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The Gourava Index of Four Operations on Graphs

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Abstract: Molecular descriptor are major in the study of QSAR/QSPR. There are numerous importance of graph theory in the field of structural chemistry. In the present paper, we study the Gourava index of four operation on graphs.

Key Words: Gourava index, Zagreb index, graph operations.

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§1. Introduction

Let ϑ denotes the collection entire graphs. A mapping $T : \vartheta \to R$ is called a topological index, if for every graph H isomorphic to G, $T(G) = T(H)$. In chemical graph theory, topological indices have several applications in isomer discrimination, QSAR/QSPR investigation, pharmaceutical drug design and many more [5]. There are few important class of topological indices that are extensively studied by a number of researchers. Out of these topological indices, the first and second Zagreb indices, first appeared in a topological structure for the total π -energy of conjugated molecules, were introduced by Gutman et.al., in [8].

The first and second Zagreb indices [3] of a molecular graph G are defined as

$$M_1(G) = \sum_{uv\in E(G)} [d(u) + d(v)].$$

and

$$M_2(G) = \sum_{uv\in E(G)} [d(u)d(v)].$$

Motivated by the definitions of the Zagreb indices and their wide applications, V. R. Kulli [10], introduced the first Gourava index of a molecular graph as follows.

The first Gourava index of a graph G is defined as

$$GO_1(G) = \sum_{uv\in E(G)} [d(u) + d(v) + d(u)d(v)].$$

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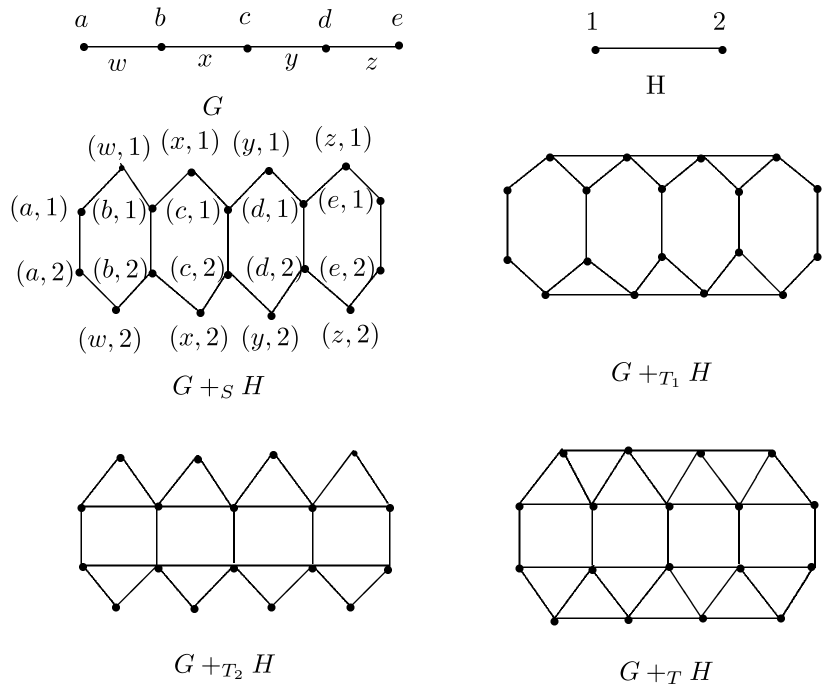

Figure 1: Graph G, H and $G +_F H$

The cartesian product is an important method to construct a ample graph and play vital role in the design and analysis the network. The cartesian product of two connected graphs G and H, which is denoted by $G \Box H$, is a graph such that the set of vertices is $V(G) \Box V(H)$ and two vertices (p_1, q_1) and (p_2, q_2) of $G \Box H$ are adjacent if and only if $p_1 = p_2$ and q_1 is adjacent with q_2 in H otherwise $q_1 = q_2$ and p_1 is adjacent with p_2 in G. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, there are four related graphs as follows:

For any connected graph G, define four operator graphs $S(G)$, $T(G)$, $Q(G) = T_1(G)$ and $R(G) = T_2(G)$ as follows:

- $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G, i.e., replacing each edge of G by a path of length 2 ([1, 18]).
- The total graph $T(G)$ of a graph G is the graph whose vertex set $V \bigcup E$, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident ([14]).
- $Q(G)$ is the graph obtained by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G, by a new edge ([15]).
- $R(G)$ is the graph obtained by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge ([15]).

Suppose that G and H are two connected graphs. M. Eliasi, B. Taeri [6] introduced four new operations named as F-sum graphs, on these graphs that are based on S, T_2, T_1, T as follows.

Let F be one of the symbols S, T_2, T_1 or $, T$. The F-sum denoted by $G +_F H$ of graphs G and H, is a graph with the set of vertices $V(G +_F H) = (V(G) \bigcup E(G)) \times V(H)$ and (p_1, p_2)

$(q_1, q_2) \in E(G +_F H)$, if and only if $p_1 = p_2 \in V(G)$ and $q_1q_2 \in E(H)$ or $q_1 = q_2$ and $(p_1, p_2) \in E(F(G))$.

Throughout this paper, we consider only simple, connected, finite and undirected graphs. For a graph G, the order and the size of graph are denoted as n_G and e_G respectively.

In mathematical chemistry, graph operations act as a very essential role, viz., as some chemically interesting graphs can be derived from some simpler graphs by operations on graphs.

In [4], H. Deng et al. computed the first and second Zagreb indices for graph operations $S(G)$, $R(G)$, $Q(G)$ and $T(G)$. Here, we extend this study by investigate the Gourava index of four operation on graphs. Investigators need to study more details on calculating topological indices of graph operations can be refer [2, 7, 9, 11, 12, 16, 17, 19].

§2. The Gourava Index of F-Sum of Graphs

In this section, we discuss main results of Gourava index of F-sum of graphs.

Theorem 2.1 *Let G and H be two connected graphs. Then,*

$$GO_1(G +_s H) = n_H GO_1(G) + n_G GO_1(H) + e_H M_1(G) + 2e_G M_1(H) + 8n_H e_G + 12e_H e_G.$$

Proof From the definition of Gourava index,

$$\begin{aligned}
GO_1(G +_s H) &= \sum_{(p_1,q_1)(p_2,q_2)\in E(G+_sH)} [d_{G+_sH}(p_1,q_1) + d_{G+_sH}(p_2,q_2) \\
&\quad + d_{G+_sH}(p_1,q_1)d_{G+_sH}(p_2,q_2)] \\
&= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)} [d_{G+_sH}(p_1,q_1) + d_{G+_sH}(p_1,q_2) \\
&\quad + d_{G+_sH}(p_1,q_1)d_{G+_sH}(p_1,q_2)] \\
&\quad + \sum_{q_1\in V(H)}\sum_{p_1p_2\in E(S(G))} [d_{G+_sH}(p_1,q_1) + d_{G+_sH}(p_1,q_2) \\
&\quad + d_{G+_sH}(p_1,q_1)d_{G+_sH}(p_1,q_2)] \\
&= I_1 + I_2, \qquad (1)
\end{aligned}$$

where I_1, I_2 are the sums of the above terms, in order.

For vertex $\forall p_1 \in V(G)$ and $q_1q_2 \in E(H)$ we get

$$\begin{aligned}
I_1 &= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)} [d_G(p_1) + d_H(q_1) + d_G(p_1) + d_H(q_2) \\
&\quad + [d_G(p_1) + d_H(q_1)][d_G(p_1) + d_H(q_2)]] \\
&= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)} \big[2d_G(p_1) + d_H(q_1) + d_H(q_2) + d_G^2(p_1) + d_G(p_1)[d_H(q_1) + d_H(q_2)] \\
&\quad + d_H(q_1)d_H(q_2)]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{p_1\in V(G)} \big[2e_H d_G(p_1) + M_1(H) + e_H d_G^2(p_1) + d_G(p_1)M_1(H) + M_2(H)\big]\\
&= 4e_He_G + n_G GO_1(H) + e_H M_1(G) + 2e_G M_1(H).
\end{aligned}$$

For edge $\forall p_1p_2 \in E(S(G)$, where the vertex $p_1 \in V(G)$, $p_2 \in V(S(G)) - V(G)$ and $q_1 \in V(H)$, since $|E(S(G)| = 2|E(G)|$,

$$\begin{aligned}
I_2 &= \sum_{q_1\in V(H)} \sum_{p_1p_2\in E(S(G))} \big[d_{S(G)}(p_1) + d_H(q_1) + d_{S(G)}(p_2)\\
&\quad +[d_{S(G)}(p_1) + d_H(q_1)]d_{S(G)}(p_2)\big]\\
&= \sum_{q_1\in V(H)} [GO_1(S(G)) + 2e_G d_H(q_1) + 2e_G d_H(q_1)]\\
&= n_H GO_1(S(G)) + 8e_He_G
\end{aligned}$$

We know that, $M_1S(G) = M_1(G) + 4e_G$ and $M_2S(G) = M_2(G) + 4e_G$. Therefore,

$$GO_1(S(G)) = GO_1(G) + 8e_G \quad \text{and} \quad I_2 = n_H GO_1(G) + 8n_He_G + 8e_He_G.$$

Substituting I_1 and I_2 in (1) we get required result

$$GO_1(G +_s H) = n_H GO_1(G) + n_G GO_1(H) + e_H M_1(G) + 2e_G M_1(H) + 8n_He_G + 12e_He_G. \quad \square$$

Theorem 2.2 *Let G and H be two connected graphs. Then,*

$$\begin{aligned}
GO_1(G +_{T_1} H) &= n_G GO_1(H) + 5e_H M_1(G) + 3e_G M_1(H) + 2n_H M_1(G) + 2e_G n_H M_1(G)\\
&\quad +10e_He_G + n_H \sum_{\substack{u_iu_j\in E(G),\\ u_ju_k\in E(G)}} [d_G(u_i)[1 + d_G(u_k)] + d_G(u_k)[1 + d_G(u_j)]\\
&\quad +d_G(u_j)[d_G(u_i) + d_G(u_j)]]
\end{aligned}$$

Proof Consider

$$\begin{aligned}
GO_1(G +_{T_1} H) &= \sum_{(p_1,q_1)(p_2,q_2)\in E(G+_{T_1}H)} \big[d_{G+_{T_1}H}(p_1,q_1) + d_{G+_{T_1}H}(p_2,q_2)\\
&\quad +d_{G+_{T_1}H}(p_1,q_1)d_{G+_{T_1}H}(p_2,q_2)\big]\\
&= \sum_{p_1\in V(G)} \sum_{q_1q_2\in E(H)} \big[d_{G+_{T_1}H}(p_1,q_1) + d_{G+_{T_1}H}(p_1,q_2)\\
&\quad +d_{G+_{T_1}H}(p_1,q_1)d_{G+_{T_1}H}(p_1,q_2)\big]\\
&\quad + \sum_{q_1\in V(H)} \sum_{p_1p_2\in E(T_1(G))} \big[d_{G+_{T_1}H}(p_1,q_1) + d_{G+_{T_1}H}(p_2,q_1)\\
&\quad +d_{G+_{T_1}H}(p_1,q_1)d_{G+_{T_1}H}(p_2,q_1)\big].
\end{aligned}$$

The edge set $E(T_1(G))$ split in to $E(S(G))$ and $E(L(G))$. Let $E(T_1(G)) = \alpha_1$, $V(G) = \beta$,

$V(T_1(G)) - V(G) = \gamma_1$. Then,

$$\begin{aligned}
GO_1(G +_{T_1} H) &= \sum_{p_1 \in V(G)} \sum_{q_1 q_2 \in E(H)} \big[d_{G+_{T_1}H}(p_1,q_1) + d_{G+_{T_1}H}(p_1,q_2) \\
&\quad + d_{G+_{T_1}H}(p_1,q_1) d_{G+_{T_1}H}(p_1,q_2)\big] \\
&\quad + \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1 \in \beta, \\ p_2 \in \gamma_1}} \big[d_{G+_{T_1}H}(p_1,q_1) + d_{G+_{T_1}H}(p_2,q_1) \\
&\quad + d_{G+_{T_1}H}(p_1,q_1) d_{G+_{T_1}H}(p_2,q_1)\big] \\
&\quad + \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1, p_2 \in \gamma_1}} \big[d_{G+_{T_1}H}(p_1,q_1) + d_{G+_{T_1}H}(p_2,q_1) \\
&\quad + d_{G+_{T_1}H}(p_1,q_1) d_{G+_{T_1}H}(p_2,q_1)\big] \\
&= J_1 + J_2 + J_3, \qquad (2)
\end{aligned}$$

where J_1, J_2, J_3 are the sums of the above terms, in order

$$\begin{aligned}
J_1 &= \sum_{p_1 \in V(G)} \sum_{q_1 q_2 \in E(H)} \big[2d_{T_1(G)}(p_1) + d_H(q_1) + d_H(q_2) \\
&\quad + [d_{T_1(G)}(p_1) + d_H(q_1)][d_{T_1(G)}(p_1) + d_H(q_2)]\big] \\
&= \sum_{p_1 \in V(G)} \sum_{q_1 q_2 \in E(H)} \Big[2d_{T_1(G)}(p_1) + d_H(q_1) + d_H(q_2) + d^2_{T_1(G)}(p_1) + d_{T_1(G)}(p_1) d_H(q_2) \\
&\quad + d_H(q_1) d_H(q_2) + d_{T_1(G)}(p_1) d_H(q_1)\Big] \\
&= \sum_{p_1 \in V(G)} \big[2e_H d_G(p_1) + GO_1(H) + e_H d^2_G(p_1) + d_G(p_1) d_H(q_2) + d_G(p_1) d_H(q_1)\big] \\
&= n_G GO_1(H) + e_H M_1(G) + e_G M_1(H) + 2e_H e_G.
\end{aligned}$$

$$\begin{aligned}
J_2 &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1 \in \beta, \\ p_2 \in \gamma_1}} \big[[d_{T_1(G)}(p_1) + 2d_H(q_1) + d_{T_1(G)}(p_2)] \\
&\quad + [d_{T_1(G)}(p_1) + d_H(q_1)][d_{T_1(G)}(p_2) + d_H(q_1)]\big] \\
&= \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1 \in \beta, \\ p_2 \in \gamma_1}} \big[[d_G(p_1) + 2d_H(q_1) + d_{T_1(G)}(p_2)] \\
&\quad + [d_G(p_1) + d_H(q_1)][d_{T_1(G)}(p_2) + d_H(q_1)]\big] \\
&= \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1 \in \beta, \\ p_2 \in \gamma_1}} \big[d_G(p_1) + 2d_H(q_1) + d_{T_1(G)}(p_2) + d_G(p_1) d_{T_1(G)}(p_2) \\
&\quad + d_G(p_1) d_H(q_1) + d_H(q_1) d_{T_1(G)}(p_2) + d^2_H(q_1)\big]
\end{aligned}$$

$$\begin{aligned}
&= \sum_{q_1\in V(H)}\sum_{p_1\in V(G)}\left[d_G(p_1)[d_G(p_1)+2d_H(q_1)+d_G(p_1)d_H(q_1)+d_H^2(q_1)]\right]\\
&\quad+\sum_{q_1\in V(H)}\sum_{\substack{p_1p_2\in\alpha_1,\\ p_1\in\beta,\\ p_2\in\gamma_1}}\left[d_{T_1(G)}(p_2)+d_G(p_1)d_{T_1(G)}(p_2)+d_H(q_1)d_{T_1(G)}(p_2)\right].
\end{aligned}$$

We observe that for $p_2\in V(T_1(G))-V(G)$, $d_{T_1(G)}(p_2)=d_G(w_i)+d_G(w_j)$, where $p_2=w_iw_j\in E(G)$. Hence,

$$\begin{aligned}
J_2 &= n_HM_1(G)+8e_He_G+2e_HM_1(G)+2e_GM_1(H)\\
&\quad+\sum_{q_1\in V(H)}\sum_{w_iw_j\in E(G)}[d_G(w_i)+d_G(w_j)+d_G(p_1)[d_G(w_i)+d_G(w_j)]\\
&\quad+d_H(q_1)[d_G(w_i)+d_G(w_j)]]\\
&= 2n_HM_1(G)+8e_He_G+4e_HM_1(G)+2e_GM_1(H)+2e_Gn_HM_1(G).
\end{aligned}$$

$$\begin{aligned}
J_3 &= \sum_{q_1\in V(H)}\sum_{p_1p_2\in\alpha_1,p_1,p_2\in\gamma_1}\left[[d_{T_1(G)}(p_1)+d_{T_1(G)}(p_2)]+[d_{T_1(G)}(p_1)d_{T_1(G)}(p_2)]\right]\\
&= n_H\sum_{\substack{u_iu_j\in E(G),\\ u_ju_k\in E(G)}}[[d_G(u_i)+d_G(u_j)+d_G(u_j)+d_G(u_k)]\\
&\quad+[d_G(u_i)+d_G(u_j)][d_G(u_j)+d_G(u_k)]]\\
&= n_H\sum_{\substack{u_iu_j\in E(G),\\ u_ju_k\in E(G)}}[d_G(u_i)[1+d_G(u_k)]+d_G(u_k)[1+d_G(u_j)]+d_G(u_j)[d_G(u_i)+d_G(u_j)]].
\end{aligned}$$

Adding J_1, J_2, J_3 in (2) we get desired result. □

Theorem 2.3 *Let G and H be two connected graphs. Then,*

$$\begin{aligned}
GO_1(G+_{T_2}H) &= 4n_HGO_1(G)+GO_1(H)+8e_HM_1(G)+5e_GM_1(H)+6n_HM_1(G)\\
&\quad+4n_HM_2(G)+24e_He_G+4n_He_G
\end{aligned}$$

Proof We know that,

$$\begin{aligned}
GO_1(G+_{T_2}H) &= \sum_{(p_1,q_1)(p_2,q_2)\in E(G+_{T_2}H)}\left[d_{G+_{T_2}H}(p_1,q_1)+d_{G+_{T_2}H}(p_2,q_2)\right.\\
&\quad\left.+d_{G+_{T_2}H}(p_1,q_1)d_{G+_{T_2}H}(p_1,q_2)\right]\\
&= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)}\left[d_{G+_{T_2}H}(p_1,q_1)+d_{G+_{T_2}H}(p_1,q_2)\right.\\
&\quad\left.+d_{G+_{T_2}H}(p_1,q_1)d_{G+_{T_2}H}(p_1,q_2)\right]\\
&+\sum_{q_1\in V(H)}\sum_{p_1p_2\in E(T_1(G))}\left[d_{G+_{T_2}H}(p_1,q_1)+d_{G+_{T_2}H}(p_2,q_1)\right.\\
&\quad\left.+d_{G+_{T_2}H}(p_1,q_1)d_{G+_{T_2}H}(p_2,q_1)\right]
\end{aligned}$$

$$= \quad K_1 + K_2, \tag{3}$$

where K_1 and K_1 are the sums of the above terms, in order

$$\begin{aligned}
K_1 &= \sum_{p_1 \in V(G)} \sum_{q_1 q_2 \in E(H)} \Big[2d_{T_2(G)}(p_1) + d_H(q_1) + d_H(q_2) \\
&\quad + d^2_{T_2(G)}(p_1) + d_{T_2(G)}(p_1)[d_H(q_1) + d_H(q_2)] + d_H(q_1)d_H(q_2)\Big] \\
&= \sum_{p_1 \in V(G)} \sum_{q_1 q_2 \in E(H)} \big[4d_{(G)}(p_1) + d_H(q_1) + d_H(q_2) \\
&\quad + 4d^2_G(p_1) + 2d_{(G)}(p_1)[d_H(q_1) + d_H(q_2)] + d_H(q_1)d_H(q_2)\big] \\
&= \sum_{p_1 \in V(G)} \big[4e_H d_G(p_1) + GO_1(H) + 4e_H d^2_G(p_1) + 2d_G(p_1)M_1(H)\big] \\
&= 8e_H e_G + GO_1(H) + 4e_H M_1(G) + 4e_G M_1(H) \qquad (3a)
\end{aligned}$$

for edge $\forall p_1p_2 \in E(T_2(G))$ and vertex $q_1 \in V(H)$. Here we denote $E(T_2(G)) = \alpha_2$, $V(G) = \beta$, $V(T_2(G)) - V(G) = \gamma_2$.

$$\begin{aligned}
K_2 &= \sum_{q_1 \in V(H)} \sum_{p_1 p_2 \in E(T_2(G))} \big[d_{G+_{T_2}H}(p_1, q_1) + d_{G+_{T_2}H}(p_2, q_1) \\
&\quad + d_{G+_{T_2}H}(p_1, q_1)d_{G+_{T_2}H}(p_2, q_1)\big] \\
&\quad + \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_2, \\ p_1 \in \beta, \\ p_2 \in \gamma_2}} \big[d_{G+_{T_2}H}(p_1, q_1) + d_{G+_{T_2}H}(p_2, q_1) + d_{G+_{T_2}H}(p_1, q_1)d_{G+_{T_2}H}(p_2, q_1)\big] \\
&= K_3 + K_4 \qquad (3b).
\end{aligned}$$

for $\forall q_1 \in V(H)$ and edge $p_1p_2 \in E(T_2(G))$ if and only if $p_1p_2 \in E(G)$.

$$\begin{aligned}
K_3 &= \sum_{q_1 \in V(H)} \sum_{p_1 p_2 \in E(G)} \Big[d_{G+_{T_2(G)}H}(p_1, q_1) + d_{G+_{T_2(G)}H}(p_2, q_1) \\
&\quad + d_{G+_{T_2(G)}H}(p_1, q_1)d_{G+_{T_2(G)}H}(p_2, q_1)\Big] \\
&= \sum_{q_1 \in V(H)} \sum_{p_1 p_2 \in E(G)} \big[d_{T_2(G)}(p_1) + d_H(q_1) + d_{T_2(G)}(p_2) + d_H(q_1) \\
&\quad + [d_{T_2(G)}(p_1) + d_H(q_1)][d_{T_2(G)}(p_2) + d_H(q_1)]\big] \\
&= \sum_{q_1 \in V(H)} \sum_{p_1 p_2 \in E(G)} [2d_G(p_1) + 2d_H(q_1) + 2d_G(p_2) + 4d_G(p_1)d_G(p_2) \\
&\quad + 2d_G(p_1)d_H(q_1) + 2d_H(q_1)d_G(p_2) + d^2_H(q_1)\big] \\
&= 4n_H GO_1(G) + 4e_H M_1(G) + e_G M_1(H) + 4n_H M_2(G) + 4e_H e_G.
\end{aligned}$$

Since we have $d_{T_2(G)}(p_1) = 2d_G(P_1)$ for each vertex $p_1 \in V(G)$ and $d_{T_2}(p_2) = 2$ for each vertex $p_2 \in V(T_2(G)) - V(G)$,

$$
\begin{aligned}
K_4 &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1p_2 \in \alpha_2, \\ p_1 \in \beta, \\ p_2 \in \gamma_2}} \big[d_{T_2(G)}(p_1) + d_H(q_1) + d_{T_2(G)}(p_2) \\
&\quad + [d_{T_2(G)}(p_1) + d_H(q_1)]d_{T_2(G)}(p_2)\big] \\
&= \sum_{q_1 \in V(H)} \sum_{\substack{p_1p_2 \in \alpha_2, \\ p_1 \in \beta, \\ p_2 \in \gamma_2}} \big[d_{T_2(G)}(p_1) + d_H(q_1) + d_{T_2(G)}(p_2) \\
&\quad + d_{T_2(G)}(p_1)d_{T_2(G)}(p_2) + d_H(q_1)d_{T_2(G)}(p_2)\big] \\
&= \sum_{q_1 \in V(H)} \sum_{\substack{p_1p_2 \in \alpha_2, \\ p_1 \in \beta, \\ p_2 \in \gamma_2}} [6d_G(p_1) + 3d_H(q_1) + 2] \\
&= \sum_{q_1 \in V(H)} \sum_{p_1 \in V(G)} d_G(p_1)\,[6d_G(p_1) + 3d_H(q_1) + 2] \\
&= 6n_H M_1(G) + 12e_G e_H + 4n_H e_G.
\end{aligned}
$$

Adding K_3 and K_4 and substitute in (3b) we get

$$
4n_H GO_1(G) + 16e_H e_G + 6n_H M_1(G) + 4e_H M_1(G) + e_G M_1(H) + 4n_H M_2(G) + 4n_H e_G. \quad (3c)
$$

Substitute $(3a)$ and (3c) in (3) we get desired results.

$$
\begin{aligned}
GO_1(G +_{T_2} H) &= 4n_H GO_1(G) + GO_1(H) + 8e_H M_1(G) + 5e_G M_1(H) + 6n_H M_1(G) \\
&\quad + 4n_H M_2(G) + 24e_H e_G + 4n_H e_G.
\end{aligned}
$$

This completes the proof. □

Theorem 2.4 *Let G and H be two connected graphs. Then,*

$$
\begin{aligned}
GO_1(G +_T H) &= 4n_H GO_1(G) + n_G GO_1(H) + 12e_H M_1(G) + 6e_G M_1(H) \\
&\quad + 2n_H M_1(G) + e_G M_2(H) + 8e_G M_1(G) + 20e_H e_G \\
&\quad + n_H \sum_{\substack{q_iq_j \in E(G), \\ q_jq_k \in E(G)}} [d_G(q_i) + 2d_G(q_j) + d_G(q_k) \\
&\quad + [d_G(q_i) + d_G(q_j)][d_G(q_j) + d_G(q_k)]]
\end{aligned}
$$

Proof Let

$$
\begin{aligned}
GO_1(G +_T H) &= \sum_{(p_1,q_1)(p_2,q_2) \in E(G +_T H)} [d_{G+_T H}(p_1, q_1) + d_{G+_T H}(p_2, q_2) \\
&\quad + d_{G+_T H}(p_1, q_1) d_{G+_T H}(p_2, q_2)]
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{p_1\in V(G)} \sum_{q_1q_2\in E(H)} [d_{G+_T H}(p_1,q_1) + d_{G+_T H}(p_1,q_2) \\
&\quad + d_{G+_T H}(p_1,q_1) d_{G+_T H}(p_1,q_2)] \\
&\quad + \sum_{q_1\in V(H)} \sum_{p_1p_2\in E(T(G))} [d_{G+_T H}(p_1,q_1) + d_{G+_T H}(p_2,q_1) \\
&\quad + d_{G+_T H}(p_1,q_1) d_{G+_T H}(p_2,q_1)].
\end{aligned}
$$

Note that $E(T(G)) = E(G)\bigcup E(S(G))\bigcup E(L(G))$. We get that

$$
\begin{aligned}
& GO_1(G+_T H) \\
&= \sum_{p_1\in V(G)} \sum_{q_1q_2\in E(H)} [d_{G+_T H}(p_1,q_1) + d_{G+_T H}(p_1,q_2) + d_{G+_T H}(p_1,q_1) d_{G+_T H}(p_1,q_2)] \\
&+ \sum_{q_1\in V(H)} \sum_{\substack{(p_1p_2)\in E(T(G)),\\ (p_1,p_2)\in V(G)}} [d_{G+_T H}(p_1,q_1) + d_{G+_T H}(p_2,q_1) + d_{G+_T H}(p_1,q_1) d_{G+_T H}(p_2,q_1)] \\
&+ \sum_{q_1\in V(H)} \sum_{\substack{(p_1p_2)\in \alpha_3,\\ p_1\in\beta,\\ p_2\in\gamma_3}} [d_{G+_T H}(p_1,q_1) + d_{G+_T H}(p_2,q_1) + d_{G+_T H}(p_1,q_1) d_{G+_T H}(p_2,q_1)] \\
&+ \sum_{q_1\in V(H)} \sum_{\substack{(p_1p_2)\in \alpha_3,\\ (p_1,p_2)\in\gamma_3}} [d_{G+_T H}(p_1,q_1) + d_{G+_T H}(p_2,q_1) + d_{G+_T H}(p_1,q_1) d_{G+_T H}(p_2,q_1)] \\
&= L_1 + L_2 + L_3 + L_4, \qquad (4)
\end{aligned}
$$

where L_1, L_2, L_3, L_4 are the sums of the above terms, in order

$$
\begin{aligned}
L_1 &= \sum_{p_1\in V(G)} \sum_{q_1q_2\in E(H)} \big[2d_{T(G)}(p_1) + d_H(q_1) + d_H(q_2) \\
&\quad + [d_{T(G)}(p_1) + d_H(q_1)][d_{T(G)}(p_1) d_H(q_2)]\big] \\
&= \sum_{p_1\in V(G)} \sum_{q_1q_2\in E(H)} \big[4d_G(p_1) + d_H(q_1) + d_H(q_2) + 4d_G^2(p_1) + 2d_G(p_1)d_H(q_1)\big] \\
&\quad + 2d_G(p_1)d_H(q_2) + d_H(q_1)d_H(q_2) \\
&= n_G GO_1(H) + 4e_H M_1(G) + 4e_G M_1(H) + 8e_G e_H.
\end{aligned}
$$

$$
\begin{aligned}
L_2 &= \sum_{q_1\in V(H)} \sum_{p_1p_2\in \alpha_3, p_1,p_2\in\beta} \big[d_{T(G)}(p_1) + 2d_H(q_1) + d_{T(G)}(p_2) \\
&\quad + [d_{T(G)}(p_1) + d_H(q_1)][d_{T(G)}(p_2) d_H(q_1)]\big] \\
&= \sum_{q_1\in V(H)} \sum_{p_1p_2\in E(G)} \big[2d_G(p_1) + 2d_G(p_2) + 2d_H(q_1) + d_H^2(q_1) + 2d_G(p_2)d_H(q_1) \\
&\quad + 4d_G(p_1)d_G(p_2) + 2d_G(p_1)d_H(q_1)] \\
&= 2n_H GO_1(G) + 4e_H M_1(G) + e_G M_2(H) + 2n_H M_2(G) + 4e_G e_H.
\end{aligned}
$$

$$\begin{aligned} L_3 &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_3, \\ p_1 \in \beta, \\ p_2 \in \gamma_3}} \left[d_{T(G)}(p_1) + d_{T(G)}(p_2) + 2d_H(q_1) \right. \\ &\quad + [d_{T(G)}(p_1) + d_H(q_1)][d_{T(G)}(p_2) d_H(q_1)]] \\ &= \sum_{q_1 \in V(H)} \sum_{(p_1 \in V(G)} \left[d_G(p_1) 2d_G(p_1) + d_H(q_1) + d_H(q_1) + d_G(p_1) d_H(q_1) + d_H^2(q_1) \right] \\ &\quad + \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_3, \\ p_1 \in \beta, \\ p_2 \in \gamma_3}} \left[d_{T(G)}(p_2) + 2d_G(p_1) d_{T(G)}(p_2) + d_H(q_1) d_{T(G)}(p_2) \right]. \end{aligned}$$

Note that $p_2 \in V(T(G)) - V(G)$, $d_{T(G)}(p_2) = d_G(p) + d_G(q)$ where $p_2 = pq \in E(G)$, we further get that

$$\begin{aligned} L_3 &= 2n_H M_1(G) + 4e_H M_1(G) + 2e_G M_1(H) + 8e_H e_G \\ &\quad + \sum_{q_1 \in V(H)} \sum_{\substack{p_1 \in \beta, \\ p_2 \in \gamma_3}} [(d_G(p) + d_G(q)) + 2d_G(p_1)(d_G(p) + d_G(q)) + d_H(q_1)(d_G(p) + d_G(q))] \\ &= 2n_H M_1(G) + 4e_H M_1(G) + 2e_G M_1(H) + 2n_H M_1(G) + 8e_G M_1(G) + 4e_H M_1(G) \\ &= 4n_H M_1(G) + 4e_H M_1(G) + 8e_G M_1(G) + 2e_G M_1(H) + 8e_H e_G. \end{aligned}$$

$$\begin{aligned} L_4 &= \sum_{q_1 \in V(H)} \sum_{(p_1, p_2) \in \gamma_3} [d_{G+_T H}(p_1, q_1) + d_{G+_T H}(p_2, q_1) + d_{G+_T H}(p_1, q_1) d_{G+_T H}(p_2, q_1)] \\ &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1, \\ p_2 \in \gamma_3}} \left[d_{T(G)}(p_1) + d_{T(G)}(p_2) + d_{T(G)}(p_1) d_{T(G)}(p_2) \right] \\ &= n_H \sum_{q_i q_j \in E(G), q_j q_k \in E(G)} [(d_G(q_i) + d_G(q_j)) + (d_G(q_j) + d_G(q_k)) \\ &\quad + [d_G(q_i) + d_G(q_j)][d_G(q_j) + d_G(q_k)]] \end{aligned}$$

Adding L_1, L_2, L_3, L_4 in (4) we get required result. □

§3. Conclusion

In this paper, we obtain explicit expression for the Gourava index of four operation on graphs in terms of first Zagreb and second Zagreb index.

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