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Algebraic Properties of the Path Complexes of Cycles

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Abstract: Let G be a simple graph and $\Delta_t(G)$ be a simplicial complex whose facets correspond to the paths of length t ($t \geq 2$) in G. It is shown that $\Delta_t(C_n)$ is matroid, vertex decomposable, shellable and Cohen-Macaulay if and only if $n = t$ or $n = t+1$, where C_n is an n-cycle. As a consequence we show that if $n = t$ or $t+1$ then $\Delta_t(C_n)$ is partitionable and Stanley's conjecture holds for $K[\Delta_t(C_n)]$.

Key Words: Vertex decomposable, simplicial complex, matroid, path.

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§1. Introduction

Let $R = K[x_1, \cdots, x_n]$, where K is a field. Fix an integer $n \geq t \geq 2$ and let G be a directed graph. A sequence $x_{i_1}, \cdots, x_{i_t}$ of distinct vertices is called a path of length t if there are $t-1$ distinct directed edges $e_1, \cdots, e_{t-1}$ where e_j is a directed edge from x_{i_j} to $x_{i_{j+1}}$. Then the path ideal of G of length t is the monomial ideal $I_t(G) = (x_{i_1} \cdots x_{i_t} : x_{i_1}, \cdots, x_{i_t}$ is a path of length t in G in the polynomial ring $R = K[x_1, \cdots, x_n]$. The distance $d(x, y)$ of two vertices x and y of a graph G is the length of the shortest path from x to y. The path complex $\Delta_t(G)$ is defined by

$$\Delta_t(G) = \langle \{x_{i_1}, \cdots, x_{i_t}\} : x_{i_1}, \cdots, x_{i_t} \text{ is a path of length t in G } \rangle.$$

Path ideals of graphs were first introduced by Conca and De Negri [3] in the context of monomial ideals of linear type. Recently the path ideal of cycles has been extensively studied by several mathematicians. In [9], it is shown that $I_2(C_n)$ is sequentially Cohen-Macaulay, if and only if, $n = 3$ or $n = 5$. Generalizing this result, in [13], it is proved that $I_t(C_n)$, $(t > 2)$, is sequentially Cohen-Macaulay, if and only if $n = t$ or $n = t+1$ or $n = 2t+1$. Also, the Betti numbers of the ideal $I_t(C_n)$ and $I_t(L_n)$ is computed explicitly in [1]. In particular, it has been shown that

Theorem 1.1(Corollary 5.15, [1]) *Let n, t, p and d be integers such that $n \geq t \geq 2$, $n = (t+1)p + d$, where $p \geq 0$ and $0 \leq d < (t+1)$. Then,*

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(*i*) *The projective dimension of the path ideal of a graph cycle* C_n *or line* L_n *is given by*

$$\operatorname{pd}(I_t(C_n)) = \begin{cases} 2p, & d \neq 0 \\ 2p-1, & d = 0 \end{cases} \qquad \operatorname{pd}(I_t(L_n)) = \begin{cases} 2p-1, & d \neq t, \\ 2p, & d = t. \end{cases}$$

(*ii*) *The regularity of the path ideal of a graph cycle* C_n *or line* L_n *is given by*

$$\operatorname{reg}(I_t(C_n)) = (t-1)p + d + 1$$
$$\operatorname{reg}(I_t(L_n)) = \begin{cases} p(t-1)+1, & d < t, \\ p(t-1)+t, & d = t. \end{cases}$$

In [8] it has been shown that, $\Delta_t(G)$ is a simplicial tree if G is a rooted tree and $t \geq 2$. One of interesting problems in combinatorial commutative algebra is the Stanley's conjectures. The Stanley's conjectures are studied by many researchers. Let R be a $\mathbb{N}^n$-graded ring and M a $\mathbb{Z}^n$- graded R- module. Then, Stanley [10] conjectured that

$$\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$$

He also conjectured in [11] that each Cohen-Macaulay simplicial complex is partitionable. Herzog, Soleyman Jahan and Yassemi in [7] showed that the conjecture about partitionability is a special case of the Stanley's first conjecture. In this work, we study algebraic properties of $\Delta_t(C_n)$. In Section 1, we recall some definitions and results which will be needed later. In Section 3, we show that the following conditions are equivalent for all $t > 2$:

(*i*) $\Delta_t(C_n)$ is matroid;
(*ii*) $\Delta_t(C_n)$ is vertex decomposable;
(*iii*) $\Delta_t(C\ _n)$ is shellable;
(*iv*) $\Delta_t(C_n)$ is Cohen-Macaulay;
(*v*) $n = t$ or $t + 1$.

(See Theorem 3.6).

In Section 4 as an application of our results we show that if $n = t$ or $t + 1$ then $\Delta_t(C_n)$ is partitionable and Stanley's conjecture holds for $K[\Delta_t(C_n)]$.

§2. Preliminaries

In this section we recall some definitions and results which will be needed later.

Definition 2.1 *A simplicial complex* Δ *over a set of vertices* $V = \{x_1, \cdots, x_n\}$*, is a collection of subsets of* V*, with the property that:*

(*a*) $\{x_i\} \in \Delta$ *for all* i*;*
(*b*) *If* $F \in \Delta$*, then all subsets of* F *are also in* Δ *(including the empty set).*

An element of Δ is called a *face* of Δ and complement of a face F is $V \setminus F$ and it is denoted by F^c. Also, the complement of the simplicial complex $\Delta = \langle F_1, \cdots, F_r \rangle$ is $\Delta^c = \langle F_1^c, \cdots, F_r^c \rangle$. The *dimension* of a face F of Δ, $\dim F$, is $|F| - 1$ where, $|F|$ is the number of elements of F and $\dim \varnothing = -1$. The faces of dimensions 0 and 1 are called *vertices* and *edges*, respectively. A *non-face* of Δ is a subset F of V with $F \notin \Delta$. we denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of Δ. The maximal faces of Δ under inclusion are called *facets* of Δ. The *dimension* of the simplicial complex Δ, $\dim\Delta$, is the maximum of dimensions of its facets. If all facets of Δ have the same dimension, then Δ is called *pure*.

Let $\mathcal{F}(\Delta) = \{F_1, \cdots, F_q\}$ be the facet set of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ completely and we write $\Delta = \langle F_1, \cdots, F_q \rangle$. A simplicial complex with only one facet is called a *simplex*. A simplicial complex Γ is called a *subcomplex* of Δ, if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.

For $v \in V$, the subcomplex of Δ obtained by removing all faces $F \in \Delta$ with $v \in F$ is denoted by $\Delta \setminus v$. That is,

$$\Delta \setminus v = \langle F \in \Delta : \ v \notin F \rangle.$$

The *link* of a face $F \in \Delta$, denoted by $\mathrm{link}_\Delta(F)$, is a simplicial complex on V with the faces, $G \in \Delta$ such that, $G \cap F = \varnothing$ and $G \cup F \in \Delta$. The link of a vertex $v \in V$ is simply denoted by $\mathrm{link}_\Delta(v)$.

$$\mathrm{link}_\Delta(v) = \big\{F \in \Delta : \ v \notin F, \ F \cup \{v\} \in \Delta \big\}.$$

Let Δ be a simplicial complex over n vertices $\{x_1, \cdots, x_n\}$. For $F \subset \{x_1, \cdots, x_n\}$, we set:

$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

We define the *facet ideal* of Δ, denoted by $I(\Delta)$, to be the ideal of S generated by $\{\mathbf{x}_F : \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{\mathbf{x}_F : \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ.

Definition 2.2 *A simplicial complex Δ on $\{x_1, \cdots, x_n\}$ is said to be a matroid if, for any two facets F and G of Δ and any $x_i \in F$, there exists a $x_j \in G$ such that $(F \setminus \{x_i\}) \cup \{x_j\}$ is a facet of Δ.*

Definition 2.3 *A simplicial complex Δ is recursively defined to be vertex decomposable, if it is either a simplex, or else has some vertex v so that*

(*a*) *Both $\Delta \setminus v$ and $\mathrm{link}_\Delta(v)$ are vertex decomposable, and*

(*b*) *No face of $\mathrm{link}_\Delta(v)$ is a facet of $\Delta \setminus v$.*

A vertex v which satisfies in condition (b) is called a shedding vertex.

Definition 2.4 *A simplicial complex Δ is shellable, if the facets of Δ can be ordered $F_1, \cdots, F_s$ such that, for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \cdots, j-1\}$ with $F_j \setminus F_l = \{v\}$.*

A simplicial complex Δ is called disconnected, if the vertex set V of Δ is a disjoint union $V = V_1 \cup V_2$ such that no face of Δ has vertices in both V_1 and V_2. Otherwise Δ is connected. It is well-known that

$$\text{matroid} \Longrightarrow \text{vertex decomposable} \Longrightarrow \text{shellable} \Longrightarrow \text{Cohen-Macaulay}$$

Definition 2.5 *For a given simplicial complex Δ on V, we define $\Delta^\vee$, the Alexander dual of Δ, by*

$$\Delta^\vee = \{V \setminus F : \ F \notin \Delta\}.$$

It is known that for the complex Δ one has $I_{\Delta^\vee} = I(\Delta^c)$. Let $I \neq 0$ be a homogeneous ideal of S and $\mathbb{N}$ be the set of non-negative integers. For every $i \in \mathbb{N} \cup \{0\}$, one defines:

$$t_i^S(I) = \max\{j : \ \beta_{i,j}^S(I) \neq 0\}$$

where $\beta_{i,j}^S(I)$ is the i, j-th graded Betti number of I as an S-module. The *Castelnuovo-Mumford regularity* of I is given by:

$$\operatorname{reg}(I) = \sup\{t_i^S(I) - i : \ i \in \mathbb{Z}\}.$$

We say that the ideal I has a *d-linear resolution*, if I is generated by homogeneous polynomials of degree d and $\beta_{i,j}^S(I) = 0$, for all $j \neq i + d$ and $i \geq 0$. For an ideal which has a d-linear resolution, the Castelnuovo-Mumford regularity would be d. If I is a graded ideal of S, we write (I_d) for the ideal generated by all homogeneous polynomials of degree d belonging to I.

Definition 2.6 *A graded ideal I is componentwise linear if (I_d) has a linear resolution for all d.*

Also, we write $I_{[d]}$ for the ideal generated by the squarefree monomials of degree d belonging to I.

Definition 2.7 *A graded S-module M is called sequentially Cohen-Macaulay (over K), if there exists a finite filtration of graded S-modules,*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

The Alexander dual, allows us to make a bridge between (sequentially) Cohen-Macaulay ideals and (componetwise) linear ideals.

Definition 2.8(Alexander Duality) *For a square-free monomial ideal $I = (M_1, \cdots, M_q) \subset S =$*

$K[x_1, \cdots, x_n]$, the Alexander dual of I, denoted by $I^\vee$, is defined to be

$$I^\vee = P_{M_1} \cap \cdots \cap P_{M_q}$$

where, P_{M_i} is prime ideal generated by $\{x_j : x_j | M_i\}$.

Theorem 2.9(Proposition 8.2.20, [6]; Theorem 3, [4]) *Let I be a square-free monomial ideal in $S = K[x_1, \cdots, x_n]$.*

(i) The ideal I is componentwise linear ideal if and only if $S/I^\vee$ is sequentially Cohen-Macaulay;

(ii) The ideal I has a q-linear resolution if and only if $S/I^\vee$ is Cohen-Macaulay of dimension $n - q$.

Remark 2.10 Two special cases, we will be considering in this paper, are when G is a cycle C_n, or a line graph L_n on vertices $\{x_1, \cdots, x_n\}$ with edges

$$E(C_n) = \big\{\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\big\};$$
$$E(L_n) = \big\{\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{n-1}, x_n\}\big\}.$$

§3. Vertex Decomposability Path Complexes of Cycles

As the main result of this section, it is shown that $\Delta_t(C_n)$ is matroid, vertex decomposable, shellable and Cohen-Macaualay if and only if $n = t$ or $n = t + 1$. For the proof we shall need the following lemmas and propositions.

Lemma 3.1 *Let $\Delta_t(L_n)$ be a simplicial complex on $\{x_1, \cdots, x_n\}$ and $2 \le t \le n$. Then $\Delta_t(L_n)$ is vertex decomposable.*

Proof If $t = n$, then $\Delta_n(L_n)$ is a simplex which is vertex decomposable. Let $2 \le t < n$ then one has

$$\Delta_t(L_n) = \langle\{x_1, \cdots, x_t\}, \{x_2, \cdots, x_{t+1}\}, \cdots, \{x_{n-t+1}, \cdots, x_n\}\rangle.$$

So $\Delta_t(L_n) \setminus x_n = \langle\{x_1, \cdots, x_t\}, \{x_2, \cdots, x_{t+1}\}, \cdots, \{x_{n-t}, \cdots, x_{n-1}\}\rangle$. Now we use induction on the number of vertices of L_n and by induction hypothesis $\Delta_t(L_n) \setminus x_n$ is vertex decomposable. On the other hand, it is clear that $\mathrm{link}_{\Delta_t(L_n)}\{x_n\} = \langle\{x_{n-t+1}, \cdots, x_{n-1}\}\rangle$. Thus $\mathrm{link}_{\Delta_t(L_n)}\{x_n\}$ is a simplex which is not a facet of $\Delta_t(L_n) \setminus x_n$. Therefore $\Delta_t(L_n)$ is vertex decomposable. □

Lemma 3.2 *Let $\Delta_2(C_n)$ be a simplicial complex on $\{x_1, \cdots, x_n\}$. Then $\Delta_2(C_n)$ is vertex decomposable.*

Proof Since $\Delta_2(C_n) = \langle\{x_1, x_2\}, \{x_2, x_3\}, \cdots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\rangle$, we have

$$\Delta_2(C_n) \setminus x_n = \langle\{x_1, , x_2\}, \{x_2, x_3\}, \cdots, \{x_{n-2}, x_{n-1}\}\rangle.$$

By lemma 3.1 $\Delta_2(C_n)\setminus x_n$ is vertex decomposable. Also it is trivial that $\mathrm{link}_{\Delta_2(C_n)}\{x_n\} = \langle\{x_{n-1}\},\{x_1\}\rangle$ is vertex decomposable and no face of $\mathrm{link}_{\Delta_2(C_n)}\{x_n\}$ is a facet of $\Delta_2(C_n)\setminus x_n$. Therefore $\Delta_2(C_n)$ is vertex decomposable. □

Lemma 3.3 *Let $\Delta_t(C_n)$ be a simplicial complex on $\{x_1,\cdots,x_n\}$ and $3\le t\le n-2$. Then $\Delta_t(C_n)$ is not Cohen-Macaulay.*

Proof It suffices to show that $I_{\Delta_t(C_n)^\vee}$ has not a linear resolution. Since $I_{\Delta_t(C_n)^\vee} = I(\Delta_t(C_n)^c)$ then one can easily check that $I_{\Delta_t(C_n)^\vee} = I_{n-t}(C_n)$. By Theorem 1.1 we have

$$\mathrm{reg}(I_{\Delta_t(C_n)^\vee}) = (n-t-1)p+d+1.$$

Since $3\le t\le n-2$ then one has $\mathrm{reg}(I_{\Delta_t(C_n)^\vee})\ne n-t$ and by Theorem 2.9 $\Delta_t(C_n)$ is not Cohen-Macaulay. □

Proposition 3.4 *Let $\Delta_t(C_n)$ be a simplicial complex on $\{x_1,\cdots,x_n\}$ and $t\ge 3$. Then $\Delta_t(C_n)$ is vertex decomposable if and only if $n=t$ or $t+1$.*

Proof By Lemma 3.3 it suffices to show that if $n=t$ or $t+1$, then $\Delta_t(C_n)$ is vertex decomposable. If $n=t$, then $\Delta_n(C_n)$ is a simplex which is vertex decomposable. If $t=n-1$, then we have

$$\Delta_{n-1}(C_n) = \langle\{x_1,\cdots,x_{n-1}\},\{x_2,\cdots,x_n\},\{x_3,\cdots,x_n,x_1\},\cdots,\{x_n,x_1,\cdots,x_{n-2}\}\rangle.$$

Now we use induction on the number of vertices of C_n and show that $\Delta_{n-1}(C_n)$ is vertex decomposable. It is clear that $\Delta_{n-1}(C_n)\setminus x_n = \langle\{x_1,\cdots,x_{n-1}\}\rangle$ is a simplex which is vertex decomposable.

On the other hand,

$$\mathrm{link}_{\Delta_{n-1}(C_n)}\{x_n\} = \langle\{x_1,\cdots,x_{n-2}\},\cdots,\{x_{n-1},x_1,\cdots,x_{n-3}\}\rangle = \Delta_{n-2}(C_{n-1}).$$

By induction hypothesis $\mathrm{link}_{\Delta_{n-1}(C_n)}\{x_n\}$ is vertex decomposable. It is easy to see that no face of $\mathrm{link}_{\Delta_{n-1}(C_n)}\{x_n\}$ is a facet of $\Delta_{n-1}(C_n)\setminus x_n$. Therefore $\Delta_{n-1}(C_n)$ is vertex decomposable. □

Proposition 3.5 *$\Delta_2(C_n)$ is a matroid if and only if $n=3$ or 4.*

Proof If $n=3$ or 4, then it is easy to see that $\Delta_2(C_n)$ is a matroid. Now we prove the converse. It suffices to show that $\Delta_2(C_n)$ is not a matroid for all $n\ge 5$. We consider two facets $\{x_1,x_2\}$ and $\{x_{n-1},x_n\}$. Then we have $(\{x_1,x_2\}\setminus\{x_1\})\cup\{x_{n-1}\} = \{x_2,x_{n-1}\}$ and $(\{x_1,x_2\}\setminus\{x_1\})\cup\{x_n\} = \{x_2,x_n\}$. Since $\{x_2,x_{n-1}\}$ and $\{x_2,x_n\}$ are not the facets of $\Delta_2(C_n)$. So $\Delta_2(C_n)$ is not matroid for all $n\ge 5$. □

For the simplicial complexes one has the following implication:

Matroid $\Rightarrow$ *vertex decomposable* $\Rightarrow$ *shellable* $\Rightarrow$ *Cohen-Macaulay*

Note that these implications are strict, but by the following theorem, for path complexes, the reverse implications are also valid.

Theorem 3.6 *Let $t \geq 3$. Then the following conditions are equivalent:*

(*i*) $\Delta_t(C_n)$ *is matroid;*

(*ii*) $\Delta_t(C_n)$ *is vertex decomposable;*

(*iii*) $\Delta_t(C_n)$ *is shellable;*

(*iv*) $\Delta_t(C_n)$ *is Cohen-Macaulay;*

(*v*) $n = t$ *or* $t + 1$.

Proof $(i) \Longrightarrow (ii)$, $(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (iv)$ is well-known.

$(iv) \Longrightarrow (v)$ follows from Lemma 3.3 and Proposition 3.4.

$(v) \Longrightarrow$ (i): If $n = t$, then $\Delta_t(C_n)$ is a simplex which is a matroid. If $n = t + 1$, then

$$\Delta_t(C_n) = \langle \{x_1, \cdots, x_t\}, \{x_2, \cdots, x_{t+1}\}, \{x_3, \cdots, x_{t+1}, x_1\}, \cdots, \{x_{t+1}, x_1, \cdots, x_{t-1}\} \rangle.$$

For any two facets F and G of $\Delta_t(C_n)$ one has $\mid F \cap G \mid = t - 1$. We claim that for any two facets F and G of $\Delta_t(C_n)$ and any $x_i \in F$, there exists a $x_j \in G$ such that $(F \setminus \{x_i\}) \cup \{x_j\}$ is a facet of $\Delta_t(C_n)$. We have to consider two cases. If $x_i \in F$ and $x_i \notin G$, then we choose $x_j \in G$ such that $x_j \notin F$. Thus $(F \setminus \{x_i\}) \cup \{x_j\} = G$ which is a facet of $\Delta_t(C_n)$.
For other case, if $x_i \in F$ and $x_i \in G$, then we choose $x_j \in G$ such that x_j is the same x_i. Therefore $(F \setminus \{x_i\}) \cup \{x_i\} = F$ is a facet of $\Delta_t(C_n)$ which completes the proof. □

§4. Stanley Decompositions

Let R be any standard graded K- algebra over an infinite field K, *i.e*, R is a finitely generated graded algebra $R = \bigoplus_{i \geq 0} R_i$ such that $R_0 = K$ and R is generated by R_1. There are several characterizations of the depth of such an algebra. We use the one that depth(R) is the maximal length of a regular R- sequence consisting of linear forms. Let $x_F = \sqcap_{i \in F} x_i$ be a squarefree monomial for some $F \subseteq [n]$ and $Z \subseteq \{x_1, \cdots, x_n\}$. The K- subspace $x_F K[Z]$ of $S = K[x_1, \cdots, x_n]$ is the subspace generated by monomials $x_F u$, where u is a monomial in the polynomial ring $K[Z]$. It is called a square free Stanley space if $\{x_i : i \in F\} \subseteq Z$. The dimension of this Stanley space is $|Z|$. Let Δ be a simplicial complex on $\{x_1, \cdots, x_n\}$. A square free Stanley decomposition $\mathcal{D}$ of $K[\Delta]$ is a finite direct sum $\bigoplus_i u_i K[Z_i]$ of squarefree Stanley spaces which is isomorphic as a $\mathbb{Z}^n$- graded K- vector space to $K[\Delta]$, *i.e.*

$$K[\Delta] \cong \bigoplus_i u_i K[Z_i].$$

We denote by sdepth$(\mathcal{D})$ the minimal dimension of a Stanley space in $\mathcal{D}$ and define sdepth$(K[\Delta])$ $= \max\{\text{sdepth}(\mathcal{D})\}$, where $\mathcal{D}$ is a Stanley decomposition of $K[\Delta]$. Stanley conjectured in [10] the upper bound for the depth of $K[\Delta]$ holding with

$$\text{depth}\,(K[\Delta]) \leq \text{sdepth}\,(K[\Delta]).$$

Also we recall another conjecture of Stanley. Let Δ be again a simplicial complex on $\{x_1, \cdots, x_n\}$ with facets $G_1, \cdots, G_t$. The complex Δ is called partitionable if there exists a partition $\Delta = \bigcup_{i=1}^{t}[F_i, G_i]$ where $F_i \subseteq G_i$ are suitable faces of Δ. Here the interval $[F_i, G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In [11] and [12] respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable. This conjecture is a special case of the previous conjecture. Indeed, Herzog, Soleyman Jahan and Yassemi [7] proved that for Cohen-Macaulay simplicial complex Δ on $\{x_1, \ldots, x_n\}$ we have that $\text{depth}\,(K[\Delta]) \leq \text{sdepth}\,(K[\Delta])$ if and only if Δ is partitionable. Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. Then as a consequence of our results, we obtain

Corollary 3.1 *If $n = t$ or $t + 1$ then $\Delta_t\,(C_n)$ is partitionable and Stanley's conjecture holds for $K[\Delta_t\,(C_n)]$.*

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