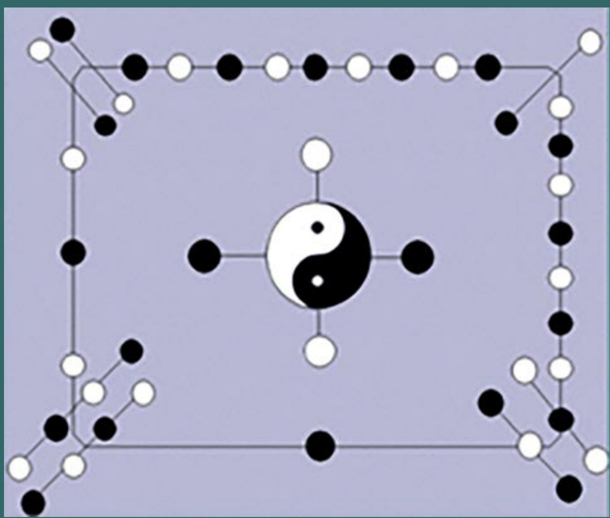
ISSN 1937 - 1055

VOLUME 4, 2020

INTERNATIONAL JOURNAL OF MATHEMATICAL COMBINATORICS



EDITED BY

THE MADIS OF CHINESE ACADEMY OF SCIENCES AND

ACADEMY OF MATHEMATICAL COMBINATORICS & APPLICATIONS, USA

DECEMBER, 2020

Vol.4, 2020 ISSN 1937-1055

International Journal of

Mathematical Combinatorics

(www.mathcombin.com)

Edited By

The Madis of Chinese Academy of Sciences and

Academy of Mathematical Combinatorics & Applications, USA

December, 2020

Aims and Scope: The *mathematical combinatorics* is a subject that applying combinatorial notion to all mathematics and all sciences for understanding the reality of things in the universe, motivated by *CC Conjecture* of Dr.Linfan MAO on mathematical sciences. The **International J.Mathematical Combinatorics** (*ISSN 1937-1055*) is a fully refereed international journal, sponsored by the *MADIS of Chinese Academy of Sciences* and published in USA quarterly, which publishes original research papers and survey articles in all aspects of mathematical combinatorics, Smarandache multi-spaces, Smarandache geometries, non-Euclidean geometry, topology and their applications to other sciences. Topics in detail to be covered are:

Mathematical combinatorics;

Smarandache multi-spaces and Smarandache geometries with applications to other sciences;

Topological graphs; Algebraic graphs; Random graphs; Combinatorial maps; Graph and map enumeration; Combinatorial designs; Combinatorial enumeration;

Differential Geometry; Geometry on manifolds; Low Dimensional Topology; Differential Topology; Topology of Manifolds;

Geometrical aspects of Mathematical Physics and Relations with Manifold Topology;

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Famous Words:

If a man empties his purse into his head, no man can take it away from him, an investment in knowledge always pays the best interest.

By Benjamin Franklin, American president

International J.Math. Combin. Vol.4(2020), 1-35

Dynamic Network with E-Index Applications

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Abstract: Unlike particles in the classical dynamics, the dynamical behavior of a complex network maybe not synchronized but fragmented, even a heterogenous moving in the eyes of human beings, which finally results in characterizing a complex network by random method or probability with statistics sometimes. However, such a dynamics on complex network is quite different from dynamics on particles because all mathematics are established on compatible systems but none on a heterogenous one. Naturally, a heterogenous system produces a contradictory system in general which was abandoned in classical mathematics but exists everywhere, i.e., it is inevitable if we would like to understand the reality of things in the world. Thus, we should establish such a mathematics on those of elements that contradictions appear together peacefully but without loss of the individual characters. For this objective, the network or in general, the continuity flow is the best candidate of the element, i.e., mathematical elements over a topological graph $\overrightarrow{G}$ in space. The main purpose of this paper is to establish such a mathematical theory on networks, including algebraic operations, differential and integral operations on networks, G-isomorphic operators, i.e., network mappings remains the unchanged underlying graph $\overrightarrow{G}$ with a generalization of the fundamental theorem of calculus, algebraic or differential equations with flow solutions and also, the dynamical equations of network with applications to other sciences by e-indexes on network. All of these results show the importance, i.e., quantitatively characterizing the reality of things by mathematical combinatorics.

Key Words: Complex network, Smarandache system, Smarandache multispace, contradictory system, continuity flow, calculus on network, dynamic equation of network, e-index, mathematical combinatorics.

AMS(2010): 05C10, 05C21, 34A12, 34D06, 35A08, 46B25, 51D20, 68M10.

§1. Introduction

Usually, standing on different viewing points brings about different models for understanding the reality of things in the world, which causes the knowledge is local or partial, not the whole on things and results in the limitation of humans. For thousands of years, one would like to divide a matter into sub-matters, i.e. its composition such as those of molecular, atoms and electrons and further, elementary particles ([25]), and a living thing into cells and genes for

[1]Received June 5, 2020, Accepted November 15, 2020.

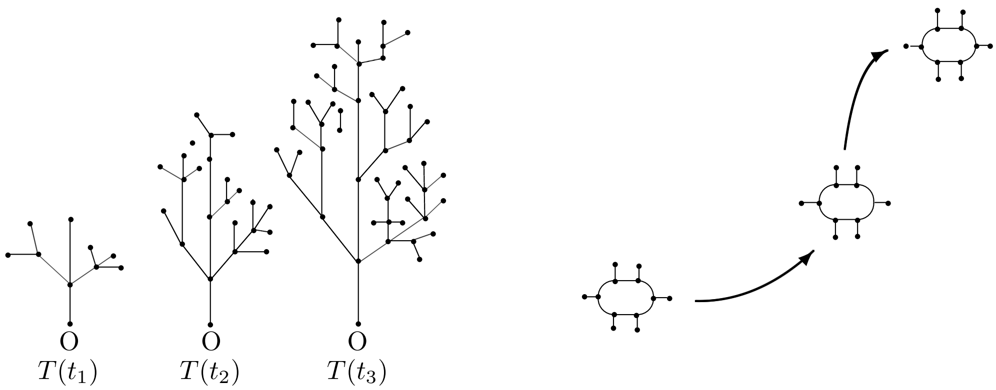
holding on its true face ([29]), which is essentially to equivalent a matter or thing to a complex network inherited by its fundamental units standing on a microscopic viewpoint. However, we are short of mathematics for characterizing the behavior of groups, particularly, a biological or adaptive system unless those of on the coordinated groups. Thus, we are more expected for establishing mathematics on groups, not only on those of isolated or ordered elements for holding on the reality of things.

According to the life cycle theory, there are series of stages for a living thing "*from birth to death*", i.e., birth, growth, maturity, decline and finally, death ([30]). Certainly, the *birth* is by chance but the *death* is inevitable, the growth, surviving and decline is the evolution or moving of a living thing such as those shown in Fig.1, where (a) is the evolution process of a tree and (b) is a mature horse runs on the earth.

(a) (b)

Fig.1

Then, *how do we characterize the evolution of the tree or moving of the horse appearing in Fig.*1? Usually, we characterize the pattern of a particle by differential equation in physics. Geometrically, we can depict the evolution of the tree or the running of the horse on the earth respectively by (a) or (b) in Fig.2.



(a) (b)

Fig.2

Certainly, the particle dynamics ignores the internal structure of the tree or the horse, abstracts them to points and characterizes their moving behavior by dynamic equations such as the Newtonian equations

$$\left(-\frac{\partial U}{\partial x_1}, -\frac{\partial U}{\partial x_2}, \cdots, -\frac{\partial U}{\partial x_n}\right) = \left(m\frac{d^2x_1}{dt^2}, m\frac{d^2x_2}{dt^2}, \cdots, m\frac{d^2x_m}{dt^2}\right), \tag{1.1}$$

where $U(\mathbf{x})$ is the potential energy of the field and $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. However, we can not apply such an equation (1.1) for establishing the evolution equations of the tree, and also the running horse in Fig.1 because they are not particles but complex networks, unless their components are all in synchronized or ignored by us. Then, *what is the right dynamic equations on the tree or the horse in Fig.*1 *by the microscopic viewing?* They must be dynamic equations on complex networks or networks, i.e, graph dynamics different from that of a particle or a rigid body ([5]). Such a dynamics is essentially on group of elements, maybe not all synchronized but with internal relationship, i.e., non-harmonious groups defined by mathematics following:

Non-Harmonious Group. *A non-harmonious group is such a group $\mathscr{T}$ consisting of elements P_i, $1 \leq i \leq p, p \geq 2$ with internal relations that P_i is constrained on equation $\mathscr{F}_i = 0$ in space on time t.*

Such a non-harmonious group is in fact a *Smarandache system* or *Smarandache multispace* because it posses *Smarandache denied axioms* (See [7], [8], [9] and [26] for details), also the parallel universe ([28]) in physical terminology. Notice that there is an inherited graph $\overrightarrow{G}_T$ defined by ([10])

$$\begin{aligned} V\left(\overrightarrow{G}_T\right) &= \{P_i \mid 1 \leq i \leq p\}, \\ E\left(\overrightarrow{G}_T\right) &= \{(P_i, P_j) \mid \text{if } P_i \text{ is interrelated with } P_j \text{ for } 1 \leq i, j \leq p\} \end{aligned}$$

However, there is naturally also a topological line graph $\overrightarrow{G}_{LT}$ inherited in a non-harmonious group $\mathscr{T}$ with respective edge and vertex sets following

$$\begin{aligned} E\left(\overrightarrow{G}_{LT}\right) &= \{P_i \mid 1 \leq i \leq p\}, \\ V\left(\overrightarrow{G}_{LT}\right) &= \{\text{maximal subsets } \{P_{i_1}, P_{i_2}, \cdots, P_{i_s}\}, 1 \leq i_1, i_2, \cdots, i_s \leq p, \\ &\qquad \text{where } P_{i_1}, P_{i_2}, \cdots, P_{i_s} \text{ have interrelation}\}, \end{aligned}$$

which is more useful for holding on the reality of mattes because nearly all living, non-living matters are non-harmonious groups with inherited line graph structures in the eyes of humans standing on a microscopic viewpoint.

Notice that such an inherited graph $\overrightarrow{G}_{LT}$ maybe more larger than the graph shown in Fig.1. For instance, we have known that a human body consists of $5 \times 10^{14} - 6 \times 10^{14}$ cells, i.e., the inherited graphs $\overrightarrow{G}_T, \overrightarrow{G}_{LT}$ of a human body by cells have respectively $5 \times 10^{14} - 6 \times 10^{14}$ vertices or edges. They are too larger graphs that nearly impossible to deal with them just by hands of humans. This fact implies that we should establish a mathematics on such non-harmonious groups for holding on the truth of matters, not only on its isolated elements but view the non-harmonious group $\mathscr{T}$ as a mathematical element entirely, i.e., mathematics over graphs or networks.

Notice that the evolution of the tree in Fig.1(a) is inclusive, i.e., the later includes the former $T(t_3) \supset T(t_2) \supset T(t_1)$ or the later develops from the former

$$T(t_3) = T(t_2) \bigcup (T(t_3) \setminus T(t_2)) = T(t_1) \bigcup (T(t_2) \setminus T(t_1)) \bigcup (T(t_3) \setminus T(t_2)) \qquad (1.2)$$

and also, all real networks such as those of internet, social relationship network, trading network, power and traffic network, $\cdots$, etc., are with the same advanced model. However, the running horse in Fig.1(b) is inclusive but unchanged, i.e., its inherited topological structure $\overrightarrow{G}_{LT}$ is invariable in running. Thus, the dynamics of the tree or the horse in Fig.1 and generally, a matter $\mathscr{T}$ can be always characterized by the motion of its inherited graph $\overrightarrow{G}_{LT}$ evolved at time t in space.

However, *can we conclude that a matter $\mathscr{T} = \overrightarrow{G}_{LT}$, the inherited graph of $\mathscr{T}$?* Certainly not because if we let $\mathscr{T}$ consisting of parts P_i, characterized by $\nu_j(P_i)$ with $i, j \geq 1$, we have

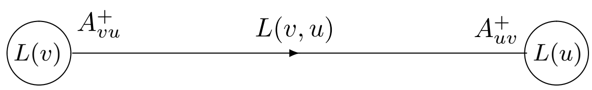
$$\mathscr{T} = \bigcup_{i \geq 1} P_i = \bigcup_{i \geq 1} \left(\bigcup_{j \geq 1} \nu_j(P_i) \right) \qquad (1.3)$$

in logic but the graph $\overrightarrow{G}_{LT}$ describes only the inherited structure but overlooked other characters of $\mathscr{T}$, which implies that a real model on $\mathscr{T}$ should retrieves all those of neglected characters on matter $\mathscr{T}$ in $\overrightarrow{G}_{LT}$, i.e., a dynamics on matters $\mathscr{T}$ should establishes on labeled graphs $\overrightarrow{G}^L_{LT}$ with labelling

$$\begin{array}{l} L : P_i \to P_i, \quad 1 \leq i \leq p, \\ L : \{P_{i_1}, P_{i_2}, \cdots, P_{i_s}\} \to \bigcap\limits_{k=1}^{s} P_{i_k} \quad \text{or} \quad L : (v, u) \to \bigcap\limits_{k=1}^{s} \left(\bigcap\limits_{l=1}^{s_l} \nu_l(P_{i_k}) \right), \end{array} \qquad (1.4)$$

where $\{i_1, i_2, \cdots, i_s\} \subset \{1, 2, \cdots, p\}$, i.e., network or its generalization, continuity flow following defined on the microscopic viewpoint, not only on particles or inherited graphs.

Definition 1.1([22-24]) *A continuity flow $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L : v \to L(v)$, $(v, u) \to L(v, u)$, 2 end-operators $A^+_{vu} : L(v, u) \to L^{A^+_{vu}}(v, u)$ and $A^+_{uv} : L(u, v) \to L^{A^+_{uv}}(u, v)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$ such as those shown in Fig.*3



v **Fig.**3 u

with $L(v, u) = -L(u, v)$, $A^+_{vu}(-L(v, u)) = -L^{A^+_{vu}}(v, u)$ for $\forall (v, u) \in E\left(\overrightarrow{G}\right)$ holding with continuity equation

$$\sum_{u \in N_G(v)} L^{A^+_{vu}}(v, u) = L(v) \quad for \quad \forall v \in V\left(\overrightarrow{G}\right) \qquad (1.5)$$

and all such continuity flows are denoted by $\mathscr{G}_{\mathscr{B}}$.

Notice that if we label edges by elements in a Banach space $\mathscr{B}$ and define the labels on vertices to be an induced labeling by

$$L(v) = \sum_{u \in N_G(v)} L^{A^+_{vu}}(v,u)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$, we can always get a continuity flow $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ on $\overrightarrow{G}$, and furthermore, if we let $\mathscr{B} = \mathbb{Z}$ and $\mathscr{A} = \{1_{\mathbb{Z}}\}$, a continuity flow $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ is nothing else but a network N.

In such induced continuity flows, the linear operator of $\mathscr{B}$, i.e., end-operators in $\mathscr{A}$ with criterion is in particular importance.

Definition 1.2([3]) *Let $\mathscr{B}$ be a Banach space over a field $\mathscr{F}$ and $\mathbf{T}: \mathscr{B} \to \mathscr{B}$ be an operator on Banach space $\mathscr{B}$ over a field $\mathscr{F}$. Then, $\mathbf{T}$ is linear if*

$$\mathbf{T}(\lambda \cdot \mathbf{A} + \mu \cdot \mathbf{B}) = \lambda \cdot \mathbf{T}(\mathbf{A}) + \mu \cdot \mathbf{T}(\mathbf{B})$$

for $\forall \mathbf{A}, \mathbf{B} \in \mathscr{B}$ and $\lambda, \mu \in \mathscr{F}$.

Theorem 1.3([3]) *Let $\mathscr{B}_1, \mathscr{B}_2$ be Banach spaces over a field $\mathbb{F}$ with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then, a linear operator $\mathbf{T}: \mathscr{B}_1 \to \mathscr{B}_2$ is continuous if and only if it is bounded, or equivalently,*

$$\|\mathbf{T}\| := \sup_{\mathbf{0} \neq \mathbf{v} \in \mathscr{B}_1} \frac{\|\mathbf{T}(\mathbf{v})\|_2}{\|\mathbf{v}\|_1} < +\infty.$$

Now, *could we establish a mathematics on continuity flows $(\overrightarrow{G}; L, \mathscr{A})$ underlying a graph in family $\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_m\}$ by viewing each of them as a mathematical elements entirely for integer $m \geq 1$?* The answer is positive, particularly for linear operators $\mathscr{A}$. In fact, the papers [6], [10] established a geometrical theory on non-harmonious groups and [11] discussed non-mathematical systems by combinatorial method, which is the fundamental of mathematics on non-harmonious groups. The papers [13]-[25] establish mathematics on continuity flows by functionals with applications to physics and biology. All the discussions are viewing continuity flows to be purely elements of Banach flow space. The main purpose of this paper is to establish such a mathematics on continuity flows paying more attentions to structure of $\overrightarrow{G}$ such as those of algebraic operations, differential and integral operations on continuity flows, G-isomorphic operators, i.e., mappings on continuity flows remains the unchanged underlying graph $\overrightarrow{G}$ with a generalization of the fundamental theorem of calculus, algebraic or differential equations with flow solutions and particularly, the dynamical equations of networks with applications to other sciences by e-indexes, which implies the truth of results appearing in [13]-[25] holds also on G-isomorphic operators.

For terminologies and notations not mentioned here, we follow references [1] for mechanics, [3] for functionals, [4] for complex network, [7] for topology and graphs, [8], [26] for Smarandache systems and multispaces.

§2. **Algebraic Operations**

Notice that a continuity flow $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ is a labelled graph. An algebraic operation on continuity flows should posses both of the algebraic and graph properties. We define the operations of addition "+" and multiplication "·" as follows:

Definition 2.1 *Let $G^L, G'^L \in \mathscr{G}^t_{\mathscr{B}}$, $\lambda \in \mathscr{F}$. Define*

$$\overrightarrow{G}^L + \overrightarrow{G'}^{L'} = \left(\overrightarrow{G} \setminus \overrightarrow{G'}\right)^L \bigcup \left(\overrightarrow{G} \bigcap \overrightarrow{G'}\right)^{L+L'} \bigcup \left(\overrightarrow{G'} \setminus \overrightarrow{G}\right)^{L'}, \tag{2.1}$$

$$\overrightarrow{G}^L \cdot \overrightarrow{G'}^{L'} = \left(\overrightarrow{G} \setminus \overrightarrow{G'}\right)^L \bigcup \left(\overrightarrow{G} \bigcap \overrightarrow{G'}\right)^{L\cdot L'} \bigcup \left(\overrightarrow{G'} \setminus \overrightarrow{G}\right)^{L'}, \tag{2.2}$$

$$\lambda \cdot \overrightarrow{G}^L = \overrightarrow{G}^{\lambda \cdot L} \tag{2.3}$$

where, $L(v,u)$ and $L'(v,u) \in \mathscr{B}$, $L + L' : (v,u) \to L(v,u) + L'(v,u)$, $L \cdot L' : (v,u) \to L(v,u) \cdot L'(v,u)$ respectively with substituting end-operators $\mathcal{A}^{*+}_{vu}$, $\mathcal{A}^{*+}_{vu}$ and $\mathcal{A}^{**+}_{vu}$ action on $(v,u) \in E\left(\overrightarrow{G}\right)$ such that

$$\begin{array}{rcl}\mathcal{A}^{*+}_{vu} & : & (L(v,u)) + L'(v,u) \to L^{A^+_{vu}}(v,u) + L'^{A'^+_{vu}}(v,u),\\ \mathcal{A}^{*+}_{vu} & : & L(v,u,) \cdot L'(v,u) \to L^{A^+_{vu}}(v,u) \cdot L'^{A'^+_{vu}}(v,u)),\\ \mathcal{A}^{**+}_{vu} & : & \lambda \cdot L(v,u) \to \lambda \cdot L^{\mathscr{A}^+_{vu}}(v,u).\end{array}$$

Let $\overrightarrow{G}^L, \overrightarrow{G'}^{L'} \in \mathscr{G}_{\mathscr{B}}$. A calculation shows that the labels on vertices of $\overrightarrow{G}$ are

$$L + L'(v) = \begin{cases} L(v) & \text{if } v \in \overrightarrow{G} \setminus \overrightarrow{G}', \\ L(v) + L'(v) & \text{if } v \in \overrightarrow{G} \bigcap \overrightarrow{G}', \\ L'(v) & \text{if } v \in \overrightarrow{G}' \setminus \overrightarrow{G} \end{cases}$$

and

$$L \cdot L'(v) = \begin{cases} L(v) & \text{if } v \in \overrightarrow{G} \setminus \overrightarrow{G}', \\ \sum\limits_{u \in N} L^{A^+_{vu}}(v,u) \cdot L'^{A'^+_{vu}}(v,u)) & \text{if } v \in \overrightarrow{G} \bigcap \overrightarrow{G}', \\ L'(v) & \text{if } v \in \overrightarrow{G}' \setminus \overrightarrow{G} \end{cases}$$

by definition. Particularly, if $\overrightarrow{G'} = \overrightarrow{G}$, we know that

$$L + L'(v) = L(v) + L'(v) \quad \text{and} \quad L \cdot L'(v) = \sum_{u \in N} L^{A^+_{vu}}(v,u) \cdot L'^{A'^+_{vu}}(v,u))$$

for $v \in \overrightarrow{G}$. The following convention are throughout in this paper.

Convention 2.2 *If $L(v,u) = \mathbf{0}$ for an edge $(v,u) \in E\left(\overrightarrow{G}^L\right)$, we always identify $\overrightarrow{G}^L$ with*

$\left(\overrightarrow{G}\setminus(v,u)\right)^L$, *i.e.*, $\overrightarrow{G}^L = \left(\overrightarrow{G}\setminus(v,u)\right)^L$.

Notice that the number of vertices of odd valency in a graph must be even. Thus, we can always transform a non-Eulerian graph to an Euleran graph by adding edges but with **0** flows between its odd vertices, which is essentially the same as the original continuity flows by Convention 2.2. We consider algebraic operations on continuity flows $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ following.

Definition 2.3 *Let* $a_1, a_2, \cdots, a_n \in \mathscr{B}$ *and* $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n} \in \mathscr{G}_{\mathscr{B}}$.

(1) Constant Elements. *Define* $a_i = \overrightarrow{G}^{I_{a_i}}$ *with* $I_{a_i} : (v,u) \to a_i$ *for* $\forall (v,u) \in E(G)$. *Particularly,* $0 = \overrightarrow{G}^{I_0} = \mathbf{O}$ *and* $1 = \overrightarrow{G}^{I_1} = \mathbf{I}$.

(2) Sum and Product. *Define*

$$\begin{aligned} a_1\overrightarrow{G}_1^{L_1} + a_2\overrightarrow{G}_2^{L_2} + \cdots + a_n\overrightarrow{G}_n^{L_n} &= \left(\bigcup_{i=1}^{n} G_i\right)^{a_1L_1+a_2L_2+\cdots+a_nL_n}, \\ \left(a_1\overrightarrow{G}_1^{L_1}\right)\cdot\left(a_2\overrightarrow{G}_2^{L_2}\right)\cdots\left(a_n\overrightarrow{G}_n^{L_n}\right) &= \left(\bigcup_{i=1}^{n} G_i\right)^{a_1L_1\cdot a_2L_2\cdot\cdots\cdot a_nL_n}. \end{aligned}$$

(3) Polynomial. *Define*

$$a_0 + a_1\overrightarrow{G}^L + a_2\overrightarrow{G}^{L^2} + \cdots + a_n\overrightarrow{G}^{L^n} = \overrightarrow{G}^{a_0+a_1L+a_2L^2+\cdots+a_nL^n}.$$

(4) Units. *Flows* $\mathbf{O}$ *and* $\mathbf{I}$ *are respectively the unit in* $(\mathscr{G}_{\mathscr{B}};+)$ *and* $(\mathscr{G}_{\mathscr{B}};\cdot)$ *because of*

$$\begin{aligned} \mathbf{O} + \overrightarrow{G}^L = \overrightarrow{G}^L + \mathbf{O} = \overrightarrow{G}^L, \\ \mathbf{I}\cdot\overrightarrow{G}^L = \overrightarrow{G}^L\cdot\mathbf{I} = \overrightarrow{G}^L. \end{aligned}$$

And we have operation properties of $\mathbf{O}$ *and* $\mathbf{I}$ *following:*

$$\begin{aligned} \mathbf{O}+\mathbf{O}=\mathbf{O}, \quad \mathbf{O}+\mathbf{I}=\mathbf{I}+\mathbf{O}=\mathbf{I}, \\ \mathbf{I}\cdot\mathbf{I}=\mathbf{I}, \quad \mathbf{O}\cdot\mathbf{O}=\mathbf{O}, \quad \mathbf{I}\cdot\mathbf{O}=\mathbf{O}\cdot\mathbf{I}=\mathbf{O}. \end{aligned}$$

(5) Inverse. *For* $\forall\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$, *if* $X + \overrightarrow{G}^L = \mathbf{O}$ *then* X *is defined to be the additive inverse of* $\overrightarrow{G}^L$. *Similarly, if* $Y\cdot\overrightarrow{G}^L = \mathbf{I}$ *then* Y *is defined to be the multiplication inverse of* $\overrightarrow{G}^L$.

Clearly,

$$X = -\overrightarrow{G}^L = \overrightarrow{G}^{-L} \quad \text{and} \quad Y = \frac{1}{\overrightarrow{G}^L} = \overrightarrow{G}^{\frac{1}{L}} = \overrightarrow{G}^{L^{-1}}.$$

We get the following equalities

$$-\overrightarrow{G}^L = \overrightarrow{G}^{-L} \qquad and \qquad \frac{1}{\overrightarrow{G}^L} = \overrightarrow{G}^{\frac{1}{L}}. \tag{2.4}$$

Applying formula (2.4), we immediately get the fraction, i.e.,

$$\frac{a_1\overrightarrow{G}_1^{L_1}+a_2\overrightarrow{G}_2^{L_2}+\cdots+a_n\overrightarrow{G}_n^{L_n}}{b_1\overrightarrow{G'}_1^{L'_1}+b_2\overrightarrow{G'}_2^{L'_2}+\cdots+b_n\overrightarrow{G'}_n^{L'_n}} = \frac{\left(\bigcup\limits_{i=1}^{n} G_i\right)^{a_1L_1+a_2L_2+\cdots+a_nL_n}}{\left(\bigcup\limits_{i=1}^{n} G'_i\right)^{b_1L'_1+b_2L'_2+\cdots+b_nL'_n}}$$

$$= \left(\bigcup_{i=1}^{n} G_i\right)^{a_1L_1+\cdots+a_nL_n}\cdot\frac{1}{\left(\bigcup\limits_{i=1}^{n} G'_i\right)^{b_1L'_1+\cdots+a_nL'_n}}$$

$$= \left(\left(\bigcup_{i=1}^{n} G_i\right)\bigcup\left(\bigcup_{i=1}^{n} G'_i\right)\right)^{\frac{a_1L_1+a_2L_2+\cdots+a_nL_n}{b_1L'_1+b_2L'_2+\cdots+b_nL'_n}}.$$

Notice that there are no the commutative laws

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$$

for $\forall\overrightarrow{G}^{L_1},\overrightarrow{G}^{L_2}\in\mathscr{G}_{\mathscr{B}}$ in general. However, we have

Theorem 2.4 *Let* $\forall\overrightarrow{G}^{L_1},\overrightarrow{G}^{L_2}\in\mathscr{G}_{\mathscr{B}}$. *Then,*

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$$

if and only if

$$L_1(v,u)\cdot L_2(v,u)=L_2(v,u)\cdot L_1(v,u)$$

and the same end-operators $\mathcal{A}_{vu}^{12+}=\mathcal{A}_{vu}^{21+}$ *for* $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, *where* $\mathcal{A}_{vu}^{12+}$ *and* $\mathcal{A}_{vu}^{21+}$ *are end-operators on* (v,u) *in* $\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}$ *or* $\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$, *respectively.*

Proof By (2.2), we know that

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_1\cdot L_2}\quad\text{and}\quad\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}=\overrightarrow{G}^{L_2\cdot L_1}.$$

Whence,

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$$

if and only if

$$L_1(v,u)\cdot L_2(v,u)=L_2(v,u)\cdot L_1(v,u)$$

and the same end-operators $\mathcal{A}_{vu}^{12+}=\mathcal{A}_{vu}^{21+}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. □

Corollary 2.5 *Let* $\mathscr{B}$ *be a commutative ring and let* $\overrightarrow{G}^{L}\in\mathscr{G}_{\mathscr{B}}$ *with* $1_{\mathscr{B}}$ *end-operator on* $(v,u)\in E\left(\overrightarrow{G}\right)$. *Then*

$$\overrightarrow{G}^{L_1}\cdot\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_2}\cdot\overrightarrow{G}^{L_1}$$

for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \mathscr{G}_{\mathscr{B}}$.

Proof It is obvious that

$$\mathcal{A}_{vu}^{12+} = \mathcal{A}_{vu}^{21+} = 1_{\mathscr{B}} \quad \text{and} \quad L_1(v,u) \cdot L_2(v,u) = L_2(v,u) \cdot L_1(v,u)$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ in this case if $\mathscr{B}$ is a commutative ring. □

Notice that if $(\mathscr{B};+,\cdot)$ is a division ring, i.e., $(\mathscr{B};+)$ and $(\mathscr{B};\cdot)$ are both of groups, then Corollary 2.5 implies the following conclusion.

Theorem 2.6 *If* $(\mathscr{B};+,\cdot)$ *is a division ring and every* $\overrightarrow{G}^{L} \in \mathscr{G}_{\mathscr{B}}$ *has* $1_{\mathscr{B}}$ *end-operator on* $(v,u) \in E\left(\overrightarrow{G}\right)$, *then* $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ *is a division ring. Furthermore,* $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ *is a field if* $(\mathscr{B};+,\cdot)$ *is a field.*

Proof Clearly, $(\mathscr{G}_{\mathscr{B}};+)$ and $(\mathscr{G}_{\mathscr{B}};\cdot)$ are both of Abelian groups with associative laws, i.e.,

$$\overrightarrow{G}^{L_1} \cdot \left(\overrightarrow{G}^{L_2} + \overrightarrow{G}^{L_3}\right) = \overrightarrow{G}^{L_1} \cdot \overrightarrow{G}^{L_2} + \overrightarrow{G}^{L_1} \cdot \overrightarrow{G}^{L_3}$$

and

$$\left(\overrightarrow{G}^{L_1} + \overrightarrow{G}^{L_2}\right) \cdot \overrightarrow{G}^{L_3} = \overrightarrow{G}^{L_1} \cdot \overrightarrow{G}^{L_3} + \overrightarrow{G}^{L_2} \cdot \overrightarrow{G}^{L_3}$$

for $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2}, \overrightarrow{G}^{L_3} \in \mathscr{G}_{\mathscr{B}}$ because of

$$L_1 \cdot (L_2 + L_3) = L_1 \cdot L_2 + L_1 \circ L_3 \quad \text{and} \quad (L_1 + L_2) \cdot L_3 = L_1 \circ L_3 + L_2 \cdot L_3,$$

i.e., $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ is a division ring.

By Corollary 2.5, $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ is commutative if $(\mathscr{B};+,\cdot)$ is commutative, i.e., $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ is a field if $(\mathscr{B};+,\cdot)$ is a field. This completes the proof. □

Example 2.7 Let U and D be 2×2 matrixes over $\mathbb{R}$ determined by

$$U = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \middle| \; a,b,c,d \in \mathbb{R} \right\}, \qquad W = \left\{ \left(\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \middle| \; a,b \in \mathbb{R} \right\}$$

and $\overrightarrow{G}$ a digraph. For continuity flows $\overrightarrow{G}^{L}$ with all end-operators being the unit 1, and

$$L: \ (v,u) \to U, \quad (v,u) \in E\left(\overrightarrow{G}\right).$$

Then,

(1) $\left\{ \overrightarrow{G}^{L} \middle| \ L:(v,u) \to U \right\}$ maybe not commutative. For example, for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ let

$$L_1(v,u) = \left(\begin{array}{cc} 1 & 1 \\ 2 & 3 \end{array} \right), \quad L_2(v,u) = \left(\begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right).$$

We have

$$L_1(v,u)\cdot L_2(v,u) = \begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}2 & 1\\ 5 & 2\end{pmatrix},$$

$$L_2(v,u)\cdot L_1(v,u) = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix} = \begin{pmatrix}3 & 4\\ 1 & 1\end{pmatrix}.$$

Thus,

$$L_1(v,u)\cdot L_2(v,u) \neq L_2(v,u)\cdot L_1(v,u),$$

i.e., $\left\{\overrightarrow{G}^L \middle| L:(v,u)\to U\right\}$ is not commutative in this case by Theorem 3.1.

(2) Let

$$L_1(v,u) = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix},\quad L_2(v,u) = \begin{pmatrix}c & 0\\ 0 & d\end{pmatrix}$$

for $(v,u)\in E\left(\overrightarrow{G}\right)$. Then,

$$L_1(v,u)\cdot L_2(v,u) = \begin{pmatrix}a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix}c & 0\\ 0 & d\end{pmatrix} = \begin{pmatrix}ac & 0\\ 0 & bd\end{pmatrix},$$

$$L_2(v,u)\cdot L_1(v,u) = \begin{pmatrix}c & 0\\ 0 & d\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & b\end{pmatrix} = \begin{pmatrix}ac & 0\\ 0 & bd\end{pmatrix},$$

i.e.,

$$L_1(v,u)\cdot L_2(v,u) = L_2(v,u)\cdot L_1(v,u)$$

for $(v,u)\in E\left(\overrightarrow{G}\right)$. We know that $\left\{\overrightarrow{G}^L \middle| L:(v,u)\to W\right\}$ is commutative by Theorem 2.4.

§3. G-isomorphic Operators

By definition, the continuity flow is vectors associating with shapes, i.e., structures. Such a kind of operators that remains the topological structure $\overrightarrow{G}$ unchanged is particularly important.

3.1 G-isomorphic Operators on Continuity Flows

Definition 3.1 *An operator $f:\overrightarrow{G}_1^{L_1}\to\overrightarrow{G}_2^{L_2}$ on $\mathscr{G}_{\mathscr{B}}$ is G-isomorphic if it holds with conditions*

(i) there is an isomorphism $\varphi:\overrightarrow{G}_1\to\overrightarrow{G}_2$ of graph;
(ii) $L_2 = f\circ\varphi\circ L_1$ for $\forall(v,u)\in E\left(\overrightarrow{G}_1\right)$.

We can therefore denote a G-isomorphic operator by $f:\overrightarrow{G}^{L_1}\to\overrightarrow{G}^{L_2}$. Particularly, let $\varphi = \mathrm{id}_{\overrightarrow{G}}$. Then, such an operator is determined by an equation

$$L_2 = f\circ L_1 \tag{3.1}$$

for $\forall(v,u)\in E(G)$ or in other words, a G-isomorphic operator is mapping on vectors with an invariant structure of graph.

Furthermore, if $\mathscr{B}$ is a function field on a variable t, i.e., $\mathscr{F}[t]$, we can therefore know such a G-isomorphic operator f holds with an equation

$$f\left(\overrightarrow{G}^{L[t]}\right)=\overrightarrow{G}^{f(L[t])}, \tag{3.2}$$

which enables us to get a few interesting equalities following.

(1) $a\left(\overrightarrow{G}^{L[t]}\right)^n=\overrightarrow{G}^{aL^n[t]}$ for $a\in\mathbb{R}$ and $n\in\mathbb{Z}^+$;

(2) $a^{\overrightarrow{G}^{L[t]}}=\overrightarrow{G}^{a^{L[t]}}$, $\log\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\log L[t]}$ for $0\neq a\in\mathbb{R}$,
$e^{\overrightarrow{G}^{L[t]}}=\overrightarrow{G}^{e^{L[t]}}$, $\ln\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\ln L[t]}$;

(3) $\sin\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\sin L[t]}$, $\cos\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\cos L[t]}$, $\tan\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\tan L[t]}$, $\cot\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\cot L[t]}$,
$\sinh\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\sinh L[t]}$, $\cosh\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\cosh L[t]}$,
$\coth\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\coth L[t]}$, $\tanh\overrightarrow{G}^{L[t]}=\overrightarrow{G}^{\tanh L[t]}$;

(4) $\left(\mathbf{I}+a\overrightarrow{G}^{L[t]}\right)^n=\overrightarrow{G}^{(1+aL[t])^n}$, $\left(\mathbf{I}+\frac{a\mathbf{I}}{\overrightarrow{G}^{L[t]}}\right)^n=\overrightarrow{G}^{\left(1+\frac{a}{L[t]}\right)^n}$ for $n\in\mathbb{Z}^+$, $a\in\mathbb{R}$;

(5) $\dfrac{\overrightarrow{G}^{nL[t]}}{\mathbf{I}-\overrightarrow{G}^{L[t]}}=\mathbf{I}+\overrightarrow{G}^{L[t]}+\overrightarrow{G}^{2L[t]}+\cdots+\overrightarrow{G}^{(n-1)L[t]}$ for $1\leq n\in\mathbb{Z}^+$.

Furthermore, we get the exponential map following.

Theorem 3.2 *Let $\overrightarrow{G}^{L[t]}\in\mathscr{G}_{\mathscr{B}}$, where $\mathscr{B}$ is a field. Then,*

$$e^{\overrightarrow{G}^{L[t]}}=\mathbf{I}+\frac{\overrightarrow{G}^{L[t]}}{1!}+\frac{\overrightarrow{G}^{2L[t]}}{2!}+\cdots+\frac{\overrightarrow{G}^{nL[t]}}{n!}+\cdots.$$

Proof Notice that

$$\begin{aligned}\mathbf{I}+\frac{\overrightarrow{G}^{L[t]}}{1!}+\frac{\overrightarrow{G}^{2L[t]}}{2!}+\cdots+\frac{\overrightarrow{G}^{nL[t]}}{n!}+\cdots &= \mathbf{I}+\overrightarrow{G}^{\frac{L[t]}{1!}}+\overrightarrow{G}^{\frac{2L[t]}{2!}}+\cdots+\overrightarrow{G}^{\frac{nL[t]}{n!}}+\cdots\\ &= \overrightarrow{G}^{1+\frac{L[t]}{1!}+\frac{2L[t]}{2!}+\cdots+\frac{nL[t]}{n!}+\cdots}=\overrightarrow{G}^{e^{L[t]}}.\end{aligned}$$

By equation (3.2), we know that $e^{\overrightarrow{G}^{L[t]}}=\overrightarrow{G}^{e^{L[t]}}$. Thus,

$$e^{\overrightarrow{G}^{L[t]}}=\mathbf{I}+\frac{\overrightarrow{G}^{L[t]}}{1!}+\frac{\overrightarrow{G}^{2L[t]}}{2!}+\cdots+\frac{\overrightarrow{G}^{nL[t]}}{n!}+\cdots. \qquad \square$$

By equation (3.2), it is clear that

$$\begin{aligned}e^{\overrightarrow{G}^{L[t]}}\cdot e^{\overrightarrow{G}^{L'[t]}} &= \overrightarrow{G}^{e^{L[t]}}\cdot\overrightarrow{G}^{e^{L'[t]}}=\overrightarrow{G}^{e^{L[t]}\cdot e^{L'[t]}}\\ &= \overrightarrow{G}^{e^{L[t]+L'[t]}}=e^{\overrightarrow{G}^{L[t]+L'[t]}},\end{aligned}$$

which is similar to that of $e^x\cdot e^y=e^{x+y}$ as the usual.

3.2 Extended Operators on Continuity Flows

Let $\overrightarrow{G}, \overrightarrow{H}$ be graphs with $\overrightarrow{G} \prec \overrightarrow{H}$. It is interesting to find an operator $f : \overrightarrow{G}^{L_1} \to \overrightarrow{H}^{L_2}$ for characterizing the trail from $\overrightarrow{G}^{L_1}$ to $\overrightarrow{H}^{L_2}$. By Convention 2.1, if $L(v,u) = \mathbf{0}$ for an edge $(v,u) \in E\left(G^L\right)$, we identify G^L with $(G\setminus(v,u))^L$ because there are no difference on flows between G^L with $(G\setminus(v,u))^L$.

Definition 3.3 *Let $\overrightarrow{G}, \overrightarrow{H}$ be graphs with $\overrightarrow{G} \prec \overrightarrow{H}$. An operator $f : \overrightarrow{G}^{L_1} \to \overrightarrow{H}^{L_2}$ on $\mathscr{G}_{\mathscr{B}}$ is extended if it holds with conditions*

(i) there is an isomorphism $\varphi : \overrightarrow{G} \to \overrightarrow{G}$ of graph;
(ii) $L_2 = f \circ \varphi \circ L_1$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ but $f : \mathbf{0} \to L_2(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{H} \setminus \overrightarrow{G}\right)$.

Certainly, such an extended operator maps a continuity flow to its extended flow. However, by Convention 2.2, we view such an extended operator f to be a H-isomorphic operator by the following ways

(1) Extend L_1 to L_1' by $L_1'(v,u) = L_1(v,u)$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ but $L_1'(v,u) = \mathbf{0}$ for $(v,u) \in E\left(\overrightarrow{H}^{L_2} \setminus \overrightarrow{G}^{L_1}\right)$;

(2) Extend $\varphi|_{\overrightarrow{G}}$ to $\varphi|_{\overrightarrow{H}}$ constraint by $\varphi|_{\overrightarrow{H}} = \varphi|_{\overrightarrow{G}}$ on graph $\overrightarrow{G}$.

By Definition 3.3, if an extended operator f exists, then its inverse f^{-1} must be existed because f is a $1-1$ mapping. Such a f^{-1} is called a contracted operator. For example, let $\overrightarrow{G}^{L_1}$, $\overrightarrow{G}^{L_2}$ be 2 continuity flows. An extended isomorphism $f(\mathbf{v_i}) = \mathbf{u_i}$ for $1 \le i \le 4$ but $f(0) = \mathbf{u_5}$ with its inverse f^{-1} is shown in Fig.4.

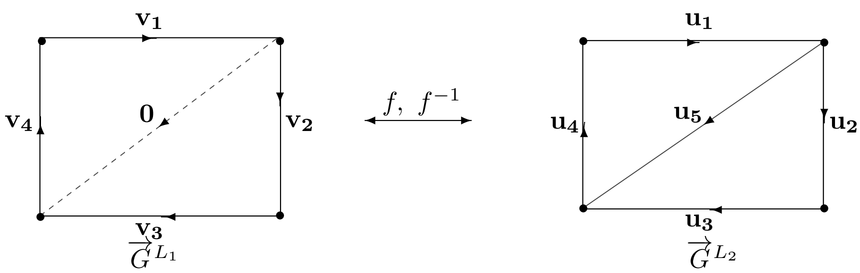


Fig.4

and if $f = ax + b$, $0 \neq a, b \in \mathbb{R}$, we can also get $\overrightarrow{G}^{L_2}$ by $\overrightarrow{G}^{L_1}$ shown in Fig.5.

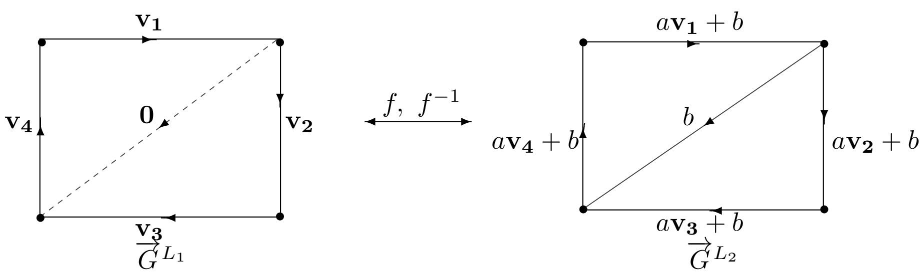


Fig.5

i.e., $\overrightarrow{G}^{L_2} = a\left(\overrightarrow{G}^{L_1}\right) + b = \overrightarrow{G}^{aL_1+b}$ with $f^{-1} = \dfrac{x}{a} - b$, both are linear G-isomorphic operators.

Generally, we get a result following.

Theorem 3.4 *Let $\emptyset \neq \overrightarrow{G}_1, \overrightarrow{G}_2 \in \mathscr{G}$, maybe with $\overrightarrow{G}_1 \simeq \overrightarrow{G}_2$ or not. There must be a G-isomorphic operator f such that*

$$f\left(\overrightarrow{G}_1^{L_1}\right) = \overrightarrow{G}_2^{L_2}$$

for $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$.

Proof Notice that $\overrightarrow{G}_1, \overrightarrow{G}_2 \neq \emptyset$. Let $G = \overrightarrow{G}_1 \bigcup \overrightarrow{G}_2$ and

$$L_1' = \begin{cases} L_1(v,u) & \text{if (v, u)} \in \mathrm{E}\left(\overrightarrow{G}_1\right), \\ \mathbf{0} & \text{otherwise;} \end{cases} \qquad L_2' = \begin{cases} L_2(v,u) & \text{if (v, u)} \in \mathrm{E}\left(\overrightarrow{G}_2\right), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Then,

$$\overrightarrow{G}_1^{L_1} = \overrightarrow{G}^{L_1'} \quad \text{and} \quad \overrightarrow{G}_2^{L_2} = \overrightarrow{G}^{L_2'}$$

by Convention 2.2. Let φ be an automorphism of $\overrightarrow{G}$ and let $f : \overrightarrow{G}^{L_1'} \to \overrightarrow{G}^{L_2'}$ be an automorphism $f : \overrightarrow{G}^{L_1} \to \overrightarrow{G}^{L_2}$ with $L_2' = f \circ \varphi \circ L_1'$. Certainly, f is a G-isomorphic operator from $\overrightarrow{G}^{L_1'}$ to $\overrightarrow{G}^{L_2'}$, i.e.,

$$f\left(\overrightarrow{G}_1^{L_1}\right) = \overrightarrow{G}_2^{L_2}.$$

This completes the proof. □

3.3 Continuous Operators

Theorem 3.4 enables us to discuss the continuity behaviours of operators on $\mathscr{G}_{\mathscr{B}}$.

Definition 3.5 *Let $(\mathscr{B}; +, \cdot)$ be a normed space over field $\mathscr{F}$ with norm $\|\mathbf{v}\|$, $\mathbf{v} \in \mathscr{B}$ and $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$. The norm of $\overrightarrow{G}^L$ is defined by*

$$\left\|\overrightarrow{G}^L\right\| = \sum_{(v,u) \in E\left(\overrightarrow{G}\right)} \|L(v,u)\|,$$

i.e., the norm $\| \ \|$ is a mapping with $\| \ \| : \mathscr{G}_{\mathscr{B}}^t \to \mathbb{R}^+$.

For example, if $\mathbf{v_1} = (0,1), \mathbf{v_2} = (1,0), \mathbf{v_3} = (1,1), \mathbf{v_4} = (1,-1)$ with $\mathbf{v_5} = \mathbf{0}$ in Fig.4, then

$$\begin{aligned} \left\|\overrightarrow{G}^{L_1}\right\| &= \|\mathbf{v_1}\| + \|\mathbf{v_2}\| + \|\mathbf{v_3}\| + \|\mathbf{v_4}\| + \|\mathbf{0}\| \\ &= \sqrt{0^2+1^2} + \sqrt{1^2+0^2} + \sqrt{1^2+1^2} + \sqrt{1^2+(-1)^2} + 0 = 2\left(1+\sqrt{2}\right). \end{aligned}$$

Certainly, we are easily known that $\mathscr{G}_{\mathscr{B}}$ is a normed space by Definition 3.5, i.e., for $\forall \overrightarrow{G}^L, \overrightarrow{G}_1^{L_1}$ and $\overrightarrow{G}_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$,

(1) $\left\|\overrightarrow{G}^L\right\| \geq 0$ and $\left\|\overrightarrow{G}^L\right\| = 0$ if and only if $\overrightarrow{G}^L = \overrightarrow{G}^{\mathbf{0}} = \mathbf{O}$;

(2) $\left\|\overrightarrow{G}^{\xi L}\right\| = \xi\left\|\overrightarrow{G}^{L}\right\|$ for any scalar $\xi \in \mathscr{F}$;

(3) $\left\|\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right\| \leq \left\|\overrightarrow{G}_1^{L_1}\right\| + \left\|\overrightarrow{G}_2^{L_2}\right\|$.

Definition 3.6 *For $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$, the distance between $\overrightarrow{G}_1^{L_1}$ and $\overrightarrow{G}_2^{L_2}$ is defined by*

$$d\left(\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}\right) = \left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}_2^{L_2}\right\|.$$

By Definition 2.1, we know that

$$\overrightarrow{G}_1^{L_1} - \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1 - L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{L_2}.$$

Therefore,

$$\begin{aligned} d\left(\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}\right) &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)} \|L_1(e)\| \\ &+ \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)} \|L_1(e)\| + \sum_{e \in E\left(\overrightarrow{G}_2 \bigcap \overrightarrow{G}_1\right)} \|L_2(e)\|. \end{aligned}$$

For example, if $\mathbf{u_1} = (1,0), \mathbf{u_2} = (0,1), \mathbf{u_3} = (-1,-1), \mathbf{u_4} = (-1,1)$ and $\mathbf{u_5} = (-2,2)$ in Fig.4, then the distance of $\overrightarrow{G}^{L_1}$ and $\overrightarrow{G}^{L_2}$ is

$$\begin{aligned} d\left(\overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2}\right) &= \sum_{i=1}^{5} \|v_i - u_i\| = \sqrt{(-1)^2 + 1^2} + \sqrt{1^1 + (-1)^2} \\ &+ \sqrt{2^2 + 2^2} + \sqrt{2^2 + (-2)^2} + \sqrt{2^2 + (-2)^2} = 8\sqrt{2}. \end{aligned}$$

Definition 3.7 *Let f be a G-isomorphic operator on $\mathscr{G}_{\mathscr{B}}$, $\overrightarrow{G}^{L}$, $\overrightarrow{G}_0^{L_0} \in \mathscr{G}_{\mathscr{B}}$ dependent on a variable t. Then, f is G-continuous at $\overrightarrow{G}_0^{L_0}$, denoted by $\lim\limits_{L \to L_0} f(\overrightarrow{G}^{L}) = f(\overrightarrow{G}_0^{L_0})$ if for any number $\epsilon > 0$, there is always a number $\delta > 0$ such that*

$$d\left(f\left(\overrightarrow{G}^{L}[t]\right),\ f\left(\overrightarrow{G}_0^{L_0}[t_0]\right)\right) < \epsilon \tag{3.3}$$

if $d(\overrightarrow{G}^{L}[t], \overrightarrow{G}_0^{L_0}[t_0]) < \delta$. Furthermore, such an operator f is completely continuous, denoted by $\lim\limits_{t \to t_0} f(\overrightarrow{G}^{L}) = f(\overrightarrow{G}_0^{L_0})$ if the inequality (3.3) holds with $|t - t_0| < \delta$.

Clearly, a completely continuous operator does not depends on the structure of graph $\overrightarrow{G}$, i.e., it is G-free or in other words, it is G-continuous over any graph G.

Theorem 3.8 *Let f be a G-isomorphic operator on $\mathscr{G}_{\mathscr{B}}$, $\overrightarrow{G}^{L}$, $\overrightarrow{G}_0^{L_0} \in \mathscr{G}_{\mathscr{B}}$, where G is the union of all graphs in $\mathscr{G}$. Then,*

$$\lim_{L \to L_0} f\left(\overrightarrow{G}^{L}\right) = f\left(\overrightarrow{G}_0^{L_0}\right) \quad or \quad \lim_{t \to t_0} f\left(\overrightarrow{G}^{L}\right) = f\left(\overrightarrow{G}_0^{L_0}\right)$$

if and only if f is continuous on L or $f\circ L$ is continuous on t for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Proof Let $\overrightarrow{H}=\bigcup\limits_{\overrightarrow{G}_i\in\mathscr{G}}\overrightarrow{G}_i$. Without loss of generality, by Convention 2.2 we can let $\overrightarrow{G}^L=\overrightarrow{H}^L$ and $\overrightarrow{G}_0^{L_0}=\overrightarrow{H}^{L_0}$. By definition, f is G-continuous or completely continuous if for a number $\epsilon>0$ there is always a number $\delta>0$ such that if $d\left(\overrightarrow{H}^L[t],\overrightarrow{H}^{L_0}[t_0]\right)<\delta$ or $|t-t_0|<\delta$ then

$$d\left(f\left(\overrightarrow{H}^L[t]\right),\ f\left(\overrightarrow{H}^{L_0}[t_0]\right)\right)\ <\ \epsilon,\quad \text{i.e.,}\quad d\left(\overrightarrow{H}^{f(L)}[t],\ \overrightarrow{H}^{f(L_0)}[t_0]\right)\ <\ \epsilon,$$

which implies that

$$\left\|\overrightarrow{H}^{f(L)}[t]-\overrightarrow{H}^{f(L_0)}[t_0]\right\|\ <\ \epsilon,\quad \text{i.e.,}\quad \sum_{e\in E\left(\overrightarrow{H}\right)}\|(f(L[t])-f(L_0[t_0]))(e)\|<\ \epsilon$$

by Definition 3.6. Notice that $\|e\|\geq 0$ for $e\in E\left(\overrightarrow{H}\right)$.

Conversely, for a number $\epsilon>0$, if there is a number $\delta>0$ such that

$$\|(f\circ L[t]-f\circ L_0[t_0])(e)\|<\ \frac{\epsilon}{\varepsilon\left(\overrightarrow{H}\right)}$$

for $\forall e\in E\left(\overrightarrow{H}\right)$ if $d\left(\overrightarrow{H}^L[t],\overrightarrow{H}^{L_0}[t_0]\right)<\delta$ or $|t-t_0|<\delta$, we get that

$$\begin{aligned}
d\left(f\left(\overrightarrow{H}^L[t]\right),\ f\left(\overrightarrow{H}^{L_0}[t_0]\right)\right) &= \left\|\overrightarrow{H}^{f\circ L}[t]-\overrightarrow{H}^{f\circ L_0}[t_0]\right\| \\
&= \sum_{e\in E\left(\overrightarrow{H}\right)}\|(f\circ L[t]-f\circ L_0[t_0])(e)\| \\
&\leq \varepsilon\left(\overrightarrow{H}\right)\times\frac{\epsilon}{\varepsilon\left(\overrightarrow{H}\right)}=\epsilon
\end{aligned}$$

where $\varepsilon\left(\overrightarrow{H}\right)$ is the size of $\overrightarrow{H}$. We therefore know that

$$d\left(f\left(\overrightarrow{H}^L[t]\right),\ f\left(\overrightarrow{H}^{L_0}[t_0]\right)\right)\ <\ \epsilon\ \Leftrightarrow\ \|(f(L[t])-f(L_0[t_0]))(e)\|<\ \epsilon \tag{3.4}$$

for $\forall e\in E\left(\overrightarrow{H}\right)$.

Similarly, we can also know that

$$d\left(\overrightarrow{H}^L[t],\ \overrightarrow{H}^{L_0}[t_0]\right)\ <\ \epsilon\ \Leftrightarrow\ \|L[t]-L_0[t_0])(e)\|<\ \epsilon \tag{3.5}$$

for $\forall e\in E\left(\overrightarrow{G}\right)$.

By the equivalences (3.4) and (3.5), we are easily knowing that

$$\lim_{L[t]\to L[t_0]}f\left(\overrightarrow{H}^{L[t]}\right)=f\left(\overrightarrow{H}^{L_0[t_0]}\right),\quad \text{i.e.,}\quad \lim_{L\to L_0}f\left(\overrightarrow{G}^L\right)=f\left(\overrightarrow{G}_0^{L_0}\right) \tag{3.6}$$

if and only if f is continuous on L by definition, and

$$\lim_{t\to t_0} f\left(\overrightarrow{H}^{L[t]}\right) = f\left(\overrightarrow{H}^{L_0[t_0]}\right), \quad \text{i.e.,} \quad \lim_{t\to t_0} f\left(\overrightarrow{G}^{L}\right) = f\left(\overrightarrow{G}_0^{L_0}\right) \tag{3.7}$$

if and only if $f \circ L$ is continuous on t for $\forall (v,u) \in E\left(\overrightarrow{H}\right)$. This completes the proof. □

Notice that the composition of continuous functions is also continuous. We therefore know the conclusion following by Theorem 3.8.

Corollary 3.9 *If f respect to L and L respect to t both are continuous, then*

$$\lim_{t\to t_0} f\left(\overrightarrow{G}^{L[t]}\right) = f\left(\overrightarrow{G}_0^{L[t_0]}\right).$$

Example 3.10 Let $f = aL^2 + b$ with $0 \neq a, b \in \mathbb{R}$ and L shown in Fig.6.

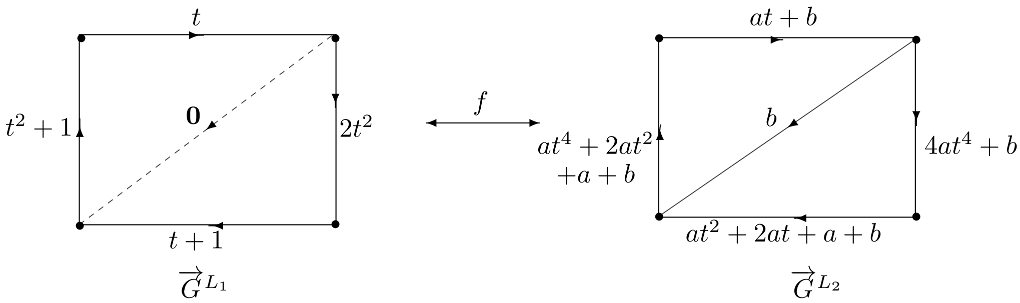


Fig.6

We know that f is G-isomorphic by Theorem 3.8 and furthermore, it is also complete.

§4. Calculus and E-index on Continuity Flows

4.1 Differential and Integral Operators

Definition 4.1 *Let $\overrightarrow{G}^{L}[t] \in \mathscr{G}_{\mathscr{B}}$ dependent on variable t and let f be a G-isomorphic operator on $\mathscr{G}_{\mathscr{B}}$ with $f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right) \to f\left(\overrightarrow{G}^{L}[t]\right)$ if $\Delta t \to 0$. Then, f is defined to be G-differential if*

$$\lim_{\Delta t\to 0} \frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right) - f\left(\overrightarrow{G}^{L}[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t] - \overrightarrow{G}^{L}[t]} \in \mathscr{G}_{\mathscr{B}},$$

denoted by

$$\frac{df}{dt} = \lim_{\Delta t\to 0} \frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right) - f\left(\overrightarrow{G}^{L}[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t] - \overrightarrow{G}^{L}[t]} \quad or \quad \dot{f} = \lim_{\Delta t\to 0} \frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right) - f\left(\overrightarrow{G}^{L}[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t] - \overrightarrow{G}^{L}[t]}.$$

Let C be a continuity flow $\overrightarrow{G}^C$ with $C:(v,u)\to constant$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. Clearly,

$$\frac{df}{dt}\left(\overrightarrow{G}^L[t]\right)=F\left(\overrightarrow{G}^L[t]\right)\ \Rightarrow\ \frac{df}{dt}\left(\overrightarrow{G}^L[t]+C\right)=F\left(\overrightarrow{G}^L[t]\right)$$

and the integral operator on $\mathscr{G}^t_{\mathscr{B}}$ is defined by

$$\int F\left(\overrightarrow{G}^L[t]\right)dt\ =\ f\left(\overrightarrow{G}^L[t]\right)+C.$$

By definition, we know formulae on differential and integral operators following.

$$\int\left(\frac{df}{dt}\left(\overrightarrow{G}^L[t]\right)\right)dt\ =\ f\left(\overrightarrow{G}^L[t]\right)+C \tag{4.1}$$

and

$$\frac{df}{dt}\left(\int\left(f\left(\overrightarrow{G}^L[t]\right)\right)dt\right)\ =\ f\left(\overrightarrow{G}^L[t]\right). \tag{4.2}$$

The following conclusion is gotten immediately by definition.

Theorem 4.2 *The differential $\frac{d}{dt}$ and integral $\int$ both are linear on $\mathscr{G}_{\mathscr{B}}$.*

Now, let $\overrightarrow{H}=\bigcup\limits_{\overrightarrow{G}_i\in\mathscr{G}}\overrightarrow{G}_i$ and $\overrightarrow{H}^{L'}=\overrightarrow{G'}^{L'},\overrightarrow{H}^L=\overrightarrow{G}^L$ by Convention 2.2. By definition, we know that

$$\begin{aligned}\lim_{\Delta t\to0}\frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right)-f\left(\overrightarrow{G}^L[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t]-\overrightarrow{G}^L[t]}&=\lim_{\Delta t\to0}\frac{\overrightarrow{H}^{f(L')[t+\Delta t]-f(L)[t]}}{\overrightarrow{H}^{L'[t+\Delta t]-L[t]}}\\&=\lim_{\Delta t\to0}\overrightarrow{H}^{\frac{f(L')[t+\Delta t]-f(L)[t]}{L'[t+\Delta t]-L[t]}}=\overrightarrow{H}^{\lim\limits_{\Delta t\to0}\frac{f(L')[t+\Delta t]-f(L)[t]}{L'[t+\Delta t]-L[t]}}.\end{aligned}$$

Thus, f is G-differential if f itself is differential on L for $\forall e\in E\left(\overrightarrow{H}\right)$.

Conversely, if f is differential on L for $\forall e\in E\left(\overrightarrow{H}\right)$, then it is clear that

$$\begin{aligned}\overrightarrow{H}^{\dot f(L)}&=\overrightarrow{H}^{\lim\limits_{\Delta t\to0}\frac{f(L')[t+\Delta t]-f(L)[t]}{L'[t+\Delta t]-L[t]}}\\&=\lim_{\Delta t\to0}\frac{f\left(\overrightarrow{G'}^{L'}[t+\Delta t]\right)-f\left(\overrightarrow{G}^L[t]\right)}{\overrightarrow{G'}^{L'}[t+\Delta t]-\overrightarrow{G}^L[t]}\in\mathscr{G}_{\mathscr{B}},\end{aligned}$$

i.e., f is G-differential. We therefore get the conclusion following.

Theorem 4.3 A G-isomorphic operator $f:\ \mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}}$ is differential if and only if $f(L)$ is differential on L.

A calculation by equations $(3.2),(4.1)$ and (4.2) shows that

1) $\frac{dC}{dt} = \mathbf{O}$, $\int \mathbf{O}dt = \mathbf{C}$;

2) $\frac{d}{dt}\left(\alpha\overrightarrow{G}^{t}\right) = \overrightarrow{G}^{\alpha}$, $\int \overrightarrow{G}^{\alpha}dt = \alpha\overrightarrow{G}^{t} + C$, where $t : (v,u) \to t$, $\alpha : (v,u) \to \alpha$ for $(v,u) \in E\left(\overrightarrow{G}\right)$, $t, \alpha \in \mathbb{R}$;

3) $\frac{d}{dt}\left(\overrightarrow{G}^{nL}\right) = n\overrightarrow{G}^{(n-1)L}$, $\int \overrightarrow{G}^{(n-1)L}dt = \frac{1}{n}\overrightarrow{G}^{nL}$, $n \in \mathbb{Z}^{+}$;

4) $\frac{d}{dt}\left(e^{\overrightarrow{G}^{L}}\right) = \overrightarrow{G}^{\frac{de^{L}}{dt}} = \overrightarrow{G}^{e^{L}} = e^{\overrightarrow{G}^{L}}$, $\int e^{\overrightarrow{G}^{L}}dt = e^{\overrightarrow{G}^{L}}$;

5) $\frac{d}{dt}\left(\ln\left|\overrightarrow{G}^{L}\right|\right) = \overrightarrow{G}^{\frac{d\ln|L|}{dt}} = \overrightarrow{G}^{\frac{1}{L}} = \frac{1}{\overrightarrow{G}^{L}}$, $\int \frac{dt}{\overrightarrow{G}^{L}} = \ln\left|\overrightarrow{G}^{L}\right|$, $L \neq \mathbf{0}$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$,

and similarly, we easily know

$$\begin{array}{llll}
\frac{d}{dt}\left(\sin\left(\overrightarrow{G}^{L}\right)\right), & \int \sin\left(\overrightarrow{G}^{L}\right)dt, & \frac{d}{dt}\left(\cos\left(\overrightarrow{G}^{L}\right)\right), & \int \cos\left(\overrightarrow{G}^{L}\right)dt, \\
\frac{d}{dt}\left(\tan\left(\overrightarrow{G}^{L}\right)\right), & \int \tan\left(\overrightarrow{G}^{L}\right)dt, & \frac{d}{dt}\left(\cot\left(\overrightarrow{G}^{L}\right)\right), & \int \tan\left(\overrightarrow{G}^{L}\right)dt, \\
\frac{d}{dt}\left(\sinh\left(\overrightarrow{G}^{L}\right)\right), & \int \sinh\left(\overrightarrow{G}^{L}\right)dt, & \frac{d}{dt}\left(\cosh\left(\overrightarrow{G}^{L}\right)\right), & \int \cosh\left(\overrightarrow{G}^{L}\right)dt, \\
\frac{d}{dt}\left(\tanh\left(\overrightarrow{G}^{L}\right)\right), & \int \tanh\left(\overrightarrow{G}^{L}\right)dt, & \frac{d}{dt}\left(\coth\left(\overrightarrow{G}^{L}\right)\right), & \int \coth\left(\overrightarrow{G}^{L}\right)dt\ .
\end{array}$$

For examples,

$$\frac{d}{dt}\left(\sin\left(\overrightarrow{G}^{L}\right)\right) = \cos\left(\overrightarrow{G}^{L}\right) \quad \text{and} \quad \int \sin\left(\overrightarrow{G}^{L}\right)dt = -\cos\left(\overrightarrow{G}^{L}\right),$$
$$\frac{d}{dt}\left(\cos\left(\overrightarrow{G}^{L}\right)\right) = -\sin\left(\overrightarrow{G}^{L}\right) \quad \text{and} \quad \int \cos\left(\overrightarrow{G}^{L}\right)dt = \sin\left(\overrightarrow{G}^{L}\right), \ \cdots.$$

Definition 4.4 *For numbers $a, b \in \mathbb{R}$, let $a = x_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of the closed interval $[a,b]$ in to subinterval, $\Delta t_i = t_i - t_{i-1}$, $\mu = \max\limits_{1\leq i\leq n} \Delta t_i$ and let $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ on variable t with assumption that $f(t)$ is bounded in $[a,b]$, only with finite non-continuous points on $[a,b]$. If*

$$\sum_{i=1}^{n} f\left(\overrightarrow{G}^{L}[\xi_i]\right) \cdot \overrightarrow{G}^{\Delta t_i} \in \mathscr{G}_{\mathscr{B}}$$

as $\mu \to 0$, where $\xi_i \in [t_{i-1}, t_i]$, then, we define

$$\int_{a}^{b} f\left(\overrightarrow{G}^{L}[t]\right)dt = \lim_{\mu\to 0}\sum_{i=1}^{n} f\left(\overrightarrow{G}^{L}[\xi_i]\right) \cdot \overrightarrow{G}^{\Delta t_i},$$

where $\overrightarrow{G}^{\Delta t_i} \in \mathscr{G}_{\mathscr{B}}$ with $\Delta t_i : (v,u) \to \Delta t_i$ for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$.

By Definition 4.4, we are easily know that

$$\int_a^a f\left(\overrightarrow{G}^L[t]\right)dt = \mathbf{O}, \quad \int_a^b f\left(\overrightarrow{G}^L[t]\right)dt + \int_b^c f\left(\overrightarrow{G}^L[t]\right)dt = \int_a^c f\left(\overrightarrow{G}^L[t]\right)dt. \tag{4.3}$$

Notice that

$$\begin{aligned}\lim_{\mu\to 0}\sum_{i=1}^{n} f\left(\overrightarrow{G}^L[\xi_i]\right)\cdot\overrightarrow{G}^{\Delta t_i} &= \lim_{\mu\to 0}\sum_{i=1}^{n}\overrightarrow{G}^{f(L)[\xi_i]}\cdot\overrightarrow{G}^{\Delta t_i}\\ &= \lim_{\mu\to 0}\sum_{i=1}^{n}\overrightarrow{G}^{f(L)[\xi_i]\Delta t_i}\\ &= \lim_{\mu\to 0}\overrightarrow{G}^{\sum\limits_{i=1}^{n} f(L)[\xi_i]\Delta t_i} = \overrightarrow{G}^{\lim\limits_{\mu\to 0}\sum\limits_{i=1}^{n} f(L)[\xi_i]\Delta t_i}.\end{aligned}$$

Whence,

$$\lim_{\mu\to 0}\sum_{i=1}^{n} f\left(\overrightarrow{G}^L[\xi_i]\right)\cdot\overrightarrow{G}^{\Delta t_i}\in\mathscr{G}_{\mathscr{B}} \quad\Leftrightarrow\quad \lim_{\mu\to 0}\sum_{i=1}^{n} f(L)[\xi_i]\Delta t_i \text{ exists}$$

as $\mu\to 0$, i.e., $f(L)$ is integral on $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Now, it should be noted that

$$\frac{d}{dt}F\left(\overrightarrow{G}^L[t]\right) = f\left(\overrightarrow{G}^L[t]\right)$$

implies that $\dfrac{dF}{dt} = f(L)$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. We know that

$$\begin{aligned}\int_a^b f\left(\overrightarrow{G}^L[t]\right)dt &= \lim_{\mu\to 0}\sum_{i=1}^{n} f\left(\overrightarrow{G}^L[\xi_i]\right)\cdot\overrightarrow{G}^{\Delta t_i} = \overrightarrow{G}^{\lim\limits_{\mu\to 0}\sum\limits_{i=1}^{n} f(L)[\xi_i]\Delta t_i}\\ &= \overrightarrow{G}^{\int_a^b f(L)[t]dt} = \overrightarrow{G}^{F(b)-F(a)} = \overrightarrow{G}^{F(b)} - \overrightarrow{G}^{F(a)}\\ &= F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=b} - F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=a}.\end{aligned}$$

We therefore get the conclusion following.

Theorem 4.5(Fundamental Theorem of Calculus) *Let $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}}$ on variable t with assumption that $f(t)$ is bounded in $[a,b]$, only with finite non-continuous points on $[a,b]$ and*

$$\frac{d}{dt}F\left(\overrightarrow{G}^L[t]\right) = f\left(\overrightarrow{G}^L[t]\right).$$

Then,

$$\int_a^b f\left(\overrightarrow{G}^L[t]\right)dt = F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=b} - F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=a}. \tag{4.4}$$

Proof Let $T\left(\overrightarrow{G}^L[t]\right)=\int\limits_a^t f\left(\overrightarrow{G}^L[x]\right)dx$. We prove that $\frac{d}{dt}\left(T\left(\overrightarrow{G}^L[t]\right)\right)=f\left(\overrightarrow{G}^L[t]\right)$. In fact,

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{T\left(\overrightarrow{G}^L[t+\Delta t]\right)-T\left(\overrightarrow{G}^L[t]\right)}{\Delta t} &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int\limits_a^{t+\Delta t} f\left(\overrightarrow{G}^L[x]\right)dx-\int\limits_a^{t} f\left(\overrightarrow{G}^L[x]\right)dx\right)\\ &= \lim_{\Delta t\to 0}\frac{1}{\Delta t}\int\limits_t^{t+\Delta t} f\left(\overrightarrow{G}^L[x]\right)dx=\lim_{\Delta t\to 0}\frac{f\left(\overrightarrow{G}^L[\xi]\right)\cdot\overrightarrow{G}^{\Delta t}}{\overrightarrow{G}^{\Delta t}}\\ &= f\left(\overrightarrow{G}^L[t]\right)\end{aligned}$$

by definition, where $\xi\in[t,t+\Delta t]$, i.e., $\frac{d}{dt}\left(T\left(\overrightarrow{G}^L[t]\right)\right)=f\left(\overrightarrow{G}^L[t]\right)$.

According to (4.1), we know that

$$F\left(\overrightarrow{G}^L[t]\right)=T\left(\overrightarrow{G}^L[t]\right)+C=\int\limits_a^t f\left(\overrightarrow{G}^L[x]\right)dx+C. \tag{4.5}$$

Now, let $t=a$ in (4.5). We get $C=F\left(\overrightarrow{G}^L[a]\right)$ by (4.1), which implies that

$$F\left(\overrightarrow{G}^L[t]\right)=\int\limits_a^t f\left(\overrightarrow{G}^L[x]\right)dx+F\left(\overrightarrow{G}^L[a]\right)\quad\text{or}\quad F\left(\overrightarrow{G}^L[b]\right)=\int\limits_a^b f\left(\overrightarrow{G}^L[x]\right)dx+F\left(\overrightarrow{G}^L[a]\right)$$

if we let $t=b$, i.e.,

$$\int\limits_a^b f\left(\overrightarrow{G}^L[t]\right)dt=F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=b}-F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=a}.$$

□

4.2 E-Index

Definition 4.6 *Let $\overrightarrow{G}^L\in\mathscr{G}_{\mathscr{B}}$ be a continuity flow. The e-index $\mathrm{ind}_e\left(\overrightarrow{G}^L\right)$ is defined by*

$$\mathrm{ind}_e\left(\overrightarrow{G}^L\right)=\frac{1}{\left|\overrightarrow{G}\right|}\sum_{v\in V\left(\overrightarrow{G}\right)}\left\|\frac{dL(v)[t]}{dt}\right\|,$$

and $L(v)$ is called the residual value of v in $\overrightarrow{G}^L$.

Particularly, if $L(v)$ is independent on time t in $\overrightarrow{G}^L$, i.e., $\left\|\frac{dL(v)}{dt}\right\|=0$, such a vertex v is said to be conserved and furthermore, if all vertices of $\overrightarrow{G}$ are conserved, $\overrightarrow{G}^L$ is called a conserved flow or A-flow.

Generally, a non-harmonious group can not be characterized by a conserved flow. Thus, the e-index surveys the deviation of $\overrightarrow{G}^L$ from conversed flows because $\mathrm{ind}_e\left(\overrightarrow{G}^L\right)=0$ if $\overrightarrow{G}^L$ is a conserved flow.

Theorem 4.7 *If $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$ is a continuity flow, then*

$$\frac{2}{\left|\overrightarrow{G}\right|}\left\|\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\frac{dL^{A_{vu}^+}(v,u)}{dt}\right\| \le \mathrm{ind}_e\left(\overrightarrow{G}^L\right) \le \frac{2}{\left|\overrightarrow{G}\right|}\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\left\|\frac{dL^{A_{vu}^+}(v,u)}{dt}\right\|.$$

Proof Notice that $L(v)=\sum\limits_{u\in N_G(v)} L^{A_{vu}^+}(v,u)$, $\dfrac{dL(v)}{dt}=\sum\limits_{u\in N_G(v)}\dfrac{dL^{A_{vu}^+}(v,u)}{dt}$ and

$$\begin{aligned}2\left\|\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\frac{dL^{A_{vu}^+}(v,u)}{dt}\right\| &\le \sum_{v\in V\left(\overrightarrow{G}\right)}\left\|\frac{dL(v)}{dt}\right\| = \sum_{v\in V\left(\overrightarrow{G}\right)}\left\|\sum_{u\in N_G(v)}\frac{dL^{A_{vu}^+}(v,u)}{dt}\right\| \\ &\le 2\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\left\|\frac{dL^{A_{vu}^+}(v,u)}{dt}\right\|,\end{aligned}$$

we get the result. □

Clearly,

$$\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\frac{dL(v,u)}{dt} \ne \sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\frac{dL^{A_{vu}^+}(v,u)}{dt}$$

and

$$\sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\left\|\frac{dL(v,u)}{dt}\right\| \ne \sum_{(v,u)\in E\left(\overrightarrow{G}\right)}\left\|\frac{dL^{A_{vu}^+}(v,u)}{dt}\right\|$$

unless $A_{vu}^+ = \mathbf{1}_{\mathscr{B}}$ or $\left\|\frac{dL(v)}{dt}\right\| = 0$ with linear operator A_{vu}^+ for $(v,u)\in E\left(\overrightarrow{G}\right)$, $\forall v\in V\left(\overrightarrow{G}\right)$, i.e., $\overrightarrow{G}^L$ is a conserved flow, and the global deviation of $\overrightarrow{G}^L$ to conserved flow is nothing else but the e-index $\mathrm{ind}_e\left(\overrightarrow{G}^L\right)$.

Theorem 4.8 *A continuity flow $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$ is conserved if and only if* $\mathrm{ind}_e\left(\overrightarrow{G}^L\right)=0$.

Proof By definition, if $\overrightarrow{G}^L$ is a conserved flow, i.e., $L(v)$ is independent on time t for $\forall v\in V\left(\overrightarrow{G}\right)$, there must be $\left\|\frac{dL(v)}{dt}\right\|=0$, i.e., $\mathrm{ind}_e\left(\overrightarrow{G}^L\right)=0$. Whence, $\mathrm{ind}_e\left(\overrightarrow{G}^L\right)=0$ by definition.

Conversely, if

$$\mathrm{ind}_e\left(\overrightarrow{G}^L\right)=\frac{1}{\left|\overrightarrow{G}\right|}\sum_{v\in V\left(\overrightarrow{G}\right)}\left\|\frac{dL(v)}{dt}\right\|=0,$$

by the definition of norm we know that $\left\|\frac{dL(v)}{dt}\right\|\ge 0$ and $\left|\overrightarrow{G}\right|>0$, i.e., there must be $\left\|\frac{dL(v)}{dt}\right\|=0$ for $\forall v\in V\left(\overrightarrow{G}\right)$, i.e., $\overrightarrow{G}^L$ is conserved flow. □

Combining Theorems 3.8 and 4.8, we get conclude results following.

Corollary 4.9 *If the sequence* $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \right\}$ *of continuity flows converges to a conserved flow* $\overrightarrow{G}^L$*, then there must be* $\lim\limits_{n\to\infty} \mathrm{ind}_e\left(\overrightarrow{G}_n^{L_n}\right) = 0$.

Corollary 4.10 *Let* $\overrightarrow{G}^L \in \mathscr{G}_{\mathscr{B}}$ *be a conserved flows and let* f *be a linear operator on* $\mathscr{G}_{\mathscr{B}}$ *commutated with all end-operators in* $\mathscr{A}$*, which induces operator* $f^* : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ *by* $f^* : \overrightarrow{G}^L \to \overrightarrow{G}^{f(L)}$*. Then,* $f^*\left(\overrightarrow{G}^L\right)$ *is a conserved flow also.*

Proof For $v \in V\left(\overrightarrow{G}\right)$, $L(v) = \sum\limits_{u\in N_G(v)} L^{A_{vu}^+}(v,u)$ by definition. Whence,

$$f(L(v)) = \sum_{u\in N_G(v)} f\left(L^{A_{vu}^+}(v,u)\right) = \sum_{u\in N_G(v)} (fL)^{A_{vu}^+}(v,u)$$

by assumption. Notice that $\overrightarrow{G}^L$ is a conserved, $L(v)$ is independent on t for $\forall v \in V\left(\overrightarrow{G}\right)$. We immediately know that $f(L(v))$, $v \in V\left(\overrightarrow{G}\right)$ are independent on t, i.e., $\left\|\frac{dL(v)}{dt}\right\| = 0$ also. By definition,

$$\mathrm{ind}_e\left(f^*\left(\overrightarrow{G}^L\right)\right) = \frac{1}{\left|f^*\left(\overrightarrow{G}\right)\right|} \sum_{v\in V\left(f^*\left(\overrightarrow{G}\right)\right)} \left\|\frac{df(L(v))[t]}{dt}\right\| = 0.$$

Whence, $f^*\left(\overrightarrow{G}^L\right)$ is conserved. □

§5. Continuity Flow Equations

5.1 Algebraic Equations

For an integer $n \geq 1$, let $\mathscr{G}$ be a graph family closed under the union operation and let $\mathscr{B}$ be a field. We consider the algebraic equation

$$\overrightarrow{G}^{L_{c_n}} \cdot X^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} = \mathbf{O} \qquad (5.1)$$

in $\mathscr{G}_{\mathscr{B}}$, where $L_{c_i}(v,u) \in \mathscr{B}$ for integers $1 \leq i \leq n$ with $L_{a_n}(v,u) \neq 0$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

If $X = \overrightarrow{G}^L$ is a solution of equation (5.1), by definition there must be

$$\overrightarrow{G}^{L_{c_n}} \cdot X^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} = \overrightarrow{G}^{L_{c_n}L^n + L_{c_{n-1}}L^{n-1} + \cdots + L_{c_1}L + L_{c_0}} = \overrightarrow{G}^{p(L)}, \qquad (5.2)$$

which implies that the equation (5.1) is equivalent to $\overrightarrow{G}^{p(L)} = \mathbf{O}$, i.e.,

$$L_{c_n}L^n(v,u) + L_{c_{n-1}}L^{n-1}(v,u) + \cdots + L_{c_n}L(v,u) + L_{c_0}(v,u) = 0 \qquad (5.3)$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ in $\mathscr{B}$, where

$$p(L) = L_{c_n} L^n + L_{c_{n-1}} L^{n-1} + \cdots + L_{c_1} L + L_{c_0}.$$

By the fundamental theorem of classical algebra, we know that there are n roots $\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}$ in $\mathscr{B}$ hold with (5.3), which implies that all of these solutions $\overrightarrow{G}^L$ of (5.1) must have

$$L: \ (v,u) \to \ \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. We therefore get the result following.

Theorem 5.1 *Let $\mathscr{G}$ be a closed graph family under union and let $\mathscr{B}$ be a field. Then, a continuity flow $\overrightarrow{G}^L$ is a solution of the algebraic equation*

$$\overrightarrow{G}^{L_{c_n}} \cdot X^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} = \mathbf{O}$$

if and only if $L: (v,u) \to \ \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, where $L_{c_i}(v,u) \in \mathscr{B}$ for integers $1 \leq i \leq n$ with $L_{c_n}(v,u) \neq 0$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and $\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}$ are the n roots of the polynomial $p(L)$ in $\mathscr{B}$.

Proof Clearly, if $L: \ (v,u) \to \ \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, then

$$L_{c_n} L^n(v,u) + L_{c_{n-1}} L^{n-1}(v,u) + \cdots + L_{c_n} L(v,u) + L_{c_0}(v,u) = 0$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ in $\mathscr{B}$, which implies that $\overrightarrow{G}^{p(L)} = \mathbf{O}$, i.e.,

$$\overrightarrow{G}^{L_{c_n}} \cdot X^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} = \mathbf{O}.$$

Conversely, by (5.2) it is clear that

$$\overrightarrow{G}^{L_{c_n}} \cdot X^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} = \mathbf{O}$$

implies $p(L) = 0$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, i.e., L must be a mapping

$$L: \ (v,u) \to \ \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. This completes the proof. □

Notice that the coefficients flows in equations (5.1) are over the same graph $\overrightarrow{G}$. We can certainly generalize it to different graphs $\overrightarrow{G}$ by Convention 2.2.

Theorem 5.2 *Let $\mathscr{G}$ be a graph family closed under union, $\overrightarrow{G}_0, \overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n \in \mathscr{G}$ and let $\mathscr{B}$ be a field. Define a graph $\widehat{G} = \bigcup\limits_{i=1}^{n} \overrightarrow{G}_i$. Then, a continuity flow $\widehat{G}^L$ is a solution of the* *algebraic equation*

$$\overrightarrow{G}_n^{L_{c_n}} \cdot X^n + \overrightarrow{G}_{n-1}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}_1^{L_{c_1}} \cdot X + \overrightarrow{G}_0^{L_{c_0}} = \mathbf{O}, \tag{5.4}$$

where $L_{c_i}(v,u) \in \mathscr{B}$ *for integers* $0 \leq i \leq n$ *with* $L_{c_n}(v,u) \neq 0$ *for* $\forall (v,u) \in E\left(\widehat{G}\right)$ *if and only if* $L: \ (v,u) \to \ \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}$ *for* $\forall (v,u) \in E\left(\widehat{G}\right)$, *where,* $\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}$ *are the* n *roots of the polynomial* $p(L)$ *in* $\mathscr{B}$.

Proof Notice that the equation (5.4) is equivalent to

$$\widehat{G}^{L'_{c_n}} \cdot X^n + \widehat{G}^{L'_{c_{n-1}}} \cdot X^{n-1} + \cdots + \widehat{G}^{L'_{c_1}} \cdot X + \widehat{G}^{L'_{c_0}} = \mathbf{O}$$

by Convention 2.2, where

$$L'_{c_i}(v,u) = \begin{cases} L_{c_i}(v,u) & \text{if } (v,u) \in \overrightarrow{G}_i, \\ 0 & \text{if } (v,u) \in \widehat{G} \setminus \overrightarrow{G}_i \end{cases}$$

for integers $0 \leq i \leq n$. Therefore, we immediately get the result by Theorem 5.1. □

We have known that an nth polynomial has n roots in an field. The next result enumerates the non-isomorphic continuity flow solutions of equation (5.1) in $\mathscr{G}_{\mathscr{B}}$.

Theorem 5.3 *Let* $\mathscr{G}$ *be a closed graph family under union and let* $\mathscr{B}$ *be a field. Then, an algebraic equation*

$$\overrightarrow{G}^{L_{c_n}} \cdot X^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} = \mathbf{O},$$

where $L_{c_i}(v,u) \in \mathscr{B}$ *for integers* $1 \leq i \leq n$ *with* $L_{c_n}(v,u) \neq 0$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *has*

$$n\left(p\left(\overrightarrow{G}^L\right), \mathscr{G}_{\mathscr{B}}\right) = \frac{n^{\varepsilon\left(\overrightarrow{G}^L\right)}}{\left|\text{Aut}\overrightarrow{G}\right|}$$

non-isomorphic solutions $\overrightarrow{G}^L$ *in* $\mathscr{G}_{\mathscr{B}}$, *where* $\text{Aut}\overrightarrow{G}$ *is the automorphism group of graph* $\overrightarrow{G}$.

Proof Notice that there are $n^{\varepsilon(\overrightarrow{G}^L)}$ ways for choice L on edges of $\overrightarrow{G}$ by Theorem 5.1 and two $\overrightarrow{G}^{L_1}$, $\overrightarrow{G}^{L_2}$ are isomorphic is and only if there is an automorphism $\varphi : \overrightarrow{G} \to \overrightarrow{G}$ such that $L_2 = L_1 \circ \varphi$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

Let $\mathscr{J}$ be all of these continuity flow $\overrightarrow{G}^L$ with

$$L: \ (v,u) \to \ \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}.$$

We consider the distinct obits in $\mathscr{J}$ acted by automorphism group $\text{Aut}\overrightarrow{G}$. Clearly, if $\varphi : \overrightarrow{G}^L \to \overrightarrow{G}^L$, there must be $\varphi = \text{id}_{\overrightarrow{G}}$, or in other words that $(\text{Aut}\overrightarrow{G})_{\overrightarrow{G}^L} = \left\{\text{id}_{\overrightarrow{G}}\right\}$.

By the Burnside lemma,

$$\left|\text{Aut}\overrightarrow{G}\right| = \left|(\text{Aut}\overrightarrow{G})_{\overrightarrow{G}^L}\right| \left|\left(\overrightarrow{G}^L\right)^{\text{Aut}\overrightarrow{G}}\right|,$$

we get that

$$\left|\left(\overrightarrow{G}^L\right)^{\text{Aut}\overrightarrow{G}}\right| = \left|\text{Aut}\overrightarrow{G}\right|,$$

i.e., each orbit of $\overrightarrow{G}^L$ acted by $\text{Aut}\overrightarrow{G}$ has the same length $\left|\text{Aut}\overrightarrow{G}\right|$. We therefore have

$$n\left(p\left(\overrightarrow{G}^L\right), \mathscr{G}_{\mathscr{B}}\right) = \frac{n^{\varepsilon\left(\overrightarrow{G}^L\right)}}{\left|\text{Aut}\overrightarrow{G}\right|}$$

non-isomorphic solutions $\overrightarrow{G}^L$ of equation (5.1) in $\mathscr{G}_{\mathscr{B}}$. □

Particularly, if $\overrightarrow{G} = C_m$, K_m or B_m for an integer $m \geq 3$, we get the conclusion following by Theorem 5.3.

Corollary 5.4 *Let C_m, B_m and K_m be respectively a bidirectional circuit, complete graph and bouquet with $m \geq 3$. Then, the numbers of non-isomorphic continuity flow solutions of equation*

$$\begin{aligned} C_m^{L_{c_n}} \cdot X^n + C_m^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + C_m^{L_{c_1}} \cdot X + C_m^{L_{c_0}} &= \mathbf{O}, \\ B_m^{L_{c_n}} \cdot X^n + B_m^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + B_m^{L_{c_1}} \cdot X + B_m^{L_{c_0}} &= \mathbf{O}, \\ K_m^{L_{c_n}} \cdot X^n + K_m^{L_{c_{n-1}}} \cdot X^{n-1} + \cdots + K_m^{L_{c_1}} \cdot X + K_m^{L_{c_0}} &= \mathbf{O} \end{aligned}$$

with $L_{c_i}(v,u) \in \mathscr{B}$ for integers $1 \leq i \leq n$, $L_{a_n}(v,u) \neq 0$ for $\forall (v,u) \in E(C_m), E(B_m)$ or $E(E_m)$ are respectively

$$\frac{n^m}{2m}, \quad \frac{n^m}{m!} \quad \textit{and} \quad \frac{n^{\frac{m(m-1)}{2}}}{m!}. \tag{5.5}$$

5.2 Differential Equations

Let $\overrightarrow{G}^{L_{c_1}}[t], \overrightarrow{G}^{L_{c_0}}[t] \in \mathscr{G}_{\mathscr{B}}$ with $L_{c_0}(v,u), L_{c_1}(v,u) \in \mathscr{B}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. Consider the differential equation

$$\frac{dX}{dt} = \overrightarrow{G}^{L_{c_1}}[t] \cdot X + \overrightarrow{G}^{L_{c_0}}[t] \tag{5.6}$$

in $\mathscr{G}_{\mathscr{B}}$. Assume $\overrightarrow{G}^{L_{c_0}}[t] = \mathbf{O}$. We have

$$\frac{dX}{dt} = \overrightarrow{G}^{L_{c_1}}[t] \cdot X, \quad \text{i.e.,} \quad \frac{dX}{X} = \overrightarrow{G}^{L_{c_1}}[t]dt. \tag{5.7}$$

Integrating (5.7) on both sides, we get

$$\ln |X| = \int \overrightarrow{G}^{L_{c_1}}[t]dt + C,$$

which implies that

$$X[t] = C \cdot e^{\int \overrightarrow{G}^{L_{c_1}}[t]dt}.$$

Now, assume C is variable on t, i.e., $C = \overrightarrow{G}^{L}[t]$ and substitute it into (5.6). We get that

$$\begin{aligned}\left(\frac{d}{dt}\left(\overrightarrow{G}^{L}[t]\right)\right)e^{\int \overrightarrow{G}^{L_{c_1}}[t]dt} &+ \overrightarrow{G}^{L}[t]\cdot e^{\int \overrightarrow{G}^{L_{c_1}}[t]dt}\cdot \overrightarrow{G}^{L_{c_1}}[t] \\ &= \overrightarrow{G}^{L_{c_1}}[t]\cdot \overrightarrow{G}^{L}[t]\cdot e^{\int \overrightarrow{G}^{L_{c_1}}[t]dt} + \overrightarrow{G}^{L_{c_0}}[t].\end{aligned}$$

Combine similar terms, we have that

$$\frac{d}{dt}\left(\overrightarrow{G}^{L}[t]\right) = \overrightarrow{G}^{L_{c_0}}\cdot e^{-\int \overrightarrow{G}^{L_{c_1}}dt} \quad \text{i.e.,} \quad \overrightarrow{G}^{L}[t] = \int \overrightarrow{G}^{L_{c_0}}\cdot e^{-\int \overrightarrow{G}^{L_{c_1}}dt}dt + C,$$

which enable us getting the solution

$$X[t] = e^{\int \overrightarrow{G}^{L_{c_1}}dt}\cdot\left(\int \overrightarrow{G}^{L_{c_0}}\cdot e^{-\int \overrightarrow{G}^{L_{c_1}}dt}dt + C\right) \tag{5.8}$$

of equation (5.6).

For the initial value problem

$$\begin{cases}\dfrac{dX}{dt} = \overrightarrow{G}^{L_{c_1}}\cdot X + \overrightarrow{G}^{L_{c_0}} \\ X|_{t=t_0} = \overrightarrow{G}^{L_0}[t_0]\end{cases} \tag{5.9}$$

of (5.6), we can determine the constant flow C in (5.8). In fact, assume $X = \overrightarrow{G}^{L}[t]\cdot e^{\int_{t_0}^{t}\overrightarrow{G}^{L_{c_1}}dx}$ and substitute it into (5.9), we similarly get that

$$\frac{d}{dt}\left(\overrightarrow{G}^{L}[t]\right) = \overrightarrow{G}^{L_{c_0}}[t]\cdot e^{-\int_{t_0}^{t}\overrightarrow{G}^{L_{c_1}}[x]dx} \quad \text{i.e.,} \quad \overrightarrow{G}^{L}[t] = \int_{t_0}^{t}\overrightarrow{G}^{L_{c_0}}[x]\cdot e^{-\int_{t_0}^{x}\overrightarrow{G}^{L_{c_1}}[s]ds}dx + C.$$

Therefore,

$$X[t] = \left(\int_{t_0}^{t}\overrightarrow{G}^{L_{c_0}}[x]\cdot e^{-\int_{t_0}^{x}\overrightarrow{G}^{L_{c_1}}[s]ds}dx + C\right)\cdot e^{\int_{t_0}^{t}\overrightarrow{G}^{L_{c_1}}[x]dx},$$

which implies that

$$X(t_0) = \left(\int_{t_0}^{t_0}\overrightarrow{G}^{L_{c_0}}[x]\cdot e^{-\int_{t_0}^{t_0}\overrightarrow{G}^{L_{c_1}}[s]ds}dx + C\right)\cdot e^{\int_{t_0}^{t_0}\overrightarrow{G}^{L_{c_1}}[x]dx}$$

if $t = t_0$. However,

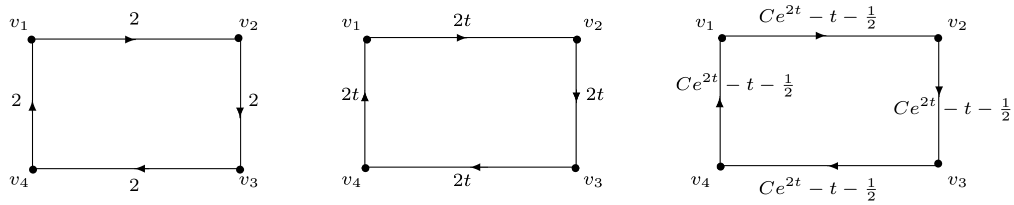
$$X|_{t=t_0} = \overrightarrow{G}^{L_0} \quad \text{and} \quad \int_{t_0}^{t_0}\overrightarrow{G}^{L_{c_0}}[x]dx = \int_{t_0}^{t_0}\overrightarrow{G}^{L_{c_1}}[x]dx = \mathbf{O}$$

by assumption. We get that $X(t_0) = (\mathbf{O} + C) \cdot e^{\mathbf{O}} = C \cdot I = C$, which concludes that $C = X(t_0) = \overrightarrow{G}^{L_0}[t]$. Consequently,

$$X[t] = \left(\int_{t_0}^{t} \overrightarrow{G}^{L_{c_0}}[x] \cdot e^{-\int_{t_0}^{x} \overrightarrow{G}^{L_{c_1}}[s]ds} dx + \overrightarrow{G}^{L_0}[t] \right) \cdot e^{\int_{t_0}^{t} \overrightarrow{G}^{L_{c_1}}[x]dx} \tag{5.10}$$

in the initial value problem (5.9).

Example 5.5 Let $\mathscr{B} = \mathbb{R}$ and $\overrightarrow{G}^{L_{c_1}}, \overrightarrow{G}^{L_{c_0}}$ shown in Fig.7(a) and (b). Then, the solution of differential equation (5.6) is the continuity flow shown in Fig.7(c),



(a) (b) (c)

Fig.7

where C is a constant. Particularly, if $X[0] = \overrightarrow{G}^{L_0}[0]$ with $L_0 : (v,u) \to te^t$ for $(v,u) \in E\left(\overrightarrow{G}\right)$, we know that the solution of the initial problem (7.9) is the continuity flow shown in Fig.4(c) with $C = \frac{1}{2}$, i.e., $X[t] = \overrightarrow{G}^{L}[t]$ with

$$L : (v,u) \to \frac{1}{2}\left(e^{2t} - 1\right) - t$$

for $(v,u) \in E\left(\overrightarrow{G}\right)$.

5.3 Linear Equation with Constant Flow Coefficients

A continuity flow $\overrightarrow{G}^{L}$ is constant if $L : (v,u) \to c_{vu}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, where $c_{vu} \in \mathscr{B}$ is a constant, denoted by $\overrightarrow{G}^{L_c}$. For an integer $n \geq 1$, a flow equation with a form

$$\frac{d^n X}{dt^n} + \overrightarrow{G}_{n-1}^{L_{c_{n-1}}} \cdot \frac{d^{n-1} X}{dt^{n-1}} + \overrightarrow{G}_{n-2}^{L_{c_{n-2}}} \cdot \frac{d^{n-2} X}{dt^{n-2}} + \cdots + \overrightarrow{G}_{1}^{L_{c_1}} \cdot \frac{dX}{dt} + \overrightarrow{G}_{0}^{L_{c_0}} = \mathbf{O} \tag{5.11}$$

is said to be a linear equation with constant flow coefficients, where $\overrightarrow{G}_{i}^{L_{c_i}}$ is constant flow for integers $0 \leq i \leq n-1$. Certainly, let $\overrightarrow{G} = \bigcup_{i=0}^{n-1} \overrightarrow{G}_i$, the equation (5.11) is equivalent to

$$\frac{d^n X}{dt^n} + \overrightarrow{G}^{L_{c_{n-1}}} \cdot \frac{d^{n-1} X}{dt^{n-1}} + \overrightarrow{G}^{L_{c_{n-2}}} \cdot \frac{d^{n-2} X}{dt^{n-2}} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot \frac{dX}{dt} + \overrightarrow{G}^{L_{c_0}} = \mathbf{O} \tag{5.12}$$

with characteristic equation

$$\Lambda^n + \overrightarrow{G}^{L_{c_{n-1}}} \cdot \Lambda^{n-1} + \overrightarrow{G}^{L_{c_{n-2}}} \cdot \Lambda^{n-2} + \cdots + \overrightarrow{G}^{L_{c_1}} \cdot \Lambda + \overrightarrow{G}^{L_{c_0}} = \mathbf{O}, \tag{5.13}$$

which is equivalent to

$$\lambda^n + L_{c_{n-1}}(v,u)\lambda^{n-1} + L_{c_{n-2}}(v,u)\lambda^{n-2} + \cdots + L_{c_1}(v,u)\lambda + L_{c_0}(v,u) = \mathbf{0} \tag{5.14}$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

For the equation (5.14), let

$$\begin{array}{l} \lambda_1^{vu} = r_1^{vu}, \lambda_2^{vu} = r_1^{vu}, \cdots, \lambda_{m_{r_1}}^{vu} = r_1^{vu}, \\ \lambda_{m_{r_1}+1}^{vu} = r_2^{vu}, \lambda_{m_{r_1}+2}^{vu} = r_2^{vu}, \cdots, \lambda_{m_{r_1}+m_{r_2}}^{vu} = r_2^{vu}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \lambda_{m_{r_1}+m_{r_2}+\cdots+m_{r_{s-1}}+1}^{vu} = r_s^{vu}, \lambda_{m_{r_1}+m_{r_2}+\cdots+m_{r_{s-1}}+2}^{vu} = r_s^{vu}, \cdots, \lambda_n^{vu} = r_s^{vu} \end{array}$$

be the n roots of (5.14), where $m_{r_1} + m_{r_2} + \cdots + m_{r_s} = n$. Then, by Theorem 5.1 we know that any solution $\Lambda = \overrightarrow{G}^{L_\lambda}$ of (5.13) must has $L_\lambda : (v,u) \to \lambda_i^{vu}$, $1 \le i \le n$. Now, we define a G-isomorphic mapping $\tau : \overrightarrow{G}^{L_\lambda} \to \overrightarrow{G}^{\tau(L_\lambda)}$ by

$$\tau: \ L_\lambda(v,u) = \lambda_i^{vu} \ \to \left\{ \begin{array}{lll} t^{i-1}e^{r_1^{vu}t} & \text{if} & 1 \le i \le m_{r_1}, \\ t^{i-1}e^{r_2^{vu}t} & \text{if} & m_{r_1}+1 \le i \le m_{r_1}+m_{r_2}, \\ \cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ t^{i-1}e^{r_s^{vu}t} & \text{if} & m_{r_1}+m_{r_2}+\cdots+m_{r_{s-1}}+1 \le i \le n \end{array} \right.$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. We therefore get the result following.

Theorem 5.6 *If the set* $\left\{\overrightarrow{G}^{L_\lambda^1}, \overrightarrow{G}^{L_\lambda^2}, \cdots, \overrightarrow{G}^{L_\lambda^m}\right\}$ *consists of all solutions of the flow equation* (5.13), *then, the set* $\left\{\overrightarrow{G}^{\tau\left(L_\lambda^1\right)}, \overrightarrow{G}^{\tau\left(L_\lambda^2\right)}, \cdots, \overrightarrow{G}^{\tau\left(L_\lambda^m\right)}\right\}$ *is a linear basis of the solution space* $\mathscr{S}$ *of flow differential equation* (5.12), *where* $m = \frac{n^{\varepsilon\left(\overrightarrow{G}^L\right)}}{\left|\mathrm{Aut}\overrightarrow{G}\right|}$.

Proof Clearly, a linear combination

$$X = \overrightarrow{G}^{L_{C_1}} \cdot \overrightarrow{G}^{\tau\left(L_\lambda^1\right)} + \overrightarrow{G}^{L_{C_2}} \cdot \overrightarrow{G}^{\tau\left(L_\lambda^2\right)} + \cdots + \overrightarrow{G}^{L_{C_m}} \cdot \overrightarrow{G}^{\tau\left(L_\lambda^m\right)}$$

is a solution of the differential equation (5.12) because of

$$X = \overrightarrow{G}^{L_{C_1}\tau\left(L_\lambda^1\right) + L_{C_2}\tau\left(L_\lambda^2\right) + \cdots + L_{C_m}\tau\left(L_\lambda^m\right)}$$

with a solution $x = L_{C_1}\tau\left(L_\lambda^1\right) + L_{C_2}\tau\left(L_\lambda^2\right) + \cdots + L_{C_m}\tau\left(L_\lambda^m\right)$ of ordinary differential equation

$$\frac{d^n x}{dt^n} + L_{c_{n-1}}(v,u)\frac{d^{n-1}x}{dt^{n-1}} + L_{c_{n-2}}(v,u)\frac{d^{n-2}x}{dt^{n-2}} + \cdots + L_{c_1}(v,u)\frac{dx}{dt} + L_{c_0}(v,u) = 0 \tag{5.15}$$

for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and a solution of differential flow equation (5.12) must be a linear combination X by the theory of ordinary differential equations, where $L_{C_i} : (v,u) \to$ constant,

$1 \leq i \leq m$.

Furthermore, $\overrightarrow{G}^{\tau(L^1_\lambda)}, \overrightarrow{G}^{\tau(L^2_\lambda)}, \cdots, \overrightarrow{G}^{\tau(L^m_\lambda)}$ is independent because if there are constant flows $\overrightarrow{G}^{L_{C_1}}, \overrightarrow{G}^{L_{C_2}}, \cdots, \overrightarrow{G}^{L_{C_m}}$ such that

$$\overrightarrow{G}^{L_{C_1}} \cdot \overrightarrow{G}^{\tau(L^1_\lambda)} + \overrightarrow{G}^{L_{C_2}} \cdot \overrightarrow{G}^{\tau(L^2_\lambda)} + \cdots + \overrightarrow{G}^{L_{C_m}} \cdot \overrightarrow{G}^{\tau(L^m_\lambda)} = \mathbf{O},$$

there must be

$$L_{C_1}(v,u)\tau\left(L^1_\lambda(v,u)\right) + L_{C_2}(v,u)\tau\left(L^2_\lambda(v,u)\right) + \cdots + L_{C_m}(v,u)\tau\left(L^m_\lambda(v,u)\right) = 0$$

hold with an edge $(v,u) \in E\left(\overrightarrow{G}\right)$, which contradicts to the fact that $\{\tau(\lambda_i^{vu}),\ 1 \leq i \leq n\}$ is the basis of ordinary differential equations (5.15) on the edge (v,u) by the theory of ordinary differential equations. □

Corollary 5.7 *The rank of the solution space $\mathscr{S}$ of flow differential equation* (5.12) *is*

$$\mathrm{rank}\mathscr{S} = \frac{n^{\varepsilon\left(\overrightarrow{G}^L\right)}}{\left|\mathrm{Aut}\overrightarrow{G}\right|}.$$

§6. Applications

Dynamic network characterizes the dynamical behavior of networks, which can be viewed as a mathematics over networks with applications to characterize the complex networks, i.e., dynamics on network and also an immediately application for revisiting the index of gross domestic product, i.e., GDP index in economy.

6.1 Dynamics on Network

Notice that the dynamic equations

$$\frac{\partial \overrightarrow{G}^{\mathscr{L}}}{\partial x_i} - \frac{d}{dt}\frac{\partial \overrightarrow{G}^{\mathscr{L}}}{\partial \dot{x}_i} = \mathbf{O}, \qquad 1 \leq i \leq n. \tag{6.1}$$

on harmonic flows $\overrightarrow{G}^L$, i.e., $L : (v,u) \to L(v,u) - iL(v,u)$ with $i^2 = -1$ are established in [21] by letting Lagrangian on edges of $\overrightarrow{G}$, where $L(t, \mathbf{x}(t), \frac{d\mathbf{x}(t)}{dt})(v,u)$ is the Lagrangian on edge (v,u) and

$$\mathscr{L} : L(v,u) \to \mathscr{L}\left[L\left(t, \mathbf{x}(t), \frac{d\mathbf{x}(t)}{dt}\right)(v,u)\right]$$

is a differentiable functional on a continuity flow $\overrightarrow{G}^L[t]$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ with $[\mathscr{L}, A] = \mathbf{0}$ for $A \in \mathscr{A}$ and particularly, the dynamic equations can be simplified to

$$\frac{\partial \overrightarrow{G}^{L^2}}{\partial x_i} - \frac{d}{dt}\frac{\partial \overrightarrow{G}^{L^2}}{\partial \dot{x}_i} = \mathbf{O}, \quad 1 \leq i \leq n. \tag{6.2}$$

if $\mathscr{L}$ is linear dependent on L, which are the second order differential equations. Then, *what is the dynamic equations of network, are they second order differential equations also*? The answer is not certain. In fact, all of these known complex models on networks such as those of ER random-graph model, small-world network model, scale-free network model can be characterized by the initial value problem

$$\begin{cases} \frac{dX}{dt} = \overrightarrow{G}^{L_{c_1}} \cdot X + \overrightarrow{G}^{L_{c_0}} \\ X|_{t=t_0} = \overrightarrow{G}^{L_0}[t_0] \end{cases} \tag{6.3}$$

of first order differential flow equation and, which can be solved by formula (5.10).

(1) ER Random-Graph Model. An ER-random model is introduced by Erdös and Rényi in 1960, generated as follows ([4]):

STEP 1. Start with N isolated vertices;

STEP 2. Pick up all possible pairs of vertices, once and only once, from the N given vertices and connect each pair of vertices by an edge with probability $p \in (0,1)$.

Without loss of generality, let $L_p : (v,u) \to p$ but $L_p : (x,y) \to 0$ if $(x,y) \neq (v,u)$ for a choice $(v,u) \in E(K_N)$. Clearly, if X is an ER-random model on N vertices, we can simulate its evolution from N isolated vertices to a random network at step t by an evolution equation

$$\begin{cases} \frac{dX}{dt} = K_N^{L_p}[t] \cdot K_N^{L}[t] \\ X[t_0] = \overline{K}_N \end{cases} \tag{6.4}$$

where K_N is a complete bidirectional graph with complement $\overline{K}_N$ of order N, and $K_N^{L_p}[t_0] = \mathbf{O}$ at the initialization t_0. By definition, we are easily know that

$$X[t] = \int_{t_0}^{t} K_N^{L_p}[s] \cdot K_N^{L}[s] ds. \tag{6.5}$$

Particularly, let $L : (v,u) \to 1$ for $\forall (v,u) \in E(K_N)$. We therefore get an ER-random model by (6.5).

(2) Small-World Network Model. The small-wold network model was discovered by Watts and Strogaz, called WS small-wold network model in 1998, which is generated by an algorithm following ([4]):

STEP 1. Start from a ring-shaped network C_N^K with N vertices, and in which each vertex is connected to its $2K$ neighbors, K vertices on each side, where $K \geq 1$ is an small integer;

STEP 2. For every pair of adjacent vertices in C_N^K, reconnected the edge in such a way that the begin end of the edge is unchanged but the the other end is disconnected with probability p and then reconnected to a vertex randomly in the network, and this process is performed edge by edge on C_N^K, once and only once, either clockwise or counterclockwise.

Notice that the WS small-wold network model may results in a non-connected network finally in the reconnecting process. For preventing the case of non-connected cases happening, Newman and Watts modified the previous algorithm by replacing STEP 2 following:

STEP 2'. For every pair of originally unconnected vertices, with probability p, $0 < p \ll 1$ add an edge to connect them.

Clearly, the union of all WS small-wold networks is $K_N - C_N^K$, and the union of all NW small-wold networks is K_N. Similar to the case of ER-random model, we know a WS small-wold network or NW small-wold network can be characterized by

$$X[t] = \int_{t_0}^{t} \left(K_N - C_N^K\right)_N^{L_p}[s] \cdot \left(K_N - C_N^K\right)^L [s]ds \quad \text{or} \quad X[t] = \int_{t_0}^{t} K_N^{L_p}[s] \cdot K_N^L[s]ds \tag{6.6}$$

with $X[t_0] = C_N^K$, respectively. Particularly, let $L : (v, u) \to 1$ for $\forall(v, u) \in E\left(K_N - C_N^K\right)$ or $E(K_N)$. We get a WS small-wold network or NW small-wold network at step t by (6.6).

(3) Scale-Free Network Model. The first scale-free network model, called BA scale-free network model is proposed by Barabási and Albert in 1999 ([2]), then a few modified BA models such as EBA model, local-world model by Albert and Barabási presented in 2000, and then other network models with the property that *preferential attachment*, i.e., the phenomenon ruler "rich gets richer" ([4]). A BA network model is generated by the algorithm following.

STEP 1. Starting from a connected network $\overrightarrow{G}_0$ of small size $m_0 \geq 1$, introduce one new vertex to the existing network each time, and this new vertex is simultaneously connected to existing m vertices in the network, where $1 \leq m \leq m_0$;

STEP 2. The incoming new vertex in STEP 1 is simultaneously connected to each of the existing vertices according to probability

$$\Pi_i = \rho_i / \sum_{j=1}^{N} \rho_j$$

for vertex v of valency ρ_i.

Notice that the union of all possible network of BA scale-free network is $G_0 + K_t$ at step t. Without loss of generality, let v be a new vertex at step t and $L_{BA} : (v, u) \to \Pi_i$ if $\rho(u) = \Pi_i$ but $L_{BA} : (x, y) \to 0$ if $x, y \neq v$ for $u, x, y \in V(G_0 + K_t)$. Clearly, if X is a BA scale-free network, we can simulate its evolution from $\overrightarrow{G}_0$ to a random network at step t by an evolution equation

$$\begin{cases} \dfrac{dX}{dt} = (G_0 + K_t)^{L_{BA}}[t] \cdot X \\ X[t_0] = G_0^{L_0} \end{cases} \tag{6.7}$$

where $(G_0 + K_t)^{L_{BA}}[t_0] = \mathbf{O}$ at the initialization t_0. By formula (5.10), we are easily know that the BA scale-free network

$$X[t] = G_0^{L_0} \cdot \int_{t_0}^{t} e^{(G_0+K_t)^{L_{BA}}[s]} ds \tag{6.8}$$

if let $L_0 : (v, u) \to 1$ for $\forall(v, u) \in E\left(G_0^{L_0}\right)$.

6.2 E-index with GDP

By the input-output model of Wassily Leontief, an economical system can be decomposed into n parts or industries $1, 2, \cdots, n$ operated with inputs in one industry produce outputs for consumption or for input into another industry, which inherits a topological graph $\overrightarrow{G}$ with vertex set $\{1, 2, \cdots, n\}$ and edge set $\{(i, j)$ if product of i input $j, 1 \leq i, j \leq n\}$ (see [30] for detils). Furthermore, such an inherited graph of the input-output model can be generalized to a continuity flow $\overrightarrow{G}^L_+$ with $L : (i, j) \to$ amount for integers $1 \leq i, j \leq n$ and end-operators $\mathscr{A} = \{1_{\mathscr{B}}\}$, such as those shown in Fig.8,

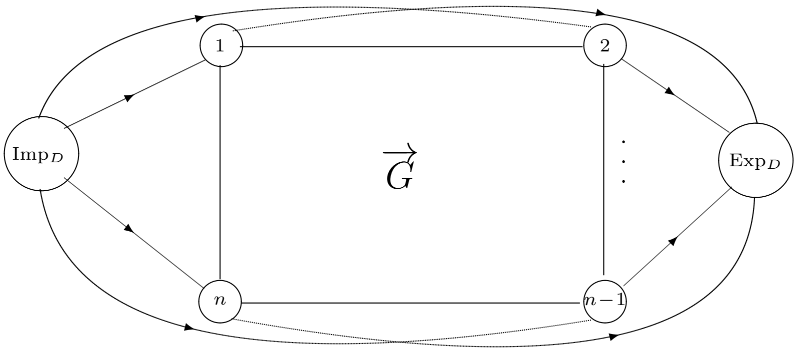


Fig.8

where $1, 2, \cdots, n$ and Imp_D, Exp_D respectively denote the n industries, the imports and the exports, $\overrightarrow{G}$ is the continuity flow inherited in the constitution of industries.

For evaluating the economic production and growth of a nation, the gross domestic product (GDP) index is a monetary measure of the market value of all the final goods and services produced in a specific time period of a nation. Then, *how to calculate the GDP of a country*? The most commonly used GDP formula based on the money spent by various groups that participate in the economy of a country is ([27])

$$\mathrm{GDP} = C + G + I + NX, \tag{6.9}$$

where, C = consumption or all private consumer spending within a countrys economy, G = total government expenditures, I = sum of a country's investments spent on capital equipment, inventories, and housing and NX = net exports, i.e., a country's total exports less total imports.

Notice that I, C and G, NX reflect respectively the investment, consumption and net export scales, the monetary measure of flows on $\overrightarrow{G}^L_+$ in Fig.8. Whence,

$$\mathrm{GDP} = \sum_{i=1}^{n} c(i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c(L(i,j)) + \sum_{i=1}^{n} c^+(i, \mathrm{Exp}_D) - \sum_{i=1}^{n} c^-(\mathrm{Imp}_D, i), \tag{6.10}$$

where, $c(i)$, $c(L(i, j))$ and $c^+(i, \mathrm{Exp}_D)$, $c^-(\mathrm{Imp}_D, i)$ are respectively the money of investment of the ith industry, the consumption of jth industry on the ith product, and the export or import of the ith industry. By definition, the real GDP growth rate is the percentage change in a countrys real GDP over time, i.e.,

$$\text{The real GDP growth rate } \kappa = \frac{\text{The final GDP} - \text{The initial GDP}}{\text{The initial GDP}} \times 100. \tag{6.11}$$

Certainly, if $\overrightarrow{G}^L_+$ is conserved by equating Imp_D with Exp_D and the price is equilibrium, there must be

$$\text{The real GDP growth rate } \kappa = \frac{d}{dt}\left(\overrightarrow{G}^L_+\right) = \frac{d}{dt}\left(L(i,j)\right), \quad \forall (i,j) \in E\left(\overrightarrow{G}_+\right),$$

i.e., the e-index $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right) = 0$ in this case. However, the continuity flow of $\overrightarrow{G}^L_+$ equated Imp_D with Exp_D is not conserved, the price of different industries is not equilibrium, even $\mathrm{Exp}_D) \neq \mathrm{Imp}_D$ in the real, i.e., the economical system of a country is a non-harmonious group, industries maybe non-synchronized. That is why the GDP doesn't add up in [27], and also alludes that the developing of humans is not harmonious with the nature. Then, *could we establish such an index that can reflects both the economic development and the damage to the nature*? The answer is positive with two indexes following:

Index 1. The revisited gross domestic product GDP_R;

Index 2. The deviation of the developing to that of the equilibrium $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right)$.

In fact, the most ideal developing of humans with the nature should be conserved, i.e., the e-index $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right) = 0$, which means the full use and the best used of resource without pollutant to the nature. However, none of the economical systems of humans coincides with this pattern because of the limitations of humans on the nature. There are some industries i with $\left|\frac{d(L(i))}{dt}\right| \neq 0$, i.e., the residue $L(i)$ is not constant on usual. *What is this case implication*? It reflects the redundancy of industry i in the developing of humans, also the harmful extent of human's activity to the nature, i.e., the contributions of $L(i)$ is negative to the developing of humans. We should revisit the classical GDP by surveying the degree of the activity of humans harmful to the nature.

Notice that

$$\left\|\frac{dc(L(i))}{dt}\right\| = \left|\frac{dc(L(i))}{dt}\right| = \frac{d}{dt}\left(|c(L(i))|\right)$$

in this case. We introduced the revisited GDP_R on continuity flow $\overrightarrow{G}^L_+$ by

$$\begin{aligned}
\mathrm{GDP}_R &= \sum_{i=1}^{n} c(i) + \sum_{i,j=1}^{n} c(L(i,j)) + \sum_{i=1}^{n}\left(c^+(i,\mathrm{Exp}_D) - c^-(\mathrm{Imp}_D,i)\right) - \sum_{i=1}^{n}|c(L(i))| \\
&= \sum_{i=1}^{n} c(i) + \sum_{i,j=1}^{n} c(L(i,j)) + \sum_{i=1}^{n}\left(c^+(i,\mathrm{Exp}_D) - c^-(\mathrm{Imp}_D,i)\right) - \sum_{i=1}^{n}\int_{t_1}^{t_2}\left|\frac{dc(L(i))}{dt}\right| dt \\
&= \sum_{i=1}^{n} c(i) + \sum_{i=1}^{n}\sum_{j=1}^{n} c(L(i,j)) + \sum_{i=1}^{n}\left(c^+(i,\mathrm{Exp}_D) - c^-(\mathrm{Imp}_D,i)\right) - \left(\left|\overrightarrow{G}\right| + 1\right) R,
\end{aligned}$$

i.e.,

$$\mathrm{GDP}_R = C + G + I + NX - \left(\left|\overrightarrow{G}\right| + 1\right) R \tag{6.12}$$

with

$$R = \int_{t_1}^{t_2} \mathrm{Ind}_e\left(\overrightarrow{G}^L_+\right) dt, \tag{6.13}$$

where, t_1, t_2 are the initial and terminal time, and R is the country's total residue in a specific time period. And *how do we evaluate the real GDP growth rate* κ? Certainly, we can also calculate κ by formula (6.11) in this case. However, the most important index is not κ but the e-index $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right)$ which surveys the degree of non-equilibrium, i.e., the more larger of $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right)$, the more we owe to the nature.

Notice that the harmonious developing of humans with the nature requires the way of humans developing must be from the non-equilibrium into an equilibrium. Consequently, a more scientific evaluation on the economical developing of humans is not only the GDP_R but also the e-index, or in other words, a pair $\left\{GDP_R, \mathrm{ind}_e\left(\overrightarrow{G}^L_+\right)\right\}$, i.e., the total economic scale and the deviation from the equilibrium but with $\mathrm{ind}_e\left(\overrightarrow{G}^L_+\right) \to 0$ if $t \to 0$, i.e., a harmonious developing of humans with the nature.

References

[1] R.Abraham and J.E.Marsden, *Foundation of Mechanics* (2nd edition), Addison-Wesley, Reading, Mass, 1978.

[2] A.L.Barabaśi and R. Albert, Emergence of scaling in random network, *Science*, Vol.286, 5439(1999), 509-520.

[3] John B.Conway, *A Course in Functional Analysis*, Springer-Verlag New York,Inc., 1990.

[4] G.R.Chen, X.F.Wang and X.Li, *Introduction to Complex Networks – Models, Structures and Dynamics* (2 Edition), Higher Education Press, Beijing, 2015.

[5] Linfan Mao, Combinatorial speculation and combinatorial conjecture for mathematics, *International J.Math. Combin.* Vol.1(2007), No.1, 1-19.

[6] Linfan Mao, Geometrical theory on combinatorial manifolds, *JP J.Geometry and Topology*, Vol.7, No.1(2007),65-114.

[7] Linfan Mao, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, First edition published by American Research Press in 2005, Second edition is as a Graduate Textbook in Mathematics, Published by The Education Publisher Inc., USA, 2011.

[8] Linfan Mao, *Smarandache Multi-Space Theory*(Second edition), First edition published by Hexis, Phoenix in 2006, Second edition is as a Graduate Textbook in Mathematics, Published by The Education Publisher Inc., USA, 2011.

[9] Linfan Mao, *Combinatorial Geometry with Applications to Field Theory*, First edition published by InfoQuest in 2009, Second edition is as a Graduate Textbook in Mathematics, Published by The Education Publisher Inc., USA, 2011.

[10] Linfan Mao, Graph structure of manifolds listing, *International J.Contemp. Math. Sciences*, Vol.5, 2011, No.2,71-85.

[11] Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.

[12] Linfan Mao, Extended Banach $\overrightarrow{G}$-flow spaces on differential equations with applications, *Electronic J.Mathematical Analysis and Applications*, Vol.3, No.2 (2015), 59-91.

[13] Linfan Mao, A new understanding of particles by $\overrightarrow{G}$-flow interpretation of differential equation, *Progress in Physics*, Vol.11, 3(2015), 193-201.

[14] Linfan Mao, A review on natural reality with physical equation, *Progress in Physics*, Vol.11, 3(2015), 276-282.

[15] Linfan Mao, Mathematics with natural reality – Action Flows, *Bull.Cal.Math.Soc.*, Vol.107, 6(2015), 443-474.

[16] Linfan Mao, Labeled graph – A mathematical element, *International J.Math. Combin.*, Vol.3(2016), 27-56.

[17] Linfan Mao, Mathematical combinatorics with natural reality, *International J.Math. Combin.*, Vol.2(2017), 11-33.

[18] Linfan Mao, Hilbert flow spaces with operators over topological graphs, *International J.Math. Combin.*, Vol.4(2017), 19-45.

[19] Linfan Mao, Complex system with flows and synchronization, *Bull.Cal.Math.Soc.*, Vol.109, 6(2017), 461-484.

[20] Linfan Mao, Mathematical 4th crisis: to reality, *International J.Math. Combin.*, Vol.3(2018), 147-158.

[21] Linfan Mao, Harmonic flow's dynamics on animals in microscopic level with balance recovery, *International J.Math. Combin.*, Vol.1(2019), 1-44.

[22] Linfan Mao, Science's dilemma – A review on science with applications, *Progress in Physics*, Vol.15, 2(2019), 78–85.

[23] Linfan Mao, A new understanding on the asymmetry of matter-antimatter, *Progress in Physics*, Vol.15, 3(2019), 78-85.

[24] Linfan Mao, Mathematical elements on natural reality, *Bull.Cal.Math.Soc.*, Vol.111, 6(2019), 597-618.

[25] Y.Nambu, *Quarks: Frontiers in Elementary Particle Physics*, World Scientific Publishing Co.Pte.Ltd, 1985.

[26] F.Smarandache, *Paradoxist Geometry*, State Archives from Valcea, Rm. Valcea, Romania, 1969, and in *Paradoxist Mathematics*, Collected Papers (Vol. II), Kishinev University Press, Kishinev, 5-28, 1997.

[27] Joseph E.Stiglitz, Amartya Sen and Jean-Paul Fitoussi, *Mismeasuring Our Lives – Why GDP Doesn't Add Up*, The New Press, 2010

[28] M.Tegmark, Parallel universes, in *Science and Ultimate Reality: From Quantum to Cosmos*, ed. by J.D.Barrow, P.C.W.Davies and C.L.Harper, Cambridge University Press, 2003.

[29] Q.Yang, *Animal Histology and Embryology*(2nd Edition), China Agricultural University Press, Beijing, 2018.

[30] Zengjun Yuan and Jun Bi, *Industrial Ecology*, Science Press, Beijing, 2010.

International J.Math. Combin. Vol.4(2020), 36-46

On the Order of a Meromomorphic Matrix Valued Function on Annuli

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Abstract: In this paper, we derive some results for meromorphic matrix valued functions on annuli and also extended some basic results of Nevanlinna theory to matrix valued meromorphic functions on annuli.

Key Words: Value distribution theory, Nevanlinna theory, annuli.

AMS(2010): 30D35.

§1. Introduction

In recent years much work has been done in generalizing theorems from complex function theory to matrix valued fuctions. A prime example is the work of Potapov [12], who provided the general formula for the factorization of a matrix valued inner function and factorization of matrix valued functions play an important role in many branches of analysis and engineering. In the year 2014, Bhoosnurmath proved some results concerning meromorphic matrix valued functions(see [14]). In 2005, A. Ya. Khrystiyanyn and A. A. Kondratyuk have proposed on the Nevanlinna Theory for meromorphic functions on annuli (see [6],[7]) and after this work others have done lot of work in this area (see [1-4], [8-22],[23-35]). Thus it is interesting to consider some results for meromorphic matrix valued functions in multiply connected domains. By Doubly connected mapping theorem [5] each doubly connected domain is conformally equivalent to the annulus $\{z : r < |z| < R\}, 0 \leq r < R \leq +\infty$. We consider only two cases : $r = 0,\ R = +\infty$ simultaneously and $0 \leq r < R \leq +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus

$$\mathbb{A} = \mathbb{A}(R_0) = \mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\},$$

where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in both cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$. In this paper we derive some results for meromorphic matrix valued functions on annulus $\mathbb{A}$. However, the methods used here are different.

First, we define the order of a matrix function which is meromorphic function on the

[1]Received August 25, 2020, Accepted November 18, 2020.

annulus

$$\mathbb{A} = \left\{ z : \frac{1}{R_0} < |z| < R_0 \right\}.$$

A complex number z is called a pole of $A(z)$ on $\mathbb{A}$ if it is a pole of one of the entries of $A(z)$ on $\mathbb{A}$, and z is called a zero of $A(z)$ on $\mathbb{A}$ if it is a pole of $A(z)^{-1}$ on $\mathbb{A}$. Let $A(z)$ be a meromorphiic $m \times m$-matrix valued function, then

$$m(R, A) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ ||A(Re^{i\theta})|| d\theta \tag{1.1}$$

and

$$m\left(\frac{1}{R}, A\right) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{||A(Re^{i\theta})||} d\theta. \tag{1.2}$$

Here $||A(z)|| = max_{||x||=1, x \in \mathbb{C}^m} ||A(z)x||$.

Set

$$N(R, A) = \int_0^R \frac{n(t, A) - n(0, A)}{t} dt + n(0, A) \log R, \tag{1.3}$$

$$N(R, f) = \int_0^R \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log R,$$

Therefore

$$T(R, A) = N(R, A) + m(R, A),$$

where $\log^+ x = \max\{\log x, 0\}$, and $n(t, A)$ is the counting function of poles of the function f in $\{z : |z| \leq t\}$. Here we show the notations of the Nevanlinna theory for meromorphiic $m \times m$-matrix valued function on annuli. Let

$$N_1(R, A) = \int_{\frac{1}{R}}^1 \frac{n_1(t, A)}{t} dt, \qquad N_2(R, A) = \int_1^R \frac{n_2(t, A)}{t} dt$$

and

$$\begin{aligned} m_0(R, A) &= m(R, A) + m\left(\frac{1}{R}, A\right) - 2m(1, A), \\ N_0(R, A) &= N_1(R, A) + N_2(R, A), \end{aligned}$$

where $n_1(t, A)$ and $n_2(t, A)$ are the counting functions of the poles of $m \times m$-matrix valued function A in $\{z : t < |z| \leq 1\}$ and $\{z : 1 < |z| \leq t\}$, respectively. The Nevanlinna charecteristic of $m \times m$-matrix valued meromorphic function $A(z)$ on the annulus $\mathbb{A}$ is defined by

$$T_0(R, A) = m_0(R, A) + N_0(R, A). \tag{1.4}$$

The order ρ of A is defined by

$$\rho = \limsup_{R \to \infty} \frac{\log T_0(R, A)}{\log R}. \tag{1.5}$$

Suppose $A(z)$ $m \times m$-matrix valued meromorphic function we can decompose $A(z)$ as follows:

$$A(z) = E(z)diag((z - z_0)^{K_1}...(z - z_0)^{K_m})F(z) \tag{1.6}$$

for each $z \in \mathbb{C}$, where $E(z)$ and $F(z)$ are analytic and invertible at z_0 on $\mathbb{A}$ and $K_m \geq \cdots \geq K_1$ are integers. The numbers $|K_j|$ for which $K_j < 0$ are called partial pole multiplicities of A at Z_0 on $\mathbb{A}$, the numbers K_j for which $K_j > 0$ are called the partial zero multiplicities of A at Z_0 on $\mathbb{A}$. The function $diag((z - z_0)^{K_j})_{j=1}^m$ is called local smith form of $A(z)$ on $\mathbb{A}$.

Throughout this paper we assume that $A(z)$ is $m \times m$-matrix valued meromorphic and regular function on the annulus $\mathbb{A}$, that is, there exist at least one point where $A(z)$ is analytic and invertible on $\mathbb{A}$. Then $A(z)^{-1}$ is also a $m \times m$-matrix valued meromorphic function $A(z)$ on the annulus $\mathbb{A}$, as can be seen by applying Cramer's rule.

Proposition 1.1 *Suppose $A(z)$ is a $m \times m$-matrix valued meromorphic function on the annulus $\mathbb{A}$ of finite order ρ. Let $\rho_{i,j}$ denote the order of the ij entry a_{ij} of $A(z)$. Then*

$$\rho = max_{1 \leq i,j \leq m} \rho_{i,j}. \tag{1.7}$$

Proof Note that

$$\begin{aligned} |a_{ij}(z)| &= |< A(z)e_j, e_i >| \\ &\leq ||A(z)e_j||||e_i|| \leq ||A(z)||. \end{aligned}$$

From this one sees that $m_0(R, a_{ij}) \leq m_0(R, A)$. Clearly $N_0(R, a_{ij}) \leq N_0(R, A)$, so that $T_0(R, a_{ij}) \leq T_0(R, A)$. This implies that

$$max_{1 \leq i,j \leq m} \rho_{i,j} \leq \rho.$$

Conversely, the local smith form shows that the highest order of a pole that $a_{ij}(z)$ can have at z_0 is $|K_1(z_0)|$ on $\mathbb{A}$ and since $E(z_0)$ and $F(z_0)$ are invertible, at least one of the $a_{ij}(z)$ will have a pole of order $|K_1(z_0)|$ at z_0 on $\mathbb{A}$. Then

$$\begin{aligned} n(t, A) &= \sum_{\{z:|z| \leq t\}} \sum_{K_j < 0} |K_j(z)| \leq \sum_{\{z:|z| \leq t\}} \{K_j < 0\}|K_1(z)| \\ &\leq m \sum_{\{z:|z| \leq t\}} |K_1(z)| \leq m \sum_{i=1}^{m} \sum_{j=1}^{m} n(t, a_{ij}), \end{aligned}$$

so that

$$N(R, A) \leq m \sum_{i,j=1}^{m} N(R, a_{ij}).$$

Similarly

$$N_1(R, A) \leq m \sum_{i,j=1}^{m} N_1(R, a_{ij}) \quad \text{and} \quad N_2(R, A) \leq m \sum_{i,j=1}^{m} N_2(R, a_{ij}).$$

Therefore

$$N_0(R,A) \leq m \sum_{i,j=1}^{m} N_0(R,a_{ij}).$$

Furthermore,

$$\begin{aligned} ||A(z)|| &= max_{||x||=1}||A(z)x|| \\ &\leq m^{\frac{1}{2}} max_{||x||=1} max_{1\leq i\leq m} \sum_{j=1}^{m} |a_{ij}(z)x_j| \\ &\leq m^{\frac{1}{2}} max_{x:|x_j|\leq 1} max_{1\leq i\leq m} \sum_{j=1}^{m} |a_{ij}(z)x_j| \\ &\leq m^{\frac{3}{2}} max_{1\leq i,j\leq m}|a_{ij}(z)|. \end{aligned}$$

Therefore

$$\begin{aligned} m_0(R,A) &\leq \log m^{\frac{3}{2}} + max_{1\leq i,j\leq m} m_0(R,a_i) \\ m_0(R,A) &\leq \log m^{\frac{3}{2}} + m \sum_{i,j=1}^{m} m_0(R,a_i). \end{aligned}$$

It follows that

$$T_0(R,A) \leq \log m^{\frac{3}{2}} + m \sum_{i,j=1}^{m} T_0(R,a_i).$$

Now for each $\epsilon > 0$, there are constants C_{ij} such that for all R sufficiently large

$$T_0(R,a_{ij}) \leq C_{ij} R^{\rho_{i,j}+\epsilon}.$$

Then for all sufficiently large R, we have

$$T_0(R,A) \leq C_{ij} R^{max\rho_{i,j}+\epsilon}.$$

Hence the order ρ of $m \times m$-matrix valued meromorphic function $A(z)$ is less than or equal to $max\rho_{i,j}$. □

Remark 1.1 Next, if $A(z)$ is $m \times m$-matrix valued entire function on $\mathbb{A}$ of order $\widehat{\rho}$ is defined as follows : it is the infimum of the numbers λ for which there exists positive constants B and C for which

$$||A(z)|| \leq Aexp(B|z|^{\lambda}) \tag{1.8}$$

for all $|z|$ sufficiently large.

Proposition 1.2 *If $A(z)$ is an $m \times m$-matrix valued entire function on $\mathbb{A}$, then $\rho = \widehat{\rho}$.*

Proof Let $\widehat{\rho}_{i,j}$ be the order of $a_{ij}(z)$ as entire matrix valued function on $\mathbb{A}$, that is defined similarly to (1.8). We claim that $\widehat{\rho} = max\widehat{\rho}_{i,j}$ for $1 \leq i,j \leq m$. Indeed, since $|a_{ij}| \leq ||A(z)||$ it follows that $max\widehat{\rho}_{i,j} \leq \widehat{\rho}$ for $1 \leq i,j \leq m$.

Conversely, suppose that $||A(z)||^2 \leq \sum_{i,j=1} |a_{ij}(z)|^2$ to see that

$$\widehat{\rho} \leq max\widehat{\rho}_{i,j} \quad for \quad 1 \leq i,j \leq m.$$

Since it is well know that for scalar functions $\widehat{\rho} = \rho_{i,j}$, it follows that we can apply Proposition 1.1 to get the desired result. □

Proposition 1.3 *Let $A(z)$ be a regular meromorphic matrix valued functions on $\mathbb{A}$ of finite order ρ. Then $A(z)^{-1}$ has order at most ρ on annuli $\mathbb{A}$.*

Proof We use the fact that if f and g are scalar meromorphic functions of order ρ_1 and ρ_2, respectively, then $f+g$, $f.g$ and $\frac{f}{g}$ are functions having order at most $max(\rho_1, \rho_2)$.

Compute $A(z)^{-1}$ by Cramers rule,

$$A(z)^{-1} = \frac{AdjA(z)}{detA(z)}$$

By Remark 1.1 and Proposition 1.1 each entry of $A(z)^{-1}$ has order at most ρ on annuli $\mathbb{A}$. Proposition 1.1 yields that $A(z)^{-1}$ has order at most ρ on $\mathbb{A}$. □

By the definition of order, one obtains the following result.

Proposition 1.4 *Let $A(z)$ and $B(z)$ be regular meromorphic matrix valued functions on $\mathbb{A}$ of finite order. Then the order of $A(z)B(z)$ is at most the maximum of the order of $A(z)$ and the order of $B(z)$ on annuli $\mathbb{A}$.*

§.2. Main Results

We use the following lemmas to prove our main result, which can be derived from the proof of Nevanlinna-Polya theorem in [13].

Lemma 2.1 *Let n be an arbitrary fixed positive integer and for each $k (k = 1, 2, \cdots, n)$. Let f_k and g_k be analytic functions of a complex variable z on a non-empty domain D.*

If f_k and g_k $(k = 1, 2, \cdots, n)$ satisfy

$$\sum_{k=1}^{n} |f_k(z)|^2 = \sum_{k=1}^{n} |g_k|^2$$

on D and if $f_1, f_2, \cdots, f_n$ are linearly independent on D, then there exists an $n \times n$ unitary *matrix C, where each of the entries of C is a complex constants such that*

$$C\begin{bmatrix} f_1(z) \\ f_2(z) \\ \cdots \\ f_n(z) \end{bmatrix} = \begin{bmatrix} g_1(z) \\ g_2(z) \\ \cdots \\ g_n(z) \end{bmatrix}$$

holds on D.

Lemma 2.2 *Let $A\begin{bmatrix} f_1(z) \\ f_2(z) \end{bmatrix}$ and $B = \begin{bmatrix} g_1(z) \\ 2_2(z) \end{bmatrix}$ be two merromorphic matrix valued functions on $\mathbb{A}$. If f_k and $g_k(k=1,2)$ satisfy*

$$|f_1(z)|^2 + |f_2(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2, \tag{2.1}$$

on $\mathbb{A}$, then there exists a 2×2 unitary matrix C where each of the entries of C is a complex constant such that

$$B = CA. \tag{2.2}$$

Proof We consider the following two cases.

Case 1. If f_1 and f_2 are linearly independent on $\mathbb{A}$, then the proof follows by Lemma 2.1.

Case 2. If f_1 and f_2 are linearly dependent on $\mathbb{A}$, then there exists two complex constants c_1 and c_2 not both zero such that

$$c_1 f_1(z) + c_2 f_2(z) = 0. \tag{2.3}$$

We discuss two subcases following.

Subcase 2.1 If $c_2 \neq 0$, then by (2.3) we get

$$f_2(z) = -\frac{c_1}{c_2} f_1(z) \tag{2.4}$$

holds on $\mathbb{A}$.

If we set $b = -\frac{c_1}{c_2}$, then by (2.4) we have

$$f_2(z) = b f_1(z) \tag{2.5}$$

on $\mathbb{A}$. Hence from (2.1), we have

$$(1 + |b|^2)|f_1(z)|^2 = |g_1(z)|^2 + |g_2(z)|^2. \tag{2.6}$$

We may assume that $f_1 \not\equiv 0$ on $\mathbb{A}$. Otherwise the proof is trivial.

Hence by (2.6), we get

$$\left|\frac{g_1(z)}{f_1(z)}\right|^2 + \left|\frac{g_2(z)}{f_2(z)}\right|^2 = 1 + |b|^2. \tag{2.7}$$

Taking the Laplacians $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ of both sides of (2.7) with respect to $z = x + iy(x, y$ real), we get

$$\left|\left(\frac{g_1(z)}{f_1(z)}\right)'\right|^2 + \left|\left(\frac{g_2(z)}{f_2(z)}\right)'\right|^2 = 0. \tag{2.8}$$

Since $\Delta|P(z)|^2 = 4|P'(z)|^2$, where $P(z)$ is an analytic function of z on $\mathbb{A}$. By (2.8), we get

$$\left(\frac{g_1(z)}{f_1(z)}\right)' = 0$$

and

$$\left(\frac{g_2(z)}{f_2(z)}\right)' = 0.$$

Hence

$$g_1(z) = cf_1(z) \text{ and } g_2(z) = df_2(z), \tag{2.9}$$

where c, d are complex constants.

Substituting (2.9) in (2.7), we get

$$|c|^2 + |d|^2 = 1 + |b|^2. \tag{2.10}$$

Let us define

$$U = \begin{bmatrix} \frac{1}{\sqrt{1+|b|^2}} & \frac{-\bar{b}}{1+\sqrt{|b|^2}} \\ \frac{b}{1+\sqrt{|b|^2}} & \frac{1}{1+\sqrt{|b|^2}} \end{bmatrix} \tag{2.11}$$

and

$$V = \begin{bmatrix} \frac{c}{1+\sqrt{|b|^2}} & \frac{-\bar{d}}{1+\sqrt{|b|^2}} \\ \frac{d}{1+\sqrt{|b|^2}} & \frac{\bar{c}}{1+\sqrt{|b|^2}} \end{bmatrix} \tag{2.12}$$

Then it is easy to prove, by using the definitions of a unitary matrix and multiplication of two 2×2 matrices, that

$$U \begin{bmatrix} \sqrt{1+|b|^2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ b \end{bmatrix} \tag{2.13}$$

and

$$U \begin{bmatrix} \sqrt{1+|b|^2} \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix} \tag{2.14}$$

Set

$$C = VU^{-1}. \tag{2.15}$$

Since all 2×2 unitary matrices form a group under the standard multiplication of matrices, by (2.15), C is a 2×2 unitary matrix.

Now, by (2.13), we have

$$U^{-1}\begin{bmatrix}1\\ b\end{bmatrix}=\begin{bmatrix}\sqrt{1+|b|^2}\\ 0\end{bmatrix}. \tag{2.16}$$

Then from (2.5), (2.9), (2.14), (2.15) and (2.16), we have

$$\begin{aligned} C\begin{bmatrix}f_1(z)\\ f_2(z)\end{bmatrix} &= f_1(z)\begin{bmatrix}1\\ b\end{bmatrix}\\ &= f_1(z)V\left(U^{-1}\begin{bmatrix}1\\ b\end{bmatrix}\right)=f_1(z)V\begin{bmatrix}\sqrt{1+|b|^2}\\ 0\end{bmatrix}\\ &= f_1(z)\begin{bmatrix}c\\ d\end{bmatrix}=\begin{bmatrix}g_1(z)\\ g_2(z)\end{bmatrix}\end{aligned}$$

Therefore (2.2) holds. Thus in this case the proof of the theorem is now completed.

Subcase 2.2 Let $c_2=0$ and $c_1\neq 0$. In this case, by (2.3) we obtain $f_1\equiv 0$.

Hence by (2.1),

$$|f_2(z)|^2=|g_1(z)|^2+|g_2(z)|^2, \tag{2.17}$$

By (2.17) and a similar discussion to that of Scubcase 1 (b becomes 0) we obtain the desired result. □

Theorem 2.1 *Let* $A\begin{bmatrix}f_1(z)\\ f_2(z)\end{bmatrix}$ *and* $B=\begin{bmatrix}g_1(z)\\ 2_2(z)\end{bmatrix}$ *be two merromorphic matrix valued functions on* $\mathbb{A}$*. If* f_k *and* $g_k(k=1,2)$ *satisfy*

$$|f_1(z)|^2+|f_2(z)|^2=|g_1(z)|^2+|g_2(z)|^2, \tag{2.18}$$

on $\mathbb{A}$*, then*

$$\rho_A=\rho_B, \tag{2.19}$$

where ρ_A *and* ρ_B *are the orders of* A *and* B *respectively.*

Proof By Lemma 2.2, we have $B=CA$ where A and B are as defined in the Theorem 2.1. Therefore

$$T_0(R,B)=T_0(R,CA).$$

Using the basics of Nevanlinna theory on annuli, we can show that

$$T_0(R,B)\le T_0(R,A)$$

as $T_0(R,C)=o(T_0(R,f))$. On further simplification, we get

$$\rho_B\le\rho_A. \tag{2.20}$$

By integer changing f_k and g_k $(k = 1, 2)$ in Lemma 2.2, we get

$$A = CB,$$

which implies

$$T_0(R, A) = T_0(R, B),$$

and hence

$$\rho_A \leq \rho_B. \tag{2.21}$$

From (2.20) and (2.21), we get

$$\rho_A = \rho_B. \tag{2.22}$$

Hence the result. □

References

[1] T. B. Cao, Z. S. Deng, On the uniqueness of meromorphic functions that share three or two finite sets on annuli, *Proceeding Mathematical Sciences*, 122(2012), 203-220.

[2] T. B. Cao, H. X. Yi,Uniqueness theorems of meromorphic functions share sets IM on Annili, *Acta Math.Sinica* (Chin.Ser.), 54(2011), 623-632.

[3] T. B. Cao, H. X. Yi, H. Y. Xu,On the multiple values and uniqueness of meromorphic functions on annuli, *Compute. Math. Appl.*,58(2009), 1457-1465.

[4] Y. X. Chen, Z. J. Wu,Exceptional values of meromorphic functions and of their derivatives on annuli, *Ann. Polon.Math.*, 105(2012), 154-165.

[5] A. Fernandez,On the value distribution of meromorphic function in the punctured plane, *Mathematychin Studii*, 34(2010), 136-144.

[6] A. Ya. Khrystiyanyn, A. A. Kondratyuk,On the Nevanlinna Theory for meromorphic functions on annuli. I, *Mathematychin Studii*, 23(2005),19-30.

[7] A. Ya. Khrystiyanyn, A. A. Kondratyuk,On the Nevanlinna Theory for meromorphic functions on annuli. II, *Mathematychin Studii*, 24(2005),57-68.

[8] Korhonen R,Nevanlinna theory in an annulus, value distribution theory and related topics, *Adv. Complex Anal Appl.*, 2004,3:547-554.

[9] H. Y. Xu, and Z. X. Xuan,The uniqueness of analytic functions on annuli sharing some values, *Abstract and Applied Analysis*, Vol.2012, Article ID 896596,13 pages,2012.

[10] Hong-Yan Xu,Exceptional values of Meromorphic Function on Annulus, *The Scientific World Journal*, Volume 2013, Article ID 937584, Hindawi Publishing Corporation, 2013.

[11] V. P. Potapov, The multiplicative structure of J-contractive matrix functions, *Trudy Moskov. Mat. Obschch*, 4(1955), 125-236; *Amer. Math. Soc. Transl.*, (2) 15(1960), 131-243.

[12] C. L. Prather, Andre C. M. Ran,Factorisation of a class of meromorphic matrix valued functions, *Journal of Mathematical Analysis and Applications*, 127(1987), 413-422.

[13] Hiroshi Haruki, Themistocles M. Rassias,A remark on the Nevanlinna-Poly theorem in analytic function theory, *Journal of Mathematical Analysis and Applications*, 200(1996), 382-387.

[14] S. S. Bhoosnurmath, K. S. L. N. Prasad,Some results concerning meromorphic matrix valued functions, *International Journal of Mathematics and Statistics Invention*, 2(2014), 28-32.

[15] R. S. Dyavanal, Ashok Rathod,Uniqueness theorems for meromorphic functions on annuli, *Indian Journal of Mathematics and Mathematical Sciences*, 12(2016), 1-10.

[16] R. S. Dyavanal, Ashok Rathod,Multiple values and uniqueness of meromorphic functions on annuli, *International Journal Pure and Applied Mathematics*, 107(2016), 983-995.

[17] R. S. Dyavanal, Ashok Rathod,On the value distribution of meromorphic functions on annuli, *Indian Journal of Mathematics and Mathematical Sciences*, 12(2016), 203-217.

[18] Ashok Rathod,The multiple values of algebroid functions and uniqueness, *Asian Journal of Mathematics and Computer Research*, 14(2016), 150-157.

[19] Ashok Rathod,The uniqueness of meromorphic functions concerning five or more values and small functions on annuli, *Asian Journal of Current Research*, 1(2016), 101-107.

[20] Ashok Rathod,Uniqueness of algebroid functions dealing with multiple values on annuli, *Journal of Basic and Applied Research International*, 19(2016), 157-167.

[21] Ashok Rathod,On the deficiencies of algebroid functions and their differential polynomials, *Journal of Basic and Applied Research International*, 1(2016), 1-11.

[22] R. S. Dyavanal, Ashok Rathod,Some generalisation of Nevanlinna's five-value theorem for algebroid functions on annuli, *Asian Journal of Mathematics and Computer Research*, 20(2017), 85-95.

[23] R. S. Dyavanal, Ashok Rathod,Nevanlinna's five-value theorem for derivatives of meromorphic functions sharing values on annuli, *Asian Journal of Mathematics and Computer Research*, 20(2017), 13-21.

[24] R. S. Dyavanal, Ashok Rathod,Unicity theorem for algebroid functions related to multiple values and derivatives on annuli, *International Journal of Fuzzy Mathematical Archive*, 13(2017), 25-39.

[25] R. S. Dyavanal, Ashok Rathod,General Milloux inequality for algebroid functions on annuli, *International Journal of Mathematics and applications*, 5(2017), 319-326.

[26] Ashok Rathod,The multiple values of algebroid functions and uniqueness on annuli, *Konuralp Journal of Mathematics* , 5(2017), 216-227.

[27] Ashok Rathod,Several uniqueness theorems for algebroid functions, *J. Anal.*, 25(2017), 203-213.

[28] Ashok Rathod,Nevanlinnas five-value theorem for algebroid functions, *Ufa Mathematical Journal*, 10(2018), 127-132.

[29] Ashok Rathod,Nevanlinnas five-value theorem for derivatives of algebroid functions on annuli, *Tamkang Journal of Mathematics*, 49(2018), 129-142.

[30] S. S. Bhoosnurmath, R. S. Dyavanal, Mahesh Barki, Ashok Rathod,Value distribution for nth difference operator of meromorphic functions with maximal deficiency sum, *J.Anal*, 27(2019), 797-811.

[31] Ashok Rathod,Characteristic function and deficiency of algebroid functions on annuli, *Ufa Mathematical Journal*, 11(2019), 121-132.

[32] Ashok Rathod,Value distribution of a algebroid function and its linear combination of

derivatives on annuli, *Electronic Journal of Mathematical Analysis and Applications*, 8(2020), 129-142.

[33] Ashok Rathod,Uniqueness theorems for meromorphic functions on annuli, *Ufa Mathematical Journal*, 12(2020), 115-121.

[34] Ashok Rathod,Exceptional values of algebroid functions on annuli, *J. Anal,(2020), https: // doi. org /10.1007/s41478-020-00251-z.*

[35] Ashok Rathod and Naveenkumar S H, On the uniqueness and value distribution of entire functions with their derivatives, *Math. Combin.Book Ser.* Vol.2(2020), 33-42.

International J.Math. Combin. Vol.4(2020), 47-65

On Some Fixed Point Theorems for Generalized ψ-Weak Contraction Mappings in Partial Metric Spaces Using C-Class Function

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Abstract: The main goal of this paper is to establish some fixed point theorems for generalized ψ-weak contraction mappings in the setting of complete partial metric spaces using C-class function. Also we give some examples in support of our results. As applications of our results, we obtain some fixed point results for contractive mappings of integral type. Our results extend, generalize and modify several results from the current existing literature regarding partial metric spaces and contractive conditions.

Key Words: Fixed point, coincidence point, generalized ψ-weak contraction mapping, partial metric space.

AMS(2010): 47H10,54H25.

§1. Introduction and Preliminaries

Let (X, d) be a metric space and let $f\colon X \to X$ be a self-mapping. Then,

(i) A point $x \in X$ is called a fixed point of f if $x = fx$;

(ii) f is called contraction if there exists a fixed constant $0 \leq c < 1$ such that

$$d(f(x), f(y)) \leq c\, d(x, y) \tag{1.1}$$

for all $x, y \in X$. If X is complete, then every contraction has a unique fixed point and that point can be obtained as a limit of repeated iteration of the mapping at any point of X (the Banach contraction principle). Obviously, every contraction is a continuous function. The Banach contraction mappings principle is the opening and vital result in the direction of fixed point theory. In this theory, contraction is one of the main tools to prove the existence and uniqueness of a fixed point. Banach's contraction principle which gives an answer to the existence and uniqueness of a solution of an operator equation $Tx = x$, is the most widely used fixed point theorem in all of analysis. This principle is constructive in nature and is one of the most useful techniques in the study of nonlinear equations. Subsequently, several authors have devoted their concentration to expanding and improving this theory (see, e.g., [3, 4, 9, 10, 12, 13, 19, 20]).

[1]Received August 12, 2020, Accepted November 19, 2020.

Partial metric spaces, introduced by Matthews ([16, 17]) are a generalizations of the notion of metric space in which, in definition of metric the condition $d(x,x) = 0$ is replaced by the condition $d(x,x) \leq d(x,y)$. In [17], Matthews discussed some properties of convergence of sequences and proved the fixed point theorem for contraction mapping on partial metric spaces: any mapping T of a complete partial metric space X onto itself that satisfies, where $0 \leq b < 1$, the inequality $p(T(x), T(y)) \leq b\, p(x,y)$ for all $x, y \in X$, has a unique fixed point. Also, the concept of PMS provides to study denotational semantics of dataflow networks [16, 17, 21, 23].

The definition of partial metric space is given by Matthews ([16]) as follows:

Definition 1.1([16]) *Let X be a nonempty set and let $p\colon X \times X \to \mathbb{R}^+$ be a function satisfy*

(pm1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$;
(pm2) $p(x,x) \leq p(x,y)$;
(pm3) $p(x,y) = p(y,x)$;
(pm4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$,

for all $x, y, z \in X$. Then p is called partial metric on X and the pair (X,p) is called partial metric space (in short PMS).

It is clear that if $p(x,y) = 0$, then from $(pm1)$ and $(pm2)$ we obtain $x = y$. But if $x = y$, $p(x,y)$ may not be zero. Various applications of this space has been extensively investigated by many authors (see [15], [22] for details).

Remark 1.2([11]) Let (X,p) be a partial metric space.

(r1) The function $d_p\colon X \times X \to \mathbb{R}^+$ defined as $d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ is a (usual) metric on X and (X, d_p) is a (usual) metric space;

(r2) The function $d_m\colon X \times X \to \mathbb{R}^+$ defined as $d_m(x,y) = \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\}$ is a (usual) metric on X and (X, d_m) is a (usual) metric space.

It is clear that d_p and d_m are equivalent. Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family of open p-balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$ where $B_p(x,\varepsilon) = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Example 1.3([6]) Let $X = \mathbb{R}^+$ and $p\colon X \times X \to \mathbb{R}^+$ given by $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Then $(\mathbb{R}^+, p)$ is a partial metric space.

Example 1.4([6]) Let $X = \{[a,b] : a, b \in \mathbb{R}, a \leq b\}$. Then $p\Big([a,b],[c,d]\Big) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on X.

On a partial metric space the notions of convergence, the Cauchy sequence, completeness and continuity are defined as follows ([16]).

Definition 1.5([16]) *Let (X,p) be a partial metric space. Then,*

(a1) *A sequence $\{x_n\}$ in (X,p) is said to be convergent to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x_n,x)$;*

(a2) *A sequence $\{x_n\}$ is called a Cauchy sequence if $\lim_{m,n\to\infty} p(x_m,x_n)$ exists and finite;*

(a3) *(X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ with respect to τ_p. Furthermore,*

$$\lim_{m,n\to\infty} p(x_m, x_n) = \lim_{n\to\infty} p(x_n, x) = p(x, x);$$

(a4) *A mapping $G\colon X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $G\Big(B_p(x_0, \delta)\Big) \subset B_p\Big(G(x_0), \varepsilon\Big)$.*

Definition 1.6([18]) *Let (X, p) be a partial metric space. Then,*

(b1) *A sequence $\{x_n\}$ in (X, p) is called 0-Cauchy if $\lim\limits_{m,n\to\infty} p(x_m, x_n) = 0$;*

(b2) *(X, p) is said to be 0-complete if every 0-Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, such that $p(x, x) = 0$.*

Definition 1.7([1], Weak Contraction Mapping) *Let (X, d) be a complete metric space. A mapping $f\colon X \to X$ is said to be weakly contractive if*

$$d(f(x), f(y)) \le d(x, y) - \psi(d(x, y)), \tag{1.2}$$

where $x, y \in X$, $\psi\colon [0, \infty) \to [0, \infty)$ is continuous and non-decreasing, $\psi(x) = 0$ if and only if $x = 0$ and $\lim\limits_{x\to\infty} \psi(x) = \infty$.

If we take $\psi(x) = cx$ where $0 < c < 1$ then (1.2) reduces to (1.1).

Definition 1.8 *Let (X, p) be a partial metric space. A point $y \in X$ is called point of coincidence of two self mappings T and f on X if there exists a point $x \in X$ such that $y = Tx = fx$.*

In 2014, Ansari [5] introduced and study C-class function and proved some fixed point theorems.

Definition 1.9([5]) *A mapping $F\colon [0, \infty) \times [0, \infty) \to R$ is called a C-class function if it is continuous and satisfies the following axioms:*

(i) $F(s, t) \le s$;

(ii) $F(s, t) = s$ *implies that either* $s = 0$ *or* $t = 0$, *for all* $s, t \in [0, \infty)$.

An extra condition on F is that $F(0, 0) = 0$ could be imposed in some cases if required. The letter $\mathcal{C}$ denotes the set of all C-class functions. The following example shows that $\mathcal{C}$ is nonempty.

Example 1.10([5]) Define a function $F\colon [0, \infty) \times [0, \infty) \to R$ by

(i) $F(s, t) = s - t$, $F(s, t) = s \Rightarrow t = 0$;

(ii) $F(s, t) = ms, 0 < m < 1$, $F(s, t) = s \Rightarrow s = 0$, ; *(iii)* $F(s, t) = \frac{s}{(1+t)^r}$, $r \in (0, \infty)$, $F(s, t) = s \Rightarrow s = 0 \,\text{or}\, t = 0$;

(iv) $F(s, t) = \frac{log(t+a^s)}{1+t}$, $a > 1$, $F(s, t) = s \Rightarrow s = 0 \,\text{or}\, t = 0$;

(v) $F(s, t) = \frac{ln(1+a^s)}{2}$, $a > e$, $F(s, 1) = s \Rightarrow s = 0$;

(vi) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;,

(vii) $F(s,t) = s log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0 \,\text{or}\, t = 0$;

$(viii)$ $F(s,t) = s - \big(\frac{1+s}{2+s}\big)\big(\frac{t}{1+t}\big)$, $F(s,t) = s \Rightarrow t = 0$;

(ix) $F(s,t) = s\beta(s)$, where $\beta\colon [0,\infty) \to [0,\infty)$ and is continuous, $F(s,t) = s \Rightarrow s = 0$;,

(x) $F(s,t) = s - \big(\frac{t}{k+t}\big)$, $F(s,t) = s \Rightarrow t = 0$;

(xi) $F(s,t) = s - \varphi(s)$, $F(s,t) = s \Rightarrow s = 0$, here $\varphi\colon [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$;

(xii) $F(s,t) = sh(s,t)$, $F(s,t) = s \Rightarrow s = 0$, here $h\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ is a continuous function such that $h(s,t) < 1$ for all $t, s > 0$;

$(xiii)$ $F(s,t) = s - \big(\frac{2+t}{1+t} t\big)$, $F(s,t) = s \Rightarrow t = 0$;

(xiv) $F(s,t) = \sqrt[n]{ln(1+s^n)}$, $F(s,t) = s \Rightarrow s = 0$;

(xv) $F(s,t) = \phi(s)$, $F(s,t) = s \Rightarrow s = 0$, here $\phi\colon [0,\infty) \to [0,\infty)$ is a upper semi-continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$;

(xvi) $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$;

$(xvii)$ $F(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t} dx$, where Γ is the Euler gamma function.

Then F are elements of $\mathcal{C}$.

Definition 1.11([5]) *A function* $\psi\colon [0,\infty) \to [0,\infty)$ *is called an altering distance function if the following properties are satisfied*

(1) ψ *is non-decreasing and continuous function*

(2) $\psi(t) = 0$ *if and only if* $t = 0$.

Remark 1.12([5]) We denote Ψ the class of all altering distance functions.

Lemma 1.13([16, 17]) *Let* (X,p) *be a partial metric space. Then,*

(c1) *A sequence* $\{x_n\}$ *in* (X,p) *is a Cauchy sequence if and only if it is a Cauchy sequence in the metric space* (X,d_p)*;*

(c2) (X,p) *is complete if and only if the metric space* (X,d_p) *is complete;*

(c3) *A subset* E *of a partial metric space* (X,p) *is closed if a sequence* $\{x_n\}$ *in* E *such that* $\{x_n\}$ *converges to some* $x \in X$*, then* $x \in E$.

Lemma 1.14([2]) *Assume that* $x_n \to z$ *as* $n \to \infty$ *in a partial metric space* (X,p) *such that* $p(z,z) = 0$. *Then* $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ *for every* $y \in X$.

The purpose of this paper is to prove a unique fixed point theorem and a coincidence point theorem under generalized ψ-weak contraction in the setting of partial metric spaces using C-class function. Our results extend, generalize and improve several results from the existing literatures.

§2. Main Results

In this section, we shall establish a unique fixed point theorem and a coincidence point theorem in a complete partial metric space. We begin with the following.

Let (X,p) be a partial metric space and $\mathcal{T}\colon X\to X$ be a mapping. We set

$$\theta_1(x,y)=\max\Big\{p(x,y),p(x,\mathcal{T}x),\frac{1}{4}[p(x,\mathcal{T}y)+p(y,\mathcal{T}x)]\Big\}, \tag{2.1}$$

$$\theta_2(x,y)=\min\Big\{p(x,\mathcal{T}x),p(y,\mathcal{T}y)\Big\}. \tag{2.2}$$

With the above setting, we introduce the following definition.

Definition 2.1 *Let (X,p) be a partial metric space. A mapping $\mathcal{T}\colon X\to X$ is called a generalized ψ-weak contraction if*

$$\psi\Big(p(\mathcal{T}x,\mathcal{T}y)\Big)\le F\Big(\psi(\theta_1(x,y)),\psi(\theta_2(x,y))\Big), \tag{2.3}$$

for all $x,y\in X$, where F is a C-class function, that is, $F\in\mathcal{C}$, $\psi\colon[0,\infty)\to[0,\infty)$ is nondecreasing and continuous function with $\psi(t)=0$ if and only if $t=0$.

Now, we are in a position to prove our main result.

Theorem 2.2 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X\to X$ be a generalized ψ-weak contraction mapping, that is, satisfying condition (2.3). Then $\mathcal{T}$ has a unique fixed point.*

Proof Let $x_0\in X$ and $\{x_n\}$ be a sequence defined as $x_{n+1}=\mathcal{T}x_n$ for any $n\in\mathbb{N}$. If for some $n\in\mathbb{N}$, $x_n=x_{n+1}=\mathcal{T}x_n$, then x_n is a fixed point of $\mathcal{T}$. So, we assume that $x_n\ne x_{n+1}$. It follows from (2.3) and $(pm4)$ that

$$\begin{aligned}\psi\Big(p(x_n,x_{n+1})\Big) &= \psi\Big(p(\mathcal{T}x_{n-1},\mathcal{T}x_n)\Big)\\ &\le F\Big(\psi\big(\theta_1(x_{n-1},x_n)\big),\psi\big(\theta_2(x_{n-1},x_n)\big)\Big),\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\theta_1(x_{n-1},x_n) &= \max\Big\{p(x_{n-1},x_n),p(x_{n-1},\mathcal{T}x_{n-1}),\frac{1}{4}[p(x_{n-1},\mathcal{T}x_n)+p(x_n,\mathcal{T}x_{n-1})]\Big\}\\ &= \max\Big\{p(x_{n-1},x_n),p(x_{n-1},x_n),\frac{1}{4}[p(x_{n-1},x_{n+1})+p(x_n,x_n)]\Big\}\\ &= \max\Big\{p(x_{n-1},x_n),p(x_{n-1},x_n),\frac{1}{4}[p(x_{n-1},x_n)+p(x_n,x_{n+1})\\ &\quad -p(x_n,x_n)+p(x_n,x_n)]\Big\}=p(x_{n-1},x_n),\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\theta_2(x_{n-1},x_n) &= \min\Big\{p(x_{n-1},\mathcal{T}x_{n-1}),p(x_n,\mathcal{T}x_n)\Big\}\\ &= \min\Big\{p(x_{n-1},x_n),p(x_n,x_{n+1})\Big\}=p(x_{n-1},x_n).\end{aligned} \tag{2.6}$$

From equations (2.4)-(2.6), we obtain

$$\begin{aligned}\psi\Big(p(x_n, x_{n+1})\Big) &\leq F\Big(\psi\big(p(x_{n-1}, x_n)\big), \psi\big(p(x_{n-1}, x_n)\big)\Big) \\ &\leq \psi\big(p(x_{n-1}, x_n)\big). \qquad (2.7)\end{aligned}$$

Hence, we have

$$p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n).$$

It follows that the sequence $\{p(x_n, x_{n+1})\}$ is monotonically decreasing. Hence

$$p(x_n, x_{n+1}) \to 0 \text{ as } n \to \infty. \tag{2.8}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in X. Suppose on the contrary that the sequence $\{x_n\}$ is not Cauchy. Then there exists $\varepsilon > 0$ and increasing sequences of integers $\{m(k)\}$ and $\{n(k)\}$ such that for all integers k,

$$n(k) > m(k) > k, \tag{2.9}$$

$$p(x_{m(k)}, x_{n(k)}) \geq \varepsilon. \tag{2.10}$$

Further corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.10). Then

$$p(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{2.11}$$

Now, we have

$$\begin{aligned}\varepsilon &\leq p(x_{m(k)}, x_{n(k)}) \\ &\leq p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\leq p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) \\ &< \varepsilon + p(x_{n(k)-1}, x_{n(k)}) \text{ (by (2.11))}. \end{aligned} \tag{2.12}$$

Letting $k \to +\infty$ in equation (2.12) and using (2.8), we get

$$\lim_{k\to\infty} p(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{2.13}$$

Again

$$\begin{aligned}p(x_{n(k)}, x_{m(k)}) &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) \\ &\quad + p(x_{m(k)-1}, x_{m(k)}) - p(x_{n(k)-1}, x_{n(k)-1}) \\ &\quad - p(x_{m(k)-1}, x_{m(k)-1}) \\ &\leq p(x_{n(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{m(k)-1}) \\ &\quad + p(x_{m(k)-1}, x_{m(k)}), \end{aligned} \tag{2.14}$$

whereas

$$\begin{aligned} p(x_{n(k)-1}, x_{m(k)-1}) &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) \\ &\quad + p(x_{m(k)}, x_{m(k)-1}) - p(x_{n(k)}, x_{n(k)}) \\ &\quad - p(x_{m(k)}, x_{m(k)}) \\ &\leq p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) \\ &\quad + p(x_{m(k)}, x_{m(k)-1}). \qquad (2.15) \end{aligned}$$

Now, on letting $k \to +\infty$ in (2.14), (2.15), using (2.8) and (2.13), we obtain

$$\lim_{k\to\infty} p(x_{m(k)-1}, x_{n(k)-1}) = \varepsilon. \tag{2.16}$$

Now setting $x = x_{m(k)-1}$ and $y = x_{n(k)-1}$ in inequality (2.3) and using $(pm4)$, we obtain

$$\begin{aligned} \psi\Big(p(x_{m(k)}, x_{n(k)})\Big) &= \psi\Big(p(\mathcal{T}x_{m(k)-1}, \mathcal{T}x_{n(k)-1})\Big) \\ &\leq F\Big(\psi\big(\theta_1(x_{m(k)-1}, x_{n(k)-1})\big), \psi\big(\theta_2(x_{m(k)-1}, x_{n(k)-1})\big)\Big), \qquad (2.17) \end{aligned}$$

where

$$\begin{aligned} \theta_1(x_{m(k)-1}, x_{n(k)-1}) &= \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, \mathcal{T}x_{m(k)-1}), \\ &\quad \frac{1}{4}\big[p(x_{m(k)-1}, \mathcal{T}x_{n(k)-1}) + p(x_{n(k)-1}, \mathcal{T}x_{m(k)-1})\big]\Big\} \\ &= \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad \frac{1}{4}\big[p(x_{m(k)-1}, x_{n(k)}) + p(x_{n(k)-1}, x_{m(k)})\big]\Big\} \\ &= \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad \frac{1}{4}\big[p(x_{m(k)-1}, x_{m(k)}) + p(x_{m(k)}, x_{n(k)}) \\ &\quad - p(x_{m(k)}, x_{m(k)}) + p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)}) \\ &\quad - p(x_{n(k)}, x_{n(k)})\big]\Big\} \\ &\leq \max\Big\{p(x_{m(k)-1}, x_{n(k)-1}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad \frac{1}{4}\big[p(x_{m(k)-1}, x_{m(k)}) + p(x_{m(k)}, x_{n(k)}) \\ &\quad + p(x_{n(k)-1}, x_{n(k)}) + p(x_{n(k)}, x_{m(k)})\big]\Big\}. \end{aligned}$$

On letting $k \to +\infty$ and using (2.8), (2.13) and (2.16), we get

$$\theta_1(x_{m(k)-1}, x_{n(k)-1}) \to \varepsilon, \tag{2.18}$$

and

$$\begin{aligned}\theta_2(x_{m(k)-1}, x_{n(k)-1}) &= \min\Big\{p(x_{m(k)-1}, \mathcal{T}x_{m(k)-1}), p(x_{n(k)-1}, \mathcal{T}x_{n(k)-1})\Big\}\\ &= \min\Big\{p(x_{m(k)-1}, x_{m(k)}), p(x_{n(k)-1}, x_{n(k)})\Big\}.\end{aligned}$$

On letting $k \to +\infty$ and using (2.8), we get

$$\theta_2(x_{m(k)-1}, x_{n(k)-1}) \to 0. \tag{2.19}$$

Thus, using equation (2.17), (2.18) and (2.19), we obtain

$$\psi(\varepsilon) \quad \leq \quad F\big(\psi(\varepsilon), \psi(0)\big) \leq \psi(\varepsilon),$$

which implies $\psi(\varepsilon) = 0$. That is $\varepsilon = 0$, which is a contradiction. Thus the sequence $\{x_n\}$ is a Cauchy sequence and hence convergent. Thus by Lemma 1.13 this sequence will also Cauchy in (X, d_p). In addition, since (X, p) is complete, (X, d_p) is also complete. Thus there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Moreover by Lemma 1.14,

$$p(z, z) = \lim_{n\to\infty} p(z, x_n) = \lim_{n,m\to\infty} p(x_n, x_m) = 0, \tag{2.20}$$

implies

$$\lim_{n\to\infty} d_p(z, x_n) = 0. \tag{2.21}$$

Now, we show that z is a fixed point of $\mathcal{T}$. Notice that due to (2.20), we have $p(z, z) = 0$. Putting $x = x_{n-1}$ and $y = z$ in equation (2.3), we obtain

$$\begin{aligned}\psi\Big(p(x_n, \mathcal{T}z)\Big) &= \psi\Big(p(\mathcal{T}x_{n-1}, \mathcal{T}z)\Big)\\ &\leq F\Big(\psi(\theta_1(x_{n-1}, z)), \psi(\theta_2(x_{n-1}, z))\Big)\\ &\leq \psi(\theta_1(x_{n-1}, z)),\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}\theta_1(x_{n-1}, z) &= \max\Big\{p(x_{n-1}, z), p(x_{n-1}, \mathcal{T}x_{n-1}), \frac{1}{4}[p(x_{n-1}, \mathcal{T}z)\\ &\quad +p(z, \mathcal{T}x_{n-1})]\Big\}\\ &= \max\Big\{p(x_{n-1}, z), p(x_{n-1}, x_n), \frac{1}{4}[p(x_{n-1}, \mathcal{T}z)\\ &\quad +p(z, x_n)]\Big\}.\end{aligned} \tag{2.23}$$

On letting $n \to +\infty$ in (2.23), using (2.20) and Lemma 1.14, we get

$$\theta_1(x_{n-1}, z) \to \frac{p(z, \mathcal{T}z)}{4}. \tag{2.24}$$

On letting $n \to +\infty$ in (2.22), using (2.24) and continuity of ψ, we get

$$\psi\Big(p(z, \mathcal{T}z)\Big) \leq \psi\Big(\frac{p(z, \mathcal{T}z)}{4}\Big).$$

The above inequality is possible only if $p(z, \mathcal{T}z) = 0$. Thus $z = \mathcal{T}z$. This shows that z is a fixed point of $\mathcal{T}$. Now to prove the uniqueness of the fixed point of $\mathcal{T}$. For this, assume that z' be another fixed point of $\mathcal{T}$ such that $z' = \mathcal{T}z'$ with $z' \neq z$. Now, using (2.3), (2.20) and condition $(pm3)$, we have

$$\begin{aligned} \psi(p(z, z')) &= \psi\Big(p(\mathcal{T}z, \mathcal{T}z')\Big) \\ &\leq F\Big(\psi\big(\theta_1(z, z')\big), \psi\big(\theta_2(z, z')\big)\Big) \leq \psi\big(\theta_1(z, z')\big), \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} \theta_1(z, z') &= \max\Big\{p(z, z'), p(z, \mathcal{T}z), \frac{1}{4}[p(z, \mathcal{T}z') + p(z', \mathcal{T}z)]\Big\} \\ &= \max\Big\{p(z, z'), p(z, z), \frac{1}{4}[p(z, z') + p(z', z)]\Big\} \\ &= p(z, z'). \end{aligned} \tag{2.26}$$

From (2.25) and (2.26), we get

$$\psi(p(z, z')) \leq \psi(p(z, z')).$$

The above inequality is possible only if $p(z, z') = 0$. Thus $z = z'$. This shows the fixed point of $\mathcal{T}$ is unique. This completes the proof. □

If we take $\max\Big\{p(x, y), p(x, \mathcal{T}x), \frac{1}{4}\big[p(x, \mathcal{T}y] + p(y, \mathcal{T}x)\big]\Big\} = p(x, y)$, $F(s, t) = ks$, $0 < k < 1$ and $\psi(t) = t$ for all $t \geq 0$ in the Theorem 2.2, then we obtain the following result in the form of a Banach contraction principle ([7]).

Corollary 2.3 *Let (X, p) be a complete partial metric space. Let $\mathcal{T} \colon X \to X$ be a mapping satisfying the inequality*

$$p(\mathcal{T}x, \mathcal{T}y) \leq k\, p(x, y)$$

for all $x, y \in X$, where $0 < k < 1$ is a constant. Then $\mathcal{T}$ has a unique fixed point in X.

Remark 2.4 Corollary 2.3 extends Banach fixed point theorem from complete metric space to the setting of complete partial metric space.

If we take $F(s, t) = ks$, $0 < k < 1$ and $\psi(t) = t$ for all $t \geq 0$ in the Theorem 2.2, then we obtain the following result.

Corollary 2.5 *Let (X, p) be a complete partial metric space. Let $\mathcal{T} \colon X \to X$ be a mapping* *satisfying the inequality*

$$p(\mathcal{T}x,\mathcal{T}y) \le k\, \max\Big\{p(x,y), p(x,\mathcal{T}x), \frac{1}{4}\big[p(x,\mathcal{T}y)+p(y,\mathcal{T}x)\big]\Big\},$$

for all $x, y \in X$, where $0 < k < 1$ is a constant. Then $\mathcal{T}$ has a unique fixed point in X.

The following result is obtain from Corollary 2.5.

Corollary 2.6 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X \to X$ be a mapping satisfying the inequality*

$$p(\mathcal{T}x,\mathcal{T}y) \le a_1\, p(x,y) + a_2\, p(x,\mathcal{T}x) + \frac{a_3}{4}\big[p(x,\mathcal{T}y)+p(y,\mathcal{T}x)\big]$$

for all $x, y \in X$, where $a_1, a_2, a_3 \ge 0$ are constants such that $a_1 + a_2 + a_3 < 1$. Then $\mathcal{T}$ has a unique fixed point in X.

Proof Follows from Corollary 2.5, by noting that

$$\begin{aligned} & a_1\, p(x,y) + a_2\, p(x,\mathcal{T}x) + \frac{a_3}{4}\big[p(x,\mathcal{T}y)+p(y,\mathcal{T}x)\big] \\ & \le (a_1+a_2+a_3)\max\Big\{p(x,y), p(x,\mathcal{T}x), \frac{1}{4}\big[p(x,\mathcal{T}y)+p(y,\mathcal{T}x)\big]\Big\}. \qquad \square \end{aligned}$$

If we take $F(s,t) = s - t$ in the Theorem 2.2, then we obtain the following result.

Corollary 2.7 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X \to X$ be a mapping satisfying the inequality*

$$\psi\Big(p(\mathcal{T}x,\mathcal{T}y)\Big) \le \psi\Big(\theta_1(x,y)\Big) - \psi\Big(\theta_2(x,y)\Big),$$

for all $x, y \in X$, where $\theta_1(x,y)$, $\theta_2(x,y)$ and ψ are as in Theorem 2.2. Then $\mathcal{T}$ has a unique fixed point in X.

If we take $\max\Big\{p(x,y), p(x,\mathcal{T}x), \frac{1}{4}\big[p(x,\mathcal{T}y]+p(y,\mathcal{T}x)\big]\Big\} = p(x,y)$, $F(s,t) = ks$, $0 < k < 1$ and $\psi(t) = t$ for all $t \ge 0$ in the Theorem 2.2, then we obtain the following result due to Matthews [17].

Corollary 2.8([17], Theorem 5.3) *Let (X,p) be a complete partial metric space. Suppose that $\mathcal{T}\colon X \to X$ be a mapping satisfying the condition*

$$p(\mathcal{T}x,\mathcal{T}y) \le k\, p(x,y), \tag{2.27}$$

for all $x, y \in X$ and $0 < k < 1$ is a constant. Then $\mathcal{T}$ has a unique fixed point.

If we take $F(s,t) = s$ and

$$\max\Big\{p(x,y), p(x,\mathcal{T}x), \frac{1}{4}\big[p(x,\mathcal{T}y]+p(y,\mathcal{T}x)\big]\Big\} = p(x,y)$$

in the Theorem 2.2, then we obtain the following result.

Corollary 2.9 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X\to X$ be a mapping satisfying the inequality:*

$$\psi\Big(p(\mathcal{T}x,\mathcal{T}y)\Big)\le\psi\big(p(x,y)\big),$$

for all $x,y\in X$, where ψ is as in Theorem 2.2. Then $\mathcal{T}$ has a unique fixed point in X.

Remark 2.10 If we take $\psi(t)=t$ for all $t\ge 0$ in Corollary 2.9, then we obtain Theorem 5.3 of Matthews [17].

If we take $F(s,t)=\frac{s}{(1+s)^r}$ for $r>0$ in the Theorem 2.2, then we obtain the following result.

Corollary 2.11 *Let (X,p) be a complete partial metric space. Let $\mathcal{T}\colon X\to X$ be a mapping satisfying the inequality*

$$\psi\Big(p(\mathcal{T}x,\mathcal{T}y)\Big)\le\frac{\theta_1(x,y)}{\Big(1+\theta_1(x,y)\Big)^r},$$

for all $x,y\in X$, where $r>0$ and $\theta_1(x,y)$ and ψ are as in Theorem 2.2. Then $\mathcal{T}$ has a unique fixed point in X.

Theorem 2.12 *Let $\mathcal{T}$ and f be two self-maps on a complete partial metric space X satisfying the inequality*

$$\psi\Big(p(\mathcal{T}x,\mathcal{T}y)\Big)\le F\Big(\psi\Big(M_1(x,y)\Big),\psi\Big(M_2(x,y)\Big)\Big),\tag{2.28}$$

where

$$M_1(x,y)=\max\Big\{p(fx,fy),p(fx,\mathcal{T}x),\frac{1}{4}[p(fx,\mathcal{T}y)+p(fy,\mathcal{T}x)]\Big\},$$

and

$$M_2(x,y)=\min\Big\{p(fx,\mathcal{T}x),p(fy,\mathcal{T}y)\Big\},$$

for all $x,y\in X$, where $F\in\mathcal{C}$ and $\psi\in\Psi$. If $\mathcal{T}(X)\subset f(X)$ and $f(X)$ is a complete subspace of X, then $\mathcal{T}$ and f have a coincidence fixed point.

Proof Let $x_0\in X$ and choose a point x_1 in X such that $\mathcal{T}x_0=fx_1,\dots,\mathcal{T}x_n=fx_{n+1}$. Then, from (2.28) and $(pm4)$, we get

$$\begin{aligned}\psi\Big(p(fx_n,fx_{n+1})\Big)&=\psi\Big(p(\mathcal{T}x_{n-1},\mathcal{T}x_n)\Big)\\&\le F\Big(\psi\Big(M_1(x_{n-1},x_n)\Big),\psi\Big(M_2(x_{n-1},x_n)\Big)\Big),\end{aligned}\tag{2.29}$$

where

$$
\begin{aligned}
M_1(x_{n-1}, x_n) &= \max\Big\{p(fx_{n-1}, fx_n), p(fx_{n-1}, \mathcal{T}x_{n-1}), \frac{1}{4}[p(fx_{n-1}, \mathcal{T}x_n) \\
&\quad +p(fx_n, \mathcal{T}x_{n-1})]\Big\} \\
&= \max\Big\{p(fx_{n-1}, fx_n), p(fx_{n-1}, fx_n), \frac{1}{4}[p(fx_{n-1}, fx_{n+1}) \\
&\quad +p(fx_n, fx_n)]\Big\} \\
&= \max\Big\{p(fx_{n-1}, fx_n), p(fx_{n-1}, fx_n), \frac{1}{4}[p(fx_{n-1}, fx_n) \\
&\quad +p(fx_n, fx_{n+1}) - p(fx_n, fx_n) + p(fx_n, fx_n)]\Big\} \\
&= p(fx_{n-1}, fx_n), \qquad (2.30)
\end{aligned}
$$

and

$$
\begin{aligned}
M_2(x_{n-1}, x_n) &= \min\Big\{p(fx_{n-1}, \mathcal{T}x_{n-1}), p(fx_n, \mathcal{T}x_n)\Big\} \\
&= \min\Big\{p(fx_{n-1}, fx_n), p(fx_n, fx_{n+1})\Big\} \\
&= p(fx_{n-1}, fx_n). \qquad (2.31)
\end{aligned}
$$

From equation (2.29)-(2.31), we get

$$
\begin{aligned}
\psi\Big(p(fx_n, fx_{n+1})\Big) &\leq F\Big(\psi\Big(p(fx_{n-1}, fx_n)\Big), \psi\Big(p(fx_{n-1}, fx_n)\Big)\Big) \\
&\leq \psi\Big(p(fx_{n-1}, fx_n)\Big). \qquad (2.32)
\end{aligned}
$$

Hence, we have

$$p(fx_n, fx_{n+1}) \leq p(fx_{n-1}, fx_n).$$

It follows that the sequence $\{p(fx_n, fx_{n+1})\}$ is monotonically decreasing. Hence

$$p(fx_n, fx_{n+1}) \to 0 \text{ as } n \to \infty. \qquad (2.33)$$

Now, we shall show that $\{fx_n\}$ is a Cauchy sequence in X. As in Theorem 2.2, we can easily show that $\{fx_n\}$ is a Cauchy sequence in X. Thus, by Lemma 1.13 this sequence will also Cauchy in (X, d_p). In addition, since (X, p) is complete, (X, d_p) is also complete. Thus there exists $u \in X$ such that $x_n \to u \Rightarrow fx_n \to fu$ as $n \to \infty$, since $f(X)$ is a complete subspace of X. Moreover, by Lemma 1.14

$$p(fu, fu) = \lim_{n\to\infty} p(fu, fx_n) = \lim_{n,m\to\infty} p(fx_n, fx_m) = 0, \qquad (2.34)$$

implies

$$\lim_{n\to\infty} d_p(fu, fx_n) = 0. \qquad (2.35)$$

Now, we show that u is a coincidence point of $\mathcal{T}$ and f. Notice that due to (2.34), we have $p(fu, fu) = 0$. Putting $x = x_{n-1}$ and $y = u$ in equation (2.28), we obtain

$$\begin{aligned} \psi\Big(p(fx_n, \mathcal{T}u)\Big) &= \psi\Big(p(\mathcal{T}x_{n-1}, \mathcal{T}u)\Big) \\ &\leq F\Big(\psi(M_1(x_{n-1}, u)), \psi(M_2(x_{n-1}, u))\Big) \leq \psi(M_1(x_{n-1}, u)), \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} M_1(x_{n-1}, u) &= \max\Big\{p(fx_{n-1}, fu), p(fx_{n-1}, \mathcal{T}x_{n-1}), \frac{1}{4}[p(fx_{n-1}, \mathcal{T}u) + p(fu, \mathcal{T}x_{n-1})]\Big\} \\ &= \max\Big\{p(fx_{n-1}, fu), p(fx_{n-1}, fx_n), \frac{1}{4}[p(fx_{n-1}, \mathcal{T}u) + p(fu, fx_n)]\Big\}. \end{aligned} \tag{2.37}$$

On letting $n \to +\infty$ in equation (2.37), using (2.34) and Lemma 1.14, we obtain

$$M_1(x_{n-1}, u) \to \frac{p(fu, \mathcal{T}u)}{4}. \tag{2.38}$$

On letting $n \to +\infty$ in equation (2.36), using (2.38) and Lemma 1.14, we obtain

$$\psi\Big(p(fu, \mathcal{T}u)\Big) \leq \psi\Big(\frac{p(fu, \mathcal{T}u)}{4}\Big). \tag{2.39}$$

The above inequality is possible only if $p(fu, \mathcal{T}u) = 0$. Thus $fu = \mathcal{T}u$. This shows that u is a coincidence point of $\mathcal{T}$ and f, that is, $fu = u = \mathcal{T}u$. This completes the proof. □

§3. Illustrations

Example 3.1 Let $X = \mathbb{R}$ and defined $p\colon X^2 \to \mathbb{R}^+$ by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then p is a partial metric on X and (X, p) is a partial metric space. Let $\mathcal{T}\colon X \to X$ be defined by $\mathcal{T}(x) = \frac{x}{7}$ and $\psi(t) = t$ for all $t \geq 0$, where $\psi\colon [0, \infty) \to [0, \infty)$ is continuous and non-decreasing function. Without loss of generality we assume that $x \geq y$. Then, choosing $x = 1$ and $y = \frac{1}{2}$, we have

$$\begin{aligned} p(x, y) = \max\{x, y\} &= x, \\ p(\mathcal{T}x, \mathcal{T}y) &= \max\Big\{\frac{x}{7}, \frac{y}{7}\Big\} = \frac{x}{7}, \\ p(x, \mathcal{T}x) &= \max\Big\{x, \frac{x}{7}\Big\} = x, \\ p(y, \mathcal{T}y) &= \max\Big\{y, \frac{y}{7}\Big\} = y, \\ p(x, \mathcal{T}y) &= \max\Big\{x, \frac{y}{7}\Big\} = x, \\ p(y, \mathcal{T}x) &= \max\Big\{y, \frac{x}{7}\Big\} = y, \\ \theta_1(x, y) &= \max\Big\{p(x, y), p(x, \mathcal{T}x), \frac{1}{4}[p(x, \mathcal{T}y) + p(y, \mathcal{T}x)]\Big\} \\ &= \max\{x, x, \frac{1}{4}(x + y)\} = x, \end{aligned}$$

$$\theta_2(x,y) = \min\big\{p(x,\mathcal{T}x), p(y,\mathcal{T}y)\big\} = \min\{x,y\} = y.$$

Result Analysis

(1) Now, consider the equation (2.27), we have

$$\begin{aligned}\psi\Big(p(\mathcal{T}(x),\mathcal{T}(y))\Big) &= \psi\Big(\frac{x}{7}\Big) = \frac{x}{7} \\ &\leq \psi(x) - \psi(y) = x - y,\end{aligned}$$

or

$$\frac{x}{7} \leq x - y.$$

Putting $x = 1$ and $y = \frac{1}{2}$, we have

$$\frac{1}{7} \leq 1 - \frac{1}{2} = \frac{1}{2},$$

which is true. Thus $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.7. Hence, by applying Corollary 2.7, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

(2) Consider the inequality (2.27), we have

$$\frac{x}{7} \leq k\,x$$

or

$$k \geq \frac{1}{7}.$$

If we take $0 < k < 1$, then $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.3 or Corollary 2.8. Hence, by applying Corollary 2.3, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

(3) Consider the inequality (2.27), we have

$$\frac{x}{7} \leq k\,x$$

or

$$k \geq \frac{1}{7}.$$

If we take $0 < k < 1$, then $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.5. Hence, by applying Corollary 2.5, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

(4) Consider the inequality (2.28), we have

$$\psi\Big(\frac{x}{7}\Big) = \frac{x}{7} \leq \psi(x) = x$$

or

$$\frac{1}{7} \leq 1,$$

which is true. Thus, $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.9. Hence, by applying Corollary 2.9, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

(5) Consider the inequality (2.28) and taking $r = 1$, we have

$$\psi\Big(\frac{x}{7}\Big) = \frac{x}{7} \leq \frac{x}{1+x}.$$

Putting $x = 1$, we get

$$\frac{1}{7} \leq \frac{1}{1+1} = \frac{1}{2},$$

which is true. Thus, $\mathcal{T}$ satisfies all the hypothesis of Corollary 2.11. Hence, by applying Corollary 2.11, $\mathcal{T}$ has a unique fixed point. It is seen that $0 \in X$ is the unique fixed point of $\mathcal{T}$.

Example 3.2 Let $X = \{1, 2, 3, 4\}$ and $p\colon X \times X \to \mathbb{R}$ be defined by

$$p(x,y) = \begin{cases} |x-y| + \max\{x,y\}, & \text{if } x \neq y, \\ x, & \text{if } x = y \neq 1, \\ 0, & \text{if } x = y = 1, \end{cases}$$

for all $x, y \in X$. Then (X, p) is a complete partial metric space.

Define the mapping $\mathcal{T}\colon X \to X$ by

$$\mathcal{T}(1) = 1,\ \mathcal{T}(2) = 1,\ \mathcal{T}(3) = 2,\ \mathcal{T}(4) = 2.$$

Now, we have

$$p(\mathcal{T}(1), \mathcal{T}(2)) = p(1,1) = 0 \leq \frac{3}{4}.3 = \frac{3}{4}p(1,2),$$

$$p(\mathcal{T}(1), \mathcal{T}(3)) = p(1,2) = 3 \leq \frac{3}{4}.5 = \frac{3}{4}p(1,3),$$

$$p(\mathcal{T}(1), \mathcal{T}(4)) = p(1,2) = 3 \leq \frac{3}{4}.7 = \frac{3}{4}p(1,4),$$

$$p(\mathcal{T}(2), \mathcal{T}(3)) = p(1,2) = 3 \leq \frac{3}{4}.4 = \frac{3}{4}p(2,3),$$

$$p(\mathcal{T}(2), \mathcal{T}(4)) = p(1,2) = 3 \leq \frac{3}{4}.6 = \frac{3}{4}p(2,4),$$

$$p(\mathcal{T}(3), \mathcal{T}(4)) = p(2,2) = 2 \leq \frac{3}{4}.5 = \frac{3}{4}p(3,4).$$

Thus, $\mathcal{T}$ satisfies all the conditions of Corollary 2.3 and Corollary 2.8 with $k = \frac{3}{4} < 1$. Now, by applying Corollary 2.3, $\mathcal{T}$ has a unique fixed point, which in this case is 1.

Example 3.3 Let $X = \{0, 1, 2, 3, \ldots\}$. Define $p\colon X \times X \to \mathbb{R}^+$ as $p(x,y) = \max\{x,y\}$ with

$\mathcal{T}, f \colon X \to X$ be defined respectively as follows: $f(x) = x$ for all $x \in X$ and

$$\mathcal{T}(x) = \begin{cases} x-1, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly (X, p) is a partial metric space. Define the mapping $\psi \colon [0, +\infty) \to [0, +\infty)$ by $\psi(t) = t$ for all $t \geq 0$ and taking $F(s, t) = s - t$. Now, let $x \leq y$. Then choose $x = \frac{1}{2}$ and $y = 1$, we have $p(\mathcal{T}x, \mathcal{T}y) = y - 1$, $p(fx, fy) = y$, $p(fx, \mathcal{T}x) = x$, $p(fy, \mathcal{T}y) = y$, $p(fx, \mathcal{T}y) = x$, $p(fy, \mathcal{T}x) = y$ and

$$\begin{aligned} M_1(x, y) &= \max\Big\{p(fx, fy), p(fx, \mathcal{T}x), \frac{1}{4}[p(fx, \mathcal{T}y) + p(fy, \mathcal{T}x)]\Big\} \\ &= \max\big\{y, x, \frac{1}{4}(x+y)\big\} = y, \end{aligned}$$

$$\begin{aligned} M_2(x, y) &= \min\Big\{p(fx, \mathcal{T}x), p(fy, \mathcal{T}y)\Big\} \\ &= \min\big\{x, y\big\} = x. \end{aligned}$$

Now, we have

$$p(\mathcal{T}x, \mathcal{T}y) = y - 1 \leq y - x.$$

Putting $x = \frac{1}{2}$ and $y = 1$ in the above inequality, we get

$$0 \leq 1 - \frac{1}{2} = \frac{1}{2}.$$

The above inequality holds good. Thus $\mathcal{T}$ and f have the properties mentioned in Theorem 2.12. Hence the conditions of Theorem 2.12 are satisfied. Here it is seen that 0 is the point of coincidence of $\mathcal{T}$ and f, that is, $f(x) = 0 = T(x)$.

§4. **Applications**

As an application of our results, we introduce some fixed point theorems of integral type. Denote Φ the set of functions $\phi \colon [0, +\infty) \to [0, +\infty)$ satisfying the following hypothesis

($\mathcal{H}_1$) ϕ is a Lebesgue-integrable mapping on each compact subset of $[0, +\infty)$;

($\mathcal{H}_2$) for any $\varepsilon > 0$ we have $\int_0^{\varepsilon} \phi(s)ds > 0$.

Now, we have the following results.

Corollary 4.1 *Let (X, p) be a complete partial metric space. Let $\mathcal{T} \colon X \to X$ be a mapping.*

Suppose that there exists $0 < k < 1$ *such that for* $\phi \in \Phi$, *we have*

$$\int_0^{p(\mathcal{T}x,\mathcal{T}y)} \phi(s)ds \leq k \int_0^{p(x,y)} \phi(s)ds \tag{4.1}$$

for all $x, y \in X$. *Then* $\mathcal{T}$ *has a unique fixed point.*

Proof Follows from Corollary 2.3 or Corollary 2.8 by taking

$$t = \int_0^t \phi(s)ds. \tag{4.2}$$

This completes the proof. □

Remark 4.2 Corollary 4.1 extends Theorem 2.1 of Branciari [8] from complete metric space to the setting of complete partial metric space.

Corollary 4.3 *Let* (X, p) *be a complete partial metric space. Let* $\mathcal{T}: X \to X$ *be a mapping. Suppose that there exists* $0 < k < 1$ *such that for* $\phi \in \Phi$, *we have*

$$\int_0^{p(\mathcal{T}x,\mathcal{T}y)} \phi(s)ds \leq k \int_0^{\max\left\{p(x,y),p(x,\mathcal{T}x),\frac{1}{4}\left[p(x,\mathcal{T}y)+p(y,\mathcal{T}x)\right]\right\}} \phi(s)ds \tag{4.3}$$

for all $x, y \in X$. *Then* $\mathcal{T}$ *has a unique fixed point.*

Proof Follows from Corollary 2.5 by taking

$$t = \int_0^t \phi(s)ds. \tag{4.4}$$

This completes the proof. □

§5. Conclusion

In this article, we establish a unique fixed point theorem and a coincidence point theorem under generalized ψ-weak contractive mappings in the framework of complete partial metric spaces and give some examples in support of our results. As application of our results, we obtain some fixed point theorems for mappings satisfying contractive condition of integral type. Our results extend, generalize and modify several results from the existing literature.

References

[1] Ya. I. Alber and S. Guerre-Delabrieer, Principles of weakly contractive maps in Hilbert spaces, new results in operator theory, *Advances and Appl.* (ed. by I. Gohberg and Yu Lyubich), Birkhauser Verlag, Basel 98 (1997), 7-22.

[2] T. Abdeljawad, E. Karapinar and K. Tas, Existence and uniqueness of common fixed point

partial metric spaces,*Appl. Math. Lett.*, 24 (2011), 1900-1904 (doi:10.1016/j.aml.2011.05.014).

[3] I. Altun, F. Sola and H. Simsek, Generalized contractions on partial metric spaces, *Topology Appl.*, 157 (2010), 2778-2785.

[4] I. Altun and A. Erduran, Fixed point theorems for monotone mappings on partial metric spaces, *Fixed Point Theory Appl.*, 2011, Article ID 508730, 10 pages.

[5] A. H. Ansari, Note on $\varphi - \psi$-contractive type mappings and related fixed points, *The 2nd Regional Conference on Math. and Appl.*, Payame Noor University, 2014, 377-380.

[6] H. Aydi, M. Abbas and C. Vetro, Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces, *Topology and Its Appl.*, 159 (2012), No. 14, 3234-3242.

[7] S. Banach, Sur les operation dans les ensembles abstraits et leur application aux equation integrals, *Fund. Math.*, 3(1922), 133-181.

[8] A. Branciari, A fixed point theorem for mapping satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.*, 29 (2002), 531-536.

[9] S. K. Chatterjea, Fixed point theorems compactes, *Rend. Acad. Bulgare Sci.*, 25 (1972), 727-730.

[10] R. Kannan, Some results on fixed point theorems, *Bull. Calcutta Math. Soc.*, 60 (1969), 71-78.

[11] E. Karapinar and U. Yüksel, Some common fixed point theorems in partial metric space, *J. Appl. Math.*, 2011, Article ID: 263621, 2011.

[12] E. Karapinar and Inci M. Erhan, Fixed point theorems for operators on partial metric spaces, *Appl. Math. Lett.*, 24 (2011), 1894-1899.

[13] E. Karapinar, Weak ϕ-contraction on partial metric spaces, *J. Comput. Anal. Appl.* (in press).

[14] M. S. Khan, M. Swaleh and S. Sessa, Fixed point theorems by altering distance between the points, *Bull. Australian Math. Soc.*, 30(1) (1984), 1-9.

[15] H. P. A. Künzi, Nonsymmetric distances and their associated topologies about the origins of basic ideas in the area of asymptotic topology, *Handbook of the History Gen. Topology* (eds. C.E. Aull and R. Lowen), Kluwer Acad. Publ., 3 (2001), 853-868.

[16] S. G. Matthews, Partial metric topology, *Research Report 2012*, Dept. Computer Science, University of Warwick, 1992.

[17] S. G. Matthews, Partial metric topology, Proceedings of the 8th summer conference on topology and its applications, *Annals of the New York Academy of Sciences*, 728 (1994), 183-197.

[18] H. K. Nashine, Z. Kadelburg, S. Radenovic and J. K. Kim, Fixed point theorems under Hardy-Rogers contractive conditions on 0-complete ordered partial metric spaces, *Fixed Point Theory Appl.*, 2012, 1-15.

[19] S. Oltra and O. Valero, Banach's fixed point theorem for partial metric spaces, *Rend. Istit. Mat. Univ. Trieste*, 36 (2004), 17-26.

[20] S. Reich, Some remarks concerning contraction mappings, *Canad. Math. Bull.*, 14 (1971), 121-124.

[21] M. Schellekens, A characterization of partial metrizibility: domains are quantifiable, *Theoretical Computer Science*, 305(1-3) (2003), 409-432.

[22] O. Vetro, On Banach fixed point theorems for partial metric spaces, *Appl. Gen. Topology*, 6 (2005), No. 12, 229-240.

[23] P. Waszkiewicz, Partial metrizibility of continuous posets, *Mathematical Structures in Computer Science*, 16(2) (2006), 359-372.

International J.Math. Combin. Vol.4(2020), 66-79

On a New Class of Harmonic p-Valent Functions Defined by Convolution Structure

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Abstract: In this paper, we investigate several properties for the harmonic classes $\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ and $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. We obtain coefficient bounds, distortion theorem, extreme points, convex compinations and integral operator for these classes.

Key Words: Multivalent harmonic function, Hadamard product, sense-preserving, distortion bounds, integral operator.

AMS(2010): 30C45.

§1. Introduction and Preliminaries

A real-valued function u is said to be harmonic in a domain $\mathfrak{D}\subset\mathbb{C}$ if it has continuous second order partial derivatives in $\mathfrak{D}$, which satisfy the Laplace equation

$$\Delta u\colon=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

We say that a complex-valued continuous function $f:\mathfrak{D}\to\mathbb{C}$ is harmonic in $\mathfrak{D}$ if both functions $u:=\mathrm{Re}f$ and $v:=\mathrm{Im}f$ are real-valued harmonic functions in $\mathfrak{D}$. We note that every complex-valued function f harmonic in $\mathfrak{D}$ with $0\in\mathfrak{D}$, can be uniquely represented as

$$f=h+\overline{g},$$

where $h,\ g$ are analytic functions in $\mathfrak{D}$ with $g(0)=0$. Then we call h the analytic part and g the co-analytic part of f (see [6]). The Jacobian of f is given by

$$J_f(z)=|h'(z)|^2-|g'(z)|^2 \qquad (z\in\mathfrak{D}).$$

The mapping f is locally univalent if $J_f(z)\neq 0$ in $\mathfrak{D}$. A result of Lewy [16] shows that the converse is true for harmonic mappings. Therefore, f is locally univalent and sense-preserving if and only if

$$|h'(z)|>|g'(z)| \qquad (z\in\mathfrak{D}).$$

Duren [12] also Ahuja [1] and Ponnusamy and Rasila [24, 25].

For $p\geq 1$, denote by $\xi(p)$ the set of all multivalent harmonic functions $f=h+\overline{g}$ defined

[1]Received August 16, 2020, Accepted November 21, 2020.

in the open unit disc $\mathcal{U}$, where h and g defined by

$$h(z) = z^p + \sum_{n=p+t}^{\infty} a_n z^n, \quad g(z) = \sum_{n=p+t-1}^{\infty} b_n z^n, \quad |b_{p+t-1}| < 1,\ t \in \mathbb{N} = \{1, 2, \cdots\} \tag{1.1}$$

are analytic functions in $\mathcal{U}$.

Let $\mathcal{F}(z) = \psi(z) + \overline{\varphi(z)}$ be a fixed multivalent harmonic function, where

$$\psi(z) = z^p + \sum_{n=p+t}^{\infty} |A_n|\, z^n,\ \varphi(z) = \sum_{n=p+t-1}^{\infty} |B_n|\, z^n,\ |B_{p+t-1}| < 1,\ t \in \mathbb{N} = \{1, 2, \cdots\}. \tag{1.2}$$

The Hadamard product (or convolution) of functions $f(z)$ and $\mathcal{F}(z)$ of the form

$$(f * \mathcal{F})(z) = z^p + \sum_{n=p+t}^{\infty} |a_n A_n|\, z^n + \sum_{n=p+t-1}^{\infty} |b_n B_n|\, \overline{z}^n. \tag{1.3}$$

Studies of convolution play an serious role in geometric function theory. It has a several researchers of this field. In 1975, Schild and Silverman [28] studied the diverse interesting results on the convolution of analytic functions. Later on, Choi et al. [5], Darwish [7], Darwish and Aouf [8], Domokos [11], Nishiwaki and Owa [20], Nishiwaki et al. [2], Owa [23] and Srivastava et al. [31] studied the generalized convolution for analytic functions only. For detailed study see the excellent text book by Ruscheweyh [27], see also [4], [9], [10], [13], [14], [15], [22], [26], [29], [30].

A function $f(z) \in \xi(p)$ is said to be in the class $\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ if

$$\Re\left\{\frac{z(f*\mathcal{F})''(z) + (f*\mathcal{F})'(z)}{(f*\mathcal{F})'(z) + \tau z(f*\mathcal{F})''(z)}\right\} \geq \rho \left|\frac{z(f*\mathcal{F})''(z) + (f*\mathcal{F})'(z)}{(f*\mathcal{F})'(z) + \tau z(f*\mathcal{F})''(z)} - p\right| + p\varkappa \tag{1.4}$$

where $0 \leq \varkappa < 1,\ \rho \geq 0,\ 0 \leq \tau < 1$ and $z \in \mathcal{U}$.

Finally, denote by $\mathcal{T}\xi(p)$ the subclass of functions $f(z) = h(z) + \overline{g(z)}$ in $\xi(p)$ where

$$h(z) = z^p - \sum_{n=p+t}^{\infty} |a_n|\, z^n, \quad g(z) = -\sum_{n=p+t-1}^{\infty} |b_n|\, z^n, \quad b_{p+t-1} < 1. \tag{1.5}$$

Let $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau) = \mathcal{T}\xi(p) \cap \xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$.

We note that

(i) $\xi_1(1, \varkappa, \rho, \tau) = KM_H(\alpha, \beta, \gamma)$

$$\Re\left\{\frac{zf''(z) + f'(z)}{f'(z) + \gamma z f''(z)}\right\} \geq \beta \left|\frac{zf''(z) + f'(z)}{f'(z) + \gamma z f''(z)} - 1\right| + \alpha \quad \text{(see [2])}.$$

(ii) $\xi_1(1, \alpha, 0, \tau) = C(\lambda, \alpha)$

$$\Re\left\{\frac{zf''(z) + f'(z)}{f'(z) + \lambda z f''(z)}\right\} > \alpha \qquad \text{(see [19])}.$$

In this paper, we obtained the coefficient bounds for the classes $\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ and $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. Further distortion theorem, extreme points, convex compinations and integral operator for the classe $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

§2. Coefficient Bounds

Now, we begin with a sufficient condition for functions in $\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Theorem 2.1 *Let $f=h+\overline{g}$ with h and g given by* (1.1). *If*

$$\sum_{n=p+t}^{\infty}\frac{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]}{p^2(1-\varkappa-\tau(p-1)(\rho+\varkappa))}\left|a_nA_n\right| + \sum_{n=p+t-1}^{\infty}\frac{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]}{p^2(1-\varkappa-\tau(p-1)(\rho+\varkappa))}\left|b_nB_n\right|\leq 1, \tag{2.1}$$

where $0\leq\varkappa<1,\ \rho\geq 0,\ 0\leq\tau<1$, then the harmonic function f is orientation preserving in $\mathcal{U}$ and $f\in\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Proof To verify that f is orientation preserving, we show

$$\begin{aligned}
\left|(h(z)*\psi(z))'\right| &= \left|pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1}\right| \\
&\geq p\left|z\right|^{p-1}-\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|\left|z\right|^{n-1} \\
&= p\left|z\right|^{p-1}\left(1-\sum_{n=p+t}^{\infty}\frac{n}{p}\left|a_nA_n\right|\left|z\right|^{n-p}\right) \\
&\geq p\left|z\right|^{p-1}\left(1-\sum_{n=p+t}^{\infty}\frac{n}{p}\left|a_nA_n\right|\right) \\
&\geq p\left|z\right|^{p-1}\left\{1-\sum_{n=p+t}^{\infty}\frac{n\left[n(1+\rho)-(p(n-1)+1)(\rho+\varkappa)\right]}{p^2(1-\varkappa-\tau(p-1)(\rho+\varkappa))}\left|a_nA_n\right|\right\} \\
&\geq p\left|z\right|^{p-1}\left\{\sum_{n=p+t-1}^{\infty}\frac{n\left[n(1+\rho)-(p(n-1)+1)(\rho+\varkappa)\right]}{p^2(1-\varkappa-\tau(p-1)(\rho+\varkappa))}\left|b_nB_n\right|\right\} \\
&\geq p\left|z\right|^{p-1}\left\{\sum_{n=p+t-1}^{\infty}\frac{n}{p}\left|b_nB_n\right|\right\} \\
&\geq \left|\sum_{n=p+t-1}^{\infty}\frac{n}{p}\left|b_nB_n\right|z^{n-1}\right| = \left|(g(z)*\varphi(z))'\right|.
\end{aligned}$$

Then, if $\psi(z)=0$ and $\varphi(z)=0$, we have $\left|h'(z)\right|=\left|g'(z)\right|$.

Next, we prove $f(z)\in\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ by establishing condition (1.4). It is sufficient to show that

$$\Re\left\{\frac{z(f*\mathcal{F})''(z)+(f*\mathcal{F})'(z)}{(f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z)}(1+\rho e^{i\theta})-p\rho e^{i\theta}\right\}\geq p\varkappa\ (-\pi\leq\theta\leq\pi),$$

or equivalently

$$\Re\left\{\frac{(1+\rho e^{i\theta})\left(z(f*\mathcal{F})''(z)+(f*\mathcal{F})'(z)\right)-p\rho e^{i\theta}\left((f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z)\right)}{(f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z)}\right\}\geq p\varkappa. \tag{2.2}$$

If we put

$$A(z)=(1+\rho e^{i\theta})\left(z(f*\mathcal{F})''(z)+(f*\mathcal{F})'(z)\right)-p\rho e^{i\theta}\left((f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z)\right)$$

and

$$B(z)=(f*\mathcal{F})'(z)+\tau z(f*\mathcal{F})''(z).$$

Since, $\Re(w)\geq p\varkappa$ if and only if $|A(z)+p(1-\varkappa)B(z)|\geq|A(z)-p(1+\varkappa)B(z)|$, it suffices to show

$$|A(z)+p(1-\varkappa)B(z)|-|A(z)-p(1+\varkappa)B(z)|\geq 0.$$

But

$$\begin{aligned}
&|A(z)+p(1-\varkappa)B(z)|\\
&=\Bigg|(1+\rho e^{i\theta})\Bigg[z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2}\\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)+pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1}\\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}\Bigg]-p\rho e^{i\theta}\Bigg[pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1}\\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}+\tau z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2}\\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)\Bigg]+p(1-\varkappa)\Bigg[pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1}\\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}+\tau z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2}\\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)\Bigg]\Bigg|\\
&=\Big|p^2\left(2+\tau(p-1)(1-\varkappa-\rho e^{i\theta})-\varkappa\right)z^{p-1}\\
&\quad+\sum_{n=p+t}^{\infty}n\left[n(1+\rho e^{i\theta})+p(\tau(n-1)+1)(1-\varkappa-\rho e^{i\theta})\right]\left|a_nA_n\right|z^{n-1}\\
&\quad+\sum_{n=p+t-1}^{\infty}n\left[n(1+\rho e^{i\theta})+p(\tau(n-1)+1)(1-\varkappa-\rho e^{i\theta})\right]\left|b_nB_n\right|\overline{z}^{n-1}\Big|.
\end{aligned}$$

Also

$$
\begin{aligned}
&|A(z)-p(1+\varkappa)B(z)| \\
&=\Bigg|(1+\rho e^{i\theta})\Bigg[z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2} \\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)+pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}\Bigg]-p\rho e^{i\theta}\Bigg[pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}+\tau z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2} \\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)\Bigg]-p(1+\varkappa)\Bigg[pz^{p-1}+\sum_{n=p+t}^{\infty}n\left|a_nA_n\right|z^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}n\left|b_nB_n\right|\overline{z}^{n-1}+\tau z\Bigg(p(p-1)z^{p-2}+\sum_{n=p+t}^{\infty}n(n-1)\left|a_nA_n\right|z^{n-2} \\
&\quad+\sum_{n=p+t-1}^{\infty}n(n-1)\left|b_nB_n\right|\overline{z}^{n-2}\Bigg)\Bigg]\Bigg| \\
&=\Big|-p^2\left[(\rho e^{i\theta}+\varkappa+1)\tau(p-1)+\varkappa\right]z^{p-1} \\
&\quad+\sum_{n=p+t}^{\infty}n\left[n(1+\rho e^{i\theta})-p(\tau(n-1)+1)(\rho e^{i\theta}+\varkappa+1)\right]\left|a_nA_n\right|z^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}n\left[n(1+\rho e^{i\theta})-p(\tau(n-1)+1)(\rho e^{i\theta}+\varkappa+1)\right]\left|b_nB_n\right|\overline{z}^{n-1}\Big|.
\end{aligned}
$$

Then

$$
\begin{aligned}
&|A(z)+p(1-\varkappa)B(z)|-|A(z)-p(1+\varkappa)B(z)|\geq 2p^2\left[1-\varkappa-\tau(p-1)(\rho e^{i\theta}+\varkappa)\right]|z|^{p-1} \\
&\quad+\sum_{n=p+t}^{\infty}2n\left[p(\tau(n-1)+1)(\rho e^{i\theta}+\varkappa)-n(1+\rho e^{i\theta})\right]\left|a_nA_n\right||z|^{n-1} \\
&\quad+\sum_{n=p+t-1}^{\infty}2n\left[p(\tau(n-1)+1)(\rho e^{i\theta}+\varkappa)-n(1+\rho e^{i\theta})\right]\left|b_nB_n\right||z|^{n-1} \\
&>2\Bigg\{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]-\sum_{n=p+t}^{\infty}n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right] \\
&\quad\left|a_nA_n\right|-\sum_{n=p+t-1}^{\infty}n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]\left|b_nB_n\right|\Bigg\}>0.
\end{aligned}
$$

The last expression is non-negative by (2.1), thus $f \in \xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

For $\sum\limits_{n=p+t}^{\infty} |x_n| + \sum\limits_{n=p+t-1}^{\infty} |y_n| = 1$, the function

$$\begin{aligned} f(z) &= z^p + \sum_{n=p+t}^{\infty} \frac{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]}{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]} x_n z^n \\ &\quad + \sum_{n=p+t-1}^{\infty} \frac{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]}{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]} \overline{y_n z^n}. \end{aligned}$$

This completes the proof. □

In the following theorem, it is shown that the condition (2.1) is also necessary for function $f = h + \overline{g}$, where h and g are of the form (1.5) and belongs to the class $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Theorem 2.2 *Let the function $f = h + \overline{g}$ be so that h and g are given by (**??**). Then $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ if and only if*

$$\begin{aligned} &\sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]}{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]} |a_n A_n| \\ &\quad + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]}{p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right]} |b_n B_n| \le 1, \end{aligned} \tag{2.3}$$

where $0 \le \varkappa < 1,\ \rho \ge 0,\ 0 \le \tau < 1,\ z \in \mathcal{U}$.

Proof Since $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau) \subset \zeta_{\mathcal{F}}(p,\varkappa,\rho,\tau)$, we need only to prove the only if part of the theorem.

Assume that $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. Then by (1.4), we have

$$\Re\left\{ \frac{z(f*\mathcal{F})''(z) + (f*\mathcal{F})'(z)}{(f*\mathcal{F})'(z) + \tau z (f*\mathcal{F})''(z)}(1+\rho e^{i\theta}) - p\rho e^{i\theta} \right\} \ge p\varkappa.$$

This is equivalent to

$$\Re\left\{ \frac{\begin{array}{l} p^2\left[1-\tau(p-1)\rho e^{i\theta}\right] z^{p-1} - \sum\limits_{n=p+t}^{\infty} n\left[n+(n-p(\tau(n-1)+1))\rho e^{i\theta}\right] \\ |a_n A_n| z^{n-1} - \sum\limits_{n=p+t-1}^{\infty} n\left[n+(n-p(\tau(n-1)+1))\rho e^{i\theta}\right] |b_n B_n| \overline{z}^{n-1} \end{array}}{\begin{array}{l} p(1+\tau(p-1))z^{p-1} - \sum\limits_{n=p+t}^{\infty} n(1+\tau(n-1)) |a_n A_n| z^{n-1} \\ \qquad - \sum\limits_{n=p+t-1}^{\infty} n(1+\tau(n-1)) |b_n B_n| \overline{z}^{n-1} \end{array}} - \alpha p \right\} \ge 0. \tag{2.4}$$

This condition must hold for all values of z, such that $|z| = r < 1$. Choosing the values of z on the positive specific values, $0 \le z = r < 1$ and noting that $\Re(-e^{i\theta}) \ge -\left|e^{i\theta}\right| = -1$, the above inequality reduces to

$$\begin{aligned} p^2\left[1-\varkappa-\tau(p-1)(\rho+\varkappa)\right] &- \sum_{n=p+t}^{\infty} n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]|a_nA_n| \\ &- \sum_{n=p+t-1}^{\infty} n\left[n(1+\rho)-p(\tau(n-1)+1)(\rho+\varkappa)\right]|b_nB_n| \geq 0. \end{aligned}$$

This gives (2.3) and the proof is complete. □

§3. Distortion Bounds

Theorem 3.1 *Let $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. Then for $|z| = r < 1$, we have*

$$\begin{aligned} |f(z)| \leq\ & (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \left(\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}\right. \\ & \left. - \frac{(p+t-1)\left[(p+t-1)(1+\rho)-p(1+\tau(p+t-2))(\varkappa+\rho)\right]}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}|a_{p+t-1}A_{p+t-1}|\right) r^{p+t} \quad (3.1) \end{aligned}$$

and

$$\begin{aligned} |f(z)| \geq\ & (1-|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} - \left(\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}\right. \\ & \left. - \frac{(p+t-1)\left[(p+t-1)(1+\rho)-p(1+\tau(p+t-2))(\varkappa+\rho)\right]}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}|a_{p+t-1}A_{p+t-1}|\right) r^{p+t}. \quad (3.2) \end{aligned}$$

Proof Assume that $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$. Then by (2.3), we get

$$\begin{aligned} |f(z)| &= \left| z^p - \sum_{n=p+t}^{\infty} |a_nA_n|\, z^n - \sum_{n=p+t-1}^{\infty} |b_nB_n|\, \overline{z}^n \right| \\ &\leq (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \sum_{n=p+t}^{\infty} (|a_nA_n|+|b_nB_n|)\, r^{p+t} \\ &\leq (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]} \\ &\quad \times \sum_{n=p+t}^{\infty} \frac{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} (|a_nA_n|+|b_nB_n|)\, r^{p+t} \\ &\leq (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]} \\ &\quad \times \sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} (|a_nA_n|+|b_nB_n|)\, r^{p+t} \\ &= (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1} + \frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]} \end{aligned}$$

$$\times\left\{1-\frac{(p+t-1)\left[(p+t-1)(1+\rho)-p(1+\tau(p-t-2))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}|a_{p+t-1}A_{p+t-1}|\right\}r^{p+t}$$

$$= (1+|b_{p+t-1}B_{p+t-1}|)r^{p+t-1}+\left\{\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}\right.$$

$$\left.-\frac{(p+t-1)\left[(p+t-1)(1+\rho)-p(1+\tau(p-t-2))(\varkappa+\rho)\right]}{(p+t)\left[(p+t)(1+\rho)-p(1+\tau(p-t-1))(\varkappa+\rho)\right]}|a_{p+t-1}A_{p+t-1}|\right\}r^{p+t}.$$

The relation (3.2) can be proved by using similar statements. So the proof is complete. □

§4. Extreme Points

In this section we determine the extreme points of the closed convex hull of the class $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Theorem 4.1 *Let $f(z)$ given by (1.5). Then $f(z)\in\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ if and only if $f(z)$ can be expressed in the form*

$$f(z)=\sum_{n=p+t-1}^{\infty}\left(\mu_n h_n(z)+\delta_n g_n(z)\right),\qquad z\in\mathcal{U}, \tag{4.1}$$

where $h_p(z)=z^p$,

$$h_n(z)=z^p-\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}z^n, n=p+t,p+t+1,\cdots$$

and

$$g_n(z)=z^p-\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\overline{z}^n, n=p+t-1,p+t\cdots,$$

$$\mu_{p+t-1}\equiv\mu_p=1-\left(\sum_{n=p+t}^{\infty}\mu_n+\sum_{n=p+t-1}^{\infty}\delta_n\right),\quad \mu_n,\delta_n\geq 0.$$

Particularly, the extreme points of $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ are $\{h_n\}$ and $\{g_n\}$.

Proof Assume that $f(z)$ can be expressed by (4.1). Then, we have

$$\begin{aligned} f(z) &= \sum_{n=p+t-1}^{\infty}(\mu_n+\delta_n)z^p-\sum_{n=p+t}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\mu_n z^n \\ &\quad -\sum_{n=p+t-1}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\delta_n\overline{z}^n \end{aligned}$$

$$f(z) = z^p-\sum_{n=p+t}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\mu_n z^n$$

$$-\sum_{n=p+t-1}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\delta_n\overline{z}^n.$$

Therefore

$$\begin{aligned}&\sum_{n=p+t}^{\infty}\frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\mu_n\\&\quad+\sum_{n=p+t-1}^{\infty}\frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\delta_n\\&=\sum_{n=p+t}^{\infty}\mu_n+\sum_{n=p+t-1}^{\infty}\delta_n=\sum_{n=p+t-1}^{\infty}(\mu_n+\delta_n)-\mu_{p+t-1}=1-\mu_p\leq 1.\end{aligned}$$

So $f(z)\in\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$.

Conversely, let $f(z)\in\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$, by putting

$$\mu_n=\frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\left|a_nA_n\right|, n=p+t,p+t+1,\cdots$$

and

$$\delta_n=\frac{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}\left|b_nB_n\right|, n=p+t-1,p+t,\cdots$$

We define $\mu_p\equiv\mu_{p+t-1}=\left(1-\sum\limits_{n=p+t}^{\infty}\mu_n-\sum\limits_{n=p+t-1}^{\infty}\delta_n\right)$.

Then, note that $0\leq\mu_n\leq 1$ $(n=p+t,p+t+1,\cdots)$, $0\leq\delta_n\leq 1$ $(n=p+t-1,p+t,\cdots)$. Hence

$$\begin{aligned}f(z)&=z^p-\sum_{n=p+t}^{\infty}\left|a_nA_n\right|z^n-\sum_{n=p+t-1}^{\infty}\left|b_nB_n\right|\overline{z}^n\\&=z^p-\sum_{n=p+t}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\mu_nz^n\\&\quad-\sum_{n=p+t-1}^{\infty}\frac{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)}{n\left[n(1+\rho)-p(1+\tau(n-1))(\varkappa+\rho)\right]}\delta_n\overline{z}^n\\&=z^p-\sum_{n=p+t}^{\infty}\left(z^p-h_n(z)\right)\mu_n-\sum_{n=p+t-1}^{\infty}\left(z^p-g_n(z)\right)\delta_n\\&=\left(1-\sum_{n=p+t}^{\infty}\mu_n-\sum_{n=p+t-1}^{\infty}\delta_n\right)z^p+\sum_{n=p+t}^{\infty}\mu_nh_n(z)+\sum_{n=p+t}^{\infty}\delta_ng_n(z)\\&=\mu_ph_p(z)+\sum_{n=p+t}^{\infty}\mu_nh_n(z)+\sum_{n=p+t-1}^{\infty}\delta_ng_n(z)=\sum_{n=p+t-1}^{\infty}\left(\mu_nh_n(z)+\delta_ng_n(z)\right),\end{aligned}$$

that is the required representation. □

§5. Convolution and Convex Combination

In this section, we determine the convolution properties and convex combination.

For harmonic function

$$f_k(z) = z^p - \sum_{n=p+t}^{\infty} |a_{n,k}|\, z^n - \sum_{n=p+t-1}^{\infty} |b_{n,k}|\, \overline{z}^n \qquad (k=1,2), \tag{5.1}$$

are in the class $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$, we denote by $(f_1 * f_2)(z)$ the Hadamard product or (the convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 * f_2)(z) = z^p - \sum_{n=p+t}^{\infty} |a_{n,1}|\,|a_{n,2}|\, z^n - \sum_{n=p+t-1}^{\infty} |b_{n,1}|\,|b_{n,2}|\, \overline{z}^n. \tag{5.2}$$

Using this definition, we show that the class $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ is closed under convolution.

Theorem 5.1 *For $0 \le \eta \le \varkappa < 1$, let the function $f_1 \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ and $f_2 \in \mathcal{T}\xi_{\mathcal{F}}(p,\eta,\rho,\tau)$. Then*

$$(f_1 * f_2)(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau) \subset \mathcal{T}\xi_{\mathcal{F}}(p,\eta,\rho,\tau). \tag{5.3}$$

Proof Since f_1 be in the class $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ and f_2 be in the class $\mathcal{T}\xi_{\mathcal{F}}(p,\eta,\rho,\tau)$ and $|a_{n,2}| < 1$ and $|b_{n,2}| < 1$. We need to prove the coefficients of $(f_1 * f_2)(z)$ satisfy the condition given by (2.1), we obtain

$$\begin{aligned}
&\sum_{n=p+t}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\eta)]}{p^2\,[1-\eta-\tau(p-1)(\rho+\eta)]}\, |a_{n,1}|\,|a_{n,2}| \\
&\quad + \sum_{n=p+t-1}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\eta)]}{p^2\,[1-\eta-\tau(p-1)(\rho+\eta)]}\, |b_{n,1}|\,|b_{n,2}| \\
&\le \sum_{n=p+t}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\eta)]}{p^2\,[1-\eta-\tau(p-1)(\rho+\eta)]}\, |a_{n,1}| \\
&\quad + \sum_{n=p+t-1}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\eta)]}{p^2\,[1-\eta-\tau(p-1)(\rho+\eta)]}\, |b_{n,1}| \\
&\le \sum_{n=p+t}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\varkappa)]}{p^2\,[1-\varkappa-\tau(p-1)(\rho+\varkappa)]}\, |a_{n,1}| \\
&\quad + \sum_{n=p+t-1}^{\infty} \frac{n\,[n(1+\rho) - p(\tau(n-1)+1)(\rho+\varkappa)]}{p^2\,[1-\varkappa-\tau(p-1)(\rho+\varkappa)]}\, |b_{n,1}| \le 1.
\end{aligned}$$

Therefore $(f_1 * f_2)(z) \in \mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau) \subset \mathcal{T}\xi_{\mathcal{F}}(p,\eta,\rho,\tau)$ for $0 \le \eta \le \varkappa < 1$. □

Next, we show that $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ is closed under convex combinations of its members.

Theorem 5.2 *The class $\mathcal{T}\xi_{\mathcal{F}}(p,\varkappa,\rho,\tau)$ is closed under convex combinations.*

Proof For $j = 1, 2, 3, \cdots$, vLet $f_j \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$, where f_j is given by

$$f_j(z) = z^p - \sum_{n=p+t}^{\infty} |a_{n,j} A_{n,j}| \, z^n - \sum_{n=p+t-1}^{\infty} |b_{n,j} B_{n,j}| \, \overline{z}^n.$$

Then, by (2.3)

$$\sum_{n=p+t}^{\infty} \frac{n\,[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |a_{n,j} A_{n,j}| + \sum_{n=p+t-1}^{\infty} \frac{n\,[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |b_{n,j} B_{n,j}| \leq 1. \tag{5.4}$$

For $\sum\limits_{j=1}^{\infty} t_j = 1, 0 \leq t_j \leq 1$, the convex combination of f_j's can be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z^p - \sum_{n=p+t}^{\infty} \left(\sum_{j=1}^{\infty} t_j \, |a_{n,j} A_{n,j}| \right) z^n - \sum_{n=p+t-1}^{\infty} \left(\sum_{j=1}^{\infty} t_j \, |b_{n,j} B_{n,j}| \right) \overline{z}^n.$$

Then by (5.4), we have

$$\begin{aligned}
&\sum_{n=p+t}^{\infty} \frac{n\,[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} \left(\sum_{j=1}^{\infty} t_j \, |a_{n,j} A_{n,j}| \right) \\
&\quad + \sum_{n=p+t-1}^{\infty} \frac{n\,[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} \left(\sum_{j=1}^{\infty} t_j \, |b_{n,j} B_{n,j}| \right) \\
&= \sum_{j=1}^{\infty} t_j \left\{ \sum_{n=p+t}^{\infty} \frac{n\,[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |a_{n,j} A_{n,j}| \right. \\
&\quad \left. + \sum_{n=p+t-1}^{\infty} \frac{n\,[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |b_{n,j} B_{n,j}| \right\} \\
&\leq \sum_{j=1}^{\infty} t_j = 1.
\end{aligned}$$

Therefore $\sum\limits_{j=1}^{\infty} t_j f_j(z) \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$.

This complete the proof. □

§6. **Integral Operator**

Finally, we examine a closure property of the class $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$ under the generalized Bernardi-Livingston integral operator (see [3, 17, 18]).

Definition 6.1 *The Bernardi operator is defined by*

$$L_{c,p}f(z) = \frac{c+p}{z^c}\int_0^\infty t^{c-1}f(t)dt, \quad c > -1. \tag{6.1}$$

If $f(z) = z^p + \sum\limits_{n=p+t}^{\infty} a_n z^n$, *then*

$$L_{c,p}f(z) = z^p + \sum_{n=p+t}^{\infty} \frac{c+p}{n+c} a_n z^n. \tag{6.2}$$

Remark 6.2 If $f = h + \overline{g}$, where

$$h(z) = z^p - \sum_{n=p+t}^{\infty} a_n z^n, \quad g(z) = -\sum_{n=p+t-1}^{\infty} b_n z^n, \quad (a_n, b_n \geq 0).$$

Then

$$L_{c,p}f(z) = L_{c,p}(h(z)) + \overline{L_{c,p}(g(z))}. \tag{6.3}$$

Theorem 6.3 *If* $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$, *then* $L_{c,p}f(z)$ $(c > -1)$ *is also in* $\mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$.

Proof By (6.2) and (6.3), we get

$$\begin{aligned} L_{c,p}f(z) &= L_{c,p}\left(z^p - \sum_{n=p+t}^{\infty} |a_n A_n| z^n - \sum_{n=p+t-1}^{\infty} |b_n B_n| \overline{z}^n\right) \\ &= z^p - \sum_{n=p+t}^{\infty} \frac{c+p}{n+c} |a_n A_n| z^n - \sum_{n=p+t-1}^{\infty} \frac{c+p}{n+c} |b_n B_n| \overline{z}^n \\ &= z^p - \sum_{n=p+t}^{\infty} x_n |a_n A_n| z^n - \sum_{n=p+t-1}^{\infty} y_n |b_n B_n| \overline{z}^n. \end{aligned}$$

Therefore,

$$\begin{aligned} &\sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} \left(\frac{c+p}{n+c} |a_n A_n|\right) \\ &\quad + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} \left(\frac{c+p}{n+c} |b_n B_n|\right) \\ &\leq \sum_{n=p+t}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |a_n A_n| \\ &\quad + \sum_{n=p+t-1}^{\infty} \frac{n\left[n(1+\rho) - p(1+\tau(n-1))(\varkappa+\rho)\right]}{p^2(1-\tau(p-1)(\rho+\varkappa)-\varkappa)} |b_n B_n| \leq 1. \end{aligned}$$

Since $f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$, by using Theorem 2.2, then $L_{c,p}f(z) \in \mathcal{T}\xi_{\mathcal{F}}(p, \varkappa, \rho, \tau)$. This complete the proof of Theorem 6.3. □

References

[1] O. P. Ahuja, Planar harmonic univalent and related mappings, *J. Inequal. Pure Appl. Math.*, 6(4) (2005), 1–18.

[2] W. G. Atshan and A. K. Wanas, On a new class of harmonic univalent functions, *Matematički Vesnik*, 65(4), 2013, 555-564.

[3] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, 135 (1969),429-446.

[4] R. W. Barnard and C. Kellog, Applications of convolution operators to problems in univalent function theory, *Michigan Math. J.*, 27 (1980), 81–94.

[5] J. H. Choi, Y. C. Kim and S. Owa, Generalizations of Hadamard products of functions with negative coefficients, *J. Math. Anal. Appl.*, 199 (1996), 495–501.

[6] J. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 9 (1984), 3–25.

[7] H. E. Darwish, On generalizations of Hadamard products of functions with negative coefficients, *Proc. Pakistan Acad. Sci.*, 43(4) (2006), 269–273.

[8] H. E. Darwish and M. K. Aouf, Generalizations of modified-Hadamard products of pvalent functions with negative coefficients, *Math. Comput. Modelling*, 49 (2009), 38–45.

[9] K. K. Dixit and S. Porwal, A convolution approach on partial sums of certain analytic and univalent functions, *J. Inequal. Pure Appl. Math.*, 10(4) (2009), 1–17.

[10] K. K. Dixit and S. Porwal, Some properties of harmonic functions defined by convolution, *Kyungpook Math. J.* , 49(4) (2009), 751–761.

[11] T. Domokos, On a subclass of certain starlike functions with negative coefficients, *Studia Univ. Babes-Bolyai Math.*, (1999), 29–36.

[12] P. L. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, (2004).

[13] B. A. Frasin, Comprehensive family of harmonic univalent functions, *SUT J. Math.*, 42(1) (2006), 145–155.

[14] S. P. Goyal, P. Goswami and N. E. Cho, Argument estimates for certain analytic functions associated with the convolution structure, *J. Inequal. Pure Appl. Math.*, 10(1) (2009), 1–13.

[15] O. P. Juneja, T. R. Reddy and M. L. Mogra, A convolution approach for analytic functions with negative coefficients, *Soochow J. Math.*, 11 (1985), 69–81.

[16] H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.*, 42 (1936), 689–692.

[17] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, 16 (1965),755–758.

[18] A. E. Livingston, On the radius of univalence of certain analytic functions, *Proc. Amer. Math.Soc.*, 17 (1966), 352-357.

[19] A. O. Mostafa, A study on starlike and convex properties for hypergeometric functions, *J. Inequal. Pure Appl. Math.*, 10(3)(2009), 1-16.

[2] J. Nishiwaki and S. Owa, An application of Hölder's inequality for convolutions, *J. Inequal. Pure Appl. Math.*, 10(4) (2009), 1–14.

[21] J. Nishiwaki, S. Owa and H. M. Srivastava, Convolution and Hölder type inequalities for a certain class of analytic functions, *Math. Inequal. Appl.*, 11 (2008), 717–727.

[22] K. I. Noor, Convolution techniques for certain classes of analytic functions, *Panamer. Math. J.*, 2(3)(1992), 73–82.

[23] S. Owa, The Quasi-Hadamard products of certain analytic functions, in: H. M. Srivastava, S. Owa (Eds.), *Current Topics in Analytic Function Theory*, World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, (1992), 234–251.

[24] S. Ponnusamy and A. Rasila, Planar harmonic mappings, *Ramanujan Mathematical Society Mathematics Newsletters*, 17(2) (2007), 40–57.

[25] S. Ponnusamy and A. Rasila, Planar harmonic and quasi-conformal mappings, *Ramanujan Mathematical Society Mathematics Newsletters*, 17(3) (2007), 85–101.

[26] R. K. Raina and D. Bansal, Some properties of a new class of analytic functions defined in terms of a Hadamard product, *J. Inequal. Pure Appl. Math.*, 9(1) (2008), 1–20.

[27] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Sem. Math. Sup., Presses Univ. de Montreal, (1982).

[28] A. Schild and H. Silverman, Convolution of univalent functions with negative coefficients, *Ann. Univ. Mariae Curie-Sklodowska*, Sect. A 29 (1975), 99–107.

[29] R. Singh and S. Singh, Convolution properties of a class of starlike functions, *Proc. Amer. Math. Soc.*, 106(1) (1989), 145–152.

[30] J. Stankiewicz and Z. Stankiewicz, Some applications of the Hadamard convolution in the theory of functions, *Ann. Univ. Mariae Curie-Sklodowska*, Sect. A 40 (1986), 251–265.

[31] H. M. Srivastava, S. Owa and S. K. Chatterjea, A note on certain classes of starlike functions, *Rend. Sem. Mat. Univ. Padova*, 77(1987), 115–124.

International J.Math. Combin. Vol.4(2020), 80-88

The Third Leap Zagreb Index of Some Graph Operations

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Abstract: Recently introducing leap Zagreb indices are a generalization of classical Zagreb indices of chemical graph theory. The third leap Zagreb index is equal to the sum of products of first and second degrees of vertices of G, where the first and second degrees of a vertex v in a graph G are equal to the number of their first and second neighbors and denoted by $d(v/G)$ and $d_2(v/G)$, respectively. In this paper, exact expression for third leap Zagreb index of some graph operations will be presented.

Key Words: Distance-degrees (of vertices), third leap Zagreb index, graph operations.

AMS(2010): 05C07, 05C12, 05C76.

§1. Introduction and Preliminaries

In this paper, we are concerned with simple graphs, i.e., finite graphs having no loops, multiple and directed edges. Let $G = (V, E)$ be such a graph with vertex set $V(G)$ and edges set $E(G)$. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. The distance $d_G(u, v)$ between any two vertices u and v of a graph G is equal to the length of (number of edges in) a shortest path connecting them. For a vertex $v \in V(G)$ and a positive integer k, the open k-neighborhood of v in a graph G is denoted by $N_k(v/G)$ and is defined as $N_k(v/G) = \{u \in V(G) : d_G(u, v) = k\}$. The k-distance degree of a vertex v in G is denoted by $d_k(v/G)$ (or simply $d_k(v)$, if no misunderstanding) and is defined as the number of k-neighbors of the vertex v in G, i.e., $d_k(v/G) = |N_k(v/G)|$. It is clearly that $d_1(v/G) = d(v/G)$ for every $v \in V(G)$.

The complement $\overline{G}$ of a graph G is a graph with vertex set $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. For a vertex v of G, the eccentricity $e(v) = \max\{d_G(v, u) : u \in V(G)\}$. The diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$ and the radius of G is $rad(G) = \min\{e(v) : v \in V(G)\}$. Let $H \subseteq V(G)$ be any subset of vertices of G. Then the induced subgraph $\langle H\rangle$ of G is the graph whose vertex set is H and whose edge set consists of all of the edges in $E(G)$ that have both endpoints in H. A graph G is called F-free graph if no induced subgraph of G is isomorphic to F.

We follow [9] for unexplained graph theoretic terminologies and notations.

[1]Received June 5, 2020, Accepted November 23, 2020.

In the interdisciplinary area where chemistry, physics and mathematics meet, molecular graph based structure descriptors, usually referred to as topological indices, are of significant importance. A topological index of a graph is a graph invariant number calculated from a graph representing a molecule. Among the most important such structure descriptors are the classical first and second Zagreb indices, which introduced, more than forty four years ago, by Gutman and Trinajestic [8], in 1972, and elaborated in [7]. They are defined as:

$$M_1(G) = \sum_{v\in V(G)} d_1^2(v/G) \qquad \text{and} \qquad M_2(G) = \sum_{uv\in E(G)} d_1(u/G)d_1(v/G).$$

For properties of the two Zagreb indices see [5, 7, 13, 18] for details of the theory of Zagreb indices see the survey [4] and the references cited therein. Recently the eccentric harmonic index is established as an eccentric version of the harmonic index, which has a huge area of applications, for more details see [14,17]. After most of the results on Zagreb indices were established, the inevitable occurred, their various modifications have been proposed, thus opening the possibility to do analogous research and publish numerous additional papers. For these modifications see the recent survey [6].

In (2017), Naji et al. [11] have been introduced a new distance-degree-based topological indices conceived depending on the second degrees of vertices, and are so-called leap Zagreb indices of a graph G and are defined as

$$\begin{aligned} LM_1(G) &= \sum_{v\in V(G)} d_2^2(v/G), \\ LM_2(G) &= \sum_{uv\in E(G)} d_2(u/G)d_2(v/G), \\ LM_3(G) &= \sum_{v\in V(G)} d(v/G)d_2(v/G). \end{aligned}$$

The leap Zagreb indices have several chemical applications. Surprisingly, the first leap Zagreb index has very good correlation with physical properties of chemical compounds like boiling point, entropy, DHVAP, HVAP and accentric factor [3].

In a later work [12], the first leap Zagreb index of graph operations was studied. In [2], the expressions for these three leap Zagreb indices of generalized xyz point line transformation graphs $T^{xyz}(G)$, when $z = 1$ are obtained. The authors in [15], generalized the results of [11], pertaining to trees and unicyclic graphs. They determined upper and lower bounds on leap Zagreb indices and characterized the extremal graphs. Leap Zagreb indices are considered in a recent survey [6].

In this paper, we present the exact expressions for the third leap Zagreb index of some graph operations containing cartesian product, composition, disjunction, symmetric difference and corona product of graphs. The following fundamental results which will be required for many of our arguments in this paper are found in Yamaguchi [19] and Soner and Naji [16].

Theorem 1.1([16,19]) *Let G be a connected graph with n vertices and m edges. Then*

$$d_2(v/G) \leq (\sum_{u\in N_1(v/G)} d_1(u/G)) - d_1(v/G).$$

and equality holds if and only if G is a $\{C_3, C_4\}$-free graph.

§2. Main Results

2.1 Cartesian Product

Definition 2.1([10]) *For given graphs G and H their cartesian product, denoted $G\square H$, is defined as the graph on the vertex set $V(G)\times V(H)$, and vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ of $V(G)\times V(H)$ are connected by an edge if and only if either ($u_1=v_1$ and $u_2v_2\in E(H)$) or ($u_2=v_2$ and $u_1v_1\in E(G)$).*

It is a well known fact that the cartesian product of graphs is commutative and associative up to isomorphism. $|V(G\square H)|=|V(G)||V(H)|$, the distance between any two vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ in $G\square H$ is given by

$$d_{G\square H}(u,v)=d_G(u_1,v_1)+d_H(u_2,v_2).$$

Lemma 2.2([12]) *Let G and H be connected graphs of orders n_1 and n_2, respectively. Then for any vertex $(u,v)\in V(G\square H)$,*

(1) $d_1((u,v)/G\square H)=d_1(u/G)+d_1(v/H)$;

(2) $d_2((u,v)/G\square H)=d_2(u/G)+d_1(u/G)d_1(v/H)+d_2(v/H)$.

Theorem 2.3 *Let G and H be two nontrivial connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\begin{aligned} LM_3(G\square H) &= n_2LM_3(G)+2m_2(M_1(G) \\ &\quad +\sum_{u\in V(G)} d_2(u/G))+n_1LM_3(H)+2m_1(M_1(H)+\sum_{v\in V(H)} d_2(v/H)). \end{aligned}$$

Proof Let G and H be two nontrivial connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then by Lemma 2.2, we obtain

$$\begin{aligned} LM_3(G\square H) &= \sum_{(u,v)\in V(G\square H)} d_1((u,v)/G\square H)d_2((u,v)/G\square H) \\ &= \sum_{u\in V(G)}\sum_{v\in V(H)} \Big[(d_1(u/G)+d_1(v/H))(d_2(u/G)+d_1(u/G)d_1(v/H)+d_2(v/H))\Big] \end{aligned}$$

$$= \sum_{u\in V(G)} \sum_{v\in V(H)} \Big[(d_1(u/G)d_2(u/G) + d_1^2(u/G)d_1(v/H) + d_1(u/G)d_2(v/H)$$
$$+ d_2(u/G)d_1(v/H) + d_1(u/G)d_1^2(v/H) + d_1(v/H)d_2(v/H))\Big]$$
$$= \sum_{u\in V(G)} \Big[n_2 d_1(u/G)d_2(u/G) + 2m_2 d_1^2(u/G) + d_1(u/G) \sum_{v\in V(H)} d_2(v/H)$$
$$+ 2m_2 d_2(u/G) + M_1(H)d_1(u/G) + LM_3(H)\Big]$$
$$= n_2 LM_3(G) + 2m_2 M_1(G) + 2m_1 \sum_{v\in V(H)} d_2(v/H) + 2m_2 \sum_{u\in V(G)} d_2(u/G)$$
$$+ 2m_1 M_1(H) + n_1 LM_3(H)$$
$$= n_2 LM_3(G) + 2m_2\Big(M_1(G) + \sum_{u\in V(G)} d_2(u/G)\Big) + n_1 LM_3(H)$$
$$+ 2m_1\Big(M_1(H) + \sum_{v\in V(H)} d_2(v/H)\Big).$$

This completes the proof. □

From Theorem 1.1, the following result follows.

Corollary 2.4 *If G and H are nontrivial connected (C_3, C_4)-free graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$LM_3(G\Box H) = n_2 LM_3(G) + 4m_2 M_1(G) + n_1 LM_3(H) + 4m_1 M_1(H) - 8m_1 m_2.$$

2.2 Composition

Definition 2.5([10]) *The composition $G[H]$ of graphs G and H with disjoint vertex sets and edge sets is a graph on vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever [u_1 is adjacent with u_2] or [$u_1 = u_2$ and v_1 is adjacent with v_2].*

The composition is not commutative. The easiest way to visualize the composition $G[H]$ is to expand each vertex of G into a copy of H, with each edge of G replaced by the set of all possible edges between the corresponding copies of H. Hence, by letting $\mathfrak{N}_1 = |E(G[H])|$, then

$$\mathfrak{N}_1 = n_1 m_2 + n_2^2 m_1. \tag{1}$$

Lemma 2.6([12]) *Let G and H be two graphs with disjoint vertex sets with n_1 and n_2 vertices and edges sets with m_1 and m_2 edges, respectively. Then*

(1) it $d_1((u, v)/G[H]) = n_2 d_1(u/G) + d_1(v/H)$;

(2) $d_2((u, v)/G[H])) = n_2 d_2(u/G) + d_1(v/\overline{H})$.

Theorem 2.7 *Let G and H be two graphs with disjoint vertex sets with n_1 and n_2 vertices and* *edges sets with m_1 and m_2 edges, respectively. Then*

$$\begin{aligned} LM_3(G[H]) &= n_2^3 LM_3(G) - n_1 M_1(H) \\ &\quad -4n_2 m_1 m_2 + 2\mathfrak{N}_1(n_2-1) + 2n_2 m_2 \sum_{u\in V(G)} d_2(u/G). \end{aligned}$$

Proof Let G and H be two graphs with disjoint vertex sets with n_1 and n_2 vertices and edges sets with m_1 and m_2 edges, respectively. Then by Lemma 2.6, we obtain

$$\begin{aligned} LM_3(G[H]) &= \sum_{(u,v)\in V(G\Box H)} d_1((u,v)/G[H]) d_2((u,v)/G[H]) \\ &= \sum_{v\in V(H)} \sum_{u\in V(G)} \Big[\Big(n_2 d_1(u/G) + d_1(v/H)\Big)\Big(n_2 d_2(u/G) + d_1(v/\overline{H})\Big)\Big] \\ &= \sum_{v\in V(H)} \sum_{u\in V(G)} \Big[n_2^2 d_1(u/G) d_2(u/G) + n_2 d_1(u/G) d_1(v/\overline{H}) + n_2 d_2(u/G) d_1(v/H) \\ &\quad + d_1(v/H) d_1(v/\overline{H})\Big] \\ &= \sum_{v\in V(H)} \Big[n_2^2 LM_3(G) + 2n_2 m_1 d_1(v/\overline{H}) + n_2 d_1(v/H) \sum_{u\in V(G)} d_2(u/G) \\ &\quad + n_1 d_1(v/H) d_1(v/\overline{H})\Big] \end{aligned}$$

$$\begin{aligned} LM_3(G[H]) &= \sum_{v\in V(H)} \Big[n_2^2 LM_3(G) + 2n_2 m_1 (n_2 - 1 - d_1(v/H)) + n_2 d_1(v/H) \sum_{u\in V(G)} d_2(u/G) \\ &\quad + n_1 d_1(v/H)(n_2 - 1 - d_1(v/H))\Big] \\ &= n_2^3 LM_3(G) + 2n_2 m_1 (n_2(n_2-1) - 2m_2) + 2n_2 m_2 \sum_{u\in V(G)} d_2(u/G) \\ &\quad + 2m_2 n_1 (n_2 - 1) - n_1 M_1(H)\Big] \\ &= n_2^3 LM_3(G) - n_1 M_1(H) + 2n_2^2 m_1 (n_2-1) - 4n_2 m_1 m_2 \\ &\quad + 2n_1 m_2 (n_2 - 1) + 2n_2 m_2 \sum_{u\in V(G)} d_2(u/G). \end{aligned}$$

By using equation 1, we get

$$\begin{aligned} LM_3(G[H]) &= n_2^3 LM_3(G) - n_1 M_1(H) \\ &\quad -4n_2 m_1 m_2 + 2\mathfrak{N}_1(n_2-1) + 2n_2 m_2 \sum_{u\in V(G)} d_2(u/G). \end{aligned}$$

This completes the proof. □

From Theorem 1.1, the following result follows.

Corollary 2.8 *If G and H are nontrivial connected (C_3, C_4)-free graphs with n_1, n_2 vertices* *and m_1, m_2 edges, respectively. Then,*

$$\begin{aligned} LM_3(G[H]) &= n_2^3 LM_3(G) + 2n_2m_2M_1(G) \\ &\quad -n_1M_1(H) + 2n_2^2m_1(n_2-1) + 2n_1m_2(n_2-1) - 8n_2m_1m_2. \end{aligned}$$

2.3 Disjunction

Definition 2.9([10]) *The disjunction $G \vee H$ of two graphs G and H with disjoint vertex sets and edge sets is the graph with vertex set $V(G)\times V(H)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever u_1 is adjacent with u_2 in G or v_1 is adjacent with v_2 in H.*

The disjunction is commutative and the number of edges of $G \vee H$ is $\mathfrak{M}_1$ ([1]) and equal to

$$\mathfrak{M}_1 = n_1^2m_2 + n_2^2m_1 - 2m_1m_2. \tag{2}$$

Lemma 2.10([12]) *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then,*

(1) $d_1((u,v)/G\vee H) = n_2d_1(u/G) + n_1d_1(v/H) - d_1(u/G)d_1(v/H)$;
(2) $d_2((u,v)/G\vee H) = (n_1n_2-1) - n_2d_1(u/G) - n_1d_1(v/H) + d_1(u/G)d_1(v/H)$.

Theorem 2.11 *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively, such that G or H not a complete graph. Then,*

$$\begin{aligned} LM_3(G\vee H) &= (4n_2m_2 - n_2^3)M_1(G) + (4n_1m_1 - n_1^3)M_1(H) \\ &\quad -M_1(G)M_1(H) + 2\mathfrak{M}_1(n_1n_2-1) - 2m_1m_2(4n_1n_2-1). \end{aligned}$$

Proof Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively, such that G or H not a complete graph. Then from Lemma 2.10, we get

$$\begin{aligned} LM_3(G\vee H) &= \sum_{(u,v)\in V(G\vee H)} d_1((u,v)/G\vee H)d_2((u,v)/G\vee H) \\ &= \sum_{v\in V(H)}\sum_{u\in V(G)} \Big[n_2(n_1n_2-1)d_1(u/G) - n_2^2d_1^2(u/G) - n_1n_2d_1(u/G)d_1(v/H) \\ &\quad + n_2d_1^2(u/G)d_1(v/H) + n_1(n_1n_2-1)d_1(v/H) - n_1n_2d_1(u/G)d_1(v/H) \\ &\quad - n_1^2d_1^2(v/H) + n_1d_1(u/G)d_1^2(v/H) - (n_1n_2-1)d_1(u/G)d_1(v/H) \\ &\quad + n_2d_1^2(u/G)d_1(v/H) + n_1d_1(u/G)d_1^2(v/H) - d_1^2(u/G)d_1^2(v/H)\Big] \\ &= 2m_1n_2^2(n_1n_2-1) - n_2^3M_1(G) - 4n_1n_2m_1m_2 + 2m_2n_2M_1(G) \\ &\quad + 2m_2n_1^2(n_1n_2-1) - 4n_1n_2m_1m_2 - n_1^3M_1(H) + 2m_1n_1M_1(H) \\ &\quad - 4m_1m_2(n_1n_2-1) + 2n_2m_2M_1(G) + 2n_1m_1M_1(H) - M_1(G)M_1(H) \\ &= (4n_2m_2 - n_2^3)M_1(G) + (4n_1m_1 - n_1^3)M_1(H) - M_1(G)M_1(H) \\ &\quad + 2(n_1n_2-1)(n_1^2m_2 + n_2^2m_1) - 4m_1m_2(3n_1n_2-1). \end{aligned}$$

By using equation 2, we get

$$\begin{aligned} LM_3(G \vee H) &= (4n_2m_2 - n_2^3)M_1(G) + (4n_1m_1 - n_1^3)M_1(H) \\ &\quad -M_1(G)M_1(H) + 2\mathfrak{M}_1(n_1n_2 - 1) - 2m_1m_2(4n_1n_2 - 1). \end{aligned}$$

This completes the proof. □

2.4 Symmetric Difference

Definition 2.12([10]) *The symmetric difference $G \oplus H$ of two graphs G and H with disjoint vertex sets and edge sets is the graph with vertex set $V(G) \times V(H)$ in which (u_1, v_1) is adjacent with (u_2, v_2) whenever u_1 is adjacent with u_2 in G or v_1 is adjacent with v_2 in H but not both.*

The symmetric difference is commutative and the number of edges of $G \oplus H$ is $\mathfrak{M}_2$ ([1]) and equal to

$$\mathfrak{M}_2 = n_1^2m_2 + n_2^2m_1 - 4m_1m_2. \tag{3}$$

Lemma 2.13([12]) *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Then,*

(1) $d_1((u,v)/G \oplus H) = n_2d_1(u/G) + n_1d_1(v/H) - 2d_1(u/G)d_1(v/H)$;

(2) $d_2((u,v)/G \oplus H) = (n_1n_2 - 1) - n_2d_1(u/G) - n_1d_1(v/H) + 2d_1(u/G)d_1(v/H)$.

Theorem 2.14 *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively, such that G or H not a complete graph. Then,*

$$\begin{aligned} LM_1(G \oplus H) &= (8n_2m_2 - n_2^3)M_1(G) \\ &\quad +(8n_1m_1 - n_1^3)M_1(H) - 4M_1(G)M_1(H) + 2(n_1n_2 - 1)(\mathfrak{M}_2 - 4m_1m_2). \end{aligned}$$

Proof The proof is similar to the proof of Theorem 2.11. □

2.5 Corona Product

Definition 2.15([10]) *Let G and H be two graphs on disjoint vertex sets with n_1 and n_2 vertices, respectively. The corona $G \circ H$ of G and H is defined as the graph obtained by taking one copy of G and n_1 copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.*

It is clear from the definition of $G \circ H$ that $n = |V(G \circ H)| = n_1 + n_1n_2$ and $m = |E(G \circ H)| = m_1 + n_1(n_2 + m_2)$, where m_1 and m_2 are the sizes of G and H, respectively. In the following results, H^j, for $1 \le j \le n_1$, denotes the copy of a graph H which joining to a vertex v_j of a graph G. Note that in general this operation is not commutative.

Lemma 2.16([12]) *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Assume that $1 \le j \le n$, then,*

$$(1)\quad d_1(v/(G\circ H)) = \begin{cases} d_1(v/G)+n_2, & if\ v\in V(G), \\ d_1(v/H)+1, & if\ v\in V(H). \end{cases};$$

$$(2)\quad d_2(v/(G\circ H)) = \begin{cases} d_2(v/G)+n_2d_1(v/G), & if\ v\in V(G), \\ d_1(v_j/G)+n_2-1+d_1(v/H^j), & if\ v\in V(H^j). \end{cases}.$$

Theorem 2.17 *Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Assuming that $1 \le j \le n$, then*

$$\begin{aligned} LM_3(G\circ H) &= LM_3(G)+n_1(M_1(G)-M_1(H))+2n_1n_2(m_1+m_2) \\ &\quad -4n_1m_2+2n_2m_1+n_1n_2(n_2-1)+4m_1m_2+n_1\sum_{v\in V(G)} d_2(v/G). \end{aligned}$$

Proof Let G and H be two graphs with n_1 and n_2 vertices and m_1 and m_2 edges, respectively. Assuming that $1 \le j \le n$, then by Lemma 2.16 we get

$$\begin{aligned} LM_3(G\circ H) &= \sum_{v\in V(G\circ H)} d_1(v/G\circ H)d_2(v/G\circ H) \\ &= \sum_{v\in V(G)} d_1(v/G\circ H)d_2(v/G\circ H) + \sum_{j=1}^{n_1}\sum_{v\in V(H^j)} d_1(v/G\circ H)d_2(v/G\circ H) \end{aligned}$$

$$\begin{aligned} LM_3(G\circ H) &= \sum_{v\in V(G)} \big[(d_1(v/G)+n_2)(d_2(v/G)+n_2d_1(v/G))\big] \\ &\quad +\sum_{j=1}^{n_1}\sum_{v\in V(H^j)} \big[(d_1(v/H)+1)(d_1(v_j/G)+n_2-1-d_1(v/H^j))\big] \\ &= \sum_{v\in V(G)} \big[d_1(v/G)d_2(v/G)+n_2d_1(v/G)^2+n_1(d_2(v/G)+n_1n_2d_1(v/G)\big] \\ &\quad +\sum_{j=1}^{n_1}\sum_{v\in V(H^j)} \big[d_1(v/H)d_1(v_j/G)+(n_2-1)d_1(v/H)+d_1(v/H^2)+d_1(v_j/G) \\ &\quad +n_2-1-d_1(v/H)\big] \\ &= LM_3(G)+n_1M_1(G)+n_1\sum_{v\in V(G)} d_2(v/G)+2n_1n_2m_1+2m_2(n_2-1) \\ &\quad +\sum_{j=1}^{n_1}\big[2m_2d_1(v_j/G)-M_1(H)+n_2d_2(v_j/G)+n_2(n_2-1)-2m_2\big] \\ &= LM_3(G)+n_1M_1(G)+n_1\sum_{v\in V(G)} d_2(v/G)+2n_1n_2m_1+2n_1m_2(n_2-1) \\ &\quad -n_1M_1(H)+2n_2m_1+n_1n_2(n_2-1)-2n_1m_2 \\ &= LM_3(G)+n_1(M_1(G)-M_1(H))+2n_1n_2(m_1+m_2)-4n_1m_2+2n_2m_1 \\ &\quad +n_1n_2(n_2-1)+4m_1m_2+n_1\sum_{v\in V(G)} d_2(v/G). \end{aligned}$$

□

References

[1] A. R. Ashrafi, T. Doslic and A. Hamzeh, The Zagreb coindices of graph operations, *Discrete Appl. Math.*, 158 (2010), 1571-1578.

[2] B. Basavanagoud and E. Chitra, On the leap Zagreb indices of generalized xyz-point-line transformation graphs $T^{xyz}(G)$ when $z = 1$, *Int. J. Math. Combin.*, 2 (2018), 44-66.

[3] B. Basavanagoud and P. Jakkannavar, Computing first leap Zagreb index of some nano structures, *Int. J. Math. And Appl.*, 6(2-B)(2018), 141-150.

[4] B. Borovicanin, K. C. Das, B. Furtula and I. Gutman, Bounds for Zagreb indices, *MATCH Commun. Math. Comput. Chem.*, 78(1) (2017), 17-100.

[5] K. C. Das and I. Gutman, Some properties of the second Zagreb index, *MATCH Commun. Math. Comput. Chem.*, 52 (2004), 103-112.

[6] I. Gutman, E. Milovanović, and I. Milovanović, Beyond the Zagreb indices, *AKCE Int. J. Graphs Combin.*, 15 (2018).

[7] I. Gutman, B. Ruscic, N. Trinajstic and C. F. Wilcox, Graph theory and molecular orbitals, *XII. Acyclic polyenes. J. Chem. Phys.*, 62 (1975), 3399-3405.

[8] I. Gutman and N. Trinajstic, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.*, 17 (1972), 535-538.

[9] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading, Mass. Menlo Park, Calif. London, 1969.

[10] W. Imrich and S. Klavzar, *Product Graphs: Structure and Recognition*, John Wiley & Sons, New York, USA, 2000.

[11] A. M. Naji, N. D. Soner and I. Gutman, On leap Zagreb indices of graphs, *Communi. Combin. Optim.*, 2(2) (2017), 99-117.

[12] A.M. Naji and N.D. Soner, The first leap Zagreb index of some graph operations, *Int. J. Appl. Graph Theory*, 2(2) (2018), 7-18.

[13] S. Nikolic, G. Kovacevic, A. Milicevic and N. Trinajstic, The Zagreb indices 30 years after, *Croat. Chem. Acta*, 76 (2003), 113-124.

[14] M. Pavithra, B. Sharada, M. I. Sowaity and A. M. Naji, Eccentric harmonic index for the subdivision of some graphs, *Proceedings Jangjeon Math. Soc.*, 23(2) (2020), 159-168.

[15] Z. Shao, I. Gutman, Z. Li, S. Wang and P. Wu, Leap Zagreb indices of trees and unicyclic graphs, *Communi. Combin. Optim.*, 3(2) (2018), 179-194.

[16] N. D. Soner and A. M. Naji, The k-distance neighborhood polynomial of a graph, *Int. J. Math. Comput. Sci. WASET Conference Proceedings*, San Francico, USA, Sep 26-27, 3(9) (2016), part XV 2359-2364.

[17] M. I. Sowaity, M. Pavithra, B. Sharada and A. M. Naji, Eccentric harmonic index of a graph, *Arab Journal of Basic and Applied Sciences*, 26.1 (2019), 497-501.

[18] K. Xu and H. Hua, A unified approach to extremal multiplicative Zagreb indices for trees, unicyclic and bicyclic graphs, *MATCH Commun. Math. Comput. Chem.*, f68 (2012), 241-256.

[19] S. Yamaguchi, Estimating the Zagreb indices and the spectral radius of triangle- and quadrangle-free connected graphs, *Chem. Phys. Letters*, 458 (2008), 396-398.

International J.Math. Combin. Vol.4(2020), 89-99

K-Banhatti Indices for Special Graphs and Vertex Gluing Graphs

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Abstract: The K-Banhatti indices was introduced by Kulli in 2016, defined as

$$B_1(G) = \sum_{ue}[d_G(u) + d_G(e)] \quad \text{and} \quad B_2(G) = \sum_{ue} d_G(u).d_G(e),$$

where ue means that the vertex u and edge e are incident and $d_G(e)$ denotes the degree of the edge e in G. In this paper, we formulate general formula for certain graphs.

Key Words: Indices, homeomorphism, graphs, bridge.

AMS(2010): 05C10, 97K30.

§1. Introduction

Topological indices is an useful tool to model physical and chemical properties of molecules to design pharmacologically active compounds, to recognize environmentally hazardous materials [1]. Applications see [7, 9, 10, 4].

Let $G(V, E)$ be a connected graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u. The edge connecting the vertices u and v will be denoted by uv. Let $d_G(e)$ denote the degree of an edge $e = uv$ in G, which is defined by $d_G(e) = d_G(u) + d_G(v) - 2$. The vertices and edges of a graph are said to be its elements [3].

The first and second Banhatti index were introduced by Kulli [2, 5] and are defined as below

$$B_1(G) = \sum_{ue}[d_G(u) + d_G(e)] \quad \text{and} \quad B_2(G) = \sum_{ue} d_G(u).d_G(e).$$

where ue means that the vertex u and edge e are incident in G.

In this paper, we studied the K-Banhatti indices of some special graphs as well as a vertex gluing of graphs by establishing general formula.

§2. Basic Definitions

A K_4-homeomorphic graph/K_4-homeomorph as $K_4(e_1, e_2, e_3, e_4, e_5, e_6)$ is the graph obtained

[1]Received August 20, 2020, Accepted November 25, 2020.

when the six edges of a complete graph with four vertices of (K_4) are subdivided edge is called a path and its length is the number of resulting segments (see Fig.1 for details).

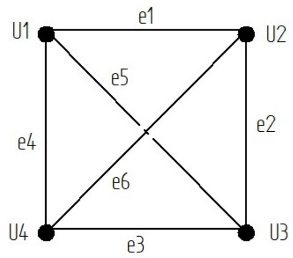

Fig.1 K_4-homeomorphic graph

A complete bipartite graph is a simple bipartite graph with partite sets U_1 and U_2, where every vertex in U_1 is adjacent with all the vertices in U_2. If $|U_1| = m$ and $|U_2| = n$, then such complete bipartite graph is denoted by $K_{m,n}$ [or $K(m, n)$]. So $K_{m,n}$ has order $m + n$ and size mn (see Fig.2 for details).

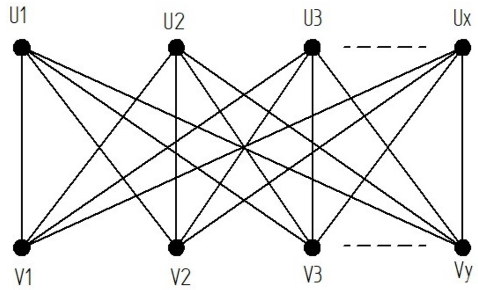

Fig.2 A complete bipartite $K_{x,y}$

A graph consisting of r paths joining two vertices is called an r-bridge graph, which is denoted by $T(e_1, e_2, \cdots, e_r)$, where $e_1, e_2, \cdots, e_r$ are the lengths of r paths. Clearly, an r-bridge graph is a generalized polygon tree (see Fig.3 following).

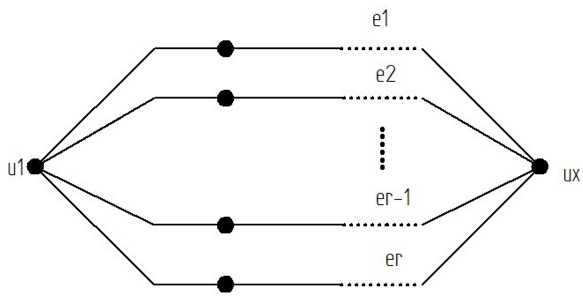

Fig.3 An r-bridge graph

A web graph $Web(r, s)$ is the graph obtained from the Cartesian product of the cycle C_r and the path P_s (see Fig.4).

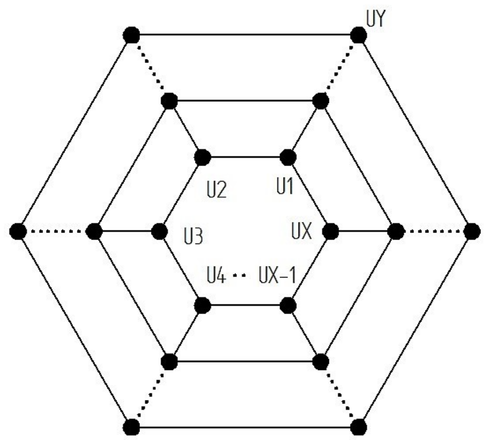

Fig.4 A web graph $Web(x, y)$

§3. K-Banhatti Indices of Some Special Graphs

This section demonstrates general formulas obtained for some special graphs.

Theorem 3.1 *Let $e_1, e_2, e_3, e_4, e_5, e_6$ be positive integers, then the K-Banhatti indices of a K_4-homeomorphism graph denoted by $K_4(e_1, e_2, e_3, e_4, e_5, e_6)$ will be as follows:*

(i) If e_1 or/and e_2 or/and e_3 or/and e_4 or/and e_5 or/and $e_6 = 1$, then the first and second Banhatti index to any one of them is, 14 and 24 respectively;

(ii) If e_1 or/and e_2 or/and e_3 or/and e_4 or/and e_5 or/and $e_6 \neq 1$ then the first and second Banhatti index to any one of them is, (number of edges) 11 and (number of edges) 15 respectively.

Proof (i) If e_1 or/and e_2 or/and e_3 or/and e_4 or/and e_5 or/and $e_6 = 1$, then any one of them will have one edge and two vertices with the same degree three. Thus the first and second Banhatti index to any one of them is 14 and 24 respectively.

(ii) If e_1 or/and e_2 or/and e_3 or/and e_4 or/and e_5 or/and e_6, then any one of them will have two or more edges and each of them will have two vertices in which at least one of the vertices is of degree two. Thus, the first and second Banhatti index to any one of them is (number of edges) 11 and (number of edges) 15 respectively. □

Example 3.2 Let $e_1, e_2, e_3, e_4, e_5, e_6$ be positive integers, the K-Banhatti indices of a K_4-homeomorphism graph denoted by $K_4(e_1, e_2, e_3, e_4, e_5, e_6)$ is,

$$B_1[K_4(e_1, e_2, e_3, e_4, e_5, e_6)] = \begin{cases} 11\sum_{i=1}^{6} e_i & \text{if } e_i \neq 1, 1 \leq i \leq 6 \\ 84 & \text{if } \sum_{i=1}^{6} e_i = 1 \\ 52 + (e_4 + e_5 + e_6)11 & \text{if } e_1 = e_2 = e_3 = 1, e_4 = e_5 = e_6 \neq 1. \end{cases} \quad (3.1)$$

$$B_2[K_4(e_1,e_2,e_3,e_4,e_5,e_6)] = \begin{cases} 15\sum_{i=1}^{6} e_i & \text{if } e_i \neq 1, 1 \leq i \leq 6 \\ 144 & \text{if } e_i \neq 1, 1 \leq i \leq 6 = 1 \\ 72 + (e_4+e_5+e_6)15 & \text{if } e_1 = e_2 = e_3 = 1, e_4 = e_5 = e_6 \neq 1. \end{cases} \quad (3.2)$$

Theorem 3.3 *Let m, n be positive integers. The first and second Banhatti index of a complete bipartite graph denoted by $K_{m,n}$ is,*

$$B_1[K_{m,n}] = mn[3m+3n-4], \quad B_2[K_{m,n}] = mn(m+n)(m+n-2).$$

Proof In complete bipartite graph having mn number of edges each one of them has two vertices that have same degree which has the first vertex of degree m and the second vertex of degree n. Hence by the definitions of first and second Banhatti index, we get that

$$B_1[K_{m,n}] = mn[3m+3n-4], \quad B_2[K_{m,n}] = mn(m+n)(m+n-2).$$

This completes the proof. □

Theorem 3.4 *Let k be a positive integer, The first and second Banhatti index of a k-bridge graph denoted by $T(e_1, e_2, \cdots, e_k)$ is,*

$$B_1[T(e_1,e_2,\cdots,e_k)] = (e_1+e_2+\cdots+e_k)8, \quad B_2[T(e_1,e_2,\cdots,e_k)] = (e_1+e_2+\cdots+e_k)8.$$

Proof This result is proving by mathematical induction. Let $K = 2$, then $G = T(e_1, e_2)$, whose graph is shown in Fig.5.

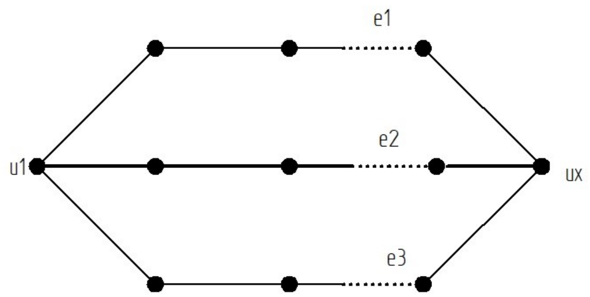

Fig.5

Thus,

$$B_1[T(e_1,e_2)] = (e_1+e_2)[3(2)+3(2)-4] = (e_1+e_2)8,$$
$$B_2[T(e_1,e_2)] = (e_1+e_2)[(2+2)^2-2(2+2)] = (e_1+e_2)8.$$

Hence, it is true for $k = 2$.

Let us assume that the result is true for $k = r$.

$$B_1[T(e_1, e_2 \cdots e_r)] = 8(e_1 + e_2 + \cdots + e_r),$$
$$B_2[T(e_1, e_2 + \cdots + e_r)] = 8(e_1 + e_2 + \cdots + e_r).$$

Now, to prove that the result is true for $k = r + 1$. Let us consider a graph with $r + 1$ bridges such as those shown in Fig.6

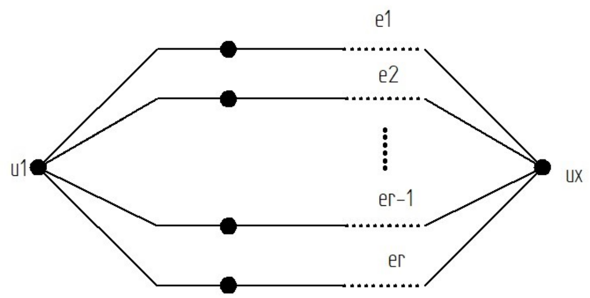

Fig.6

where e_i denotes the position of the edges of graph $T(e_1, e_2, \cdots, e_r)$ at the i^{th} position. The graph H is the path which contains endings V_1 and V_2 and e_{r+1} is the number of edges in H as follows (see Fig.7 for details).

Fig.7

Connect the graph $T(e_1, e_2, \cdots, , e_r)$ with the graph H such that $V_1 = U_1$ and $V_2 = U_2$. the vertices $V_1 = U_1$ and $V_2 = U_2$ are of degree $r + 1$, as shown in Fig.8 following.

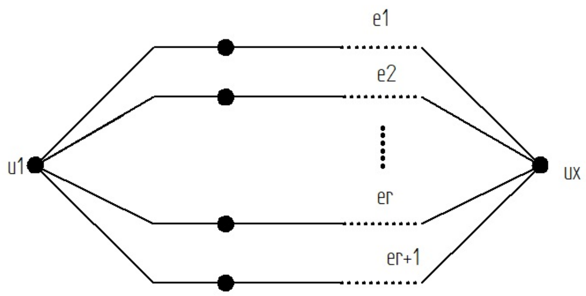

Fig.8

Thus,

$$\begin{aligned} B_1(T_{r+1}) &= B_1(T_r) + B_1(H) \\ &= 8(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) + 8e_{r+1} = (e_1 + e_2 + \cdots + e_{r+1})8. \\ B_2(T_{r+1}) &= B_2(T_r) + B_2(H) \\ &= 8(e_1 + e_2 + e_3 + e_4 + e_5 + e_6) + 8e_{r+1} = (e_1 + e_2 + \cdots + 8e_{r+1}). \end{aligned}$$

Therefore, the result is also true for $k = r + 1$.

Hence, the result is true for all k by the induction principle.

$$B_1(T_{r+1}) = 8(e_1 + e + 2 + \cdots + e_r) = B_2(T_{r+1}).$$ □

§4. K.Banhatti Indices of Certain Vertex Gluing Graphs

This section contains the general formulas for first and second Banhatti index of certain vertex gluing graphs. Let K_4^2- homeomorphism be a graph obtained from two different K_4- homeomorphism graphs $K_4(e_1, e_2, e_3, e_4, e_5, e_6)$ and $K_4(e_7, e_8, e_9, e_{10}, e_{11}, e_{12})$ with one common vertex U_1 (vertex gluing of graph) (see Fig.9 for details).

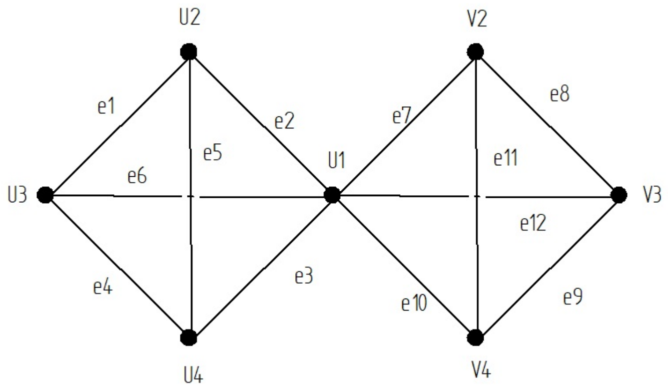

Fig.9 A graph K_4^2 - homeomorphism

Theorem 4.1 *If e_i be a positive integer such that $1 \leq i \leq 12$, then the first and second Banhatti index of K_4^2-homeomorphism graph are respectively*

(1) *If $e_i = 1$ then $B_1(e_i) = 14$, $B_2(e_i) = 24$ for $i = 1, 4, 5, 8, 9, 11$;*

(2) *If $e_i \geq 2$ then $B_1(e_i) = 11$, $B_2(e_i) = 15$ for $1 \leq i \leq 12$;*

(3) *If $e_i = 1$ then $B_1(e_i) = 23, B_2(e_i) = 63$ for $i = 2, 3, 6, 7, 10, 12$,*

then,

$$B_1(K_4^2 - homeomorphism) = \sum_{i=1}^{12} B_1(e_i)$$

$$B_2(K_4^2 - homeomorphism) = \sum_{i=1}^{12} B_2(e_i)$$

Proof The proof is divided into three cases following.

Case 1. If $e_i = 1$ then $B_1(e_i) = 14$ and $B_2(e_i) = 24$, $i = 1, 4, 5, 8, 9, 11$ and any edge e_i has two vertices having the same degree three, then

$$B_1(e_i) = 3(3) + 3(3) - 4 = 14,$$
$$B_2(e_i) = (3+3)^2 - 2(3+3) = 24.$$

Case 2. If $e_i \geq 2$, then

$$B_1(e_i) = 11 \text{ and } B_2(e_i) = 15, 1 \leq i \leq 12$$

and all edges in this case has at least one vertex of degree two, then

$$B_1(e_i) = 3(3) + 3(2) - 4 = 1$$
$$B_2(e_i) = (3+2)^2 - 2(3+2) = 15.$$

Case 3. If $e_i = 1$ then $B_1(e_i) = 23$ and $B_2(e_i) = 63$, $i = 2, 3, 6, 7, 10, 12$, and all edges in this case have two vertices, the first one of degree three and second one of degree six. Then

$$B_1(e_i) = 3(3) + 3(6) - 4 = 23$$
$$B_2(e_i) = (3+6)^2 - 2(3+6) = 63.$$

This completes the proof. □

Let u_1-gluing of complete bipartite graph be a graph obtained from two different complete bipartite graphs $K_{x,y}$ and $K_{p,q}$ with common one vertex u_1 denoted by $K^{p;q}_{x,y}(u_1)$, a vertex gluing of graph (see Fig.10 for details).

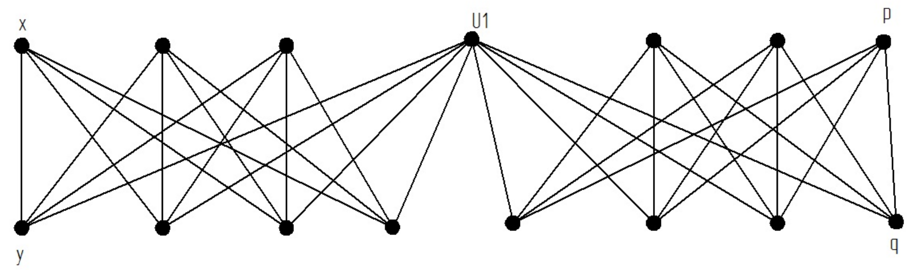

Fig.10 A u_1 - gluing of complete bipartite graph $K^{y;q}_{x,p}(u_1)$

Theorem 4.2 *Let x, y, p and q be positive integers. The first and second Banhatti index of the u_1- gluing of complete bipartite graph $K^{p;q}_{x,y}(u_1)$ is*

(*i*) $B_1[K^{p;q}_{x,y}(u_1)] = y(x-1)(3x+3y-4) + y(3x+3y+3q-4) + q(3p+3y+3q-4) + q(p-1)(3p+3q-4)$;

(*ii*) $B_2[K^{p;q}_{x,y}(u_1)] = y(x-1)(x+y)(x+y-2) + y(x+y+q)(x+y+q-2) + q(y+p+q)(y+p+q-2) + q(p-1)(p+q)(p+q-2)$.

Proof We consider two cases following.

Case 1. In complete bipartite graph $K_{x,y}$ there are xy edges. $y(x-1)$ of them are incident on two vertices of degree x and y. The remaining y will incidents on two vertices of degree x and $(y+q)$.

Case 2. In complete bipartite graph $K_{p,q}$ there are pq edges. $q(p-1)$ of them will incidents on two vertices of degree p and q. The remaining q will incidents on two vertices of degree p and $(y+q)$.

From Cases 1 and 2 we get that

$$\begin{aligned} B_1[K_{x,y}^{p,q}(u_1)] &= y(x-1)(3x+3y-4) \\ &\quad +y(3x+3y+3q-4)+q(3p+3y+3q-4)+q(p-1)(3p+3q-4), \\ B_2[K_{x,y}^{p,q}(u_1)] &= y(x-1)(x+y)(x+y-2)+y(x+y+q)(x+y+q-2) \\ &\quad +q(y+p+q)(y+p+q-2)+q(p-1)(p+q)(p+q-2). \end{aligned}$$

This completes the proof. □

Let u_1-gluing of $x,y-$ bridge graph be a graph obtained from two different $k-$bridge graphs T_1 and T_2 with common one vertex u_1 denoted by $K_x^y(u_1)$, a vertex gluing of graph (see Fig.11 for details).

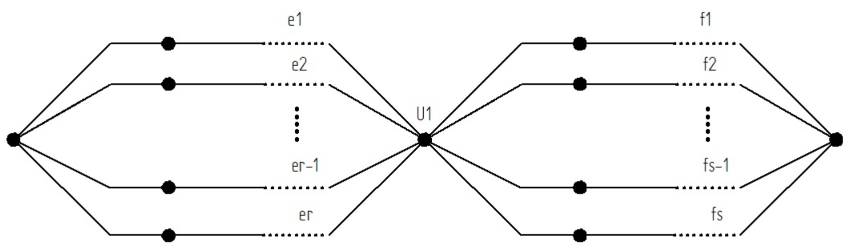

Fig.11 A u_1- gluing of $x,y-$bridge graph $T_x^y(u_1)$

Theorem 4.3 *Let x and y be positive integers. The first and second Banhatti index of the u_1- bridge graph $T_x^y(u_1)$ is*

$$B_1[T_x^y(u_1)] = \sum_{i=1}^{x} e_i(8) + \sum_{j=1}^{y} f_j(8) = B_2[T_x^y(u_1)].$$

Proof We have e_i, $i=1,2,3\cdots x$ and f_j, $j=1,2,3\cdots y$, the numbers of edges, all of them have atleast one vertex of degree two, then

$$B_1[T_x^y(u_1)] = \sum_{i=1}^{x} e_i(8) + \sum_{j=1}^{y} f_j(8) = B_2[T_x^y(u_1)].$$ □

Let u_1-gluing of web graph be a graph obtained from two different web graphs. Web(x,p) and web(y,q) with one common vertex u_1 denoted by $W_{x,p}^{y,q}(u_1)$, a vertex gluing of graph (see Fig.12 for details).

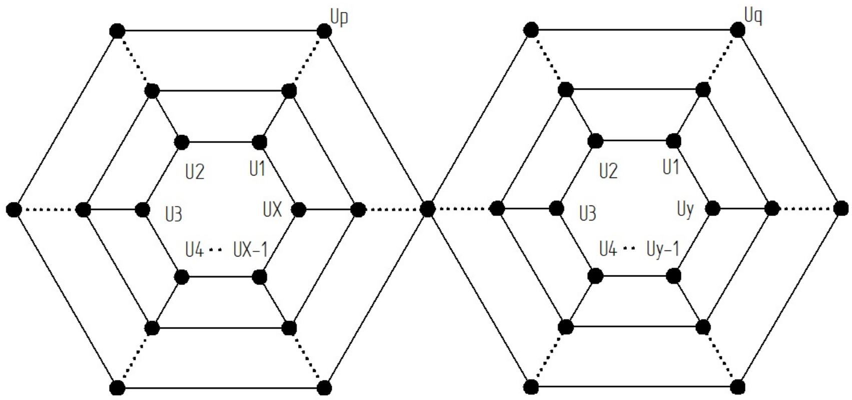

Fig.12 A U_1- gluing of web graph $W_{x,p}^{y;q}(u_1)$

Theorem 4.4 *Let x, p, y and q be positive integers. Then the first and second Banhatti index of the u_1- gluing of Web graph $W_{x,p}^{y;q}(u_1)$ is*

$$B_1[W_{x,p}^{y;q}(u_1)] = \begin{cases} a & if\ p, q = 2 \\ b & if\ p = 2 \\ c & if\ p, q \neq 2. \end{cases} \tag{4.1}$$

where $a = 52(x + y - 2) + 138,\ b = 14(3x + 2y - 5) + 17(2y - 1) + 20y(2q - 5) + 141,\ c = 28(x + y - 2) + 34(x + y - 1) + 20[x(2p - 5) + y(2q - 5)] + 144$ *and*

$$B_2[W_{x,p}^{y;q}(u_1)] = \begin{cases} d & if\ p, q = 2 \\ e & if\ p = 2 \\ f & if\ p, q \neq 2. \end{cases} \tag{4.2}$$

where $d = 72(x + y - 2) + 378,\ e = 24(3x + 2y - 5) + 35(2y - 1) + 48y(2q - 5) + 395$ *and* $f = 48(x + y - 2) + 70(x + y - 1) + 48[x(2p - 5) + y(2q - 5)] + 412.$

Proof We consider three cases and their edge and vertex partition of above web graph as follow.

Case 1. If

(3,3)	(3,6)
3(x+y-2)	6

Then, by definitions of K.Banhatti indices, we get

$$\begin{aligned} B_1[W_{x,p}^{y;q}(u_1)] &= 52(x + y - 2) + 138 \quad \text{and} \\ B_2[W_{x,p}^{y;q}(u_1)] &= 72(x + y - 2) + 378. \end{aligned}$$

Case 2. If

(3,3)	(3,4)	(3,6)	(4,4)	(4,6)
(3x+2y-5)	(2y-1)	5	y(2q-5)	1

Then, by definitions of K.Banhatti indices, we get

$$\begin{aligned} B_1[W^{y,q}_{x,p}(u_1)] &= 14(3x+2y-5)+17(2y-1)+20y(2q-5)+141,\\ B_2[W^{y,q}_{x,p}(u_1)] &= 24(3x+2y-5)+35(2y-1)+48y(2q-5)+395. \end{aligned}$$

Case 3. If

(3,3)	(3,4)	(3,6)	(4,4)	(4,6)
2(x+y-2)	2(x+y-1)	4	x(2p-5)+y(2q-5)	2

Then, by definitions of K.Banhatti indices, we get

$$\begin{aligned} B_1[W^{y,q}_{x,p}(u_1)] &= 28(x+y-2)+34(x+y-1)+20[x(2p-5)+y(2q-5)]+144,\\ B_2[W^{y,q}_{x,p}(u_1)] &= 48(x+y-2)+70(x+y-1)+48[x(2p-5)+y(2q-5)]+412. \end{aligned}$$

Hence, by combining all the three cases we get

$$B_1[W^{y,q}_{x,p}(u_1)] = \begin{cases} a & \text{if } p,q=2 \\ b & \text{if } p=2 \\ c & \text{if } p,q\neq 2. \end{cases}$$

where $a = 52(x+y-2)+138$, $b = 14(3x+2y-5)+17(2y-1)+20y(2q-5)+141$, $c = 28(x+y-2)+34(x+y-1)+20[x(2p-5)+y(2q-5)]+144$ and

$$B_2[W^{y,q}_{x,p}(u_1)] = \begin{cases} d & \text{if } p,q=2 \\ e & \text{if } p=2 \\ f & \text{if } p,q\neq 2. \end{cases}$$

where $d = 72(x+y-2)+378$, $e = 24(3x+2y-5)+35(2y-1)+48y(2q-5)+395$ and $f = 48(x+y-2)+70(x+y-1)+48[x(2p-5)+y(2q-5)]+412$. □

§5. Conclusions

Here, the general formula for K.Banhatti indices of certain graphs namely K_4- homeomorphism, complete bipartite, k-bridge graphs and vertex gluing of graphs are established.

Acknowledgments

We are thankful to the referee/editor for his valuable suggestions.

References

[1] M.V.Diudea, I.Gutman, L.Jntschi, *Molecular Topology*, Nova Huntington, 2002.

[2] I.Gutman, V.R.Kulli et.al, On Banhatti and Zagreb indices, *Journal of Virtul Institute*, Vol.7 (2017), 53-67.

[3] I.Gutman, V.R.Kulli, B.Chaluvaraju, On Banhatti and Zagreb indices, *Mathematical virtual Institute*, 7(2017), 53-67.

[4] Harisha, P.S. Ranjini, V. Lokesha, K-Banhatti Indices, K-Hyper Banhatti indices, forgotten index, first hyper Zagreb index of generalized transformation graphs, *Electronic Journal of Mathematical Analysis and Applications*, 9(1) (2021) 334-344 (Appearing).

[5] V.R.Kulli, On K.Banhatti indices of graphs, *J.Comput Math.Sci.*, 7(2016), 213-218.

[6] V.R.Kulli, On K hyper-Banhatti indices and coindices of graphs, *Int. Res.J.Pure Algebra*, 6(2016), 300-304.

[7] B.Liu, I.Gutman, On general Randic indices, *MATCH Commun. Math. Comput. Chem.*, 58(2007) 147-154.

[8] Mohanad A.Mohammed et.al, The atom bond connectivity index of certain graphs, *International Journal of Pure and Applied Math.*, Volume 106 (2)(2016) 415-427.

[9] P.S.Ranjini, V.Lokesha and M.A.Rajan, On the Zagreb indices of the line graphs of the subdivision graphs, *Applied Mathematics and Computation*, (2010)218(3): 699-702.

[10] M.Randic, On characterization of molecular branching, *J.Am. Chem. Soc.*, 97(1975) 6609-6615.

[11] R.Todeschini and V.Consonni, *Molecular Descriptor for Chemoinformatics*, Wiley-VCH, Weinheim, 2009.

International J.Math. Combin. Vol.4(2020), 100-107

On Right Distributive Torian Algebras

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Abstract: Torian algebras were introduced in [7]. In this paper, torian algebras $(X;*,0)$ which satisfy the condition $(y*z)*x=(y*x)*(z*x)$ for all $x,y,z\in X$ (called right distributive torian algebras) are studied. Their properties are investigated. It is shown that every right distributive torian algebra fixes its zero element. Moreover, necessary and sufficient conditions for a torian algebra to be right distributive are also presented.

Key Words: Torian algebras, right distributivity, Smarandachely torian algebra.

AMS(2010): 20N02, 20N05, 06F35.

§1. Introduction

In recent times, the study of algebras of type (2,0) has generated interest among mathematicians. Kim and Kim, in [1] introduced the notion of BE-algebras. In [2] and [3], Ahn and So introduced the notions of ideals and upper sets in BE-algebras and investigated related properties. In [6] and [7], Ilojide introduced the notions of obic algebras and torian algebras. The notion of ideals in torian algebras was also introduced and studied in [8]. In this paper, torian algebras $(X;*,0)$ which satisfy the condition $(y*z)*x=(y*x)*(z*x)$ for all $x,y,z\in X$ (called right distributive torian algebras) are studied. Their properties are investigated. It is shown that every right distributive torian algebra fixes its zero element. Moreover, necessary and sufficient conditions for a torian algebra to be right distributive are also presented.

§2. Preliminaries

Definition 2.1([6]) *A triple $(X;*,0)$; where X is a non-empty set, $*$ a binary operation on X, and 0 a constant element of X is called an obic algebra if the following axioms hold for all $x,y,z\in X$:*

(1) $x*0=x$;
(2) $[x*(y*z)]*x=x*[y*(z*x)]$;
(3) $x*x=0$.

Example 2.1([6]) Consider the multiplicative group $G=\{1,-1,i,-i\}$. Define a binary operation $*$ on G by $a*b=ab^{-1}$. Then $(G;*,1)$ is an obic algebra.

[1]Received July 13, 2020, Accepted November 27, 2020.

Lemma 2.1([6]) *Let X be an obic algebra. Then for all $x, y \in X$, the following hold:*

$$x * y = [x * (y * x)] * x.$$

Definition 2.2([7]) *An obic algebra X is called torian if $[(x * y) * (x * z)] * (z * y) = 0$ for all $x, y, z \in X$. Otherwise, if there are $x, y, z \in X$, such that $[(x * y) * (x * z)] * (z * y) \neq 0$, such an obic algebra X is called Smarandachely torian.*

Lemma 2.2([7]) *Let X be a torian algebra. Then the following hold for all $x, y, z \in X$:*

$$(x * y) * z = (x * z) * y.$$

Definition 2.3([7]) *Let X be a torian algebra. An element $x \in X$ is said to fix 0 if $0 * x = 0$. If every element in X fixes 0, then X is said to fix 0.*

Lemma 2.3([7]) *Let X be a torian algebra. Define the relation $\sim$ on X by $x \sim y \Leftrightarrow x * y = 0$ for all $x, y \in X$. Then $(X; \sim)$ is a partially ordered set.*

Lemma 2.4([8]) *Let X be a torian algebra with the partial ordering $\sim$. Then, $[(x*y)*(z*y)] \sim (x * z)$ for all $x, y, z \in X$.*

Definition 2.4([7]) *A torian algebra X is called a weak property torian algebra (WPTA) if $x * y = 0$ and $y * x = 0$ imply that $x = y$ for all $x, y \in X$.*

Proposition 2.1([7]) *Let X be a WPTA. Then for all $x, y, z \in X$, the following hold:*

$$x * [x * (x * y)] = x * y.$$

Lemma 2.5 *Let X be a torian algebra with partial ordering $\sim$. Then $(x * y) \sim z \Leftrightarrow (x * z) \sim y$ for all $x, y, z \in X$.*

From now on, X will denote a weak property torian algebra.

§3. Main Results

Definition 3.1 *Let X be a torian algebra. An element $x \in X$ is said to be right distributive in X if $(y * z) * x = (y * x) * (z * x)$ for all $y, z \in X$.*

Example 3.1 For any torian algebra X, 0 is right distributive in X.

Remark 3.1 If every element in a torian algebra X is right distributive in X, then X is said to be a right distributive torian algebra.

The following Lemma follows from definition.

Lemma 3.1 *Let X be a right distributive toran algebra. Then the following hold for all*

$x, y, z \in X$:

(1) $(0 * z) * x = (0 * x) * (z * x)$;
(2) $y * x = (y * x) * (0 * x)$;
(3) $0 * x = 0$;
(4) $(x * z) * x = 0 * (z * x)$;
(5) $0 * z = 0 * (z * x)$;
(6) $(y * z) * z = y * z$;
(7) $(y * x) * z = (y * x) * (z * x)$;
(8) $[(0 * x) * z] * x = [(0 * x) * x] * (z * x)$;
(9) $(y * x) = (y * x) * [(0 * x) * x]$;
(10) $(x * z) * x = (0 * x) * (z * x)$;
(11) $(0 * x) * z = (0 * x) * (z * x)$;
(12) $(x * z) * x = 0$.

Proposition 3.1 *Let X be a right distributive torian algebra. Then the following hold for all $x, y, z \in X$:*

(1) $(0 * x) * [[z * (x * z)] * z] = (0 * z) * x$;
(2) $[y * (x * y)] * y = [[y * (x * y)] * y] * (0 * x)$;
(3) $[[x * (z * x)] * x] * x = 0 * [[z * (x * z)] * z]$;
(4) $0 * z = 0 * [[z * (x * z)] * z]$;
(5) $[[y * (z * y)] * y] * z = [y * (z * y)] * y$;
(6) $[[y * (x * y)] * y] * z = [[y * (x * y)] * y] * [z * (x * z)] * z$;
(7) $[(0 * x) * x] * [[z * (x * z)] * z] = [(0 * x) * z] * x$;
(8) $[y * (x * y)] * y = [[y * (x * y)] * y] * [(0 * x) * x]$;
(9) $[[x * (z * x)] * x] * x = (0 * x) * [[z * (x * z)] * z]$;
(10) $(0 * x) * z = (0 * x) * [[z * (x * z)] * z]$;
(11) $[[x * (z * x)] * x] * x = 0$.

Proof The proof follows from Lemmas 2.1 and 3.1. □

Proposition 3.2 *Let X be a right distributive torian algebra. Then the following hold for all $x, y, z \in X$:*

(1) $(0 * x) * [z * [z * (z * x)]] = (0 * z) * x$;
(2) $y * [y * (y * x)] = [y * [y * (y * x)]] * (0 * x)$;
(3) $[x * [x * (x * z)]] * x = 0 * [z * [z * (z * x)]]$;
(4) $0 * z = 0 * [z * [z * (z * x)]]$;
(5) $[y * [y * (y * z)]] * z = y * [y * (y * z)]$;
(6) $[y * [y * (y * x)]] * z = [y * [y * (y * x)]] * [z * [z * (z * x)]]$;
(7) $[(0 * x) *] * [z * [z * (z * x)]] = [(0 * x) * z] * x$;
(8) $y * [y * (y * x)] = [y * [y * (y * x)]] * [(0 * x) * x]$;
(9) $[x * [x * (x * z)]] * x = (0 * x) * [z * [z * (z * x)]]$;
(10) $(0 * x) * [z * [z * (z * x)]] = (0 * x) * z$;

(11) $[x * [x * (x * z)]] * x = 0$.

Proof The proof follows from Proposition 2.1 and Lemma 3.1. □

The following proposition follows from Lemma 3.1.

Proposition 3.3 *Every right distributive torian algebra fixes* 0.

Example 3.2 Consider the set $\mathbb{R}$ of real numbers. Define a binary operation $*$ on $\mathbb{R}$ by

$$x * y = \begin{cases} 0, & x \leq y \\ x, & x > y \end{cases}$$

Then, $(\mathbb{R}; *, 0)$ is a right distributive torian algebra.

Theorem 3.1 *Let X be a torian algebra such that $[(x * z) * y] * [(x * z) * (y * z)] = 0$ for all $x, y, z \in X$. Then X is right distributive if and only if $(x * y) * y = x * y$ for all $x, y \in X$.*

Proof Suppose $(x*y)*y = x*y$. Notice that $(x*z)*(y*z) = [(x*z)*z]*(y*z) \sim (x*z)*y$(by Lemma 2.4). So, $[(x * z) * (y * z)] * [(x * z) * y] = 0$. Now, by the hypothesis, we have $(x * z) * y = (x * z) * (y * z)$; giving us $(x * y) * z = (x * z) * (y * z)$ as required.

The converse is obvious from Lemma 3.1(6). The proof is complete. □

Corollary 3.1 *Let X be a torian algebra such that $[[x*(z*x)]*y]*[[[x*[(z*x)]*x]]*[[y*[(z*y)]*y]]] = 0$ for all $x, y, z \in X$. Then X is right distributive if and only if $[x*[(y*x)]*x]*y = [x*(y*x)]*x$ for all $x, y \in X$.*

Proof The proof follows from Theorem 3.1 and Lemma 2.1. □

Corollary 3.2 *Let X be a torian algebra such that $[[x * [x * (x * z)]] * y] * [[x * [x * (x * z)]] * [y * [y * (y * z)]]] = 0$ for all $x, y, z \in X$. Then X is right distributive if and only if $[x * [*(x * y)]] * y = x * [x * (x * y)]$ for all $x, y \in X$.*

Proof The proof follows from Theorem 3.1 and Proposition 2.1. □

Theorem 3.2 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z, p, v \in X$:*

(1) $[x * (y * z)] * [x * (y * p)] \sim (z * p)$;
(2) $x \sim y \Rightarrow (z * y) \sim (z * x)$;
(3) $(x * y) \sim v \Rightarrow (x * v) \sim [x * (x * y)]$;
(4) $[(x * z) * y] * [(x * z) * (y * z)] = 0$.

Then, $[x * [x * [y * (y * x)]]] = [x * (x * y)] * (y * x)$ *for all* $x, y \in X$.

Proof Notice that $[x * (x * y)] * [x * [x * [y * (y * x)]]] \sim [y * [y * (y * x)]] = y * x$. Hence, $[x * (x * y)] * (y * x) \sim [x * [x * [y * (y * x)]]]$. Now let $[x * [y * (y * x)] = v$. Then we have $(x*v) \sim [y*(y*x)]$. Notice that $[y*(y*x)] \sim y$. So, $(x*y) \sim [x*[y*(y*x)]]$; giving us $(x*y) \sim v$;

so that $(x*v) \sim [x*(x*y)]$. Now notice also that $[y*(y*x)] = [y*(y*x)]*(y*x) \sim [x*(y*x)]$. Since $(x*v) \sim [y*(y*x)]$ and $[y*(y*x)] \sim [x*(y*x)]$, we have $(x*v) \sim [x*(y*x)]$.

Now, multiply both sides of the last relation on the right by v to get $[(x*v)*v] \sim [x*(y*x)]*v$. That is, $[(x*v)*v] \sim (x*v)*(y*x)$; giving us $(x*v) \sim [(x*v)*(y*x)]$; leading to $(x*v) \sim [[x*(x*y)]*(y*x)]$. Substituting back for v, we have $[x*[x*[y*(y*x)]]] \sim [x*(x*y)]*(y*x)$. Since $[x*(x*y)]*(y*x) \sim [x*[x*[y*(y*x)]]]$ and $[x*[x*[y*(y*x)]]] \sim [x*(x*y)]*(y*x)$, we conclude that $[x*[x*[y*(y*x)]]] = [x*(x*y)]*(y*x)$ as required. □

Corollary 3.3 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x,y,z,p,v \in X$:*

(1) $[x*[[y*(z*y)]*y]]*[[x*[y*(p*y)]*y]] \sim [[z*(p*z)]*z]$;

(2) $x \sim y \Rightarrow [[z*(y*z)]*z] \sim [[z*(x*z)]*z]$;

(3) $[[x*(y*x)]*x] \sim v \Rightarrow [[x*(v*x)]*x] \sim [x*[[x*(y*x)]*x]]$;

(4) $[[[x*(z*x)]*x]*y]*[[[x*[(z*x)]*x]]*[[y*[(z*y)]*y]]] = 0$.

Then, $[x*[x*[y*[y*(x*y)]*y]]] = [[x*[x*(y*x)]*x]]*[[y*[(x*y)]*x]]$ *for all $x,y \in X$.*

Proof The proof follows from Theorem 3.2 and lemma 2.1. □

Corollary 3.4 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x,y,z,p,v \in X$:*

(1) $[x*[y*[y*(y*z)]]]*[x*[y*[y*(y*p)]]] \sim [z*[z*(z*p)]]$;

(2) $x \sim y \Rightarrow [z*[z*(z*y)]] \sim [z*[z*(z*x)]]$;

(3) $[x*[x*(x*y)]] \sim v \Rightarrow [x*[x*(x*v)]] \sim [x*[x*[x*(x*y)]]]$;

(4) $[[x*[x*(x*z)]]*y]*[[x*[x*(x*z)]]*[y*[y*(y*z)]]] = 0$.

Then, $[x*[x*[y*[y*[y*(y*x)]]]]] = [x*[x*[x*(x*y)]]]*[y*[y*(y*x)]]$ *for all $x,y \in X$.*

Proof The Proof follows from Theorem 3.2 and Proposition 2.1. □

Theorem 3.3 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x,y,z,p,v \in X$:*

(1) $[x*(y*z)]*[x*(y*p)] \sim (z*p)$;

(2) $x \sim y \Rightarrow (z*y) \sim (z*x)$;

(3) $(x*y) \sim v \Rightarrow (x*v) \sim [x*(x*y)]$;

(4) $[(x*z)*y]*[(x*z)*(y*z)] = 0$.

Then $(x*y)*[x*(x*y)] = x*y$ *for all $x,y \in X$.*

Proof From Theorem 3.2, for all $x,y \in X$, we have

$$[x*(x*y)]*(y*x) = [x*[x*[y*(y*x)]]] \tag{1}$$

Put $x*y$ for x, and put x for y in expression (1). Then, the left hand side becomes

$$\begin{aligned}[(x*y)*[(x*y)*x]]*[x*(x*y) &= [(x*y)*[(x*x)*y]*[x*(x*y)\\ &= [(x*y)*(0*y)]*[x*(x*y)\\ &= (x*y)*[x*(x*y)].\end{aligned}$$

Also, the right hand side becomes

$$(x*y)*[(x*y)*[x*[x*(x*y)]]]] = (x*y)*[(x*y)*(x*y)] = x*y.$$

Hence, equating the left and right hand sides, we have $(x*y)*[x*(x*y)] = x*y$ as required. The proof is complete. □

Corollary 3.5 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z, p, v \in X$:*

(1) $[x*[[y*(z*y)]*y]]*[x*[[y*(p*y)]*y]] \sim [[z*(p*z)]*z]$;
(2) $x \sim y \Rightarrow [[z*(y*z)]*z] \sim [[[z*(x*z)]*z]$;
(3) $[[x*(y*x)]*x] \sim v \Rightarrow [[x*(v*x)]*x] \sim [x*[x*(y*x)]*x]$;
(4) $[[[x*(z*x)]*x]*y]*[[[x*(z*x)]*x]*[[y*(z*y)]*y]] = 0$.

Then, $[[x*(y*x)]*x]*[[x*[[x*(y*x)]*x] = [[x*(y*x)]*x]$ *for all* $x, y \in X$.

Proof The proof follows from Theorem 3.3 and Lemma 2.1. □

Corollary 3.6 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z, p, v \in X$:*

(1) $[x*[y*[y*(y*z)]]]*[x*[y*[y*(y*p)]]] \sim [z*[z*(z*p)]]$;
(2) $x \sim y \Rightarrow [z*[z*(z*y)]] \sim [z*[z*(z*x)]]$;
(3) $[x*[x*(x*y)]] \sim v \Rightarrow [x*[x*(x*v)]] \sim [[x*[x*[x*(x*y)]]]]$;
(4) $[[x*[x*(x*z)]]*y]*[[x*[x*(x*z)]]*[y*[y*(y*z)]]] = 0$.

Then, $[x*[x*(x*y)]]*[x*[x*(x*y)]]$ *for all* $x, y \in X$.

Proof The proof follows from Theorem 3.3 and Proposition 2.1. □

Remark 3.2 Let X be a torian algebra. We define $x*y^k = [(x*y)*y]*\cdots]*y$ (k times); where k is a natural number.

Theorem 3.4 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z \in X$:*

(1) $x \sim y \Rightarrow (x*z) \sim (y*z)$;
(2) $x*y^k = x*y^{k+1}$, *where* $k \in \mathbb{N}$; *the set of natural numbers;*
(3) $x*y^k = x*y^l$ *for all* $l \geq k \in \mathbb{N}$;
(4) $(x*z^k)*(y*z^k \sim (x*y)$.

Then, $(x*y)*z^k = (x*z^k) = (x*z^k)*(y*z^k)$ *for all* $x, y, z \in X$.

Proof By hypothesis, we have $x * z^k = x * z^{2k}$. Since, $(x * z^k) * (y * z^k) \sim (x * y)$, we have $[(x * z^k) * (y * z^k)] * z^k \sim (x * y) * z^k$; which gives $[(x * z^k) * z^k] * (y * z^k) \sim (x * y) * z^k$; which results to $(x * z^{2k}) * (y * z^k) \sim (x * y) * z^k$. Since $x * z^k = x * z^{2k}$, we now have

$$(x * z^k) * (y * z^k) \sim (x * y) * z^k \tag{1}$$

Notice that $(y * z^k) * y = 0$. So, $(y * z^k) \sim y$. We therefore have $[(x * z^k) * y] \sim [(x * z^k) * (y * z^k)]$; which gives

$$[(x * y) * z^k] \sim [(x * z^k) * (y * z^k)] \tag{2}$$

By expressions (1) and (2), we have $(x * y) * z^k = (x * z^k) * (y * z^k)$ as required. The proof is complete. □

Proposition 3.4 *Let X be a right distributive torian algebra. If $(x * y) * z^k = (x * z^k) * (y * z^k)$, then $x * z^k = x * z^{k+1}$ for all $x, y, z \in X; k \in \mathbb{N}$.*

Proof By hypothesis, we have $(x * z) * z^k = (x * z^k) * (z * z^k)$, which gives $x * z^{k+1} = x * z^k$ as required. The proof is complete. □

Theorem 3.5 *Let X be a right distributive torian algebra with partial ordering $\sim$ such that the following hold for all $x, y, z \in X$:*

(1) $x \sim y \Rightarrow (x * z) \sim (y * z)$;

(2) $x * y^k = x * y^{k+1}$; *where $k \in \mathbb{N}$, the set of natural numbers;*

(3) $x * y^k = x * y^l$ *for all $l \geq k \in \mathbb{N}$.*

Then, $[y * (y * x)^k] * (x * y)^k = [x * (x * y)^k] * (y * x)^k$ for all $x, y \in X$.

Proof By hypothesis, we have

$$x * (x * y)^{k_1} = x * (x * y)^{k_1} \tag{3}$$

and

$$y * (y * x)^{k_2} = y * (y * x)^{k_2} \tag{4}$$

Let k be the maximum of k_1 and k_2. Then

$$x * (x * y)^k = x * (x * y)^{k+1} \tag{5}$$

and

$$y * (y * x)^k = y * (y * x)^{k+1} \tag{6}$$

Notice that $[x * (x * y)] * y = 0$. So, $x * (x * y) \sim y$ and from expression (5), we have

$$x * [(x * y)^k \sim y * (x * y)^k \tag{7}$$

Now, multiply expression (7) on both sides on the right by $y * x$ (k times) to get

$$[x * (x * y)^k] * (y * x)^k \sim [y * (x * y)^k] * (y * x)^k \tag{8}$$

Now apply Lemma 2.2 to expression (8) to get

$$[x * (x * y)^k] * (y * x)^k \sim [y * (y *)^k] * (x * y)^k \tag{9}$$

Also notice that $[y * (y * x)] * x = 0$. So, $[y * (y * x)] \sim x$; and so from expression (6), we have

$$[y * (y * x)^k] \sim [x * (y * x)^k] \tag{10}$$

Multiply both sides of expression (10) on the right by $x * y$ (k times) to get

$$[y * (y * x)^k] * (x * y)^k \sim [x * (y * x)^k] * (x * y)^k \tag{11}$$

Now apply Lemma 2.2 to expression (11) to get

$$[y * (y * x)^k] * (x * y)^k \sim [x * (x * y)^k] * (y * x)^k \tag{12}$$

From expressions (9) and (12), we have $[y * (y * x)^k] * (x * y)^k$
$= [x * (x * y)^k] * (y * x)^k$ as required. The proof is complete. □

References

[1] H. S. Kim and Y. H. Kim, On BE-algebras, *Sci. Math. Jpn.*, 66(2007), 113–116.

[2] S.S. Ahn and K. S. So, On ideals and upper sets in BE-algebras, *Sci. Math. Jpn.*, 68(2008), 351–357.

[3] S.S. Ahn and K. S. So, On generalized upper sets in BE-algebras, *Bull. Korean Math. Soc.*, 46(2009), 281–287.

[4] R. H. Bruck, *A Survey of Binary Systems*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1966, 185pp.

[5] J. Dene and A. D. Keedwell, *Latin Squares and Their Applications*, the English University press Ltd, 1974, 549pp.

[6] E. Ilojide, On obic algebras, *International J. Math. Combin.*, 4(2019), 80–88.

[7] E. Ilojide, A note on torian algebras, *International J. Math. Combin.*, 2(2020), 80–87.

[8] E. Ilojide, On ideals of torian algebras, *International J. Math. Combin.*, 2(2020), 101–108.

International J.Math. Combin. Vol.4(2020), 108-113

E-Super Arithmetic Graceful Labelling of Some Special Classes of Cubic Graphs Related to Cycles

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Abstract: We introduce a new concept called E-Super arithmetic graceful graphs.A (p, q) - graph G is said to be E-Super arithmetic graceful if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \cdots, p+q\}$ such that $f(E(G)) = \{1, 2, \cdots, q\}$, $f(V(G)) = \{q+1, q+2, \cdots, q+p\}$ and the induced mapping f^* given by $f^*(uv) = f(u) + f(v) - f(uv)$ for $uv \in E(G)$ has the range $\{p+q+1, p+q+2, \cdots, p+2q\}$. In this paper we prove that $W(C_n)$,$D(C_{2n})$, $D_1(C_{2n})$, $D_2(C_{4n})$ are E-Super arithmetic graceful.

Key Words: E-Super arithmetic graceful graph, Smarandachely edge magic, $W(C_n)$, $D(C_{2n})$, $D_1(C_{2n})$, $D_2(C_{4n})$.

AMS(2010): 05C78.

§1. Introduction

Acharya and Hegde [1] have defined $(k, d)-$ arithmetic graphs. Let G be a graph with q edges and let k and d be positive integers.A labelling f of G is said to be $(k, d)-$ arithmetic if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x) + f(y)$ for each edge xy are $k, k+d, k+2d, \cdots, k+(q-1)d$. The case where $k = 1$ and $d = 1$ was called additively graceful by Hegde [3].

A labelling of $G(V, E)$ is said to be E-Super if $f(E(G)) = \{1, 2, 3, \cdots, |E(G)|\}$. A labelling of $G(V, E)$ is said to be E-Super if $f(E(G)) = \{1, 2, 3, \cdots, |E(G)|\}$. Marimuthu and Balakrishnan [5] defined a graph $G(V, E)$ to be edge magic graceful if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \cdots, p+q\}$ such that $|f(u)+f(v)-f(uv)|$ is a constant for all edges uv of G. Otherwise, it is said to be *Smarandachely edge magic*, i.e., $|\{|f(u) + f(v) - f(uv)|, uv \in E(G)\}| \geq 2$.

We introduce a new concept called E-Super arithmetic graceful graphs. We define a graph $G(p, q)$ to be *E-Super arithmetic graceful* if there exists a bijection f from $V(G) \cup E(G)$ to $\{1, 2, \cdots, p+q\}$ such that $f(E(G)) = \{1, 2, \cdots, q\}$, $f(V(G)) = \{q+1, q+2, \cdots, q+p\}$ and the induced mapping f^* given by $f^*(uv) = f(u) + f(v) - f(uv)$ for $uv \in E(G)$ has the range $\{p+q+1, p+q+2, \cdots, p+2q\}$. In this paper, we prove that graphs $W(C_n)$,$D(C_{2n})$, $D_1(C_{2n})$, $D_2(C_{4n})$ are E-Super arithmetic graceful.

[1]Received June 14, 2020, Accepted November 29, 2020.

§2. Preliminaries

Definition 2.1 *Let C_n denote the cycle for $n \geq 3$. Let $W(C_n)$ denote the graph with vertices $\{u_1, u_2, \cdots, u_n\}$ and $\{v_1, v_2, \cdots, v_n\}$ and edges $\{u_iu_{i+1}\}$,$\{u_iv_i\}$ and $\{v_iv_{i+1}\}$ where addition is modulo n.*

$W(C_n)$ is a cubic graph.

Illustration 2.1 The cubic graph $W(C_4)$ is shown in Fig.2.1.

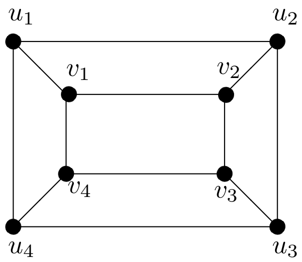

Fig.2.1

Definition 2.2 *Let C_{2n}, $n \geq 2$ denote the even cycle with $2n$ vertices $\{u_1, u_2, \cdots, u_{2n}\}$. By drawing n diagonals suitably we obtain cubic graphs related to even cycles. $D(C_{2n})$ denotes the cubic graph with vertices $\{u_1, u_2, \cdots, u_{2n}\}$ and edges $\{u_iu_{i+1} | i = 1, 2, \cdots, 2n,$ where $u_{2n+1} = u_1\}$ and $\{u_iu_{n+i} | i = 1, 2, \cdots, n\}$, $D(C_{2n})$ has $2n$ vertices and $3n$ edges. Particularly, $D(C_4)$ is the complete graph K_4.*

Illustration 2.2 The cubic graph $D(C_8)$ is shown in Fig.2.2.

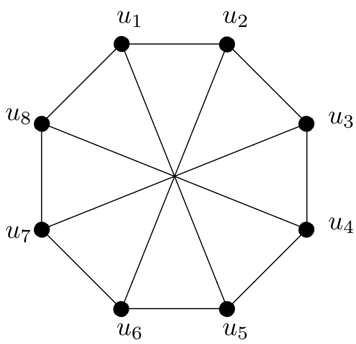

Fig.2.2

Definition 2.3 *$D_1(C_{2n})$ denotes the cubic graph with vertices $\{u_1, u_2, \cdots, u_{2n}\}$ and edges $\{u_iu_{i+1} | i = 1, 2, \cdots, 2n$ where $u_{2n+1} = u_1\}$, u_1u_{n+1} and $\{u_iu_{2n+2-i} | i = 2, 3, \cdots, n\}$. $D_1(C_{2n})$ is a cubic graph with $2n$ vertices and $3n$ edges.*

Illustration 2.3 The cubic graph $D_1(C_6)$ is shown in Fig.2.3.

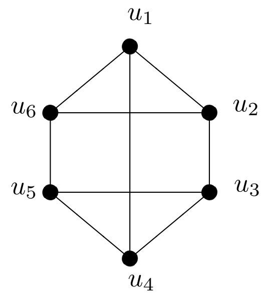

Fig.2.3

Definition 2.4 *$D_2(C_{4n})$ denotes the cubic graph with vertices $\{u_1, u_2, \cdots, u_{4n}\}$ and edges $\{u_iu_{i+1}|i = 1, 2, \cdots, 4n$ where $u_{n+1} = u_1\}$,$\{u_iu_{3n+1+i}|i = 1, 2, \cdots, n\}$ and $\{u_iu_{5n+1-i}|i = n+1, n+2, \cdots, 2n\}$. $D_2(C_{4n})$ has $4n$ vertices and $6n$ edges.*

Illustration 2.4 The cubic graph $D_2(C_{12})$ is shown in Fig.2.4.

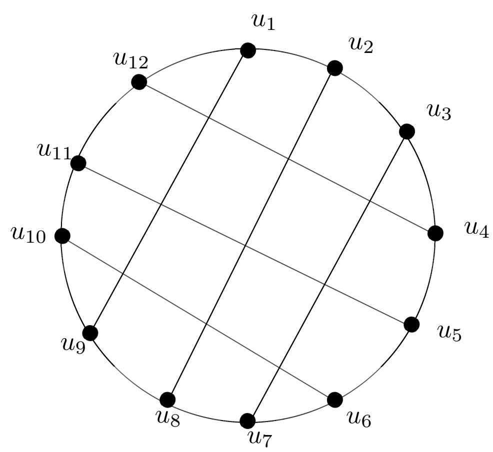

Fig.2.4

§3. Main Results

Theorem 3.1 *$W(C_n)$ is E-Super arithmetic graceful for all $n \geq 3$.*

Proof $W(C_n)$ has $2n$ vertices and $3n$ edges. Define $f : V \cup E \longrightarrow \{1, 2, \ldots, 5n\}$ as follows:

$f(u_i) = 3n + i, \quad i = 1, 2, \cdots, n,$
$f(v_i) = 4n + i, \quad i = 1, 2, \cdots, n,$
$f(u_iu_{i+1}) = n + i, \quad i = 1, 2, \cdots, n$ where $u_{n+1} = u_1$,
$f(u_iv_i) = i, \quad i = 1, 2, \cdots, n,$
$f(v_iv_{i+1}) = 2n + i, \quad i = 1, 2, \cdots, n$ where $v_{n+1} = v_1$.

Clearly, f is a bijection and $f^*(E(W(C_n))) = \{5n + 1, \cdots, 8n\}$. Therefore, $W(C_n)$ is E-Super arithmetic graceful for $n \geq 3$. □

Example 3.2 A E-Super arithmetic graceful labelling of $W(C_5)$ is shown in Fig.3.1.

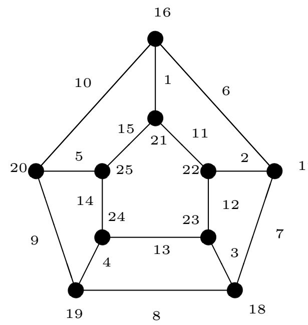

Fig.3.1

Theorem 3.3 *$D(C_{2n})$ is E-Super arithmetic graceful for all $n \geq 2$.*

Proof Let $\{u_1, u_2, \ldots, u_{2n}\}$ be the vertices of $D(C_n)$. Define $f : V \cup E \longrightarrow \{1, 2, \cdots, 5n\}$ as follows:

$f(u_i) = 3n + i, \quad i = 1, 2, \cdots, 2n,$

$f(u_i u_{i+1}) = i, \quad i = 1, 2, \cdots, 2n$ where $u_{2n+1} = u_1$,

$f(u_i u_{n+i}) = 2n + i, \quad i = 1, 2, \cdots, n.$

Clearly, f is a bijection and $f^*(E(D(C_{2n}))) = \{5n + 1, \cdots, 8n\}$. Therefore $D(C_{2n})$ is E-Super arithmetic graceful for $n \geq 2$. □

Example 3.4 An E-Super arithmetic graceful labelling of $D(C_6)$ is shown in Fig.3.2.

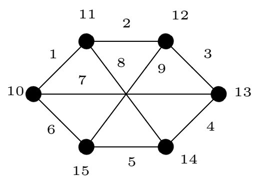

Fig.3.2

Theorem 3.5 *$D_1(C_{2n})$ for $n \geq 3$ is E-Super arithmetic graceful.*

Proof Let $u_1, u_2, \cdots, u_{2n}$ be the vertices of $D_1(C_{2n})$. Define $f : V \cup E \longrightarrow \{1, 2, \ldots, 5n\}$ as follows:

$f(u_i) = 3n + i, \quad i = 1, 2, \cdots, 2n,$

$f(u_i u_{i+1}) = i, \quad i = 1, 2, \cdots, 2n$ where $u_{2n+1} = u_1$,

$f(u_1 u_{n+1}) = 2n + 1,$

$f(u_i u_{2n+2-i}) = 2n + i, \quad i = 2, 3, \cdots, n.$

Clearly, f is a bijection and

$$f^*(E(D_1(C_{2n}))) = \{5n + 1, \cdots, 8n\}.$$

Therefore, $E(D_1(C_{2n}))$ is E-Super arithmetic graceful for $n \geq 3$. □

Example 3.6 An E-Super arithmetic graceful labelling of $D_1(C_8)$ is shown in Fig.3.3.

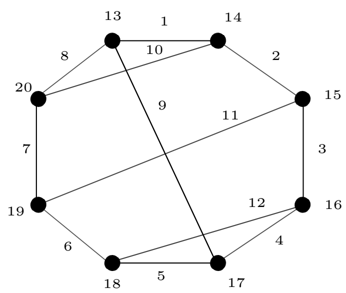

Fig.3.3

Theorem 3.7 *$D_2(C_{4n})$ for $n \geq 2$ is E-Super arithmetic graceful.*

Proof Define $f : V \cup E \longrightarrow \{1, 2, \cdots, 10n\}$ as follows:

$f(u_i) = 6n + i, \quad i = 1, 2, \cdots, 4n,$

$f(u_i u_{i+1}) = i, \quad i = 1, 2, \cdots, 4n$ where $u_{4n+1} = u_1$,

$f(u_i u_{3n+1-i}) = 4n + i, \quad i = 1, 2, \cdots, n,$

$f(u_i u_{5n+1-i}) = 4n + i, \quad i = n + 1, \cdots, 2n.$

Clearly, f is a bijection and

$$f^*(E(D_2(C_{4n}))) = \{10n + 1, 10n + 2, \cdots, 16n\}.$$

Therefore $D_2(C_{4n})$ is E-Super arithmetic graceful for $n \geq 2$. □

Example 3.8 An E-Super arithmetic graceful labelling of $D_2(C_{16})$ is shown in Fig.3.4.

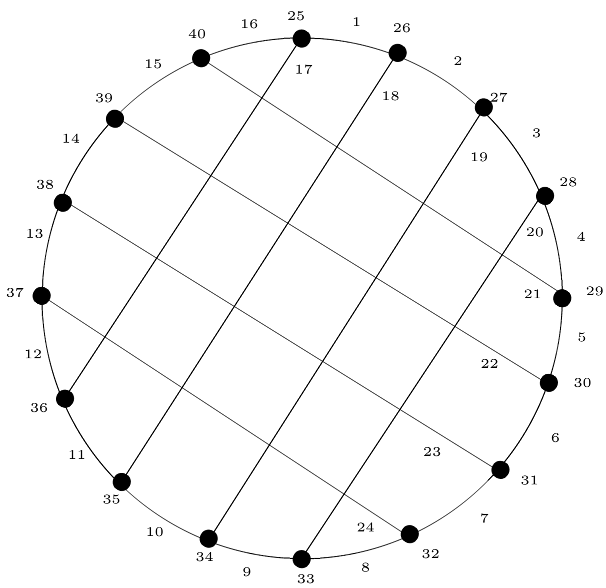

Fig.3.4

References

[1] B.D.Acharya and S.M.Hedge, Arithmetic graphs, *J. Graph Theory*, 14 (1990) 275-299.

[2] S.W.Golomb, How to number a graph, in *Graph Theory and Computing*, R.C.Reed ed., Academic press, New York(1972) 23-37.

[3] S.M.Hedge, Additively graceful graphs, *Mat. Acad,Sci.Lett.,* 12(1989) 387-390.

[4] Joseph A.Gallian, A Dynamic Survey of Graph Labelling, *The Electronic Journal of combinatorics*, DS6 (2016).

[5] A Kotzig and A.Rosa, Magic valuation of finite graphs, *Canad.Math.Bull*, 13(1970) 451-456.

[6] J.A.MacDougall, M.Miller, Slamin and W.D.Walls, Vertex - magic total labelling of graphs, *Util.Math.*61(2002) 3-21.

[7] G.Marimuthu and M.Balakrishnan, Super edge magic graceful graphs, *Information Sciences*, 287 (10)(2014) 140-151.

[8] V.Ramachandran, C.Sekar, (1,N)-arithmetic graphs, *International Journal of Computers and Applications*, Vol.38 (1) (2016) 55-59.

[9] A.Rosa, On certain valuations of the vertices of a graph, *Theory of graphs* (International Symposium, Rome, July 1966), Gordon and Breach, N.Y and Dunod Paris (1967) 349-355.

Papers Published in IJMC, 2020

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Famous Words

We have only an incomplete or non-comprehensive science for things in the universe which is the limitation of humans science. In this case, we can hardly conclude that a scientific conclusion is true in the whole universe because it is understanding only by humans ourself, an intelligent creature happily born on the earth. (Extracted from the paper: Science's Dilemma - a Review on Science with Applications, *Progress in Physics*, Vol.15, 2(2019), 78-85.)

By Dr.Linfan MAO, a Chinese mathematician, philosophical critic.

Author Information

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[12]W.S.Massey, *Algebraic topology: an introduction*, Springer-Verlag, New York 1977.

Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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December 2020

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An International Journal on Mathematical Combinatorics

ISSN 1937-1055