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Some Classes of Analytic Functions with q-Calculus

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Abstract: We study the estimates for the Second q-Hankel determinant of analytic functions in a class which unifies a number of classes studied previously by Darus, Ramreddy, Ravichandran, Yang and others. Our class includes q-convex and q-starlike functions. Also we study the estimate for q-Toeplitz determinants whose elements are the coefficients a_n for f in close-to-q-convex functions.

Key Words: Second Hankel determinant, subordination, q-starlike and q-convex functions, close-to-q-convex, Toeplitz matrices.

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§1. Introduction

The Hankel determinants $H_q(n)$ of Taylor's coefficients of function $f \in \mathcal{A}$ where $\mathcal{A}$ denotes the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \ \ and \ \ |z| < 1\}$. is defined by

$$\mathbf{H_q(n)} := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}$$

where $(a_1 = 1, \ \ n = q \in \mathbb{N})$. Hankel matrices (and determinants) play an important role in several branches of mathematics and have many applications [11]. $H_2(1)$ is the classical Fekete-Szegö functional. Fekete-Szegö in [4] found the maximum value of $H_2(1)$. Pommerenke in [16] proved that the Hankel determinant of univalent functions satisfy

$$|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}} \quad (n = 1, 2, \cdots, q = 2, 3, \cdots),$$

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where $\beta > \frac{1}{4000}$ and K depends only on q.

Hayman [8] showed that

$$|H_2(n)| = |a_n a_{n+2} - a_{n+1}^2| < An^{\frac{1}{2}} \quad , n = 2, 3, \cdots ,$$

where A is an absolute constant for a really mean univalent functions. Hankel determinants are useful in showing that a function of bounded characteristic in $\mathcal{U}$, i.e, a function which is ratio of tow bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational. In recent years, several authors investigated bounds for the Hankel determinant belonging tow various subclasses of univalent and multivalent functions in a class which unifies a number of classes studied earlier by Deepak Bansal, K. I. Noor, T. Yavuz, Sarika Verma, Shigeyoshi Owa and others. Closely related to Hankel determinants are the Toepliz determinants. A Toeplitz matrix $T_q(n)$ defined by

$$\mathbf{T_q(n)} := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix}$$

A Toeplitz matrix can be thought of as an upside-down Hankel matrix, in that Hankel matrices have constant entries along the reverse diagonal, whereas Toeplitz matrices have constant entries along the diagonal. A good summary of the applications of Toeplitz matrices to a wide range of areas of pure and applied mathematics can also be found in [11].

We aim to define q-starlike, q-convex functions and Ma-Minda starlike and convex functions. We use the concept of principle of subordination and q-calculus to define our classes. Recently in the second half of the twentieth century q-calculus aroused interest due to lot of applications in the various mathematical fields such as combinatorics, number theory, quantum theory and the theory of relativity. The q-derivative of a function is defined in the following.

Definition 1.1([9]) *The q-derivative of f is given by*

$$\partial_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad , \quad \textit{where } 0 < q < 1. \tag{1.2}$$

Equivalently, (1.2) may be written as

$$\partial_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & \text{q} \neq 1 \\ n, & \text{q} = 1. \end{cases}$$

Note that as $q \to 1^-$, $[n]_q \to n$.

Definition 1.2([7]) *Let f be analytic in $\mathcal{U}$ and be given by (1.1). Then a function f is starlike if and only if, $\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0$. We denote the class of starlike functions by S^*.*

The class of functions with positive real part plays a significant role in complex function theory. Using principle of subordination we define the functions with positive real part.

Definition 1.3([17]) *Let f and g be analytic in $\mathcal{U}$, then f is said to be subordinate to the function g, written $f(z) \prec g(z)$, if there exists an analytic function $\omega : \mathcal{U} \to \mathcal{U}$ satisfying $\omega(0) = 0$ and $|\omega(z)| < 1$ such that $f(z) = g(\omega(z))$, $z \in \mathcal{U}$.*

Definition 1.4([3]) *Let $\mathcal{P}$ denote the class of analytic functions $p : \mathcal{U} \to \mathbb{C}$, $p(0) = 1$, and $\Re\{p(z)\} > 0$, then $p(z) \prec \frac{1+z}{1-z}$.*

The class $\mathcal{P}$ can be completely characterized in terms of subordination. We need the following lemmas to derive our results.

Lemma 1.5([3]) *If the function $p \in \mathcal{P}$ is given by the series*

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots , \tag{1.3}$$

then the following sharp estimate holds:

$$|c_n| \leq 2 \quad (n = 1, 2, \cdots).$$

Lemma 1.6([6]) *If the function $p \in \mathcal{P}$ is given by the series (1.3), then*

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{1.4}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z, \tag{1.5}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

§2. Main Results

Definition 2.1 *Let $\varphi : U \to \mathbb{C}$ be analytic, and let the Maclaurin series of φ is given by*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1, B_2 \in \mathbb{R}, B_1 > 0). \tag{2.1}$$

Let $0 \leq \gamma \leq 1$ and $\tau \in \mathbb{C}\backslash\{0\}$. A function $f \in \mathcal{A}$ is in the class $\mathcal{R}^{\tau}_{q,\gamma}(\varphi)$ if it satisfies the following subordination:

$$1 + \frac{1}{\tau}(\partial_q f(z) + \gamma z \partial_q^2 f(z) - 1) \prec \varphi(z).$$

Theorem 2.2 *Let $0 \leq \gamma \leq 1$, $\tau \in \mathbb{C}\backslash\{0\}$ and let the function f as in (1.1) be in the class $\mathcal{R}^{\tau}_{q,\gamma}(\varphi)$. Also let $p = \frac{[2]_q[4]_q(1+\gamma)(1+[3]_q\gamma)}{([3]_q)^2(1+[2]_q\gamma)^2}$.*

(1) If B_1, B_2 and B_3 satisfy the conditions $2|B_2|(1-p)+B_1(1-2p) \leq 0$, $|B_1B_3 - pB_2^2| - pB_1^2 \leq 0$, then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{|\tau|^2 B_1^2}{([3]_q)^2(1+[2]_q\gamma)^2}.$$

(2) If B_1, B_2 and B_3 satisfy the conditions $2|B_2|(1-p)+B_1(1-2p) \geq 0$, $2|B_1B_3 - pB_2^2| - 2(1-p)B_1|B_2| - B_1 \geq 0$, or the conditions $2|B_2|(1-p)+B_1(1-2p) \leq 0$, $|B_1B_3 - pB_2^2| - pB_1^2 \geq 0$, then the second Hankel determinant satisfies

$$|a_2a_4 - a_3^2| \leq \frac{|\tau|^2}{[2]_q[4]_q(1+\gamma)(1+[3]_q\gamma)}|B_1B_3 - pB_2^2|.$$

(3) If B_1, B_2 and B_3 satisfy the conditions $2|B_2|(1-p)+B_1(1-2p) > 0$, $|B_1B_3 - pB_2^2| - pB_1^2 \leq 0$, then the second Hankel determinant satisfies

$$\begin{aligned}|a_2a_4 - a_3^2| &\leq \frac{|\tau|^2 B_1^2}{4[4]_q[2]_q(1+\gamma)(1+[3]_q\gamma)} \\ &\times \left[\frac{4p|B_1B_3 - pB_2^2| - 4B_1(1-p)[|B_2|(3-2p)+B_1] - 4B_2^2(1-p)^2 - B_1^2(1-2p)^2}{|B_1B_3 - pB_2^2| - B_1(1-p)(2|B_2|+B_1)}\right].\end{aligned}$$

Proof Since $f \in \mathcal{R}^{\tau}_{q,\gamma}(\varphi)$, there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ in $\mathcal{U}$ such that

$$1 + \frac{1}{\tau}(\partial_q f(z) + \gamma z \partial_q^2 f(z) - 1) = \varphi(w(z)). \tag{2.2}$$

Define the function p_1 by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + \cdots,$$

or equivalently,

$$w(z) = \frac{p_1(z)-1}{p_1(z)+1} = \frac{1}{2}\left(c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right) \tag{2.3}$$

Then p_1 is analytic in $\mathcal{U}$ with $p_1(0) = 0$ and has a positive real part in $\mathcal{U}$. By using (2.3) together with (2.1), it is evident that

$$\varphi\left(\frac{p_1(z)-1}{p_1(z)+1}\right) = 1 + \frac{1}{2}B_1c_1z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2c_1^2\right)z^2 + \cdots \tag{2.4}$$

since f has the Maclaurin series given by (1.1), a computation shows that

$$\begin{aligned} 1+\frac{1}{\tau}(\partial_q f(z)+\gamma z\partial_q^2 f(z)-1) &= 1+\frac{[2]_q a_2(1+\gamma)}{\tau}z+\frac{[3]_q a_3(1+[2]_q\gamma)}{\tau}z^2 \\ &\quad +\frac{[4]_q a_4(1+[3]_q\gamma)}{\tau}z^3+\cdots . \end{aligned} \tag{2.5}$$

It follows from (2.2), (2.4) and (2.5) that

$$\begin{aligned} a_2 &= \frac{\tau B_1 c_1}{2[2]_q(1+\gamma)}, \\ a_3 &= \frac{\tau B_1}{4[3]_q(1+[2]_q\gamma)}\left[2c_2+c_1^2(\frac{B_2}{B_1}-1)\right], \\ a_4 &= \frac{\tau}{8[4]_q(1+[3]_q\gamma)}\left[B_1(4c_3-4c_1c_2+c_1^3)+2B_2c_1(2c_2-c_1^2)+B_3c_1^3\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} & a_2a_4-a_3^2 \\ & =\frac{\tau^2 B_1 c_1}{16([4]_q[2]_q)(1+\gamma)(1+[3]_q\gamma)}\left[B_1(4c_3-4c_1c_2+c_1^3)+2B_2c_1(2c_2-c_1^2)+B_3c_1^3\right] \\ & -\frac{\tau^2 B_1^2}{16([3]_q)^2(1+[2]_q\gamma)^2}\left[4c_2^2+c_1^4(\frac{B_2}{B_1}-1)^2+4c_2c_1^2(\frac{B_2}{B_1}-1)\right] \\ = & \frac{\tau^2 B_1 c_1}{16([4]_q[2]_q)(1+\gamma)(1+[3]_q\gamma)}\left\{\left[\left(4c_1c_3-4c_2c_1^2+c_1^4\right)+\frac{2B_2c_1^2}{B_1}\left(2c_2-c_1^2\right)+\frac{B_3}{B_1}c_1^4\right]\right. \\ & \left.-\frac{[4]_q[2]_q(1+\gamma)(1+[3]_q\gamma)}{([3]_q)^2(1+[2]_q\gamma)^2}\left[4c_2^2+c_1^4(\frac{B_2}{B_1}-1)^2+4c_2c_1^2(\frac{B_2}{B_1}-1)\right]\right\}, \end{aligned}$$

which yields

$$\begin{aligned} |a_2a_4-a_3^2| &= T\left|4c_1c_3+c_1^4\left[1-2\frac{B_2}{B_1}-p(\frac{B_2}{B_1}-1)^2+\frac{B_3}{B_1}\right]\right. \\ &\quad \left.-4pc_2^2-4c_1^2c_2\left[1-\frac{B_2}{B_1}+p(\frac{B_2}{B_1}-1)\right]\right|, \end{aligned} \tag{2.6}$$

where,

$$T=\frac{|\tau|^2B_1^2}{16([4]_q[2]_q)(1+\gamma)(1+[3]_q\gamma)} \quad \text{and} \quad p=\frac{[4]_q[2]_q(1+\gamma)(1+[3]_q\gamma)}{([3]_q)^2(1+[2]_q\gamma)^2}.$$

It can be easily verified that $p\in[\frac{64}{81},\frac{8}{9}]$ for $0\leq\gamma\leq 1$ and $0\leq q\leq 1$. Let

$$\begin{aligned} d_1 &= 4, \qquad d_2=-4\left[1-\tfrac{B_2}{B_1}+p(\tfrac{B_2}{B_1}-1)\right], \\ d_3 &= -4p, \quad\ d_4=\left[1-2\tfrac{B_2}{B_1}-p(\tfrac{B_2}{B_1}-1)^2+\tfrac{B_3}{B_1}\right]. \end{aligned} \tag{2.7}$$

Then (2.6) becomes

$$\left|a_2a_4-a_3^2\right|=T\left|d_1c_1c_3+d_2c_1^2c_2+d_3c_2^2+d_4c_1^4\right|. \tag{2.8}$$

Since the function $p(e^{i\theta}z)(\theta \in \mathbb{R})$ is in the class $\mathcal{P}$ for any $p \in \mathcal{P}$, there is no loss of generality in assuming $c_1 > 0$. Write $c_1 = c, c \in [0,2]$. Substituting the values of c_2 and c_3 respectively from (1.6) and (1.5) in (2.8), we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| &= \frac{T}{4}\left|c^4\left(d_1+2d_2+d_3+4d_4\right)+2xc^2(4-c^2)\left(d_1+d_2+d_3\right)\right. \\ &\quad \left. +\,(4-c^2)x^2\left(-d_1c^2+d_3(4-c^2)\right)+2d_1c(4-c^2)(1-|x|^2z)\right|. \end{aligned}$$

Replacing $|x|$ by μ and substituting the values of d_1, d_2, d_3 and d_4 from (2.7) yields

$$\begin{aligned} |a_2a_4 - a_3^2| &\le \frac{T}{4}\left[4c^4\left|\frac{B_3}{B_1}-p\frac{B_2^2}{B_1^2}\right|+8\left|\frac{B_2}{B_1}\right|\mu c^2(4-c^2)(1-p)\right. \\ &\quad \left. +(4-c^2)\mu^2(4c^2+4p(4-c^2))+8c(4-c^2)(1-\mu^2)\right] \\ &= T\left[c^4\left|\frac{B_3}{B_1}-p\frac{B_2^2}{B_1^2}\right|+2c(4-c^2)+2\mu\left|\frac{B_2}{B_1}\right|c^2(4-c^2)(1-p)\right. \\ &\quad \left. +\mu^2(4-c^2)(1-p)(c-\alpha)(c-\beta)\right] \equiv F(c,\mu), \end{aligned} \tag{2.9}$$

where $\alpha = 2,\ \beta = 2p/(1-p) > 2$.

Note that for $(c,\mu) \in [0,2]\times[0,1]$, differentiating $F(c,\mu)$ in (2.9) partially with respect to μ yields

$$\frac{\partial F}{\partial \mu} = T\left[2\left|\frac{B_2}{B_1}\right|c^2(4-c^2)(1-p)+2\mu(4-c^2)(1-p)(c-\alpha)(c-\beta)\right]. \tag{2.10}$$

Then, for $0<\mu<1,\ 0<q<1$ and any fixed c with $0<c<2$, it is clear from (2.10) that $\frac{\partial F}{\partial \mu} > 0$, that is, $F(c,\mu)$ is an increasing function of μ. Hence, for fixed $c \in [0,2]$, the maximum of $F(c,\mu)$ occurs at $\mu = 1$, and

$$\max F(c,\mu) = F(c,1) \equiv G(c),$$

which is

$$G(c) = T\left\{c^4\left[\left|\frac{B_3}{B_1}-p\frac{B_2^2}{B_1^2}\right|-(1-p)\left(2\left|\frac{B_2}{B_1}\right|+1\right)\right]+4c^2\left[2\left|\frac{B_2}{B_1}\right|(1-p)+1-2p\right]+16p\right\}.$$

Let

$$\begin{aligned} X &= \left|\frac{B_3}{B_1}-p\frac{B_2^2}{B_1^2}\right|-(1-p)\left(2\left|\frac{B_2}{B_1}\right|+1\right), \\ Y &= 4\left[2\left|\frac{B_2}{B_1}\right|(1-p)+1-2p\right], \\ Z &= 16p. \end{aligned} \tag{2.11}$$

Since

$$\max(Xt^2+Yt+Z) = \begin{cases} Z, & Y \le 0, X \le \frac{-Y}{4}; \\ 16X+4Y+Z, & Y \ge 0, X \ge \frac{-Y}{8} or Y \le 0, X \ge \frac{-Y}{4}; \\ \frac{4XZ-Y^2}{4X}, & Y > 0, X \le \frac{-y}{8}, \end{cases} \tag{2.12}$$

where $0 \le t \le 4$. Then we have

$$
\begin{aligned}
|a_2a_4 - a_3^2| \quad &\le \quad \frac{B_1}{16([4]_q-1)([3]_q-1)([2]_q-1)} \\
&\times \begin{cases} Z, & Y \le 0, X \le \frac{-Y}{4}; \\ 16X+4Y+Z, & Y \ge 0, X \ge \frac{-Y}{8} \text{ or } Y \le 0, X \ge \frac{-Y}{4}; \\ \frac{4XZ-Y^2}{4X}, & Y > 0, X \le \frac{-y}{8}, \end{cases}
\end{aligned}
$$

where X, Y and Z are given by (2.11). $\square$

Remark 2.3 notice that

(1) As $q \to 1^-$ Theorem 2.2 reduces to Theorem 3 in [12].

(2) As $q \to 1^-$ for the choice $\varphi(z) := (1+Az)/(1+Bz)$ with $-1 \le B < A \le 1$ Theorem 2.2 reduces to Theorem 2.1 in [12].

Definition 2.4 *An analytic function f is close-to-q-convex in $\mathcal{U}$, if and only if, there exists $g \in S_q^*$ such that*

$$\Re\left\{\frac{z\partial_q f(z)}{g(z)}\right\} > 0.$$

We denote the class of close-to-q-convex functions by K_q.

For $f \in S^*$, we can write $z\partial_q f(z) = f(z)h(z)$, where $h \in P$, the class of function satisfying $\Re h(z) > 0$ for $z \in \mathcal{U}$ and

$$h(z) = 1 + \sum_{n=2}^{\infty} c_n z^n.$$

For $f \in K_q$, we can write $z\partial_q f(z) = g(z)p(z)$, where $p \in P$ and

$$p(z) = 1 + \sum_{n=2}^{\infty} p_n z^n.$$

Theorem 2.5 *Let $f \in K_q$ and given by* (1.1) *with associated starlike function g define by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then

$$T_2(2) = |a_3^2 - a_2^2| \le [5]_q, \qquad (b_2 \in \mathbb{R})$$

and the inequality is sharp.

Proof Write $z\partial_q f(z) = g(z)h(z)$ and $zg'(z) = g(z)p(z)$, with

$$h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$$

and

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n.$$

Then equating the coefficients in $z\partial_q f(z) = g(z)h(z)$ where coefficients' relations from $zg'(z) = g(z)p(z)$ is also used, we obtain

$$\begin{aligned} a_2 &= \frac{c_1 + p_1}{[2]_q}, \\ a_3 &= \frac{p_1^2 + p_2 + 2p_1c_1 + 2c_2}{2[3]_q} \end{aligned}$$

so that

$$\begin{aligned} |a_3^2 - a_2^2| &= \left| \frac{-1}{[2]_q^2}c_1^2 + \frac{1}{[3]_q^2}c_2^2 - \frac{2}{[2]_q^2}c_1p_1 + \frac{2}{[3]_q^2}c_1c_2p_1 - \frac{1}{[2]_q^2}p_1^2 + \frac{1}{[3]_q^2}c_1^2p_1^2 + \frac{1}{[3]_q^2}c_2p_1^2 \right. \\ & \left. + \frac{1}{[3]_q^2}c_1p_1^3 + \frac{1}{4[3]_q^2}p_1^4 + \frac{1}{[3]_q^2}c_2p_2 + \frac{1}{[3]_q^2}c_1p_1p_2 + \frac{1}{2[3]_q^2}p_1^2p_2 + \frac{1}{4[3]_q^2}p_2^2 \right|. \end{aligned}$$

We now use Lemma 1.6 to express c_2 and p_2 in terms of c_1 and p_1 and writing$X = 4 - c_1^2$ and $Y = 4 - p_1^2$ for simplicity to get

$$\begin{aligned} & |a_3^2 - a_2^2| \\ &= \left| \frac{-1}{[2]_q^2}c_1^2 + \frac{1}{4[3]_q^2}c_1^4 - \frac{2}{[2]_q^2}c_1p_1 + \frac{1}{[3]_q^2}c_1^3p_1 - \frac{1}{[2]_q^2}p_1^2 + \frac{7}{4[3]_q^2}c_1^2p_1^2 \right. \\ & \quad + \frac{3}{2[3]_q^2}c_1p_1^3 + \frac{9}{16[3]_q^2}p_1^4 + \frac{1}{2[3]_q^2}c_1^2xX + \frac{1}{[3]_q^2}c_1p_1xX + \frac{3}{4[3]_q^2}p_1^2xX \\ & \quad + \frac{1}{4[3]_q^2}x^2X^2 + \frac{1}{4[3]_q^2}c_1^2yY + \frac{1}{2[3]_q^2}c_1p_1yY \\ & \quad \left. + \frac{3}{8[3]_q^2}p_1^2yY + \frac{1}{4[3]_q^2}xXyY + \frac{1}{16[3]_q^2}y^2Y^2 \right|. \end{aligned}$$

Without loss in generality we can assume that $c_1 = c$ where $0 \le c \le 2$. Also since we are assuming $b_2 = p_1$ to be real, we can write $p_1 = r$, with $0 \le |r| \le 2$, and write $|r| = p$. We note at this point a further normalisation of p_1 to be real would remove the requirement that $p_1 = b_2$ is real, but such normalisation does not appear to be justified. It follows from Lemma 1.6 that with now $X = 4 - c^2$ and $Y = 4 - p^2$. So,

$$\begin{aligned} & |a_3^2 - a_2^2| \\ & \le \left| \frac{-1}{[2]_q^2}c^2 + \frac{1}{4[3]_q^2}c^4 - \frac{2}{[2]_q^2}cp + \frac{1}{[3]_q^2}c^3p - \frac{1}{[2]_q^2}p^2 + \frac{7}{4[3]_q^2}c^2p^2 + \frac{3}{2[3]_q^2}cp^3 + \frac{9}{16[3]_q^2}p^4 \right| \\ & \quad + \frac{1}{2[3]_q^2}c^2|x|X + \frac{1}{[3]_q^2}cp|x|X + \frac{3}{4[3]_q^2}p^2|x|X + \frac{1}{4[3]_q^2}|x|^2X^2 + \frac{1}{4[3]_q^2}c^2|y|Y \\ & \quad + \frac{1}{2[3]_q^2}cp|y|Y + \frac{3}{8[3]_q^2}p^2|y|Y + \frac{1}{4[3]_q^2}|x|X|y|Y + \frac{1}{16[3]_q^2}|y|^2Y^2. \end{aligned}$$

Now we assume $|x| \leq 1$ and $|y| \leq 1$ and simplify to obtain

$$\begin{aligned}|a_3^2 - a_2^2| \quad &\leq \quad \left|\frac{-1}{[2]_q^2}c^2 + \frac{1}{4[3]_q^2}c^4 - \frac{2}{[2]_q^2}cp + \frac{1}{[3]_q^2}c^3p - \frac{1}{[2]_q^2}p^2 + \frac{7}{4[3]_q^2}c^2p^2 + \frac{3}{2[3]_q^2}cp^3 + \frac{9}{16[3]_q^2}p^4\right| \\ &+\frac{9}{[3]_q^2} - \frac{1}{4[3]_q^2}c^4 + \frac{6}{[3]_q^2}cp - \frac{1}{[3]_q^2}c^3p + \frac{3}{[3]_q^2}p^2 - \frac{3}{4[3]_q^2}c^2p^2 - \frac{1}{2[3]_q^2}cp^3 - \frac{5}{16[3]_q^2}p^4.\end{aligned}$$

Suppose that the expression between the modulus signs is positive, then

$$\begin{aligned}|a_3^2 - a_2^2| \leq \psi_1(c,p) \quad &= \quad \frac{9}{[3]_q^2} - \frac{1}{[2]_q^2}c^2 + \frac{2(3[2]_q^2 - [3]_q^2)}{[2]_q^2[3]_q^2}cp \\ &+\frac{2(3[2]_q^2 - [3]_q^2)}{[2]_q^2[3]_q^2}p^2 + \frac{1}{[3]_q^2}c^2P^2 + \frac{1}{[3]_q^2}cp^3 + \frac{1}{4[3]_q^2}p^4.\end{aligned}$$

Then for $0 \leq c \leq 2$ and $0 \leq p \leq 2$ and fixed q with $0 < q < 1$ and calculus we get that $\psi_1(c,p)$ has a maximum value of $[5]_q$ at [0,2].

If the expression between the modulus signs is negative, then

$$\begin{aligned}|a_3^2 - a_2^2| \leq \psi_2(c,p) \quad &= \quad \frac{9}{[3]_q^2} + \frac{1}{[2]_q^2}c^2 - \frac{1}{2[3]_q^2}c^4 + \frac{2(3[2]_q^2 + [3]_q^2)}{[2]_q^2[3]_q^2}cp \\ &-\frac{2}{[3]_q^2}c^3p\frac{3[2]_q^2 + [3]_q^2}{[2]_q^2[3]_q^2}p^2 - \frac{5}{2[3]_q^2}c^2P^2 - \frac{2}{[3]_q^2}cp^3 - \frac{7}{8[3]_q^2}p^4.\end{aligned}$$

Then for $0 \leq c \leq 2$ and $0 \leq p \leq 2$ and fixed q with $0 < q < 1$ and calculus we get that $\psi_2(c,p)$ has a maximum value less than $[3]_q$. Thus the proof is complete. □

As $q \to 1^-$, we have following result due to D. K. Thomas and S. Abdul Halim [18].

Corollary 2.6 *Let $f \in K$ and be given by* (**??**) *with the associated starlike function g be defined by*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

Then

$$T_2(2) = |a_3^2 - a_2^2| \leq 5,$$

provided b_2 is real. The inequality is sharp.

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