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On Obic Algebras

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Abstract: In this paper, obic algebras are introduced and their properties are investigated. Homomorphisms and krib maps as well as monics of obic algebras are studied. Properties of implicative obic algeras are also investigated.

Key Words: Obic algebras, krib maps, monics.

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§1. Introduction

Algebras of type (2,0) are well known types of algebraic structures. They comprise non-empty sets, some constant element together with a binary operation. In [1], Kim and Kim introduced the notion of BE-algebras. Ahn and so, in [2] and [3] introduced the notions of ideals and upper sets in BE-algebras and investigated related properties. In this paper, a new class of algebras called obic algebras are introduced. Their properties are investigated. Homomorphisms and krib maps as well as monics of obic algebras are studied. Moreover, translations in obic algebras are investigated as well as properties of implicative obic algebras.

Definition 1.1 *A non-empty set X together with a binary operation $*$ defined on X is called a groupoid.*

Definition 1.2 *A triple $(X;*,0)$, where X is a non-empty set, $*$ a binary operation on X and 0 a constant element of X is called an obic algebra if the following axioms hold for all $x, y, z \in X$:*

(1) $x * 0 = x$;

(2) $[x * (y * z)] * x = x * [y * (z * x)]$;

(3) $x * x = 0$.

Example 1.1 Consider the multiplicative group $G = \{1, -1, i, -i\}$. Define a binary operation $*$ on G by $a * b = ab^{-1}$. Then $(G;*,1)$ is an obic algebra.

Example 1.2 Let $\mathbb{Z}$ denote the set of integers. Then $(\mathbb{Z};-,0)$ is an obic algebra.

Example 1.3 Let $X = \{0,1\}$. Define a binary operation $*$ on X in Table 1.

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$*$	0	1
0	0	1
1	1	0

Table 1

Then, $(X, *, 0)$ is an obic algebra.

We shall adopt the notation X for an obic algebra $(X; *, 0)$ unless stated otherwise.

Definition 1.3 *An obic algebra is called simple if* $y * (z * x) = x * (y * z)$ *for all* $x, y, z \in X$.

Definition 1.4 *An obic algebra is called plain if* $0 * (y * z) = (0 * y) * z$ *for all* $y, z \in X$.

Definition 1.5 *An obic algebra* X *is said to have the weak property (WP) if* $x * y = 0$ *and* $y * x = 0$ *imply that* $x = y$.

Definition 1.6 *An obic algebra* X *is called prime if* $0 * x = 0$ *for all* $x \in X$.

Lemma 1.1 *Let* X *be an obic algebra. Then for all* $x, y \in X$, *the following hold:*
$x * y = [x * (y * x)] * x$.

Definition 1.7 *A non-empty subset* S *of an obic algebra* X *is called a subalgebra if* S *is an obic algebra with respect to the binary operation in* X.

Example 1.4 Let X be an obic algebra. Then X and $\{0\}$ are subalgebras of X.

Example 1.5 Let X be the obic algebra in example 1.1. Then the subset $\{1, -1\}$ is a subalgebra of X.

The following results are immediately obtained by the definition.

Proposition 1.1 *A non-empty subset* S *of an obic algebra is a subalgebra if and only if the following hold:*

(1) $0 \in S$;

(2) $x * y \in S$ *for all* $x, y \in S$.

Proposition 1.2 *Let* X *be a plain obic algebra. Then, the subset* $S = \{x \in X : 0 * x = 0\}$ *is a subalgebra of* X.

§2. Obic Homomorphisms

Definition 2.1 *Let* $(X; *, 0)$ *and* $(Y; \circ, 0')$ *be obic algebras. A function* $f : X \to Y$ *is called an obic homomorphism if* $f(a * b) = f(a) \circ f(b)$ *for all* $a, b \in X$.

Definition 2.2 *Let* $f : X \to Y$ *be an obic homomorphism. The set* $\{x \in X : f(x) = 0'\}$ *is called the kernel of* f.

Proposition 2.1 *Let $f: X \to Y$ be an obic homomorphism. Then the kernel of f is a subalgebra of X.*

Then, we get conclusions following by definition.

Proposition 2.2 *Let $f: X \to Y$ be an obic homomorphism. Then,*

(1) $f(0) = 0'$;

(2) $x * y = 0 \Rightarrow f(x) \circ f(y) = 0'$ *for all* $x, y \in X$.

Let $f: X \to Y$ be an obic homomorphism. Define a relation $\sim$ by $(x \sim y) \Leftrightarrow f(x) = f(y)$. Then, we know

Lemma 2.1 *Let $f: X \to Y$ be an obic homomorphism. The relation $\sim$ defined by $(x \sim y) \Leftrightarrow f(x) = f(y)$ is an equivalence relation.*

Definition 2.3 *An equivalence relation $\sim$ on an obic algebra X is called a congruence if $(x \sim y)$ and $(u \sim v) \Rightarrow (x * u) \sim (y * v)$.*

We have the following result by definition.

Lemma 2.2 *Let $f: X \to Y$ be an obic homomorphism. The equivalence relation $\sim$ defined by $(x \sim y) \Rightarrow f(x) = f(y)$ is a congruence.*

Let $[x]$ be the equivalence class of $x \in X$ and let $\overline{X}$ denote the collection of equivalence classes in the equivalence relation $\sim$. Define a binary operation $\diamond$ on $\overline{X}$ by $[x] \diamond [y] = [x * y]$.

Theorem 2.1 *Let $f: X \to Y$ be an obic homomorphism. Then $(\overline{X}; \diamond, [0])$ is an obic algebra.*

Proof By Lemma 2.2, the binary operation $\diamond$ is well-defined. Now, let $[x], [y], [z] \in \overline{X}$. Consider $[x] \diamond [0] = [x * 0] = [x]$. Also,

$$\begin{aligned}([x] \diamond ([y] \diamond [z])) \diamond [x] &= ([x] \diamond [y * z]) \diamond [x] = ([x * (y * z)]) \diamond [x] \\ &= [(x * (y * z)) * x] = [x * (y * (z * x))] \\ &= [x] \diamond ([y] \diamond ([z] \diamond [x])).\end{aligned}$$

Also, $[x] \diamond [x] = [x * x] = [0]$. □

Theorem 2.2 *Let $f: X \to X$ be an endomorphism. Then $f(X)$ is isomorphic to $\overline{X}$.*

Proof Consider the map $\phi: f(X) \to \overline{X}$ such that $\phi(y) = [y]$. Let $y_1, y_2 \in f(X)$. Then $\phi(y_1 * y_2) = [y_1 * y_2] = [y_1] \diamond [y_2] = \phi(y_1) \diamond \phi(y_2)$. Also, ϕ is one to one and onto. □

Theorem 2.3 *Let $\phi: X \to X$ be an obic homomorphism; where X has the weak property. Then ϕ is one to one if and only if $ker(\phi) = \{0\}$.*

Proof Suppose ϕ is one to one. Let $x \in ker(\phi)$. Then $\phi(x) = 0 = \phi(0)$. So, $ker(\phi) = \{0\}$.

Conversely, suppose $ker(\phi) = \{0\}$. Let $x, y \in X$ such that $\phi(x) = \phi(y)$. Then $\phi(x * y) = \phi(x) * \phi(y) = 0$. Also, $\phi(y * x) = 0$. So, $(x * y), (y * x) \in ker(\phi)$. Hence ϕ is one to one. □

Definition 2.4 *An obic homomorphism $f : X \to X$ is called idempotent if $f(f(x)) = f(x)$ for all $x \in X$.*

Theorem 2.4 *Let X be an obic algebra with weak property. Let ϕ be an idempotent endomorphism on X. Then ϕ is one to one if and only if ϕ is the identity map.*

Proof Suppose ϕ is one to one. Let $x \in X$. Then $\phi((x * \phi(x))) = \phi(x) * \phi(\phi(x)) = \phi(x) * \phi(x) = 0 = \phi(0)$.So, $x * \phi(x) = 0$. Similarly argument gives $\phi(x) * x = 0$. And so $\phi(x) = x$. Hence ϕ is the identity map.

The converse is obvious. □

§3. Implicative Obic Algebras

Definition 3.1 *An obic algebra X is called implicative if $x * (y * x) = x$ for all $x, y \in X$.*

The following conclusion can be obtained by the definition.

Lemma 3.1 *Let X be an implicative obic algebra. Then the following hold:*

(1) $0 * 0 = 0$;

(2) $x * y = (x * y) * (0 * y)$;

(3) $x * y = (x * (y * x)) * y$;

(4) $y * x = y * (x * (y * x))$.

Definition 3.2 *Let X be an obic algebra. Let x be a fixed element of X. The map $L_x : X \to X$ such that $L_x(a) = x * a$ for all $a \in X$ is called a left translation on X. Similarly, the map $R_x : X \to X$ such that $R_x(a) = a * x$ for all $a \in X$ is called a right translation on X.*

Theorem 3.1 *Let $L_x : X \to X$ be an endomorphism. Then $x = 0$. Moreover, if X is implicative, then $x = x * (x * y)$.*

Proof Consider $x = x * 0 = L_x(0) = L_x(0 * 0) = L_x(0) * L_x(0) = (x * 0) * (x * 0) = 0$. Now suppose X is implicative. Let $y \in X$. Then $x = x * 0 = L_x(0) = L_x(0 * y) = L_x(0) * L_x(y) = x * (x * y)$. □

Denote the collection of left translations on an obic algebra X by $L(X)$ and define a binary operation $\odot$ on $L(X)$ by $(L_a \odot L_b)(x) = L_a(x) * L_b(x)$ for all $x \in X$.

Theorem 3.2 *Let X be an implicative obic algebra. Then $(L(X); \odot, L_0)$ is an obic algebra.*

Proof Let $L_a, L_b, L_c \in L(X)$. For every $x \in X$, consider $(L_a \odot L_0)(x) = L_a(x) * L_0(x) = (a * x) * (0 * x) = (a * x) = L_a(x)$. So, $L_a \odot L_0 = L_a$.

Also consider

$$\begin{aligned}(L_a \odot (L_b \odot L_c) \odot L_a)(x) &= (a * x) * ((b * x) * ((c * x) * (a * x))) \\ &= (L_a \odot (L_b \odot (L_c \odot L_a)))(x).\end{aligned}$$

So,

$$(L_a \odot (L_b \odot L_c) \odot L_a) = (L_a \odot (L_b \odot (L_c \odot L_a))).$$

And clearly,

$$(L_a \odot L_a)(x) = L_a(x) * L_a(x) = (a * x) * (a * x) = 0 = L_0(x). \qquad \square$$

Corollary 3.1 *Let X be a prime obic algebra. Then $(L(X); \odot, L_0)$ is an obic algebra.*

Corollary 3.2 *$(L(X); \odot, L_0)$ is prime if and only if X is prime.*

We therefore know that

Proposition 3.1 *Let X be an obic algebra. Then the translation $L_0 : X \to X$ commutes with any endomorphism on X.*

Definition 3.3 *An obic algebra X is said to have the distributive property if $0 * (x * y) = (0 * x) * (0 * y)$ for all $x, y \in X$.*

Proposition 3.2 *Let X be an obic algebra with distributive property. Then the translation $L_0 : X \to X$ is the only homomorphism in the collection $L(X)$.*

Proof Clearly, L_0 is a homomorphism. Let $x \in X$ such that $x \neq 0$. Let Suppose L_x is a homomorphism on X. Consider $x = (x * 0) = L_x(0) = L_x(0 * 0) = L_x(0) * L_x(0) = 0$; which is a contradiction. $\square$

§4. Krib Maps in Obic Algebras

Definition 4.1 *Let X be an obic algebra. A self map $\alpha : X \to X$ is called a right krib map if $\alpha(x * y) = x * \alpha(y)$ for all $x, y \in X$.*

If $\alpha(x * y) = \alpha(x) * y$ for all $x, y \in X$, then α is called a left krib map. α is called a krib map if it is both a right and a left krib map.

Example 4.1 Consider the obic algebra X in Example 1.3. Define $\alpha : X \to X$ by $\alpha(1) = 0, \alpha(0) = 1$. Then α is a right krib map of X.

Denote by $D(X)$ the collection of right krib maps on an obic algebra X. We know the next result by definition.

Proposition 4.1 *Let X be an obic algebra. Then $D(X)$ is a monoid.*

Definition 4.2 *Let X be an obic algebra. A map $\alpha : X \to X$ is called regular if $\alpha(0) = 0$. If $\alpha(0) \neq 0$, then α is called irregular.*

The following results can be verified immediately.

Lemma 4.1 *Let α be an irregular right krib map of an obic algebra X. Then the following hold for all $x \in X$:*

(1) $x * \alpha(x) \neq 0$;
(2) $\alpha(x) = x * \alpha(0)$.

Proposition 4.2 *Every right krib map of a prime obic algebra is regular.*

Corollary 4.1 *Let α be a krib map on a prime obic algebra X. Then α is regular.*

Proposition 4.3 *A right krib map α of an obic algebra X is regular if and only if $x * \alpha(x) = 0$ for all $x \in X$.*

Proof Suppose α is regular. Then $0 = \alpha(0) = x\alpha(x)$. Conversely, suppose $x * \alpha(x) = 0$. Then $\alpha(0) = x * \alpha(x) = 0$. □

Proposition 4.4 *A left krib map α of an obic algebra X is regular if and only if $\alpha(x) * x = 0$ for all $x \in X$.*

Theorem 4.1 *Let α be a krib map of an obic algebra X. Then the following are equivalent:*

(1) $x * \alpha(x) = 0$;
(2) α *is regular;*
(3) $\alpha(x) * x = 0$.

Proof The proof is straightforward by definition. □

§5. Monics of Obic Algebras

Let X be an obic algebra. Define '$\wedge$' by $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Definition 5.1 *Let X be an obic algebra. A function $\theta : X \to X$ is called a left (resp. right)monic if $\theta(x * y) = (\theta(x) * y) \wedge (x * \theta(y))$ (resp. $\theta(x * y) = (x * \theta(y)) \wedge (\theta(x) * y)$) for all $x, y \in X$.*

If $\theta : X \to X$ is both a left and a right monic, then θ is called a monic.

Example 5.1 Let X be the obic algebra given by Table 2.

$*$	0	1
0	0	1
1	1	0

Table 2

The map $\theta : X \to X$ such that $\theta(1) = 0, \theta(0) = 1$ is a left monic.

Definition 5.2 *Let X be an obic algebra. A map $\alpha : X \to X$ is called regular if $\alpha(0) = 0$.*

Definition 5.3 *Let X be an obic algebra. A self map θ on X is called self preserving if $\theta(x) * x = x$ for all $x \in X$.*

Definition 5.4 *Let X be an obic algebra. A self map θ on X is called anti-self preserving if $x * \theta(x) = x$ for all $x \in X$.*

Definition 5.5 *Let X be an obic algebra. A self map θ on X is called preserving if it is both self-preserving and ant-self-preserving.*

Proposition 5.1 *Let θ be a regular left monic on an obic algebra X. Then,*

$$(x * \theta(x)) * [(x * \theta(x)) * (\theta(x) * x)] = (y * \theta(y)) * [(y * \theta(y)) * (\theta(y) * y)]$$

for all $x, y \in X$.

Proof Now, $0 = \theta(0) = \theta(x * x) = (x * \theta(x)) * [(x * \theta(x)) * (\theta(x) * x)]$. Similar argument gives also $(y * \theta(y)) * [(y * \theta(y)) * (\theta(y) * y)] = 0$. Hence, the conclusion follows. □

Proposition 5.2 *Let X be a regular left monic on an associative obic algebra X. Then $0 * [\theta(x) * x] = 0 * [\theta(y) * y]$ for all $x, y \in X$.*

Proof By proposition 5.1,

$$\begin{aligned} 0 &= [x * \theta(x)] * [(x * \theta(x)) * (\theta(x) * x)] \\ &= [(x * \theta(x)) * (x * \theta(x))] * [\theta(x) * x] \\ &= 0 * [\theta(x) * x]. \end{aligned}$$

Similarly, we have $0 * [\theta(y) * y] = 0$. The conclusion follows. □

Proposition 5.3 *Let θ be a self preserving left monic on an obic algebra X. Then $[x * \theta(x)] * [(x * \theta(x)) * x)] = \theta(0)$ for all $x \in X$.*

Proof Now,

$$\begin{aligned} \theta(0) &= \theta(x * x) \\ &= [\theta(x) * x] \wedge [x * \theta(x)] \\ &= [x * \theta(x))] * [(x * \theta(x)) * x]. \end{aligned} \qquad \square$$

Corollary 5.1 *Let θ be a regular self preserving left monic on an obic algebra X. Then $[x * \theta(x))] * [(x * \theta(x)) * x] = 0$ for all $x \in X$.*

The following propositions can be immediately verified.

Proposition 5.4 *Let θ be a regular self preserving left monic on an obic algebra X. Then $[x * \theta(x))] * [(x * \theta(x)) * x] = [y * \theta(y))] * [(y * \theta(y)) * y]$ for all $x, y \in X$.*

Proposition 5.5 *Let θ be a self preserving left monic on an associative obic algebra X. Then $0 * x = \theta(0)$ for all $x \in X$.*

Proposition 5.6 *Let θ be a regular self preserving left monic on an associative obic algebra X. Then $0 * [\theta(x) * x] = 0 * [\theta(y) * y]$ for all $x, y \in X$.*

Proposition 5.7 *Let θ be a regular self preserving left monic on an associative obic algebra X. Then $0 * x = 0$ for all $x \in X$.*

Theorem 5.1 *Let X be an associative obic algebra with a self preserving left monic θ. Then X is prime if and only if θ is regular.*

Proof Suppose X is prime. Then,

$$\begin{aligned} 0 = 0 * x &= [(x * \theta(x)) * (x * \theta(x))] * x \\ &= (x * \theta(x)) * [(x * \theta(x)) * x] \\ &= x \wedge [x * \theta(x)] \\ &= \theta(x * x) = \theta(0). \end{aligned}$$

Conversely, suppose θ is regular. Then,

$$\begin{aligned} 0 = \theta(0) &= \theta(x * x) = [(\theta(x) * x)] \wedge [x * \theta(x)] \\ &= x \wedge [x * \theta(x)] = 0 * x. \qquad \square \end{aligned}$$

The two conclusions following can be easily verified by definition.

Proposition 5.8 *Let θ be an anti-self preserving left monic on an obic algebra X. Then $x * [x * \theta(x)] = \theta(0)$ for all $x \in X$.*

Proposition 5.9 *Let θ be a regular anti-self preserving left monic on an obic algebra X. Then $y * [y * \theta(y)] = x * [x * \theta(x)]$ for all $x, y \in X$.*

Theorem 5.2 *Let θ be an anti-self preserving left monic on an associative obic algebra X. Then $0 * [\theta(x) * x] = \theta(0)$. Moreover, if θ is regular, then $0 * [\theta(x) * x] = 0$ for all $x \in X$.*

Proof Notice that

$$\begin{aligned} \theta(0) = \theta(x * x) &= [\theta(x) * x] \wedge [x * \theta(x)] \\ &= [\theta(x) * x] \wedge x \\ &= 0 * [\theta(x) * x]. \end{aligned}$$

The second part of the theorem is obvious. $\square$

Proposition 5.10 *Let X be an obic algebra with a left monic θ. Then $\theta(x) = x * [x * \theta(x)]$ for all $x \in X$.*

Proof Notice that

$$\begin{aligned} \theta(x) &= \theta(x * 0) = \theta(x) \\ &= [\theta(x) * 0] \wedge [x * \theta(0)] \\ &= x * [x * \theta(x)]. \end{aligned}$$

This completes the proof. □

Corollary 5.2 *Let θ be an anti-self preserving regular left monic on an obic algebra X. Then $\theta(x) = 0$ for all $x \in X$.*

Corollary 5.3 *Let θ be a regular left monic on an associative obic algebra X. Then $\theta(x) = 0 * \theta(x)$ for all $x \in X$.*

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