{G}^L\right)\right), & \int \tan\left(\overrightarrow{G}^L\right)dt, & \frac{d}{dt}\left(\cot\left(\overrightarrow{G}^L\right)\right), & \int \tan\left(\overrightarrow{G}^L\right)dt, \\
\frac{d}{dt}\left(\sinh\left(\overrightarrow{G}^L\right)\right), & \int \sinh\left(\overrightarrow{G}^L\right)dt, & \frac{d}{dt}\left(\cosh\left(\overrightarrow{G}^L\right)\right), & \int \cosh\left(\overrightarrow{G}^L\right)dt, \\
\frac{d}{dt}\left(\tanh\left(\overrightarrow{G}^L\right)\right), & \int \tanh\left(\overrightarrow{G}^L\right)dt, & \frac{d}{dt}\left(\coth\left(\overrightarrow{G}^L\right)\right), & \int \coth\left(\overrightarrow{G}^L\right)dt\ .
\end{array}$$

For examples,

$$\frac{d}{dt}\left(\sin\left(\overrightarrow{G}^L\right)\right) = \cos\left(\overrightarrow{G}^L\right) \quad \text{and} \quad \int \sin\left(\overrightarrow{G}^L\right)dt = -\cos\left(\overrightarrow{G}^L\right),$$
$$\frac{d}{dt}\left(\cos\left(\overrightarrow{G}^L\right)\right) = -\sin\left(\overrightarrow{G}^L\right) \quad \text{and} \quad \int \cos\left(\overrightarrow{G}^L\right)dt = \sin\left(\overrightarrow{G}^L\right), \ \cdots.$$

Definition 4.4 *For numbers $a, b \in \mathbb{R}$, let $a = x_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of the closed interval $[a,b]$ in to subinterval, $\Delta t_i = t_i - t_{i-1}$, $\mu = \max\limits_{1\leq i\leq n} \Delta t_i$ and let $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ on variable t with assumption that $f(t)$ is bounded in $[a,b]$, only with finite non-continuous points on $[a,b]$. If*

$$\sum_{i=1}^{n} f\left(\overrightarrow{G}^L[\xi_i]\right)\cdot\overrightarrow{G}^{\Delta t_i} \in \mathscr{G}_{\mathscr{B}}$$

as $\mu \to 0$, where $\xi_i \in [t_{i-1}, t_i]$, then, we define

$$\int_a^b f\left(\overrightarrow{G}^L[t]\right)dt = \lim_{\mu\to 0}\sum_{i=1}^{n} f\left(\overrightarrow{G}^L[\xi_i]\right)\cdot\overrightarrow{G}^{\Delta t_i},$$

where $\overrightarrow{G}^{\Delta t_i} \in \mathscr{G}_{\mathscr{B}}$ with $\Delta t_i :\ (v,u) \to \Delta t_i$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$.

By Definition 4.4, we are easily know that

$$\int_a^a f\left(\overrightarrow{G}^L[t]\right)dt = \mathbf{O}, \quad \int_a^b f\left(\overrightarrow{G}^L[t]\right)dt + \int_b^c f\left(\overrightarrow{G}^L[t]\right)dt = \int_a^c f\left(\overrightarrow{G}^L[t]\right)dt. \tag{4.3}$$

Notice that

$$\begin{aligned}
\lim_{\mu\to 0}\sum_{i=1}^n f\left(\overrightarrow{G}^L[\xi_i]\right)\cdot \overrightarrow{G}^{\Delta t_i} &= \lim_{\mu\to 0}\sum_{i=1}^n \overrightarrow{G}^{f(L)[\xi_i]}\cdot \overrightarrow{G}^{\Delta t_i}\\
&= \lim_{\mu\to 0}\sum_{i=1}^n \overrightarrow{G}^{f(L)[\xi_i]\Delta t_i}\\
&= \lim_{\mu\to 0}\overrightarrow{G}^{\sum\limits_{i=1}^n f(L)[\xi_i]\Delta t_i} = \overrightarrow{G}^{\lim\limits_{\mu\to 0}\sum\limits_{i=1}^n f(L)[\xi_i]\Delta t_i}.
\end{aligned}$$

Whence,

$$\lim_{\mu\to 0}\sum_{i=1}^n f\left(\overrightarrow{G}^L[\xi_i]\right)\cdot \overrightarrow{G}^{\Delta t_i} \in \mathscr{G}_{\mathscr{B}} \quad \Leftrightarrow \quad \lim_{\mu\to 0}\sum_{i=1}^n f(L)[\xi_i]\Delta t_i \ \text{ exists}$$

as $\mu \to 0$, i.e., $f(L)$ is integral on $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

Now, it should be noted that

$$\frac{d}{dt}F\left(\overrightarrow{G}^L[t]\right) = f\left(\overrightarrow{G}^L[t]\right)$$

implies that $\dfrac{dF}{dt} = f(L)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. We know that

$$\begin{aligned}
\int_a^b f\left(\overrightarrow{G}^L[t]\right)dt &= \lim_{\mu\to 0}\sum_{i=1}^n f\left(\overrightarrow{G}^L[\xi_i]\right)\cdot \overrightarrow{G}^{\Delta t_i} = \overrightarrow{G}^{\lim\limits_{\mu\to 0}\sum\limits_{i=1}^n f(L)[\xi_i]\Delta t_i}\\
&= \overrightarrow{G}^{\int_a^b f(L)[t]dt} = \overrightarrow{G}^{F(b)-F(a)} = \overrightarrow{G}^{F(b)} - \overrightarrow{G}^{F(a)}\\
&= F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=b} - F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=a}.
\end{aligned}$$

We therefore get the conclusion following.

Theorem 4.5(Fundamental Theorem of Calculus) *Let $f: \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ on variable t with assumption that $f(t)$ is bounded in $[a,b]$, only with finite non-continuous points on $[a,b]$ and*

$$\frac{d}{dt}F\left(\overrightarrow{G}^L[t]\right) = f\left(\overrightarrow{G}^L[t]\right).$$

Then,

$$\int_a^b f\left(\overrightarrow{G}^L[t]\right)dt = F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=b} - F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=a}. \tag{4.4}$$

Proof Let $T\left(\overrightarrow{G}^L[t]\right)=\int\limits_a^t f\left(\overrightarrow{G}^L[x]\right)dx$. We prove that $\frac{d}{dt}\left(T\left(\overrightarrow{G}^L[t]\right)\right)=f\left(\overrightarrow{G}^L[t]\right)$. In fact,

$$\begin{aligned}
\lim_{\Delta t\to 0}\frac{T\left(\overrightarrow{G}^L[t+\Delta t]\right)-T\left(\overrightarrow{G}^L[t]\right)}{\Delta t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int\limits_a^{t+\Delta t} f\left(\overrightarrow{G}^L[x]\right)dx-\int\limits_a^t f\left(\overrightarrow{G}^L[x]\right)dx\right)\\
&= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int\limits_t^{t+\Delta t} f\left(\overrightarrow{G}^L[x]\right)dx=\lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G}^L[\xi]\right)\cdot\overrightarrow{G}^{\Delta t}}{\overrightarrow{G}^{\Delta t}}\\
&= f\left(\overrightarrow{G}^L[t]\right)
\end{aligned}$$

by definition, where $\xi\in[t,t+\Delta t]$, i.e., $\frac{d}{dt}\left(T\left(\overrightarrow{G}^L[t]\right)\right)=f\left(\overrightarrow{G}^L[t]\right)$.

According to (4.1), we know that

$$F\left(\overrightarrow{G}^L[t]\right)=T\left(\overrightarrow{G}^L[t]\right)+C=\int\limits_a^t f\left(\overrightarrow{G}^L[x]\right)dx+C. \tag{4.5}$$

Now, let $t=a$ in (4.5). We get $C=F\left(\overrightarrow{G}^L[a]\right)$ by (4.1), which implies that

$$F\left(\overrightarrow{G}^L[t]\right)=\int\limits_a^t f\left(\overrightarrow{G}^L[x]\right)dx+F\left(\overrightarrow{G}^L[a]\right)\quad\text{or}\quad F\left(\overrightarrow{G}^L[b]\right)=\int\limits_a^b f\left(\overrightarrow{G}^L[x]\right)dx+F\left(\overrightarrow{G}^L[a]\right)$$

if we let $t=b$, i.e.,

$$\int\limits_a^b f\left(\overrightarrow{G}^L[t]\right)dt=F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=b}-F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=a}.$$

□

4.2 E-Index

Definition 4.6 *Let $\overrightarrow{G}^L\in\mathscr{G}_{\mathscr{B}}$ be a continuity flow. The e-index* $\mathrm{ind}_e\left(\overrightarrow{G}^L\right)$ *is defined by*

$$\mathrm{ind}_e\left(\overrightarrow{G}^L\right)=\frac{1}{\left|\overrightarrow{G}\right|}\sum_{v\in V\left(\overrightarrow{G}\right)}\left\|\frac{dL(v)[t]}{dt}\right\|,$$

and $L(v)$ is called the residual value of v in $\overrightarrow{G}^L$.

Particularly, if $L(v)$ is independent on time t in $\overrightarrow{G}^L$, i.e., $\left\|\frac{dL(v)}{dt}\right\|=0$, such a vertex v is said to be conserved and furthermore, if all vertices of $\overrightarrow{G}$ are conserved, $\overrightarrow{G}^L$ is called a conserved flow or A-flow.

Generally, a non-harmonious group can not be characterized by a conserved flow. Thus, the e-index surveys the deviation of $\overrightarrow{G}^L$ from conversed flows because $\mathrm{ind}_e\left(\overrightarrow{G}^L\right)=0$ if $\overrightarrow{G}^L$ is a conserved flow.

Theorem 4.7 *If $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$ is a continuity flow, then*

$$\frac{2}{\left|\overrightarrow{G}\right|}\left\|\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\frac{dL^{A^+_{vu}}(v,u)}{dt}\right\| \leq \mathrm{ind}_e\left(\overrightarrow{G}^L\right) \leq \frac{2}{\left|\overrightarrow{G}\right|}\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\left\|\frac{dL^{A^+_{vu}}(v,u)}{dt}\right\|.$$

Proof Notice that $L(v) = \sum\limits_{u\in N_G(v)} L^{A^+_{vu}}(v,u)$, $\dfrac{dL(v)}{dt} = \sum\limits_{u\in N_G(v)} \dfrac{dL^{A^+_{vu}}(v,u)}{dt}$ and

$$\begin{aligned} 2\left\|\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\frac{dL^{A^+_{vu}}(v,u)}{dt}\right\| &\leq \sum_{v\in V\left(\overrightarrow{G}\right)}\left\|\frac{dL(v)}{dt}\right\| = \sum_{v\in V\left(\overrightarrow{G}\right)}\left\|\sum_{u\in N_G(v)}\frac{dL^{A^+_{vu}}(v,u)}{dt}\right\| \\ &\leq 2\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\left\|\frac{dL^{A^+_{vu}}(v,u)}{dt}\right\|, \end{aligned}$$

we get the result. □

Clearly,

$$\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\frac{dL(v,u)}{dt} \neq \sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\frac{dL^{A^+_{vu}}(v,u)}{dt}$$

and

$$\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\left\|\frac{dL(v,u)}{dt}\right\| \neq \sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\left\|\frac{dL^{A^+_{vu}}(v,u)}{dt}\right\|$$

unless $A^+_{vu} = \mathbf{1}_{\mathscr{B}}$ or $\left\|\frac{dL(v)}{dt}\right\| = 0$ with linear operator A^+_{vu} for $(v,u) \in E\left(\overrightarrow{G}\right)$, $\forall v \in V\left(\overrightarrow{G}\right)$, i.e., $\overrightarrow{G}^L$ is a conserved flow, and the global deviation of $\overrightarrow{G}^L$ to conserved flow is nothing else but the e-index $\mathrm{ind}_e\left(\overrightarrow{G}^L\right)$.

Theorem 4.8 *A continuity flow $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$ is conserved if and only if* $\mathrm{ind}_e\left(\overrightarrow{G}^L\right) = 0$.

Proof By definition, if $\overrightarrow{G}^L$ is a conserved flow, i.e., $L(v)$ is independent on time t for $\forall v \in V\left(\overrightarrow{G}\right)$, there must be $\left\|\frac{dL(v)}{dt}\right\| = 0$, i.e., $\mathrm{ind}_e\left(\overrightarrow{G}^L\right) = 0$. Whence, $\mathrm{ind}_e\left(\overrightarrow{G}^L\right) = 0$ by definition.

Conversely, if

$$\mathrm{ind}_e\left(\overrightarrow{G}^L\right) = \frac{1}{\left|\overrightarrow{G}\right|}\sum_{v\in V\left(\overrightarrow{G}\right)}\left\|\frac{dL(v)}{dt}\right\| = 0,$$

by the definition of norm we know that $\left\|\frac{dL(v)}{dt}\right\| \geq 0$ and $\left|\overrightarrow{G}\right| > 0$, i.e., there must be $\left\|\frac{dL(v)}{dt}\right\| = 0$ for $\forall v \in V\left(\overrightarrow{G}\right)$, i.e., $\overrightarrow{G}^L$ is conserved flow. □

Combining Theorems 3.8 and 4.8, we get conclude results following.

Corollary 4.9 *If the sequence* $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots,\right\}$ *of continuity flows converges to a conserved flow* $\overrightarrow{G}^L$*, then there must be* $\lim\limits_{n\to\infty} \text{ind}_e\left(\overrightarrow{G}_n^{L_n}\right) = 0$.

Corollary 4.10 *Let* $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$ *be a conserved flows and let* f *be a linear operator on* $\mathscr{G}_{\mathscr{B}}$ *commutated with all end-operators in* $\mathscr{A}$*, which induces operator* $f^* : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ *by* $f^* : \overrightarrow{G}^L \to \overrightarrow{G}^{f(L)}$*. Then,* $f^*\left(\overrightarrow{G}^L\right)$ *is a conserved flow also.*

Proof For $v \in V\left(\overrightarrow{G}\right)$, $L(v) = \sum\limits_{u\in N_G(v)} L^{A_{vu}^+}(v,u)$ by definition. Whence,

$$f(L(v)) = \sum_{u\in N_G(v)} f\left(L^{A_{vu}^+}(v,u)\right) = \sum_{u\in N_G(v)} (fL)^{A_{vu}^+}(v,u)$$

by assumption. Notice that $\overrightarrow{G}^L$ is a conserved, $L(v)$ is independent on t for $\forall v \in V\left(\overrightarrow{G}\right)$. We immediately know that $f(L(v))$, $v \in V\left(\overrightarrow{G}\right)$ are independent on t, i.e., $\left\|\frac{dL(v)}{dt}\right\| = 0$ also. By definition,

$$\text{ind}_e\left(f^*\left(\overrightarrow{G}^L\right)\right) = \frac{1}{\left|f^*\left(\overrightarrow{G}\right)\right|} \sum_{v\in V\left(f^*\left(\overrightarrow{G}\right)\right)} \left\|\frac{df(L(v))[t]}{dt}\right\| \ = \ 0.$$

Whence, $f^*\left(\overrightarrow{G}^L\right)$ is conserved. □

§5. Continuity Flow Equations

5.1 Algebraic Equations

For an integer $n \geq 1$, let $\mathscr{G}$ be a graph family closed under the union operation and let $\mathscr{B}$ be a field. We consider the algebraic equation

$$\overrightarrow{G}^{L_{c_n}} \cdot X^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} = \mathbf{O} \tag{5.1}$$

in $\mathscr{G}_{\mathscr{B}}$, where $L_{c_i}(v,u) \in \mathscr{B}$ for integers $1 \leq i \leq n$ with $L_{a_n}(v,u) \neq 0$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

If $X = \overrightarrow{G}^L$ is a solution of equation (5.1), by definition there must be

$$\begin{aligned}&\overrightarrow{G}^{L_{c_n}} \cdot X^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}}\\ &\qquad = \overrightarrow{G}^{L_{c_n}L^n + L_{c_{n-1}}L^{n-1} + \cdots + L_{c_1}L + L_{c_0}} \ = \ \overrightarrow{G}^{p(L)},\end{aligned} \tag{5.2}$$

which implies that the equation (5.1) is equivalent to $\overrightarrow{G}^{p(L)} = \mathbf{O}$, i.e.,

$$L_{c_n}L^n(v,u) + L_{c_{n-1}}L^{n-1}(v,u) + \cdots + L_{c_n}L(v,u) + L_{c_0}(v,u) = 0 \tag{5.3}$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ in $\mathscr{B}$, where

$$p(L)=L_{c_n}L^n+L_{c_{n-1}}L^{n-1}+\cdots+L_{c_1}L+L_{c_0}.$$

By the fundamental theorem of classical algebra, we know that there are n roots $\lambda_1^{vu},\lambda_2^{vu}$, $\cdots,\lambda_n^{vu}$ in $\mathscr{B}$ hold with (5.3), which implies that all of these solutions $\overrightarrow{G}^L$ of (5.1) must have

$$L:\ (v,u)\to\ \{\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_n^{vu}\}$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. We therefore get the result following.

Theorem 5.1 *Let $\mathscr{G}$ be a closed graph family under union and let $\mathscr{B}$ be a field. Then, a continuity flow $\overrightarrow{G}^L$ is a solution of the algebraic equation*

$$\overrightarrow{G}^{L_{c_n}}\cdot X^n+\overrightarrow{G}^{L_{c_{n-1}}}\cdot X^{n-1}+\cdots+\overrightarrow{G}^{L_{c_1}}\cdot X+\overrightarrow{G}^{L_{c_0}}=\mathbf{O}$$

if and only if $L:(v,u)\to\ \{\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_n^{vu}\}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, where $L_{c_i}(v,u)\in\mathscr{B}$ for integers $1\le i\le n$ with $L_{c_n}(v,u)\neq 0$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ and $\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_n^{vu}$ are the n roots of the polynomial $p(L)$ in $\mathscr{B}$.

Proof Clearly, if $L:\ (v,u)\to\ \{\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_n^{vu}\}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, then

$$L_{c_n}L^n(v,u)+L_{c_{n-1}}L^{n-1}(v,u)+\cdots+L_{c_n}L(v,u)+L_{c_0}(v,u)=0$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ in $\mathscr{B}$, which implies that $\overrightarrow{G}^{p(L)}=\mathbf{O}$, i.e.,

$$\overrightarrow{G}^{L_{c_n}}\cdot X^n+\overrightarrow{G}^{L_{c_{n-1}}}\cdot X^{n-1}+\cdots+\overrightarrow{G}^{L_{c_1}}\cdot X+\overrightarrow{G}^{L_{c_0}}=\mathbf{O}.$$

Conversely, by (5.2) it is clear that

$$\overrightarrow{G}^{L_{c_n}}\cdot X^n+\overrightarrow{G}^{L_{c_{n-1}}}\cdot X^{n-1}+\cdots+\overrightarrow{G}^{L_{c_1}}\cdot X+\overrightarrow{G}^{L_{c_0}}=\mathbf{O}$$

implies $p(L)=0$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, i.e., L must be a mapping

$$L:\ (v,u)\to\ \{\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_n^{vu}\}$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. This completes the proof. □

Notice that the coefficients flows in equations (5.1) are over the same graph $\overrightarrow{G}$. We can certainly generalize it to different graphs $\overrightarrow{G}$ by Convention 2.2.

Theorem 5.2 *Let $\mathscr{G}$ be a graph family closed under union, $\overrightarrow{G}_0,\overrightarrow{G}_1,\overrightarrow{G}_2,\cdots,\overrightarrow{G}_n\in\mathscr{G}$ and let $\mathscr{B}$ be a field. Define a graph $\widehat{G}=\bigcup\limits_{i=1}^{n}\overrightarrow{G}_i$. Then, a continuity flow $\widehat{G}^L$ is a solution of the* *algebraic equation*

$$\overrightarrow{G}_n^{L_{c_n}} \cdot X^n + \overrightarrow{G}_{n-1}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}_1^{L_{c_1}} \cdot X + \overrightarrow{G}_0^{L_{c_0}} = \mathbf{O}, \tag{5.4}$$

where $L_{c_i}(v,u) \in \mathscr{B}$ *for integers* $0 \le i \le n$ *with* $L_{c_n}(v,u) \ne 0$ *for* $\forall (v,u) \in E\left(\widehat{G}\right)$ *if and only if* $L: \ (v,u) \to \ \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}$ *for* $\forall (v,u) \in E\left(\widehat{G}\right)$*, where,* $\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}$ *are the* n *roots of the polynomial* $p(L)$ *in* $\mathscr{B}$*.*

Proof Notice that the equation (5.4) is equivalent to

$$\widehat{G}^{L'_{c_n}} \cdot X^n + \widehat{G}^{L'_{c_{n-1}}} \cdot X^{n-1} + \cdots + \widehat{G}^{L'_{c_1}} \cdot X + \widehat{G}^{L'_{c_0}} = \mathbf{O}$$

by Convention 2.2, where

$$L'_{c_i}(v,u) = \begin{cases} L_{c_i}(v,u) & \text{if } (v,u) \in \overrightarrow{G}_i, \\ 0 & \text{if } (v,u) \in \widehat{G} \setminus \overrightarrow{G}_i \end{cases}$$

for integers $0 \le i \le n$. Therefore, we immediately get the result by Theorem 5.1. □

We have known that an nth polynomial has n roots in an field. The next result enumerates the non-isomorphic continuity flow solutions of equation (5.1) in $\mathscr{G}_{\mathscr{B}}$.

Theorem 5.3 *Let* $\mathscr{G}$ *be a closed graph family under union and let* $\mathscr{B}$ *be a field. Then, an algebraic equation*

$$\overrightarrow{G}^{L_{c_n}} \cdot X^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} = \mathbf{O},$$

where $L_{c_i}(v,u) \in \mathscr{B}$ *for integers* $1 \le i \le n$ *with* $L_{c_n}(v,u) \ne 0$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *has*

$$n\left(p\left(\overrightarrow{G}^L\right), \mathscr{G}_{\mathscr{B}}\right) = \frac{n^{\varepsilon\left(\overrightarrow{G}^L\right)}}{\left|\mathrm{Aut}\overrightarrow{G}\right|}$$

non-isomorphic solutions $\overrightarrow{G}^L$ *in* $\mathscr{G}_{\mathscr{B}}$*, where* $\mathrm{Aut}\overrightarrow{G}$ *is the automorphism group of graph* $\overrightarrow{G}$*.*

Proof Notice that there are $n^{\varepsilon(\overrightarrow{G}^L)}$ ways for choice L on edges of $\overrightarrow{G}$ by Theorem 5.1 and two $\overrightarrow{G}^{L_1}$, $\overrightarrow{G}^{L_2}$ are isomorphic is and only if there is an automorphism $\varphi: \overrightarrow{G} \to \overrightarrow{G}$ such that $L_2 = L_1 \circ \varphi$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

Let $\mathscr{J}$ be all of these continuity flow $\overrightarrow{G}^L$ with

$$L: \ (v,u) \to \ \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}.$$

We consider the distinct obits in $\mathscr{J}$ acted by automorphism group $\mathrm{Aut}\overrightarrow{G}$. Clearly, if $\varphi: \overrightarrow{G}^L \to \overrightarrow{G}^L$, there must be $\varphi = \mathrm{id}_{\overrightarrow{G}}$, or in other words that $(\mathrm{Aut}\overrightarrow{G})_{\overrightarrow{G}^L} = \left\{\mathrm{id}_{\overrightarrow{G}}\right\}$.

By the Burnside lemma,

$$\left|\mathrm{Aut}\overrightarrow{G}\right| = \left|(\mathrm{Aut}\overrightarrow{G})_{\overrightarrow{G}^L}\right| \left|\left(\overrightarrow{G}^L\right)^{\mathrm{Aut}\overrightarrow{G}}\right|,$$

we get that

$$\left|\left(\overrightarrow{G}^L\right)^{\mathrm{Aut}\overrightarrow{G}}\right| = \left|\mathrm{Aut}\overrightarrow{G}\right|,$$

i.e., each orbit of $\overrightarrow{G}^L$ acted by $\mathrm{Aut}\overrightarrow{G}$ has the same length $\left|\mathrm{Aut}\overrightarrow{G}\right|$. We therefore have

$$n\left(p\left(\overrightarrow{G}^L\right), \mathscr{G}_{\mathscr{B}}\right) = \frac{n^{\varepsilon\left(\overrightarrow{G}^L\right)}}{\left|\mathrm{Aut}\overrightarrow{G}\right|}$$

non-isomorphic solutions $\overrightarrow{G}^L$ of equation (5.1) in $\mathscr{G}_{\mathscr{B}}$. $\square$

Particularly, if $\overrightarrow{G} = C_m$, K_m or B_m for an integer $m \geq 3$, we get the conclusion following by Theorem 5.3.

Corollary 5.4 *Let C_m, B_m and K_m be respectively a bidirectional circuit, complete graph and bouquet with $m \geq 3$. Then, the numbers of non-isomorphic continuity flow solutions of equation*

$$\begin{aligned}
C_m^{L_{c_n}} \cdot X^n + C_m^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + C_m^{L_{c_1}} \cdot X + C_m^{L_{c_0}} &= \mathbf{O}, \\
B_m^{L_{c_n}} \cdot X^n + B_m^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + B_m^{L_{c_1}} \cdot X + B_m^{L_{c_0}} &= \mathbf{O}, \\
K_m^{L_{c_n}} \cdot X^n + K_m^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + K_m^{L_{c_1}} \cdot X + K_m^{L_{c_0}} &= \mathbf{O}
\end{aligned}$$

with $L_{c_i}(v,u) \in \mathscr{B}$ for integers $1 \leq i \leq n$, $L_{a_n}(v,u) \neq 0$ for $\forall (v,u) \in E(C_m), E(B_m)$ or $E(E_m)$ are respectively

$$\frac{n^m}{2m}, \quad \frac{n^m}{m!} \quad \textit{and} \quad \frac{n^{\frac{m(m-1)}{2}}}{m!}. \tag{5.5}$$

5.2 Differential Equations

Let $\overrightarrow{G}^{L_{c_1}}[t], \overrightarrow{G}^{L_{c_0}}[t] \in \mathscr{G}_{\mathscr{B}}$ with $L_{c_0}(v,u), L_{c_1}(v,u) \in \mathscr{B}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. Consider the differential equation

$$\frac{dX}{dt} = \overrightarrow{G}^{L_{c_1}}[t] \cdot X + \overrightarrow{G}^{L_{c_0}}[t] \tag{5.6}$$

in $\mathscr{G}_{\mathscr{B}}$. Assume $\overrightarrow{G}^{L_{c_0}}[t] = \mathbf{O}$. We have

$$\frac{dX}{dt} = \overrightarrow{G}^{L_{c_1}}[t] \cdot X, \quad \text{i.e.,} \quad \frac{dX}{X} = \overrightarrow{G}^{L_{c_1}}[t]dt. \tag{5.7}$$

Integrating (5.7) on both sides, we get

$$\ln|X| = \int \overrightarrow{G}^{L_{c_1}}[t]dt + C,$$

which implies that

$$X[t] = C \cdot e^{\int \overrightarrow{G}^{L_{c1}}[t]dt}.$$

Now, assume C is variable on t, i.e., $C = \overrightarrow{G}^{L}[t]$ and substitute it into (5.6). We get that

$$\begin{aligned}\left(\frac{d}{dt}\left(\overrightarrow{G}^{L}[t]\right)\right) e^{\int \overrightarrow{G}^{L_{c1}}[t]dt} \quad &+ \quad \overrightarrow{G}^{L}[t] \cdot e^{\int \overrightarrow{G}^{L_{c1}}[t]dt} \cdot \overrightarrow{G}^{L_{c1}}[t] \\ &= \quad \overrightarrow{G}^{L_{c1}}[t] \cdot \overrightarrow{G}^{L}[t] \cdot e^{\int \overrightarrow{G}^{L_{c1}}[t]dt} + \overrightarrow{G}^{L_{c0}}[t].\end{aligned}$$

Combine similar terms, we have that

$$\frac{d}{dt}\left(\overrightarrow{G}^{L}[t]\right) = \overrightarrow{G}^{L_{c0}} \cdot e^{-\int \overrightarrow{G}^{L_{c1}}dt} \quad \text{i.e.,} \quad \overrightarrow{G}^{L}[t] = \int \overrightarrow{G}^{L_{c0}} \cdot e^{-\int \overrightarrow{G}^{L_{c1}}dt} dt + C,$$

which enable us getting the solution

$$X[t] = e^{\int \overrightarrow{G}^{L_{c1}}dt} \cdot \left(\int \overrightarrow{G}^{L_{c0}} \cdot e^{-\int \overrightarrow{G}^{L_{c1}}dt} dt + C\right) \tag{5.8}$$

of equation (5.6).

For the initial value problem

$$\begin{cases} \dfrac{dX}{dt} = \overrightarrow{G}^{L_{c1}} \cdot X + \overrightarrow{G}^{L_{c0}} \\ X|_{t=t_0} = \overrightarrow{G}^{L_0}[t_0] \end{cases} \tag{5.9}$$

of (5.6), we can determine the constant flow C in (5.8). In fact, assume $X = \overrightarrow{G}^{L}[t] \cdot e^{\int_{t_0}^{t} \overrightarrow{G}^{L_{c1}}dx}$ and substitute it into (5.9), we similarly get that

$$\frac{d}{dt}\left(\overrightarrow{G}^{L}[t]\right) = \overrightarrow{G}^{L_{c0}}[t] \cdot e^{-\int_{t_0}^{t} \overrightarrow{G}^{L_{c1}}[x]dx} \quad \text{i.e.,} \quad \overrightarrow{G}^{L}[t] = \int_{t_0}^{t} \overrightarrow{G}^{L_{c0}}[x] \cdot e^{-\int_{t_0}^{x} \overrightarrow{G}^{L_{c1}}[s]ds} dx + C.$$

Therefore,

$$X[t] = \left(\int_{t_0}^{t} \overrightarrow{G}^{L_{c0}}[x] \cdot e^{-\int_{t_0}^{x} \overrightarrow{G}^{L_{c1}}[s]ds} dx + C\right) \cdot e^{\int_{t_0}^{t} \overrightarrow{G}^{L_{c1}}[x]dx},$$

which implies that

$$X(t_0) = \left(\int_{t_0}^{t_0} \overrightarrow{G}^{L_{c0}}[x] \cdot e^{-\int_{t_0}^{t_0} \overrightarrow{G}^{L_{c1}}[s]ds} dx + C\right) \cdot e^{\int_{t_0}^{t_0} \overrightarrow{G}^{L_{c1}}[x]dx}$$

if $t = t_0$. However,

$$X|_{t=t_0} = \overrightarrow{G}^{L_0} \quad \text{and} \quad \int_{t_0}^{t_0} \overrightarrow{G}^{L_{c0}}[x]dx = \int_{t_0}^{t_0} \overrightarrow{G}^{L_{c1}}[x]dx = \mathbf{O}$$

by assumption. We get that $X(t_0)=(\mathbf{O}+C)\cdot e^{\mathbf{O}}=C\cdot I=C$, which concludes that $C=X(t_0)=\overrightarrow{G}^{L_0}[t]$. Consequently,

$$X[t]=\left(\int\limits_{t_0}^{t}\overrightarrow{G}^{L_{c_0}}[x]\cdot e^{-\int_{t_0}^{x}\overrightarrow{G}^{L_{c_1}}[s]ds}dx+\overrightarrow{G}^{L_0}[t]\right)\cdot e^{\int_{t_0}^{t}\overrightarrow{G}^{L_{c_1}}[x]dx} \tag{5.10}$$

in the initial value problem (5.9).

Example 5.5 Let $\mathscr{B}=\mathbb{R}$ and $\overrightarrow{G}^{L_{c_1}},\overrightarrow{G}^{L_{c_0}}$ shown in Fig.7(a) and (b). Then, the solution of differential equation (5.6) is the continuity flow shown in Fig.7(c),

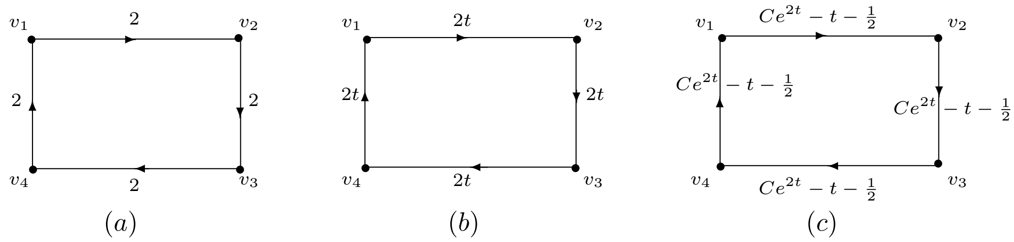


Fig.7

where C is a constant. Particularly, if $X[0]=\overrightarrow{G}^{L_0}[0]$ with $L_0:(v,u)\to te^t$ for $(v,u)\in E\left(\overrightarrow{G}\right)$, we know that the solution of the initial problem (7.9) is the continuity flow shown in Fig.4(c) with $C=\frac{1}{2}$, i.e., $X[t]=\overrightarrow{G}^{L}[t]$ with

$$L:(v,u)\to\frac{1}{2}\left(e^{2t}-1\right)-t$$

for $(v,u)\in E\left(\overrightarrow{G}\right)$.

5.3 Linear Equation with Constant Flow Coefficients

A continuity flow $\overrightarrow{G}^{L}$ is constant if $L:(v,u)\to c_{vu}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, where $c_{vu}\in\mathscr{B}$ is a constant, denoted by $\overrightarrow{G}^{L_c}$. For an integer $n\geq 1$, a flow equation with a form

$$\frac{d^nX}{dt^n}+\overrightarrow{G}_{n-1}^{L_{c_{n-1}}}\cdot\frac{d^{n-1}X}{dt^{n-1}}+\overrightarrow{G}_{n-2}^{L_{c_{n-2}}}\cdot\frac{d^{n-2}X}{dt^{n-2}}+\cdots+\overrightarrow{G}_{1}^{L_{c_1}}\cdot\frac{dX}{dt}+\overrightarrow{G}_{0}^{L_{c_0}}=\mathbf{O} \tag{5.11}$$

is said to be a linear equation with constant flow coefficients, where $\overrightarrow{G}_{i}^{L_{c_i}}$ is constant flow for integers $0\leq i\leq n-1$. Certainly, let $\overrightarrow{G}=\bigcup\limits_{i=0}^{n-1}\overrightarrow{G}_i$, the equation (5.11) is equivalent to

$$\frac{d^nX}{dt^n}+\overrightarrow{G}^{L_{c_{n-1}}}\cdot\frac{d^{n-1}X}{dt^{n-1}}+\overrightarrow{G}^{L_{c_{n-2}}}\cdot\frac{d^{n-2}X}{dt^{n-2}}+\cdots+\overrightarrow{G}^{L_{c_1}}\cdot\frac{dX}{dt}+\overrightarrow{G}^{L_{c_0}}=\mathbf{O} \tag{5.12}$$

with characteristic equation

$$\Lambda^n+\overrightarrow{G}^{L_{c_{n-1}}}\cdot\Lambda^{n-1}+\overrightarrow{G}^{L_{c_{n-2}}}\cdot\Lambda^{n-2}+\cdots+\overrightarrow{G}^{L_{c_1}}\cdot\Lambda+\overrightarrow{G}^{L_{c_0}}=\mathbf{O}, \tag{5.13}$$

which is equivalent to

$$\lambda^n + L_{c_{n-1}}(v,u)\lambda^{n-1} + L_{c_{n-2}}(v,u)\lambda^{n-2} + \cdots + L_{c_1}(v,u)\lambda + L_{c_0}(v,u) = \mathbf{0} \tag{5.14}$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

For the equation (5.14), let

$$\begin{array}{l}
\lambda_1^{vu} = r_1^{vu}, \lambda_2^{vu} = r_1^{vu}, \cdots, \lambda_{m_{r_1}}^{vu} = r_1^{vu}, \\
\lambda_{m_{r_1}+1}^{vu} = r_2^{vu}, \lambda_{m_{r_1}+2}^{vu} = r_2^{vu}, \cdots, \lambda_{m_{r_1}+m_{r_2}}^{vu} = r_2^{vu}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
\lambda_{m_{r_1}+m_{r_2}+\cdots+m_{r_{s-1}}+1}^{vu} = r_s^{vu}, \lambda_{m_{r_1}+m_{r_2}+\cdots+m_{r_{s-1}}+2}^{vu} = r_s^{vu}, \cdots, \lambda_n^{vu} = r_s^{vu}
\end{array}$$

be the n roots of (5.14), where $m_{r_1} + m_{r_2} + \cdots + m_{r_s} = n$. Then, by Theorem 5.1 we know that any solution $\Lambda = \overrightarrow{G}^{L_\lambda}$ of (5.13) must has $L_\lambda : (v,u) \to \lambda_i^{vu}$, $1 \le i \le n$. Now, we define a G-isomorphic mapping $\tau : \overrightarrow{G}^{L_\lambda} \to \overrightarrow{G}^{\tau(L_\lambda)}$ by

$$\tau: \ L_\lambda(v,u) = \lambda_i^{vu} \ \to \left\{\begin{array}{lll}
t^{i-1}e^{r_1^{vu}t} & \text{if} & 1 \le i \le m_{r_1}, \\
t^{i-1}e^{r_2^{vu}t} & \text{if} & m_{r_1}+1 \le i \le m_{r_1}+m_{r_2}, \\
\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\
t^{i-1}e^{r_s^{vu}t} & \text{if} & m_{r_1}+m_{r_2}+\cdots+m_{r_{s-1}}+1 \le i \le n
\end{array}\right.$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. We therefore get the result following.

Theorem 5.6 *If the set* $\left\{\overrightarrow{G}^{L_\lambda^1}, \overrightarrow{G}^{L_\lambda^2}, \cdots, \overrightarrow{G}^{L_\lambda^m}\right\}$ *consists of all solutions of the flow equation* (5.13), *then, the set* $\left\{\overrightarrow{G}^{\tau\left(L_\lambda^1\right)}, \overrightarrow{G}^{\tau\left(L_\lambda^2\right)}, \cdots, \overrightarrow{G}^{\tau\left(L_\lambda^m\right)}\right\}$ *is a linear basis of the solution space* $\mathscr{S}$ *of flow differential equation* (5.12), *where* $m = \dfrac{n^{\varepsilon\left(\overrightarrow{G}^L\right)}}{\left|\mathrm{Aut}\overrightarrow{G}\right|}$.

Proof Clearly, a linear combination

$$X = \overrightarrow{G}^{L_{C_1}} \cdot \overrightarrow{G}^{\tau\left(L_\lambda^1\right)} + \overrightarrow{G}^{L_{C_2}} \cdot \overrightarrow{G}^{\tau\left(L_\lambda^2\right)} + \cdots + \overrightarrow{G}^{L_{C_m}} \cdot \overrightarrow{G}^{\tau\left(L_\lambda^m\right)}$$

is a solution of the differential equation (5.12) because of

$$X = \overrightarrow{G}^{L_{C_1}\tau\left(L_\lambda^1\right) + L_{C_2}\tau\left(L_\lambda^2\right) + \cdots + L_{C_m}\tau\left(L_\lambda^m\right)}$$

with a solution $x = L_{C_1}\tau\left(L_\lambda^1\right) + L_{C_2}\tau\left(L_\lambda^2\right) + \cdots + L_{C_m}\tau\left(L_\lambda^m\right)$ of ordinary differential equation

$$\frac{d^n x}{dt^n} + L_{c_{n-1}}(v,u)\frac{d^{n-1}x}{dt^{n-1}} + L_{c_{n-2}}(v,u)\frac{d^{n-2}x}{dt^{n-2}} + \cdots + L_{c_1}(v,u)\frac{dx}{dt} + L_{c_0}(v,u) = 0 \tag{5.15}$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and a solution of differential flow equation (5.12) must be a linear combination X by the theory of ordinary differential equations, where $L_{C_i} : (v,u) \to$ constant,

$1 \leq i \leq m$.

Furthermore, $\overrightarrow{G}^{\tau(L_\lambda^1)}, \overrightarrow{G}^{\tau(L_\lambda^2)}, \cdots, \overrightarrow{G}^{\tau(L_\lambda^m)}$ is independent because if there are constant flows $\overrightarrow{G}^{L_{C_1}}, \overrightarrow{G}^{L_{C_2}}, \cdots, \overrightarrow{G}^{L_{C_m}}$ such that

$$\overrightarrow{G}^{L_{C_1}} \cdot \overrightarrow{G}^{\tau(L_\lambda^1)} + \overrightarrow{G}^{L_{C_2}} \cdot \overrightarrow{G}^{\tau(L_\lambda^2)} + \cdots + \overrightarrow{G}^{L_{C_m}} \cdot \overrightarrow{G}^{\tau(L_\lambda^m)} = \mathbf{O},$$

there must be

$$L_{C_1}(v,u)\tau\left(L_\lambda^1(v,u)\right) + L_{C_2}(v,u)\tau\left(L_\lambda^2(v,u)\right) + \cdots + L_{C_m}(v,u)\tau\left(L_\lambda^m(v,u)\right) = 0$$

hold with an edge $(v,u) \in E\left(\overrightarrow{G}\right)$, which contradicts to the fact that $\{\tau(\lambda_i^{vu}),\ 1 \leq i \leq n\}$ is the basis of ordinary differential equations (5.15) on the edge (v,u) by the theory of ordinary differential equations. $\square$

Corollary 5.7 *The rank of the solution space $\mathscr{S}$ of flow differential equation* (5.12) *is*

$$\mathrm{rank}\mathscr{S} = \frac{n^{\varepsilon\left(\overrightarrow{G}^L\right)}}{\left|\mathrm{Aut}\overrightarrow{G}\right|}.$$

§6. Applications

Dynamic network characterizes the dynamical behavior of networks, which can be viewed as a mathematics over networks with applications to characterize the complex networks, i.e., dynamics on network and also an immediately application for revisiting the index of gross domestic product, i.e., GDP index in economy.

6.1 Dynamics on Network

Notice that the dynamic equations

$$\frac{\partial \overrightarrow{G}^{\mathscr{L}}}{\partial x_i} - \frac{d}{dt}\frac{\partial \overrightarrow{G}^{\mathscr{L}}}{\partial \dot{x}_i} = \mathbf{O}, \qquad 1 \leq i \leq n. \tag{6.1}$$

on harmonic flows $\overrightarrow{G}^L$, i.e., $L : (v,u) \to L(v,u) - iL(v,u)$ with $i^2 = -1$ are established in [21] by letting Lagrangian on edges of $\overrightarrow{G}$, where $L(t, \mathbf{x}(t), \frac{d\mathbf{x}(t)}{dt})(v,u)$ is the Lagrangian on edge (v,u) and

$$\mathscr{L} : L(v,u) \to \mathscr{L}\left[L\left(t, \mathbf{x}(t), \frac{d\mathbf{x}(t)}{dt}\right)(v,u)\right]$$

is a differentiable functional on a continuity flow $\overrightarrow{G}^L[t]$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ with $[\mathscr{L}, A] = \mathbf{0}$ for $A \in \mathscr{A}$ and particularly, the dynamic equations can be simplified to

$$\frac{\partial \overrightarrow{G}^{L^2}}{\partial x_i} - \frac{d}{dt}\frac{\partial \overrightarrow{G}^{L^2}}{\partial \dot{x}_i} = \mathbf{O}, \quad 1 \leq i \leq n. \tag{6.2}$$

if $\mathscr{L}$ is linear dependent on L, which are the second order differential equations. Then, *what is the dynamic equations of network, are they second order differential equations also*? The answer is not certain. In fact, all of these known complex models on networks such as those of ER random-graph model, small-world network model, scale-free network model can be characterized by the initial value problem

$$\begin{cases} \dfrac{dX}{dt} = \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} \\ X|_{t=t_0} = \overrightarrow{G}^{L_0}[t_0] \end{cases} \tag{6.3}$$

of first order differential flow equation and, which can be solved by formula (5.10).

(1) ER Random-Graph Model. An ER-random model is introduced by Erdös and Rényi in 1960, generated as follows ([4]):

STEP 1. Start with N isolated vertices;

STEP 2. Pick up all possible pairs of vertices, once and only once, from the N given vertices and connect each pair of vertices by an edge with probability $p \in (0,1)$.

Without loss of generality, let $L_p : (v,u) \to p$ but $L_p : (x,y) \to 0$ if $(x,y) \neq (v,u)$ for a choice $(v,u) \in E(K_N)$. Clearly, if X is an ER-random model on N vertices, we can simulate its evolution from N isolated vertices to a random network at step t by an evolution equation

$$\begin{cases} \dfrac{dX}{dt} = K_N^{L_p}[t] \cdot K_N^{L}[t] \\ X[t_0] = \overline{K}_N \end{cases} \tag{6.4}$$

where K_N is a complete bidirectional graph with complement $\overline{K}_N$ of order N, and $K_N^{L_p}[t_0] = \mathbf{O}$ at the initialization t_0. By definition, we are easily know that

$$X[t] = \int_{t_0}^{t} K_N^{L_p}[s] \cdot K_N^{L}[s] ds. \tag{6.5}$$

Particularly, let $L : (v,u) \to 1$ for $\forall (v,u) \in E(K_N)$. We therefore get an ER-random model by (6.5).

(2) Small-World Network Model. The small-wold network model was discovered by Watts and Strogaz, called WS small-wold network model in 1998, which is generated by an algorithm following ([4]):

STEP 1. Start from a ring-shaped network C_N^K with N vertices, and in which each vertex is connected to its $2K$ neighbors, K vertices on each side, where $K \geq 1$ is an small integer;

STEP 2. For every pair of adjacent vertices in C_N^K, reconnected the edge in such a way that the begin end of the edge is unchanged but the the other end is disconnected with probability p and then reconnected to a vertex randomly in the network, and this process is performed edge by edge on C_N^K, once and only once, either clockwise or counterclockwise.

Notice that the WS small-wold network model may results in a non-connected network finally in the reconnecting process. For preventing the case of non-connected cases happening, Newman and Watts modified the previous algorithm by replacing STEP 2 following:

STEP 2'. For every pair of originally unconnected vertices, with probability p, $0 < p \ll 1$ add an edge to connect them.

Clearly, the union of all WS small-wold networks is $K_N - C_N^K$, and the union of all NW small-wold networks is K_N. Similar to the case of ER-random model, we know a WS small-wold network or NW small-wold network can be characterized by

$$X[t] = \int_{t_0}^{t} \left(K_N - C_N^K\right)_N^{L_p}[s] \cdot \left(K_N - C_N^K\right)^L [s]ds \quad \text{or} \quad X[t] = \int_{t_0}^{t} K_N^{L_p}[s] \cdot K_N^L[s]ds \tag{6.6}$$

with $X[t_0] = C_N^K$, respectively. Particularly, let $L : (v,u) \to 1$ for $\forall (v,u) \in E\left(K_N - C_N^K\right)$ or $E(K_N)$. We get a WS small-wold network or NW small-wold network at step t by (6.6).

(3) Scale-Free Network Model. The first scale-free network model, called BA scale-free network model is proposed by Barabási and Albert in 1999 ([2]), then a few modified BA models such as EBA model, local-world model by Albert and Barabási presented in 2000, and then other network models with the property that *preferential attachment*, i.e., the phenomenon ruler "rich gets richer" ([4]). A BA network model is generated by the algorithm following.

STEP 1. Starting from a connected network $\overrightarrow{G}_0$ of small size $m_0 \geq 1$, introduce one new vertex to the existing network each time, and this new vertex is simultaneously connected to existing m vertices in the network, where $1 \leq m \leq m_0$;

STEP 2. The incoming new vertex in STEP 1 is simultaneously connected to each of the existing vertices according to probability

$$\Pi_i = \rho_i / \sum_{j=1}^{N} \rho_j$$

for vertex v of valency ρ_i.

Notice that the union of all possible network of BA scale-free network is $G_0 + K_t$ at step t. Without loss of generality, let v be a new vertex at step t and $L_{BA} : (v,u) \to \Pi_i$ if $\rho(u) = \Pi_i$ but $L_{BA} : (x,y) \to 0$ if $x, y \neq v$ for $u, x, y \in V(G_0 + K_t)$. Clearly, if X is a BA scale-free network, we can simulate its evolution from $\overrightarrow{G}_0$ to a random network at step t by an evolution equation

$$\begin{cases} \dfrac{dX}{dt} = (G_0 + K_t)^{L_{BA}}[t] \cdot X \\ X[t_0] = G_0^{L_0} \end{cases} \tag{6.7}$$

where $(G_0 + K_t)^{L_{BA}}[t_0] = \mathbf{O}$ at the initialization t_0. By formula (5.10), we are easily know that the BA scale-free network

$$X[t] = G_0^{L_0} \cdot \int_{t_0}^{t} e^{(G_0 + K_t)^{L_{BA}}[s]} ds \tag{6.8}$$

if let $L_0 : (v,u) \to 1$ for $\forall (v,u) \in E\left(G_0^{L_0}\right)$.

6.2 E-index with GDP

By the input-output model of Wassily Leontief, an economical system can be decomposed into n parts or industries $1, 2, \cdots, n$ operated with inputs in one industry produce outputs for consumption or for input into another industry, which inherits a topological graph $\overrightarrow{G}$ with vertex set $\{1, 2, \cdots, n\}$ and edge set $\{(i, j)$ if product of i input $j, 1 \leq i, j \leq n\}$ (see [30] for detils). Furthermore, such an inherited graph of the input-output model can be generalized to a continuity flow $\overrightarrow{G}^L_+$ with $L : (i, j) \to$ amount for integers $1 \leq i, j \leq n$ and end-operators $\mathscr{A} = \{1_{\mathscr{B}}\}$, such as those shown in Fig.8,

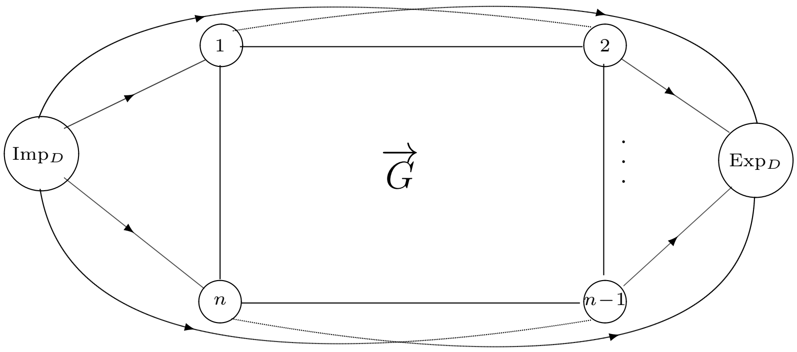


Fig.8

where $1, 2, \cdots, n$ and Imp_D, Exp_D respectively denote the n industries, the imports and the exports, $\overrightarrow{G}$ is the continuity flow inherited in the constitution of industries.

For evaluating the economic production and growth of a nation, the gross domestic product (GDP) index is a monetary measure of the market value of all the final goods and services produced in a specific time period of a nation. Then, *how to calculate the GDP of a country*? The most commonly used GDP formula based on the money spent by various groups that participate in the economy of a country is ([27])

$$\mathrm{GDP} = C + G + I + NX, \tag{6.9}$$

where, C = consumption or all private consumer spending within a countrys economy, G = total government expenditures, I = sum of a country's investments spent on capital equipment, inventories, and housing and NX = net exports, i.e., a country's total exports less total imports.

Notice that I, C and G, NX reflect respectively the investment, consumption and net export scales, the monetary measure of flows on $\overrightarrow{G}^L_+$ in Fig.8. Whence,

$$\mathrm{GDP} = \sum_{i=1}^{n} c(i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c(L(i,j)) + \sum_{i=1}^{n} c^+(i, \mathrm{Exp}_D) - \sum_{i=1}^{n} c^-(\mathrm{Imp}_D, i), \tag{6.10}$$

where, $c(i)$, $c(L(i, j))$ and $c^+(i, \mathrm{Exp}_D)$, $c^-(\mathrm{Imp}_D, i)$ are respectively the money of investment of the ith industry, the consumption of jth industry on the ith product, and the export or import of the ith industry. By definition, the real GDP growth rate is the percentage change in a countrys real GDP over time, i.e.,

$$\text{The real GDP growth rate } \kappa = \frac{\text{The final GDP} - \text{The initial GDP}}{\text{The initial GDP}} \times 100. \tag{6.11}$$

Certainly, if $\overrightarrow{G}^L_+$ is conserved by equating Imp_D with Exp_D and the price is equilibrium, there must be

$$\text{The real GDP growth rate } \kappa = \frac{d}{dt}\left(\overrightarrow{G}^L_+\right) = \frac{d}{dt}\left(L(i,j)\right), \quad \forall (i,j) \in E\left(\overrightarrow{G}_+\right),$$

i.e., the e-index $\mathrm{ind}_\mathrm{e}\left(\overrightarrow{G}^L_+\right) = 0$ in this case. However, the continuity flow of $\overrightarrow{G}^L_+$ equated Imp_D with Exp_D is not conserved, the price of different industries is not equilibrium, even $\mathrm{Exp}_D) \neq \mathrm{Imp}_D$ in the real, i.e., the economical system of a country is a non-harmonious group, industries maybe non-synchronized. That is why the GDP doesn't add up in [27], and also alludes that the developing of humans is not harmonious with the nature. Then, *could we establish such an index that can reflects both the economic development and the damage to the nature*? The answer is positive with two indexes following:

Index 1. The revisited gross domestic product GDP_R;
Index 2. The deviation of the developing to that of the equilibrium $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right)$.

In fact, the most ideal developing of humans with the nature should be conserved, i.e., the e-index $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right) = 0$, which means the full use and the best used of resource without pollutant to the nature. However, none of the economical systems of humans coincides with this pattern because of the limitations of humans on the nature. There are some industries i with $\left|\frac{d(L(i))}{dt}\right| \neq 0$, i.e., the residue $L(i)$ is not constant on usual. *What is this case implication*? It reflects the redundancy of industry i in the developing of humans, also the harmful extent of human's activity to the nature, i.e., the contributions of $L(i)$ is negative to the developing of humans. We should revisit the classical GDP by surveying the degree of the activity of humans harmful to the nature.

Notice that

$$\left\|\frac{dc(L(i))}{dt}\right\| = \left|\frac{dc(L(i))}{dt}\right| = \frac{d}{dt}\left(|c(L(i))|\right)$$

in this case. We introduced the revisited GDP_R on continuity flow $\overrightarrow{G}^L_+$ by

$$\begin{aligned}
\mathrm{GDP}_R &= \sum_{i=1}^{n} c(i) + \sum_{i,j=1}^{n} c(L(i,j)) + \sum_{i=1}^{n}\left(c^+(i,\mathrm{Exp}_D) - c^-(\mathrm{Imp}_D, i)\right) - \sum_{i=1}^{n}|c(L(i))| \\
&= \sum_{i=1}^{n} c(i) + \sum_{i,j=1}^{n} c(L(i,j)) + \sum_{i=1}^{n}\left(c^+(i,\mathrm{Exp}_D) - c^-(\mathrm{Imp}_D, i)\right) - \sum_{i=1}^{n}\int_{t_1}^{t_2}\left|\frac{dc(L(i))}{dt}\right| dt \\
&= \sum_{i=1}^{n} c(i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c(L(i,j)) + \sum_{i=1}^{n}\left(c^+(i,\mathrm{Exp}_D) - c^-(\mathrm{Imp}_D, i)\right) - \left(\left|\overrightarrow{G}\right| + 1\right)R,
\end{aligned}$$

i.e.,

$$\mathrm{GDP}_R = C + G + I + NX - \left(\left|\overrightarrow{G}\right| + 1\right)R \tag{6.12}$$

with

$$R = \int_{t_1}^{t_2} \mathrm{Ind}_e\left(\overrightarrow{G}^L_+\right) dt, \tag{6.13}$$

where, t_1, t_2 are the initial and terminal time, and R is the country's total residue in a specific time period. And *how do we evaluate the real GDP growth rate* κ? Certainly, we can also calculate κ by formula (6.11) in this case. However, the most important index is not κ but the e-index $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right)$ which surveys the degree of non-equilibrium, i.e., the more larger of $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right)$, the more we owe to the nature.

Notice that the harmonious developing of humans with the nature requires the way of humans developing must be from the non-equilibrium into an equilibrium. Consequently, a more scientific evaluation on the economical developing of humans is not only the GDP_R but also the e-index, or in other words, a pair $\left\{GDP_R, \mathrm{ind}_e\left(\overrightarrow{G}^L_+\right)\right\}$, i.e., the total economic scale and the deviation from the equilibrium but with $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right) \to 0$ if $t \to 0$, i.e., a harmonious developing of humans with the nature.

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