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Graphs, Networks and Natural Reality
– from Intuitive Abstracting to Theory

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Abstract: In the view of modern science, a matter is nothing else but a complex network $\overrightarrow{G}$, i.e., the reality of matter is characterized by complex network. However, there are no such a mathematical theory on complex network unless local and statistical results. *Could we establish such a mathematics on complex network*? The answer is affirmative, i.e., *mathematical combinatorics* or mathematics over topological graphs. Then, *what is a graph*? *How does it appears in the universe*? *And what is its role for understanding of the reality of matters*? The main purpose of this paper is to survey the progressing process and explains the notion from graphs to complex network and then, abstracts mathematical elements for understanding reality of matters. For example, L.Euler's solving on the problem of Königsberg seven bridges resulted in graph theory and embedding graphs in compact n-manifold, particularly, compact 2-manifold or surface with combinatorial maps and then, complex networks with reality of matters. We introduce 2 kinds of mathematical elements respectively on living body or non-living body for self-adaptive systems in the universe, i.e., continuity flow and harmonic flow $\overrightarrow{G}^L$ which are essentially elements in Banach space over graphs with operator actions on ends of edges in graph $\overrightarrow{G}$. We explain how to establish mathematics on the 2 kinds of elements, i.e., vectors underling a combinatorial structure $\overrightarrow{G}$ by generalize a few well-known theorems on Banach or Hilbert space and contribute mathematics on complex networks. All of these imply that graphs expand the mathematical field, establish the foundation on holding on the nature and networks are closer more to the real but without a systematic theory. However, its generalization enables one to establish mathematics over graphs, i.e., mathematical combinatorics on reality of matters in the universe.

Key Words: Graph, 2-cell embedding of graph, combinatorial map, complex network, reality, mathematical element, Smarandache multispace, mathematical combinatorics.

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§1. Introduction

What is the role of mathematics to natural reality? Certainly, as the science of quantity, mathematics is the main tool for humans understanding matters, both for the macro and the micro in the universe. Generally, it builds a model and characterizes the behavior of a matter

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for holding on reality and then, establishes a theory, such as those shown in Fig.1.

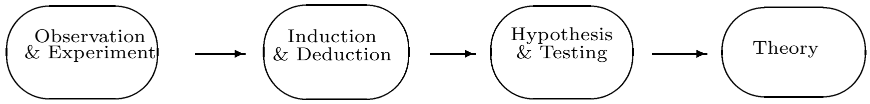


Fig.1

This scientific method on matters in the universe is completely reflected in the solving process of L.Euler on the problem of Königsberg seven bridges. Geographically, the city of Königsberg is located on both sides of Pregel River, including two large islands which were connected to each other and the mainland by seven bridges, such as those shown in Fig.2. The residents of Königsberg usually wished to pass through each bridge once without repeat, initialing at point of the mainland or islands.

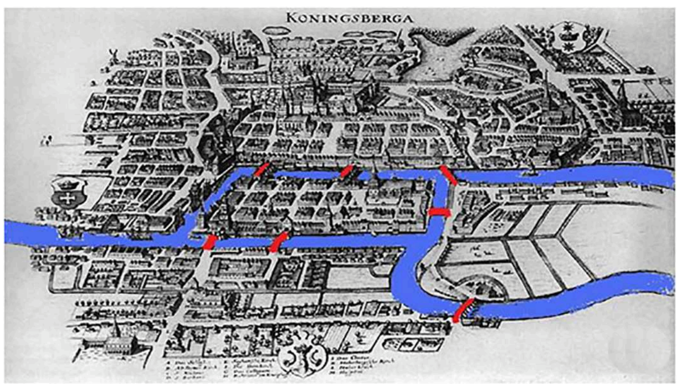
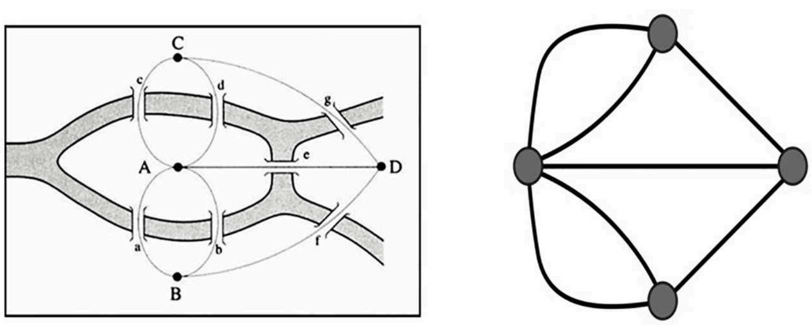


Fig.2

However, no one traveled in such a way once. Then, *a resident should how to travel for such a walk*? L.Euler solved this problem, and answered it had no solution in 1736. *How did he do it*? Let A,B,C,D be the two sides and islands. Then, he abstracted this problem on (a) equivalent to finding a traveling passing through each lines on (b) without repeating.



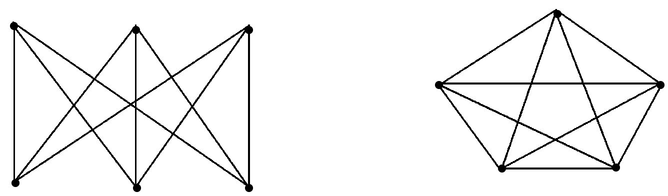
(a) (b)

Fig.3

Clearly, such a traveling must be with the same in and out times at each point A,B,C or D. But, (b) is not fitted with such conditions. So, there are no such a traveling in the problem on

Königsberg seven bridges.

Euler's solving method on the problem of Königsberg seven bridges finally resulted graph theory into beings today. A *graph* G is an ordered 3-tuple $(V, E; I)$, where V, E are finite sets, $V \neq \emptyset$ and $I : E \to V \times V$. Call V the *vertex set* and E the *edge set* of G, denoted by $V(G)$ and $E(G)$, respectively. For example, two graphs $K(3,3)$ and K_5 are shown in Fig.4.



$K(3,3)$ K_5

Fig.4

Usually, if $(u, v) = (v, u)$ for $\forall u, v \in V(G)$, then G is called a graph. Otherwise, it is called a directed graph with an orientation $u \to v$ on each edge (u, v), denoted by $\overrightarrow{G}$.

Let $G_1 = (V_1, E_1, I_1)$, $G_2 = (V_2, E_2, I_2)$ be 2 graphs. If there exists a $1-1$ mapping $\phi : V_1 \to V_2$ and $\phi : E_1 \to E_2$ such that $\phi I_1(e) = I_2\phi(e)$ for $\forall e \in E_1$ with the convention that $\phi(u, v) = (\phi(u), \phi(v))$, then we say that G_1 is *isomorphic* to G_2, denoted by $G_1 \cong G_2$ and ϕ an *isomorphism* between G_1 and G_2. Clearly, all automorphisms $\phi : V(G) \to V(G)$ of graph G form a group under the composition operation, and denoted by Aut$G$ the automorphism group of graph G. A few automorphism groups of well-known graphs are listed in Table 1.

G	Aut$G$	order
P_n	Z_2	2
C_n	D_n	$2n$
K_n	S_n	$n!$
$K_{m,n}(m \neq n)$	$S_m \times S_n$	$m!n!$
$K_{n,n}$	$S_2[S_n]$	$2n!^2$

Table 1

Certainly, an edge $e = uv \in E(G)$ can be divided into two semi-arcs e_u, e_v such as those shown in Fig.5.

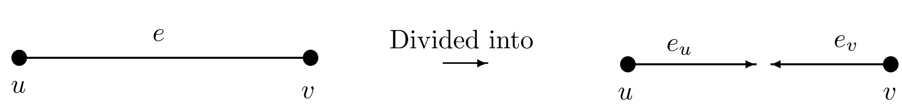


Fig.5

Similarly, two semi-arcs e_u, f_v are called v-incident or e-incident if $u = v$ or $e = f$. Denote all semi-arcs of a graph G by $X_{\frac{1}{2}}(G)$. A $1-1$ mapping ξ on $X_{\frac{1}{2}}(G)$ such that $\forall e_u, f_v \in X_{\frac{1}{2}}(G)$, $\xi(e_u)$ and $\xi(f_v)$ are $v-$incident or $e-$incident if e_u and f_v are $v-$incident or $e-$incident, is called a semi-arc automorphism of the graph G. Clearly, all semi-arc automorphisms of a graph also form a group, denoted by $\mathrm{Aut}_{\frac{1}{2}}G$.

Certainly, *graph theory* studies properties of graphs. A property is nothing else but a family of graph, i.e., $\mathscr{P} = \{G_1, G_2, \cdots, G_n, \cdots\}$ but closed under isomorphisms ϕ of graphs, i.e., $G^\phi \in \mathscr{P}$ if $G \in \mathscr{P}$. For example, hamiltonian graphs, Euler graphs and also interesting parameters, such as those of connectivity, independent number, covering number, girth, level number, $\cdots$ of a graph.

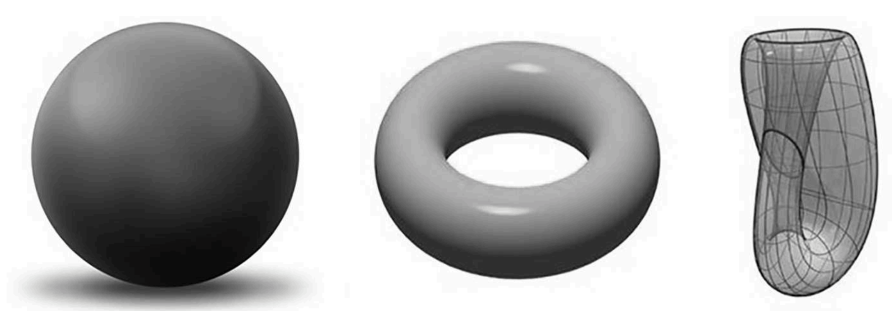
The main purpose of this paper is to survey the progressing process and explains the notion from graphs to complex network and then, abstracts mathematical elements for understanding reality of matters. For example, L.Euler's solving on the problem of Königsberg seven bridges resulted in graph theory and embedding graphs in compact n-manifold, particularly, compact 2-manifold or surface with combinatorial maps and then, complex networks with reality of matters. We introduce 2 kinds of mathematical elements respectively on living or non-living body in the universe, i.e., continuity and harmonic flows $\overrightarrow{G}^L$ which are essentially elements in Banach space over graphs with operator actions on ends of edges in graph $\overrightarrow{G}$. We explain how to establish mathematics on the 2 kinds of elements, i.e., vectors underling a combinatorial structure $\overrightarrow{G}$ by generalize a few well-known theorems on Banach or Hilbert space and contribute a mathematics on complex networks.

For terminologies and notations not mentioned here, we follow references [1],[2] and [4] for graphs, [3] for complex network, [6] for automorphisms of graph, [24] for algebraic topology, [25] for elementary particles and [6],[26] for Smarandache systems and multispaces.

§2. Embedding Graphs on Surfaces

2.1 Surface

A surface is a 2-dimensional compact manifold without boundary. For example, a few surfaces are shown in Fig.6.



Sphere Torus Klein bottle

Fig.6

Clearly, the intuition imagination is difficult for determining surface of higher genus. However, T.Radó showed the following result, which is the fundamental of combinatorial topology, ro topological graphs on surfaces.

Theorem 2.1(T.Radó 1925,[24]) *For any compact surface* S*, there exist a triangulation* $\{T_i, i \geq 1\}$ *on* S.

T.Radó's result on triangulation of surface enables one to present a surface by listing every triangle with each side a label and a direction, i.e., the polygon representation. Then, the surface is assembled by identifying the two sides with the same label and direction. This way results in a polygon representation on a surface finally. For examples, the polygon representations on surfaces in Fig.6 are shown in Fig.7.

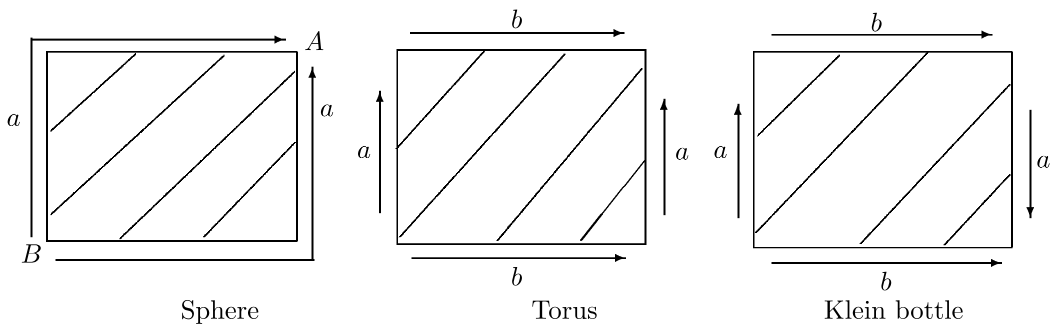


Fig.7

We know the classification theorem of surfaces following.

Theorem 2.2([24]) *Any connected compact surface* S *is either homeomorphic to a sphere, or to a connected sum of tori, or to a connected sum of projective planes, i.e., its surface presentation* $\mathcal{S}$ *is elementary equivalent to one of the standard surface presentations following:*

(1) *The sphere* $S^2 = \langle a | aa^{-1} \rangle$;

(2) *The connected sum of* p *tori*

$$\underbrace{T^2 \# T^2 \# \cdots \# T^2}_{p} = \left\langle a_i, b_i, 1 \leq i \leq p \mid \prod_{i=1}^{p} a_i b_i a_i^{-1} b_i^{-1} \right\rangle;$$

(3) *The connected sum of* q *projective planes*

$$\underbrace{P^2 \# P^2 \cdots \# P^2}_{q} = \left\langle a_i, 1 \leq i \leq q \mid \prod_{i=1}^{q} a_i \right\rangle.$$

A combinatorial proof on Theorem 2.2 can be found in [6]. By definition, the *Euler characteristic* of $\mathcal{S}$ is

$$\chi(\mathcal{S}) = |V(\mathcal{S})| - |E(\mathcal{S})| + |F(\mathcal{S})|,$$

where $V(\mathcal{S}), E(\mathcal{S})$ and $F(\mathcal{S})$ are respective the set of vertex set, edge set and face set of the polygon representation of surface $\mathcal{S}$. Then, we know the next result.

Theorem 2.3([24]) *Let S be a connected compact surface with a presentation $\mathcal{S}$. Then*

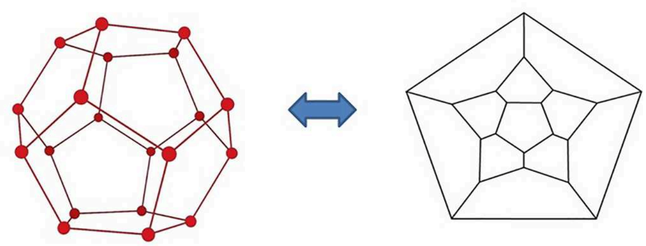
$$\chi(S) = \begin{cases} 2, & \text{if } \mathcal{S} \sim_{El} S^2, \\ 2-2p, & \text{if } \mathcal{S} \sim_{El} \underbrace{T^2\#T^2\#\cdots\#T^2}_{p}, \\ 2-q, & \text{if } \mathcal{S} \sim_{El} \underbrace{P^2\#P^2\#\cdots\#P^2}_{q}. \end{cases}$$

Theorem 2.3 enables one to define the genus of orientable or non-orientable surface S by numbers p and q, respectively, and the genus of sphere is defined to be 0.

2.2 Embedding Graph

A graph G is said to be embeddable into a topological space $\mathscr{T}$ if there is a $1-1$ continuous mapping $\phi : G \to \mathscr{T}$ with $\phi(p) \neq \phi(q)$ if p, q are different points on graph G. Particularly, if $\mathscr{T} = \mathbb{R}^2$ is a Euclidean plane, we say that G is a planar graph.

A most interesting case on the embedding problem of graphs is the case of surface, which is essentially to search the polyhedral structures on surfaces. Clearly, many results on embedding graphs is on these surfaces with small genus. For example, embedding results on $p = 0$ the sphere, $p = 1$ the torus, $\cdots$ of orientable surfaces, or on $q = 1$ the projective plane, $q = 2$ the Kein bottle, $\cdots$ of non-orientable surfaces. The most simple case is embedding graphs on sphere which is equivalent to a planar graph, such as the dodecahedron shown in Fig.8.



Dodecahedron Planar graph

Fig.8

We have known a few criterions on planar graphs following.

Theorem 2.4(Euler,1758, [2]) *Let G be a planar graph with p vertices, q edges and r faces. Then, $p - q + r = 2$.*

Theorem 2.5(Kuratowski,1930, [1]) *A graph is planar if and only if it contains no subgraphs homeomorphic with K_5 or $K(3,3)$.*

A 2-cell embedding of G on surface S is defined to be a continuous 1-1 mapping $\tau : G \to S$ such that each component in $S \setminus \tau(G)$ homeomorphic to an open 2-disk $\{\ (x,y) \mid x^2 + y^2 < 1\}$. Certainly, the image $\tau(G)$ is contained in the 1-skeleton of a triangulation on the surface S.

For example, the embedding of K_4 on the sphere and Klein bottle are shown in Fig.9 (a) and (b), respectively.

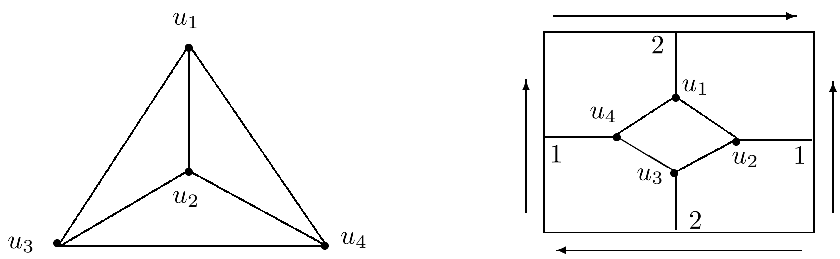


Fig.9

There is an algebraic representation for characterizing the 2-cell embedding of graphs. For $v \in V(G)$, denote by $N^e_G(v) = \{e_1, e_2, \cdots, e_{\rho(v)}\}$ all the edges incident with the vertex v. A permutation on $e_1, e_2, \cdots, e_{\rho(v)}$ is said to be a *pure rotation* and all pure rotations incident with v is denoted by $\varrho(v)$. Generally, a *pure rotation system* of the graph G is defined to be $\rho(G) = \{\varrho(v) | v \in V(G)\}$ which was observed and used by Dyck in 1888, Heffter in 1891 and then formalized by Edmonds in 1960. For example,

$$\rho(K_4) = \{(u_1u_4, u_1u_3, u_1u_2), (u_2u_1, u_2u_3, u_2u_4), (u_3u_1, u_3u_4, u_3u_2), (u_4u_1, u_4u_2, u_4u_3)\},$$
$$\rho(K_4) = \{(u_1u_2, u_1u_3, u_1u_4), (u_2u_1, u_2u_3, u_2u_4), (u_3u_2, u_3u_4, u_3u_1), (u_4u_1, u_4u_2, u_4u_3)\}$$

are respectively the pure rotation systems for embeddings of K_4 on the sphere and Klein bottle shown in Fig.9.

Theorem 2.6(Heffter 1891, Edmonds 1960, [4]) *Every embedding of a graph G on an orientable surface S induces a unique pure rotation system $\rho(G)$. Conversely, Every pure rotation system $\rho(G)$ of a graph G induces a unique embedding of G on an orientable surface S.*

Clearly, an embedding of graph G can be associated 0, 1 or 2-band respectively with vertices, edges and face on its surface. A band decomposition is called *locally orientable* if each 0-band is assigned an orientation, and a 1-band is called *orientation-preserving* if the direction induced on its ends by adjoining 0-bands are the same as those induced by one of the two possible orientations of the 1-band. Otherwise, *orientation-reversing*. An edge e in a graph G embedded on a surface S associated with a locally orientable band decomposition is said to be *type* 0 if its corresponding 1-band is orientation-preserving and otherwise, *type* 1. A *rotation system* $\rho^L(v)$ of $v \in V(G)$ to be a pair $(\mathcal{J}(v), \lambda)$, where $\mathcal{J}(v)$ is a pure rotation system and $\lambda : E(G) \rightarrow \mathbb{Z}_2$ is determined by $\lambda(e) = 0$ or $\lambda(e) = 1$ if e is *type* 0 or *type* 1 edge, respectively.

Theorem 2.7(Ringel 1950s, Stahl 1978,[4]) *Every rotation system on a graph G defines a unique locally orientable 2-cell embedding of $G \rightarrow S$. Conversely, every 2-cell embedding of a graph $G \rightarrow S$ defines a rotation system for G.*

A 2-cell embedding of a connected graph on surface is called map by W.T.Tutte. He characterized embeddings by purely algebra in 1973 ([5], [26]). By his definition, an embedding $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a basic permutation $\mathcal{P}$, i.e, for any $x \in \mathcal{X}_{\alpha,\beta}$, no integer k exists such that $\mathcal{P}^k x = \alpha x$, acting on $\mathcal{X}_{\alpha,\beta}$, the disjoint union of quadricells Kx of $x \in X$ (the base set), where $K = \{1, \alpha, \beta, \alpha\beta\}$ is the Klein group, satisfying the following two conditions:

(1) $\alpha\mathcal{P} = \mathcal{P}^{-1}\alpha$;

(2) the group $\Psi_J = \langle \alpha, \beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$.

Furthermore, if the group $\Psi_I = \langle \alpha\beta, \mathcal{P} \rangle$ is transitive on $\mathcal{X}_{\alpha,\beta}$, then M is non-orientable. Otherwise, orientable.

For example, the embedding $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ of graph K_4 on torus shown in Fig.10.

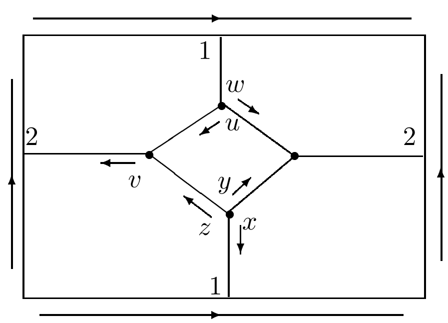


Fig.10

can be algebraic represented by

$$\begin{aligned} \mathcal{X}_{\alpha,\beta} &= \{x, y, z, u, v, w, \alpha x, \alpha y, \alpha z, \alpha u, \alpha v, \alpha w, \beta x, \beta y, \\ &\quad \beta z, \beta u, \beta v, \beta w, \alpha\beta x, \alpha\beta y, \alpha\beta z, \alpha\beta u, \alpha\beta v, \alpha\beta w\}, \\ \mathcal{P} &= (x, y, z)(\alpha\beta x, u, w)(\alpha\beta z, \alpha\beta u, v)(\alpha\beta y, \alpha\beta v, \alpha\beta w) \\ &\quad \times(\alpha x, \alpha z, \alpha y)(\beta x, \alpha w, \alpha u)(\beta z, \alpha v, \beta u)(\beta y, \beta w, \beta v), \end{aligned}$$

with vertices

$$\begin{aligned} &v_1 = \{(x, y, z), (\alpha x, \alpha z, \alpha y)\}, \qquad v_2 = \{(\alpha\beta x, u, w), (\beta x, \alpha w, \alpha u)\}, \\ &v_3 = \{(\alpha\beta z, \alpha\beta u, v), (\beta z, \alpha v, \beta u)\}, \quad v_4 = \{(\alpha\beta y, \alpha\beta v, \alpha\beta w), (\beta y, \beta w, \beta v)\}, \end{aligned}$$

edges

$$\{e, \alpha e, \beta e, \alpha\beta e\},\ e \in \{x, y, z, u, v, w\}$$

and faces

$$\begin{aligned} f_1 &= \{(x, u, v, \alpha\beta w, \alpha\beta x, y, \alpha\beta v, \alpha\beta z), (\beta x, \alpha z, \alpha v, \beta y, \alpha x, \alpha w, \beta v, \beta u)\}, \\ f_2 &= \{(z, \alpha\beta u, w, \alpha\beta y), (\beta z, \alpha y, \beta w, \alpha u)\}. \end{aligned}$$

Two embeddings $M_1 = (\mathcal{X}^1_{\alpha,\beta}, \mathcal{P}_1)$ and $M_2 = (\mathcal{X}^2_{\alpha,\beta}, \mathcal{P}_2)$ are said to be *isomorphic* if there exists a bijection ξ

$$\xi : \mathcal{X}^1_{\alpha,\beta} \longrightarrow \mathcal{X}^2_{\alpha,\beta}$$

such that for $\forall x \in \mathcal{X}^1_{\alpha,\beta}$, $\xi\alpha(x) = \alpha\xi(x), \xi\beta(x) = \beta\xi(x), \xi\mathcal{P}_1(x) = \mathcal{P}_2\xi(x)$. Particularly, if $M_1 = M_2 = M$, an isomorphism between M_1 and M_2 is then called an *automorphism* of embedding M. Clearly, all automorphisms of a embedding M form a group, called the *automorphism group* of M, denoted by $\mathrm{Aut}M$.

There are two main problems on embedding of graphs on surfaces following.

Problem 2.1 *Let G be a graph and S a surface. Whether or not G can be embedded on S?*

This problem had been solved by Duke on orientable case in 1966, and Stahl on non-orientable case in 1978. They obtained the result following.

Theorem 2.8(Duke 1966, Stahl 1978,[4]) *Let G be a connected graph and let $GR(G), CR(G)$ be the respective genus range of G on orientable or non-orientable surfaces. Then, $GR(G)$ and $CR(G)$ both are unbroken interval of integers.*

Theorem 2.8 bring about to determine the minimum and maximum genus $\gamma(G)$, $\gamma_M(G)$ of graph G on surfaces. Among them, the most simple case is to determine the maximum genus $\gamma_M(G)$ on non-orientable case, which was obtained by Edmonds in 1965. It is the Betti number $\beta(G) = |E(G)| - |V(G)| + 1$. The maximum genus $\gamma_M(G)$ of G on orientable case is determined by Xuong with the deficiency $\xi(G)$, i.e., the minimum number of components in $G \setminus T$ for all spanning trees T in G in 1979. However, it is difficult for the minimum genus $\gamma(G)$, only a few results on typical graphs. For example, the genus of K_n and Km, n are listed following.

Theorem 2.9(Ringel and Youngs 1968, [4]) *The minimum genus of a complete graph is given by*

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil, \qquad n \geq 3.$$

Theorem 2.10(Ringel 1965, [4]) *The minimum genus of a complete bipartite graph is given by*

$$\gamma(K(m,n)) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil, \qquad m, n \geq 2.$$

Problem 2.2 *Let G be a graph and S a surface. How many non-isomorphic embeddings of G on S?*

This problem is difficult, only be partially solved until today. However, the following simple result enables one to enumerate rooted embeddings, where an embedding $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is rooted on an element $r \in \mathcal{X}_{\alpha,\beta}$ if r is marked beforehand.

Theorem 2.11([5],[6]) *The auotmorphism group of a rooted embedding M, i.e.,* $\mathrm{Aut}M^r$ *is trivial.*

Theorem 2.12([5],[6]) $|\mathrm{Aut}M| \mid |\mathcal{X}_{\alpha,\beta}| = 4\varepsilon(M)$.

A root r in an embedding M is called an *i-root* if it is incident to a vertex of valency i. Two i-roots r_1, r_2 are *transitive* if there exists an automorphism $\tau \in$ AutM such that $\tau(r_1) = r_2$. Define the *enumerator* $v(D,x)$ and the *root polynomials* $r(M,x)$, $r(\mathcal{M}(D),x)$ as follows:

$$v(D,x) = \sum_{i\geq 1} i v_i x^i; \qquad r(M,x) = \sum_{i\geq 1} r(M,i)x^i,$$

where $r(M,i)$ denotes the number of non-transitive i-roots in M.

Theorem 2.12 enables us to get the following results by applying the enumerator and root polynomial of M.

Theorem 2.13(Mao and Liu, [21]) *The number $r^O(\Gamma)$ of non-isomorphic rooted maps on orientable surfaces underlying a simple graph Γ is*

$$r^O(\Gamma) = \frac{2\varepsilon(\Gamma)\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{|\mathrm{Aut}\Gamma|},$$

where $\varepsilon(\Gamma)$,$\rho(v)$ denote the size of Γ and the valency of the vertex v, respectively.

Theorem 2.14(Mao and Liu, [22]) *The number $r^N(\Gamma)$ of rooted maps on non-orientable surfaces underlying a graph Γ is*

$$r^N(\Gamma) = \frac{(2^{\beta(\Gamma)+1}-2)\varepsilon(\Gamma)\prod\limits_{v\in V(\Gamma)}(\rho(v)-1)!}{\left|Aut_{\frac{1}{2}}\Gamma\right|}.$$

For a few well-known graphs, Theorems 2.13 and 2.14 enables us to get Table 2.

G	$r^O(G)$	$r^N(G)$
P_n	$n-1$	0
C_n	1	1
K_n	$(n-2)!^{n-1}$	$(2^{\frac{(n-1)(n-2)}{2}}-1)(n-2)!^{n-1}$
$K_{m,n}(m\neq n)$	$2(m-1)!^{n-1}(n-1)!^{m-1}$	$(2^{mn-m-n+2}-2)(m-1)!^{n-1}(n-1)!^{m-1}$
$K_{n,n}$	$(n-1)!^{2n-2}$	$(2^{n^2-2n+2}-1)(n-1)!^{2n-2}$
B_n	$\frac{(2n)!}{2^n n!}$	$(2^{n+1}-1)\frac{(2n)!}{2^n n!}$
Dp_n	$(n-1)!$	$(2^n-1)(n-1)!$
$Dp_n^{k,l}(k\neq l)$	$\frac{(n+k+l)(n+2k-1)!(n+2l-1)!}{2^{k+l-1}n!k!l!}$	$\frac{(2^{n+k+l}-1)(n+k+l)(n+2k-1)!(n+2l-1)!}{2^{k+l-1}n!k!l!}$
$Dp_n^{k,k}$	$\frac{(n+2k)(n+2k-1)!^2}{2^{2k}n!k!^2}$	$\frac{(2^{n+2k}-1)(n+2k)(n+2k-1)!^2}{2^{2k}n!k!^2}$

Table 2

Apply the Burnside Lemma in permutation groups, we got the numbers of unrooted maps of complete graph K_n on orientable or non-orientable surfaces by calculating the stabilizer of each automorphism of complete maps.

Theorem 2.15(Mao, Liu and Tian, [23]) *The number* $n^O((K_n)$ *of complete maps of order* $n \geq 5$ *on orientable surfaces is*

$$n^O(K_n) = \frac{1}{2}(\sum_{k|n} + \sum_{k|n,k\equiv 0(mod2)})\frac{(n-2)!^{\frac{n}{k}}}{k^{\frac{n}{k}}(\frac{n}{k})!} + \sum_{k|(n-1),k\neq 1}\frac{\phi(k)(n-2)!^{\frac{n-1}{k}}}{n-1}.$$

and $n(K_4) = 3$.

Theorem 2.16(Mao, Liu and Tian, [23]) *The number* $n^N(K_n)$ *of complete maps of order* $n, n \geq 5$ *on non-orientable surfaces is*

$$\begin{aligned} n^N(K_n) &= \frac{1}{2}(\sum_{k|n} + \sum_{k|n,k\equiv 0(mod2)})\frac{(2^{\alpha(n,k)}-1)(n-2)!^{\frac{n}{k}}}{k^{\frac{n}{k}}(\frac{n}{k})!} \\ &\quad + \sum_{k|(n-1),k\neq 1}\frac{\phi(k)(2^{\beta(n,k)}-1)(n-2)!^{\frac{n-1}{k}}}{n-1}, \end{aligned}$$

and $n^N(K_4) = 8$*, where,*

$$\alpha(n,k) = \begin{cases} \dfrac{n(n-3)}{2k}, & \text{if} \quad k \equiv 1(mod2); \\ \dfrac{n(n-2)}{2k}, & \text{if} \quad k \equiv 0(mod2), \end{cases} \qquad \beta(n,k) = \begin{cases} \dfrac{(n-1)(n-2)}{2k}, & \text{if} \quad k \equiv 1(mod2); \\ \dfrac{(n-1)(n-3)}{2k}, & \text{if} \quad k \equiv 0(mod2). \end{cases}$$

§3. Complex Networks with Reality

A network is a directed graph G associated with a non-negative integer-valued function c on edges and conserved at each vertex, which are abstracting of practical networks, for instance, the electricity, communication and transportation networks such as those shown in Fig.11 for the high-speed rail network in China planed a few years ago.

Fig.11

Clearly, a network is nothing else but a labeled graph G^L with $L : E(G) \to \mathbb{Z}^+$. generally, a *labeled graph* on a graph $G = (V, E; I)$ is a mapping $\theta_L : V \cup E \to L$ for a label set L, denoted by G^L. If $\theta_L : E \to \emptyset$ or $\theta_L : V \to \emptyset$, then G^L is called a *vertex labeled graph* or an *edge labeled* *graph*, denoted by G^V or G^E, respectively. Otherwise, it is called a *vertex-edge labeled graph*. Similarly, two networks $G_1^{L_1}, G_2^{L_2}$ are equivalent, if there is an isomorphism $\phi : G_1 \to G_2$ such that $\phi(L_1(x)) = L_2(\phi(x))$ for $x \in V(G_1)\bigcup E(G_2)$.

It should be noted that labeled graphs are more useful in understanding matters in the universe. For example, there is a famous story, i.e., the blind men with an elephant. In this story, 6 blind men were be asked to determine what is an elephant looks like. The man touched the elephant's leg, tail, trunk, ear, belly or tusk respectively claims it's like a pillar, a rope, a tree branch, a hand fan, a wall or a solid pipe. Each of them insisted on his own and not accepted others.

Fig.12

They then entered into an endless argument. *All of you are right*! A wise man explained to them: why are you telling it differently is because each one of you touched the different part of the elephant. *What is the meaning of the wise man*? He claimed nothing else but the looks like of an elephant, i.e.,

$$\begin{aligned}\text{An elephant} &= \{4 \text{ pillars}\}\bigcup\{1 \text{ rope}\}\bigcup\{1 \text{ tree branch}\}\\ &\bigcup \{2 \text{ hand fans}\}\bigcup\{1 \text{ wall}\}\bigcup\{1 \text{ solid pipe}\}.\end{aligned}$$

Usually, a thing T is identified with known characters on it at one time, and this process is advanced gradually by ours. For example, let $\mu_1, \mu_2, \cdots, \mu_n$ be the known and $\nu_i, i \geq 1$ the unknown characters at time t. Then, the thing T is understood by

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right)\bigcup\left(\bigcup_{k\geq 1}\{\nu_k\}\right)$$

in logic and with an approximation $T^\circ = \bigcup\limits_{i=1}^{n}\{\mu_i\}$ at time t, which are both Smarandache multispace ([7],[26]).

What is the implications of this story for understanding matters in the universe? It lies in the situation that humans knowing matters in the universe is analogous to these blind men. However, if the wise man were L.Euler, a mathematician he would tell these blind men that an elephant looks like nothing else but a tree labeled by sets as shown in Fig.13, where, $\{a\}$ =tusk, $\{b_1, b_2\}$ =ears, $\{c\}$ =head, $\{d\}$ =belly, $\{e_1, e_2, e_3, e_4\}$ =legs and $\{f\}$ =tail with their intersection sets labeled on edges.

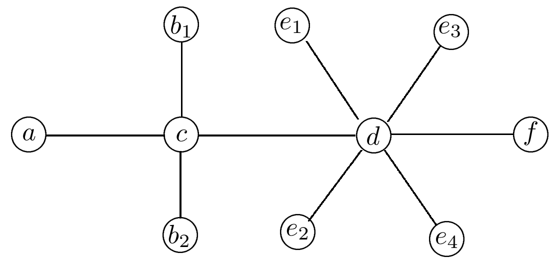


Fig.13

For the case of Euclidean space with dimension≥ 3, the intuition tells us that to embed a graph in a of dimension is not difficult by the result following. However, it is obvious but a universal skeleton inherited in all matters.

Theorem 3.1 *A simple graph G can be rectilinear embedded, i.e., all edge are segment of straight line in a Euclidean space $\mathbb{R}^n$ with $n \geq 3$.*

In fact, we can choose n district points in curve (t, t^2, t^3) of Euclidean space $\mathbb{R}^3$ on n different values of t. Then, it can be easily show that all these straight lines are never intersecting. Whence, it is a trivial problem on embedding graphs of $\mathbb{R}^3$. However, all matters are in 3-dimensional Euclidean space in the eyes of humans, i.e., the reality of a matter in the universe should be understanding on its 1-dimensional skeleton in the space.

Then, *what is the reality of a matter*? The word *reality* of a matter T is its state as it actually exist, including everything that is and has been, no matter it is observable or comprehensible by humans. *How can we hold on the reality of matters*? Usually, a matter T is multilateral, i.e., Smarandache multispace or complex one and so, hold on its reality is difficult for humans in logic, such as the meaning in the story of the blind men with an elephant.

For hold on the reality of matters, a general notion is

$$\text{Matter} \stackrel{Decompose}{\longrightarrow} \text{Microcosmic Particles} \stackrel{Abstract}{\longrightarrow} \text{Complex Network}.$$

For example, the physics determine the nature of matters by subdividing a matter to an irreducibly smallest detectable particle ([28]), i.e., elementary particles, which is essentially transfer the matter to a complex network such as those meson's and baryon's composition by quarks.

Similarly, the basic unit of life or the basic unit of heredity are cells and genes in biology which also enables us to get the life networks of cell or genes. This notion can be found in all modern science with an conclusion that *a matter = a complex network*. Its essence of this notion is to determine the nature of irreducibly smallest detectable units and then, holds on reality of the matter. However, a matter can be always divided into submatters, then sub-submatters and so on. A natural question on this notion is whether it has a terminal point or not. On the other hand, it is a very large complex network in general. For example, the complex network of a human body consists of $5\times 10^{14}--6\times 10^{14}$ cells. *Are we really need such a large and complex network for the reality of matters*? Certainly not! *How can we hold on the reality of matters* *by such a complex network*? And *do we have mathematical theory on complex network*? The answer is not certain because although we have established a theory on complex network but it is only a local theory by combination of the graphs and statistics with the help of computer ([3]), can not be used for the reality of matters.

However, we find a beacon light inspired by the *traditional Chinese medicine*. There are 12 meridians which completely reflects the physical condition of human body in traditional Chinese medicine: LU, LI, ST, SP, HT, SI, BL, KI, PC, SJ, GB, LR. For example, the LI and GB meridians are shown in Fig.14.

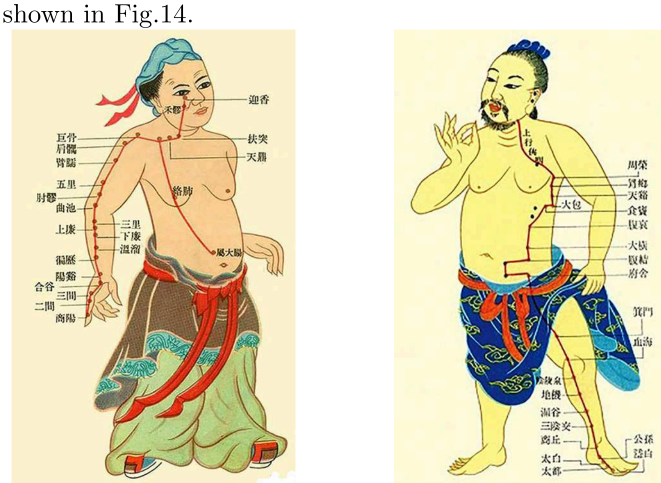


Fig.14

All of these 12 meridians can be classified into 3 classes following:

Class 1. *Paths*, including LU, LI, SP, HT, SI, KI, PC and LR meridians;

Class 2. *Trees*, including GB, ST and SJ meridians;

Class 3. *Gluing Product of circuit with paths* $C_n \bigodot P_{m_1} \bigodot P_{m_2}$, including BL meridian.

According to the Standard China National Standard (GB 12346-90), the inherited graph of the 12 meridians on a human body is shown in Fig.15.

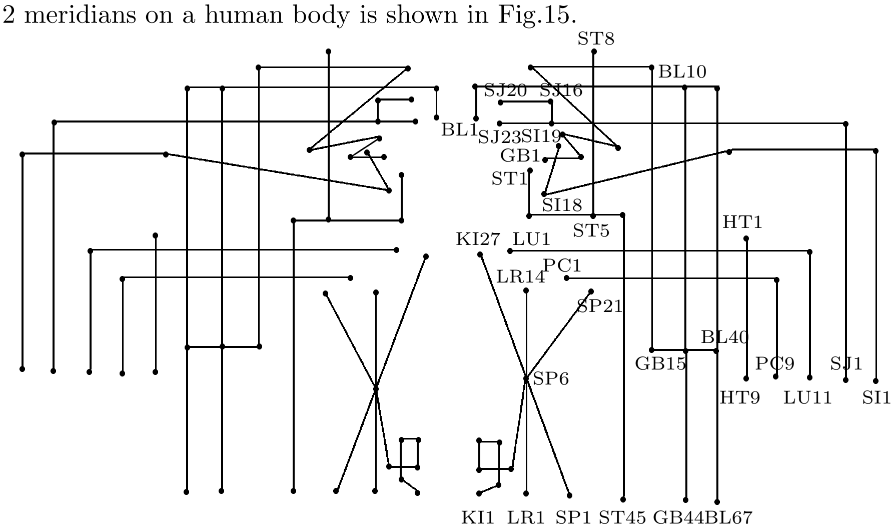


Fig.15 12 Meridian graph on a human body

By the traditional Chinese medicine ([28]), if there exists an imbalanced acupoint on one of the 12 meridians, this person must has illness and in turn, there must be imbalance acupoints on the 12 meridians for a patient. Thus, finding out which acupoint on which meridian is in imbalance with Yin more than Yang or Yang more than Yin is the main duty of a Chinese doctor. Then, the doctor regulates the meridian by acupuncture or drugs so that the balance on the imbalance acupoints recovers again, and then the patient recovers.

Then, *what is the significance of the treatment theory in traditional Chinese medicine to science*? It implies we are not need a large complex network for holding on the body of human. *Whether or not classically mathematical elements enough for understanding complex networks, i.e., matters in the universe*? The answer is negative because all of them are local. Then, *could we establish a mathematics over elements underlying combinatorial structures*? The answer is affirmative, i.e., *mathematical combinatorics* discussed in this paper. Certainly, we can introduce 2 kinds of elements respectively on living of non-living matters.

Element 1(Non-Living Body). A *continuity flow* $\overrightarrow{G}^L$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L: v \to L(v),\ (v,u) \to L(v,u)$, 2 end-operators $A^+_{vu}: L(v,u) \to L^{A^+_{vu}}(v,u)$ and $A^+_{uv}: L(u,v) \to L^{A^+_{uv}}(u,v)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$ such as those shown in Fig.16,

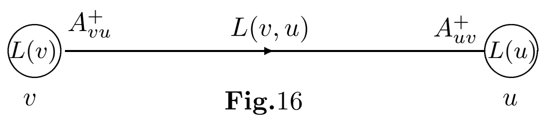


Fig.16

with $L(v,u) = -L(u,v)$, $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and holding with continuity equation

$$\sum_{u \in N_G(v)} L^{A^+_{vu}}(v,u) = L(v) \quad \text{for} \quad \forall v \in V\left(\overrightarrow{G}\right).$$

Element 2(Living Body). A *harmonic flow* $\overrightarrow{G}^L$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L: v \to L(v) - iL(v)$ for $v \in E\left(\overrightarrow{G}\right)$ and $L: (v,u) \to L(v,u) - iL(v,u)$, 2 end-operators $A^+_{vu}: L(v,u) - iL(v,u) \to L^{A^+_{vu}}(v,u) - iL^{A^+_{vu}}(v,u)$ and $A^+_{uv}: L(v,u) - iL(v,u) \to L^{A^+_{uv}}(v,u) - iL^{A^+_{uv}}(v,u)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$ such as those shown in Fig.17,

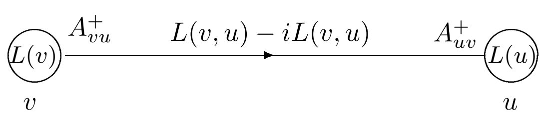


Fig.17

where $i^2 = -1$, $L(v,u) = -L(u,v)$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and holding with continuity equation

$$\sum_{u \in N_G(v)} \left(L^{A^+_{vu}}(v,u) - iL^{A^+_{vu}}(v,u)\right) = L(v) - iL(v)$$

for $\forall v \in V\left(\overrightarrow{G}\right)$.

Let $\mathscr{G}$ be a closed family of graphs $\overrightarrow{G}$ under the union operation and let $\mathscr{B}$ be a linear space $(\mathscr{B};+,\cdot)$, or furthermore, a commutative ring, a Banach or Hilbert space $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$. Denoted by $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ and $\left(\mathscr{G}_{\mathscr{B}}^{\pm};+,\cdot\right)$ the respectively elements 1 and 2 form by graphs $G\in\mathscr{G}$. Then, elements 1 and 2 can be viewed as vectors underlying an embedded graph G in space, which enable us to establish mathematics on complex networks and get results following.

Theorem 3.2([9-10,14-18]) *If $\mathscr{G}$ is a closed family of graphs $\overrightarrow{G}$ under the union operation and $\mathscr{B}$ a linear space $(\mathscr{B};+,\cdot)$, then, $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ and $\left(\mathscr{G}_{\mathscr{B}}^{\pm};+,\cdot\right)$ with linear operators A_{vu}^{+}, A_{uv}^{+} for $\forall(v,u)\in E\left(\bigcup\limits_{G\in\mathscr{G}}\overrightarrow{G}\right)$ under operations $+$ and $\cdot$ form respectively a linear space, and furthermore, a commutative ring if $\mathscr{B}$ is a commutative ring $(\mathscr{B};+,\cdot)$ over a field $\mathscr{F}$.*

Theorem 3.3([9-10,14-18]) *If $\mathscr{G}$ is a closed family of graphs under the union operation and $\mathscr{B}$ a Banach space $(\mathscr{B};+,\cdot)$, then, $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ and $\left(\mathscr{G}_{\mathscr{B}}^{\pm};+,\cdot\right)$ with linear operators A_{vu}^{+}, A_{uv}^{+} for $\forall(v,u)\in E\left(\bigcup\limits_{G\in\mathscr{G}}\overrightarrow{G}\right)$ under operations $+$ and $\cdot$ form respectively a Banach or Hilbert space respect to that $\mathscr{B}$ is a Banach or Hilbert space.*

A few well-known results such as those of Banach theorem, closed graph theorem and Hahn-Banach theorem are also generalized on elements 1 and 2. For example, we obtained results following.

Theorem 3.4(Taylor, [15]) *Let $\overrightarrow{G}^L\in\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ and there exist kth order derivative of L to t on a domain $\mathscr{D}\subset\mathbb{R}$, where $k\geq 1$. If A_{vu}^{+}, A_{uv}^{+} are linear for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, then*

$$\overrightarrow{G}^L=\overrightarrow{G}^{L(t_0)}+\frac{t-t_0}{1!}\overrightarrow{G}^{L'(t_0)}+\cdots+\frac{(t-t_0)^k}{k!}\overrightarrow{G}^{L^{(k)}(t_0)}+o\left((t-t_0)^{-k}\,\overrightarrow{G}\right),$$

for $\forall t_0\in\mathscr{D}$, where $o\left((t-t_0)^{-k}\,\overrightarrow{G}\right)$ denotes such an infinitesimal term $\widehat{L}$ of L that

$$\lim_{t\to t_0}\frac{\widehat{L}(v,u)}{(t-t_0)^k}=0\quad for\quad \forall(v,u)\in E\left(\overrightarrow{G}\right).$$

Particularly, if $L(v,u)=f(t)c_{vu}$, where c_{vu} is a constant, denoted by $f(t)\overrightarrow{G}^{L_C}$ with $L_C: (v,u)\to c_{vu}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ and

$$f(t)=f(t_0)+\frac{(t-t_0)}{1!}f'(t_0)+\frac{(t-t_0)^2}{2!}f''(t_0)+\cdots+\frac{(t-t_0)^k}{k!}f^{(k)}(t_0)+o\left((t-t_0)^k\right),$$

then

$$f(t)\overrightarrow{G}^{L_C}=f(t)\cdot\overrightarrow{G}^{L_C}.$$

Theorem 3.5(Hahn-Banach, [19]) *Let $\mathscr{H}_{\mathscr{B}}^{\pm}$ be an element 2 subspace of $\mathscr{G}_{\mathscr{B}}^{\pm}$ and let $F:\mathscr{H}_{\mathscr{B}}^{\pm}\to\mathbb{C}$ be a linear continuous functional on $\mathscr{H}_{\mathscr{B}}^{\pm}$. Then, there is a linear continuous functional $\widetilde{F}:\mathscr{G}_{\mathscr{B}}^{\pm}\to\mathbb{C}$ hold with*

(1) $\widetilde{F}\left(\overrightarrow{G}^{L^2}\right) = F\left(\overrightarrow{G}^{L^2}\right)$ *if* $\overrightarrow{G}^{L^2} \in \mathscr{H}_{\mathscr{B}}^{\pm}$;

(2) $\left\|\widetilde{F}\right\| = \|F\|$.

For applications of elements 1 and 2 to physics and other sciences such as those of elementary particles, gravitations, ecological system, $\cdots$ etc., the reader is refereed to references [11]-[13] and [18]-[20] for details.

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