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Independent Open Irredundant Colorings of Graphs

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Abstract: A vertex $v \in V - S$ is an external private neighbor of u with respect to S if v is adjacent to u but no other vertex in S. A set $S \subseteq V$ is *open irredundant* if every vertex in S has an external private neighbor with respect to S. A set S is called an *independent open irredundant set* or *ioir*-set if S is an independent set and every vertex in S has an external private neighbor with respect to S. An *independent open irredundant coloring* of a graph G is a partition of $V(G)$ into *independent open irredundant* sets. In this paper, we introduce the study of *independent open irredundant colorings* of graphs.

Key Words: Independence, irredundance, open irredundant coloring, independent open irredundant coloring, Smarandachely k-independent open irredundant set.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

Domination is a well studied concept in graph theory. For an excellent treatment of fundamentals of domination we refer to the book by Haynes et al. [6]. Several advanced topics in domination are given in the book edited by Haynes et al. [7].

The neighbourhood of a vertex $x \in V(G)$ in the graph G is denoted by $N(x)$ and the closed neighbourhood $\{x\} \cup N(x)$ by $N[x]$. If X is a subset of $V(G)$, then $N[X] = \bigcup_{x \in X} N[x]$ and the subgraph induced by X is denoted by $G[X]$.

In 1999, Cockayne [3] introduced the study of a large class of generalized irredundant sets in graphs. Each type of a generalized irredundant set $S \subset V$ is defined by the types of private neighbors (i.e self, internal or external) that each vertex in the set must have. A subset S of V in a graph G is said to be *independent* if no two vertices in S are adjacent. Let $u \in S$. A vertex $v \in V - S$ is an external private neighbor of u with respect to S if v is adjacent to u but no other vertex in S. A vertex $u \in S$ is its own private neighbor if it is not adjacent to any vertex in S. A set S is called *irredundant* if every vertex in S is either its own private neighbor or has an external private neighbor, with respect to S. A set S is called an *independent open*

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irredundant set or *ioir*-set if S is an independent set and every vertex in S has an external private neighbor.

Generally, a set S is called a *Smarandachely k-independent open irredundant set* if there is a subset $V_0 \subset V$ with $|V_0| = k$ such that S is an independent set and every vertex in S has an external private neighbor in V_0. Clearly, if $V_0 = V$, a Smarandachely $|G|$-independent open irredundant set is nothing else but an *ioir*-set.

In [3], Cockayne identifies 12 types of generalised irredundant sets the properties of which are hereditary. Perhaps the most interesting of these are the *ioir*-sets. One can therefore define $ioir(G)$ to equal the minimum size of a maximal *ioir*-set and $IOIR(G)$ to equal the maximum size of an *ioir*-set. These generalized irredundant sets are also studied by Finbow in [5] and Cockayne and Finbow in [4].

If a collection of edges between two sets of vertices, say A and B, define a bijection between A and B, then we call such a perfect matching a bijective matching.

A proper k-coloring of a graph G is a partition $\pi = \{V_1, V_2, \cdots, V_k\}$ of V into k non-empty independent sets. The chromatic number $\chi(G)$ equals the minimum integer k for which G has a k-coloring. More generally given a property P concerning subsets of V, a P-coloring is a partition $\pi = \{V_1, V_2, \cdots, V_k\}$ of V into sets, such that each V_i has the property P. If the property P is independence, the P-coloring is the usual coloring and if the property P is domination, the corresponding P-coloring gives the concept of domatic partition. Haynes et al. [8] introduced the concept of irredundant colorings and open irredundant colorings of graphs. Arumugam et al. [1] initiate a study of open irredundant colorings and obtain some results on irredundant colorings and open irredundant colorings. Motivated by the work on [1,8], we initiate a study of *independent open irredundant colorings*. An *independent open irredundant coloring* of a graph G is a partition of V into nonempty independent open irredundant sets. The *independent open irratic number* is the minimum order of an independent open irredundant coloring of G, and it is denoted by $\chi_{ioir}(G)$. In section 2, we obtain some results on independent open irredundant colorings. A study of harmonious, achromatic coloring on middle graph, central graph, total graph, line graph of various classes of graphs can be found in [10, 11, 12, 13]. In Section 3, we investigate the *independent open irratic number* for the middle graph, central graph, total graph, line graph of double star graph families.

We need the following theorems.

Theorem 1.1([6]) *If a graph G has no isolated vertices, then G has a minimum dominating set which is also open irredundant.*

Theorem 1.2([8]) *For any graph G, $n/IR(G) \leq \chi_{ir}(G) \leq n - IR(G) + 1$.*

Observation 1.3([1]) Since any *oir*-coloring of G is an *ir*-coloring of G, it follows that $\chi_{ir}(G) \leq \chi_{oir}(G)$.

Theorem 1.4([8]) *For any graph G, $\chi_{oir}(G) = 2$ if and only if $V(G)$ can be partitioned into two subsets V_1 and V_2 such that there exists a bijective matching between V_1 and V_2.*

Throughout, we assume that G is a graph without isolated vertices.

§2. Independent Open Irredundant Colorings

Observation 2.1 Since any *ioir*-coloring of G is an *oir*-coloring and χ-coloring of G, it follows that $\chi_{ir}(G) \leq \chi_{oir}(G) \leq \chi_{ioir}(G)$ and $\chi_{ir}(G) \leq \chi(G) \leq \chi_{ioir}(G)$.

Observation 2.1 Since $V(G)$ is not an *ioir*-set of G, it follows that $2 \leq \chi_{ioir}(G) \leq n$.

Theorem 2.3 *For any graph G, $\chi_{ioir}(G) = 2$ if and only if $V(G)$ can be partitioned into two independent subsets V_1 and V_2 such that there exists a bijective matching between V_1 and V_2.*

Proof The proof follows from Theorem 1.4. □

Theorem 2.4 *Let G be a graph of order n. Then $\chi_{ioir}(G) = n$ if and only if for any independent set $S \subset V$, there exists $v, w \in S$ such that $N(v) \subseteq N(w)$ or $N(w) \subseteq N(v)$.*

Proof Assume that $\chi_{ioir}(G) = n$. Suppose there is an independent set $S \subset V$ such that $N(v) \nsubseteq N(w)$ and $N(w) \nsubseteq N(v)$ $\forall v, w \in S$. Then there exists a vertex $z_1 \in N(v)$ such that z_1 is not adjacent to w and there exists a vertex $z_2 \in N(w)$ such that z_2 is not adjacent to v. Hence $\{v, w\}$ is an *ioir*-set and $IOIR(G) \geq 2$. Therefore $\chi_{ioir}(G) \leq n - 1$ which is a contradiction. The converse is obvious. □

Observation 2.5 For any complete graph K_n and complete bipartite graph $K_{m,n}$, we have $\chi_{ioir}(K_n) = n$ and $\chi_{ioir}(K_{m,n}) = m + n$.

Observation 2.6 For any tree T, $\chi_{ioir}(T) = n$ if and only if T is a star.

Theorem 2.7 *For the path $P_n = (v_1, v_2, \cdots, v_n)$, we have $\chi_{ioir}(P_n) = 3$.*

Proof Let $V_1 = \{v_1, v_4, v_7, v_{10}, \cdots\}$, $V_2 = \{v_2, v_5, v_8, v_{11}, \cdots\}$, $V_3 = \{v_3, v_6, v_9, v_{12}, \cdots\}$. Clearly $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ into independent open irredundant sets. Hence $\chi_{ioir}(P_n) \leq 3$. By Theorem 2.3, $\chi_{ioir}(P_n) \geq 3$ and so $\chi_{ioir}(P_n) = 3$. □

Theorem 2.8 *For the cycle $C_n = (v_1, v_2, \cdots, v_n)$, we have*

$$\chi_{ioir}(C_n) = \begin{cases} 4 & \text{if } n = 4 \text{ or } n = 7 \\ 3 & \text{otherwise} \end{cases}$$

Proof We can easily observe that $\chi_{ioir}(C_4) = 4$. We now prove that $\chi_{ioir}(C_n) = 3$ for $n \neq 4$ or 7. By Theorem 2.3, $\chi_{ioir}(C_n) \geq 3$. Now we consider three cases.

Case 1. $n \equiv 0(mod 3)$.

Let $V_1 = \{v_1, v_4, v_7, v_{10}, \cdots, v_{n-2}\}$, $V_2 = \{v_2, v_5, v_8, v_{11}, \cdots, v_{n-1}\}$ and $V_3 = \{v_3, v_6, v_9, v_{12}, \cdots, v_n\}$. Clearly $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ into independent open irredundant sets since any three consecutive vertices in the cycle receives distinct colors. Hence $\chi_{ioir}(C_n) \leq 3$.

Case 2. $n \equiv 1(mod 3)$.

Let $V_1 = \{v_1, v_3, v_6, v_8, v_{11}, v_{14}, v_{17}, \cdots, v_{l-3}, v_l, v_{l+3}, \cdots, v_{n-2}\}$, $V_2 = \{v_2, v_4, v_7, v_9, v_{12},$

$v_{15}, v_{18}, \cdots, v_{l-3}, v_l, v_{l+3}, \cdots, v_{n-1}\}$, $V_3 = \{v_5, v_{10}, v_{13}, v_{16}, v_{19}, \cdots, v_{l-3}, v_l, v_{l+3}, \cdots, v_n\}$. We now prove that $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ into independent open irredundant sets. Clearly the sets V_i, $i = 1, 2, 3$ are independent. Hence it is enough to prove that every vertex in the set V_i has an external private neighbour with respect to V_i, $i = 1, 2, 3$. Note that v_1, v_5, v_6 are the external private neighbors of v_2, v_4, v_7 respectively and v_n, v_4, v_7 and v_{10} are the external private neighbors of v_1, v_3, v_8 and v_9 respectively. All other remaining vertices v_i have external private neighbor v_{i-1}.

Case 3. $n \equiv 2 (mod 3)$.

Let $V_1 = \{v_1, v_4, v_7, v_{10}, \cdots, v_{n-1}\}$, $V_2 = \{v_2, v_5, v_8, v_{11}, \cdots, v_n\}$ and $V_3 = \{v_3, v_6, v_9, v_{12}, \cdots, v_{n-2}\}$. Since v_2, v_{n-1}, v_{n-2} are the external private neighbors of v_1, v_n, v_{n-1} respectively and remaining vertices v_i have external private neighbor v_{i+1}, $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ into independent open irredundant sets. Hence $\chi_{ioir}(C_n) \leq 3$. Now we prove that $\chi_{ioir}(C_7) = 4$. Since any independent open irredundant set of C_7 has at most two vertices, minimum four colors are required to color the vertices of C_7. Let $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_6\}$, $V_3 = \{v_3, v_5\}$ and $V_4 = \{v_7\}$. Clearly $\{V_1, V_2, V_3, V_4\}$ is an *ioir*-coloring of C_7. Hence $\chi_{ioir}(C_7) = 4$. □

Proposition 2.9 *For any graph G, $n/IOIR(G) \leq \chi_{ioir}(G) \leq n - IOIR(G) + 1$, where $IOIR(G)$ is the upper independent open irredundance number of G.*

Proof Let $\chi_{ioir}(G) = k$. Let $\{V_1, V_2, \cdots, V_k\}$ be an *ioir*-coloring of G. Since $|V_i| \leq IOIR(G)$, it follows that $n = \sum_{i=1}^{k} |V_i| \leq k.IOIR(G)$. Hence $n/IOIR(G) \leq \chi_{ioir}(G)$. Now, let S be an independent open irredundant set of G with $|S| = IOIR(G)$. Then $\{S\} \cup \{\{v\} : v \in V - S\}$ is an *ioir*-coloring of G. Hence $\chi_{ioir}(G) \leq n - IOIR(G) + 1$. □

Theorem 2.10 *Let G be a connected graph with $\delta = 1$ and let r denote the maximum number of leaves adjacent to a support vertex v of G. Then $\chi_{ioir}(G) \geq r + 2$.*

Proof Let $v_1, v_2, \cdots, v_r$ be the leaves adjacent to v. Since any independent open irredundant set in G contains at most one of the leaves v_i, the result follows. □

Observation 2.11 Let $T \neq K_{1,n}$ be any tree and let r denote the maximum number of leaves adjacent to a support vertex v of T. Then $\chi_{ioir}(T) \geq r + 2$.

§3. IOIR-Coloring on Double Star Graph Families

In this section we investigate the independent open irratic number for the central graph, middle graph, total graph, line graph of star graph $K_{1,n}$ and double star graph $K_{1,n,n}$.

The central graph $C(G)$ of a graph G is formed by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i)x, y are in $E(G)$ and x, y are adjacent in G. (ii)x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The total graph of G has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in G.

The line graph of G denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

A star is a complete bipartite graph $K_{1,m}$ with $m \geq 2$, and the unique vertex v of this star of degree m is called the center.

Double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $2n+1$ vertices and $2n$ edges. Let $V(K_{1,n,n}) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and $E(K_{1,n,n}) = \{e_1, e_2, \cdots, e_n\} \cup \{s_1, s_2, \cdots, s_n\}$.

Proposition 3.1 *For any star graph $K_{1,n}$, we have*

(*i*) $\chi_{ioir}(M(K_{1,n})) = n + 2$;
(*ii*) $\chi_{ioir}(C(K_{1,n})) = n + 1$;
(*iii*) $\chi_{ioir}(T(K_{1,n})) = n + 2$;
(*iv*) $\chi_{ioir}(L(K_{1,n})) = n$.

Proof (*i*) By the definition of middle graph, each edge vv_i in $K_{1,n}$ is subdivided by the vertex e_i in $M(K_{1,n})$ and the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $n+1$ in $M(K_{1,n})$. i.e $V(M(K_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Hence $n+1$ distinct colors are required to color the vertices $v, e_1, e_2, \cdots, e_n$. Note that e_i is the only external private neighbour of v_i with respect to any set $S \subseteq V$. Therefore we assign the color which is different from the already assigned colors to v_i. Hence $\chi_{ioir}(M(K_{1,n})) \geq n + 2$. Assign *ioir*-coloring as follows: For $1 \leq i \leq n$, assign the color c_i for e_i and assign the color c_{n+1} to v. For $1 \leq i \leq n$, assign the color c_{n+2} to all the vertices $v_1, v_2, \cdots, v_n$. Hence $\chi_{ioir}(M(K_{1,n})) \leq n + 2$.

(*ii*) By the definition of central graph, each edge vv_i in $K_{1,n}$ is subdivided by the vertex e_i in $C(K_{1,n})$ and the vertices $v_1, v_2, \cdots, v_n$ induce a clique of order n in $C(K_{1,n})$. i.e $V(C(K_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Since v_i $(1 \leq i \leq n)$ induce a clique of order n, we have $\chi_{ioir}(C(K_{1,n})) \geq n$. We now prove that $\chi_{ioir}(C(K_{1,n})) \geq n + 1$. Suppose $\chi_{ioir}(C(K_{1,n})) = n$. Let V_i be the set of vertices which are colored with c_i, $i = 1$ to n. Let we assign the color c_i to v_i $(1 \leq i \leq n)$ and assign the color c_1 to v. Therefore the vertices $e_1, e_2, \cdots, e_n$ are colored by $c_2, c_3, \cdots, c_{n-1}, c_n$ in some arrangement. Hence at least two of the vertices e_i and e_j are colored with the same color c_m. Clearly any vertex adjacent to vertices e_i and e_j is also joined to vertex of color c_m. It follows that there is no external private neighbour for the vertices e_i and e_j with respect to V_m. This is a contradiction. Hence $\chi_{ioir}(C(K_{1,n})) \geq n + 1$. Assign *ioir*-coloring as follows: For $1 \leq i \leq n$, assign the color c_i for v_i and assign the color c_{n+1} for each e_i. Finally we assign the color c_1 to v. Hence $\chi_{ioir}(C(K_{1,n})) \leq n + 1$.

(*iii*) By the definition of total graph, we have $V(T(K_{1,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\}$ $\cup \{e_i : 1 \leq i \leq n\}$, in which the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $n+1$. Clearly $\chi_{ioir}(T(K_{1,n})) \geq n + 1$. Let we assign the color c_i to e_i $(1 \leq i \leq n)$ and assign the color c_{n+1} to v. Since e_i and v are the external private neighbors of v_i with respect to V_i and V_{n+1}, we need one more color to v_i. Hence $\chi_{ioir}(T(K_{1,n})) \geq n + 2$. Assign *ioir*-coloring as follows: For $1 \leq i \leq n$, assign the color c_i for e_i and assign the color c_{n+1} to v. Finally we assign the color c_{n+2} to each v_i. Hence $\chi_{ioir}(T(K_{1,n})) \leq n + 2$.

(iv) Since $L(K_{1,n}) \cong K_n$, $\chi_{ioir}(L(K_{1,n})) = n$. □

Proposition 3.2 *For any double star graph $K_{1,n,n}$, we have*

$$\chi_{ioir}(M(K_{1,n,n})) = \begin{cases} n+1 & \forall n \geq 3 \\ 4 & n = 2 \end{cases}$$

Proof Clearly we observe that $\chi_{ioir}(M(K_{1,2,2})) = 4$. By the definition of middle graph, each edge vv_i and v_iu_i $(1 \leq i \leq n)$ in $K_{1,n,n}$ are subdivided by the vertices e_i and s_i in $M(K_{1,n,n})$ and the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $n+1$ (say K_{n+1}) in $M(K_{1,n,n})$. i.e $V(M(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. Clearly $\chi_{ioir}(M(K_{1,n,n})) \geq n+1$. Assign *ioir*-coloring as follows: For $1 \leq i \leq n$, assign the color c_i for e_i and assign the color c_{n+1} to v. For $1 \leq i \leq n$, assign two distinct colors c_l and c_m other than c_{n+1} and c_i to the vertices v_i and s_i. Finally, assign the color c_{n+1} to each $u_i (1 \leq i \leq n)$. Let V_i be the set of vertices which are colored with c_i, $i = 1$ to $n+1$. Note that v is the external private neighbor of all the vertices e_i with respect to V_i, $1 \leq i \leq n$ and e_i's are the external private neighbors of v with respect to V_{n+1}. For $1 \leq i \leq n$, s_i is the external private neighbor of u_i and v_i with respect to V_{n+1} and V_l. Finally, v_i is the external private neighbor of s_i with respect to V_m. Hence $\chi_{ioir}(M(K_{1,n,n})) \leq n+1$. □

Proposition 3.3 *For any double star graph $K_{1,n,n}$, we have $\chi_{ioir}(C(K_{1,n,n})) = n+2$.*

Proof By the definition of central graph, each edge vv_i and v_iu_i $(1 \leq i \leq n)$ in $K_{1,n,n}$ are subdivided by the vertices e_i and s_i in $C(K_{1,n,n})$. The vertices $v, u_1, u_2, \cdots, u_n$ induce a clique of order $n+1$ (say K_{n+1}) and the vertices $v_i (1 \leq i \leq n)$ induce a clique of order n in $C(K_{1,n,n})$. i.e $V(C(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. Clearly $\chi_{ioir}(C(K_{1,n,n})) \geq n+1$. We now prove that $\chi_{ioir}(C(K_{1,n,n})) \geq n+2$. Suppose $\chi_{ioir}(C(K_{1,n,n})) = n+1$. Since v, u_i $(1 \leq i \leq n)$ induce a clique of order $n+1$, let us assign the color c_{n+1} to v and assign the color c_i to $u_i (1 \leq i \leq n)$. Since e_i has degree 2 and v is adjacent to the vertex of color c_i $\forall i$, v_i is the only external private neighbour of e_i. But v_i is adjacent to the vertex of color c_j, $\forall j \neq i$. Therefore e_i must be colored only with c_i and v_i must be colored only with c_{n+1}. Since v_i $(1 \leq i \leq n)$ induce a clique of order n, v_l, it leads to a contradiction. Hence $\chi_{ioir}(C(K_{1,n,n})) \geq n+2$. Consider the colors $c_1, c_2, \cdots, c_{n+2}$. Assign *ioir*-coloring as follows: Assign the colour c_{n+1} to v and assign the color c_i to u_i, where $1 \leq i \leq n$. Assign the color c_{n+1} to all the vertices $s_1, s_2, \cdots, s_n$ and assign the color c_{n+2} to all the vertices $e_1, e_2, \cdots, e_n$. Finally, we assign the color c_i to v_i for $1 \leq i \leq n$. Let V_i be the set of vertices which are colored with c_i, $i = 1$ to $n+2$. For $1 \leq i \leq n$, e_i is the external private neighbor of v with respect to V_{n+1} and v_i is the external private neighbor of e_i with respect to V_{n+2}. For $1 \leq i \leq n$, e_i is the external private neighbor of v_i with respect to V_i and v_i is the external private neighbor of s_i with respect to V_{n+1}. Finally, v is the external private neighbor of all the vertices u_i with respect to V_i. Hence $\chi_{ioir}(C(K_{1,n,n})) \leq n+2$. □

Proposition 3.4 *For any double star graph $K_{1,n,n}$, we have $\chi_{ioir}(T(K_{1,n,n})) = n+1$.*

Proof By the definition of total graph, we have $V(T(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup$

$\{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$ in which the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $n+1$. Clearly $\chi_{ioir}(T(K_{1,n,n})) \geq n+1$. Consider the colors $c_1, c_2, \cdots, c_{n+1}$. Assign *ioir*-coloring as follows: Assign the color c_{n+1} to v and assign the colour c_i to e_i, where $1 \leq i \leq n$. For $1 \leq i \leq n$, assign two distinct colors other than c_{n+1} and c_i to the vertices v_i and s_i. Finally, assign the color c_{n+1} to each $u_i (1 \leq i \leq n)$. Hence $\chi_{ioir}(T(K_{1,n,n})) \leq n+1$. □

Proposition 3.5 *For any double star graph $K_{1,n,n}$, we have $\chi_{ioir}(L(K_{1,n,n})) = n+1$.*

Proof By the definition of line graph, each edge of $K_{1,n,n}$ taken to be as vertex in $(L(K_{1,n,n}))$. The vertices $e_1, e_2, \cdots, e_n$ induce a clique of order n and the vertices $s_1, s_2, \cdots, s_n$ are all pendant in $(L(K_{1,n,n}))$. i.e $V(L(K_{1,n,n})) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. From Theorem 2.10, we have $\chi_{ioir}(L(K_{1,n,n})) \geq n+1$. Assign *ioir*-coloring as follows: Assign the color c_{n+1} to all the vertices s_i, where $1 \leq i \leq n$ and assign the color c_i to e_i, where $1 \leq i \leq n$. Hence $\chi_{ioir}(L(K_{1,n,n})) \leq n+1$. □

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