International J.Math. Combin. Vol.4(2019), 112-121

On the Eccentric Sequence of Composite Graphs

Yaser Alizadeh

(Department of Mathematics, Hakim Sabzevari University, Sabzevar, Iran)

E-mail: y.alizadeh.s@gmail.com

Abstract: The eccentric sequence of a graph is defined as list of eccentricity of its vertices. Eccentric sequence of composite graphs under seven graph products: line graph, sum, cartesian product, disjunction, symmetric difference, lexicographic product and corona product is investigated. Also some family of non vertex transitive graphs that are self centered are determined as product of graphs. It is proved that for any positive integer d, there is an infinite family of non-vertex transitive self centered graphs with diameter d. The relation between total eccentricity of a tree and total eccentricity of its line graph is given.

Key Words: Eccentricity, eccentric sequence, graph product, self centered graph.

AMS(2010): 05C76.

§1. Introduction

We consider only simple connected graphs in this paper. Let $G = (V(G), E(G))$ be a graph and u, v be two vertices of G. The *distance* between u and v, $d_G(u, v)$ (simply $d(u, v)$) is the length of shortest path connecting u and v. For a vertex $v \in V(G)$, the *eccentricity* of v, $\varepsilon_G(v)$ is the maximum distance from v to other vertices in G. The maximum and the minimum eccentricity among all vertices of G are called *diameter* $\mathrm{diam}(G)$ and *radius* $\mathrm{rad}(G)$ of G respectively. The *center* of G, $C(G)$ is the set of vertices whose eccentricity is equal to $\mathrm{rad}(G)$. A graph G is called *self centered* if all of its vertices have a same eccentricity. Let $V(G) = \{v_1,\ v_2, \cdots, v_n\}$ be the vertex set of G. *Eccentric sequence* of G is the sequence $\varepsilon_1,\ \varepsilon_2, \cdots, \varepsilon_n$ where ε_i is the eccentricity of vertex v_i. The *minimum eccentric sequence* of G, $es(G)$ is the sequence $\{\varepsilon_1^{t_1},\ \varepsilon_2^{t_2}, \cdots,\ \varepsilon_k^{t_k}\}$ where $\varepsilon_1,\ \varepsilon_2 \cdots, \varepsilon_k$ are the different eccentricities of vertices and t_i denotes the number of vertices with eccentricity ε_i and more over $\varepsilon_{i+1} = \varepsilon_i + 1$ for $1 \leqslant i \leqslant k-1$. Note that $\varepsilon_1 = \mathrm{rad}(G)$ and $\varepsilon_k = \mathrm{diam}(G)$. Eccentric sequence is interesting since it provides information on the vertex eccentricities and some structural properties of the graph such as diameter, radius and variability of vertex eccentricities. Call a sequence of positive integer eccentric if it is eccentric sequence of a graph. In a series of papers several properties of eccentric sequences are studied. For instance see surveys [3, 7, 11, 14, 17]. Characterization of eccentric sequences of graphs was first considered by Lesniak [11] who characterized sequences which are the eccentricity sequences of trees.

[1]Received July 23, 2019, Accepted December 8, 2019.

Study of graph invariants specially topological indices under graph products is very interested in mathematical literature. Some properties and application are reported in surveys [1, 2, 4 C 6, 8 C 10, 12, 15, 18]. In this paper, we study the eccentric sequence of composite graphs. We obtain explicit formulas of eccentric sequence for some graph product such as: line graph, sum, cartesian product, disjunction, symmetric difference, lexicographic product and corona product. Two important topological indices based on eccentricity of vertices are the total eccentricity and eccentric connectivity index. The *total eccentricity* of a graph G, $\xi(G)$ is the sum of eccentricities of its vertices. Clearly, if

$$es(G) = \{\varepsilon_1^{t_1}, \cdots, \varepsilon_k^{t_k}\}$$

then,

$$\xi(G) = \sum_{i=1}^{k} t_i \varepsilon(v_i).$$

The *eccentric connectivity index* of graph G, ECI(G), introduced by Sharma et al. [16], is defined as

$$\mathrm{ECI}(G) = \sum_{v \in V(G)} \varepsilon(v)\deg(v),$$

where $\deg(v)$ denotes degree of vertex v. These topological indices have been used as mathematical models for the prediction of biological activities of diverse nature. The automorphism group of G is denoted with Aut(G). A graph is called *vertex transitive* if for any pair of vertices u and v, there is an automorphism α such that $\alpha(u) = v$. It is known that an automorphism of a graph preserve the distance function. It follows that vertex-transitive graphs are always regular and self centered graph.

In this paper we construct an infinite family of non-vertex transitive graphs which are self centered. In the rest of the section, some standard graph products are introduced, then eccentric sequence of graphs under these graph products is verified. First, we start with line graph. *Line graph* of G, $L(G)$ is a graph which each vertex of $L(G)$ is associated with an edge of G and two vertices in $L(G)$ is adjacent if and only if the corresponding edges of G have an end vertex in common. The *sum* of two graphs G_1 and G_2, $G_1 + G_2$ is defined as the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{u_1u_2 \,|u_1 \in V(G_1), u_2 \in V(G_2)\}$. The next binary graph product is cartesian product. The *Cartesian product* $G_1 \Box G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and (u_1, u_2) is adjacent to (v_1, v_2) if $u_1 = v_1$ and $(u_2v_2) \in E(G_2)$, or $u_2 = v_2$ and $(u_1v_1) \in E(G_1)$.

The *disjunction* $G_1 \vee G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and

$$E(G_1 \vee G_2) = \{(u_1, u_2)(v_1, v_2) \,|u_1v_1 \in E(G_1) \text{ or } u_2v_2 \in E(G_2) \ \}.$$

For given graphs G_1 and G_2, their *symmetric difference* $G_1 \bigoplus G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1v_1 \in E(G_1)$ or $u_2v_2 \in E(G_2)$

The diameter of disjunction and symmetric difference of two graphs when both of them contain more than one vertex do not exceed of 2. The next binary operation is the *lexicographic product*. The lexicographic product of two graphs G_1 and G_2, $G_1[G_2]$ is the graph with vertex set $V(G_1) \times V(G_2)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent if (u_1 is adjacent with v_1) or ($u_1 = v_1$ and u_2 and v_2 are adjacent).The operations sum, disjunction and symmetric difference are symmetric operation and this fact implies that they have symmetric eccentric sequence. But the lexicographic product do not have such property. Let n_i, $i = 1, 2$ denotes the order of G_i. The *corona product* of two graphs is denoted by $G_1 \circ G_2$ and is obtained from one copy of G_1 and n_1 copies of G_2, and then joining all vertices of the i-th copy of G_2 to the i-th vertex of G_1 for $i = 1, 2, \cdots, n_1$. In [13], application of coronas in chemical modeling was reported.

§2. Main Result

In this section, explicit formulas for eccentric sequence of some composite graphs is given.

2.1 Line Graph of Trees

Eccentric sequence of line graph of a tree can be determined by its eccentric sequence. We present a relation between of total eccentricity of a tree and total eccentricity of its line graph.

Theorem 2.1 *Let T be a tree with* rad(T) = r. *If $es(T) = \{r^{n_0}, (r+1)^{n_1}, \cdots, (2r)^{n_r}\}$, then eccentric sequence of $L(T)$ is obtained as*

$$es(L(T)) = \begin{cases} \{r^{n_1}, (r+1)^{n_2}, \cdots, (2r-1)^{n_r}\} & \text{if } C(T) = K_1 \\ \{(r-1)^1, r^{n_1}, \cdots, (2r-2)^{n_{r-1}}\} & \text{if } C(T) = K_2 \end{cases}$$

where $n_r \geqslant 0$ and for $0 \leqslant i \leqslant r-1$, $n_i \geqslant 1$.

Proof We must to consider two cases.

Case 1. $C(T) = K_1$.

Let p be the unique central vertex. Let $N_i(p) = \{v \in V(T) | d(p, v) = i\}$. Since T is a tree, for a vertex $u \in N_i(p)$, $\varepsilon_T(u) = i + r$. Also there is a unique vertex $v \in N_{i-1}(p)$ that is adjacent to u. Now consider the bijection $f : V(T) - \{p\} \to E(T)$, if $u \in N_i(p)$, $i \geqslant 1$, then $f(u) = uv$ where $v \in N_{i-1}(p)$. For a vertex $u \in N_i(p)$, we have $\varepsilon_T(u) = r + i$ and $\varepsilon_{L(T)}(f(u)) = r + i - 1$. Thus if

$$es(T) = \{r^1, (r+1)^{n_1}, \cdots, (2r)^{n_r}\},$$

the eccentric sequence of $L(T)$ is obtained as

$$es(L(T)) = \{r^{n_1}, (r+1)^{n_2}, \ldots, (2r-1)^{n_r}\}.$$

Case 2. $C(T) = K_2$.

Let p_1 and p_2 be two adjacent central vertices of T. It is easy to see $\varepsilon_{L(T)}(p_1p_2) = r-1$. Let $T-\{p_1p_2\} = T_1 \cup T_2$ which $p_j \in T_j$ for $j = 1, 2$. Clearly if $u \in T_j$, clearly $\varepsilon(u) = r + d(u, p_j)$. By a similar argument to Case 1, if $u \in N_i(P_j) \cap T_j$, $j = 1, 2$, then there is a unique vertex $v \in N_{i-1}(P_j)$ that $uv \in E(T)$. Thus we get again a bijection $f : V(T) - \{p_1, p_2\} \to E(T) - \{p_1p_2\}$. Also if $u \in N_i(P_j) \cap T_j$, then $\varepsilon_{L(T)}(f(u)) = \varepsilon_T(u) - 1$. This implies that if

$$es(T) = \{r^2, (r+1)^{n_1}, \cdots, (2r-1)^{n_{r-1}}\},$$

then

$$es(L(T)) = \{(r-1)^1, (r)^{n_1}, \cdots, (2r-2)^{n_{r-1}}\}. \qquad \square$$

Corollary 2.2 *Let T be a tree. $L(T)$ is self centered graph if and only if T is a star graph.*

Proof Clearly the line graph of a star graph is complete graph and then is self centered. Let T be a tree of order n and rad(T) = r. If $L(T)$ is self center graph, by Theorem 2.1, eccentric sequence of T has form $es(T) = \{(r-1)^{n-1}\}$ or $\{r^1, (r+1)^{n-2}\}$. This means that the variability of eccentricity in T is 1 or 2. Since T is a tree, this implies that rad(T) = 1 and the proof is completed. $\square$

Corollary 2.3 *Let T be a tree of order n and radius r. Then*

$$\xi(T) = \xi(L(T)) + n + r - 1.$$

Proof We consider two cases with respect to $es(T)$ and $es(L(T))$.

Case 1. $es(T) = \{r^1, (r+1)^{n_1}, \cdots, (2r)^{n_r}\}$ and $es(L(T)) = \{r^{n_1}, (r+1)^{n_2}, \cdots, (2r-1)^{n_r}\}$.

By a straight calculation we get

$$\xi(T) - \xi(L(T)) = r + \sum_{i=1}^{r} n_i = r + n - 1.$$

Case 2. $es(T) = \{r^2, (r+1)^{n_1}, \cdots, (2r-1)^{n_{r-1}}\}$ and $es(L(T)) = \{r-1^1, (r)^{n_1}, \cdots, (2r-2)^{n_{r-1}}\}$.

Again, in this case a same result is obtained as well.

$$\xi(T) - \xi(L(T)) = 2r - (r-1) + \sum_{i=1}^{r-1} n_i = r + 1 + n - 2 = n + r - 1. \qquad \square$$

2.2 Sum

Theorem 2.4 *Let G_1 and G_2 be simple connected graphs. Then,*

$$es(G_1 + G_2) = \{1^{c_1+c_2}, 2^{n_1+n_2-c_1-c_2}\},$$

where c_i is the number of vertices of eccentricity 1 in G_i and n_i is the number of vertices of G_i; $i = 1, 2$.

Proof It is not difficult to see that $\mathrm{diam(G_1 + G_2)} \leqslant 2$. For vertex $x \in V(G_i)$ we have $\varepsilon_{G_1+G_2}(x) = 1$ if and only if $\varepsilon_{G_i}(x) = 1$. Let c_i denotes the number vertices of eccentricity 1 in G_i, $i = 1, 2$. Then the eccentric sequence of $G_1 + G_2$ is obtained as $es(G_1 + G_2) = \{1^{c_1+c_2}, 2^{n_1+n_2-c_1-c_2}\}$. □

The eccentric sequence of sum of more than two graphs can be obtained by a reasoning similar to the above.

Corollary 2.5 *Let $G_1, G_2, \cdots, G_k$ be simple connected graphs. Then*

$$es(G_1 + G_2 + \cdots + G_k) = \{1^{\sum_{i=1}^{k} c_i}, 2^{\sum_{i=1}^{k} n_i - c_i}\},$$

where c_i is the number of vertices of eccentricity 1 (or 0) in G_i and n_i is the number of vertices of G_i; $i = 1, 2, \cdots n$.

Corollary 2.6 *For any integer $n \geqslant 5$, there is a self centered graph and non vertex transitive of diameter 2 and order n.*

Proof It is sufficient to consider the complete bipartite graph $K_{2,n-2} = \bar{K}_2 + \bar{K}_{n-2}$. □

2.3 Cartesian Product

Theorem 2.7 *Let $es(G_1) = \{{\varepsilon_1}^{t_1}, {\varepsilon_2}^{t_2}, \cdots, {\varepsilon_k}^{t_k}\}$ and $es(G_2) = \{{\delta_1}^{s_1}, {\delta_2}^{s_2}, \cdots, {\delta_m}^{s_m}\}$. Then,*

$$es(G_1 \Box G_2) = \left\{(\varepsilon_i + \delta_j)^{t_i s_j}\right\}_{1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant m.}$$

Proof It is known that $d_{G_1 \Box G_2}((x, y), (u, v)) = d_{G_1}(x, u) + d_{G_2}(y, v)$, this implies that $\varepsilon_{G_1 \Box G_2}(x, y) = \varepsilon_{G_1}(x) + \varepsilon_{G_2}(y)$.

Let m_i and n_j be the number of vertices of eccentricity ε_i and δ_j in G_1 and G_2 respectively. Then, $m_i n_j$ vertices of $G_1 \Box G_2$ have eccentricity $\varepsilon_i + \delta_j$. Therefore $es(G_1 \Box G_2) = \left\{(\varepsilon_i + \delta_j)^{t_i s_j}\right\}_{1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant m}$. □

Corollary 2.8 *There are infinite family of non-vertex transitive self centered graph.*

Proof It is sufficient to consider the powers of a non-vertex transitive self centered graph such as $K_{m,n}$ where $m \neq n$ and $m, n \geqslant 2$. □

2.4 Disjunction

First note that if $G = K_1$ then $G \vee H \cong H$ and $G \oplus H \cong H$ as well. Therefore the considered graph for these two graph products are except K_1.

Theorem 2.9 *Let $G_1 \neq K_1 \neq G_2$. Then*

$$es(G_1 \vee G_2) = \{1^{c_1c_2}, 2^{n_1n_2-c_1c_2}\}$$

where c_i is the number of vertices of eccentricity 1 in G_i and n_i is the number of vertices of G_i; $i = 1, 2$.

Proof Let (x, y) and (u, v) be two vertices of $G_1 \vee G_2$ and $xx' \in E(G_1)$ and $vv' \in E(G_2)$. Since (x, y) and (u, v) both are adjacent to (x', v') then $d((x, y), (u, v)) \leqslant 2$. Therefore for each vertex $(x, y) \in G_1 \vee G_2$, we have $\varepsilon_{G_1 \vee G_2}(x, y) \leqslant 2$. If $\varepsilon_{G_1}(x) > 1$, and $d_{G_1}(x, y) \geqslant 2$ then $d((x, u), (y, u)) = 2$. Hence, $\varepsilon_{G_1 \vee G_2}(x, y) = 1$ if and only if $\varepsilon_{G_1}(x) = 1 = \varepsilon_{G_2}(y)$. Let c_i vertices of G_i are of eccentricity 1 for $i = 1, 2$. Then c_1c_2 vertices of $G_1 \vee G_2$ have eccentricity 1 and the other vertices are of eccentricity 2. This implies that $es(G_1 \vee G_2) = \{1^{c_1c_2}, 2^{n_1n_2-c_1c_2}\}$. □

Corollary 2.10 *Let G_1 and G_2 be two graphs of radius at least 2. Then $G_1 \vee G_2$ is a self centered graph and $es(G_1 \vee G_2) = \{2^{n_1n_2}\}$.*

2.5 Symmetric Difference

Lemma 2.11([9]) *Let G_1 and G_2 be two simple connected graphs. The number of vertices of G_i is denoted by n_i for $i = 1, 2$. Then $deg_{G_1 \oplus G_2}((u, v)) = n_2deg_{G_1}(u) + n_1deg_{G_2}(v) - 2deg_{G_1}(u)deg_{G_2}(v)$.*

Theorem 2.12 *Let $G_1 \neq K_1 \neq G_2$. Then $es(G_1 \oplus G_2) = \{2^{n_1n_2}\}$.*

Proof Let (x, y) and (u, v) be two vertices of $G_1 \vee G_2$ and $xx' \in E(G_1)$ and $vv' \in E(G_2)$. Since (x, y) and (u, v) are adjacent to (x, v') then $d((x, y), (u, v)) \leqslant 2$. On the other hand, (x, y) and (x', v') are not adjacent in $G_1 \oplus G_2$. Therefore each vertex $(x, y) \in G_1 \oplus G_2$ we have $\varepsilon_{G_1 \oplus G_2}(x, y) = 2$. This concludes that $es(G_1 \oplus G_2) = \{2^{n_1n_2}\}$. □

Corollary 2.13 *For any positive integer d there is an infinite family of non vertex transitive graphs which are self centered with diameter d.*

Proof Let $\oplus_{i=1}^{k} G_i = G_1 \oplus G_2 \oplus \cdots \oplus G_k$. For any positive integers $n, k \geqslant 3$, let $G_{n,k} = \oplus_{i=1}^{k} P_n$. Using Lemma 2.11 and Theorem 2.12 we get that $G_{n,k}$ is a self centered graph of diameter 2 and it is non vertex transitive because it is not regular. Now consider the graph $H_{n,k,d} = G_{n,k} \Box C_{2(d-2)}$ which is self center graph of diameter d. Since $G_{n,k}$ is not regular graph then $H_{n,k,d}$ is not regular and consequently is not vertex transitive graph as well. Clearly diameter of $H_{n,k,d}$ is d and the proof is completed. □

2.6 Lexicographic Product

For the lexicographic product of graphs, the distance of pair vertices is determined by the following lemma.

Lemma 2.14([9]) *The distance of pair vertices in* $G_1[G_2]$ *is*

$$d_{G_1[G_2]}((u_1,v_1),(u_2,v_2)) = \begin{cases} d_{G_1}(u_1,u_2) & v_1=v_2 \\ 1 & u_1=u_2, v_1v_2 \in E(G_2) \\ 2 & otherwise \end{cases}$$

Therefore, the eccentricity of vertex $(u,v) \in V(G_1[G_2])$ is determined by

$$\varepsilon(u,v) = \begin{cases} 1 & \text{if } \varepsilon_{G_1}(u) = \varepsilon_{G_2}(v) = 1 \\ 2 & \text{if } \varepsilon_{G_1}(u) = 1 \ \text{ and } \varepsilon_{G_2}(v) \geqslant 2 \\ \varepsilon_{G_1}(u) & \text{if } \varepsilon(u) \geqslant 2 \end{cases}$$

Now, all conditions are ready to obtain the eccentric sequence of $G_1[G_2]$.

Theorem 2.15 *Let* $es(G_1) = \{\varepsilon_1^{t_1}, \ldots, \varepsilon_k^{t_k}\}$ *and* n_i *and* c_i, $i=1,2$ *be the order and the number of vertices of eccentricity* 1 *of* G_i *respectively. Then*

$$es(G_1[G_2]) = \{1^{c_1c_2}, 2^{c_1(n_2-c_2)+t_2n_2}, \varepsilon_3^{t_3n_2}, \cdots, \varepsilon_k^{t_kn_2}\}.$$

2.7 Corona

Remark 2.16 If $G_1 = K_1$, then $G_1oG_2 = K_1 + G_2$ and

$$es(G_1oG_2) = \{1^{1+c_2}, 2^{n_2-c_2}\}$$

.

Theorem 2.17 *Let* $G_1 \neq K_1$ *and* $es(G_1) = \{\varepsilon_i^{t_i}\}_{i=1}^{k}$. *Then*

$$es(G_1oG_2) = \left\{\varepsilon_2^{t_1}, \varepsilon_3^{t_2+n_2t_1}, \varepsilon_4^{t_3+n_2t_2}, \cdots, \varepsilon_{k+1}^{t_k+n_2t_{k-1}}, \varepsilon_{k+2}^{n_2t_k}\right\},$$

where $n_2 = |V(G_2)|$ *and* $\varepsilon_{i+1} = \varepsilon_i + 1$.

Proof Let $V(G_1) = \{v_1, v_2, \ldots, v_n\}$ and $G_{2,i}$ be the copy of G_2 associated to v_i. From the structure of corona product of graphs one can see that $\varepsilon_{G_1oG_2}(v_i) = \varepsilon_{G_1}(v_i) + 1$ and if $x \in V(G_{2,i})$, $\varepsilon_{G_1oG_2}(x) = \varepsilon_{G_1}(v_i) + 2, 1 \leqslant i \leqslant n$. Then for $x \in V(G_1oG_2)$,

$$\varepsilon_2 = \varepsilon_1 + 1 \leqslant \varepsilon(x) \leqslant \varepsilon_k + 2 = \varepsilon_{k+2}.$$

Let v be a central vertex of G_1 and $\varepsilon_{G_1}(v) = \varepsilon_1$, then $\varepsilon_{G_1oG_2}(v) = \varepsilon_1 + 1 = \varepsilon_2$. This follows that center of G_1 coincides center of G_1oG_2. For $i \geqslant 2$, the set of vertices of G_1 having eccentricity ε_i and the vertices of $G_{2,t}$ which $\varepsilon(V_t) = \varepsilon_{i-1}$ are of eccentricity ε_{i+1} in G_1oG_2. Thus for $i \geqslant 2$, the number of vertices that have eccentricity ε_{i+1} is $t_i + n_2t_{i-1}$. The proof is completed. □

§3. Examples and Concluding Remarks

In this section, our theorems for eccentric sequence are illustrated for several more particular composite graphs. We first give the expressions for suspensions.

Corollary 2.18 *Let G be a graph on n vertices. Then*

$$es(K_1+G)=\{1^{c+1},2^{n-c}\},$$

where c is the number of vertices of eccentricity 1 in G.

Next, the eccentric sequence for the fan graph K_1+P_n and the wheel graph $W_n=K_1+C_n$ are presented by

Corollary 2.19 *For the fan graph K_1+P_n and the wheel graph $W_n=K_1+C_n$,*

$$es(K_1+P_n)=\begin{cases}\{1^2\} & \text{if } n=1,\\ \{1^3\} & \text{if } n=2,\\ \{1^2,2^2\} & \text{if } n=3,\\ \{1^1,2^n\} & \text{if } n\geqslant 4\end{cases}$$

and

$$es(W_n)=\begin{cases}\{1^4\} & \text{if } n=3\\ \{1^1,2^n\} & \text{if } n\geqslant 4.\end{cases}$$

By composing paths and cycles with various small graphs, we can obtain different classes of polymer like graphs. For example, we state the eccentric sequence for the fence graph $P_n[K_2]$ and the closed fence $C_n[K_2]$ in the following conclusion.

Corollary 2.20 *For the fence graph $P_n[K_2]$ and the closed fence $C_n[K_2]$,*

$$es(P_n[K_2])=\begin{cases}\{1^2\} & \text{if } n=1,\\ \{1^4\} & \text{if } n=2,\\ \{1^2,2^4\} & \text{if } n=3,\\ \{2^{2t_2},3^{2t_3},\cdots k^{2t_k}\} & \text{if } n\geqslant 4,\end{cases}$$

where $es(P_n)=\{2^{t_2},3^{t_3},\cdots k^{t_k}\}$ for $n\geqslant 4$ and

$$es(C_n[K_2])=\begin{cases}\{1^6\} & \text{if } n=3,\\ \{[\frac{n}{2}]^{2n}\} & \text{if } n\geqslant 4.\end{cases}$$

The t-thorny graph of a given graph G is obtained as $Go\bar{K_n}$, where $\bar{K_n}$ denotes the empty graph on n vertices. For the t-thorny path and t-thorny cycle we get the following eccentric sequence.

Corollary 2.21 *For the t-thorny graph $P_n o\bar{K}_t$,*

$$es(P_n o\bar{K}_t) = \begin{cases} \{1^1, 2^t\} & n = 1, \\ \{2^2, 3^{2t}\} & n = 2, \\ \{2^1, 3^{t+2}, 4^{2t}\} & n = 3. \end{cases}$$

If $n \geqslant 4$ and it is even

$$es(P_n o\bar{K}_t) = \{(\frac{n}{2}+1)^2, (\frac{n}{2}+2)^{2t+2}, \cdots, n^{2t+2}, (n+1)^{2t}\}$$

and if $n \geqslant 5$ and it is odd then

$$es(P_n o\bar{K}_t) = \{(\frac{n+1}{2})^1, (\frac{n+1}{2}+1)^{t+2}, (\frac{n+1}{2}+2)^{2t+2}, \cdots, n^{2t+2}, (n+1)^{2t}\}.$$

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