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On $1RJ$ Moves in Cartesian Product Graphs

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Abstract: Let G be an undirected graph with n vertices in which a robot is placed at a vertex say v, and a hole at vertex u and in all other $(n-2)$ vertices are obstacles. We refer to this assignment of robot and obstacles as a configuration C_u^v of G. Suppose we have a one player game in which an obstacle can be slide to an adjacent vertex if it is empty i.e. if it has a hole and the robot can move from vertex u to an empty vertex v if $d(u,v) \leq 2$ where $d(u,v)$ is the distance between vertex u and v. The goal is to take the robot to a particular destination vertex by using a sequence of mRJ moves of the robot for $m = 1$ and simple moves of the robot as well as obstacles as the case may be. The results of this paper, which is an extension of the work [Motion planning in Cartesian product graphs, *Discussiones Mathematicae Graph Theory* 34 (2014) 207-221] gives the minimum number of moves required for the motion planning problem in Cartesian product of two graphs each having girth six or more.

Key Words: Robot motion in a graph, Cartesian product of graphs, 1RJ move.

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§1. Introduction

Given a graph G, with a robot placed at one of it's vertices and movable obstacles at some other vertices. Assuming that we are allowed to slide the obstacles to an adjacent vertex if it is empty and the robot can move from vertex u to an empty vertex v if $d(u,v) \leq 2$. Let $u, v \in V(G)$, and suppose that the robot is at v and the hole at u and obstacles at other vertices we refer to this as a configuration C_u^v. The number of edges in a path is called its length. The girth of a graph G, denoted by $g(G)$, is the length of a shortest cycle contained in the graph. A simple move is referred to as moving an obstacle or the robot to an adjacent empty vertex. A graph G is k-reachable if there exists a k-configuration such that the robot can reach any vertex of the graph in a finite number of simple moves. Let u and v be two vertices having a robot and a hole, respectively. Further let $[u, d_1, d_2, d_3, \cdots, d_m, v]$ be a path having obstacles at the vertices $d_1, d_2, d_3, \cdots, d_m$. An mRJ move from the vertex u to the

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empty vertex v is defined as movement of the robot to an empty vertex v by jumping over m obstacles $d_1, d_2, d_3, \cdots, d_m$. Although throughout this paper we would only consider the case where $m = 1$ (i.e. $1RJ$moves only) and simple moves of the robot as well as obstacles as the case may be. Let $[u, d_1, d_2, d_3, \cdots, d_m, v]$ be a path in a graph such that u and v have a hole and a robot respectively, and $d_1, d_2, d_3, ...d_m$ have obstacles. An mRJ move from vertex u to v is denoted by $v \overset{r}{\underset{1}{\leftarrow}} u$. Similarly we use $v \overset{r}{\leftarrow} u$ and $v \overset{o}{\leftarrow} u$ to denote respectively, the robot move and the obstacle move from vertex u to an adjacent vertex v where $u, v \in E(G)$. The objective is to find a minimum sequence of moves that takes the robot from (source) vertex u to a (destination) vertex v. The vertex set and edge set of a graph G is denoted by $V(G)$ and $E(G)$ respectively. We refer to $|V(G)|$ and $|E(G)|$ as the order and the size of G, respectively. A graph G is said to be non-trivial if $|V(G)| > 1$. In this article, we restrict our study to simple finite non-trivial graphs. For two vertices $u, v \in V(G)$, let $d_G(u, v)$ denotes the distance between u and v in G. We use $d(u, v)$ instead of $d_G(u, v)$ to represent the distance between the vertices u and v in the graph G. We denote the path, the cycle and the complete graph on n vertices by P_n, C_n and K_n respectively.

The motion planning problem in graph was proposed by Papadimitriou et al. [9] where it was shown that with arbitrary number of holes, the decision version of such problem is NP-complete and that the problem is complex even when it is restricted to planar graphs. They also gave time algorithm for trees. The result in [9] was improve in [3]. Robot motion planning on graphs (RMPG) is a graph with a robot placed at one of its vertices and movable obstacle at some of the other vertices while generalization of RMPG problem is the Multiple robot motion planning in graph (MRMPG) whereby we have k different robots with respective destinations. Ellips and Azadeh [6] studied MRMPG on trees and introduced the concept of minimal solvable trees. Auletta et al. [2] also studied the feasibility of MRMPG problem on trees and gave an algorithm that, on input of two arrangements of k robots on a tree of order n, decides in time $O(n)$ whether the two arrangements are reachable from one another. Parberry [8] worked on grid of order n^2 with multiple robots while Deb and Kapoor [5, 4] generalized and apply the technique used in [8] to calculate the minimum number of moves for the motion planning problem for the cartesian product of two given graphs. A recent work is by the present authors [1] whereby they gave the minimum number of moves required for the motion planning problem in some lexicographic product graphs.

The MRMPG problem of grid graph of order n^2 with $n^2 - 1$ robots is known as $(n^2 - 1)$-puzzle. The objective of $(n^2 - 1)$-puzzle is to verify whether two given configurations of the grid graph of order n^2 are reachable from each other and if they are reachable then to provide a sequence of minimum number of moves that takes one configuration to the other. The $(n^2 - 1)$-puzzle have been studied extensively in [7, 8, 10, 11].

Our work was motivated by Deb and Kapoor [4] whereby they gave minimum sequence of moves required for the motion planning problem in Cartesian product of two graphs having girth 6 or more. They also proved that the path traced by the robot coincides with a shortest path in case of Cartesian product of graphs. In this paper we extend the work in [4] by considering the case in which the robot can jump one obstacle at a move (or time) and thus we give the minimum number of moves required for the motion planning problem in Cartesian product of two graphs say G and H.

Definition 1.1 *The Cartesian product $G\Box H$ of two graphs G and H is a graph with vertex set $V(G)\times V(H)$ in which (u_i,v_j) and (u_p,v_q) are adjacent if one of the following condition holds:*

(1) $u_i = u_p$ *and* $\{v_j, v_q\} \in E(H)$;
(2) $v_j = v_q$ *and* $\{u_i, u_p\} \in E(G)$.

The graphs G and H are known as the factors of $G\Box H$. Now onwards G and H are simple graphs with $V(G) = \{1, 2, 3, ...m\}$ unless otherwise stated.

Suppose we are dealing with r-copies of a graph G and we are denoting these r-copies of G by G^i, where $i = \{1, 2, 3, \cdots, r\}$. Then for each vertex $u \in V(G)$ we denote the corresponding vertex in the i^{th} copy G^i by u^i. The girth of a graph G, denoted by $g(G)$ is the length of the shortest cycle contained in graph G. Now we refer to the work of Deb and Kapoor [4] for a good pre-knowledge of this work.

§2. Local Moves of the Hole

Definition 2.1 *An edge u^i, v^j in $G\Box H$ is said to be a G-edge (respectively, H-edge) if $u = v$ and $\{i, j\} \in E(G)$ (respectively, if $i = j$ and $\{u, v\} \in E(H)$).*

Definition 2.2 *For any path P in $G\Box H$, by G-length and H-length of P we mean the number of G-edges and H-edges in P, respectively. We use $l_G(P)$ and $l_H(P)$ to denote the G-length and H-length of P, respectively.*

Definition 2.3 *Given two graphs G and H. For any $u^i, v^j \in V(G\Box H)$, we call the distance between u and v in H to be the H-distance between u^i and v^j in $G\Box H$, and the distance between i and j in G to be the G-distance between u^i and v^j in $G\Box H$. We use $d_G(u^i, v^j)$ and $d_H(u^i, v^j)$ to denote the G-distance and H-distance between u^i and v^j in $G\Box H$, respectively.*

Now, we use $d(u, v)$ instead of $d_G(u, v)$ to represent the distance between u and v in G.

Proposition 2.1 *Given two graphs G and H. Let $\{i, j\}, \{j, k\}, \{k, l\}, \{l, m\} \in E(G)$ and $u \in V(H)$. Then (i) $d_{G\Box H-u^k}(u^i, u^l) = min\{d_{G-k}(i, l), 5\}$ and (ii) $d_{G\Box H-u^k}(u^i, u^m) = min\{d_{G-k}(i, m), 6\}$.*

Proof (i) Let Q be a shortest path connecting u^i and u^l in $G\Box H - u^k$. We need to show that $|Q| = min\{d_{G-k}(i, l), 5\}$. We consider the following cases.

Case 1. $V(Q)\cap V(G^i) = V(Q)$ which implies that $V(Q) \subseteq V(G^i - u^k)$ and so $|Q| = d_{G-k}(i, l)$.

Case 2. $V(Q) \cap V(G^i) \neq V(Q)$. We claim that $|Q| = 5$. From the Cartesian product of graphs, notice that for any $u, v \in E(H)$, the vertices u^x, v^y are adjacent in $G\Box H$ if and only if $x = y$. Therefore if we are moving away from the copy G^i using the path Q we must also come back to the copy G^i. Hence G-distance covered along the path Q must be at least two. Also $d(i, l) = 3$, otherwise i, k or $j, l \in E(G)$ and this implies $|Q| = 2$, which is not possible.

So G-distance traveled along the path Q must be at least three. Hence $|Q| \geq 5$. Now for any $u, v \in E(H)$ the path $[u^i, u^j, v^j, v^k, v^l, u^l]$ connects u^i and u^l in $G\square H$.

(ii) Since we have established that $d_{G\square H-u^k}(u^i, u^l) = min\{d_{G-k}(i,l), 5\}$ and $k, l \in E(G)$ we then conclude that $d_{G\square H-u^k}(u^i, u^m) = min\{d_{G-k}(i,m), 6\}$. This proves our claim. $\square$

Corollary 2.2 *Given two graphs G and H. Let $\{i,j\}, \{j,k\}, \{k,l\} \in E(G)$ and $u \in V(H)$. Then starting from the configuration $C_{u^k}^{u^i}$ of $G\square H$ we require at least $min\{1 + d_{G-k}(i,l), 6\}$ moves to move the robot to u^l. In particular, if $g(G) \geq 6$, then we need at least 6 moves to move the robot to u^l.*

Proof Notice that, $\{u^i, u^j\}, \{u^j, u^k\}, \{u^k, u^l\} \in E(G\square H)$. In order to move the robot from u^k to u^l, before it, the hole must be moved from u^i to u^l. This would take $min\{d_{G-k}(i,l), 5\}$ moves. Since $d_{G\square H-u^k}(u^i, u^l) = min\{d_{G-k}(i,l), 5\}$. Then the simple move $u^l \xleftarrow{r} u^k$ takes the robot from u^k to u^l. Hence the result follows.

If $g(H) \geq 6$ then $d_{G-k}(i,l) \geq 5$ and so $min\{1 + d_{G-k}(i,l), 6\} = 6$. Thus, at least six moves are required to take the robot from u^k to u^l. $\square$

Corollary 2.3 *Given two graphs G and H. Let $\{i,j\}, \{j,k\}, \{k,l\}, \{l,m\} \in E(G)$ and $u \in V(H)$. Then starting from the configuration $C_{u^k}^{u^i}$ of $G\square H$ we require at least $min\{1+d_{G-k}(i,m), 7\}$ moves to move the robot to u^m. In particular, if $g(G) \geq 6$, then we need at least 7 moves to move the robot to u^m.*

Proof Just as in Corollary 2.2, in order to move the robot from u^k to u^m, before it, the hole must be moved from u^i to u^m. This would take $min\{d_{G-k}(i,m), 6\}$ moves. Since $d_{G\square H-u^k}(u^i, u^m) = min\{d_{G-k}(i,m), 6\}$. Then the $1RJ$ move $u^m \underset{1}{\xleftarrow{r}} u^k$ takes the robot from u^k to u^m. Hence the result follows.

If $g(H) \geq 6$ then $d_{G-k}(i,m) \geq 6$ and so $min\{1 + d_{G-k}(i,m), 6\} = 6$. Therefore at least seven moves are required to take the robot from u^k to u^m. $\square$

As Cartesian product of graphs is commutative, so the proof of the following proposition can be drawn in the same line as that of Proposition 2.1.

Proposition 2.4 *Given two non-trivial graphs G and H. Let $\{u,v\}, \{v,w\}, \{w,x\}, \{x,y\} \in E(H)$ and $i \in V(G)$. Then (i) $d_{G\square H-v^i}(u^i, x^i) = min\{d_{H-v}(u,x), 5\}$ and (ii) $d_{G\square H-v^i}(u^i, y^i) = min\{d_{H-v}(u,y), 6\}$.*

Corollary 2.5 *Given two graphs G and H. Let $\{u,v\}, \{v,w\}, \{w,x\} \in E(H)$ and $i \in V(G)$. Then starting from the configuration $C_{v^i}^{u^i}$ of $G\square H$ we require at least $min\{1 + d_{H-v}(u,x), 6\}$ moves to move the robot to x^i. In particular, if $g(G) \geq 6$, then we need at least 6 moves to move the robot to x^i.*

Corollary 2.6 *Given two graphs G and H. Let $\{u,v\}, \{v,w\}, \{w,x\}, \{x,y\} \in E(H)$ and $i \in V(G)$. Then starting from the configuration $C_{v^i}^{u^i}$ of $G\square H$ we require at least $min\{1 + d_{H-v}(u,y), 7\}$ moves to move the robot to y^i. In particular, if $g(G) \geq 6$, then we need at least 7 moves to move the robot to y^i.*

The theorem below gives the advantage of a $1RJ$ move of the robot over a simple move.

Theorem 2.7 *Given two graphs G and H. Let $\{u,v\},\{v,w\} \in E(H)$ and $i \in V(G)$. Then starting from the configuration $C_{v^i}^{u^i}$ of $G\square H$ we require at least 3 moves to move the robot to w^i.*

Proof Since $\{u^i,v^i\} \in E(G\square H)$. First we would require the move $u^i \xleftarrow{r} v^i$ which would take the robot from v^i to u^i. In order to move the robot to w^i, before it, the hole must be moved from v^i to w^i. This take $d_{G\square H}(v^i,w^i) = 1$. Then the move $w^i \xleftarrow[1]{r} u^i$ takes the robot from u^i to w^i. Hence the result follows. $\square$

Proposition 2.8 *Given two graphs G and H. Let $\{i,j\},\{j,k\} \in E(G)$ and $\{u,v\} \in E(H)$. Then, starting from the configuration $C_{u^j}^{u^i}$ of $G\square H$, we need at least four moves to move the robot to v^k.*

Proof To move the robot from u^j to v^k before it, the hole must be moved from u^i to v^k. This takes three steps (or moves), since $d_{G\square H-u^j}(u^i,v^k) = 3$. Then the move $v^k \xleftarrow[1]{r} u^i$ takes the robot to v^k. Hence the result follows. $\square$

As Cartesian product of graphs is commutative, so the proof of the following proposition can be drawn in the same line as that of Proposition 2.8.

Proposition 2.9 *Given two graphs G and H. Let $\{i,j\} \in E(G)$ and $\{u,v\} \in E(H)$. Then, starting from the configuration $C_{v^i}^{u^i}$ of $G\square H$, we need at least four moves to move the robot to v^k.*

Definition 2.4 *A robot move in $G\square H$ is called a G-move (respectively,H-move) if the edge along which the move took place is a G-edge(respectively,H-edge).*

Definition 2.5 *Let T be a sequence of moves that take the robot from u^i to v^j in $G\square H$. An H-move (respectively, G-move) in T of the robot is said to be a secondary H-move (respectively, G-move) if it is preceded by an H-move (respectively, G-move). An H-move (respectively, G-move) in T of the robot is said to be a primary H-move (respectively, G-move) if it is preceded by a G-move (respectively, H-move). Also the edge corresponding to a primary G-move (respectively, H-move) in T is said to be a primary G-edge (respectively, H-edge).*

Definition 2.6 *A simple move $G\square H$ is said to be a G-simple move (respectively, H-simple move) if the edge along which the simple move took place is a G-edge(respectively, H-edge). Also, a $1RJ$-move in $G\square H$ is said to be a G-$1RJ$-move (respectively, H-$1RJ$-move) if the edge along which the $1RJ$-move took place is a G-edge(respectively, H-edge).*

Definition 2.7 *Let T be a sequence of moves that take the robot from u^i to v^j in $G\square H$. A G-simple move (respectively, H-simple move) in T of the robot preceded by a G-$1RJ$-move (respectively, H-$1RJ$-move) is said to be a G-primary simple move (respectively, H-primary simple move). A G-$1RJ$-move (respectively, H-$1RJ$-move) in T of the robot preceded by another G-$1RJ$-move (respectively, H-$1RJ$-move) is said to be a G-secondary $1RJ$-move (respectively,*

H-secondary $1RJ$-move).

In view of the above definitions we summarize the results of this section in terms of the following remark.

Remark 2.10 Given two graphs G and H, each having girth six or more.

(1) In view of Corollaries 2.2 and 2.5, to perform each G-primary simple (or H-primary simple) move of the robot we require at least 6 moves.

(2) In view of Corollaries 2.3 and 2.6, to perform each G-secondary (or H-secondary) $1RJ$ move of the robot we require at least 7 moves.

(3) In view of Propositions 2.8 and 2.9, to perform each G-primary (or H-primary) $1RJ$-move of the robot we require at least 4 moves.

(4) In a minimum sequence of moves, the robot should take as many primary moves as possible.

§3. Trace of the Robot

To begin this section, we now state the following lemma without proof. This lemma gives the least (or minimum) number of H-moves and G-moves a sequence can have in $G\Box H$.

Lemma 3.1 *Let G and H be two graphs such that $i, j \in V(G)$ and $u, v \in V(H)$. Further, let T be a sequence of moves that take the robot from u^i to v^j in $G\Box H$. Then the minimum number of H-moves (respectively,G-moves) of the robot in T is*

(1) $\frac{p}{2}$ *(respectively,$\frac{k}{2}$)moves, if p is even (respectively, k is even);*

(2) —it $\frac{p+1}{2}$ (respectively,$\frac{k+1}{2}$) moves, if p is odd (respectively, k is odd). Where $d_G(i, j) = k$ and $d_H(u, v) = p$.

Lemma 3.2 *Consider the graphs G and H each having girth six or more. Let $i, j \in V(G)$ and $\{u, v\}, \{u, w\} \in E(H)$. Then each robot move in a minimum sequence of moves that takes $C^{u^i}_{v^i}$ to $C^{u^j}_{w^j}$ in $G\Box H$ is a $G - 1RJ$-move. Also such a minimum sequence involves exactly $\frac{k}{2}$ number of $G - 1RJ$-moves of the robot and $\frac{7k}{2}$ moves in total, where $k = d(i, j) \geq 1$ and k is even.*

Proof Let T be a sequence of moves that takes $C^{u^i}_{v^i}$ to $C^{u^j}_{w^j}$ in $G\Box H$. First assume that the number of robot moves in T is z and each of these robot moves in T is a $G - 1RJ$-move. By Proposition 2.9, we need at least four moves to accomplish the first $G - 1RJ$-move of the robot. Notice that each remaining $z - 1$ robot moves in T is a G-secondary $1RJ$-move. So by Remark 2.10, we need minimum of $7(z - 1)$ G-secondary $1RJ$-moves. Now, if $u^j \overset{r}{\underset{1}{\leftarrow}} u^q$ is the z^{th} robot move in T, it will leave the graph $G\Box H$ with the configuration $C^{u^j}_{u^q}$. Since $d_{G\Box H-u^j}(u^q, w^j) = 3$, so we need minimum of three more move to take the hole from u^q to w^j. Hence T involves minimum $7z$ moves. Notice that, the expression $7z$ takes the minimum value when z is minimum. Next, let $d(i, j) = k$ and $[i = i_0, i_2, i_4, \cdots, i_k]$ be a path of length $\frac{k}{2}$

connecting i and j in G. Then $[u^i = u^{i_0}, u^{i_2}, u^{i_4}, \cdots, u^{i_k} = u^j]$is a path of length $\frac{k}{2}$ in $G\Box H$ joining u^i to u^j. So the sequence of moves

$$v^i \xleftarrow{o^*} u^{i_2} \xleftarrow[1]{r} u^{i_0} \xleftarrow{o^*} u^{i_4} \xleftarrow[1]{r} u^{i_2} \xleftarrow{o^*} u^{i_6} \xleftarrow[1]{r} u^{i_4} \xleftarrow{o^*} u^{i_8} ... \xleftarrow[1]{r} u^{i_k} \xleftarrow[1]{r} u^{i_{k-2}} \xleftarrow{o^*} w^j$$

takes the robot from u^i to u^j along this path and each move in this sequence is a $G-1RJ$-move. Also it involves exactly $\frac{k}{2}$ number of $G-1RJ$-moves of the robot. Therefore by Lemma 3.1, a minimum sequence of moves in T (not involving H-moves of the robot) that takes the configuration $C^{u^i}_{v^i}$ to $C^{u^j}_{w^j}$ involves exactly $7\frac{k}{2}$ moves.

Finally, assume that the sequence T involves H-moves also. If the sequence involves H-moves then we would require at least two H-moves. The first H-move of the robot in T would take it away from copy G^u and the other would bring it back to G^u. Note here that T would still require additional $\frac{k}{2}$ $G-1RJ$ moves. Thus we conclude that T is not minimum. This completes the proof. □

Lemma 3.3 *Consider the graphs G and H each having girth six or more. Let $i, j \in V(G)$ and $\{u, v\}, \{u, w\} \in E(H)$. Then each robot move in a minimum sequence of moves that takes $C^{u^i}_{v^i}$ to $C^{u^j}_{w^j}$ in $G\Box H$ is a G-move. Also such a minimum sequence involves exactly $\frac{k+1}{2}$ number of G moves of the robot and $\frac{7k+3}{2}$ moves in total, where $k = d(i, j) \geq 1$ and k is odd.*

Proof Let T be a sequence of moves that takes $C^{u^i}_{v^i}$ to $C^{u^j}_{w^j}$ in $G\Box H$. First assume that the number of robot moves in T in z and each of these robot moves in T is a G-move. By Proposition 2.9, we need at least four moves to accomplish the first $G-1RJ$-move of the robot. Notice that each succeeding $z-2$ robot moves in T is a G-secondary $1RJ$-move. So by Remark 2.10, we need minimum of $7(z-2)$ G-secondary $1RJ$ moves. Clearly, the z^{th} move of the robot is a G-primary simple move. Thus by Remark 2.10, we require at least six moves to perform this G-primary simple move. Now, if $u^j \xleftarrow{r} u^s$ is the z^{th} robot move in T, it will leave the graph $G\Box H$ with the configuration $C^{u^j}_{u^s}$. Since $d_{G\Box H-u^j}(u^s, w^j) = 2$, so we need minimum of two more moves to take the hole from u^s to w^j. Hence T involves minimum $7z-2$ moves. The expression $7z-2$ takes the minimum value when z is minimum. Next, let $d(i, j) = k$ and $[i = i_0, i_2, i_4, ..., i_{k-1}]$ be a path of length $\frac{k+1}{2}$ connecting i and j in G. Then $[u^i = u^{i_0}, u^{i_2}, u^{i_4}, \cdots, u^{i_{k-1}} = u^j]$ is a path of length $\frac{k+1}{2}$ in $G\Box H$ joining u^i to u^j. So the sequence of moves

$$v^i \xleftarrow{o^*} u^{i_2} \xleftarrow[1]{r} u^{i_0} \xleftarrow{o^*} u^{i_4} \xleftarrow[1]{r} u^{i_2} \xleftarrow{o^*} u^{i_6} \xleftarrow[1]{r} u^{i_4} \xleftarrow{o^*} u^{i_8} ... u^{i_{k-1}} \xleftarrow{r} u^{i_{k-3}} \xleftarrow{o^*} w^j$$

takes the robot from u^i to u^j along this path and each move in this sequence is a G-move. Also it involves exactly $\frac{k+1}{2}$ number of G-moves of the robot. Therefore by Lemma 3.1, a minimum sequence of moves in T (not involving H-moves of the robot) that takes the configuration $C^{u^i}_{v^i}$ to $C^{u^j}_{w^j}$ involves exactly $\frac{7k+3}{2}$ moves.This completes the proof. □

Since the Cartesian product of graphs is commutative, so the proof of the next two lemmas can be drawn in the same as line as that of Lemmas 3.2 and 3.3.

Lemma 3.4 *Consider the graphs G and H each having girth six or more. Let $\{i,j\},\{i,k\} in E(G)$ and $u,v \in V(H)$. Then each robot move in a minimum sequence of moves that takes $C_{u^j}^{u^i}$ to $C_{v^k}^{v^i}$ in $G\Box H$ is an $H-1rJ$move. Also such a minimum sequence involves exactly $\frac{p}{2}$ number of $H-1rJ$moves of the robot and $\frac{7p}{2}$ moves in total, where $p = d(u,v) \geq 1$ and p is even.*

Lemma 3.5 *Consider the graphs G and H each having girth six or more. Let $\{i,j\},\{i,k\} \in E(G)$ and $u,v \in V(H)$. Then each robot move in a minimum sequence of moves that takes $C_{u^j}^{u^i}$ to $C_{v^k}^{v^i}$ in $G\Box H$ is a H-move. Also such a minimum sequence involves exactly $\frac{p+1}{2}$ number of Hmoves of the robot and $\frac{7p+3}{2}$ moves in total, where $p = d(u,v) \geq 1$ and p is odd.*

In view of the results obtained in this section we have the following theorem.

Theorem 3.6 *Given two connected graphs G and H each having girth six or more. Consider the configuration $C_{v^i}^{u^i}$ of $G\Box H$.Then to move the robot from*

(1) *G^u to G^v we require at least $(p-1)+\frac{7}{2}(p-2)$ moves or $(p-1)+\frac{7}{2}(p-3)+6$ moves according as p is even or odd respectively;*

(2) *H^i to H^j we require at least $(k+2)+\frac{7}{2}(k-2)$ moves or $(k+2)+\frac{7}{2}(k-3)+6$ moves according as k is even or odd respectively.*

§4. Minimum Number of Moves

Definition 4.1 *Given a path P connecting u^i and v^j in $G\Box H$. By a minimal sequence of moves with trace P we mean a sequence with minimum number of moves that takes the robot from u^i to v^j along the path P in $G\Box H$.*

Definition 4.2 *By a minimal u^iv^j-path in $G\Box H$ we mean a u^iv^j-path P such that the G-edges in P induces a ij-path in G and the H-edges in P induces a uv-path in H.*

Definition 4.3 *Give two graphs G, H and a path P in $G\Box H$. By a primary edge in P we mean an H-edge that is preceded by a G-edge or a G-edge that is preceded by an H-edge. By a secondary edge in P we mean an H-edge that is preceded by an H-edge or a G-edge that is preceded by a G-edge.*

In view of the definitions above we now state the following lemma without proof. This lemma gives the maximum number of primary edges that a path can have in $G\Box H$ with given H-length and G-length respectively.

Lemma 4.1 *Given two graphs G and H. Let P be a path connecting u^i and v^j in $G\Box H$ such that $l_G(P) = a$ and $l_H(P) = b$. Then, the maximum number of primary edges P can have when*

(1) $a = b$ *is* $a - 1$, *if* a *and* b *are both even;*

(2) $a = b$ *is* a, *if* a *and* b *are both odd;*

(3) $a > b$ *is* $b - 1$, *if* a *is odd and* b *is even and the first edge in* P *is an* H*-edge;*

(4) $a > b$ *is* b *or* $b + 1$, *if* a *and* b *are positive integers with opposite parity and the first edge in* P *is a* G*-edge according as* $a = b + 1$ *or otherwise respectively;*

(5) $a > b$ *is* b, *if* a *is even and* b *is odd and the first edge in* P *is an* H*-edge;*

(6) $a > b$ *is* $b - 1$, *if both* a *and* b *is even and the first edge in* P *is an* H*-edge;*

(7) $a > b$ *is* b, *if both* a *and* b *is even (odd) and the first edge in* P *is a* G*-edge (*H*-edge);*

(8) $a > b$ *is* $b + 1$, *if both* a *and* b *is odd and the first edge in* P *is a* G*-edge;*

(9) $a < b$ *is* a, *if* a *is even and* b *is a positive integer and the first edge in* P *is an* H*-edge;*

(10) $a < b$ *is* $a - 1$, *if* a *is even and* b *is a positive integer and the first edge in* P *is a* G*-edge;*

(11) $a < b$ *is* a *or* $a + 1$, *if* a *is odd and* b *is even and the first edge in* P *is an* H*-edge according as* $a = b - 1$ *or otherwise respectively;*

(12) $a < b$ *is* a, *if* a *is odd and* b *is a positive integer and the first edge in* P *is a* G*-edge;*

(13) $a < b$ *is* $a + 1$, *if both* a *and* b *is odd and the first edge in* P *is an* H*-edge.*

In order to prove our result we need the following.

Remark 4.1 (See [5]) *Given two graphs* G *and* H *each having girth six or more. To perform each primary* G*-move (or* H*-move) of the robot we require at least* 3 *moves.*

Proposition 4.3 *Given two graphs* G *and* H*. Let* $\{i,j\},\{j,k\},\{k,l\},\{l,m\} \in E(G)$ *and* $\{u,v\},\{v,w\},\{w,x\},\{x,y\} \in E(H)$*. Then, starting from the configuration*

(*i*) $C_{u^k}^{w^k}$ *of* $G\Box H$*, we need at least five moves to move the robot to* w^m*;*

(*ii*) $C_{w^k}^{w^m}$ *of* $G\Box H$*, we need at least five moves to move the robot to* y^m*;*

(*iii*) $C_{u^i}^{u^k}$ *of* $G\Box H$*, we need at least four moves to move the robot to* v^k*;*

(*iv*) $C_{u^k}^{w^k}$ *of* $G\Box H$*, we need at least four moves to move the robot to* w^l*.*

Proof (*i*) To move the robot from w^k to w^m, before it, the hole must be moved from u^k to w^m. This takes $d_{G\Box H-u^k}(u^k, w^m) = 4$. Then the $1RJ$-move $w^m \xleftarrow[1]{r} w^k$ takes the robot to w^m. Hence the result follows.

(*ii*) As Cartesian product of graphs is commutative, the proof can be drawn in the same line as (*i*) above.

(*iii*) To move the robot from u^k to v^k, before it, the hole must be moved from u^i to v^k. This takes $d_{G\Box H-u^k}(u^i, v^k) = 3$. Then the $1RJ$-move $v^k \xleftarrow[1]{r} u^k$ takes the robot to v^k. Hence the result follows.

(*iv*) As Cartesian product of graphs is commutative, the proof can be drawn in the same line as (*iii*) above. □

Definition 4.4 *Let* T *be a sequence of moves that takes the robot from* u^i *to* v^j *in* $G\Box H$*. A* $G-1RJ$*-move (respectively,* $H-1RJ$*-move) that is preceded by an* $H-1RJ$*-move (respectively,* $G-1RJ$*-move)is said to be a primary* $G-1RJ$*-move (respectively,primary* $H-1RJ$*-move).*

Also, a G simple move (respectively, H simple move)preceded by a $H-1RJ$-move (respectively, $G-1RJ$-move) is said to be a strong-primary G-move (respectively,H-move).

In view of the above definitions we have this remark.

Remark 4.4 Given two graphs each having girth six or more, in view of Proposition 4.4, to perform each

(1) Primary $G-1RJ$-move (respectively, primary $H-1RJ$-move)of the robot we require at least 5 moves;

(2) Strong-primary or weak-secondary G-move (respectively,H-move) of the robot we require at least 4 moves.

Theorem 4.5 *Given two graphs G and H each having girth six or more. Consider the configuration $C_{v^i}^{u^i}$ of $G\Box H$. For some $j \in G\Box H$, let P be a minimal path connecting u^i and v^j in $G\Box H$. Let T be a minimal sequence with trace P. Where $l_G(P)=a$ and $l_H(P)=b$. Suppose that the first move of the robot is an H-move then T involves at least*

(i) $k-2m+\frac{7}{2}(a+b)-8$ *moves if a and b are both even;*
(ii) $k-2m-3n-q-4r+\frac{7}{2}(a+b)-1$ *moves if a and b are both odd;*
(iii) $k-2m-3n-q+\frac{7}{2}(a+b)-\frac{9}{2}$ *moves if otherwise.*
Furthermore, suppose that the first move of the robot is a G-move then T involves at least

(i) $k-2m+\frac{7}{2}(a+b)-4$ *moves if a and b are both even;*
(ii) $k-2m-3n-q-4r+\frac{7}{2}(a+b)+3$ *moves if a and b are both odd;*
(iii) $k-2m-3n-q+\frac{7}{2}(a+b)-\frac{1}{2}$ *moves if otherwise,*

where m is the number of primary $G-1RJ$ (or primary $H-1RJ$)- moves, n is the number of strong-primary G (or H)-move, q is the number of G-primary (or H-primary) simple moves and r is the number of primary moves of the robot in T and $k=d(u,v)$.

Proof We consider cases following.

Case 1. The first edge in P is an H-edge.

Subcase 1.1 Since T is minimal so it involves exactly $\frac{a+b}{2}$ robot moves. In this case the first robot move is an $H-1RJ$-move, say $w^i \xleftarrow[1]{r} u^i$. In order to realize this move, before it, the hole must move from v^i to w^i. Therefore, we require $k-1$ moves to realize the first robot move, since $d_{G\Box H-u^i}(v^i,w^i)=k-2$ ($k-2$ moves to bring the hole at w^i plus the robot move $w^i \xleftarrow[1]{r} u^i$. Since m is the number of primary G(or H)-$1RJ$-moves in T, so the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b}{2}-m-1$. Hence, by Remark 2.10, the number of moves in T is $k-1+5m+\frac{7}{2}(a+b-2m-2)$, i.e., $k-2m+\frac{7}{2}(a+b)-8$ moves.

Subcase 1.2 Since T is minimal so it involves exactly $\frac{a+b+2}{2}$ robot moves. Just as in Subcase 1.1 above, we require $k-1$ moves to realize the first robot move. By definition of m,n,q and r in T the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b+2}{2}-m-n-q-r-1$. Hence, by Remarks 2.10, 4.2 and 4.4 the number of moves in T is $k-1+5m+4n+6q+3r+7(\frac{a+b+2}{2}-m-n-q-r-1)$, i.e., $k-2m-3n-q-4r+\frac{7}{2}(a+b)-1$ moves.

Subcase 1.3 Since T is minimal so it involves exactly $\frac{a+b+1}{2}$ robot moves. Similarly as in Subcase 1.1 above, we require $k-1$ moves to realize the first robot move. By definition of m, n and q in T the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b+1}{2}-m-n-q-1$. Hence, by Remarks 2.10 and 4.4 the number of moves in T is $k-1+5m+4n+6q+7(\frac{a+b+1}{2}-m-n-q-1)$, i.e., $k-2m-3n-q+\frac{7}{2}(a+b)-\frac{9}{2}$ moves.

Case 2. The first edge in P is a G-edge.

Subcase 2.1 Since T is minimal so it involves exactly $\frac{a+b}{2}$ robot moves. In this case the first robot move is a $G-1RJ$-move. Let this move be $u^k \overset{r}{\underset{1}{\leftarrow}} u^i$. So to perform this move we must first move the hole from v^i to u^k. Clearly $d_{G\Box H-u^i}(v^i, u^k) = k+2$. Therefore, we require $k+3$ moves to perform the first robot move ($k+2$ moves to bring the hole at u^k plus the robot move $u^k \overset{r}{\underset{1}{\leftarrow}} u^i$. Since m is the number of primary G(or H)-$1RJ$-moves in T, so the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b}{2}-m-1$. Hence, by Remark 2.10, the number of moves in T is $k+3+5m+\frac{7}{2}(a+b-2m-2)$, i.e., $k-2m+\frac{7}{2}(a+b)-4$moves.

Subcase 2.2 Since T is minimal so it involves exactly $\frac{a+b+2}{2}$ robot moves. Just as in Subcase 2.1 above, we require $k+3$ moves to realize the first robot move. By definition of m, n, q and r in T the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b+2}{2}-m-n-q-r-1$. Hence, by Remarks 2.10, 4.2 and 4.4 the number of moves in T is $k+3+5m+4n+6q+3r+7(\frac{a+b+2}{2}-m-n-q-r-1)$, i.e., $k-2m-3n-q-4r+\frac{7}{2}(a+b)+3$ moves.

Subcase 2.3 Since T is minimal so it involves exactly $\frac{a+b+1}{2}$ robot moves. Similarly as in Subcase 2.1 above, we require $k+3$ moves to realize the first robot move. By definition of m, n and q in T the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b+1}{2}-m-n-q-1$. Hence, by Remark 2.10 and 4.4 the number of moves in T is $k+3+5m+4n+6q+7(\frac{a+b+1}{2}-m-n-q-1)$, i.e., $k-2m-3n-q+\frac{7}{2}(a+b)-\frac{1}{2}$ moves.

This completes the proof. $\square$

§5. Conclusion and Future Work

In this article, we have been able to investigate the minimum number of moves required for the motion planning of Cartesian product of graphs whereby the robot/object can jump an obstacle. It is clear that the path traced by the robot moves of such motions is less than the minimal path in particular for some cases it is half of the minimal path and off course this path is along the shortest path.

As future work, we plan to investigate this kind of motion in other product graphs, in particular strong and modular product.

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