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Strongly 2-Multiplicative Graphs

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Abstract: Since the year 2000 a number of authors have studied strongly multiplicative graphs. In this vein we introduce the concept of strongly k-multiplicative graph and prove that certain class of graphs such as paths, binary tree, cycle etc. are strongly 2-multiplicative.

Key Words: Strongly 2-multiplicative, graph labelling, paths, star, fan graph, binary tree, comb graph, triangular snake, ladder.

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§1. Introduction

A graph G consists of a nonempty set $V = V(G)$ of points called vertices and another set $E = E(G)$ whose elements are called edges where each edge is identified with an unordered pair of vertices in V. Each pair $e = (u, v)$ in E of points of V is an edge of G and is said to be incident with u and v. In this case u and v are said to be adjacent to each other. The number of vertices in G is called the order of G.

We begin with some basic definitions and notations [7], [12], [6].

Definition 1.1 *A walk of a graph G is a finite, alternative sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \cdots, v_{n-1}, e_n, v_n$, beginning with v_0 and ending with v_n such that each edge e_i is incident with v_{i-1} and v_i. The number of edges is called the length of the walk. A walk is called a path if all its vertices (and thus necessarily all the edges) are distinct. A path on n vertices is denoted by P_n.*

Definition 1.2 *A walk in a graph is closed if its initial and terminal vertices are identical. A closed walk is called a cycle. A cycle on $n(\geq 3)$ vertices is denoted by C_n.*

Definition 1.3 *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 1.4 *A bigraph or bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. (V_1, V_2) is a bipartition of G.*

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A complete bipartite graph is a bipartite graph with bipartition (V_1, V_2) such that every vertex of V_1 joined to all the vertices of V_2. If V_1 contains m points and V_2 contains n points then the complete bipartite graph is denoted by $K_{m,n}$. A star $K_{1,n}$ is a complete bipartite graph.

Definition 1.5 *A graph is acyclic if it has no cycles. A tree is a connected acyclic graph.*

Definition 1.6 *The wheel $W_n (n \geq 4)$ is the graph obtained from the join of K_1 and C_{n-1}.*

Definition 1.7 *A fan $F_n (n \geq 2)$ is the graph obtained from the join of the path P_n and K_1.*

Definition 1.8 *A ladder L_n is a graph with vertex set $V(L_n) = \{v_i : 1 \leq i \leq 2n\}$ and edge set $E(L_n) = \{v_{2i}v_{2i+2}, v_{2i-1}v_{2i+1} : 1 \leq i \leq n-1\} \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq n\}$.*

Definition 1.9 *A triangular ladder is a graph T_n, whose vertex set is $V(T_n) = \{v_i : 1 \leq i \leq 2n\}$ and whose edge set is $E(T_n) = E(L_n) \bigcup \{v_{2i}v_{2i+1} : 1 \leq i \leq n-1\}$.*

Definition 1.10 *A complete n-ary tree is a tree in which every internal vertex is of degree $n+1$, the root vertex is of degree n and the pendent vertices are of degree 1 and have the same depth.*

Definition 1.11 *A chord of a cycle C_n is an edge joining two non-adjacent vertices of the cycle C_n.*

Definition 1.12 *The graph obtained by joining a single pendent edge to each vertex of a path is called a comb.*

Definition 1.13 *Duplication of a vertex v by a new edge $e = uw$ in a graph G produces a new graph $G^{'}$ such that $N(u) = \{v, w\}$ and $N(w) = \{u, v\}$.*

Definition 1.14 *Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph $G^{'}$ such that $N(w) = \{u, v\}$.*

Definition 1.15 *A triangular snake is a graph obtained from the duplication of each edge of a path by a new vertex.*

Definition 1.16 *The windmill graph $K_n^m, (n > 3)$ consists of m copies of K_n with a vertex in common.*

Consider a graph G of order n. Let P_1 and P_2 be two paths in G with the same vertex set V. Then we say that P_1 and P_2 are path homotopic with respect to V. We denote this by $P_1 \simeq_V P_2$. One can easily prove that this relation is an equivalence relation. Let $\mathcal{P}$ be the path homotopy class consisting of those paths which are path homotopic to the path P with a given vertex set and let $\mathcal{A}$ denote the set of all distinct path homotopy classes in G.

Definition 1.17 *A graph G of order n is said to be strongly k-multiplicative if there is an injective mapping $f : V(G) \to \{1, 2, \cdots, n\}$ such that the induced mapping $h : \mathcal{A} \to \mathbb{Z}^+$ defined by $h(\mathcal{P}) = \prod_{i=1}^{k+1} f(v_{j_i})$, where $j_1, j_2, \cdots, j_{k+1} \in \{1, 2, \cdots, n\}$, $k+1 \leq n$ and $\mathcal{P}$ is the path* *homotopy class of paths having the vertex set* $\{v_{j_1}, v_{j_2}, \cdots, v_{j_{k+1}}\}$*, is injective.*

In particular, if k=2 we call G, strongly 2- multiplicative and if k=1, then we call G, strongly 1- multiplicative or simply strongly multiplicative.

In 2001, L. W. Beineke and S. M. Hegde [5] have introduced the concept of strongly multiplicative graphs. Since then many authors including C. Adiga, H. N. Ramaswamy and D. D. Somashekara [2],[3], [4], M. A. Seoud and A. Zid [9], B. D. Acharya, Germina and Ajitha [1], S. K. Vaidya and K. K. Kanani [10], [11] and M. Muthusamy, K. C. Raajasekar and J. Basker Babujee [8] have also studied and contributed to the concept of strongly multiplicative graphs. For more details one may refer the survey article "*A dynamic survey of graph labeling*" by J. A. Gallian [6].

In the next section we prove our main results.

§2. Main Results

We first note that for a graph to be strongly 2-multiplicative, it has to have at least 3 vertices.

Theorem 2.1 *The path* P_n *is strongly* 2*-multiplicative.*

Proof Consider a path P_n of length $n-1$. We label the vertices as follows: $v_i = i$ for all i. Then $\mathcal{A}$ consists of $n-2$ distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots, \mathcal{P}_{n-2}$, where $\mathcal{P}_i$ is the path homotopy class of paths having the vertex set $\{v_i, v_{i+1}, v_{i+2}\}$, for $1 \leqslant i \leq n-2$. Then $h(\mathcal{P}_i)$=$(i)(i+1)(i+2)$, for $1 \leq i \leq n-2$. Since $i(i+1)(i+2) < (i+1)(i+2)(i+3)$, for $1 \leq i \leq n-3$, it follows that $h(\mathcal{P}_i) < h(\mathcal{P}_{i+1})$, for $1 \leq i \leq n-3$. Hence h is injective and P_n is strongly 2-multiplicative. □

Theorem 2.2 *Every cycle* C_n*, is strongly* 2*-multiplicative.*

Proof Consider a cycle C_n=$(v_1, v_2, v_3, \cdots, v_n, v_1)$ of order n and let p be the largest prime less than n. We label the vertices as follows: v_i=i, for $1 \leq i \leq p-1$, $v_i = i+1$, for $p \leq i \leq n-1$ and $v_n = p$. If $n = 3$, then $\mathcal{A}$ consists of only one path homotopy class and is trivially strongly 2-multiplicative. If $n > 3$, then $\mathcal{A}$ consists of n distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots, \mathcal{P}_n$, where $\mathcal{P}_i$ is the path homotopy classes of paths having the vertex sets $\{v_i, v_{i+1}, v_{i+2}\}$, for $1 \leq i \leq n-2$, $\mathcal{P}_{n-1}$ is the path homotopy class of paths having the vertex set $\{v_{n-1}, v_n, v_1\}$ and $\mathcal{P}_n$ is the path homotopy class of paths having the vertex set $\{v_n, v_1, v_2\}$. Then $h(\mathcal{P}_i) = (i)(i+1)(i+2)$, for $1 \leq i \leq p-3$, $h(\mathcal{P}_{p-2}) = (p-2)(p-1)(p+1)$, $h(\mathcal{P}_{p-1}) = (p-1)(p+1)(p+2)$, $h(\mathcal{P}_i) = (i+1)(i+2)(i+3)$, for $p \leq i \leq n-3$, $h(\mathcal{P}_{n-2}) = (n-1)(n)(p)$ or $h(\mathcal{P}_{n-2}) = (n-2)(n)(p)$, if p is the immediate predecessor of n, $h(\mathcal{P}_{n-1}) = n \cdot p \cdot 1$ and $h(\mathcal{P}_n) = p \cdot 1 \cdot 2$. Then from the definition of h it follows that $h(\mathcal{P}_i) < h(\mathcal{P}_{i+1})$, $1 \leq i \leq n-3$ and $h(\mathcal{P}_n) < h(\mathcal{P}_{n-1}) < h(\mathcal{P}_{n-2})$, also $h(\mathcal{P}_i) \neq h(\mathcal{P}_j)$, $n-2 \leq j \leq n$ and $1 \leq i \leq n-3$. Since $h(\mathcal{P}_j)$ is divisible by p, where as $h(\mathcal{P}_i)$ is not, h is injective and the graph C_n is strongly 2-multiplicative. □

Theorem 2.3 *Every cycle with one chord is strongly 2-multiplicative.*

Proof First, consider a cycle C_4 with vertices v_1, v_2, v_3, v_4. Let the chord be $e = v_1v_3$. We label the vertices as follows: $v_1 = 1$, $v_2 = 4$, $v_3 = 2$ and $v_4 = 3$. Then $\mathcal{A}$ consists of 4 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$, corresponding to the path homotopy classes of paths having the vertex sets $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_1\}$ and $\{v_4, v_1, v_2\}$ respectively. Then $h(\mathcal{P}_1) = 8$, $h(\mathcal{P}_2) = 24$, $h(\mathcal{P}_3) = 6$, $h(\mathcal{P}_4) = 12$. Clearly h is injective and C_4 with one chord is strongly 2-multiplicative.

Second, consider a cycle C_5 with vertices v_1, v_2, v_3, v_4, v_5. Let the chord be $e = v_1v_3$. We label the vertices as follows: $v_1 = 1$, $v_2 = 4$, $v_3 = 2$, $v_4 = 5$ and $v_5 = 3$. Then $\mathcal{A}$ consists of 7 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, $\mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6$ and $\mathcal{P}_7$, corresponding to path homotopy classes of paths having the vertex sets $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_1\}$, $\{v_5, v_1, v_2\}$, $\{v_5, v_1, v_3\}$ and $\{v_4, v_3, v_1\}$ respectively. Then $h(\mathcal{P}_1)$=8, $h(\mathcal{P}_2) = 40$, $h(\mathcal{P}_3) = 30$, $h(\mathcal{P}_4) = 15$, $h(\mathcal{P}_5) = 12$, $h(\mathcal{P}_6) = 6$ and $h(\mathcal{P}_7) = 10$. Clearly h is injective and C_5 with one chord is strongly 2-multiplicative.

Finally, let $n > 5$. Consider a cycle $C_n = (v_1, v_2, v_3, \cdots, v_n, v_1)$ of order n and let p_1 and p_2 be the two consecutive primes such that $0 < p_2 < p_1 < n$ and that p_1 is the largest. Let $e = v_1v_{p_2}$ be the chord of the cycle C_n. We label the vertices as follows: $v_i = i$, for $1 \le i \le p_1-1$, $v_i = i+1$, for $p_1 \le i \le n-1$ and v_n=p_1. Then $\mathcal{A}$ consists of $n+4$ ($n+2$, in case $n = 6$ and $n = 7$) distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots, \mathcal{P}_n$, $\mathcal{P}_{n+1}$, $\mathcal{P}_{n+2}$, $\mathcal{P}_{n+3}$, $\mathcal{P}_{n+4}$, where $\mathcal{P}_i$ is the path homotopy class of paths having the vertex sets $\{v_i, v_{i+1}, v_{i+2}\}$, for $1 \le i \le n-2$ and $\mathcal{P}_{n-1}$, $\mathcal{P}_n$, $\mathcal{P}_{n+1}$, $\mathcal{P}_{n+2}$, $\mathcal{P}_{n+3}$ and $\mathcal{P}_{n+4}$ are the path homotopy classes of paths having the vertex set $\{v_{n-1}, v_n, v_1\}$, $\{v_n, v_1, v_2\}$, $\{v_n, v_1, v_{p_2}\}$, $\{v_{p_2+1}, v_{p_2}, v_1\}$, $\{v_2, v_1, v_{p_2}\}$ and $\{v_{p_2-1}, v_{p_2}, v_1\}$ respectively. Then $h(\mathcal{P}_i)$=$(i)(i+1)(i+2)$, for $1 \le i \le p_1-3$, $h(\mathcal{P}_{p_1-2}) = (p_1-2)(p_1-1)(p_1+1)$, $h(\mathcal{P}_{p_1-1}) = (p_1 - 1)(p_1 + 1)(p_1 + 2)$, $h(\mathcal{P}_i) = (i + 1)(i + 2)(i + 3)$, for $p_1 \le i \le n - 3$, $h(\mathcal{P}_{n-2}) = (n - 1)(n)(p_1)$ or $h(\mathcal{P}_{n-2}) = (n - 2)(n)(p_1)$, if p_1 is the immediate predecessor of n, $h(\mathcal{P}_{n-1})$=$n.p_1.1$, $h(\mathcal{P}_n)$=$p_1.1.2$, $h(\mathcal{P}_{n+1})$=$p_1.1.p_2$, $h(\mathcal{P}_{n+2})$=$(p_2 + 1).p_2.1$ $h(\mathcal{P}_{n+3})$=$2.1.p_2$ and $h(\mathcal{P}_{n+4})$=$(p_2 - 1).p_2.1$. Then from the definition of h it follows that $h(\mathcal{P}_i)$<$h(\mathcal{P}_{i+1})$, for 1≤i≤p_2-3 and p_2+1≤i≤$n-3$ and $h(\mathcal{P}_n)$<$h(\mathcal{P}_{n-1})$< $h(\mathcal{P}_{n-2})$, also $h(\mathcal{P}_i)$≠ $h(\mathcal{P}_j)$, 1≤i≤p_2-3, p_2+1≤i≤$n-3$ and $n-2$≤j≤n. Since $h(\mathcal{P}_j)$ is divisible by p_1, where as $h(\mathcal{P}_i)$ is not. $h(\mathcal{P}_{n+3}) < h(\mathcal{P}_{n+4}) < h(\mathcal{P}_{n+2}) < h(\mathcal{P}_{n+1}) < h(\mathcal{P}_{p_2-2}) < h(\mathcal{P}_{p_2-1}) < h(\mathcal{P}_{p_2})$ and these are not equal to $h(\mathcal{P}_i)$ and $h(\mathcal{P}_j)$, where 1≤i≤$p_2 - 3$, $p_2 + 1$≤i≤$n - 3$ and $n - 2$≤j≤n, since these are divisible by p_2 whereas $h(\mathcal{P}_i)$ and $h(\mathcal{P}_j)$ are not. Hence h is injective and $C_n, n > 5$ with one chord is strongly 2-multiplicative. □

Remark 2.4 (1) In general, a cycle C_n=$(v_1, v_2, v_3, \cdots, v_n, v_1)$ with one chord joining any two non adjacent vertices, can be shown to be strongly 2-multiplicative.

(2) A cycle with twin chords can be shown to be strongly 2-multiplicative.

Theorem 2.5 *The graph obtained by duplication of an arbitrary vertex of a cycle by a new edge is strongly 2-multiplicative.*

Proof Consider a cycle C_n=$(v_1, v_2, v_3, \cdots, v_n, v_1)$. We duplicate the vertex v_n by an edge e with end vertices v_{n+1} and v_{n+2}. Let the graph so obtained be G. Then $| V(G) |= n + 2$ and $| E(G) |= n + 3$. Let p be the largest prime less than n. We label the vertices as follows: $v_i=i$, for $1 \leq i \leq p-1$ and for $n < i \leq n+2$, $v_i = i+1$, for $p \leq i \leq n-1$ and $v_n = p$. If $n = 3$, then $\mathcal{A}$ consists of 6 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$, corresponding to the path homotopy classes of paths having the vertex sets $\{v_1, v_2, v_3\}$, $\{v_3, v_4, v_5\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_3, v_5\}$, $\{v_1, v_3, v_4\}$ and $\{v_1, v_3, v_5\}$ respectively. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 40$, $h(\mathcal{P}_3) = 24$, $h(\mathcal{P}_4) = 30$, $h(\mathcal{P}_5) = 8$, $h(\mathcal{P}_4) = 10$. If $n > 3$, then $\mathcal{A}$ consists of $n+5$ distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots, \mathcal{P}_n$, $\mathcal{P}_{n+1}$, $\mathcal{P}_{n+2}$, $\mathcal{P}_{n+3}$, $\mathcal{P}_{n+4}$, $\mathcal{P}_{n+5}$, where $\mathcal{P}_i$ is the path homotopy class of paths having the vertex sets $\{v_i, v_{i+1}, v_{i+2}\}$, for $1 \leq i \leq n-2$ and $\mathcal{P}_{n-1}$, $\mathcal{P}_n$, $\mathcal{P}_{n+1}$, $\mathcal{P}_{n+2}$, $\mathcal{P}_{n+3}$, $\mathcal{P}_{n+4}$ and $\mathcal{P}_{n+5}$ are the path homotopy classes of paths having the vertex sets $\{v_{n-1}, v_n, v_1\}$, $\{v_n, v_1, v_2\}$, $\{v_n, v_{n+1}, v_{n+2}\}$, $\{v_{n+1}, v_n, v_{n-1}\}$, $\{v_{n+1}, v_n, v_1\}$, $\{v_{n+2}, v_n, v_{n-1}\}$ and $\{v_{n+2}, v_n, v_1\}$ respectively. Then $h(\mathcal{P}_i) = (i)(i+1)(i+2)$, for $1 \leq i \leq p-3$, $h(\mathcal{P}_{p-2}) = (p-2)(p-1)(p+1)$, $h(\mathcal{P}_{p-1}) = (p-1)(p+1)(p+2)$, $h(\mathcal{P}_i) = (i+1)(i+2)(i+3)$, for $p \leq i \leq n-3$, $h(\mathcal{P}_{n-2}) = (n-1)(n)(p)$ or $h(\mathcal{P}_{n-2}) = (n-2)(n)(p)$, if p is the immediate predecessor of n, $h(\mathcal{P}_{n-1}) = n \cdot p \cdot 1$, $h(\mathcal{P}_n) = p \cdot 1 \cdot 2$, $h(\mathcal{P}_{n+1}) = p \cdot (n+1) \cdot (n+2)$, $h(\mathcal{P}_{n+2}) = n \cdot p \cdot (n+1)$, $h(\mathcal{P}_{n+3}) = (n+1) \cdot p \cdot 1$, $h(\mathcal{P}_{n+4}) = (n+2) \cdot p \cdot n$ and $h(\mathcal{P}_{n+5}) = (n+2) \cdot p \cdot 1$. Then from the definition of h it follows that $h(\mathcal{P}_i) < h(\mathcal{P}_{i+1})$, $1 \leq i \leq n-3$ and $h(\mathcal{P}_n) < h(\mathcal{P}_{n-1}) < h(\mathcal{P}_{n+3}) < h(\mathcal{P}_{n+5}) < h(\mathcal{P}_{n-2}) < h(\mathcal{P}_{n+2}) < h(\mathcal{P}_{n+4}) < h(\mathcal{P}_{n+1})$ and these not equal to $h(\mathcal{P}_k)$ where $1 \leq k \leq n-3$, since these are divisible by p whereas $h(\mathcal{P}_k)$ is not. Hence h is injective and the graph obtained by duplication of an arbitrary vertex of a cycle by a new edge is strongly 2-multiplicative. $\square$

Remark 2.6 If we duplicate an edge in a cycle of an order n by a new vertex, then we obtain a cycle of order $n+1$ with one chord. Hence by Theorem 2.3 the graph obtained by duplication of an arbitrary edge of cycle by a new vertex is strongly 2-multiplicative.

Theorem 2.7 *The comb graph is strongly 2-multiplicative*

Proof Consider the comb graph G of order $2n (n \geq 2)$ with vertex set $G = \{v_1, v_2, v_3, \cdots, v_{2n}\}$ as shown below.

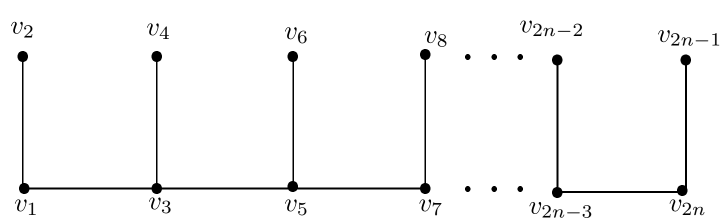

Figure 1

Then $\mathcal{A}$ consists of $3n-4$ distinct path homotopy classes $\mathcal{P}_{2i-1,2i+1,2i+3}$, $\mathcal{P}_{2i-1,2i,2i+1}$, $\mathcal{P}_{2i-1,2i+1,2i+2}$, corresponding to path homotopy classes of paths having vertex sets $\{v_{2i-1}, v_{2i+1}, v_{2i+3}\}$, $\{v_{2i-1}, v_{2i}, v_{2i+1}\}$ and $\{v_{2i-1}, v_{2i+1}, v_{2i+2}\}$ respectively, for $1 \leq i \leq n-2$ and path homotopy classes $\mathcal{P}_{2n-3,2n-2,2n-1}$, $\mathcal{P}_{2n-3,2n-1,2n}$ corresponding to path homotopy classes of paths having the vertex sets $\{v_{2n-3}, v_{2n-2}, v_{2n-1}\}$ and $\{v_{2n-3}, v_{2n-1}, v_{2n}\}$ respectively. We label the vertices as follows: $v_i = i$, for all i. Then $h(\mathcal{P}_{i,j,k}) = i \cdot j \cdot k$. Since $(2i-1) \cdot (2i) \cdot (2i+1) < (2i-1) \cdot (2i+1) \cdot (2i+2) < (2i-1) \cdot$

$(2i+1)\cdot(2i+3)$, $(2i-1)\cdot(2i+1)\cdot(2i+3) < (2i+1)\cdot(2i+2)\cdot(2i+3)$, for $1 \le i \le n-2$ and $(2i-1)\cdot(2i)\cdot(2i+1) < (2i-1)\cdot(2i+1)\cdot(2i+2)$ for $i = n-1$, it follows that $h(\mathcal{P}_{1,2,3}) < h(\mathcal{P}_{1,3,4}) < \cdots < h(\mathcal{P}_{2n-3,2n-1,2n})$. Therefore h is injective and the comb graph is strongly 2-multiplicative. □

Theorem 2.8 *The triangular snake graph is strongly 2-multiplicative.*

Proof Consider the triangular snake graph $T_n (n \ge 2)$ with vertex set $V(T_n) = \{v_1, v_2, v_3, \cdots, v_{2n-1}\}$ as shown below.

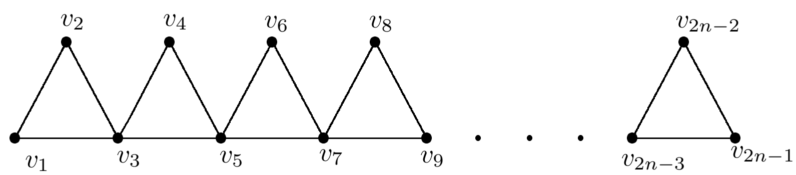

Figure 2

Then $\mathcal{A}$ consists of $5n-9$ distinct path homotopy classes

$$\mathcal{P}_{2i-1,2i,2i+1},\ \mathcal{P}_{2i-1,2i+1,2i+2},\ \mathcal{P}_{2i-1,2i+1,2i+3},\ \mathcal{P}_{2i,2i+1,2i+2},\ \mathcal{P}_{2i,2i+1,2i+3}$$

corresponding to path homotopy classes of paths having vertex sets

$$\{v_{2i-1}, v_{2i}, v_{2i+1}\},\ \{v_{2i-1}, v_{2i+1}, v_{2i+2}\},\ \{v_{2i-1}, v_{2i+1}, v_{2i+3}\},\ \{v_{2i}, v_{2i+1}, v_{2i+2}\}$$

and $\{v_{2i}, v_{2i+1}, v_{2i+3}\}$ respectively, for $1 \le i \le n-2$ and path homotopy classes $\mathcal{P}_{2n-3,2n-2,2n-1}$ corresponding to path homotopy class of paths having the vertex set $\{v_{2n-3}, v_{2n-2}, v_{2n-1}\}$. We label the vertices as follows: $v_i = i$, for all i. Then $h(\mathcal{P}_{i,j,k}) = i\cdot j\cdot k$. Since $(2i-1)\cdot(2i)\cdot(2i+1) < (2i-1)\cdot(2i+1)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+3) < (2i)\cdot(2i+1)\cdot(2i+2) < (2i)\cdot(2i+1)\cdot(2i+3)$, $(2i)\cdot(2i+1)\cdot(2i+3) < (2i+1)\cdot(2i+2)\cdot(2i+3)$, for $1 \le i \le n-2$ and $(2n-3)\cdot(2n-1)\cdot(2n-4) < (2n-3)\cdot(2n-2)\cdot(2n-1)$ it follows that $h(\mathcal{P}_{1,2,3}) < h(\mathcal{P}_{1,3,4}) < \ldots < h(\mathcal{P}_{2n-3,2n-2,2n-1})$. Therefore h is injective and the triangular snake graph is strongly 2-multiplicative. □

Theorem 2.9 *The ladder graph L_n is strongly 2-multiplicative.*

Proof Consider the ladder graph L_n with vertex set $V(L_n) = \{v_1, v_2, v_3, \cdots, v_{2n}\}$ as shown below.

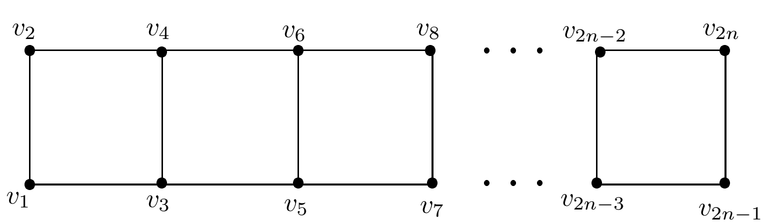

Figure 3

Then $\mathcal{A}$ consists of $6n-8$ distinct path homotopy classes

$\mathcal{P}_{2i,2i-1,2i+1}$, $\mathcal{P}_{2i-1,2i,2i+2}$, $\mathcal{P}_{2i-1,2i+1,2i+2}$, $\mathcal{P}_{2i-1,2i+1,2i+3}$, $\mathcal{P}_{2i,2i+2,2i+4}$, $\mathcal{P}_{2i,2i+2,2i+1}$, corresponding to path homotopy classes of paths having vertex sets $\{v_{2i}, v_{2i-1}, v_{2i+1}\}$, $\{v_{2i-1}, v_{2i}, v_{2i+2}\}$, $\{v_{2i-1}, v_{2i+1}, v_{2i+2}\}$, $\{v_{2i-1}, v_{2i+1}, v_{2i+3}\}$, $\{v_{2i}, v_{2i+2}, v_{2i+4}\}$ and $\{v_{2i}, v_{2i+2}, v_{2i+1}\}$ respectively, for $1 \le i \le n-2$ and path homotopy classes $\mathcal{P}_{2n-2,2n-3,2n-1}$, $\mathcal{P}_{2n-3,2n-2,2n}$, $\mathcal{P}_{2n-3,2n-1,2n}$, $\mathcal{P}_{2n-2,2n,2n-1}$ corresponding to path homotopy classes of paths having the vertex sets $\{v_{2n-2}, v_{2n-3}, v_{2n-1}\}$, $\{v_{2n-3}, v_{2n-2}, v_{2n}\}$, $\{v_{2n-3}, v_{2n-1}, v_{2n}\}$ and $\{v_{2n-2}, v_{2n}, v_{2n-1}\}$ respectively. We label the vertices as follows: v_i=i, for all i. Then $h(\mathcal{P}_{i,j,k}) = i \cdot j \cdot k$. Since $(2i)\cdot(2i-1)\cdot(2i+1) < (2i-1)\cdot(2i)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+3) < (2i)\cdot(2i+2)\cdot(2i+1) < (2i)\cdot(2i+2)\cdot(2i+4)$, $(2i)\cdot(2i+2)\cdot(2i+4) < (2i+2)\cdot(2i+1)\cdot(2i+3)$, for $1 \le i \le n-2$ and $(2i)\cdot(2i-1)\cdot(2i+1) < (2i-1)\cdot(2i)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+2) < (2i)\cdot(2i+2)\cdot(2i+1)$ for $i = n-1$ it follows that $h(\mathcal{P}_{2,1,3}) < h(\mathcal{P}_{1,2,4}) < \cdots < h(\mathcal{P}_{2n-2,2n,2n-1})$. Therefore h is injective and the graph L_n is strongly 2-multiplicative. □

Theorem 2.10 *The binary tree is strongly 2-multiplicative.*

Proof Consider the binary tree G consisting of $2^{n+1} - 1$ vertices with n levels. We label the vertices, using breadth-first search method as follows $v_i = i$, for $1 \le i \le 2^{n+1} - 1$ as shown in the figure.

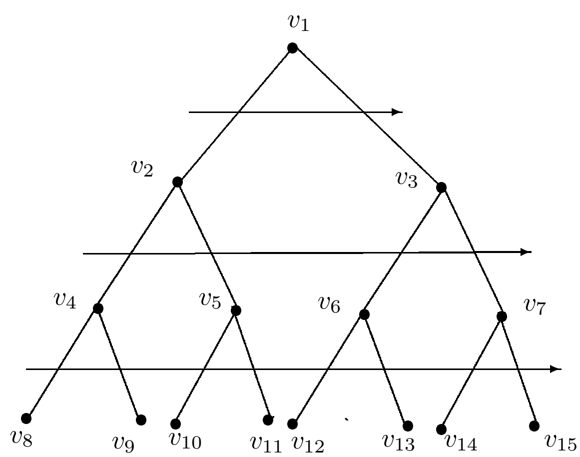

Figure 4

If $n = 1$ then the tree becomes a path with 3 vertices and is trivially strongly 2-multiplicative. So, let $n > 1$. Then for each m, consisting of the edges of level $m-1$ and of the level m, $1 < m \le n-1$, there are 5.2^{m-2} distinct path homotopy classes consisting of 2^{m-2} bunches of 5 path homotopy classes $\mathcal{P}_{m,r,1}$, $\mathcal{P}_{m,r,2}$, $\mathcal{P}_{m,r,3}$, $\mathcal{P}_{m,r,4}$, $\mathcal{P}_{m,r,5}$ corresponding to path homotopy classes of paths having vertex sets

$$\{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)}, v_{2(2^{m-2}+r-1)+1}\},\ \{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)}, v_{4(2^{m-2}+r-1)}\},$$
$$\{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)}, v_{4(2^{m-2}+r-1)+1}\},\ \{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)+1}, v_{4(2^{m-2}+r-1)+2}\}$$

and $\{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)+1}, v_{4(2^{m-2}+r-1)+3}\}$ respectively, where $1 \le r \le 2^{m-2}$ and if $m = n$, in addition to $5 \cdot 2^{m-2}$ distinct path homotopy classes described above we have 2^{n-1} distinct path homotopy classes $\mathcal{P}_{n+1,r,1}$ corresponding to the paths having vertex sets

$\{v_{2^{n-1}+r-1}, v_{2(2^{n-1}+r-1)}, v_{2(2^{n-1}+r-1)+1}\}$, where $1 \leq r \leq 2^{n-1}$. Then $h(\mathcal{P}_{m,r,1}) = (2^{m-2}+r-1) \cdot (2(2^{m-2}+r-1)) \cdot (2(2^{m-2}+r-1)+1)$, $h(\mathcal{P}_{m,r,2}) = (2^{m-2}+r-1) \cdot (2(2^{m-2}+r-1)) \cdot (4(2^{m-2}+r-1))$, $h(\mathcal{P}_{m,r,3}) = (2^{m-2}+r-1) \cdot (2(2^{m-2}+r-1)) \cdot (4(2^{m-2}+r-1)+1)$, $h(\mathcal{P}_{m,r,4}) = (2^{m-2}+r-1) \cdot (2(2^{m-2}+r-1)+1) \cdot (4(2^{m-2}+r-1)+2)$, $h(\mathcal{P}_{m,r,5}) = (2^{m-2}+r-1) \cdot (2(2^{m-2}+r-1)+1) \cdot (4(2^{m-2}+r-1)+3)$, for $1 < m \leq n$, $1 \leq r \leq 2^{m-2}$ and $h(\mathcal{P}_{n+1,r,1}) = (2^{n-1}+r-1) \cdot (2(2^{n-1}+r-1)) \cdot (2(2^{n-1}+r-1)+1)$, for $1 \leq r \leq 2^{n-1}$. Then to show h is injective, consider the following cases:

Case 1. Let $k = 2^{m-2}+r-1$. Then $h(\mathcal{P}_{m,r,2}) = k \cdot 2k \cdot 4k$, $h(\mathcal{P}_{m,r,3}) = k \cdot 2k \cdot (4k+1)$, $h(\mathcal{P}_{m,r,4}) = k \cdot (2k+1) \cdot (4k+2)$ and $h(\mathcal{P}_{m,r,5}) = k \cdot (2k+1) \cdot (4k+3)$. Since $2k < 2k+1$ and $4k < 4k+1 < 4k+2 < 4k+3$, we have $k \cdot 2k \cdot 4k < k \cdot 2k \cdot (4k+1) < k \cdot (2k+1) \cdot (4k+2) < k \cdot (2k+1) \cdot (4k+3)$. Hence $h(\mathcal{P}_{m,r,2}) < h(\mathcal{P}_{m,r,3}) < h(\mathcal{P}_{m,r,4}) < h(\mathcal{P}_{m,r,5})$.

Case 2. Let $k = 2^{m-1}-1$. Then $h(\mathcal{P}_{m,2^{m-2},5}) = k \cdot (2k+1) \cdot (4k+3)$, $h(\mathcal{P}_{m+1,1,2}) = (k+1) \cdot (2(k+1)) \cdot (4(k+1))$. Since $k < k+1$, $2k+1 < 2k+2$ and $4k+3 < 4k+4$, we have $k \cdot 2k+1 \cdot 4k+3 < k+1 \cdot 2k+2 \cdot 4k+4$. Hence $h(\mathcal{P}_{m,2^{m-2},5}) < h(\mathcal{P}_{m+1,1,2})$.

Case 3. Let $k = 2^{m-2}+r-1$. Then $h(\mathcal{P}_{m,r,5}) = k \cdot (2k+1) \cdot (4k+3)$, $h(\mathcal{P}_{m,r+1,1}) = (k+1) \cdot (2(k+1)) \cdot (4(k+1))$. Since $k < k+1$, $2k+1 < 2k+2$ and $4k+3 < 4k+4$, we have $k \cdot 2k+1 \cdot 4k+3 < k+1 \cdot 2k+2 \cdot 4k+4$. Hence $h(\mathcal{P}_{m,r,5}) < h(\mathcal{P}_{m,r+1,2})$, for $1 \leq r \leq 2^{m-2}-1$.

Case 4. Since $r-1 < r$, we have $(2^{m-2}+r-1) \cdot (2(2^{m-2}+r-1)) \cdot (2(2^{m-2}+r-1)+1) < ((2^{m-2})+r) \cdot (2(2^{m-2}+r)) \cdot (2(2^{m-2}+r)+1)$, which is same as $h(\mathcal{P}_{m,r,1}) < h(\mathcal{P}_{m,r+1,1})$, for $1 \leq r \leq 2^{m-2}-1$.

Case 5. Let $k = 2^{m-1}-1$. Then $h(\mathcal{P}_{m,2^{m-2},1}) = k \cdot 2k \cdot (2k+1)$, $h(\mathcal{P}_{m+1,1,1}) = (k+1) \cdot (2(k+1)) \cdot (2(k+1)+1)$. Since $k < k+1$, $2k < 2k+2$ and $2k+1 < 2k+3$, we have $k \cdot 2k \cdot (2k+1) < (k+1) \cdot (2k+2) \cdot (2k+3)$. Hence $h(\mathcal{P}_{m,2^{m-2},1}) < h(\mathcal{P}_{m+1,1,1})$.

Case 6. For given m and r, we have $h(\mathcal{P}_{m,r,1}) = (2^{m-2}+r-1) \cdot (2(2^{m-2}+r-1)) \cdot (2(2^{m-2}+r-1)+1)$ in which one of the three factors differs from the three factors of $h(\mathcal{P}_{s,t,i})$ for $s < m$, $1 \leq t \leq 2^{s-1}$, $1 \leq i \leq 5$ and for $s = m$, $1 \leq t < r \leq 2^{m-2}$, $1 \leq i \leq 5$. Hence $h(\mathcal{P}_{m,r,1}) \neq h(\mathcal{P}_{s,t,i})$ for $s < m$, $1 \leq t \leq 2^{s-1}$, $1 \leq i \leq 5$ and for $s = m$, $1 \leq t < r \leq 2^{m-2}$, $1 \leq i \leq 5$.

Thus, by Cases (1)-(6) it follows that h is injective and G is strongly 2-multiplicative. $\square$

Theorem 2.11 *The complete graph K_n is strongly 2-multiplicative if and only if* $3 \leq n \leq 5$.

Proof First, consider K_3 with vertices v_1, v_2 and v_3. Then there is only one path homotopy class and is trivially strongly 2-multiplicative.

Second, consider K_4 with vertices v_1, v_2, v_3 and v_4. Then there are four distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$, corresponding to paths having vertex sets $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 4$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 12$, $h(\mathcal{P}_4) = 24$. Clearly h is injective and K_4 is strongly 2-multiplicative.

Third, consider K_5 with vertices v_1, v_2, v_3, v_4 and v_5. Then there are ten distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$ and $\mathcal{P}_{10}$, corresponding to paths having vertex sets $\{v_1, v_2, v_3\}$, $\{v_1, v_3, v_4\}$, $\{v_1, v_4, v_5\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_4, v_5\}$, $\{v_2, v_5, v_1\}$, $\{v_3, v_4, v_5\}$, $\{v_3, v_5, v_1\}$, $\{v_4, v_1, v_2\}$ and $\{v_5, v_2, v_3\}$ respectively. We label the vertices as follows : v_i=i, $1 \leq i \leq 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 20$, $h(\mathcal{P}_4) = 24$, $h(\mathcal{P}_5) = 40$, $h(\mathcal{P}_6) = 10$, $h(\mathcal{P}_7) = 60$, $h(\mathcal{P}_8) = 15$, $h(\mathcal{P}_9) = 8$ and $h(\mathcal{P}_{10}) = 30$. Clearly h is injective and K_5 is strongly 2-multiplicative.

Finally, consider a complete graph K_n, where $n \geq 6$. Clearly corresponding to each triangle, one can always find a path homotopy class of paths of length 2 having the vertex set, the vertices of triangle. In any labelling of the vertices, we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ where $\mathcal{P}$ consisting of paths having the vertices labelled $1, 3$ and 4 and $\mathcal{P}'$ consisting of paths having the vertices labelled $1, 2$ and 6. Clearly $\mathcal{P} \neq \mathcal{P}'$, but $h(\mathcal{P}) = 12 = h(\mathcal{P}')$. Hence for $n \geq 6$, K_n is not strongly 2-multiplicative. □

Theorem 2.12 *The star graph S_n is strongly 2-multiplicative if and only if* $3 \leq n \leq 7$.

Proof First, consider S_3 with vertices v_1,v_2 and v_3. Here v_2,v_3 are pendent vertices. Then there is only one path homotopy class and is trivially strongly 2-multiplicative.

Second, consider S_4 with vertices v_1,v_2,v_3 and v_4. Here v_2, v_3, v_4 are pendent vertices. Then there are three distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$ and $\{v_3, v_1, v_4\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 4$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 12$. Clearly h is injective and S_4 is strongly 2-multiplicative.

Third, consider S_5 with vertices v_1, v_2, v_3, v_4 and v_5. Here v_2, v_3, v_4, v_5 are pendent vertices. Then there are six distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$ and $\{v_4, v_1, v_5\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 15$, $h(\mathcal{P}_6) = 20$. Clearly h is injective and S_5 is strongly 2-multiplicative.

Fourth, consider S_6 with vertices v_1, v_2, v_3, v_4, v_5 and v_6. Here v_2, v_3, v_4, v_5, v_6 are pendent vertices. Then there are ten distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$ and $\mathcal{P}_{10}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$ and $\{v_5, v_1, v_6\}$ respectively. We label the vertices as follows: $v_1 = 2$, $v_2 = 1$, $v_i = i$, $3 \leq i \leq 6$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 24$, $h(\mathcal{P}_6) = 30$, $h(\mathcal{P}_7) = 36$, $h(\mathcal{P}_8) = 40$, $h(\mathcal{P}_9) = 48$, $h(\mathcal{P}_{10}) = 60$. Clearly h is injective and S_6 is strongly 2-multiplicative.

Fifth, consider S_7 with vertices $v_1, v_2, v_3, v_4, v_5, v_6$ and v_7. Here $v_2, v_3, v_4, v_5, v_6, v_7$ are pendent vertices. Then there are fifteen distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{14}$ and $\mathcal{P}_{15}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_2, v_1, v_7\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_3, v_1, v_7\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_4, v_1, v_7\}$, $\{v_5, v_1, v_6\}$, $\{v_5, v_1, v_7\}$ and $\{v_6, v_1, v_7\}$ respectively. We label the vertices as follows : v_1=2, $v_2 = 1$, v_i=i, $3 \leq i \leq 7$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 14$, $h(\mathcal{P}_6) = 24$, $h(\mathcal{P}_7) = 30$, $h(\mathcal{P}_8) = 36$, $h(\mathcal{P}_9) = 42$, $h(\mathcal{P}_{10}) = 40$, $h(\mathcal{P}_{11}) = 48$, $h(\mathcal{P}_{12}) = 56$, $h(\mathcal{P}_{13}) = 60$, $h(\mathcal{P}_{14}) = 70$,

$h(\mathcal{P}_{15}) = 84$. Clearly h is injective and S_7 is strongly 2-multiplicative.

Finally, consider a star graph S_n, where $n \geq 8$. In any labeling of the vertices we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$ but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 8$, S_n is not strongly 2-multiplicative. $\square$

Theorem 2.13 *The fan graph F_n is strongly 2-multiplicative if and only if $3 \leq n \leq 6$.*

Proof First, consider $F_2 = K_1 + P_2$. Let the vertex of K_1 be v_1 and the vertices of P_2 be v_2 and v_3. Then there is only one path homotopy class and is trivially strongly 2-multiplicative.

Second, consider $F_3 = K_1 + P_3$. Let the vertex of K_1 be v_1 and the vertices of P_3 be v_2, v_3 and v_4. Then there are four distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_3, v_1, v_4\}$ and $\{v_2, v_3, v_4\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 4$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 12$, $h(\mathcal{P}_4) = 24$. Clearly h is injective and F_3 is strongly 2-multiplicative.

Third, consider $F_4 = K_1 + P_4$. Let the vertex of K_1 be v_1 and the vertices of P_4 be v_2, v_3, v_4 and v_5. Then there are eight distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6, \mathcal{P}_7$ and $\mathcal{P}_8$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_4, v_1, v_5\}$, $\{v_2, v_3, v_4\}$ and $\{v_3, v_4, v_5\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 15$, $h(\mathcal{P}_6) = 20$, $h(\mathcal{P}_7) = 24$, $h(\mathcal{P}_8) = 60$. Clearly h is injective and F_4 is strongly 2-multiplicative.

Fourth, consider $F_5 = K_1 + P_5$. Let the vertex of K_1 be v_1 and the vertices of P_5 be v_2, v_3, v_4,v_5 and v_6. Then there are thirteen distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, $\mathcal{P}_4, \mathcal{P}_5$, $\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8, \mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11}, \mathcal{P}_{12}$ and $\mathcal{P}_{13}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_5, v_1, v_6\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$ and $\{v_4, v_5, v_6\}$ respectively. We label the vertices as follows: $v_1 = 3$, $v_i = i - 1$, $i = 2, 3$, $v_i = i$, $4 \leq i \leq 6$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 15$, $h(\mathcal{P}_4) = 18$, $h(\mathcal{P}_5) = 24$, $h(\mathcal{P}_6) = 30$, $h(\mathcal{P}_7) = 36$, $h(\mathcal{P}_8) = 60$, $h(\mathcal{P}_9) = 72$, $h(\mathcal{P}_{10}) = 90$, $h(\mathcal{P}_{11}) = 8$, $h(\mathcal{P}_{12}) = 40$, $h(\mathcal{P}_{13}) = 120$. Clearly h is injective and F_5 is strongly 2-multiplicative.

Fifth, consider $F_6 = K_1 + P_6$. Let the vertex of K_1 be v_1 and the vertices of P_6 be v_2, v_3, v_4, v_5, v_6 and v_7. Then there are nineteen distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{14}$, $\mathcal{P}_{15}$, $\mathcal{P}_{16}$, $\mathcal{P}_{17}$, $\mathcal{P}_{18}$ and $\mathcal{P}_{19}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_2, v_1, v_7\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_3, v_1, v_7\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_4, v_1, v_7\}$, $\{v_5, v_1, v_6\}$, $\{v_5, v_1, v_7\}$, $\{v_6, v_1, v_7\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$ and $\{v_5, v_6, v_7\}$ respectively. We label the vertices as follows : $v_1 = 3$, $v_i = i - 1$, $i = 2, 3$, $v_i = i$, $4 \leq i \leq 7$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 15$, $h(\mathcal{P}_4) = 18$, $h(\mathcal{P}_5) = 21$, $h(\mathcal{P}_6) = 24$, $h(\mathcal{P}_7) = 30$, $h(\mathcal{P}_8) = 36$, $h(\mathcal{P}_9) = 42$, $h(\mathcal{P}_{10}) = 60$, $h(\mathcal{P}_{11}) = 72$, $h(\mathcal{P}_{12}) = 84$, $h(\mathcal{P}_{13}) = 90$, $h(\mathcal{P}_{14}) = 105$, $h(\mathcal{P}_{15}) = 126$,$h(\mathcal{P}_{16}) = 8$, $h(\mathcal{P}_{17}) = 40$, $h(\mathcal{P}_{18}) = 120$, $h(\mathcal{P}_{19}) = 210$. Clearly h is injective and F_6 is strongly 2-multiplicative.

Finally, consider a fan graph F_n, where $n \geq 7$. In any labeling of the vertices we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$ but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 7$, F_n is not strongly 2-multiplicative. $\square$

Theorem 2.14 *The wheel graph W_n is strongly 2-multiplicative if and only if $4 \leq n \leq 7$.*

Proof First, consider $W_4 = K_1 + C_3$. Let the vertex of K_1 be v_1 and the vertices of C_3 be v_2, v_3 and v_4. Then there are four distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_3, v_1, v_4\}$ and $\{v_2, v_3, v_4\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 4$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 12$, $h(\mathcal{P}_4) = 24$. Clearly h is injective and W_4 is strongly 2-multiplicative.

Second, consider $W_5 = K_1 + C_4$. Let the vertex of K_1 be v_1 and the vertices of C_4 be v_2, v_3, v_4 and v_5. Then there are ten distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$ and $\mathcal{P}_{10}$ corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_4, v_1, v_5\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_2\}$ and $\{v_5, v_2, v_3\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 15$, $h(\mathcal{P}_6) = 20$, $h(\mathcal{P}_7) = 24$, $h(\mathcal{P}_8) = 60$, $h(\mathcal{P}_9) = 40$, $h(\mathcal{P}_{10}) = 30$. Clearly h is injective and W_5 is strongly 2-multiplicative.

Third, consider $W_6 = K_1 + C_5$. Let the vertex of K_1 be v_1 and the vertices of C_5 be v_2, v_3, v_4, v_5 and v_6. Then there are fifteen distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{14}$ and $\mathcal{P}_{15}$ corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_5, v_1, v_6\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$, $\{v_5, v_6, v_2\}$ and $\{v_6, v_2, v_3\}$ respectively. We label the vertices as follows: $v_1 = 2$, $v_2 = 1$, $v_3 = 3$, $v_4 = 6$, $v_5 = 4$, $v_6 = 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 8$, $h(\mathcal{P}_4) = 10$, $h(\mathcal{P}_5) = 36$, $h(\mathcal{P}_6) = 24$, $h(\mathcal{P}_7) = 30$, $h(\mathcal{P}_8) = 48$, $h(\mathcal{P}_9) = 60$, $h(\mathcal{P}_{10}) = 40$, $h(\mathcal{P}_{11}) = 18$, $h(\mathcal{P}_{12}) = 72$, $h(\mathcal{P}_{13}) = 120$, $h(\mathcal{P}_{14}) = 20$, $h(\mathcal{P}_{15}) = 15$. Clearly h is injective and W_6 is strongly 2-multiplicative.

Fourth, consider $W_7 = K_1 + C_6$. Let the vertex of K_1 be v_1 and the vertices of C_6 be v_2, v_3, v_4, v_5, v_6 and v_7. Then there are twenty one distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{14}$, $\mathcal{P}_{15}$, $\mathcal{P}_{16}$, $\mathcal{P}_{17}$, $\mathcal{P}_{18}$, $\mathcal{P}_{19}$, $\mathcal{P}_{20}$ and $\mathcal{P}_{21}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_2, v_1, v_7\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_3, v_1, v_7\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_4, v_1, v_7\}$, $\{v_5, v_1, v_6\}$, $\{v_5, v_1, v_7\}$, $\{v_6, v_1, v_7\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$, $\{v_5, v_6, v_7\}$, $\{v_6, v_7, v_2\}$ and $\{v_7, v_2, v_3\}$ respectively. We label the vertices as follows : $v_1 = 2$, $v_2 = 1$, $v_i = i$, for $i = 3, 7$, $v_4 = 6$, $v_5 = 4$, $v_6 = 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 8$, $h(\mathcal{P}_4) = 10$, $h(\mathcal{P}_5) = 14$, $h(\mathcal{P}_6) = 36$, $h(\mathcal{P}_7) = 24$, $h(\mathcal{P}_8) = 30$, $h(\mathcal{P}_9) = 42$, $h(\mathcal{P}_{10}) = 48$, $h(\mathcal{P}_{11}) = 60$, $h(\mathcal{P}_{12}) = 84$, $h(\mathcal{P}_{13}) = 40$, $h(\mathcal{P}_{14}) = 56$, $h(\mathcal{P}_{15}) = 70$, $h(\mathcal{P}_{16}) = 18$, $h(\mathcal{P}_{17}) = 72$, $h(\mathcal{P}_{18}) = 120$, $h(\mathcal{P}_{19}) = 140$, $h(\mathcal{P}_{20}) = 35$, $h(\mathcal{P}_{21}) = 21$. Clearly h is injective and W_7 is strongly 2-multiplicative.

Finally, consider a wheel graph W_n, where $n \geq 8$. In any labeling of the vertices we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$ but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 8$, W_n is not strongly 2-multiplicative. □

Theorem 2.15 *The complete bipartite graph $K_{2,n}$ is strongly 2-multiplicative if and only if* $2 \leq n \leq 3$.

Proof First, consider complete bipartite graph $K_{2,2}$. Let $A = \{v_1, v_2\}$ and $B = \{v_3, v_4\}$ be the two partitions of vertex set of $K_{2,2}$. Then $\mathcal{A}$ consists of 4 distinct path homotopy classes

$\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$, corresponding to paths having vertex sets $\{v_3, v_1, v_4\}$, $\{v_3, v_2, v_4\}$, $\{v_1, v_3, v_2\}$ and $\{v_1, v_4, v_2\}$ respectively. We label the vertices as follows: $v_i = i$, for all $i \in \{1, 2, 3, 4\}$. Then $h(\mathcal{P}_1) = 12$, $h(\mathcal{P}_2) = 24$, $h(\mathcal{P}_3) = 6$, $h(\mathcal{P}_4) = 8$. Clearly h is injective $K_{2,2}$ is strongly 2-multiplicative.

Second, consider complete bipartite graph $K_{2,3}$. Let $A = \{v_1, v_2\}$ and B =$\{v_3, v_4, v_5\}$ be the two partitions of vertex set of $K_{2,3}$. Then $\mathcal{A}$ consists of 9 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ $\mathcal{P}_4$,$\mathcal{P}_5, \mathcal{P}_6$,$\mathcal{P}_7, \mathcal{P}_8$ and $\mathcal{P}_9$, corresponding to paths having vertex sets $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_2, v_4\}$, $\{v_3, v_2, v_5\}$, $\{v_2, v_5, v_1\}$, $\{v_2, v_4, v_1\}$, $\{v_4, v_2, v_5\}$, $\{v_1, v_4, v_5\}$ and $\{v_2, v_3, v_1\}$ respectively. We label the vertices as follows: $v_i = i$, for all $i \in \{1, 2, 3, 4, 5\}$. Then $h(\mathcal{P}_1) = 12$, $h(\mathcal{P}_2) = 15$, $h(\mathcal{P}_3) = 24$, $h(\mathcal{P}_4) = 30$, $h(\mathcal{P}_5) = 10$, $h(\mathcal{P}_6) = 8$, $h(\mathcal{P}_7) = 40$, $h(\mathcal{P}_8) = 20$, $h(\mathcal{P}_9) = 6$. Clearly h is injective $K_{2,3}$ is strongly 2-multiplicative.

Finally, consider a complete bipartite graph $K_{2,4}$. In any labelling of the vertices we get as the value of $h(\mathcal{P})$ one of 12,24 and 30. Since $12 = 1 \cdot 3 \cdot 4 = 1 \cdot 6 \cdot 2$, $24 = 1 \cdot 6 \cdot 4 = 3 \cdot 4 \cdot 2$ and $30 = 3 \cdot 2 \cdot 5 = 1 \cdot 6 \cdot 5$, we get two distinct path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ with $h(\mathcal{P})$=$h(\mathcal{P}')$. Hence $K_{2,4}$ is not strongly 2-multiplicative. Like this one can show that, a complete bipartite graph $K_{2,n}$, for $n > 4$ is not strongly 2-multiplicative. □

Theorem 2.16 *The graph $P_2 + P_n$ is strongly 2-multiplicative if and only if $n \leq 3$.*

Proof First, consider the graph $P_2 + P_2$. This is same as K_4, which is strongly 2-multiplicative by Theorem 2.11.

Second, consider the graph $P_2 + P_3$. Then $\mathcal{A}$ consists of 10 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ $\mathcal{P}_4$,$\mathcal{P}_5, \mathcal{P}_6$,$\mathcal{P}_7, \mathcal{P}_8$, $\mathcal{P}_9$ and $\mathcal{P}_{10}$ corresponding to paths having vertex sets $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_2, v_4\}$, $\{v_3, v_2, v_5\}$, $\{v_2, v_5, v_1\}$, $\{v_2, v_4, v_1\}$, $\{v_4, v_2, v_5\}$, $\{v_1, v_4, v_5\}$, $\{v_2, v_3, v_1\}$ and $\{v_3, v_4, v_5\}$ respectively. We label the vertices as follows: $v_i = i$, for all $i \in \{1, 2, 3, 4, 5\}$. Then $h(\mathcal{P}_1) = 12$, $h(\mathcal{P}_2) = 15$, $h(\mathcal{P}_3) = 24$, $h(\mathcal{P}_4) = 30$, $h(\mathcal{P}_5) = 10$, $h(\mathcal{P}_6) = 8$, $h(\mathcal{P}_7) = 40$, $h(\mathcal{P}_8) = 20$, $h(\mathcal{P}_9) = 6$, $h(\mathcal{P}_{10}) = 60$. Clearly h is injective and $P_2 + P_3$ is strongly 2-multiplicative.

Finally, consider graph $P_2 + P_n$ where $n \geq 4$. In any labeling of the vertices we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$ but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 4$, $P_2 + P_n$ is not strongly 2-multiplicative. □

Theorem 2.17 *The peterson graph is strongly 2-multiplicative.*

Proof Consider a peterson graph with vertices $v_1, v_2, v_3, v_4, \cdots, v_{10}$.

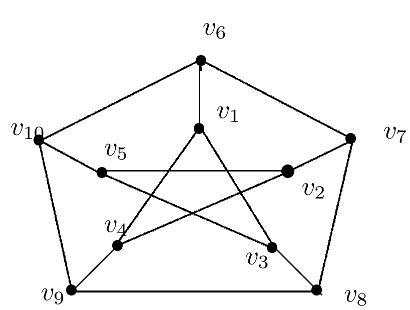

Figure 5

Then $\mathcal{A}$ consists of 21 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots, \mathcal{P}_{21}$, corresponding to paths having vertex sets $\{v_4, v_1, v_3\}$, $\{v_5, v_2, v_4\}$, $\{v_5, v_3, v_1\}$, $\{v_1, v_4, v_2\}$, $\{v_2, v_5, v_3\}$, $\{v_6, v_7, v_8\}$, $\{v_7, v_8, v_9\}$, $\{v_8, v_9, v_{10}\}$, $\{v_9, v_{10}, v_6\}$, $\{v_9, v_{10}, v_4\}$, $\{v_{10}, v_6, v_7\}$, $\{v_6, v_1, v_4\}$, $\{v_6, v_1, v_3\}$, $\{v_7, v_2, v_5\}$, $\{v_7, v_2, v_3\}$, $\{v_8, v_3, v_1\}$, $\{v_8, v_3, v_5\}$, $\{v_9, v_4, v_2\}$, $\{v_9, v_4, v_1\}$, $\{v_{10}, v_5, v_3\}$ and $\{v_{10}, v_5, v_2\}$ respectively. We label the vertices as follows: $v_i = i$, for $1 \leq i \leq 7$, $v_8 = 9$, $v_9 = 8$, $v_{10} = 10$. Then $h(\mathcal{P}_1) = 12$, $h(\mathcal{P}_2) = 40$, $h(\mathcal{P}_3) = 15$, $h(\mathcal{P}_4) = 8$, $h(\mathcal{P}_5) = 30$, $h(\mathcal{P}_6) = 378$, $h(\mathcal{P}_7) = 504$, $h(\mathcal{P}_8) = 720$, $h(\mathcal{P}_9) = 480$, $h(\mathcal{P}_{10}) = 320$, $h(\mathcal{P}_{11}) = 420$, $h(\mathcal{P}_{12}) = 24$, $h(\mathcal{P}_{13}) = 18$, $h(\mathcal{P}_{14}) = 70$, $h(\mathcal{P}_{15}) = 42$, $h(\mathcal{P}_{16}) = 27$, $h(\mathcal{P}_{17}) = 135$, $h(\mathcal{P}_{18}) = 64$, $h(\mathcal{P}_{19}) = 32$, $h(\mathcal{P}_{20}) = 150$, $h(\mathcal{P}_{21}) = 100$. Clearly h is injective peterson graph is strongly 2-multiplicative. $\square$

Theorem 2.18 *The windmill K_n^m is strongly 2-multiplicative if and only if $m \leq 3, n \leq 3$.*

Proof First, if $m = 2$, then the proof follows from the proof of Theorem 2.5, with $n = 3$.

Second, consider the K_3^3 with vertices v_1, v_2, v_3, v_4, v_5, v_6 and v_7 such that v_1 be the common vertex as shown in the figure.

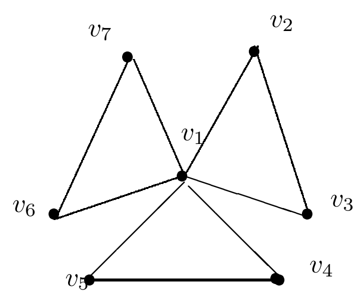

Figure 6

Then $\mathcal{A}$ consists of 15 distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4, \ldots$, $\mathcal{P}_{15}$ corresponding to paths having vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_2, v_1, v_7\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_3, v_1, v_7\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_4, v_1, v_7\}$, $\{v_5, v_1, v_6\}$, $\{v_5, v_1, v_7\}$ and $\{v_6, v_1, v_7\}$ respectively. We label the vertices as follows: $v_1 = 2$, $v_2 = 1$ and $v_i = i$ for all i for $3 \leq i \leq 7$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 14$, $h(\mathcal{P}_6) = 24$, $h(\mathcal{P}_7) = 30$, $h(\mathcal{P}_8) = 36$, $h(\mathcal{P}_9) = 42$, $h(\mathcal{P}_{10}) = 40$, $h(\mathcal{P}_{11}) = 48$, $h(\mathcal{P}_{12}) = 56$, $h(\mathcal{P}_{13}) = 60$, $h(\mathcal{P}_{14}) = 70$, $h(\mathcal{P}_{15}) = 84$. Clearly h is injective K_3^3 is strongly 2-multiplicative.

Finally, consider a windmill K_n^m for $m \geq 3$, $n > 3$. In any labelling of the vertices, we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$, but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 3$,$m \geq 3$, K_n^m is not strongly 2-multiplicative. $\square$

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