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Product Cordial Labeling of Extensions of Barbell Graph

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Abstract: A barbell graph $B(r, n)$ is a graph consists of path P_n joining two complete graphs K_r. This paper deals with study of the product cordial labeling of graphs that are obtained by applying various graph operations on barbell graph.

Key Words: Barbell graph, product cordial labeling, Smarandachely product cordial labeling, duplication, switching, degree splitting.

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§1. Introduction

All the graphs considered in this paper are finite, simple, connected and undirected. Through out this work, $|X|$ denotes the cardinality of the set X. By order and size of a graph we means the cardinality of vertex set and the cardinality of edge set respectively. For various graph theoretic notations and terminology we follow [1].

A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices(or edges) then the labeling is called vertex labeling(or edge labeling). A mapping $f : V(G) \to \{0, 1\}$ is called *binary vertex labeling* of a graph $G = (V(G), E(G))$. Also the number of vertices(or edges) having label i under the map f are denoted by $v_f(i)$(or $e_f(i)$) and the set of all vertices adjacent to v are denoted by $N(v)$.

A *product cordial labeling* of a graph $G = (V(G), E(G))$ is a function f from $V(G)$ to $\{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, the number $v_f(0)$ of vertices labeled with 0 and the number $v_f(1)$ of vertices labeled with 1 differ by at most 1, and the number $e_f(0)$ of edges labeled with 0 and the number $e_f(1)$ of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a *product cordial graph*. Opposed to the product cordial labeling, a *Smarandachely product cordial labeling* on G is such a labeling $f : V(G) \to \{0, 1\}$ with induced labeling $f(u)f(v)$ on edge $uv \in E(G)$ that $|v_f(0) - v_f(1)| \geq 2$ or $|e_f(0) - e_f(1)| \geq 2$.

The product cordial labeling was introduced by Sundaram et. al. [3], [4]. They proved that many graphs are product cordial: trees; unicyclic graphs of odd order; triangular snakes;

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dragons; helms; path and cycle related graphs. They also proved that a graph having p vertices and q edges is product cordial, then $q \leq \frac{(p-1)(p+1)}{4} + 1$. For further results on product cordial labeling we refer to the dynamic survey of graph labeling by Gallian [2].

A *barbell graph* consists of a path graph of order n connecting two complete graphs of order $r \geq 3$ each and it is denoted by $B(r, n)$. S K Vaidya and Chirag Barasara [5] proved that if G and G' are the graphs such that their orders or sizes differ at most by 1, then the new graph obtained by joining G and G' by a path P_k of $k \in \mathbb{N}$ length is product cordial. This result along with the definition of barbell graph shows that barbell graph is product cordial. In this paper we study the product cordial labeling of graphs that are obtained by performing certain operations on barbell graph. We first define these operations.

Definition 1.1 *The duplication of a vertex v of graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G'.*

Definition 1.2 *The duplication of vertex v_k by a new edge $e = v'_k v''_k$ in a graph G produce a new graph G' such that $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$.*

Definition 1.3 *The duplication of an edge $e = uv$ by a new vertex w in a graph G produce a new graph G' such that $N(w) = \{u, v\}$.*

Definition 1.4 *The duplication of an edge $e = uv$ of a graph G produce a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = \{N(u) \cup \{v'\}\} \setminus \{v\}$ and $N(v') = \{N(v) \cup \{u'\}\} \setminus \{u\}$.*

Definition 1.5 *A vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.*

Definition 1.6 *Let $G = (V(G), E(G))$ be a graph with $V(G) = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of vertices having at least two vertices and having the same degree and $T = V(G) \setminus \cup S_i$. Then the degree splitting graph of G is a graph obtained from G by adding vertices $w_1, w_2, \cdots w_t$ and joining w_i to each vertex of $S_i (1 \leq i \leq t)$.*

In the present work we proved that graphs obtained from barbell graph $B(r, n)$ by duplicating all vertices by edges and duplicating all edges by vertices in path joining complete graphs are product cordial for all r and n. We also show that a graph obtained by switching a vertex of path in barbell graph B(r; n) admits product cordial labeling for all r and n. We also derive partial results for the product cordial labeling of graphs that are obtained from barbell graph $B(r, n)$ by duplicating vertex by vertex and edge by edge in the path joining complete graphs. Further we show that for certain values of r and n the degree splitting graph of barbell graph as well as degree splitting graph of path in barbell graph are product cordial.

§2. Main Results

Theorem 2.1 *A barbell graph $B(r, n)$ with duplication of edges of path joining complete graphs by vertices, is product cordial for all possible values of r and n.*

Proof In a barbell graph $G = B(r,n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from Barbell graph by taking duplication of edges of path by vertices and also $v'_1, v'_2, \cdots, v'_{n-1}$ be vertices of duplication of path edges $v_1v_2,\ v_2v_3, \cdots, v_{n-1}v_n$ respectively. Then $|V(G')| = 2r+2n-1$ and $|E(G')| = r(r-1)+3n-1$. We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r$$
$$f(u'_i) = 0; 1 \leqslant i \leqslant r$$
$$f(v_j) = \begin{cases} 1, & 1 \le j \le \lceil \frac{n}{2} \rceil ; \\ 0, & \lceil \frac{n}{2} \rceil + 1 \le j \le n. \end{cases}$$
$$f(v'_j) = \begin{cases} 1, & 1 \le j \le \lceil \frac{n}{2} \rceil ; \\ 0, & \lceil \frac{n}{2} \rceil + 1 \le j \le n-1. \end{cases}$$

According to above definition of f, we have $v_f(0)+1 = r+n = v_f(1)$. Thus $|v_f(0)-v_f(1)| \le 1$. For the edges labeled with 0 and 1 consider the following cases.

Case 1. n is odd.

In this case we have $e_f(0) = \frac{r(r-1)}{2} + \frac{3n-1}{2} = e_f(1)$. So, $|e_f(0) - e_f(1)| \le 1$.

Case 2. n is even

In this case we have $e_f(0) = \frac{r(r-1)}{2} + \frac{3n-2}{2} = e_f(1) + 1$. Hence, $|e_f(0) - e_f(1)| \le 1$.

Thus G' has product cordial labeling. □

Example 2.1 A barbell graph $B(5,4)$ with duplication of edges of path joining complete graphs by vertices and its product cordial labeling is shown in Figure 1.

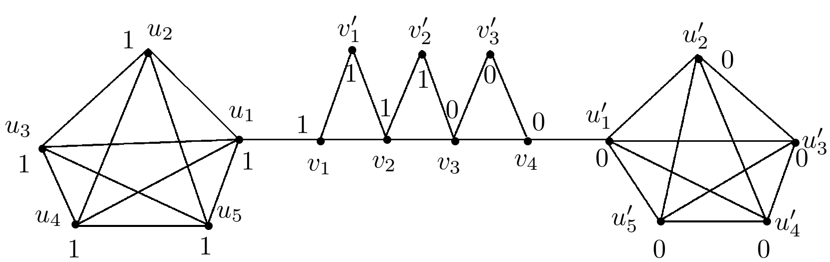

Figure 1 Barbell graph $B(5,4)$ with duplication of edges of path by vertices

Theorem 2.2 *A barbell graph $B(r,n)$ with duplication of vertices of path joining complete graphs by edges, is product cordial for all r and n.*

Proof In a barbell graph $G = B(r,n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from barbell graph by taking duplication of vertices of path by edges and also $v_1'v_2',\ v_2'v_3',\cdots,v_{2n-1}'v_{2n}'$ be edges of duplication of path vertices $v_1,v_2,\cdots,v_n$. Then $|V(G')|=2r+3n$ and $|E(G')|=r(r-1)+4n+1$. We define $f:V(G')\to\{0,1\}$ as

$$f(u_i)=1;1\leqslant i\leqslant r$$
$$f(u_i')=0;1\leqslant i\leqslant r$$
$$f(v_j)=\begin{cases}1, & 1\le j\le\lceil\frac{n}{2}\rceil;\\ 0, & \lceil\frac{n}{2}\rceil+1\le j\le n.\end{cases}$$
$$f(v_j')=\begin{cases}1, & 1\le j\le n;\\ 0, & n+1\le j\le 2n.\end{cases}$$

According to above definition of f, we have $e_f(0)=\frac{r(r-1)}{2}+2n+1=e_f(1)+1$. Thus $|e_f(0)-e_f(1)|\le 1$. For the vertices labeled with 0 and 1 consider the following cases.

Case 1. n is odd

In this case we have $v_f(0)=r+\frac{3n-1}{2}=v_f(1)+1$. So $|v_f(0)-v_f(1)|\le 1$.

Case 2. n is even

In this case we have $v_f(0)=r+\frac{3n}{2}=v_f(1)$. Thus $|v_f(0)-v_f(1)|\le 1$.

And hence G' is product cordial. □

Example 2.2 A barbell graph $B(5,6)$ with duplication of vertices of path joining complete graphs by edges and its product cordial labeling is shown in Figure 2.

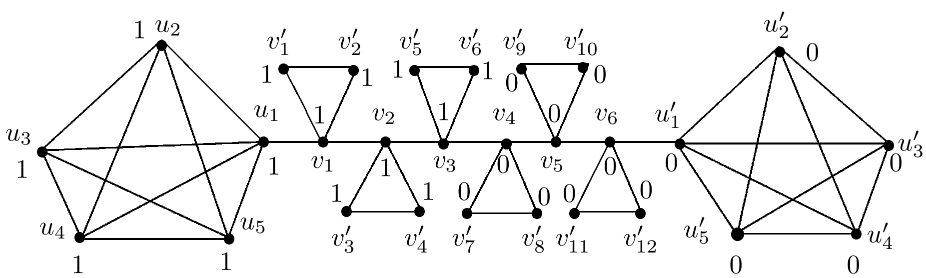

Figure 2 Barbell graph $B(5,6)$ with duplication of vertices by edges

Theorem 2.3 *A barbell graph $B(r,n)$ with switching of a vertex of path joining complete graphs is product cordial for all possible values of r and n.*

Proof Let G be a barbell graph and let $u_1,u_2,\cdots,u_r$ and $u_1',u_2',\cdots,u_r'$ be vertices of complete graphs and $v_1,v_2,\cdots,v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from G by switching vertex v of path. Here for v we have two choices either v is end vertex of path or internal vertex of path.

Case 1. v is end vertex say v_1.

In this case we have $|V(G')| = 2r + n$ and $|E(G')| = r(r-1) + 2n - 3$. Define $f : V(G') \rightarrow \{0, 1\}$ as

$$f(u_i) = 0; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & j = 1, n, n-1, \cdots, \lceil \frac{n}{2} \rceil + 2; \\ 0, & j = 2, 3, \cdots, \rceil \frac{n}{2} \rceil + 1. \end{cases}$$

Subcase 1.1 is n odd.

In this case we have $e_f(1) = \frac{r(r-1)}{2} + n - 1 = e_f(0) + 1$ and $v_f(1) = r + \frac{n+1}{2} = v_f(0) + 1$.

Subcase 1.2 n is even.

In this case we have $e_f(1) + 1 = \frac{r(r-1)}{2} + n - 1 = e_f(0)$ and $v_f(1) = r + \frac{n}{2} = v_f(0)$.

Thus from both the sub cases we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 2. v is internal vertex say v_2.

In this case we have $|V(G')| = 2r + n$ and $|E(G')| = r(r-1) + 2n - 4$. Define $f : V(G') \rightarrow \{0, 1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = 0; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & j = 2, n, n-1, \cdots, \rceil \frac{n}{2} \rceil + 2; \\ 0, & j = 1, 3, 4, \cdots, \rceil \frac{n}{2} \rceil + 1. \end{cases}$$

Then we have $e_f(1) = \frac{r(r-1)}{2} + n - 2 = e_f(0)$ and $v_f(1) = r + \frac{n+1}{2} = v_f(0) + 1$. Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Thus G' is product cordial graph. □

Example 2.3 Consider a barbell graph $B(6, 6)$ with switching of end vertex of path joining complete graphs. Then it is product cordial and its labeling is as shown in Figure 3.

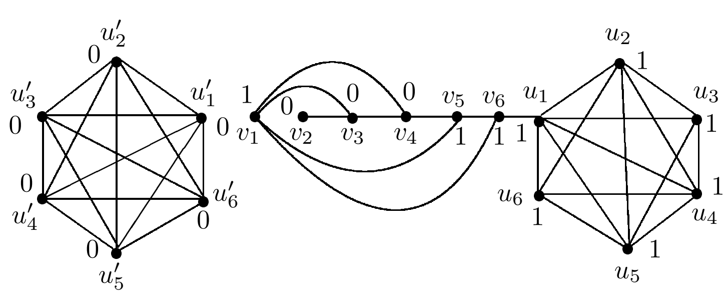

Figure 3 Barbell graph $B(6, 6)$ with switching of end vertex of path

Theorem 2.4 *A barbell graph with duplication of vertices of path joining complete graphs by vertices is product cordial for the following choices of r and n:*

(1) $r \geqslant 3$ *and* $n = 4$;

(2) $r \geq 5$ *and* $n \geq 6$.

Proof In a barbell graph $G = B(r,n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from barbell graph by taking duplication of edges of path by vertices and also $v'_1, v'_2, \cdots, v'_n$ be vertices of duplication of path edges $v_1,\ v_2, \cdots, v_n$ respectively. Then $|V(G')| = 2r + 2n$ and $|E(G')| = r(r-1) + 3n + 1$.

Case 1. $r \geqslant 3$ and $n = 4$.

We consider the following sub cases for r to define the function on $V(G')$.

Subcase 1.1 $r = 3$.

We define $f : V(G') \to \{0, 1\}$ as

$$f(u_i) = 0; 1 \leqslant i \leqslant 3,$$
$$f(u'_i) = 0; 1 \leqslant i \leqslant 3,$$
$$f(v_j) = \begin{cases} 1, & 1 \leqslant i \leqslant 3; \\ 0, & i = 4, \end{cases}$$
$$f(v'_j) = 1; 1 \leqslant j \leqslant 4.$$

Subcase 1.2 $r \geqslant 4$.

We define $f : V(G') \to \{0, 1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = \begin{cases} 1, & 1 \leqslant i \leqslant 4; \\ 0, & 5 \leqslant i \leqslant r, \end{cases}$$
$$f(v_j) = 0; 1 \leqslant j \leqslant 4,$$
$$f(v'_j) = 0; 1 \leqslant j \leqslant 4.$$

According to above definitions of f in different sub cases, we have $v_f(0) = r + 4 = v_f(1)$ and $e_f(0) = \frac{r(r-1)}{2} + 7 = e_f(1) + 1$. So, Thus $|v_f(0) - v_f(1)| \leq 1$. $|e_f(0) - e_f(1)| \leq 1$.

Case 2. $r \geq 5$ and $n \geq 6$.

We consider the following sub cases for r to define the function on $V(G')$.

Subcase 2.1 $n = 6$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u_i') = \begin{cases} 1, & 1 \leqslant i \leqslant 5; \\ 0, & 6 \leqslant i \leqslant r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & j = 2; \\ 0, & 3 \leq j \leq n, \end{cases}$$
$$f(v_j') = 0; 3 \leq j \leq n.$$

Subcase 2.2 $n = 7$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u_i') = \begin{cases} 1, & 1 \leqslant i \leqslant 5; \\ 0, & 6 \leqslant i \leqslant r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & j = 2; \\ 0, & 3 \leq j \leq n, \end{cases}$$
$$f(v_j') = 0; 3 \leq j \leq n.$$

Subcase 2.3 $n \geq 8$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u_i') = \begin{cases} 1, & 1 \leqslant i \leqslant 5; \\ 0, & 6 \leqslant i \leqslant r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & 2 \leq j \leq \lceil \frac{n}{2} \rceil - 1; \\ 0, & j = 1, \ \rceil \frac{n}{2} \rceil \leq j \leq n, \end{cases}$$
$$f(v_j') = \begin{cases} 1, & 3 \leq j \leq \lceil \frac{n}{2} \rceil - 1; \\ 0, & j = 1, 2, \ \lceil \frac{n}{2} \rceil \leq j \leq n. \end{cases}$$

According to above definitions of f in different subcases, we have $v_f(0) = r + n = v_f(1)$. Thus $|v_f(0) - v_f(1)| \leq 1$. For the number of edges labeled with 0 and 1 consider the following cases.

Case 1. n is odd.

In this case we have $e_f(0) = \frac{r(r-1)}{2} + \frac{3n+1}{2} = e_f(1)$. So, $|e_f(0) - e_f(1)| \leq 1$.

Case 2. n is even.

In this case we have $e_f(0) = \frac{r(r-1)}{2} + \frac{3n}{2} + 1 = e_f(1) + 1$. Hence, $|e_f(0) - e_f(1)| \leq 1$. Thus G' has product cordial labeling. □

Example 2.4 A barbell graph $B(5,6)$ with duplication of vertices of path joining complete graphs by vertices is product cordial and its product cordial labeling is shown in Figure 4.

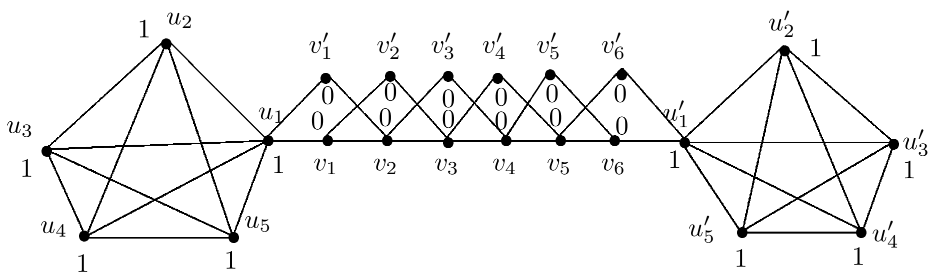

Figure 4 Barbell graph $B(5,6)$ with duplication of vertices by vertices

Theorem 2.5 *A barbell graph $B(r,n)$ with duplication of edges of path joining complete graphs by edges is product cordial for*

(1) $r \geq 4$ *and* $n = 4$;
(2) $r \geq 4$ *and n is odd with* $n \geq 5$.

Proof In a barbell graph $G = B(r,n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from barbell graph by taking duplication of edges of path by edges and also $v'_1v'_2,\ v'_2v'_3, \cdots, v'_{2n-3}v'_{2n-2}$ be edges of duplication of path vertices $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$ respectively. Then $|V(G)| = 2r + 3n - 2$ and $|E(G)| = r(r-1) + 4n - 2$.

Case 1. $r \geq 4$ and $n = 4$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$

$$f(u'_i) = \begin{cases} 1, & 1 \leqslant i \leqslant 4; \\ 0, & 5 \leqslant i \leqslant r, \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1; \\ 0, & 2 \leq j \leq 4, \end{cases}$$

$$f(v'_j) = 0; 1 \leqslant j \leqslant 6.$$

According to above definitions of f in different subcases, we have $v_f(0) = r + 5 = v_f(1)$ and $e_f(0) = \frac{r(r-1)}{2} + 7 = e_f(1)$. So, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 2. $r \geq 4$ and odd $n \geq 5$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$

$$f(u'_i) = \begin{cases} 1, & 1 \leqslant i \leqslant 4; \\ 0, & 5 \leqslant i \leqslant r, \end{cases}$$

$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \frac{n+1}{2}; \\ 0, & j = 1,\ \frac{n+3}{2} \leq j \leq n, \end{cases}$$

$$f(v'_j) = \begin{cases} 1, & 3 \leq j \leq n-5; \\ 0, & n-4 \leq j \leq 2n-2. \end{cases}$$

According to above definitions of f in different subcases, we have $v_f(0) = r + 3\left(\frac{n-1}{2}\right) = v_f(1)$ and $e_f(0) = \frac{r(r-1)}{2} + 2n - 1 = e_f(1)$. So, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Thus G' has product cordial labeling. □

Example 2.5 A barbell graph $B(5,5)$ with duplication of edges of path joining complete graphs by edges is product cordial and its product cordial labeling is shown in Figure 5.

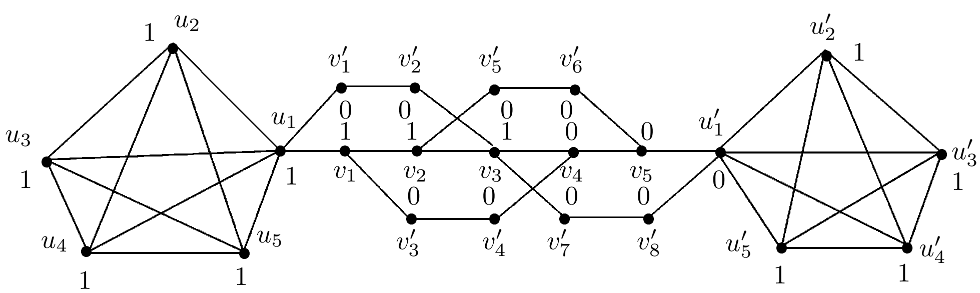

Figure 5 Barbell graph $B(5,5)$ with duplication of edges by edges

Theorem 2.6 *A degree splitting graph of barbell graph $B(r,n)$ is product cordial for $r = 3$ and n is odd.*

Proof For a barbell graph $G = B(r,n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1.

Let G' be degree splitting graph of G and w_1, w_2 be inserting vertices with the properties. $N(w_1) = \{v \in V(G) : d(v) = r\}$, $N(w_2) = \{v \in V(G) : d(v) = 2\}$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u_i') = 0; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \lceil \frac{n}{2} \rceil; \\ 0, & \lceil \frac{n}{2} \rceil + 1 \leq j \leq n, \end{cases}$$
$$f(w_1) = 0,$$
$$f(w_2) = 1.$$

Then we have $e_f(1) = 6 + n = e_f(0) - 1$ and $v_f(1) - 1 = 4 + \frac{n}{2} - \frac{1}{2} = v_f(0)$.

Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus G' is product cordial graph. □

Example 2.6 Consider the degree splitting graph of $B(3,5)$.Then it is product cordial and its product cordial labeling is shown in Figure 6.

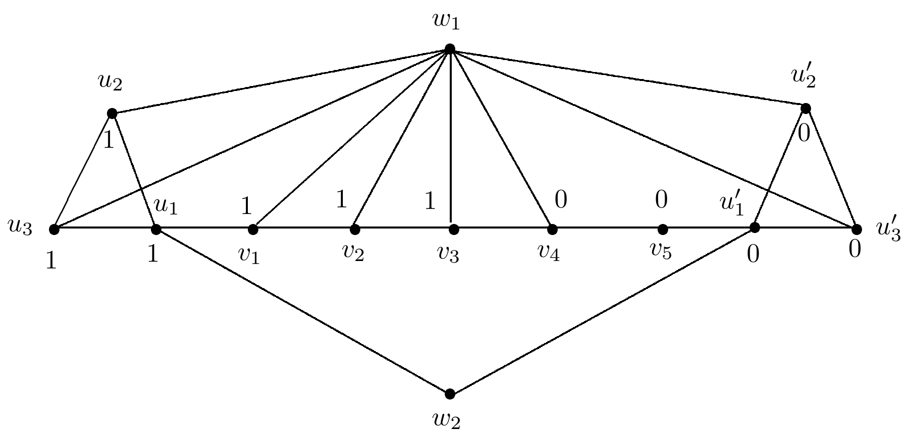

Figure 6 Degree splitting graphs of $B(3,5)$

Theorem 2.7 *A graph obtained by taking degree splitting graph of path joining complete graphs in barbell graph $B(r,n)$ is product cordial for*

(1) $r \geq 3$ *and n is even;*

(2) $r = 3$ *and n is odd with $n \neq 1$;*

(3) $r = 4$ *and n is odd with $n \neq 1,3,5$;*

(4) $r \geq 5$ *and n is odd with $n \neq 1,3,5,7,13$.*

Proof For a barbell graph $G = B(r,n)$, let $u_1, u_2, \cdots, u_r$ and $u_1', u_2', \cdots, u_r'$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from $B(r,n)$ by taking degree splitting graph of path joining complete graphs and v' be the inserting vertex. Then we have $|V(G)| = 2r + n + 1$ and $|E(G)| = r(r-1) + 2n + 1$.

Case 1. $r \geq 3$ and n is even.

We define $f : V(G') \to \{0, 1\}$ as

$$f(u_i) = 0; 1 \leqslant i \leqslant r,$$
$$f(u_i') = 1; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \lceil \frac{n}{2} \rceil; \\ 0, & \lceil \frac{n}{2} \rceil + 1 \leq j \leq n, \end{cases}$$
$$f(v') = 1.$$

Then we have $e_f(1) = \frac{r(r-1)}{2} + n = e_f(0) + 1$ and $v_f(1) - 1 = r + \frac{n}{2} = v_f(0)$. Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus G' is product cordial graph in this case.

Case 2. $r = 3$ and n is odd with $n \neq 1$.

Subcase 2.1 $n = 3$.

We define $f : V(G') \to \{0, 1\}$ as

$$f(u_i) = \begin{cases} 1, & i = 1; \\ 0, & 2 \leq i \leq r, \end{cases}$$
$$f(u_i') = 0; 1 \leqslant i \leqslant r;$$
$$f(v_j) = 1; 1 \leq j \leq n,$$
$$f(v') = 1.$$

Subcase 2.2 $n \geq 5$.

We define $f : V(G') \to \{0, 1\}$ as

$$f(u_i) = f(u_i') = 0; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \frac{n+5}{2}; \\ 0, & \frac{n+7}{2} \leq j \leq n, \end{cases}$$
$$f(v') = 1.$$

According to the above definitions of f in different sub cases, we have $e_f(1) - 1 = n + 3 = e_f(0)$ and $v_f(1) = \frac{n+7}{2} = v_f(0)$. Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 3. $r = 4$ and n is odd with $n \neq 1, 3, 5$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = f(u_i') = 0; 1 \leqslant i \leqslant rm$$
$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \frac{n+7}{2}; \\ 0, & \frac{n+9}{2} \leq j \leq nm, \end{cases}$$
$$f(v') = 1.$$

Then we have $e_f(1) - 1 = n + 6 = e_f(0)$ and $v_f(1) = \frac{n+9}{2} = v_f(0)$. Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 4. $r \geq 5$ and n is odd with $n \neq 1, 3, 5, 7, 13$.

Subcase 4.1 $n = 9$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u_i') = \begin{cases} 1, & 1 \leq i \leq 5; \\ 0, & 6 \leq i \leq r, \end{cases}$$
$$f(v_j) = 0; 1 \leqslant j \leqslant n,$$
$$f(v') = 0.$$

Subcase 4.2 $n = 11$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u_i') = \begin{cases} 1, & 1 \leq i \leq 5; \\ 0, & 6 \leq i \leq r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & j = 1; \\ 0, & 2 \leq j \leq n, \end{cases}$$
$$f(v') = 0.$$

Subcase 4.3 $n = 15$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = \begin{cases} 1, & 1 \leq i \leq 6; \\ 0, & 7 \leq i \leq r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & j = 2, 4; \\ 0, & \text{otherwise}, \end{cases}$$
$$f(v') = 0.$$

Subcase 4.4 $n \geq 17$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = \begin{cases} 1, & 1 \leq i \leq 6; \\ 0, & 7 \leq i \leq r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & j = 1, 3 \leq j \leq \frac{n-11}{2}; \\ 0, & j = 2, \frac{n-9}{2} \leqslant j \leqslant n, \end{cases}$$
$$f(v') = 1.$$

According to the above definitions of f in different sub cases, we have $e_f(1) - 1 = \frac{r(r-1)}{2} + n = e_f(0)$ and $v_f(1) = r + \frac{n+1}{2} = v_f(0)$. Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. □

Example 2.7 Consider the degree splitting graphs of path joining complete graphs in barbell graphs $B(3,5), B(4,4)$ and $B(7,9)$. Then they are product cordials and their labeling are as shown in Figures 7, 8 and 9.

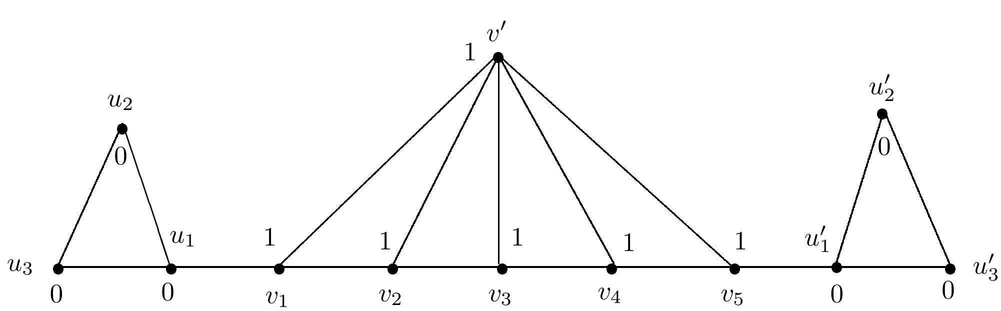

Figure 7 Degree splitting graph of path joining complete graphs in barbell graph $B(3,5)$

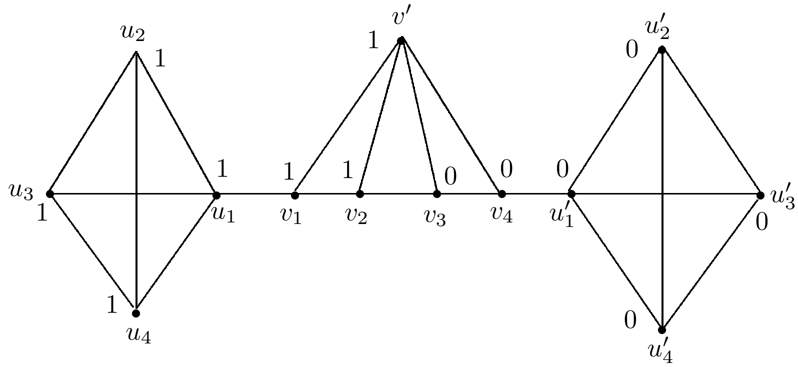

Figure 8 Degree splitting graph of path joining complete graphs in barbell graph $B(4,4)$

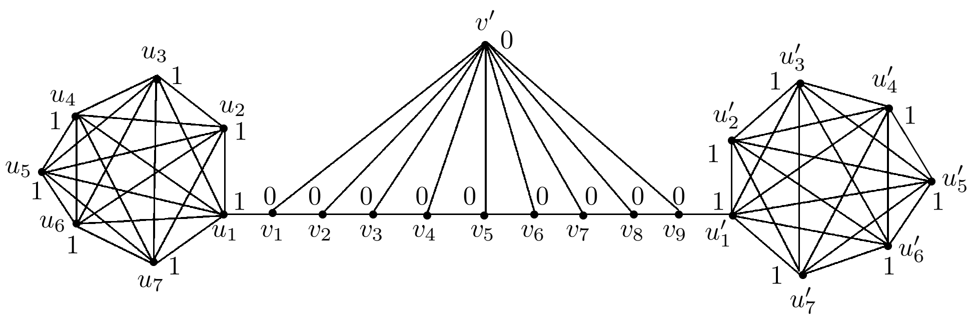

Figure 9 Degree splitting graph of path joining complete graphs in barbell graph $B(7,9)$

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