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On Isomorphism Theorems of Neutrosophic R-Modules

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Abstract: This work deals with the isomorphism theorems of Neutrosophic R- modules. In this work, we assumed all rings to be commutative rings, we studied neutrosophic module [2], neutrosophic submodule, pseudo neutrosophic module and pseudo neutrosophic submodule. We considered the concept of Lagrange theorem [11] and discovered that in case of finite neutrosophic modules, the order of both neutrosophic submodules and pseudo neutrosophic submodules do not generally divide the order of neutrosophic module. The concept of cosets in general does not partition the neutrosophic module, even the pseudo neutrosophic submodules do not in general partition the neutrosophic module. This work also shows that the neutrosophic module is also a module and we considered the isomorphism theorem for modules [8] and extended it to Neutrosophic R modules and discovered that the isomorphism theorem for R modules also hold for neutrosophic R modules but where the order of a neutrosophic submodule divides the order of a neutrosophic module, the theorem may fail. We also stated and proved the isomorphism theorems of neutrosophic R-modules.

Key Words: Neutrosophy, module, neutrosophic R-module, neutrosophic group, ring, neutrosophic R-submodule, partition, coset, isomorphism.

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§1. Introduction

In 1980 [1], Florentin Smarandache introduced the notion of neutrosophy as a new branch of philosophy. Neutrosophy is the base of neutrosophic logic which is an extension of the fuzzy logic in which indeterminacy is included [2]. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. Since the world is full of indeterminacy, several real world problems involving indeterminacy arising from law, medicine, sociology, psychology, politics, engineering, industry, economics, management and decision making, finance, stocks and share, meteorology, artificial intelligence, IT, communication etc can be solved by neutrosophic logic. Using Neutrosophic theory, Vasantha Kandasamy and Florentin Smaran-

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dache introduced the concept of neutrosophic algebraic structures [13]. Some of the neutrosophic algebraic structures introduced and studied include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and so on. Neutrosophic module was defined by Florentin and Vasantha in [11].

In section two of this work, we present some elementary properties of neutrosophic R-modules and section three is devoted to the study of the isomorphism theorems of neutrosophic R-modules.

§2. Some Elementary Properties of Neutrosophic R-module

We begin this section with the following definitions.

Definition 2.1([11]) *Let R be a commutative ring. An R-module is an (additive) abelian group M equipped with scalar multiplication $R \times M \to M$ such that the following axioms hold for all $m, n \in M$ and all $r, s, 1 \in R$:*

(1) $r(m+n) = rm + rn$;
(2) $(r+s)m = rm + sm$;
(3) $(rs)m = r(sm)$;
(4) $1 \cdot m = m$.

Remark 2.2 This definition also makes sense for non commutative rings R in which in this case, M is called a left R-module. If R is a commutative ring, then a neutrosophic left R-module $\langle M \cup I \rangle$ becomes a neutrosophic right R-module and we simply call $\langle M \cup I \rangle$ a neutrosophic R-module.

Remark 2.3 In the definition of neutrosophic R-module, we replaced the abelian group by a neutrosophic abelian group, all other factors remain the same.

Definition 2.4 *Let $\langle M \cup I \rangle$ be a neutrosophic module. Let H and K be any two neutrosophic submodules of $\langle M \cup I \rangle$, we say H and K are neutrosophic conjugates if we can find $x, y \in \langle M \cup I \rangle$ such that $xH = Ky$.*

We illustrate this with the following example.

Example 2.5 Let $R = \{0, 1, 2\}$ be the ring of integers and let $Z_6 \cup I = \{0, 1, 2, 3, 4, 5, I, 2I, 3I, 4I, 5I, 1+I, 1+2I, 1+3I, 1+4I, \cdots, 5+5I\}$ be a neutrosphic group under addition modulo 6. Then $R \times \langle Z_6 \cup I \rangle \to \langle Z_6 \cup I \rangle = \{0, 1, 2, 3, 4, 5, I, 2I, 3I, 4I, 1+I, \cdots, 5+5I\} = \langle Z_6 \cup I \rangle$. This is a neutrosophic module.

$H = \{0, 3, 3I, 3+3I\}$ is a neutosophic submodule of $\langle Z_6 \cup I \rangle K = \{0, 2, 4, 2+2I, 4+4I, 2I, 4I\}$ is a neutrosophic sub module of $\langle Z_6 \cup I \rangle$. For $2, 3$ in $\langle M \cup I \rangle$, we have $2H = 3K = \{0\}$, so H and K are neutrosophic conjugates. In case of neutrosophic conjugate, we do not demand $O(H) = O(K)$.

Definition 2.6 *Let $\langle M \cup I \rangle$ be a neutrosophic module and H a neutrosophic sub module of $\langle M \cup I \rangle$ for $n \in \langle M \cup I \rangle$, then $H + n = \{h + n/h \in H\}$ is called a coset of H in $\langle M \cup I \rangle$. As neutrosophic modules are formed from neutrosophic abelian groups, we do not talk about left and right cosets as the left and right cosets coincide.*

Example 2.7 Let $\langle M \cup I \rangle = \langle Z_2 \cup I \rangle = \{0, 1, I, 1 + I\}$ be a neutrosophic module and let $H = \{0, I\}$ be a neutrosophic sub module. The cosets of H are $H+0 = \{0, I\}$, $H+1 = \{1, 1+I\}$, $H + I = \{I, 0\}$ and $H + \{1 + I\} = \{1 + I, 1\}$.

Definition 2.8 *The cosets of a neutrosophic module do not generally partition the neutrosophic module.*

Example 2.9 Let $\langle M \cup I \rangle = \{0, 1, I, 1 + I\}$ be a neutrosophic module and let $H = \{0, I\}$ be a neutrosophic sub module. Then the cosets are $H+0 = \{0, I\}$, $H+1 = \{1, 1+I\}$, $H+I = \{I, 0\}$ and $H + \{1 + I\} = \{1 + I, 1\}$.

Therefore the classes are $[0] = [I] = \{0, I\}$ and $[I] = [1 + I] = \{1, 1 + I\}$. Here, we see the cosets do not partition the neutrosophic module.

Example 2.10 Let $\langle Z_3 \cup I \rangle = \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$ be a neutrosophic module and let $P = \{0, 2, I, 2I\}$ be a neutrosophic submodule, then the cosets of P are $P+0 = \{0, 2, I, 2I\}$, $P + 1 = \{1, 0, I, 2I\}$, $P + 2 = \{2, 1, I + 2, 2 + 2I\}$, $P + I = \{I, I + 2, 2I, 0\}$, $P+2I = \{2I, 2+2I, 0, I\}$, $P+\{1+I\} = \{1+I, I, 1+2I, 1\}$, $P+\{1+2I\} = \{1+2I, 2I, 1, 1+I\}$,$P + \{2 + I\} = \{2 + I, 1 + I, 2 + 2I, 2\}$ and $P + \{2 + 2I\} = \{2 + 2I, 1 + 2I, 2, 2 + I\}$. The cosets partition the neutrosophic module. Therefore, we see that the cosets do not generally partition the neutrosophic module.

Theorem 2.1 *The neutrosophic module is indeed a module.*

Proof Suppose that the neutrosophic module $\langle M \cup I, + \rangle$ is an (additive) Abelian neutrosophic group. Every (additive) Abelian neutrosophic group is a group. We know that a module is an Abelian group over a ring. Therefore a neutrosophic module is a module. We illustrate with an example.

Consider $R = \langle Z_3 \rangle = \{0, 1, 2\}$ is a ring and let $N(M) = \langle M \cup I \rangle = \langle Z_3 \cup I \rangle$, then $\langle Z_3 \cup I \rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$. Let $R \times N(M) = \{0, 1, 2\} \times \{0, 1, 2, I, 2I, 1+I, 1 + 2I, 2 + I, 2 + 2I\} = \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$.

Clearly, this is an additive Abelian neutrosophic group which is also a group. Also, an Abelian group over a ring gives a module, which is also a group. Therefore a neutrosophic module is a module. □

Definition 2.11 *Let $\langle M \cup I \rangle$ be a neutrosophic Abelian group and R a commutative ring. Let $R \times \langle M \cup I \rangle \to \langle M \cup I \rangle$ be a neutrosophic R-module. A proper subset P of $\langle M \cup I \rangle$ is said to be a neutrosophic submodule of the R-module if P is a non-empty set which is closed under addition and scalar multiplication.*

Definition 2.12([11]) *A pseudo neutrosophic group is a neutrosophic group which has no proper* *subset which is a group.*

Definition 2.13([11]) *Let $N(M) = \langle M \cup I \rangle$ be a neutrosophic module, a proper subset P of $N(M)$ which is a pseudo neutrosophic subgroup is called a pseudo neutrosophic submodule.*

Example 2.14 Let $R = \{0,1\}$ be a ring and let $N(M) = \langle Z_4 \cup I \rangle = \{0,1,2,3,I,2I,3I,1+I,1+2I,1+3I,2+I,2+2I,2+3I,3+I,3+2I,3+3I\}$, be a neutrosophic group. The neutrosophic R-module $R \times \langle Z_4 \cup I \rangle = \{0,1\} \times \{Z_4 \cup I\} = \{0,1,2,3,I,2I,3I,1+I,1+2I,1+3I,2+I,2+2I,2+3I,3+I,3+2I,3+3I\}$. Let $P = \{0, 3+3I\}$ be a pseudo neutrosophic subgroup of $\langle M \cup I \rangle$. Thus P is a pseudo neutrosophic submodule.

Theorem 2.2([8]) *The lagrange theorem for classical module states that the order of any submodule of a finite module is a factor of the order of the module.*

Definition 2.15 *The order of a neutrosophic submodule does not in general divide the order of the neutrosophic module.*

Example 2.16 Let us consider an example of Lagrange theorem on Neutrosophic module Let $\langle Z_3 \cup I \rangle = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$ be a neutrosophic module and let $P = \{0,2,I,2I\}$ be a neutrosophic submodule, let us bear in mind that the order of the neutrosophic submodule need not divide the order of the neutrosophic module, then the cosets of P are $P+0 = \{0,2,I,2I\}$, $P+1 = \{1,0,I,2I\}$, $P+2 = \{2,1,I+2,2+2I\}$, $P+I = \{I,I+2,2I,0\}$, $P+2I = \{2I,2+2I,0,I\}$, $P+\{1+I\} = \{1+I,I,1+2I,1\}$, $P+\{1+2I\} = \{1+2I,2I,1,1+I\}$, $P+\{2+I\} = \{2+I,1+I,2+2I,2\}$, $P+\{2+2I\} = \{2+2I,1+2I,2,2+I\}$.

The order of the neutrosophic module is nine and the order of the neutrosophic submodule is four, the number of elements in each coset is four as well. There are nine cosets. Therefore, we have $9 \neq 4.9$, four is not a factor of nine.

In general, the neutrosophic modules do not satisfy Lagrange theorem on finite modules.

§3. Isomorphism Theorems of Neutrosophic R-modules

Theorem 3.1 *Let $f : M \cup I \to N \cup I$ be a neutrosophic R module homomorphism. Then,*

(1) $\ker f$ *is a neutrosophic submodule of* $\langle N \cup I \rangle$*;*

(2) Imf *is a neutrosophic submodule of* $\langle N \cup I \rangle$*.*

Proof Let $\langle M \cup I \rangle \in \ker f$ and $r \in R$. Then $f\langle rm \rangle = rf\langle m \rangle = r\langle 0 \rangle = 0$. So $\langle rm \rangle \in \ker f$. Thus, $\ker f$ is a neutrosophic R submodule of $\langle M \cup I \rangle$.

In addition, suppose $m \in \langle M \cup I \rangle$ and $r \in R$, we have $rf\langle m \rangle = f\langle rm \rangle \in Imf$. So, Imf is a neutrosophic R submodule of $\langle N \cup I \rangle$. □

Example 3.1 Let $f : Z_4 \cup I \to Z_3 \cup I$ defined by $f : \{a\}_4 \to \{2a\}_3$ where $\{a\}_4$ means $a$mod 4 and $\{2a\}_3$ means $2a$mod 3. The kernel are $\{0,3,3I,3+3I\}$ mapped to $Z_3 \cup I$ under the operation $a\text{mod } 4 \xrightarrow{f} 2a\text{mod } 3$. The image of $\langle Z_4 \cup I \rangle$ are $\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$ which is the neutrosophic submodule of $\langle Z_3 \cup I \rangle$.

Corollary 3.2 *If M_1 and M_2 are R submodules of the neutrosophic R module $\langle M \cup I\rangle$ in Theorem* 3.1*, then*

$$M_1 + M_2/M_1 \cong M_2/M_1 \cap M_2.$$

Proof This is a corollary to Theorem 3.1. Notice that $\langle M \cup I\rangle = \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I, 2 + 3I, 3 + I, 3 + 2I, 3 + 3I\}$, $M_2 = \{0, 1, 2\}$, $M_1 = \{0, 1\}$, $M_2/M_1 = \{0, 1, 2\} + \{0, 1\} = \{\{0, 1\}, \{1, 2\}, \{2, 0\}\} = \{0, 1, 2\}$, $M_1 + M_2/M_1 = \{0, 1\} + \{0, 1, 2\} = \{\{0, 1, 2\}, \{1, 2, 0\}\} = \{0, 1, 2\}$, $M_1 \cap M_2 = \{0, 1\}$, $M_2/M_1 \cap M_2 = \{0, 1, 2\}/\{0, 1\} = \{0, 1, 2\} + \{0, 1\} = \{\{0, 1\}, \{1, 2\}, \{2, 0\}\} = \{0, 1, 2\}$ and $M_1 + M_2/M_1 \cong M_2/M_1 \cap M_2$. It is noteworthy to mention that Theorem 3.1 holds even when the submodules are not neutrosophic submodules but just submodules. □

Theorem 3.3 *If $\langle M_1\cup I\rangle \subseteq \langle M_2\cup I\rangle \subseteq \langle M\cup I\rangle$ are neutrosophic R-modules, then $M_2 \cup I\ /M_1 \cup I$ is a neutrosophic submodule of $\langle M \cup I\rangle\ /\langle M_1 \cup I\rangle$ and*

$$\langle M \cup I\rangle/\langle M_1 \cup I\rangle \Big/ \langle M_2 \cup I\rangle\ /\langle M_1 \cup I\rangle \cong \langle M \cup I\rangle\ /\langle M_2 \cup I\rangle.$$

Proof Define $\theta : M \cup I\ /M_1 \cup I \to M \cup I\ /M_2 \cup I$ by $\theta\langle x + M_1 \cup I\rangle = x + M_2 \cup I$. We have to check whether it is well defined. If we have two different representatives for $x + M_2 \cup I$, it means $x + M_1 \cup I = y + M_1 \cup I$ which is the same as saying $x - y \in M_1 \cup I$ but $\langle M_1 \cup I\rangle \subset \langle M_2 \cup I\rangle$, therefore, $x - y \in \langle M_2 \cup I\rangle$, hence $x + M_2 \cup I$ is the same as $y + M_2 \cup I$. θ is well defined and θ is a neutrosophic R module homomorphism. Now, what is the kernel of θ? Cleraly,

$$\ker\theta = \{\bar{x} \in M \cup I\ /M_1 \cup I : x + M_2 \cup I = 0 + M_2 \cup I\},$$

i.e.,

$$\ker\theta = \{x + M_1 \cup I \in M \cup I\ /M_1 \cup I : x \in M_2 \cup I\} = M_2 \cup I\ /M_1 \cup I.$$

If you take any $x + M_2 \cup I$ in $M \cup I/M_2 \cup I$, look at $x + M_1 \cup I$ and $\theta(x + M_1 \cup I) = x + M_2 \cup I$. Therefore, it is surjective. □

Example 3.2 Let $\langle M \cup I\rangle = \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$, $\langle M_2 \cup I\rangle = \{0, 1, I, 1 + I\}$, $\langle M_1 \cup I\rangle = \{0, I\}$, $M \cup I/M_1 \cup I = \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\} + \{0, I\} = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$, $M_2 \cup I/M_1 \cup I = \{0, 1, I, 1+I\}+\{0, I\} = \{0, 1, I, 2I, 1+I, 1 + 2I\}$, $M_2 \cup I/M_1 \cup I$ is a neutrosophic submodule of $M \cup I/M_1 \cup I$.

$$\begin{aligned}&\langle M \cup I\rangle/\langle M_1 \cup I\rangle \Big/ \langle M_2 \cup I\rangle/\langle M_1 \cup I\rangle\\ &= \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}/\{0, 1, I, 2I, 1 + I, 1 + 2I\}\\ &= \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\},\end{aligned}$$

$$
\begin{aligned}
M \cup I/M_2 \cup I &= \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}\big/\{0,1,I,1+I\} \\
&= \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\},
\end{aligned}
$$

Whence,

$$\langle M \cup I\rangle/\langle M_1 \cup I\rangle \Big/ \langle M_2 \cup I\rangle/\langle M_1 \cup I\rangle \cong \langle M \cup I\rangle/\langle M_2 \cup I\rangle.$$

Corollary 3.4 *Let $M \cup I$ be a neutrosophic module. Let M_1 and M_2 be submodules of $\langle M \cup I\rangle$ and let $M_1 \subseteq M_2 \subseteq \langle M \cup I\rangle$, then $\langle M \cup I\rangle/M_1 \Big/ M_2/M_1 \cong \langle M \cup I\rangle/M_2$.*

This is a corollary of Theorem 3.3.

Example 3.3 We consider the following example Let $\langle M \cup I\rangle = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$, $M_2 = \{0,1,2\}$, $M_1 = \{0,1\}$. Then $M_2 \Big/ M_1 = \{0,1,2\} \Big/ \{0,1\} = \{0,1,2\} + \{0,1\} = \{\{0,1\},\{1,2\},\{2,1\}\} = \{0,1,2\}$, $\langle M \cup I\rangle/M_1 = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\} + \{0,1\}, = \{\{0,1\},\{1,2\},\{2,0\},\{I,1+I\},\{2I,2I+1\},\{1+I,2+I\},\{1+2I,2+2I\},\{2+I,I\},\{2+2I,2I\}\} = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$ and $\langle M \cup I\rangle/M_1 \Big/ M_2/M_1 = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\} \Big/ \{0,1,2\} = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\} + \{0,1,2\} = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$, $\langle M \cup I\rangle/M_2 = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}/\{0,1,2\} = \{0,1,2,I,2I,1+I,1+I,1+2I,2+I,2+2I\} + \{0,1,2\} = \{\{0,1,2\},\{1,2,0\},\{2,0,1\},\{I,1+I,2+I\},\{2I,2I+1,2I+2\},\{1+I,2+I,I\},\{1+2I,2+2I,2I\},\{2+I,I,1+I\},\{2+2I,2I,1+2I\}\} = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$. Whence,

$$\langle M \cup I\rangle/M_1 \Big/ M_2/M_1 \cong \langle M \cup I\rangle/M_2.$$

Theorem 3.5 *Let $f : \langle M \cup I\rangle \to N \cup I$ be a neutrosophic R module homomorphism, then $Imf \cong M \cup I \big/ \ker f$*

Proof Define $\theta : M \cup I \big/ \ker f \to Imf$, $\theta\bar{x} = f(x)$. We want to prove that it is well-defined since there could be many representatives of $\bar{x}$. If $\bar{x} = \bar{y} \to x - y \in \ker f \to f(x-y) = 0$. Since f is a neutrosophic module homomorphism $f(x) = f(y) \to \theta(\bar{x}) = \theta(\bar{y}) \to \theta$ is well defined. θ is a homomorphism since f is a homomorphism for all $\bar{x}, \bar{y} \in M \cup I \big/ \ker f$. $\theta(\bar{x} + \bar{y}) = \theta(\overline{x+y}) = f(x+y) = f(x) + f(y) = \theta(\bar{x}) = \theta(\bar{y})$ for all $r \in R$ and $\bar{x} \in M \cup I \big/ \ker f$. By definition of scalar multiplication on $M \cup I \big/ \ker f$, $\theta(r.\bar{x}) = \theta(\overline{rx}) = f(rx) = rf(x) = r\theta(\bar{x})$, θ is a neutrosophic R module homomorphism. Now, let $y \in Imf \to x \in M \cup I$ such that $f(x) = y \to \theta(\bar{x}) = y$ this implies θ is surjective. If $\theta(\bar{x}) = 0$, then $f(x) = 0 \to x \in \ker f \to \bar{x} = 0$. This implies θ is injective and it implies θ is an isomorphism. □

Example 3.4 Let $f : Z_3 \cup I \to Z_3 \cup I$ be defined by f: $[a]_3 \to [4a]_3$ where $[a]_3$ means $a$mod 3 and $[4a]_3$ means $4a$mod 3. The image of $f = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$, $\ker f = \{0\}$, $M \cup I \big/ \ker f = \{0,1,2,I,2I,1+I,1+2\}/\{0\} = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$. $Imf = \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$. Hence, $Imf \cong M \cup I \big/ \ker f$.

Theorem 3.6 *If $\langle M_1 \cup I\rangle$ and $\langle M_2 \cup I\rangle$ are neutrosophic R submodules of $\langle M \cup I\rangle$, then* $\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle \cong \langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle$.

Proof Define $\theta : \langle M_2 \cup I\rangle \to \langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle$ by $\theta(x) = \bar{x}$. Note that we do not have to worry about well definiteness. There is no representative issue, every element has its own existence $\theta(x+y) = \overline{x+y} = \bar{x} + \bar{y} = \theta(x) + \theta(y)$. $\ker\theta = \{x \in M_2 \cup I : \bar{x} = 0\} = \{x \in M_2 \cup I : x \in M_1 \cup I\} = \langle M_2 \cup I\rangle \cap \langle M_1 \cup I\rangle$. It is injective. $\overline{x+y}$ or $\bar{x} + M_1 \cup I$ is a coset, $y \in M_1 \cup I$ and $x \in M_2 \cup I$, $\overline{x+y} = (x+y) + M_1 = (x + M_1 \cup I) + (y + M_1 \cup I)$, $y + M_1 \cup I = 0$ (in neutrosophic quotient module) $= x + M_1 \cup I = \bar{x} \to \theta(x) = \overline{x+y} \to \theta$ is surjective. □

The next example is an illustration of Theorem 3.6.

Example 3.5 Let $M \cup I = \{0, 1, 2, I, 2I, 3I, 1+I, 1+2I, 1+3I, 2+I, 2+2I, 2+3I, 3+I, 3+2I, 3+3I\}$, $M_2 \cup I = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$, $M_1 \cup I = \{0, 2, 2I, 2+2I\}$. We show that $\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle \cong \langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle$. Notice that $\langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\} \Big/ \{0, 2, 2I, 2+2I\}$, $\langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$ and $\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle = \{0, 2, 2I, 2+2I\} + \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$. Therefore, $\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$, $\langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle = \{0, 2, 2I, 2+2I\}$, $\langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\} \Big/ \{0, 2, 2I, 2+2I\} = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\} + \{0, 2, 2I, 2+2I\}$, $\langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$. Therefore, we know that

$$\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle \cong \langle M_2 \cup I\rangle \Big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle.$$

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