International J.Math. Combin. Vol.4(2018), 53-64

Spectra of a New Join in Duplication Graph

K.Reji Kumar

(Department of Mathematics, N. S. S. College, Cherthala, Kerala)

Renny P. Varghese

(Department of Mathematics, Catholicate College, Pathanamthitta, Kerala)

E-mail: rkkmaths@yahoo.co.in, rennypv1@gmail.com

Abstract: The *duplication graph* $D_G(G)$ of a graph G is obtained by inserting new vertices corresponding to each vertex of G and making the vertex adjacent to the neighborhood of the corresponding vertex of G and deleting the edges of G. Let G_1 and G_2 be two graph with vertex sets $V(G_1)$ and $V(G_2)$ respectively. The *D_G-vertex join* of G_1 and G_2 is denoted by $G_1 \sqcup G_2$ and it is the graph obtained from $D_G(G_1)$ and G_2 by joining every vertex of $V(G_1)$ to every vertex of $V(G_2)$. The DG-*add vertex join* of G_1 and G_2 is denoted by $G_1 \bowtie G_2$ and is the graph obtained from $D_G(G_1)$ and G_2 by joining every additional vertex of $D_G(G_1)$ to every vertex of $V(G_2)$. In this paper we determine the A-spectra and L-spectra of the two new joins of graphs G_1 and G_2 when G_1 is a regular graph and G_2 is an arbitrary graph. As an application we give the number of spanning tree, the Kirchhoff index and Laplace energy like invariant of the new join. Also we obtain some infinite family of new class of integral graphs.

Key Words: Spectrum, cospectral graphs, Join of graphs, spanning tree, Kirchhoff index, Laplace-energy like invariant.

AMS(2010): 05C50.

§1. Introduction

All graphs described in this paper are simple and undirected. Let G be a graph with vertex set $V(G_1) = \{v_1, v_2, \cdots v_n\}$. The adjacency matrix of G, denoted by $A(G) = (a_{ij})_{n\times n}$ is an $n \times n$ symmetric matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Let d_i be the degree of the vertex v_i in G and $D(G) = diag(d_1, d_2, \cdots d_n)$ be the diagonal matrix of G. The Laplacian matrix is defined as $L(G) = D(G) - A(G)$. The characteristic polynomial of $A(G)$ is defined as $f_G(A : x) = det(xI_n - A)$, where I_n is the identity matrix of order n. The roots of the characteristic equation of $A(G)$ are called the *eigenvalues* of G. It is

[1]Received January 23, 2018, Accepted November 26, 2018.

denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. It is called the *A - Spectrum* of G. The eigen values of $L(G)$ is denoted by $0 = \mu_1(G) \leq \mu_2(G), \cdots \leq \mu_n(G)$ and it is called the *L - Spectrum* of G. Since $A(G)$ and $L(G)$ are real and symmetric, their eigen values are all real numbers. A graph is *A - integral*, if the A - spectrum consists only of integers [4,14]. Two graphs are said to be *A - Cospectral* if they have the same A - spectrum.

The characteristic polynomial and spectra of graphs help to investigate some properties of graphs such as energy [8,16], number of spanning trees [18, 9,1], the Kirchhoff index [2, 5, 11], Laplace energy like invariants [7] etc.

The first result on Laplacian matrix, which was discovered by Kirchhoff, appeared in a paper published in the year 1847 is related to electrical network. There exists a vast literature that studies the Laplacian eigen values and their relationship with various properties of graphs [12,13]. Most of the studies of the Laplacian eigen values has naturally concentrated on external non trivial eigen values. Gutman et al. [16] discovered the connection between photoelectron spectra of standard hydrocarbons and the Laplacian eigen values of the underlying molecular graphs.

In a recent paper Reji Kumar and Renny P. Varghese [18] introduced subdivision graph vertex join of two given graphs and studies its spectral properties. They also studied [19] the spectral properties of some classes of hypergraphs.

In the next section we define DG - vertex join and DG - add vertex join of two graphs and discuss some important results, which are found essential to prove the results given in the subsequent sections. In the third section we find the A - spectrum and the L - spectrum of the new join and prove some related results. As an application, we find the number of spanning trees, Kirchhoff index and Laplacian - energy like invariant. Fourth section contains a discussion on some infinite family of integral graphs.

§2. Preliminaries

In a paper published in 1973 on duplicate graphs, which appeared in the *Journal of Indian Mathematical Society*, Sampathkumar [10] defined duplicate graphs. Let G be a graph with vertex set V(G) $= \{v_1, v_2, \cdots, v_n\}$. Take another set $U = \{u_1, u_2, \cdots, u_n\}$. Make u_i adjacent to all the vertices in $N(v_i)$, the neighbourhood set of v_i, in G for each i and remove all edges of G. The resulting graph is called the *duplication graph* of G and is denoted by $D(G)$. The following result tells us an easy way to find the determinant of a bigger matrix using the determinant of relatively smaller matrices.

Proposition 2.1 *Let M_1, M_2, M_3, M_4 be respectively $p \times p, p \times q, q \times p, q \times q$ matrix with M_1 and M_4 are invertible then*

$$det\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = det(M_1)det(M_4 - M_3M_1^{-1}M_2)$$

$$= det(M_4)det(M_1 - M_2M_4^{-1}M_3),$$

where $M_4 - M_3M_1^{-1}M_2$ and $M_1 - M_2M_4^{-1}M_3$ are called the Schur complements of M_1 and M_4 respectively.

Let G be a graph on n vertices, with the adjacency matrix A. The characteristic matrix $xI - A$ of A has determinant $det(xI - A) = f_G(A : x) \neq 0$, so is invertible. The A - *coronal* ([6]), $\Gamma_A(x)$ of G is defined to be the sum of the entries of the matrix $(xI - A)^{-1}$. This can be calculated as

$$\Gamma_A(x) = \mathbf{1}_n^T(xI - A)^{-1}\mathbf{1}_n.$$

The A - coronal of some classes of graphs are given here.

Lemma 2.2([6]) *Let G be r - regular on n vertices. Then*

$$\Gamma_A(x) = \frac{n}{x - r}.$$

Since for any graph G with n vertices, each row sum of the Laplacian matrix $L(G)$ is equal to 0, we have $\Gamma_L(x) = \frac{n}{x}$.

Lemma 2.3([6]) *Let G be the bipartite graph K_{pq}, where $p + q = n$. Then*

$$\Gamma_A(x) = \frac{nx + 2pq}{x^2 - pq}.$$

The following results on an $n \times n$ real matrix is useful in this context.

Proposition 2.4([15]) *Let A be an $n \times n$ real matrix, and $J_{s\times t}$ denote the $s \times t$ matrix with all entries equal to one. Then*

$$det(A + \alpha J_n \times n) = det(A) + \alpha \mathbf{1}_n^T adj(A)\mathbf{1}^n.$$

Here α is a real number and $adj(A)$ is the adjugate matrix of A.

Corollary2.5([15]) *Let A be an $n \times n$ real matrix. Then*

$$det(xI_n - A - \alpha J_{n\times n}) = (1 - \alpha\Gamma_A(x))\ det(xI_n - A).$$

Next we proceed to define the DG - vertex join and the DG - advertex join of two graphs.

Definition 2.6 *Let G_1 be a graph on n_1 vertices and m_1 edges. G_2 be an arbitrary graph on n_2 vertices The $DG -$ vertex join of G_1 and G_2 is denoted by $G_1 \sqcup G_2$ and is the graph obtained from $D(G_1)$ and G_2 by joining every vertex of $V(G_1)$ to every vertex of $V(G_2)$. Where $D(G_1)$ is the duplication graph of G_1.*

In Figure 1 an example of DG - vertex join of the graphs C_4 and K_2 is given.

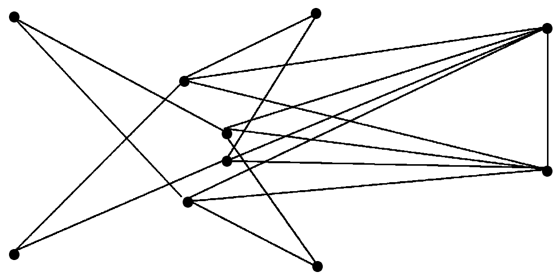

Figure 1 $C_4 \sqcup K_2$

Definition 2.7 *The $DG-$ addvertex join of G_1 and G_2 is denoted by $G_1 \bowtie G_2$ and is the graph obtained from $D(G_1)$ and G_2 by joining the additional vertices of $D(G_1)$ corresponding to the vertices of G_1 with every vertex of $V(G_2)$.*

In Figure 2 an example of DG - advertex join of the graphs C_4 and K_2 is given.

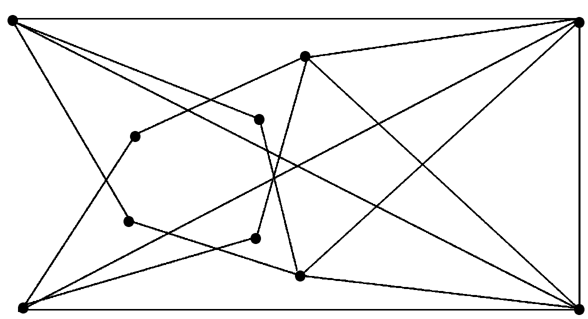

Figure 2 $C_4 \bowtie K_2$

§3. Spectrum of $G_1 \sqcup G_2$ for Some Classes of Graphs G_1 and G_2

In this section we study the spectrum of DG - vertex join of some classes of graphs G_1 and G_2. We prove the following results in this connection.

Theorem 3.1 *Let G_1 be an r_1 - regular graph on n_1 vertices and m_1 edges. G_2 be an arbitrary graph on n_2 vertices. Then, the Characteristic polynomial of $G_1 \sqcup G_2$ is*

$$f_{G_1 \sqcup G_2}(A:x) = (x^2 - n_1 x \Gamma_{A_2}(x) - r_1^2) \prod_{i=2}^{n_2} (x - \lambda_i(G_2)) \prod_{i=2}^{n_1} (x^2 - \lambda_i(G_1)^2).$$

Proof The adjacency matrix of $G_1 \sqcup G_2$ is

$$A = \begin{bmatrix} 0 & A_1 & J_{n_1\times n_2} \\ A_1 & 0_{n_1} & 0_{n_1\times n_2} \\ J_{n_2\times n_1} & 0_{n_2\times n_1} & A_2 \end{bmatrix}$$

where A_1 and A_2 are the adjacency matrix of G_1 and G_2 respectively and J is a matrix with each entries 1.

The characteristic polynomial of $G_1 \sqcup G_2$ is

$$\begin{aligned} f_{G_1\sqcup G_2}(A:x) &= \begin{vmatrix} xI_{n_1} & -A_1 & -J \\ -A_1 & xI_{n_1} & 0 \\ -J & 0 & xI_{n_2}-A_2 \end{vmatrix} \\ &= det(xI_{n_2}-A_2)\ det\ S, \end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n_1} \end{pmatrix} - \begin{pmatrix} -J_{n_1\times n_2} \\ 0 \end{pmatrix} (xI_{n_2}-A_2)^{-1} \begin{pmatrix} -J_{n_2\times n_1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n1} \end{pmatrix} - \begin{pmatrix} \Gamma_{A_2}(x)J_{n_1\times n_1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} xI-\Gamma_{A_2}(x)J_{n_1\times n_1} & -A_1 \\ -A_1 & xI \end{pmatrix} \end{aligned}$$

Whence,

$$\begin{aligned} det\ S &= det(xI)\ det\left((xI-\Gamma_{A_2}(x)J-\frac{A_1^2}{x}\right) \\ &= x^{n_1}\ det\left(xI-\Gamma_{A_2}(x)J-\frac{A_1^2}{x})\right) \\ &= x^{n_1}\ det\left(xI-\frac{A_1^2}{x}-\Gamma_{A_2}(x)J\right) \\ &= x^{n_1}\ det\left(xI-\frac{A_1^2}{x}\right)\left(1-\Gamma_{A_2}(x)\Gamma_{\frac{A_1^2}{x}}(x)\right), \end{aligned}$$

Notice that G_1 is r_1 - regular and the row sum of A_1^2 is r_1^2. We get

$$\Gamma_{\frac{A_1^2}{x}} = \frac{n_1}{x-\frac{r_1^2}{x}} = \frac{n_1 x}{x^2-r_1^2}$$

and

$$\begin{aligned} det\ S &= x^{n_1} det\left(xI - \frac{A_1^2}{x})\right)\left(1 - \frac{n_1 x}{x^2 - r_1^2}\Gamma_{A_2}(x)\right) \\ &= det(x^2 I - A^2)\left(\frac{x^2 - r_1^2 - n_1 x \Gamma_{A_2}(x)}{x^2 - r_1^2}\right). \end{aligned}$$

Hence

$$det(xI - A) = (x^2 - n_1 x \Gamma_{A_2}(x) - r_1^2)\prod_{i=1}^{n_2}(x - \lambda_i(G_2))\prod_{i=2}^{n_1}(x^2 - \lambda_i(G_1)^2). \qquad \square$$

Corollary 3.2 *Let G_1 be an r_1 - regular graph on n_1 vertices, G_2 be r_2 - regular graph on n_2 vertices. Then the $A -$ Spectrum of $G_1 \sqcup G_2$ consists of*

(*i*) $\lambda_i(G_2)$, *for* $i = 2, 3, \cdots, n_2$;

(*ii*) $\pm\lambda_i(G_1)$, *for* $i = 2, 3, \cdots, n_1$;

(*iii*) *Three roots of the equation*

$$x^3 - r_2 x^2 - (n_1 n_2 + r_1^2)x + r_1^2 r_2.$$

Proof If G_2 is r_2 - regular then

$$\Gamma_{A_2}(x) = \frac{n_2}{x - r_2}.$$

We get

$$\begin{aligned} det(xI - A) &= (x^3 - r_2 x^2 - (n_1 n_2 + r_1^2)x + r_1^2 r_2) \\ &\quad \times \prod_{i=2}^{n_2}(x - \lambda_i(G_2))\prod_{i=2}^{n_1}(x^2 - \lambda_i(G_1)^2). \qquad \square \end{aligned}$$

Corollary 3.3 *Let G_1 be an r_1 - regular graph on n_1 vertices, $A -$ Spectrum of $G_1 \sqcup \overline{K_n}$ consists of*

(*i*) *0, repeats n_2 times;*

(*ii*) $\pm\lambda_i(G_1)$, *for* $i = 2, 3, \cdots, n_1$;

(*iii*) $\pm\sqrt{n_1 n_2 + r_1^2}$.

Corollary 3.4 *Let G_1 be an r_1 - regular graph on n_1 vertices. $A -$ Spectrum of $G_1 \sqcup K_{pq}$ consists of*

(*i*) *0, repeats $p + q - 2$ times;*

(*ii*) $\pm\lambda_i(G_1)$, *for* $i = 2, 3, \cdots, n_1$;

(*iii*) *Four roots of the equation*

$$x^4 - (pq + r_1^2 + n_1 p + n_1 q)x^2 - 2pqn_1 x + r_1^2 pq.$$

3.1 Laplacian Spectrum of $G_1 \sqcup G_2$ for Some Classes of Graphs G_1 and G_2

Theorem 3.5 *Let G_1 be an r_1 - regular graph on n_1 vertices and m_1 edges. G_2 be an arbitrary graph on n_2 vertices. then,*

$$\begin{aligned} f_{G_1 \sqcup G_2}(L:x) &= x(x^2-(n_1+n_2+2r_1)x+r_1(2n_1+n_2)) \\ &\quad \times \prod_{i=2}^{n_2}(x-n_1-\mu_i(G_2)) \prod_{i=2}^{n_1}(x^2-(2r_1+n_2)x+n_2r_1+r_1^2-\lambda_i(G_1)^2). \end{aligned}$$

Proof The Laplace adjacency matrix of $G_1 \sqcup G_2$ is

$$L = \begin{bmatrix} (r_1+n_2)I & -A_1 & J_{n_1\times n_2} \\ -A_1 & r_1 I & 0_{n_1\times n_2} \\ -J_{n_2\times n_1} & 0_{n_1\times n_1} & n_1 I_{n_2}+L_2 \end{bmatrix}$$

where L_2 is the Laplacian adjacency matrix of G_2

The Laplacian characteristic polynomial of $G_1 \sqcup G_2$ is

$$f_{G_1 \sqcup G_2}(L:x) = \begin{vmatrix} (x-r_1-n_2)I_{n_1} & A_1 & J \\ A_1 & (x-r_1)I_{n_1} & 0 \\ J & 0 & (x-n_1)I_{n_2}-L_2 \end{vmatrix}.$$

Using proposition 2.2 we get

$$f_{G_1 \sqcup G_2}(L:x) = det((x-n_1)I_{n_2}-L_2)\ detS,$$

where

$$\begin{aligned} S &= \begin{pmatrix} (x-r_1-n_2)I_{n_1} & A_1 \\ A_1 & (x-r_1)I_{n_1} \end{pmatrix} - \begin{pmatrix} J \\ 0 \end{pmatrix} ((x-n_1)I_{n_1}-L_2)^{-1} \begin{pmatrix} J & 0 \end{pmatrix} \\ &= \begin{pmatrix} (x-r_1-n_2)I & A_1 \\ A_1 & (x-r_1)I \end{pmatrix} - \begin{pmatrix} \Gamma_{L_2}(x-n_1)J_{n_1\times n_1} & 0 \\ 0 & o \end{pmatrix} \\ &= \begin{pmatrix} (x-r_1-n_2)I-\Gamma_{L_2}(x-n_1)J & A_1 \\ A_1 & (x-r_1)I \end{pmatrix} \end{aligned}$$

Therefore,

$$det\ S = (x-r_1)^{n_1} det\left((x-r_1-n_2)I-\Gamma_{L_2}(x-n_1)J-\frac{A_1^2}{x-r_1}\right).$$

By Corollary 2.7

$$\begin{aligned} det\ S &= (x-r_1)^{n_1} det\left((x-r_1-n_2)I - \frac{A_1^2}{x-r_1}\right) \\ &\quad \times \left(1-\Gamma_{L_2}(x-n_1)\Gamma_{\frac{A_1^2}{x-r_1}}(x-r_1-n_2)\right) \\ &= det\left((x-r_1-n_2)(x-r_1)I - A^2\right)\left(1-\Gamma_{L_2}(x-n_1)\Gamma_{\frac{A_1^2}{x-r_1}}(x-r_1-n_2)\right). \end{aligned}$$

Since G_1 is r_1 regular graph, the row sum of $\frac{A_1^2}{x-r_1}$ is $\frac{r_1^2}{x-r_1}$. Therefore,

$$\begin{aligned} \Gamma_{\frac{A_1^2}{x-r1}}(x-r_1-n_2) &= \frac{n_1(x-r_1)}{x^2-(2r_1+n_2)x+n_2r_1}, \\ 1-\Gamma_{L_2}(x-n_1)\Gamma_{\frac{A_1^2}{x-r_1}}(x-r_1-n_2) &= \frac{x(x^2-(n_1+n_2+2r_1)x+r_1(2n_1+n_2))}{(x-n_1)(x^2-(2r_1+n_2)x+n_2r_1)}. \end{aligned}$$

Hence

$$\begin{aligned} f_{G_1\sqcup G_2}(L:x) &= x(x^2-(n_1+n_2+2r_1)x+r_1(2n_1+n_2)) \\ &\quad \times \prod_{i=2}^{n_2}(x-n_1-\mu_i(G_2))\prod_{i=2}^{n_1}(x^2-(2r_1+n_2)x+n_2r_1+r_1^2-\lambda_i(G_1)^2). \quad \square \end{aligned}$$

Let $t(G)$ denote the number of spanning tree of the graph G, the total number of distinct spanning subgraphs of G that are trees. The number of spanning trees of the graph describe the network which is one of the natural characteristics of its reliability. If G is a connected graph with n vertices and the Laplacian spectrum $0=\mu_1(G)\le\mu_2(G),\cdots\le,\mu_n(G)$ then ([17])

$$t(G) = \frac{\mu_2(G)\mu_3(G)\cdots\mu_n(G)}{n}$$

Corollary 3.6 *Let G_1 be an r_1 - regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices. Then*

$$t(G_1\sqcup G_2) = \frac{r_1(2n_1+n_2)\prod_{i=2}^{n_1}(n_1+\mu_i(G_2))\prod_{i=2}^{n_2}(r_1^2+n_2r_1-\lambda_i^2(G_1))}{2n_1+n_2}.$$

Proof By Theorem 3.5 the roots of $f_{G_1\sqcup G_2}(L:x)$ are as follows:

(i) 0;

(ii) $n_1+\mu_i(G_2)$ for $i=2,3,\cdots,n_2$;

(iii) Two roots say x_1 and x_2 of the equation $x^2-(n_1+n_2+2r_1)x+r_1(2n_1+n_2)$;

(iv) Two roots say x_{i1} and x_{i2} of the equation $x^2-(2r_1+n_2)x+n_2r_1+r_1^2-\lambda_i(G_1)^2$ for $i=2,3,\cdots,n_2$.

For Case (iii), $x_1x_2=r_1(2n_1+n_2)$, and for Case (iv), $x_{i1}x_{i2}=n_2r_1+r_1^2-\lambda_i(G_1)^2$,

$i=2,3,\cdots,n_2$. Then, we get that

$$t(G_1 \sqcup G_2) = \frac{r_1(2n_1+n_2)\prod_{i=2}^{n_1}(n_1+\mu_i(G_2))\prod_{i=2}^{n_2}(r_1^2+n_2r_1-\lambda_i^2(G_1))}{2n_1+n_2}. \qquad \square$$

Another Laplacian spectrum based on graph invariant was defined by Liu and Liu [3] called the Laplacian - energy - like invariant. The Laplacian - energy - like invariant(LEL) of a graph G of n vertices is defined as

$$LEL(G) = \sum_{i=2}^{n}\sqrt{\mu_i}.$$

Corollary 3.7 *Let G_1 be an r_1 - regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices. Then Laplace - energy - like invariant*

$$\begin{aligned} LEL &= \left(n_1+n_2+2r_1+2\sqrt{r_1(2n_1+n_2)}\right)^{1/2} + \sum_{i=2}^{n_2}\left(n_1+\mu_i(G_1)^2\right)^{1/2} \\ &\quad + \sum_{i=2}^{n_1}\left(\frac{2r_1+n_2+\sqrt{r_1^2+n_2r_1-\lambda_i(G_1)^2}}{r_1^2+n_2r_1-\lambda_i(G_1)^2}\right)^{1/2}. \end{aligned}$$

Proof Using Theorem 3.5 and Corollary 3.6 we have

$$\begin{aligned} \sqrt{x_1}+\sqrt{x_2} &= (x_1+x_2+2\sqrt{x_1x_2})^{1/2} \\ &= \left(n_1+n_2+2\sqrt{r_1(2n_1+n_2)}\right)^{1/2}, \\ \frac{1}{\sqrt{x_{i1}}}+\frac{1}{\sqrt{x_{i2}}} &= \frac{\sqrt{x_{i1}}+\sqrt{x_{i2}}}{2\sqrt{x_{i1}x_{i2}}} \\ &= \left(\frac{x_1+x_2+\sqrt{x_1x_2}}{x_{i1}x_{i2}}\right)^{1/2} \\ &= \left(\frac{2r_1+n_2+\sqrt{r_1^2+n_2r_1-\lambda_i(G_1)^2}}{r_1^2+n_2r_1-\lambda_i(G_1)^2}\right)^{1/2}. \end{aligned}$$

Hence the required result is obtained using the formula for LEL. $\square$

Klein [5] propounder of *resistance distance* defined electric resistance in network corresponding to the considered graph as the resistance distance between any two adjacent nodes is 1 ohm. The sum of the resistance distance between all pairs of the vertices of a graph is conceived as a new graph invariant. The electric resistance is calculated by means of the Kirchhoff laws called *kirchhoff index*.

Kirchhoff index of a connected graph G with $n(n \geq 2)$ vertices is defined as

$$Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\mu_i}$$

Corollary 3.8 *Let G_1 be an r_1 - regular graph on n_1 vertices. G_2 be an arbitrary graph on n_2*

vertices. Then

$$Kf(G_1 \sqcup G_2) = (2n_1 + n_2)\left[\frac{n_1 + n_2 + 2r_1}{r_1(2n_1 + n_2)} + \sum_{i=2}^{n_2} \frac{1}{n_1 + \mu_i(G_2)} + \sum_{i=2}^{n_1} \frac{2r_1 + n_2}{r_1^2 + n_2 r_1 - \lambda_i(G_1)^2}\right].$$

Proof Using Theorem 3.5, Corollary 3.7 and the formula for Kirchhoff index we obtain the required result. □

3.2 Spectra of DG - add Vertex Graph of Some Classes of Graphs

Next we discuss some spectral properties of the DG - add vertex graph of some classes of graphs.

Proposition 3.9 *Let G_1 be an r_1 - regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices. Then $G_1 \sqcup G_2$ and $G_1 \bowtie G_2$ are A - cospectral*

Proof Notice that the characteristic polynomials of $G_1 \sqcup G_2$ and $G_1 \bowtie G_2$ are same. Hence we get the result. □

Proposition 3.10 *Let G_1 be an r_1 - regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices then $G_1 \sqcup G_2$ and $G_1 \bowtie G_2$ are L - cospectral.*

§4. Infinite Families of Integral Graphs

The following properties give a necessary and sufficient condition for DG - vertex join and DG - add vertex join of G_1 and G_2 to be integral.

Proposition 4.1 *Let G_1 be r_1 - regular graph on n_1 vertices and G_2 be r_2 - regular graph on n_2 vertices. $G_1 \sqcup G_2$ (respectively $G_1 \bowtie G_2$) is an integral graph if and only if G_1 and G_2 are integral graphs and the roots of $x^3 - r_2x^2 - (n_1n_2 + r_1^2)x + r_1^2r_2$ are integers.*

In particular if $G_2 = \overline{K_n}$ (totally disconnected) then $r_2 = 0$ then $G_1 \sqcup G_2$ (respectively $G_1 \bowtie G_2$) is integral iff G_1 is an integral graph and $n_1n_2 + r_1^2$ is a perfect square.

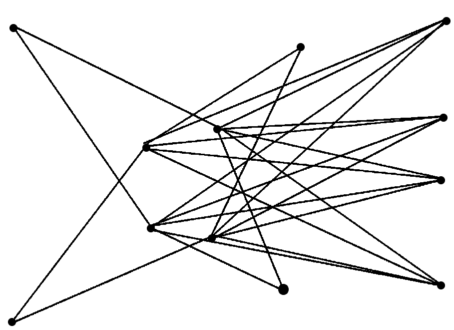

Figure 3 $K_4 \sqcup \overline{K_4}$ with spectrum $\{-5, -1^3, 0^4, 1^3, 5\}$

Proposition 4.2 *Let G_1 be r_1 - regular graph on n_1. $G_1 \sqcup K_{pq}$ (respectively $G_1 \bowtie K_{pq}$) is an integral graph if and only if G_1 is an integral graph and the roots of $x^4 - (pq + r_1^2 + n_1 p + n_1 q)x^2 - 2pqn_1 x + r_1^2 pq$ are integers.*

References

[1] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs, Theory and Applications*(Third Edition), Johann Ambrosius Barth Heidelberg, 1995.

[2] X. Gao, Y.F. Luo, W.W. Liu, Kirchhoff index in line, subdivision and total graphs of a regular graph, *Discrete Appl. Math.*, 160(2012), 560 - 565.

[3] J.Liu, B. Liu, A Laplacian - energy - like invarient of a graph, *MATCH Commun. Math. comput. Chem.*, 59(2008), 397 - 413.

[4] G. Indulal, Spectrum of two new joins of graphs and infinite families of integral graphs, *Kragujevac J. Math.*, 36 (2012), 133 - 139.

[5] D. J. Klein, M. Randic, Resistance distance, *J. Math. Chem.*, 12 (1993), 81 - 95.

[6] C. McLeman, E. McNicholas, Spectra of coronae, *Linear Algebra Appl.*, 435 (2011), 998 - 1007.

[7] B.Arsic,I. Gutman, K. Ch. Das, K. Xu, Relation between Kirchoff Index and Laplacian - Energy - like Invarient, *Bulletin T.CXLIV de l'Academic Serbe Des Sciences et Des Arts - 2012 Classe Des Sciences Matematiques et Naturelles Sciences Mathematiques*, No 37.

[8] X.L. Li, Y.T. Shi, I. Gutman, *Graph Energy*, Springer-Verlag New York Inc., 2012.

[9] K. Ozeki, T. Yamashita, Spanning trees: A survey, *Graphs and Combinatorics*, 27(1) (2011), 1 - 26.

[10] E. Sampathkumar, On duplicate graphs, *Journal of the Indian Math. Soc.*, 37(1973) 285 - 293.

[11] Y.P. Zhang, X.R. Yong, M. J. Golin, Chebyshev polynomials and spanning tree formulas for circulant and related graphs, *Discrete Math.* 298(1) (2005), 334 - 364.

[12] K.C. Das, An improved upper bound for Laplacian graph eigenvalues, *Linear Algebra Appl.* 368 (2003), 269 - 278.

[13] K.C. Das, Proof of conjectures involving algebraic connectivity of graphs, *Linear Algebra Appl.*, 438 (2013), 3291 - 3302.

[14] K. Balinska, D. Cvetkovic, Z. Radosavljevic, S. Simic, and D. Stevanovic, A survey on integral graphs, *Publ. Elektrotehn. Fak. Ser. Mat.*, 13 (2002), 42 - 65.

[15] Liu, X.G. and Zhang, Z.H., Spectra of subdivision-vertex and subdivision-edge joins of graphs, 2014, *arXiv: 1212.0619v2* [math.CO].

[16] I. Gutman, D. Babic, V. Gineityte, Degeneracy in the equivalent bond orbital model for highenergy band in the photoelectron spectra of saturated hydrocarbons, *ACH Models Chem.*, 135 (1998), 901 - 909.

[17] Jianxi Li, Wai Chee Shiu, An Chang, The number of spanning tree of a graph, *Applied Mathematics Letters*, 23 (2010), 286 - 290.

[18] Renny P Varghese and K Reji Kumar, Spectra of a new join of two graphs, *Advances in Theoretical and Applied Mathematics*, Vol.11, 4 (2016), 459 - 470.

[19] K Reji Kumar and Renny P.Varghese, Spectrum of (k, r) - regular hypergraphs, *International J.Math. Combin.*, Vol.2 (2017), 52 - 59.

[20] C.D. Godsil, Inverses of trees, *Combinatorica*, 5 (1985) 33 - 39.