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Prime BI-Ideals of Po-Γ-Groupoids

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Abstract: In this paper, we have studied the notion of prime bi-ideals and semi prime bi-ideals of Po-Γ-groupoids and explored the various properties of prime bi-ideals in Po-Γ-groupoids. Also we obtained the condition for Po-Γ-groupoids to be regular.

Key Words: Partially ordered Γ-groupoid (po-Γ-groupoid), left (right, two-sided) ideal, quasi-ideal, bi-ideal, prime (semi prime) bi-ideal, regular partially ordered -Γ-groupoid.

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§1. Introduction

In 1983, A.P.J. van der Walt [5] introduced the interesting concepts of prime and semi prime bi-ideals for an associative ring with unity. In 1995, using the concepts defined by A. P. J. van der Walt, the structure of a ring containing prime and semi prime bi-ideals were studied by H. J. le Roux [2]. In 2001, Kehayopulu and Tsingelis [1] studied prime ideals of groupoids. Following [1], in 2005, S.K. Lee developed prime left (right) ideals of groupoids [3] and obtained some results on prime bi-ideals of groupoids [4]. In this paper we have studied the notion of prime bi-ideals and semi prime bi-ideals of Po-Γ-groupoids. Let M be a non empty set. M is called Γ-groupoid if for all $a, b \in M$ and $\gamma \in \Gamma$, $a\gamma b \in M$.

A set $(G, \Gamma, \leq)$ is called a partial order-Γ-groupoid(or simply Po-Γ-groupoid) if

(i) $(G, \leq)$ is a partial ordered set;

(ii) (G, Γ) is a Γ-groupoid such that $a \leq b \Rightarrow a\gamma x \leq b\gamma x$ and $x\gamma_1 a \leq x\gamma_1 b$ for all $a, b, x \in G; \gamma, \gamma_1 \in \Gamma$.

Throughout this paper G denotes a Po-Γ-groupiod.

A non empty subset A of G is called right(resp.left) ideal of G if

(i) $A\Gamma G \subseteq A$ (resp.$G\Gamma A \subseteq A$);

(ii) $a \in A$, $b \leq a$ for $b \in G$ implies $b \in A$.

A non empty subset A is called an ideal of G if it is a right and left ideal of G.

For non-empty subsets A and B of a po-Γ-groupoid G, the product $A\Gamma B$ of A and B and the subset $(A]$ of G are defined by $A\Gamma B = \{a\gamma b \in S : a \in A, b \in B, \gamma \in \Gamma\}$; $(A]=$ $\{x \in G : \exists a \in A(x \leq a)\}$.

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A non empty subset Q of G is called a quasi ideal if

(i) $(Q\Gamma G] \cap (G\Gamma Q] \subseteq Q$;

(ii) $a \leq q; q \in Q$ implies $a \in Q$.

A non empty subset B of G is called a bi-ideal if

(i) $(B\Gamma G\Gamma B] \subseteq B$;

(ii) $a \leq b; b \in B$ implies $a \in B$.

Every quasi ideal is a bi-ideal. But the converse need not be true. A bi-ideal B of G is prime, for $x, y \in G, (x\Gamma G\Gamma y] \subseteq B$ implies $x \in B$ or $y \in B$. A bi-ideal B of G is semi-prime, for $x \in G, (x\Gamma G\Gamma x] \subseteq B$ implies $x \in B$. A non-empty subset I of G is prime if I is an ideal of G such that for any ideals A, B of G , $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. It is clear that $(x)_l = (x \cup G\Gamma x]$(resp.$(x)_r = (x \cup x\Gamma G]$) is the principle left(resp.right) ideal generated by x.

§2. Main Results

Theorem 2.1 *A bi-ideal B of G is prime if and only if for a right ideal R and a left ideal L of G $(R\Gamma L] \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof Suppose that $(R\Gamma L] \subseteq B$ for a right ideal R and a left ideal L of G and $R \nsubseteq B$. Then there exists $x \in R\backslash B$ such that $(x\Gamma G\Gamma y] \subseteq (R\Gamma G\Gamma L] \subseteq (R\Gamma L] \subseteq B$ for any $y \in L$ which implies $y \in B$. So $L \subseteq B$.

Conversely, let $(x\Gamma G\Gamma y] \subseteq B$ for $x, y \in G$. Then $(x\Gamma G]\Gamma(G\Gamma y] \subseteq (x\Gamma G\Gamma G\Gamma y] \subseteq (x\Gamma G\Gamma y] \subseteq B$. By hypothesis, we have $(x\Gamma G] \subseteq B$ or $G\Gamma y] \subseteq B$. If $(x\Gamma G] \subseteq B$, then $x\Gamma x \in (x\Gamma G]\Gamma G \subseteq B$. Now, $(x)_r(x)_l = (x \cup x\Gamma G]\Gamma(x \cup G\Gamma x] = (x\Gamma x \cup x\Gamma G\Gamma x \cup x\Gamma G\Gamma x \cup x\Gamma G\Gamma G\Gamma x] \subseteq (x\Gamma x \cup x\Gamma G] \subseteq B$ which implies $(x)_r \subseteq B$ or $(x)_l \subseteq B$. Therefore $x \in B$. If $(G\Gamma y] \subseteq B$, then by the similar method $y \in B$. □

Theorem 2.2 *If a bi-ideal B of G is prime, then B is a left or right ideal of G.*

Proof Let B be a prime bi-ideal of G.Then $(B\Gamma G] \subseteq B$ or $(G\Gamma B] \subseteq B$ as $(B\Gamma G]\Gamma(G\Gamma B] \subseteq (B\Gamma G\Gamma B] \subseteq B$ and by Theorem 2.1. So B is a a left ideal or right ideal of G. □

Theorem 2.3 *Let G be a po-Γ-groupoid. Then the following statements are hold:*

(i) *Any left/right/both sided ideal of G is a bi-ideal of G;*

(ii) *Intersection of right and left ideals of G is a bi-ideal of G;*

(iii) *Arbitrary intersection of bi-ideals of G is also a bi-ideal of G;*

(iv) *If B is a bi-ideal of G, then $B\Gamma r$ and $r\Gamma B$ are bi-ideals of G, for any $r \in G$.*

Proof This result can be immediately verified by definition. □

Notation 1 For a bi-ideal of B of G, we define $L_B = \{x \in B : G\Gamma x \subseteq B\}$, $R_B = \{x \in B : x\Gamma G \subseteq B\}$, $I_L = \{y \in L_B : y\Gamma G \subseteq L_B\}$ and $I_R = \{y \in R_B : G\Gamma y \subseteq R_B\}$.

Theorem 2.4 *Let B be bi-ideal of G . Then L_B is a left ideal of G contained in B if L_B is non empty.*

Proof Let $x \in L_B$. Let $g \in G$ and $\gamma \in \Gamma$. Then $g\gamma x \in G\Gamma x \subseteq B$. Now $G\Gamma g\gamma x \subseteq G\Gamma G\Gamma x \subseteq G\Gamma x \subseteq B$ which implies $g\gamma x \in L_B$. $G\Gamma L_B \subseteq L_B$. hence L_B is a left ideal. □

Theorem 2.5 *Let B be bi-ideal of G. Then I_L is the largest ideal of G contained in B if I_L is non empty. Furthermore, I_L coincides with I_R.*

Proof Let $x \in I_L$.Then $x\Gamma G \subseteq L_B$. For any $g \in G$ and $\gamma \in \Gamma$, we have $x\gamma g \in x\Gamma G \subseteq L_B$ and $x\gamma g\Gamma G \subseteq x\Gamma G\Gamma G \subseteq x\Gamma G \subseteq L_B$, So I_L is a right ideal of G.

Since $I_L \subseteq L_B \subseteq B$, we have $x \in L_B$ which implies $x\gamma g \in I_L$ and $G\Gamma x \subseteq B$.

Now, $G\Gamma g\gamma x \subseteq G\Gamma G\Gamma x \subseteq G\Gamma x \subseteq B$. So $g\gamma x \in L_B$. By Theorem 2.4 and $x \in I_L$, we have $x\Gamma G \subseteq L_B$. Then $g\gamma x\Gamma G \subseteq G\Gamma L_B \subseteq L_B$, and we have $g\gamma x \in I_L$. Therefore I_L is a left ideal.

Let A be an ideal of G such that $A \subseteq B$. If $x \in A$, then $x \in B$ and $G\Gamma \subseteq x \subseteq A \subseteq B$ which implies $x \in L_B$ and $A \subseteq L_B$.

Let $x \in A$. Then $x\Gamma G \subseteq A \subseteq L_B$. Hence $x \in I_L$ and $A \subseteq I_L$ which implies I_L is the largest ideal of G contained in B. Similarly I_R is the largest ideal of G contained in B. □

Notation 2 We denote I_B as $I_B :\equiv I_R = I_L$ by Theorem 2.5.

Theorem 2.6 *If B is a prime bi-ideal of G, then I_B is a prime ideal of G contained in B.*

Proof Let B be a prime bi-ideal of G. Then by Theorem 2.5, I_B is an ideal of G.

Suppose $X\Gamma Y \subseteq I_B$ for any ideals X, Y of G. Since $I_B \subseteq L_B \subseteq B$, we have $X\Gamma Y \subseteq B$. By Theorem 2.1, $X \subseteq B$ or $Y \subseteq B$. But I_B is the largest ideal contained in B, so $X \subseteq I_B$ or $Y \subseteq I_B$ which implies I_B is a prime ideal of G. □

Corollary 2.7 *If B be a semi-prime bi-ideal of G, then I_B is a semi-prime ideal of G if I_B is non empty.*

Theorem 2.8 *If a bi-ideals B of G is semi-prime, then*

(i) *for any left ideal L of G, $L\Gamma L \subseteq B$ implies $L \subseteq B$;*
(ii) *for any right ideal R of G, $R\Gamma R \subseteq B$ implies $R \subseteq B$.*

Proof Suppose $L\Gamma L \subseteq B$ for a left ideal L of G and $L \nsubseteq B$. Then there exists $x \in L\backslash B$, $x\Gamma G\Gamma x \subseteq L\Gamma G\Gamma L \subseteq L\Gamma L \subseteq B$. Since B is a semi-prime we have $x \in B$, a contradiction.

The second assertion can be proved similarly. □

Theorem 2.9 *If a bi-ideal B of G is semi-prime, then B is a quasi-ideal of G.*

Proof Let $y \in (B\Gamma G] \cap (G\Gamma B]$. Then $(y\Gamma G\Gamma y] \subseteq ((B\Gamma G]\Gamma G\Gamma(G\Gamma B]] \subseteq (B\Gamma G\Gamma B] \subseteq B$. Since B is a semi prime, we have $y \in B$. Hence B is a quasi-ideal of G. □

Remark 2.10 For a Po-Γ-groupoid G,

(a) The set of all prime ideal of G is denoted by *spec*(G);

(b) $Bspec(G)$ denotes the set of prime bi-ideals of G;

(c) $Sspec(G)$ denotes the set of all semi prime bi-ideals of G.

Then we know conclusions following easily by definition.

Theorem 2.11 *If G is finite, then $\bigcap spec(G) = \bigcap Bspec(G)$.*

Theorem 2.12 *A bi-ideal B of G is semi prime if and only if for a right ideal (left ideal) A of G $(A\Gamma A] \subseteq B$ implies $A \subseteq B$.*

Theorem 2.13 *The intersection of any family of prime bi-ideals of G is a semi prime bi-ideal of G.*

Theorem 2.14 *If G is finite, then $\bigcap spec(G) = \bigcap Sspec(G)$.*

We note that G is regular if for any $x \in G$, there exist $a \in G$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x \leq x\gamma_1 a\gamma_2 x$.

The following results shows the necessary and sufficient condition for a Po-Γ-groupiod to be regular.

Theorem 2.15 *Let G be Po-gamma-groupiod. Then G is regular if and only if every bi-ideal of G is semi-prime.*

Proof Let G be regular and B a bi-ideal of G. Suppose that $x\Gamma G\Gamma x \subseteq B$ for $x \in G$. Then there exist $a \in G$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $x \leq x\gamma_1 a\gamma_2 x \in x\Gamma a\Gamma x \in x\Gamma G\Gamma x \subseteq B$ which implies $x \in B$. Hence B is semi prime.

Conversely , assume that every bi-ideal of G is semi-prime. Let $B = (a\Gamma G\Gamma a]$ for $a \in G$. Then $B\Gamma G\Gamma B = (a\Gamma G\Gamma a]\Gamma G\Gamma(a\Gamma G\Gamma a] \subseteq (a\Gamma G\Gamma a] = B$, which implies B is a bi-ideal of G and by assumption $(a\Gamma G\Gamma a]$ is semi-prime. Since $a\Gamma G\Gamma a \subseteq (a\Gamma G\Gamma a] = B$, we get $a \in (a\Gamma G\Gamma a] = B$. Then there exist $x \in G$ and $\gamma_1, \gamma_2 \in \Gamma$ such that $a \leq a\gamma_1 x\gamma_2 a$. □

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