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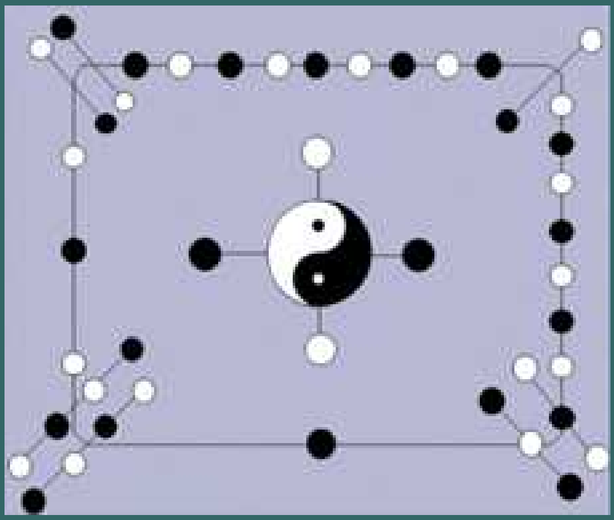

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The mathematician lives long and lives young; the wings of his soul do not early drop off, nor do its pores become clogged with the earthy particles blown from the dusty highways of vulgar life.

By James Joseph Sylvester, a British mathematician.

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A Combinatorial Approach For the Spanning Tree Entropy in Complex Network

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Abstract: The goal of this paper is to propose the combinatorial method to facilitate the calculation of the number of spanning trees for complex networks. In particular, we derive the explicit formulas for the triangular snake, double triangular snake, four triangular snake, the total graph of path, the generalized friendship graphs and the subdivision of double triangular snake. Finally, we calculate their spanning trees entropy and we compare it between them.

Key Words: Entropy, cyclic snakes, total graph, number of spanning trees.

AMS(2010): 05C05, 05C30.

§1. Introduction

In real life, most of the systems are represented by graphs, such that the nodes denote the basic constituents of the system and edges describe their interaction. The Internet, electric, bioinformatics, telephone calls, social networks and many other systems are now represented by complex graphs [1].

There are many different types of networks and their classification depends on the properties such as nodes degrees, clustering coefficients, shortest paths. Another concern in studying complex network is how to evaluate the robustness of a network and its ability to adapt to changes [21]. The robustness of a network is correlated to its ability to deal with internal feedbacks within the network and to avoid malfunctioning when a fraction of its constituents is damaged. We use the entropy of spanning trees or what is called the asymptotic complexity [4] in order to quantify the robustness and to characterize the structure. The number of spanning trees in G, also called, the complexity of the graph is a well-studied quantity (for long time) and appear in a number of applications. Most notable application fields are network reliability [15, 16, 17], enumerating certain chemical isomers [18] and counting the number of Eulerian circuits in a graph [19].

[1]Received March 7, 2018, Accepted November 15, 2018.

A graph G has different subgraphs. In fact a graph having $|V(G)|$ nodes has

$$2^{(\frac{|V(G)|(|V(G)-1|)}{2})}$$

possible distinct subgraphs. Some of these subgraphs are trees and the others are not trees. We are focused certain kinds of trees called spanning trees. The history of determining the number of spanning trees $\tau(G)$ of a graph G, dates back to the year 1842 in which the German Mathematician Gustav Kirchhoff [2] introduced a relation between the number of spanning trees of a graph G, and the determinant of a specific submatrix associated with G. This method is infeasible for large graphs. For this reason scientists have developed techniques to get around the difficulties and have paid more attention to deriving explicit and simple formulas for special classes, see [3 - 13].

The basic combinatorial idea, Feussners recursive formula [20], for counting $\tau(G)$ in a graph G is quite intuitive. For an undirected simple graph G, let e be any edge of G. All spanning trees in G can be separated into two parts: one part contains all spanning trees without e as a tree edge; the other part contains all spanning trees with e as a tree edge. The first part has the same number of spanning trees as graph $G - e$, but leaving all other edges and vertices as they are. The second part has the same number of spanning trees as graph $G \odot e$, where $G \odot e$is the graph (not a subgraph) obtained from G by contracting the edge $e = \{u, v\}$ until the two vertices u and v coincide. Call this new vertex uv. Both $G - e$ and $G \odot e$ have fewer edges, than G. So the number of spanning trees in G can be counted recursively in this way.
In this paper, we propose the combinatorial method to facilitate the calculation of the number of spanning trees for complex networks. In particular, we derive the explicit formulas for the triangular snake ($\Delta_k - snake$), double triangular snake ($2\Delta_k - snake$), four triangular snake ($4\Delta_k - snake$), the total graph of path $P_n(T(P_n))$, the graph $nC_4 \odot 2P_n$, the generalized friendship graphs ${}^kF_n$ and the subdivision of double triangular snake ($S(2\Delta_n - snake)$). Finally, we calculate their spanning trees entropy and we compare it between them.

§2. Preliminary Notes

The combinatorial method involves the operation of contraction of an edge. An edge e of a graph G is said to be contracted if it is deleted and its ends are identified. The resulting graph is denoted by $G \bullet e$. Also we denote by $G - e$ the graph obtained from G by deleting the edge e.

Theorem 2.1([13-20]) *Let G be a planar graph (multiple edges are allowed in here). Then for any edge $\tau(G) = \tau(G - e) + \tau(G \bullet e)$.*

Definition 2.2([22]) *A triangular snake($\Delta - snake$) is a connected graph in which all blocks are triangles and the block-cut-point graph is a path, as shown in Figure 1.*

Definition 2.3 *For an integer number m, an m-triangular snake is a graph formed by m triangular snakes having a common path. If $m = 2$ that graph is called the double triangular* *snake is denoted by* $2\Delta - snake$*, as shown in Figure 1.*

Definition 2.4 *The friendship graph* $F_{n,k}$ *is a collection of* k*-cycles (all of order* n*), meeting at a common vertex, as shown in Figure 1.*

Definition 2.5 *The graph* $nC_m \odot 2P_n$ *is a connected graph obtained from* n *copies of* C_m *(* nC_m *is a disconnected graph) and two paths where each path connects with one vertex* u_i $(i = 1, 2, \cdots, 2n)$ *of each copy of* C_m *. All the vertices* u_i $(i = 1, 2, \cdots, 2n)$ *are distinct as shown in Figure 1.*

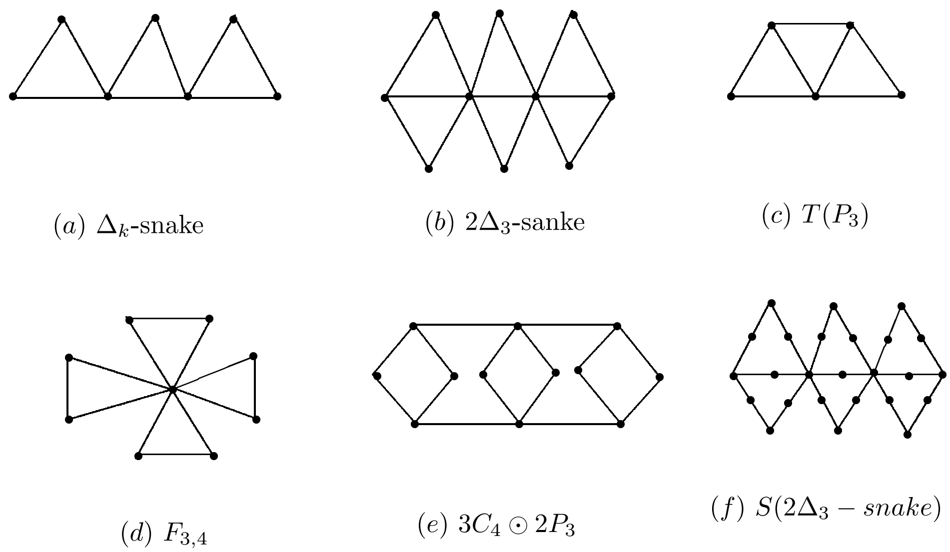


Figure 1 Triangular snake, double triangular snake, four triangular snake, total graph of path, generalized friendship and subdivision of double triangular snake

Definition 2.6 *The total graph of a graph* G *is the graph whose vertex set is* $V(G) \cup E(G)$ *and two vertices are adjacent whenever they are either adjacent or incident in* G*. The total graph of* G *denoted by* $T(G)$*.*

§3. Main Results

Theorem 3.1 *The number of spanning trees of triangular snake graph is*

$$\tau(\Delta_n) = 3^n.$$

Proof Consider a triangular snake graph Δ_n' constructed from Δ_n by deleting one edge. See Figure 2.

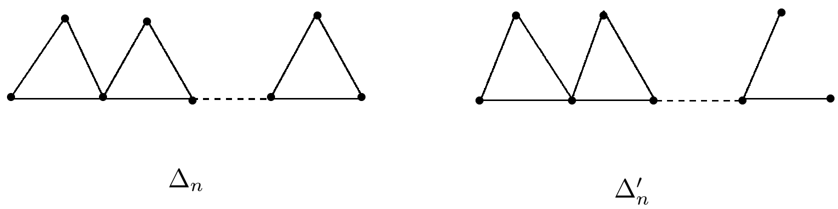


Figure 2 Triangular snake graph (Δ_n)

We put

$$\Delta_n = \tau(\Delta_n) \text{ and } \Delta_n^{'} = \tau(\Delta_n^{'}).$$

It is clear that

$$\Delta_n = 2(\Delta_{n-1}) + 3(\Delta_{n-1}^{'}) \text{ and } \Delta_n^{'} = 2(\Delta_{n-1}) - 3(\Delta_{n-1}^{'})$$

with initial conditions $\Delta_1 = 3, \Delta_n^{'} = 1$ thus we have

$$\begin{pmatrix} \Delta_n \\ \Delta_n^{'} \end{pmatrix} = A \begin{pmatrix} \Delta_n \\ 4\Delta_n^{'} \end{pmatrix},$$

where,

$$A = \begin{pmatrix} 2 & 3 \\ 2 & -3 \end{pmatrix}; \quad \begin{pmatrix} \Delta_n \\ \Delta_n^{'} \end{pmatrix} = A \begin{pmatrix} \Delta_{n-1} \\ \Delta_{n-1}^{'} \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} \Delta_1 \\ \Delta_1^{'} \end{pmatrix},$$

we compute A^{n-1}as follows:

$$\det(A - \lambda I_2) = \lambda^2 - \lambda - 12 = 0,\ \lambda_1 = -4 \text{ and } \lambda_2 = 3, \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that $A = MBM^{-1}$, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{3} \end{pmatrix}; \quad M^{-1} \begin{pmatrix} \frac{1}{7} & \frac{-3}{7} \\ \frac{6}{7} & \frac{3}{7} \end{pmatrix}; \quad A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (-4)^{n-1} & 0 \\ 0 & (3)^{n-1} \end{pmatrix}$$

From which, we obtain

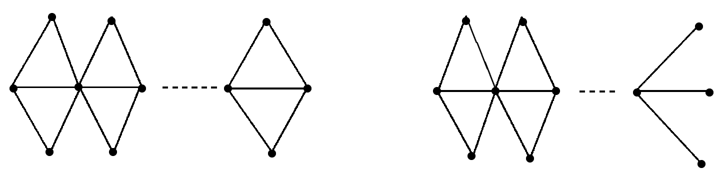
$$A^{n-1} = \begin{pmatrix} \frac{(-4)^{n-1}}{7} + \frac{2*3^n}{7} & \frac{-3*(-4)^{n-1}}{7} + \frac{3^n}{7} \\ \frac{-2*(-4)^{n-1}}{7} + \frac{2*(3)^{n-1}}{7} & \frac{6*(-4)^{n-1}}{7} + \frac{3^{n-1}}{7} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.2 *The number of spanning trees of the double triangular snake is*

$$\tau(2\Delta_n - snake) = 8^n.$$

Proof Consider a double triangular snake graph $2\Delta'_n$-snake constructed from $2\Delta_n$-snake by deleting two edges. See Figure 3.



$2\Delta_n$-snake $2\Delta'_n$-snake

Figure 3 Triangular snake graph (Δ_n)

We put

$$2\Delta_n - snake = \tau(2\Delta_n - snake) \text{ and } 2\Delta_2^{'} - snake = \tau(2\Delta_2^{'} - snake).$$

It is clear that

$$\begin{aligned} 2\Delta_n - snake &= 7(2\Delta_{n-1} - snake) + 8(2\Delta_2^{'} - snake) \\ 2\Delta_2^{'} - snake &= 2(2\Delta_{n-1} - snake) - 8(2\Delta_{n-1}^{'} - snake) \end{aligned}$$

with initial conditions $2\Delta_1 - snake = 8$, $2\Delta_1^{'} - snake = 1$. Thus we have

$$\begin{pmatrix} 2\Delta_n - snake \\ 2\Delta_n^{'} - snake \end{pmatrix} = A \begin{pmatrix} 2\Delta_{n-1} - snake \\ 2\Delta_n^{'} - snake \end{pmatrix}, \text{ where } A = \begin{pmatrix} 7 & 8 \\ 2 & -8 \end{pmatrix},$$

$$\begin{pmatrix} 2\Delta_n - snake \\ 2\Delta_n^{'} - snake \end{pmatrix} = A \begin{pmatrix} 2\Delta_{n-1} - snake \\ 2\Delta_n^{'} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} 2\Delta_1 - snake \\ 2\Delta_1^{'} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 - \lambda - 72 = 0, \ \lambda_1 = -9 \text{ and } \lambda_2 = 8, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that $A = MBM^{-1}$, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{8} \end{pmatrix}; \quad M^{-1} = \begin{pmatrix} \frac{-1}{7} & \frac{8}{7} \\ \frac{8}{7} & \frac{-8}{7} \end{pmatrix}; \quad A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (8)^{n-1} & 0 \\ 0 & (-9)^{n-1} \end{pmatrix}.$$

From which, we obtain

$$A^{n-1} = \begin{pmatrix} \frac{(-8)^{n-1}}{7} + \frac{8*(-9)^{n-1}}{7} & \frac{8^n}{7} + \frac{-8*(-9)^{n-1}}{7} \\ \frac{-2*(8)^{n-1}}{7} + \frac{(-9)^{n-1}}{7} & \frac{-2*(8)^n}{7} + \frac{-(-9)^{n-1}}{7} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.3 *The number of spanning trees in* $4\Delta_n - snake$ *is* $\tau(2\Delta_n - snake)$=48^n*, where* n *is the number of blocks.*

Proof Consider a double triangular snake graph $2\Delta_2^{'} - snake$ constructed from $2\Delta_n - snake$ by deleting four edges. See Figure 4.

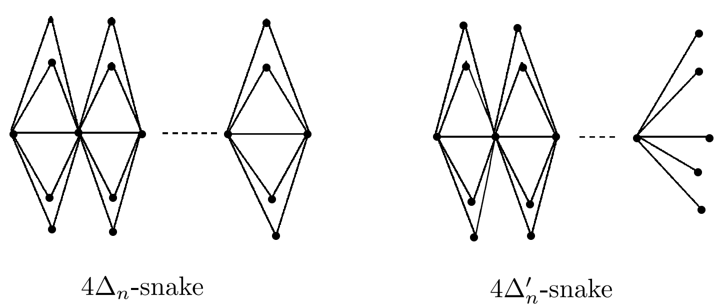


Figure 4 Friendship graph $F_{4,k}$

We put

$$4\Delta_n - snake = \tau(4\Delta_n - snake) \text{ and } 4\Delta_n^{'} - snake = \tau(4\Delta_n^{'} - snake).$$

It is clear that

$$4\Delta_n - snake = 47(4\Delta_{n-1} - snake) + 48(4\Delta_2^{'} - snake)$$

and

$$4\Delta_n^{'} - snake = 2(4\Delta_{n-1} - snake) - 48(4\Delta_{n-1}^{'} - snake)$$

with initial conditions $4\Delta_1 - snake = 48$, $4\Delta_1^{'} - snake = 1$. Thus, we have

$$\begin{pmatrix} 4\Delta_n - snake \\ 4\Delta_n^{'} - snake \end{pmatrix} = A \begin{pmatrix} 4\Delta_{n-1} - snake \\ 4\Delta_{n-1}^{'} - snake \end{pmatrix},$$

where

$$A = \begin{pmatrix} 47 & 48 \\ 2 & -48 \end{pmatrix}, \quad \begin{pmatrix} 4\Delta_n - snake \\ 4\Delta_n^{'} - snake \end{pmatrix} = A \begin{pmatrix} 4\Delta_{n-1} - snake \\ 4\Delta_{n-1}^{'} - snake \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} 4\Delta_1 - snake \\ 4\Delta_1^{'} - snake \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + 4\lambda - 2352 = 0, \ \lambda_1 = 48 \text{ and } \lambda_2 = -49, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ \frac{1}{48} & -2 \end{pmatrix}; \ M^{-1} = \begin{pmatrix} \frac{96}{97} & \frac{48}{97} \\ \frac{1}{97} & \frac{-48}{97} \end{pmatrix}; \ A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (48)^{n-1} & 0 \\ 0 & (-49)^{n-1} \end{pmatrix}.$$

From which, we obtain

$$A^{n-1} = \begin{pmatrix} \frac{2*(48)^n}{97} + \frac{(-49)^{n-1}}{97} & \frac{48^n}{97} + \frac{-48}{97} * (-49)^{n-1} \\ \frac{2*(48)^{n-1}}{97} + \frac{-2}{97} * (-49)^{n-1} & \frac{(48)^n}{97} + \frac{96}{97} * (-49)^{n-1} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.4 *The number of spanning trees of the total graph of path P_n is*

$$\tau(T(P_n)) = \frac{1}{\sqrt{5}} \left[(\frac{7+3\sqrt{5}}{2})^n - (\frac{7-3\sqrt{5}}{2})^n \right].$$

Proof Consider a total graph of path $P_nT(P_n^{'})$ constructed from $T(P_n)$ by deleting one edge. See Figure 5.

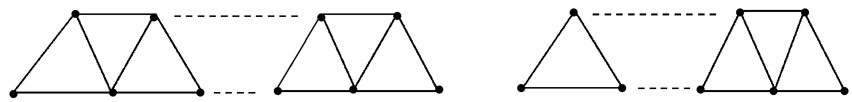


Figure 5 Total graph of path

We put

$$T(P_n) = \tau(T(P_n)) \text{ and } T(P_n^{'}) = \tau(T(P_n^{'}).$$

It is clear that

$$T(P_n) = 7T(P_{n-1}) - T(P_{n-2}^{'}),$$

where $T(P_n)$ is the number of even block and

$$T(P_n^{'}) = 48T(P_{n-2}) - 7T(P_{n-3}^{'}),$$

where $T(P_n^{'})$ is the number of odd block with initial conditions $T(P_2) = 3, T(P_2^{'}) = 1$. Thus, we have

$$\begin{pmatrix} T(P_n) \\ T(P_n^{'}) \end{pmatrix} = A \begin{pmatrix} T(P_{n-1}) \\ T(P_{n-1}^{'}) \end{pmatrix},$$

where

$$A = \begin{pmatrix} 7 & -1 \\ 48 & -7 \end{pmatrix}, \quad \begin{pmatrix} T(P_n) \\ T(P_n^{'}) \end{pmatrix} = A \begin{pmatrix} T(P_{n-1}) \\ T(P_{n-1}^{'}) \end{pmatrix} = \cdots = A^{n-2} \begin{pmatrix} T(P_2) \\ T(P_2^{'}) \end{pmatrix},$$

$\lambda_1 = 1$ and $\lambda_2 = -1$, $\lambda_1 \neq \lambda_2$. Then there is a matrix M is invertible such that A=MDM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ 6 & 8 \end{pmatrix}; \quad M^{-1} \begin{pmatrix} 4 & \frac{-1}{2} \\ & \\ -3 & \frac{1}{2} \end{pmatrix}; \quad A^{n-2} = MB^{n-2}M^{-1},$$

where

$$B^{n-2} = \begin{pmatrix} (1)^{n-2} & 0 \\ 0 & (-1)^{n-2} \end{pmatrix}.$$

From which, we obtain

$$A^{n-2} = \begin{pmatrix} 4*(1)^{n-2} - 3*(-1)^{n-2} & (\frac{-1}{2})*(1)^{n-2} + (\frac{1}{2})*(-1)^{n-2} \\ 24*(1)^{n-2} - 24*(-1)^{n-2} & -3*(1)^{n-2} + 4*(-1)^{n-2} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.5 *The number of spanning trees in the graph* $nC_4 \circ 2P_n$ *is* $\tau(nC_4 \circ 2P_n) = 4^n$.

Proof Consider a graph B_n constructed from $nC_4 \circ 2P_n = A_n$ by deleting two edges. See Figure 6.

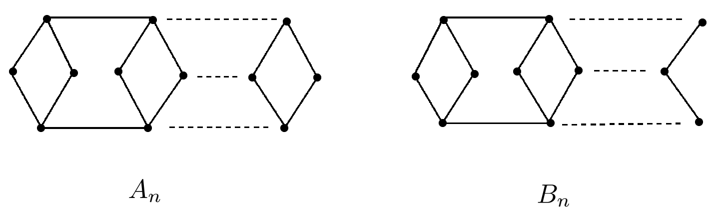


Figure 6 $nC_4 \circ 2P_n$ graph

We put

$$A_n = \tau(A_n) \text{ and } B_n = \tau_(B_n).$$

It is clear that

$$A_n = 3A_{n-1} + 4B_{n-1} \text{ and } B_n = 2A_{n-1} - 4B_{n-1}$$

with initial conditions $A_1 = 4$ and $B_1 = 1$ thus we have

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = A \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} 3 & 4 \\ 2 & -4 \end{pmatrix}, \quad \begin{pmatrix} A_n \\ B_n \end{pmatrix} = A \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 20 = 0, \ \lambda_1 = -5 \text{ and } \lambda_2 = 4, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{4} \end{pmatrix};\ M^{-1} = \frac{\frac{1}{9}}{4} \begin{pmatrix} \frac{1}{4} & -1 \\ 2 & 1 \end{pmatrix};\ A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (-5)^{n-1} & 0 \\ 0 & (4)^{n-1} \end{pmatrix}.$$

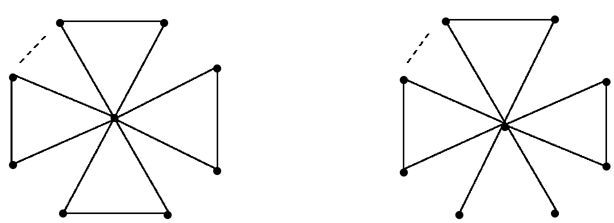
From which, we obtain

$$A^{n-1} = \begin{pmatrix} \frac{(-5)^{n-1}}{9} + \frac{2*(4)^n}{9} & \frac{-4*(-5)^{n-1}}{9} + \frac{4^n}{9} \\ \frac{-2*(-5)^{n-1}}{9} + \frac{2*4^{n-1}}{9} & \frac{8*(-5)^{n-1}}{9} + \frac{4^{n-1}}{9} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.6 *The number of spanning trees of friendship graph $F_{3,k}$ is $\tau(F_{3,k})$=3^k.*

Proof Consider a friendship graph $F^{'}_{3,k}$ constructed from $F_{3,k}$ by deleting one edge. See Figure 7.



$F_{3,k}$ $F'_{3,k}$

Figure 7 Friendship graph $F_{3,k}$

We put

$$F_{3,k} = \tau(F_{3,k}) \text{ and } F^{'}_{3,k} = \tau(F^{'}_{3,k}).$$

It is clear that

$$\tau(F_{3,k}) = 2\tau(F_{3,k-1}) + 3\tau(F^{'}_{3,k-1}) \text{ and } \tau(F^{'}_{3,k}) = 2\tau(F_{3,k-1}) - 3\tau(F^{'}_{3,k-1})$$

with initial conditions $(F_{3,1}) = 3$, $(F^{'}_{3,1}) = 1$. Thus we have

$$\begin{pmatrix} F_{3,k} \\ F^{'}_{3,k} \end{pmatrix} = A \begin{pmatrix} F_{3,k-1} \\ F^{'}_{3,k-1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} 2 & 3 \\ 2 & -3 \end{pmatrix}, \quad \begin{pmatrix} F_{3,k} \\ F'_{3,k} \end{pmatrix} = A \begin{pmatrix} F_{3,k-1} \\ F'_{3,k-1} \end{pmatrix} = \cdots = A^{k-1} \begin{pmatrix} F_{3,1} \\ F'_{3,1} \end{pmatrix}.$$

We compute A^{k-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 - \lambda - 12 = 0,\ \lambda_1 = -4 \text{ and } \lambda_2 = 3,\ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{3} \end{pmatrix};\ M^{-1} = \frac{\frac{1}{9}}{4} \begin{pmatrix} \frac{1}{7} & \frac{-3}{7} \\ \frac{6}{7} & \frac{3}{7} \end{pmatrix};\ A^{k-1} = MB^{k-1}M^{-1},$$

where

$$B^{k-1} = \begin{pmatrix} (-4)^{k-1} & 0 \\ 0 & (3)^{k-1} \end{pmatrix}.$$

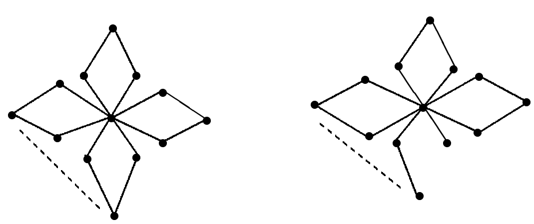
From which, we obtain

$$A^{k-1} = \begin{pmatrix} \frac{(-4)^{k-1}}{7} + \frac{2*(3)^k}{7} & \frac{-3*(-4)^{k-1}}{7} + \frac{3^k}{7} \\ \frac{-2*(-4)^{k-1}}{7} + \frac{2*3^{k-1}}{7} & \frac{6*(-4)^{k-1}}{7} + \frac{3^{k-1}}{7} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.7 *The number of spanning trees of friendship graph $F_{4,k}$ is $\tau(F_{4,k})$=4^k.*

Proof Consider a friendship graph $F'_{4,k}$ constructed from $F_{4,k}$ by deleting one edge. See Figure 8.



$F_{4,k}$ $F'_{4,k}$

Figure 8 Friendship graph $F_{4,k}$

We put

$$\tau(F_{4,k}) = 3\tau(F_{4,k-1}) + 4\tau(F^{'}_{4,k-1}) \text{ and } tau(F^{'}_{4,k}) = 2\tau(F_{4,k-1}) - 4\tau(F^{'}_{4,k-1})$$

with initial conditions $(F_{4,1}) = 4$, $(F^{'}_{4,1}) = 1$. Thus, we have

$$\begin{pmatrix} F_{4,k} \\ F^{'}_{4,k} \end{pmatrix} = A \begin{pmatrix} F_{4,k-1} \\ F^{'}_{4,k-1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} 3 & 4 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} F_{4,k} \\ F^{'}_{4,k} \end{pmatrix} = A \begin{pmatrix} F_{4,k-1} \\ F^{'}_{4,k-1} \end{pmatrix} = \cdots = A^{k-1} \begin{pmatrix} F_{4,1} \\ F^{'}_{4,1} \end{pmatrix}.$$

We compute A^{k-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 20 = 0, \ \lambda_1 = -5 \text{ and } \lambda_2 = 4, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that $A = MBM^{-1}$, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{4} \end{pmatrix}; \ M^{-1} = \frac{4}{9} \begin{pmatrix} \frac{1}{4} & -1 \\ 2 & 1 \end{pmatrix}; \ A^{k-1} = MB^{k-1}M^{-1},$$

where

$$B^{k-1} = \begin{pmatrix} (-5)^{k-1} & 0 \\ 0 & (4)^{k-1} \end{pmatrix}.$$

From which, we obtain

$$A^{k-1} = \begin{pmatrix} \frac{(-5)^{k-1}}{9} + \frac{2*(4)^k}{9} & \frac{-4*(-5)^{k-1}}{9} + \frac{4^k}{9} \\ \frac{-2*(-5)^{k-1}}{9} + \frac{2*4^{k-1}}{9} & \frac{8*(-5)^{k-1}}{9} + \frac{4^{k-1}}{9} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.8 *The number of spanning trees of friendship graph $F_{n,k}$ is $\tau(F_{n,k})$=n^k.*

Proof Consider a friendship graph $F^{'}_{n,k}$ constructed from $F_{n,k}$ by deleting one edge. See Figure 9.

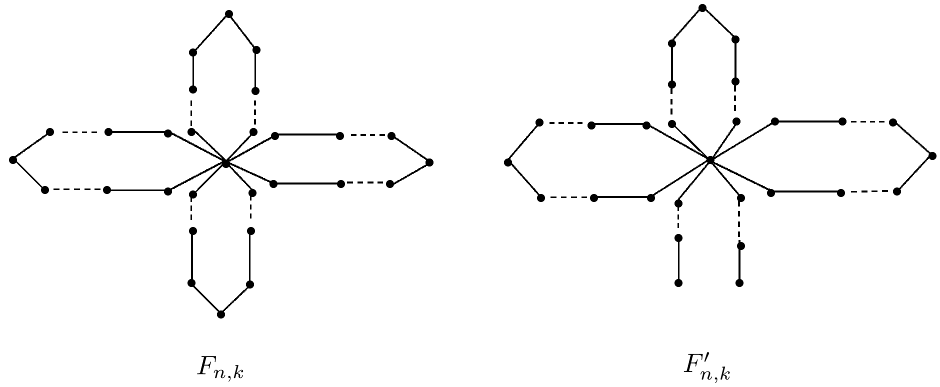


Figure 9 Friendship graph $F_{4,k}$

We put

$$F_{n,k} = \tau(F_{n,k}) \text{ and } F^{'}_{n,k} = \tau(F^{'}_{n,k}).$$

It is clear that

$$\tau(F_{n,k}) = (n-1)\tau(F_{n,k-1}) + n\tau(F^{'}_{n,k-1}) \text{ and} \tau(F^{'}_{n,k}) = 2\tau(F_{n,k-1}) - n\tau(F^{'}_{n,k-1})$$

with initial conditions $(F_{n,1}) = n$, $(F^{'}_{n,1}) = 1$. Thus, we have

$$\begin{pmatrix} F_{n,k} \\ F^{'}_{n,k} \end{pmatrix} = A \begin{pmatrix} F_{n,k-1} \\ F^{'}_{n,k-1} \end{pmatrix},$$

where

$$A = \begin{pmatrix} n-1 & n \\ 2 & -n \end{pmatrix}, \begin{pmatrix} F_{n,k} \\ F^{'}_{n,k} \end{pmatrix} = A \begin{pmatrix} F_{n,k-1} \\ F^{'}_{n,k-1} \end{pmatrix} = \cdots = A^{k-1} v \begin{pmatrix} n-1 & n \\ 2 & -n \end{pmatrix}.$$

We compute A^{k-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - n(n-1) = 0, \ \lambda_1 = -(n+1) \text{ and } \lambda_2 = n, \ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{n} \end{pmatrix}; \ M^{-1} = \frac{n}{2n+1} \begin{pmatrix} \frac{1}{n} & -1 \\ 2 & 1 \end{pmatrix}; \ A^{k-1} = MB^{k-1}M^{-1},$$

where

$$B^{k-1} = \begin{pmatrix} -(n+1)^{k-1} & 0 \\ 0 & (n)^{k-1} \end{pmatrix}.$$

From which, we obtain

$$A^{k-1} = \begin{pmatrix} \frac{(-n-1)^{k-1}}{2n+1} + \frac{2*(n)^k}{2n+1} & \frac{-n*(-n-1)^{k-1}}{2n+1} + \frac{n^k}{2n+1} \\ \frac{-2*(-n-1)^{k-1}}{2n+1} + \frac{2*n^{k-1}}{2n+1} & \frac{2n*(-n-1)^{k-1}}{2n+1} + \frac{n^{k-1}}{2n+1} \end{pmatrix}$$

and hence the result follows. □

Theorem 3.9 *The number of spanning trees of the subdivision of double triangular snake graph is* $\tau(S(2\Delta_n - snake)) = 32^n$.

Proof Consider a double triangular snake graph $S(2\Delta_n^{'} - snake)$ constructed from $S(2\Delta_n - snake)$ by deleting one edges. See Figure 10,

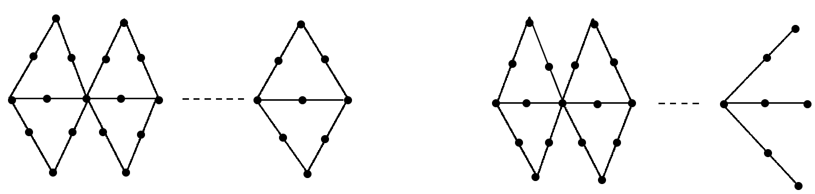


Figure 10 Friendship graph $F_{4,k}$

We put

$$S(2\Delta_n - snake) = \tau(S(2\Delta_n - snake)) \text{ and } S(2\Delta_n^{'} - snake) = \tau(S(2\Delta_n^{'} - snake)).$$

It is clear that

$$S(2\Delta_n - snake) = 31(S(2\Delta_{n-1} - snake)) + 32(S(2\Delta_2^{'} - snake))$$

and

$$S(2\Delta_2^{'} - snake) = 2(S(2\Delta_{n-1} - snake)) - 32(S(2\Delta_{n-1}^{'} - snake))$$

with initial conditions $S(2\Delta_1 - snake) = 32$, $S(2\Delta_1^{'} - snake) = 1$. Thus, we have

$$\begin{pmatrix} S(2\Delta_n - snake) \\ S(2\Delta_n^{'} - snake) \end{pmatrix} = A \begin{pmatrix} S(2\Delta_{n-1} - snake) \\ S(2\Delta_n^{'} - snake) \end{pmatrix},$$

where

$$A = \begin{pmatrix} 31 & 32 \\ 2 & -32 \end{pmatrix},$$

$$\begin{pmatrix} S(2\Delta_n - snake) \\ S(2\Delta_n^{'} - snake) \end{pmatrix} = A \begin{pmatrix} S(2\Delta_{n-1} - snake) \\ S(2\Delta_n^{'} - snake) \end{pmatrix} = \cdots = A^{n-1} \begin{pmatrix} S(2\Delta_1 - snake) \\ S(2\Delta_1^{'} - snake) \end{pmatrix}.$$

We compute A^{n-1} as follows:

$$\det(A - \lambda I_2) = \lambda^2 + \lambda - 1056 = 0,\ \lambda_1 = -33 \text{ and } \lambda_2 = 32,\ \lambda_1 \neq \lambda_2.$$

Then there is a matrix M is invertible such that A=MBM^{-1}, where

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and M is an invertible transformation matrix formed by eigenvectors

$$M = \begin{pmatrix} 1 & 1 \\ -2 & \frac{1}{32} \end{pmatrix};\ M^{-1} = \begin{pmatrix} \frac{1}{65} & \frac{-32}{65} \\ \\ \frac{64}{65} & \frac{32}{65} \end{pmatrix};\ A^{n-1} = MB^{n-1}M^{-1},$$

where

$$B^{n-1} = \begin{pmatrix} (32)^{n-1} & 0 \\ 0 & (-33)^{n-1} \end{pmatrix}.$$

From which, we obtain

$$A^{n-1} = \begin{pmatrix} \frac{(32)^{n-1}}{65} + \frac{64*(-33)^{n-1}}{65} & \frac{(-32)^n}{65} + \frac{-32*(-33)^{n-1}}{65} \\ \frac{-2*(32)^{n-1}}{65} + \frac{2*(-33)^{n-1}}{65} & \frac{2*(32)^n}{65} + \frac{(-33)^{n-1}}{65} \end{pmatrix}$$

and hence the result follows. □

§4. Spanning Tree Entropy

The entropy of spanning trees of a network or the asymptotic complexity is a quantitative measure of the number of spanning trees and it characterizes the network structure. We use this entropy to quantify the robustness of networks. The most robust network is the network that has the highest entropy. We can calculate its spanning tree entropy which is a finite number and a very interesting quantity characterizing the network structure, defined in [15, 16] as

$$Z(G) = \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|};$$

$$
\begin{aligned}
Z(\Delta_k - snake) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{3^n}{2n+1} = 0.5493;\\
Z(2\Delta_k - snake) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{\ln(8^n)}{3n+1} = 0.6931;\\
Z(4\Delta_k - snake) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{\ln(48^n)}{5n+1} = 0.7742;\\
Z(T(P_n)) &= \lim_{n\to\infty} \frac{\ln \frac{1}{\sqrt{5}}[(\frac{7+3\sqrt{5}}{2})^n - (\frac{7-3\sqrt{5}}{2})]}{2n-1} = \ln(\sqrt{\frac{7+3\sqrt{5}}{2}}) = 0.7650;\\
Z(nC_4 \odot 2P_n) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{\ln(4^n)}{4n} = \frac{\ln 4}{4} = 0.3466;\\
Z(F_3^k) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{k\to\infty} \frac{\ln(3^k)}{2k+1} = 0.5493;\\
Z(F_4^k) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{k\to\infty} \frac{\ln(4^k)}{3k+1} = 0.4621;\\
Z(F_n^k) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{k\to\infty} \frac{\ln(n^k)}{(n-1)k+1} = \ln \frac{(n)}{n-1};\\
Z(S(2\Delta_k - snake)) &= \lim_{V(G)\to\infty} \frac{\ln \tau(G)}{|V(G)|} = \lim_{n\to\infty} \frac{\ln(32^n)}{8n+1} = \ln \frac{(32)}{8} = 0.4332.
\end{aligned}
$$

§5. Conclusion

In this paper, we proposed the combinatorial method to facilitate the calculation of the number of spanning trees for complex networks. In particular, we derive the explicit formulas for the triangular snake ($\Delta_k - snake$), double triangular snake ($2\Delta_k - snake$), four triangular snake ($4\Delta_k - snake$), the total graph of path $P_n(T(P_n))$, the graph $nC_4 \odot 2P_n$, the generalized friendship graphs F_n^k and the subdivision of double triangular snake ($S(2\Delta_n - snake)$). Finally, we calculate their spanning trees entropy and we compare it between them.

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On Isomorphism Theorems of Neutrosophic R-Modules

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Abstract: This work deals with the isomorphism theorems of Neutrosophic R- modules. In this work, we assumed all rings to be commutative rings, we studied neutrosophic module [2], neutrosophic submodule, pseudo neutrosophic module and pseudo neutrosophic submodule. We considered the concept of Lagrange theorem [11] and discovered that in case of finite neutrosophic modules, the order of both neutrosophic submodules and pseudo neutrosophic submodules do not generally divide the order of neutrosophic module. The concept of cosets in general does not partition the neutrosophic module, even the pseudo neutrosophic submodules do not in general partition the neutrosophic module. This work also shows that the neutrosophic module is also a module and we considered the isomorphism theorem for modules [8] and extended it to Neutrosophic R modules and discovered that the isomorphism theorem for R modules also hold for neutrosophic R modules but where the order of a neutrosophic submodule divides the order of a neutrosophic module, the theorem may fail. We also stated and proved the isomorphism theorems of neutrosophic R-modules.

Key Words: Neutrosophy, module, neutrosophic R-module, neutrosophic group, ring, neutrosophic R-submodule, partition, coset, isomorphism.

AMS(2010): 03B60,12E05,97H40.

§1. Introduction

In 1980 [1], Florentin Smarandache introduced the notion of neutrosophy as a new branch of philosophy. Neutrosophy is the base of neutrosophic logic which is an extension of the fuzzy logic in which indeterminacy is included [2]. In the neutrosophic logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. Since the world is full of indeterminacy, several real world problems involving indeterminacy arising from law, medicine, sociology, psychology, politics, engineering, industry, economics, management and decision making, finance, stocks and share, meteorology, artificial intelligence, IT, communication etc can be solved by neutrosophic logic. Using Neutrosophic theory, Vasantha Kandasamy and Florentin Smaran-

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dache introduced the concept of neutrosophic algebraic structures [13]. Some of the neutrosophic algebraic structures introduced and studied include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and so on. Neutrosophic module was defined by Florentin and Vasantha in [11].

In section two of this work, we present some elementary properties of neutrosophic R-modules and section three is devoted to the study of the isomorphism theorems of neutrosophic R-modules.

§2. Some Elementary Properties of Neutrosophic R-module

We begin this section with the following definitions.

Definition 2.1([11]) *Let R be a commutative ring. An R-module is an (additive) abelian group M equipped with scalar multiplication $R \times M \to M$ such that the following axioms hold for all $m, n \in M$ and all $r, s, 1 \in R$:*

(1) $r(m+n) = rm + rn$;
(2) $(r+s)m = rm + sm$;
(3) $(rs)m = r(sm)$;
(4) $1 \cdot m = m$.

Remark 2.2 This definition also makes sense for non commutative rings R in which in this case, M is called a left R-module. If R is a commutative ring, then a neutrosophic left R-module $\langle M \cup I \rangle$ becomes a neutrosophic right R-module and we simply call $\langle M \cup I \rangle$ a neutrosophic R-module.

Remark 2.3 In the definition of neutrosophic R-module, we replaced the abelian group by a neutrosophic abelian group, all other factors remain the same.

Definition 2.4 *Let $\langle M \cup I \rangle$ be a neutrosophic module. Let H and K be any two neutrosophic submodules of $\langle M \cup I \rangle$, we say H and K are neutrosophic conjugates if we can find $x, y \in \langle M \cup I \rangle$ such that $xH = Ky$.*

We illustrate this with the following example.

Example 2.5 Let $R = \{0, 1, 2\}$ be the ring of integers and let $Z_6 \cup I = \{0, 1, 2, 3, 4, 5, I, 2I, 3I, 4I, 5I, 1+I, 1+2I, 1+3I, 1+4I, \cdots, 5+5I\}$ be a neutrosphic group under addition modulo 6. Then $R \times \langle Z_6 \cup I \rangle \to \langle Z_6 \cup I \rangle = \{0, 1, 2, 3, 4, 5, I, 2I, 3I, 4I, 1+I, \cdots, 5+5I\} = \langle Z_6 \cup I \rangle$. This is a neutrosophic module.

$H = \{0, 3, 3I, 3+3I\}$ is a neutosophic submodule of $\langle Z_6 \cup I \rangle K = \{0, 2, 4, 2+2I, 4+4I, 2I, 4I\}$ is a neutrosophic sub module of $\langle Z_6 \cup I \rangle$. For $2, 3$ in $\langle M \cup I \rangle$, we have $2H = 3K = \{0\}$, so H and K are neutrosophic conjugates. In case of neutrosophic conjugate, we do not demand $O(H) = O(K)$.

Definition 2.6 *Let $\langle M \cup I\rangle$ be a neutrosophic module and H a neutrosophic sub module of $\langle M \cup I\rangle$ for $n \in \langle M \cup I\rangle$, then $H + n = \{h + n/h \in H\}$ is called a coset of H in $\langle M \cup I\rangle$. As neutrosophic modules are formed from neutrosophic abelian groups, we do not talk about left and right cosets as the left and right cosets coincide.*

Example 2.7 Let $\langle M \cup I\rangle = \langle Z_2 \cup I\rangle = \{0, 1, I, 1 + I\}$ be a neutrosophic module and let $H = \{0, I\}$ be a neutrosophic sub module. The cosets of H are $H+0 = \{0, I\}$, $H+1 = \{1, 1+I\}$, $H + I = \{I, 0\}$ and $H + \{1 + I\} = \{1 + I, 1\}$.

Definition 2.8 *The cosets of a neutrosophic module do not generally partition the neutrosophic module.*

Example 2.9 Let $\langle M \cup I\rangle = \{0, 1, I, 1+I\}$ be a neutrosophic module and let $H = \{0, I\}$ be a neutrosophic sub module. Then the cosets are $H+0 = \{0, I\}$, $H+1 = \{1, 1+I\}$, $H+I = \{I, 0\}$ and $H + \{1 + I\} = \{1 + I, 1\}$.

Therefore the classes are $[0] = [I] = \{0, I\}$ and $[I] = [1 + I] = \{1, 1 + I\}$. Here, we see the cosets do not partition the neutrosophic module.

Example 2.10 Let $\langle Z_3 \cup I\rangle = \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$ be a neutrosophic module and let $P = \{0, 2, I, 2I\}$ be a neutrosophic submodule, then the cosets of P are $P+0 = \{0, 2, I, 2I\}$, $P + 1 = \{1, 0, I, 2I\}$, $P + 2 = \{2, 1, I + 2, 2 + 2I\}$, $P + I = \{I, I + 2, 2I, 0\}$, $P+2I = \{2I, 2+2I, 0, I\}$, $P+\{1+I\} = \{1+I, I, 1+2I, 1\}$, $P+\{1+2I\} = \{1+2I, 2I, 1, 1+I\}$,$P + \{2 + I\} = \{2 + I, 1 + I, 2 + 2I, 2\}$ and $P + \{2 + 2I\} = \{2 + 2I, 1 + 2I, 2, 2 + I\}$. The cosets partition the neutrosophic module. Therefore, we see that the cosets do not generally partition the neutrosophic module.

Theorem 2.1 *The neutrosophic module is indeed a module.*

Proof Suppose that the neutrosophic module $\langle M \cup I, +\rangle$ is an (additive) Abelian neutrosophic group. Every (additive) Abelian neutrosophic group is a group. We know that a module is an Abelian group over a ring. Therefore a neutrosophic module is a module. We illustrate with an example.

Consider $R = \langle Z_3\rangle = \{0, 1, 2\}$ is a ring and let $N(M) = \langle M \cup I\rangle = \langle Z_3 \cup I\rangle$, then $\langle Z_3 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$. Let $R\times N(M) = \{0, 1, 2\}\times\{0, 1, 2, I, 2I, 1+I, 1 + 2I, 2 + I, 2 + 2I\} = \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$.

Clearly, this is an additive Abelian neutrosophic group which is also a group. Also, an Abelian group over a ring gives a module, which is also a group. Therefore a neutrosophic module is a module. □

Definition 2.11 *Let $\langle M \cup I\rangle$ be a neutrosophic Abelian group and R a commutative ring. Let $R \times \langle M \cup I\rangle \to \langle M \cup I\rangle$ be a neutrosophic R-module. A proper subset P of $\langle M \cup I\rangle$ is said to be a neutrosophic submodule of the R-module if P is a non-empty set which is closed under addition and scalar multiplication.*

Definition 2.12([11]) *A pseudo neutrosophic group is a neutrosophic group which has no proper* *subset which is a group.*

Definition 2.13([11]) *Let $N(M) = \langle M \cup I \rangle$ be a neutrosophic module, a proper subset P of $N(M)$ which is a pseudo neutrosophic subgroup is called a pseudo neutrosophic submodule.*

Example 2.14 Let $R = \{0, 1\}$ be a ring and let $N(M) = \langle Z_4 \cup I \rangle = \{0, 1, 2, 3, I, 2I, 3I, 1 + I, 1 + 2I, 1 + 3I, 2 + I, 2 + 2I, 2 + 3I, 3 + I, 3 + 2I, 3 + 3I\}$, be a neutrosophic group. The neutrosophic R-module $R \times \langle Z_4 \cup I \rangle = \{0, 1\} \times \{Z_4 \cup I\} = \{0, 1, 2, 3, I, 2I, 3I, 1 + I, 1 + 2I, 1 + 3I, 2 + I, 2 + 2I, 2 + 3I, 3 + I, 3 + 2I, 3 + 3I\}$. Let $P = \{0, 3 + 3I\}$ be a pseudo neutrosophic subgroup of $\langle M \cup I \rangle$. Thus P is a pseudo neutrosophic submodule.

Theorem 2.2([8]) *The lagrange theorem for classical module states that the order of any submodule of a finite module is a factor of the order of the module.*

Definition 2.15 *The order of a neutrosophic submodule does not in general divide the order of the neutrosophic module.*

Example 2.16 Let us consider an example of Lagrange theorem on Neutrosophic module Let $\langle Z_3 \cup I \rangle = \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\}$ be a neutrosophic module and let $P = \{0, 2, I, 2I\}$ be a neutrosophic submodule, let us bear in mind that the order of the neutrosophic submodule need not divide the order of the neutrosophic module, then the cosets of P are $P+0 = \{0, 2, I, 2I\}$, $P+1 = \{1, 0, I, 2I\}$, $P+2 = \{2, 1, I+2, 2+2I\}$, $P+I = \{I, I+2, 2I, 0\}$, $P+2I = \{2I, 2+2I, 0, I\}$, $P+\{1+I\} = \{1+I, I, 1+2I, 1\}$, $P+\{1+2I\} = \{1+2I, 2I, 1, 1+I\}$, $P + \{2 + I\} = \{2 + I, 1 + I, 2 + 2I, 2\}$, $P + \{2 + 2I\} = \{2 + 2I, 1 + 2I, 2, 2 + I\}$.

The order of the neutrosophic module is nine and the order of the neutrosophic submodule is four, the number of elements in each coset is four as well. There are nine cosets. Therefore, we have $9 \neq 4.9$, four is not a factor of nine.

In general, the neutrosophic modules do not satisfy Lagrange theorem on finite modules.

§3. Isomorphism Theorems of Neutrosophic R-modules

Theorem 3.1 *Let $f : M \cup I \to N \cup I$ be a neutrosophic R module homomorphism. Then,*

(1) $\ker f$ *is a neutrosophic submodule of* $\langle N \cup I \rangle$*;*
(2) Imf *is a neutrosophic submodule of* $\langle N \cup I \rangle$*.*

Proof Let $\langle M \cup I \rangle \in \ker f$ and $r \in R$. Then $f\langle rm \rangle = rf\langle m \rangle = r\langle 0 \rangle = 0$. So $\langle rm \rangle \in \ker f$. Thus, $\ker f$ is a neutrosophic R submodule of $\langle M \cup I \rangle$.

In addition, suppose $m \in \langle M \cup I \rangle$ and $r \in R$, we have $rf\langle m \rangle = f\langle rm \rangle \in Imf$. So, Imf is a neutrosophic R submodule of $\langle N \cup I \rangle$. □

Example 3.1 Let $f : Z_4 \cup I \to Z_3 \cup I$ defined by $f : \{a\}_4 \to \{2a\}_3$ where $\{a\}_4$ means $a$mod 4 and $\{2a\}_3$ means $2a$mod 3. The kernel are $\{0, 3, 3I, 3+3I\}$ mapped to $Z_3 \cup I$ under the operation $a\text{mod } 4 \xrightarrow{f} 2a\text{mod } 3$. The image of $\langle Z_4 \cup I \rangle$ are $\{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$ which is the neutrosophic submodule of $\langle Z_3 \cup I \rangle$.

Corollary 3.2 *If M_1 and M_2 are R submodules of the neutrosophic R module $\langle M\cup I\rangle$ in Theorem* 3.1*, then*

$$M_1+M_2/M_1\cong M_2/M_1\cap M_2.$$

Proof This is a corollary to Theorem 3.1. Notice that $\langle M\cup I\rangle=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I,2+3I,3+I,3+2I,3+3I\}$, $M_2=\{0,1,2\}$, $M_1=\{0,1\}$, $M_2/M_1=\{0,1,2\}+\{0,1\}=\{\{0,1\},\{1,2\},\{2,0\}\}=\{0,1,2\}$, $M_1+M_2/M_1=\{0,1\}+\{0,1,2\}=\{\{0,1,2\},\{1,2,0\}\}=\{0,1,2\}$, $M_1\cap M_2=\{0,1\}$, $M_2/M_1\cap M_2=\{0,1,2\}/\{0,1\}=\{0,1,2\}+\{0,1\}=\{\{0,1\},\{1,2\},\{2,0\}\}=\{0,1,2\}$ and $M_1+M_2/M_1\cong M_2/M_1\cap M_2$. It is noteworthy to mention that Theorem 3.1 holds even when the submodules are not neutrosophic submodules but just submodules. □

Theorem 3.3 *If $\langle M_1\cup I\rangle\subseteq\langle M_2\cup I\rangle\subseteq\langle M\cup I\rangle$ are neutrosophic R-modules, then $M_2\cup I\ /M_1\cup I$ is a neutrosophic submodule of $\langle M\cup I\rangle\ /\langle M_1\cup I\rangle$ and*

$$\langle M\cup I\rangle/\langle M_1\cup I\rangle\Big/\langle M_2\cup I\rangle\ /\langle M_1\cup I\rangle\cong\langle M\cup I\rangle\ /\langle M_2\cup I\rangle.$$

Proof Define $\theta: M\cup I\ /M_1\cup I\to M\cup I\ /M_2\cup I$ by $\theta\langle x+M_1\cup I\rangle=x+M_2\cup I$. We have to check whether it is well defined. If we have two different representatives for $x+M_2\cup I$, it means $x+M_1\cup I=y+M_1\cup I$ which is the same as saying $x-y\in M_1\cup I$ but $\langle M_1\cup I\rangle\subset\langle M_2\cup I\rangle$, therefore, $x-y\in\langle M_2\cup I\rangle$, hence $x+M_2\cup I$ is the same as $y+M_2\cup I$. θ is well defined and θ is a neutrosophic R module homomorphism. Now, what is the kernel of θ? Cleraly,

$$\ker\theta=\{\bar{x}\in M\cup I\ /M_1\cup I: x+M_2\cup I=0+M_2\cup I\},$$

i.e.,

$$\ker\theta=\{x+M_1\cup I\in M\cup I\ /M_1\cup I: x\in M_2\cup I\}=M_2\cup I\ /M_1\cup I.$$

If you take any $x+M_2\cup I$ in $M\cup I/M_2\cup I$, look at $x+M_1\cup I$ and $\theta(x+M_1\cup I)=x+M_2\cup I$. Therefore, it is surjective. □

Example 3.2 Let $\langle M\cup I\rangle=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$, $\langle M_2\cup I\rangle=\{0,1,I,1+I\}$, $\langle M_1\cup I\rangle=\{0,I\}$, $M\cup I/M_1\cup I=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}+\{0,I\}=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$, $M_2\cup I/M_1\cup I=\{0,1,I,1+I\}+\{0,I\}=\{0,1,I,2I,1+I,1+2I\}$, $M_2\cup I/M_1\cup I$ is a neutrosophic submodule of $M\cup I/M_1\cup I$.

$$\begin{aligned}&\langle M\cup I\rangle/\langle M_1\cup I\rangle\Big/\langle M_2\cup I\rangle/\langle M_1\cup I\rangle\\&=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}/\{0,1,I,2I,1+I,1+2I\}\\&=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\},\end{aligned}$$

$$\begin{aligned} M\cup I/M_2\cup I &= \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}/\{0,1,I,1+I\} \\ &= \{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}, \end{aligned}$$

Whence,

$$\langle M\cup I\rangle/\langle M_1\cup I\rangle \Big/ \langle M_2\cup I\rangle/\langle M_1\cup I\rangle \cong \langle M\cup I\rangle/\langle M_2\cup I\rangle.$$

Corollary 3.4 *Let $M\cup I$ be a neutrosophic module. Let M_1 and M_2 be submodules of $\langle M\cup I\rangle$ and let $M_1\subseteq M_2\subseteq\langle M\cup I\rangle$, then $\langle M\cup I\rangle/M_1 \Big/ M_2/M_1\cong\langle M\cup I\rangle/M_2$.*

This is a corollary of Theorem 3.3.

Example 3.3 We consider the following example Let $\langle M\cup I\rangle=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$, $M_2=\{0,1,2\}$, $M_1=\{0,1\}$. Then $M_2\Big/M_1=\{0,1,2\}\Big/\{0,1\}=\{0,1,2\}+\{0,1\}=\{\{0,1\},\{1,2\},\{2,1\}\}=\{0,1,2\}$, $\langle M\cup I\rangle/M_1=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}+\{0,1\},=\{\{0,1\},\{1,2\},\{2,0\},\{I,1+I\},\{2I,2I+1\},\{1+I,2+I\},\{1+2I,2+2I\},\{2+I,I\},\{2+2I,2I\}\}=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$ and $\langle M\cup I\rangle/M_1\Big/M_2/M_1=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}\Big/\{0,1,2\}=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}+\{0,1,2\}=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$, $\langle M\cup I\rangle/M_2=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}/\{0,1,2\}=\{0,1,2,I,2I,1+I,1+I,1+2I,2+I,2+2I\}+\{0,1,2\}=\{\{0,1,2\},\{1,2,0\},\{2,0,1\},\{I,1+I,2+I\},\{2I,2I+1,2I+2\},\{1+I,2+I,I\},\{1+2I,2+2I,2I\},\{2+I,I,1+I\},\{2+2I,2I,1+2I\}\}=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$. Whence,

$$\langle M\cup I\rangle/M_1\Big/M_2/M_1\cong\langle M\cup I\rangle/M_2.$$

Theorem 3.5 *Let $f:\langle M\cup I\rangle\to N\cup I$ be a neutrosophic R module homomorphism, then $Imf\cong M\cup I\big/\ker f$*

Proof Define $\theta: M\cup I\big/\ker f\to Imf$, $\theta\bar{x}=f(x)$. We want to prove that it is well-defined since there could be many representatives of $\bar{x}$. If $\bar{x}=\bar{y}\to x-y\in\ker f\to f(x-y)=0$. Since f is a neutrosophic module homomorphism $f(x)=f(y)\to\theta(\bar{x})=\theta(\bar{y})\to\theta$ is well defined. θ is a homomorphism since f is a homomorphism for all $\bar{x},\bar{y}\in M\cup I\big/\ker f$. $\theta(\bar{x}+\bar{y})=\theta(\overline{x+y})=f(x+y)=f(x)+f(y)=\theta(\bar{x})=\theta(\bar{y})$ for all $r\in R$ and $\bar{x}\in M\cup I\big/\ker f$. By definition of scalar multiplication on $M\cup I\big/\ker f$, $\theta(r.\bar{x})=\theta(\overline{rx})=f(rx)=rf(x)=r\theta(\bar{x})$, θ is a neutrosophic R module homomorphism. Now, let $y\in Imf\to x\in M\cup I$ such that $f(x)=y\to\theta(\bar{x})=y$ this implies θ is surjective. If $\theta(\bar{x})=0$, then $f(x)=0\to x\in\ker f\to\bar{x}=0$. This implies θ is injective and it implies θ is an isomorphism. □

Example 3.4 Let $f:Z_3\cup I\to Z_3\cup I$ be defined by f: $[a]_3\to[4a]_3$ where $[a]_3$ means $a$mod 3 and $[4a]_3$ means $4a$mod 3. The image of $f=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$, $\ker f=\{0\}$, $M\cup I\big/\ker f=\{0,1,2,I,2I,1+I,1+2\}/\{0\}=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$. $Imf=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\}$. Hence, $Imf\cong M\cup I\big/\ker f$.

Theorem 3.6 *If $\langle M_1 \cup I\rangle$ and $\langle M_2 \cup I\rangle$ are neutrosophic R submodules of $\langle M \cup I\rangle$, then* $\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle \cong \langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle$.

Proof Define $\theta : \langle M_2 \cup I\rangle \to \langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle$ by $\theta(x) = \bar{x}$. Note that we do not have to worry about well definiteness. There is no representative issue, every element has its own existence $\theta(x+y) = \overline{x+y} = \bar{x} + \bar{y} = \theta(x) + \theta(y)$. $\ker\theta = \{x \in M_2 \cup I : \bar{x} = 0\} = \{x \in M_2 \cup I : x \in M_1 \cup I\} = \langle M_2 \cup I\rangle \cap \langle M_1 \cup I\rangle$. It is injective. $\overline{x+y}$ or $\bar{x} + M_1 \cup I$ is a coset, $y \in M_1 \cup I$ and $x \in M_2 \cup I$, $\overline{x+y} = (x+y) + M_1 = (x + M_1 \cup I) + (y + M_1 \cup I)$, $y + M_1 \cup I = 0$ (in neutrosophic quotient module) $= x + M_1 \cup I = \bar{x} \to \theta(x) = \overline{x+y} \to \theta$ is surjective. □

The next example is an illustration of Theorem 3.6.

Example 3.5 Let $M \cup I = \{0, 1, 2, I, 2I, 3I, 1+I, 1+2I, 1+3I, 2+I, 2+2I, 2+3I, 3+I, 3+2I, 3+3I\}$, $M_2 \cup I = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$, $M_1 \cup I = \{0, 2, 2I, 2+2I\}$. We show that $\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle \cong \langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle$. Notice that $\langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\} \big/ \{0, 2, 2I, 2+2I\}$, $\langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$ and $\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle = \{0, 2, 2I, 2+2I\} + \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$. Therefore, $\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$, $\langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle = \{0, 2, 2I, 2+2I\}$, $\langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\} \big/ \{0, 2, 2I, 2+2I\} = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\} + \{0, 2, 2I, 2+2I\}$, $\langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle = \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$. Therefore, we know that

$$\langle M_1 \cup I\rangle + \langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle \cong \langle M_2 \cup I\rangle \big/ \langle M_1 \cup I\rangle \cap \langle M_2 \cup I\rangle.$$

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On $1RJ$ Moves in Cartesian Product Graphs

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Abstract: Let G be an undirected graph with n vertices in which a robot is placed at a vertex say v, and a hole at vertex u and in all other $(n-2)$ vertices are obstacles. We refer to this assignment of robot and obstacles as a configuration C_u^v of G. Suppose we have a one player game in which an obstacle can be slide to an adjacent vertex if it is empty i.e. if it has a hole and the robot can move from vertex u to an empty vertex v if $d(u,v) \leq 2$ where $d(u,v)$ is the distance between vertex u and v. The goal is to take the robot to a particular destination vertex by using a sequence of mRJ moves of the robot for $m = 1$ and simple moves of the robot as well as obstacles as the case may be. The results of this paper, which is an extension of the work [Motion planning in Cartesian product graphs, *Discussiones Mathematicae Graph Theory* 34 (2014) 207-221] gives the minimum number of moves required for the motion planning problem in Cartesian product of two graphs each having girth six or more.

Key Words: Robot motion in a graph, Cartesian product of graphs, 1RJ move.

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§1. Introduction

Given a graph G, with a robot placed at one of it's vertices and movable obstacles at some other vertices. Assuming that we are allowed to slide the obstacles to an adjacent vertex if it is empty and the robot can move from vertex u to an empty vertex v if $d(u,v) \leq 2$. Let $u, v \in V(G)$, and suppose that the robot is at v and the hole at u and obstacles at other vertices we refer to this as a configuration C_u^v. The number of edges in a path is called its length. The girth of a graph G, denoted by $g(G)$, is the length of a shortest cycle contained in the graph. A simple move is referred to as moving an obstacle or the robot to an adjacent empty vertex. A graph G is k-reachable if there exists a k-configuration such that the robot can reach any vertex of the graph in a finite number of simple moves. Let u and v be two vertices having a robot and a hole, respectively. Further let $[u, d_1, d_2, d_3, \cdots, d_m, v]$ be a path having obstacles at the vertices $d_1, d_2, d_3, \cdots, d_m$. An mRJ move from the vertex u to the

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empty vertex v is defined as movement of the robot to an empty vertex v by jumping over m obstacles $d_1, d_2, d_3, \cdots, d_m$. Although throughout this paper we would only consider the case where $m = 1$ (i.e. $1RJ$moves only) and simple moves of the robot as well as obstacles as the case may be. Let $[u, d_1, d_2, d_3, \cdots, d_m, v]$ be a path in a graph such that u and v have a hole and a robot respectively, and $d_1, d_2, d_3, ...d_m$ have obstacles. An mRJ move from vertex u to v is denoted by $v \overset{r}{\underset{1}{\leftarrow}} u$. Similarly we use $v \overset{r}{\leftarrow} u$ and $v \overset{o}{\leftarrow} u$ to denote respectively, the robot move and the obstacle move from vertex u to an adjacent vertex v where $u, v \in E(G)$. The objective is to find a minimum sequence of moves that takes the robot from (source) vertex u to a (destination) vertex v. The vertex set and edge set of a graph G is denoted by $V(G)$ and $E(G)$ respectively. We refer to $|V(G)|$ and $|E(G)|$ as the order and the size of G, respectively. A graph G is said to be non-trivial if $|V(G)| > 1$. In this article, we restrict our study to simple finite non-trivial graphs. For two vertices $u, v \in V(G)$, let $d_G(u, v)$ denotes the distance between u and v in G. We use $d(u, v)$ instead of $d_G(u, v)$ to represent the distance between the vertices u and v in the graph G. We denote the path, the cycle and the complete graph on n vertices by P_n, C_n and K_n respectively.

The motion planning problem in graph was proposed by Papadimitriou et al. [9] where it was shown that with arbitrary number of holes, the decision version of such problem is NP-complete and that the problem is complex even when it is restricted to planar graphs. They also gave time algorithm for trees. The result in [9] was improve in [3]. Robot motion planning on graphs (RMPG) is a graph with a robot placed at one of its vertices and movable obstacle at some of the other vertices while generalization of RMPG problem is the Multiple robot motion planning in graph (MRMPG) whereby we have k different robots with respective destinations. Ellips and Azadeh [6] studied MRMPG on trees and introduced the concept of minimal solvable trees. Auletta et al. [2] also studied the feasibility of MRMPG problem on trees and gave an algorithm that, on input of two arrangements of k robots on a tree of order n, decides in time $O(n)$ whether the two arrangements are reachable from one another. Parberry [8] worked on grid of order n^2 with multiple robots while Deb and Kapoor [5, 4] generalized and apply the technique used in [8] to calculate the minimum number of moves for the motion planning problem for the cartesian product of two given graphs. A recent work is by the present authors [1] whereby they gave the minimum number of moves required for the motion planning problem in some lexicographic product graphs.

The MRMPG problem of grid graph of order n^2 with $n^2 - 1$ robots is known as $(n^2 - 1)$-puzzle. The objective of $(n^2 - 1)$-puzzle is to verify whether two given configurations of the grid graph of order n^2 are reachable from each other and if they are reachable then to provide a sequence of minimum number of moves that takes one configuration to the other. The $(n^2 - 1)$-puzzle have been studied extensively in [7, 8, 10, 11].

Our work was motivated by Deb and Kapoor [4] whereby they gave minimum sequence of moves required for the motion planning problem in Cartesian product of two graphs having girth 6 or more. They also proved that the path traced by the robot coincides with a shortest path in case of Cartesian product of graphs. In this paper we extend the work in [4] by considering the case in which the robot can jump one obstacle at a move (or time) and thus we give the minimum number of moves required for the motion planning problem in Cartesian product of two graphs say G and H.

Definition 1.1 *The Cartesian product $G\Box H$ of two graphs G and H is a graph with vertex set $V(G)\times V(H)$ in which (u_i,v_j) and (u_p,v_q) are adjacent if one of the following condition holds:*

(1) $u_i=u_p$ *and* $\{v_j,v_q\}\in E(H)$;

(2) $v_j=v_q$ *and* $\{u_i,u_p\}\in E(G)$.

The graphs G and H are known as the factors of $G\Box H$. Now onwards G and H are simple graphs with $V(G)=\{1,2,3,...m\}$ unless otherwise stated.

Suppose we are dealing with r-copies of a graph G and we are denoting these r-copies of G by G^i, where $i=\{1,2,3,\cdots,r\}$. Then for each vertex $u\in V(G)$ we denote the corresponding vertex in the i^{th} copy G^i by u^i. The girth of a graph G, denoted by $g(G)$ is the length of the shortest cycle contained in graph G. Now we refer to the work of Deb and Kapoor [4] for a good pre-knowledge of this work.

§2. Local Moves of the Hole

Definition 2.1 *An edge u^i,v^j in $G\Box H$ is said to be a G-edge (respectively, H-edge) if $u=v$ and $\{i,j\}\in E(G)$ (respectively, if $i=j$ and $\{u,v\}\in E(H)$).*

Definition 2.2 *For any path P in $G\Box H$, by G-length and H-length of P we mean the number of G-edges and H-edges in P, respectively. We use $l_G(P)$ and $l_H(P)$ to denote the G-length and H-length of P, respectively.*

Definition 2.3 *Given two graphs G and H. For any $u^i,v^j\in V(G\Box H)$, we call the distance between u and v in H to be the H-distance between u^i and v^j in $G\Box H$, and the distance between i and j in G to be the G-distance between u^i and v^j in $G\Box H$. We use $d_G(u^i,v^j)$ and $d_H(u^i,v^j)$ to denote the G-distance and H-distance between u^i and v^j in $G\Box H$, respectively.*

Now, we use $d(u,v)$ instead of $d_G(u,v)$ to represent the distance between u and v in G.

Proposition 2.1 *Given two graphs G and H. Let $\{i,j\},\{j,k\},\{k,l\},\{l,m\}\in E(G)$ and $u\in V(H)$. Then (i) $d_{G\Box H-u^k}(u^i,u^l)=min\{d_{G-k}(i,l),5\}$ and (ii) $d_{G\Box H-u^k}(u^i,u^m)=min\{d_{G-k}(i,m),6\}$.*

Proof (i) Let Q be a shortest path connecting u^i and u^l in $G\Box H-u^k$. We need to show that $|Q|=min\{d_{G-k}(i,l),5\}$. We consider the following cases.

Case 1. $V(Q)\cap V(G^i)=V(Q)$ which implies that $V(Q)\subseteq V(G^i-u^k)$ and so $|Q|=d_{G-k}(i,l)$.

Case 2. $V(Q)\cap V(G^i)\neq V(Q)$. We claim that $|Q|=5$. From the Cartesian product of graphs, notice that for any $u,v\in E(H)$, the vertices u^x, v^y are adjacent in $G\Box H$ if and only if $x=y$. Therefore if we are moving away from the copy G^i using the path Q we must also come back to the copy G^i. Hence G-distance covered along the path Q must be at least two. Also $d(i,l)=3$, otherwise i,k or $j,l\in E(G)$ and this implies $|Q|=2$, which is not possible.

So G-distance traveled along the path Q must be at least three. Hence $|Q| \geq 5$. Now for any $u, v \in E(H)$ the path $[u^i, u^j, v^j, v^k, v^l, u^l]$ connects u^i and u^l in $G\Box H$.

(ii) Since we have established that $d_{G\Box H-u^k}(u^i, u^l) = min\{d_{G-k}(i,l), 5\}$ and $k, l \in E(G)$ we then conclude that $d_{G\Box H-u^k}(u^i, u^m) = min\{d_{G-k}(i,m), 6\}$. This proves our claim. □

Corollary 2.2 *Given two graphs G and H. Let $\{i,j\},\{j,k\},\{k,l\} \in E(G)$ and $u \in V(H)$. Then starting from the configuration $C^{u^i}_{u^k}$ of $G\Box H$ we require at least $min\{1 + d_{G-k}(i,l), 6\}$ moves to move the robot to u^l. In particular, if $g(G) \geq 6$, then we need at least 6 moves to move the robot to u^l.*

Proof Notice that, $\{u^i, u^j\}, \{u^j, u^k\}, \{u^k, u^l\} \in E(G\Box H)$. In order to move the robot from u^k to u^l, before it, the hole must be moved from u^i to u^l. This would take $min\{d_{G-k}(i,l), 5\}$ moves. Since $d_{G\Box H-u^k}(u^i, u^l) = min\{d_{G-k}(i,l), 5\}$. Then the simple move $u^l \overset{r}{\leftarrow} u^k$ takes the robot from u^k to u^l. Hence the result follows.

If $g(H) \geq 6$ then $d_{G-k}(i,l) \geq 5$ and so $min\{1+d_{G-k}(i,l), 6\} = 6$. Thus, at least six moves are required to take the robot from u^k to u^l. □

Corollary 2.3 *Given two graphs G and H. Let $\{i,j\},\{j,k\},\{k,l\},\{l,m\} \in E(G)$ and $u \in V(H)$. Then starting from the configuration $C^{u^i}_{u^k}$ of $G\Box H$ we require at least $min\{1+d_{G-k}(i,m), 7\}$ moves to move the robot to u^m. In particular, if $g(G) \geq 6$, then we need at least 7 moves to move the robot to u^m.*

Proof Just as in Corollary 2.2, in order to move the robot from u^k to u^m, before it, the hole must be moved from u^i to u^m. This would take $min\{d_{G-k}(i,m), 6\}$ moves. Since $d_{G\Box H-u^k}(u^i, u^m) = min\{d_{G-k}(i,m), 6\}$. Then the $1RJ$ move $u^m \underset{1}{\overset{r}{\leftarrow}} u^k$ takes the robot from u^k to u^m. Hence the result follows.

If $g(H) \geq 6$ then $d_{G-k}(i,m) \geq 6$ and so $min\{1 + d_{G-k}(i,m), 6\} = 6$. Therefore at least seven moves are required to take the robot from u^k to u^m. □

As Cartesian product of graphs is commutative, so the proof of the following proposition can be drawn in the same line as that of Proposition 2.1.

Proposition 2.4 *Given two non-trivial graphs G and H. Let $\{u,v\},\{v,w\},\{w,x\},\{x,y\} \in E(H)$ and $i \in V(G)$. Then (i) $d_{G\Box H-v^i}(u^i, x^i) = min\{d_{H-v}(u,x), 5\}$ and (ii) $d_{G\Box H-v^i}(u^i, y^i) = min\{d_{H-v}(u,y), 6\}$.*

Corollary 2.5 *Given two graphs G and H. Let $\{u,v\},\{v,w\},\{w,x\} \in E(H)$ and $i \in V(G)$. Then starting from the configuration $C^{u^i}_{v^i}$ of $G\Box H$ we require at least $min\{1 + d_{H-v}(u,x), 6\}$ moves to move the robot to x^i. In particular, if $g(G) \geq 6$, then we need at least 6 moves to move the robot to x^i.*

Corollary 2.6 *Given two graphs G and H. Let $\{u,v\},\{v,w\},\{w,x\},\{x,y\} \in E(H)$ and $i \in V(G)$. Then starting from the configuration $C^{u^i}_{v^i}$ of $G\Box H$ we require at least $min\{1 + d_{H-v}(u,y), 7\}$ moves to move the robot to y^i. In particular, if $g(G) \geq 6$, then we need at least 7 moves to move the robot to y^i.*

The theorem below gives the advantage of a $1RJ$ move of the robot over a simple move.

Theorem 2.7 *Given two graphs G and H. Let $\{u, v\}, \{v, w\} \in E(H)$ and $i \in V(G)$. Then starting from the configuration $C_{v^i}^{u^i}$ of $G\square H$ we require at least 3 moves to move the robot to w^i.*

Proof Since $\{u^i, v^i\} \in E(G\square H)$. First we would require the move $u^i \xleftarrow{r} v^i$ which would take the robot from v^i to u^i. In order to move the robot to w^i, before it, the hole must be moved from v^i to w^i. This take $d_{G\square H}(v^i, w^i) = 1$. Then the move $w^i \xleftarrow[1]{r} u^i$ takes the robot from u^i to w^i. Hence the result follows. □

Proposition 2.8 *Given two graphs G and H. Let $\{i, j\}, \{j, k\} \in E(G)$ and $\{u, v\} \in E(H)$. Then, starting from the configuration $C_{u^j}^{u^i}$ of $G\square H$, we need at least four moves to move the robot to v^k.*

Proof To move the robot from u^j to v^k before it, the hole must be moved from u^i to v^k. This takes three steps (or moves), since $d_{G\square H - u^j}(u^i, v^k) = 3$. Then the move $v^k \xleftarrow[1]{r} u^i$ takes the robot to v^k. Hence the result follows. □

As Cartesian product of graphs is commutative, so the proof of the following proposition can be drawn in the same line as that of Proposition 2.8.

Proposition 2.9 *Given two graphs G and H. Let $\{i, j\} \in E(G)$ and $\{u, v\} \in E(H)$. Then, starting from the configuration $C_{v^i}^{u^i}$ of $G\square H$, we need at least four moves to move the robot to v^k.*

Definition 2.4 *A robot move in $G\square H$ is called a G-move (respectively,H-move) if the edge along which the move took place is a G-edge(respectively,H-edge).*

Definition 2.5 *Let T be a sequence of moves that take the robot from u^i to v^j in $G\square H$. An H-move (respectively, G-move) in T of the robot is said to be a secondary H-move (respectively, G-move) if it is preceded by an H-move (respectively, G-move). An H-move (respectively, G-move) in T of the robot is said to be a primary H-move (respectively, G-move) if it is preceded by a G-move (respectively, H-move). Also the edge corresponding to a primary G-move (respectively, H-move) in T is said to be a primary G-edge (respectively, H-edge).*

Definition 2.6 *A simple move $G\square H$ is said to be a G-simple move (respectively, H-simple move) if the edge along which the simple move took place is a G-edge(respectively, H-edge). Also, a $1RJ$-move in $G\square H$ is said to be a G-$1RJ$-move (respectively, H-$1RJ$-move) if the edge along which the $1RJ$-move took place is a G-edge(respectively, H-edge).*

Definition 2.7 *Let T be a sequence of moves that take the robot from u^i to v^j in $G\square H$. A G-simple move (respectively, H-simple move) in T of the robot preceded by a G-$1RJ$-move (respectively, H-$1RJ$-move) is said to be a G-primary simple move (respectively, H-primary simple move). A G-$1RJ$-move (respectively, H-$1RJ$-move) in T of the robot preceded by another G-$1RJ$-move (respectively, H-$1RJ$-move) is said to be a G-secondary $1RJ$-move (respectively,*

H-secondary $1RJ$-move).

In view of the above definitions we summarize the results of this section in terms of the following remark.

Remark 2.10 Given two graphs G and H, each having girth six or more.

(1) In view of Corollaries 2.2 and 2.5, to perform each G-primary simple (or H-primary simple) move of the robot we require at least 6 moves.

(2) In view of Corollaries 2.3 and 2.6, to perform each G-secondary (or H-secondary) $1RJ$ move of the robot we require at least 7 moves.

(3) In view of Propositions 2.8 and 2.9, to perform each G-primary (or H-primary) $1RJ$-move of the robot we require at least 4 moves.

(4) In a minimum sequence of moves, the robot should take as many primary moves as possible.

§3. Trace of the Robot

To begin this section, we now state the following lemma without proof. This lemma gives the least (or minimum) number of H-moves and G-moves a sequence can have in $G\Box H$.

Lemma 3.1 *Let G and H be two graphs such that $i, j \in V(G)$ and $u, v \in V(H)$. Further, let T be a sequence of moves that take the robot from u^i to v^j in $G\Box H$. Then the minimum number of H-moves (respectively,G-moves) of the robot in T is*

(1) $\frac{p}{2}$ *(respectively,$\frac{k}{2}$)moves, if p is even (respectively, k is even);*

(2) —it $\frac{p+1}{2}$ (respectively,$\frac{k+1}{2}$) moves, if p is odd (respectively, k is odd). Where $d_G(i, j) = k$ and $d_H(u, v) = p$.

Lemma 3.2 *Consider the graphs G and H each having girth six or more. Let $i, j \in V(G)$ and $\{u, v\}, \{u, w\} \in E(H)$. Then each robot move in a minimum sequence of moves that takes $C_{v^i}^{u^i}$ to $C_{w^j}^{u^j}$ in $G\Box H$ is a $G-1RJ$-move. Also such a minimum sequence involves exactly $\frac{k}{2}$ number of $G-1RJ$-moves of the robot and $\frac{7k}{2}$ moves in total, where $k = d(i, j) \geq 1$ and k is even.*

Proof Let T be a sequence of moves that takes $C_{v^i}^{u^i}$ to $C_{w^j}^{u^j}$ in $G\Box H$. First assume that the number of robot moves in T is z and each of these robot moves in T is a $G-1RJ$-move. By Proposition 2.9, we need at least four moves to accomplish the first $G-1RJ$-move of the robot. Notice that each remaining $z-1$ robot moves in T is a G-secondary $1RJ$-move. So by Remark 2.10, we need minimum of $7(z-1)$ G-secondary $1RJ$-moves. Now, if $u^j \overset{r}{\underset{1}{\leftarrow}} u^q$ is the z^{th} robot move in T, it will leave the graph $G\Box H$ with the configuration $C_{u^q}^{u^j}$. Since $d_{G\Box H-u^j}(u^q, w^j) = 3$, so we need minimum of three more move to take the hole from u^q to w^j. Hence T involves minimum $7z$ moves. Notice that, the expression $7z$ takes the minimum value when z is minimum. Next, let $d(i, j) = k$ and $[i = i_0, i_2, i_4, \cdots, i_k]$ be a path of length $\frac{k}{2}$

connecting i and j in G. Then $[u^i = u^{i_0}, u^{i_2}, u^{i_4}, \cdots, u^{i_k} = u^j]$is a path of length $\frac{k}{2}$ in $G\Box H$ joining u^i to u^j. So the sequence of moves

$$v^i \overset{o^*}{\longleftarrow} u^{i_2} \underset{1}{\overset{r}{\leftarrow}} u^{i_0} \overset{o^*}{\longleftarrow} u^{i_4} \underset{1}{\overset{r}{\leftarrow}} u^{i_2} \overset{o^*}{\longleftarrow} u^{i_6} \underset{1}{\overset{r}{\leftarrow}} u^{i_4} \overset{o^*}{\longleftarrow} u^{i_8} ... \underset{1}{\overset{r}{\leftarrow}} u^{i_k} \underset{1}{\overset{r}{\leftarrow}} u^{i_{k-2}} \overset{o^*}{\longleftarrow} w^j$$

takes the robot from u^i to u^j along this path and each move in this sequence is a $G-1RJ$-move. Also it involves exactly $\frac{k}{2}$ number of $G-1RJ$-moves of the robot. Therefore by Lemma 3.1, a minimum sequence of moves in T (not involving H-moves of the robot) that takes the configuration $C_{v^i}^{u^i}$ to $C_{w^j}^{u^j}$ involves exactly $7\frac{k}{2}$ moves.

Finally, assume that the sequence T involves H-moves also. If the sequence involves H-moves then we would require at least two H-moves. The first H-move of the robot in T would take it away from copy G^u and the other would bring it back to G^u. Note here that T would still require additional $\frac{k}{2}$ $G-1RJ$ moves. Thus we conclude that T is not minimum. This completes the proof. □

Lemma 3.3 *Consider the graphs G and H each having girth six or more. Let $i, j \in V(G)$ and $\{u, v\}, \{u, w\} \in E(H)$. Then each robot move in a minimum sequence of moves that takes $C_{v^i}^{u^i}$ to $C_{w^j}^{u^j}$ in $G\Box H$ is a G-move. Also such a minimum sequence involves exactly $\frac{k+1}{2}$ number of G moves of the robot and $\frac{7k+3}{2}$ moves in total, where $k = d(i, j) \geq 1$ and k is odd.*

Proof Let T be a sequence of moves that takes $C_{v^i}^{u^i}$ to $C_{w^j}^{u^j}$ in $G\Box H$. First assume that the number of robot moves in T in z and each of these robot moves in T is a G-move. By Proposition 2.9, we need at least four moves to accomplish the first $G-1RJ$-move of the robot. Notice that each succeeding $z-2$ robot moves in T is a G-secondary $1RJ$-move. So by Remark 2.10, we need minimum of $7(z-2)$ G-secondary $1RJ$ moves. Clearly, the z^{th} move of the robot is a G-primary simple move. Thus by Remark 2.10, we require at least six moves to perform this G-primary simple move. Now, if $u^j \overset{r}{\leftarrow} u^s$ is the z^{th} robot move in T, it will leave the graph $G\Box H$ with the configuration $C_{u^s}^{u^j}$. Since $d_{G\Box H - u^j}(u^s, w^j) = 2$, so we need minimum of two more moves to take the hole from u^s to w^j. Hence T involves minimum $7z-2$ moves. The expression $7z-2$ takes the minimum value when z is minimum. Next, let $d(i, j) = k$ and $[i = i_0, i_2, i_4, ..., i_{k-1}]$ be a path of length $\frac{k+1}{2}$ connecting i and j in G. Then $[u^i = u^{i_0}, u^{i_2}, u^{i_4}, \cdots, u^{i_{k-1}} = u^j]$ is a path of length $\frac{k+1}{2}$ in $G\Box H$ joining u^i to u^j. So the sequence of moves

$$v^i \overset{o^*}{\longleftarrow} u^{i_2} \underset{1}{\overset{r}{\leftarrow}} u^{i_0} \overset{o^*}{\longleftarrow} u^{i_4} \underset{1}{\overset{r}{\leftarrow}} u^{i_2} \overset{o^*}{\longleftarrow} u^{i_6} \underset{1}{\overset{r}{\leftarrow}} u^{i_4} \overset{o^*}{\longleftarrow} u^{i_8} ... u^{i_{k-1}} \overset{r}{\leftarrow} u^{i_{k-3}} \overset{o^*}{\longleftarrow} w^j$$

takes the robot from u^i to u^j along this path and each move in this sequence is a G-move. Also it involves exactly $\frac{k+1}{2}$ number of G-moves of the robot. Therefore by Lemma 3.1, a minimum sequence of moves in T (not involving H-moves of the robot) that takes the configuration $C_{v^i}^{u^i}$ to $C_{w^j}^{u^j}$ involves exactly $\frac{7k+3}{2}$ moves.This completes the proof. □

Since the Cartesian product of graphs is commutative, so the proof of the next two lemmas can be drawn in the same as line as that of Lemmas 3.2 and 3.3.

Lemma 3.4 *Consider the graphs G and H each having girth six or more. Let $\{i,j\},\{i,k\} in E(G)$ and $u,v \in V(H)$. Then each robot move in a minimum sequence of moves that takes $C_{u^j}^{u^i}$ to $C_{v^k}^{v^i}$ in $G\Box H$ is an $H-1rJ$move. Also such a minimum sequence involves exactly $\frac{p}{2}$ number of $H-1rJ$moves of the robot and $\frac{7p}{2}$ moves in total, where $p=d(u,v)\geq 1$ and p is even.*

Lemma 3.5 *Consider the graphs G and H each having girth six or more. Let $\{i,j\},\{i,k\}\in E(G)$ and $u,v\in V(H)$. Then each robot move in a minimum sequence of moves that takes $C_{u^j}^{u^i}$ to $C_{v^k}^{v^i}$ in $G\Box H$ is a H-move. Also such a minimum sequence involves exactly $\frac{p+1}{2}$ number of Hmoves of the robot and $\frac{7p+3}{2}$ moves in total, where $p=d(u,v)\geq 1$ and p is odd.*

In view of the results obtained in this section we have the following theorem.

Theorem 3.6 *Given two connected graphs G and H each having girth six or more. Consider the configuration $C_{v^i}^{u^i}$ of $G\Box H$.Then to move the robot from*

(1) G^u to G^v we require at least $(p-1)+\frac{7}{2}(p-2)$ moves or $(p-1)+\frac{7}{2}(p-3)+6$ moves according as p is even or odd respectively;

(2) H^i to H^j we require at least $(k+2)+\frac{7}{2}(k-2)$ moves or $(k+2)+\frac{7}{2}(k-3)+6$ moves according as k is even or odd respectively.

§4. Minimum Number of Moves

Definition 4.1 *Given a path P connecting u^i and v^j in $G\Box H$. By a minimal sequence of moves with trace P we mean a sequence with minimum number of moves that takes the robot from u^i to v^j along the path P in $G\Box H$.*

Definition 4.2 *By a minimal u^iv^j-path in $G\Box H$ we mean a u^iv^j-path P such that the G-edges in P induces a ij-path in G and the H-edges in P induces a uv-path in H.*

Definition 4.3 *Give two graphs G, H and a path P in $G\Box H$. By a primary edge in P we mean an H-edge that is preceded by a G-edge or a G-edge that is preceded by an H-edge. By a secondary edge in P we mean an H-edge that is preceded by an H-edge or a G-edge that is preceded by a G-edge.*

In view of the definitions above we now state the following lemma without proof. This lemma gives the maximum number of primary edges that a path can have in $G\Box H$ with given H-length and G-length respectively.

Lemma 4.1 *Given two graphs G and H. Let P be a path connecting u^i and v^j in $G\Box H$ such that $l_G(P)=a$ and $l_H(P)=b$. Then, the maximum number of primary edges P can have when*

(1) *$a = b$ is $a-1$, if a and b are both even;*

(2) *$a = b$ is a, if a and b are both odd;*

(3) *$a > b$ is $b-1$, if a is odd and b is even and the first edge in P is an H-edge;*

(4) *$a > b$ is b or $b+1$, if a and b are positive integers with opposite parity and the first edge in P is a G-edge according as $a = b+1$ or otherwise respectively;*

(5) *$a > b$ is b, if a is even and b is odd and the first edge in P is an H-edge;*

(6) *$a > b$ is $b-1$, if both a and b is even and the first edge in P is an H-edge;*

(7) *$a > b$ is b, if both a and b is even (odd) and the first edge in P is a G-edge (H-edge);*

(8) *$a > b$ is $b+1$, if both a and b is odd and the first edge in P is a G-edge;*

(9) *$a < b$ is a, if a is even and b is a positive integer and the first edge in P is an H-edge;*

(10) *$a < b$ is $a-1$, if a is even and b is a positive integer and the first edge in P is a G-edge;*

(11) *$a < b$ is a or $a+1$, if a is odd and b is even and the first edge in P is an H-edge according as $a = b-1$ or otherwise respectively;*

(12) *$a < b$ is a, if a is odd and b is a positive integer and the first edge in P is a G-edge;*

(13) *$a < b$ is $a+1$, if both a and b is odd and the first edge in P is an H-edge.*

In order to prove our result we need the following.

Remark 4.1 (See [5]) *Given two graphs G and H each having girth six or more. To perform each primary G-move (or H-move) of the robot we require at least* 3 *moves.*

Proposition 4.3 *Given two graphs G and H. Let $\{i,j\},\{j,k\},\{k,l\},\{l,m\} \in E(G)$ and $\{u,v\},\{v,w\},\{w,x\},\{x,y\} \in E(H)$. Then, starting from the configuration*

(i) $C_{u^k}^{w^k}$ of $G\Box H$, we need at least five moves to move the robot to w^m;

(ii) $C_{w^k}^{w^m}$ of $G\Box H$, we need at least five moves to move the robot to y^m;

(iii) $C_{u^i}^{u^k}$ of $G\Box H$, we need at least four moves to move the robot to v^k;

(iv) $C_{u^k}^{w^k}$ of $G\Box H$, we need at least four moves to move the robot to w^l.

Proof (i) To move the robot from w^k to w^m, before it, the hole must be moved from u^k to w^m. This takes $d_{G\Box H-u^k}(u^k, w^m) = 4$. Then the $1RJ$-move $w^m \overset{r}{\underset{1}{\leftarrow}} w^k$ takes the robot to w^m. Hence the result follows.

(ii) As Cartesian product of graphs is commutative, the proof can be drawn in the same line as (i) above.

(iii) To move the robot from u^k to v^k, before it, the hole must be moved from u^i to v^k. This takes $d_{G\Box H-u^k}(u^i, v^k) = 3$. Then the $1RJ$-move $v^k \overset{r}{\underset{1}{\leftarrow}} u^k$ takes the robot to v^k. Hence the result follows.

(iv) As Cartesian product of graphs is commutative, the proof can be drawn in the same line as (iii) above. □

Definition 4.4 *Let T be a sequence of moves that takes the robot from u^i to v^j in $G\Box H$. A $G-1RJ$-move (respectively, $H-1RJ$-move) that is preceded by an $H-1RJ$-move (respectively, $G-1RJ$-move)is said to be a primary $G-1RJ$-move (respectively,primary $H-1RJ$-move).*

Also, a G simple move (respectively, H simple move)preceded by a $H-1RJ$-move (respectively, $G-1RJ$-move) is said to be a strong-primary G-move (respectively,H-move).

In view of the above definitions we have this remark.

Remark 4.4 Given two graphs each having girth six or more, in view of Proposition 4.4, to perform each

(1) Primary $G-1RJ$-move (respectively, primary $H-1RJ$-move)of the robot we require at least 5 moves;

(2) Strong-primary or weak-secondary G-move (respectively,H-move) of the robot we require at least 4 moves.

Theorem 4.5 *Given two graphs G and H each having girth six or more. Consider the configuration $C_{v^i}^{u^i}$ of $G\square H$. For some $j \in G\square H$, let P be a minimal path connecting u^i and v^j in $G\square H$. Let T be a minimal sequence with trace P. Where $l_G(P) = a$ and $l_H(P) = b$. Suppose that the first move of the robot is an H-move then T involves at least*

(i) $k-2m+\frac{7}{2}(a+b)-8$ moves if a and b are both even;
(ii) $k-2m-3n-q-4r+\frac{7}{2}(a+b)-1$ moves if a and b are both odd;
(iii) $k-2m-3n-q+\frac{7}{2}(a+b)-\frac{9}{2}$ moves if otherwise.
Furthermore, suppose that the first move of the robot is a G-move then T involves at least

(i) $k-2m+\frac{7}{2}(a+b)-4$ moves if a and b are both even;
(ii) $k-2m-3n-q-4r+\frac{7}{2}(a+b)+3$ moves if a and b are both odd;
(iii) $k-2m-3n-q+\frac{7}{2}(a+b)-\frac{1}{2}$ moves if otherwise,

where m is the number of primary $G-1RJ$ (or primary $H-1RJ$)- moves, n is the number of strong-primary G (or H)-move, q is the number of G-primary (or H-primary) simple moves and r is the number of primary moves of the robot in T and $k = d(u,v)$.

Proof We consider cases following.

Case 1. The first edge in P is an H-edge.

Subcase 1.1 Since T is minimal so it involves exactly $\frac{a+b}{2}$ robot moves. In this case the first robot move is an $H-1RJ$-move, say $w^i \overset{r}{\underset{1}{\leftarrow}} u^i$. In order to realize this move, before it, the hole must move from v^i to w^i. Therefore, we require $k-1$ moves to realize the first robot move, since $d_{G\square H-u^i}(v^i, w^i) = k-2$ ($k-2$ moves to bring the hole at w^i plus the robot move $w^i \overset{r}{\underset{1}{\leftarrow}} u^i$. Since m is the number of primary G(or H)-$1RJ$-moves in T, so the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b}{2}-m-1$. Hence, by Remark 2.10, the number of moves in T is $k-1+5m+\frac{7}{2}(a+b-2m-2)$, i.e., $k-2m+\frac{7}{2}(a+b)-8$ moves.

Subcase 1.2 Since T is minimal so it involves exactly $\frac{a+b+2}{2}$ robot moves. Just as in Subcase 1.1 above, we require $k-1$ moves to realize the first robot move. By definition of m, n, q and r in T the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b+2}{2}-m-n-q-r-1$. Hence, by Remarks 2.10, 4.2 and 4.4 the number of moves in T is $k-1+5m+4n+6q+3r+7(\frac{a+b+2}{2}-m-n-q-r-1)$, i.e., $k-2m-3n-q-4r+\frac{7}{2}(a+b)-1$ moves.

Subcase 1.3 Since T is minimal so it involves exactly $\frac{a+b+1}{2}$ robot moves. Similarly as in Subcase 1.1 above, we require $k-1$ moves to realize the first robot move. By definition of m, n and q in T the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b+1}{2} - m - n - q - 1$. Hence, by Remarks 2.10 and 4.4 the number of moves in T is $k - 1 + 5m + 4n + 6q + 7(\frac{a+b+1}{2} - m - n - q - 1)$, i.e., $k - 2m - 3n - q + \frac{7}{2}(a+b) - \frac{9}{2}$ moves.

Case 2. The first edge in P is a G-edge.

Subcase 2.1 Since T is minimal so it involves exactly $\frac{a+b}{2}$ robot moves. In this case the first robot move is a $G-1RJ$-move. Let this move be $u^k \xleftarrow[1]{r} u^i$. So to perform this move we must first move the hole from v^i to u^k. Clearly $d_{G\Box H-u^i}(v^i, u^k) = k+2$. Therefore, we require $k+3$ moves to perform the first robot move ($k+2$ moves to bring the hole at u^k plus the robot move $u^k \xleftarrow[1]{r} u^i$. Since m is the number of primary G(or H)-$1RJ$-moves in T, so the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b}{2} - m - 1$. Hence, by Remark 2.10, the number of moves in T is $k + 3 + 5m + \frac{7}{2}(a + b - 2m - 2)$, i.e., $k - 2m + \frac{7}{2}(a+b) - 4$moves.

Subcase 2.2 Since T is minimal so it involves exactly $\frac{a+b+2}{2}$ robot moves. Just as in Subcase 2.1 above, we require $k+3$ moves to realize the first robot move. By definition of m, n, q and r in T the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b+2}{2} - m - n - q - r - 1$. Hence, by Remarks 2.10, 4.2 and 4.4 the number of moves in T is $k + 3 + 5m + 4n + 6q + 3r + 7(\frac{a+b+2}{2} - m - n - q - r - 1)$, i.e., $k - 2m - 3n - q - 4r + \frac{7}{2}(a+b) + 3$ moves.

Subcase 2.3 Since T is minimal so it involves exactly $\frac{a+b+1}{2}$ robot moves. Similarly as in Subcase 2.1 above, we require $k+3$ moves to realize the first robot move. By definition of m, n and q in T the number of G(or H)-secondary $1RJ$ robot moves in T is $\frac{a+b+1}{2} - m - n - q - 1$. Hence, by Remark 2.10 and 4.4 the number of moves in T is $k + 3 + 5m + 4n + 6q + 7(\frac{a+b+1}{2} - m - n - q - 1)$, i.e., $k - 2m - 3n - q + \frac{7}{2}(a+b) - \frac{1}{2}$ moves.

This completes the proof. □

§5. Conclusion and Future Work

In this article, we have been able to investigate the minimum number of moves required for the motion planning of Cartesian product of graphs whereby the robot/object can jump an obstacle. It is clear that the path traced by the robot moves of such motions is less than the minimal path in particular for some cases it is half of the minimal path and off course this path is along the shortest path.

As future work, we plan to investigate this kind of motion in other product graphs, in particular strong and modular product.

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Characteristic Properties of the Indicatrix Under a Kropina Change of Finsler Metric

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Abstract: The theory of $\beta-$change in Finsler geometry was first introduced by C. Shibata in [13]. In this paper, we study the behaviour of Indicatrices under a special $\beta-$change, known as Kropina change of Finsler metric.

Key Words: Indicatrix, $\beta-$change, Kropina change, curvature tensor, conformal flatness.

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§1. Introduction

The notion of $\beta-$change in Finsler spaces was introduced by C. Shibata in [13]. Since then so many results have been obtained using this theory. In [1], S. H. Abed generalized the theory of $\beta-$change and introduced a new change, called conformal $\beta-$change. In differential geometry, the theory of indicatrices has been very interesting topic for geometers from all over the world for both pure mathematical and applied reasons. The theory of indicatries and its properties have been studied by so many authors (([7], [10], $\cdots$, [14]) In the present paper we study the behavior of the indicatrices given by a particular $\beta-$change, known as Kropina change.

This paper is organized as follows:

In the second section, we discuss the basic definitions and examples of some special Finsler spaces. In Section 3, we consider the Indicatrices given by a $\beta-$change, called Kropina change and study its properties in detail. The terminologies and notations are referred to Matsumoto's monograph [11] in this paper.

§2. Preliminaries

Let M be an $n-$ dimensional smooth manifold, T_xM, the tangent space at $x \in M$, and TM the tangent bundle, the disjoint union of tangent spaces, i.e.,

$$TM := \bigsqcup_{x\in M} T_xM.$$

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The elements of TM are denoted by (x, y), where $x = (x^i) \in M$ and $y \in T_xM$, called supporting element. The slit tangent bundle TM_0 is defined as $TM \setminus \{0\}$.

A Finsler metric on a smooth manifold M is a function $F : TM \longrightarrow [0, \infty)$ satisfying the following properties:

(1) F is smooth on TM_0,

(2) F is positively 1−homogeneous on the fibers of tangent bundle TM and

(3) the hession of F^2 with elements $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positively defined on TM.

A smooth manifold M equipped with the Finsler metric F is called Finsler manifold and the corresponding space, denoted by $F^n = (M, F)$ is called a Finsler space. F is called fundamental function and g_{ij} is called fundamental metric tensor of the Finsler space F^n. The normalized supporting element ℓ_i, angular metric tensor h_{ij}, and the metric tensor g_{ij} of F^n are defined respectively as:

$$\ell_i = \frac{\partial F}{\partial y^i}, \quad h_{ij} = \frac{\partial^2 F}{\partial y^i \partial y^j} \ \ \& \ g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{2.1}$$

Finsler metrics were introduced in order to generalize the Riemannian ones in the sense that metric should not depend only on the point, but also on the direction. In Finsler geometry, (α, β) metrics, introduced in [12], form a very important and rich class of Finsler metrics which can be expressed in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1−form and ϕ is a positive smooth function on the domain of definition. The notable (α, β) metrics are Randers metric, Kropina metric, generalized Kropina metric, Z. Shen's square metric and Matsumoto metric. If $\phi(s) = 1 + s$, we get $F = \alpha + \beta$, called Randers metric. In particular, when $\phi(s) = \frac{1}{s}$, we get $F = \frac{\alpha^2}{\beta}$, called Kropina metric. Kropina metrics were induced by V. K. Kropina [8]. Kropina metrics seem to be among the simplest non-trivial Finsler metrics with many interesting applications in physics, electron optics with a magnatic field etc.([2], [3], [6]). Now we give some definitions and results that have been used in the next section.

Definition 2.1 *A Finsler space $F^n = (M, F)(n > 2)$ is called $P2-$like, if there exist a covariant vector field P_i such that the hv curvature tensor P_{hijk} of F^n can be written in the form*

$$P_{hijk} = P_h C_{ijk} - P_i C_{hjk}.$$

Let the Finsler space $F^n(n > 2)$ is $P2-$like. Then we have the result following.

Theorem 2.1([9]) *For a $P2-$like Finsler space $F^n = (M, F)(n > 2)$, the hv curvature tensor P_{hijk} vanishes, or the $v-$ curvature tensor S_{hijk} of F^n vanishes.*

Definition 2.2 *A Finsler space $F^n = (M, F)(n > 3)$ is called $R3-$like, if the third curvature tensor R_{hijk} of Cartan is expressible in the form $R_{hijk} = g_{hj}L_{ik} + g_{ik}L_{hj} - g_{hk}L_{ij} - g_{ij}L_{hk}$, where $L_{ik} = \frac{1}{n-2}\left(R_{ik} - \frac{r}{2}g_{ik}\right)$, $R_{hj} = R^m_{h\ jm}$ and $r = \frac{1}{n-1}R^m_m$.*

For the $(v)hv-$torsion tensor P_{hij} and the $(h)hv-$ torsion tensor C_{hij}, we define

$$^*P_{hij} = P_{hij} - \lambda C_{hij},$$

where the scalar λ is homogeneous of degree one with respect to y^i and is given by $\dfrac{P_i C^i}{C_j C^j}$ for $C_j \neq 0$.

Definition 2.3 *A Finsler space $F^n = (M, F)(n > 2)$ is called a $^*P-$Finsler space, if the torsion tensor $^*P_{hij} = 0$.*

Definition 2.4 *A Finsler space $F^n = (M, F)$ is called a Landsberg space, if the $(v)hv-$torsion tensor $P_{hij} = 0$.*

Definition 2.5([5]) *A non-Riemannian Finsler space $F^n = (M, F)(n > 4)$ is called $S4-$like, if the $v-$curvature tensor S_{hijk} is written in the form*

$$L^2 S_{hijk} = h_{hj} M_{ik} + h_{ik} M_{hj} - h_{hk} M_{ij} - h_{ij} M_{hk},$$

where M_{ij} is symmetric and indicatory tensor given by $M_{ij} = \dfrac{1}{n-3}\left[S_{ij} - \dfrac{S h_{ij}}{2(n-2)}\right]$.

Theorem 2.2([15]) *Let $F^n = (M, F)(n > 4)$ be a $R3-$like (non-Landsberg) $^*P-$Finsler space. Then F^n is $S4-$like.*

Theorem 2.3([15]) *An $R3-$like Landsberg space $F^n = (M, F)(n > 3)$ is a Finsler space satisfying $S_{hijk} = 0$, or a Riemannian space of constant curvature.*

After some calculation, we find the following result.

Theorem 2.4([15]) *If a Finsler space $F^n = (M, F)(n > 4)$ is $S4-$like, then the Finsler space $\bar{F}^n = (M, \bar{F})$, obtained from F^n by a Kropina change, is also $S4-$like.*

§3. Indicatrices Given by a Kropina Change

Let $F^n = (M, F)$ be a Finsler space. For any $x \in M$, the tangent space T_xM is regarded as an $n-$ dimensional Riemannian space with the fundamental tensor $g_{ij}(x, y)$, where $x = (x^i)$ is fixed. In terms of the Cartan connection $C\Gamma$ of F^n, components C^i_{jk} of the $(h)hv-$torsion tensor are christoffel symbols of T_xM and the $v-$curvature tensor S^i_{hjk} is the Riemannian curvature tensor of T_xM. The indicatrix I_x at a point x is a hypersurface of the Riemannian space T_xM which is defined by the equation $F(x, y) = 1$, where x is fixed. Consequently, I_x is regarded as an $(n-1)-$dimensional Riemannian space.

Now, we consider a special $\beta-$change, called Kropina change, defined by

$$\bar{F} = \frac{F^2}{\beta} = f(F, \beta), \tag{3.1}$$

where $\beta = b_i(x)y^i$ is a non-zero $1-$form on M.

Differentiation of (3.1) with respect t0 F and β gives us the following relations:

$$\begin{aligned} f_1 &= \frac{\partial \bar{F}}{\partial F} = \frac{2F}{\beta},\ f_2 = \frac{\partial \bar{F}}{\partial \beta} = -\frac{F^2}{\beta^2}, \\ f_{11} &= \frac{\partial^2 \bar{F}}{\partial F^2} = \frac{2}{\beta},\ f_{22} = \frac{\partial^2 \bar{F}}{\partial \beta^2} = \frac{2F^2}{\beta^3},\ f_{12} = \frac{\partial^2 \bar{F}}{\partial \beta \partial F} = -\frac{2F}{\beta^2} \end{aligned} \tag{3.2}$$

$$\bar{F} = f_1 + f_2\beta = \frac{F^2}{\beta},\ Ff_{12} + \beta f_{22} = 0,\ Ff_{11} + \beta f_{12} = 0. \tag{3.3}$$

$$p = ff_1/F = \frac{2F^2}{\beta^2},\ q = ff_2 = -\frac{F^4}{\beta^3},\ q_{0} = ff_{22} = \frac{2F^4}{\beta^4}. \tag{3.4}$$

Further, $\bar{\ell}_i = \bar{F}_{y^i}$ gives

$$\bar{\ell}_i = f_1\ell_i + f_2 b_i = -\frac{F^2}{\beta^2}\left(b_i - \frac{2\beta}{F^2}y_i\right) \tag{3.5}$$

$$\bar{h}_{ij} = \bar{F}\dot{\partial}_i\dot{\partial}_j\bar{F} \ \text{ gives}$$

$$\bar{h}_{ij} = ph_{ij} + q_{0}m_im_j = \frac{2F^2}{\beta^2}h_{ij} + \frac{2F^4}{\beta^4}m_im_j, \quad m_i = b_i - \frac{\beta}{F^2}y_i. \tag{3.6}$$

Furthermore, we find

$$\begin{aligned} p_{0} = q_{0} + {f_2}^2 &= \frac{3F^4}{\beta^4},\ q_{-1} = ff_{12}/F = -\frac{2F^2}{\beta^3},\ p_{-1} = q_{-1} + pf_2/f = -\frac{4F^2}{\beta^3}, \\ q_{-2} &= \frac{f\left(f_{11} - f_1/F\right)}{F^2} = 0,\ p_{-2} = q_{-2} + p^2/f^2 = \frac{4}{\beta^2}. \end{aligned} \tag{3.7}$$

Notice that $\bar{g}_{ij} = \frac{1}{2}\left(\bar{F}^2\right)_{y^iy^j}$ gives

$$\begin{aligned} \bar{g}_{ij} &= pg_{ij} + p_{0}b_ib_j + p_{-1}\left(b_iy_j + b_jy_i\right) + p_{-2}y_iy_j \\ &= \frac{2F^2}{\beta^2}g_{ij} + \frac{3F^4}{\beta^4}b_ib_j - \frac{4F^2}{\beta^3}\left(b_iy_j + b_jy_i\right) + \frac{4}{\beta^2}y_iy_j. \end{aligned} \tag{3.8}$$

By the Kropina change $F_{ij} = \dfrac{h_{ij}}{F}$ is invariant under certain conditions, where $h_{ij} = g_{ij} - \ell_i\ell_j$ is the angular metric tensor.

From now on, we shall call a tensor which is invariant under the Kropina change a K-invariant tensor. For the v-curvature tensor S_{hijk}, putting

$$LS^*_{hijk} = S_{hijk} + \frac{1}{n-3}\,\mathfrak{U}_{jk}\ \left\{h_{ij}S_{hk} + h_{hk}S_{ij} - Sh_{ij}h_{hk}/\left(n-2\right)\right\}, \tag{3.9}$$

we find that S^*_{hijk} is K-invariant under certain restrictions, where we use the notation $\mathfrak{U}_{jk}$ to denote the interchange of indices j, k and subtraction.

For a $S4-$like Finsler space, we have the following result.

Theorem 3.1([14]) *Let $F^n = (M, F)(n > 4)$ be a $S4-$like Finsler space. Then the indicatrix I_x is conformally flat.*

Also, we can easily prove the result following.

Theorem 3.2) *A non-Riemannian Finsler space $F^n = (M, F)(n > 4)$ is $S4-$like if and only if the K-invariant tensor S^*_{hijk} vanishes.*

From equation (3.1), Theorems 2.1, 2.4, 3.1 and 3.2, we find the following result.

Theorem 3.3 *For a $P2-$like Finsler space $F^n = (M, F)(n > 4)$, the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a Kropina change is conformally flat provided that $P_{hijk} \neq 0$.*

From Theorems 2.2, 2.4 and 3.1, we immediately find the following theorem.

Theorem 3.4 *Let $F^n = (M, F)(n > 4)$, be a $R3-$ like (non-Landsberg) $^*P-$ Finsler space. Then the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a Kropina change, is conformally flat.*

From equation (3.1), Theorems 2.3, 2.4, 3.1 and 3.2, we immediately find the next result.

Theorem 3.5 *Let $F^n = (M, F)(n > 4)$, be an $R3-$ like Landsberg space. If F^n is not a Riemannian space of constant curvature, then the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a Kropina change, is conformally flat.*

Theorem 3.6([4]) *Let $F^n = (M, F)(n > 2)$, be a $^*P-$ Finsler space. If the $hv-$curvature tensor P_{hijk} is symmetric in $j\&k$, then $P_{hijk} = 0$, or the $v-$curvature tensor $S_{hijk} = 0$.*

Therefore, by, equation (3.1) and Theorems 2.1, 2.43.13.23.6 we immediately get the following conclusion.

Theorem 3.7 *Let $F^n = (M, F)(n > 2)$, be a $^*P-$ Finsler space. If the $hv-$curvature tensor P_{hijk} is symmetric in $j\&k$, then the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a Kropina change, is conformally flat provided that $P_{hijk} \neq 0$.*

According to the $\beta-$ change of a Finsler metric, the $v-$curvature tensor S^i_{hjk} changes as follows ([13]):

$$\bar{S}^i_{hjk} = S^i_{hjk} + \mathfrak{U}_{jk}\ \left(C^i_{mk}V^m_{hj} - C^m_{hk}V^i_{mj} - V^i_{mk}V^m_{hj}\right), \quad V^h_{ij} = C^h_{ij} - \bar{C}^h_{ij}. \tag{3.10}$$

In case of Kropina change, from (3.2), we get a conclusion following.

Theorem 3.8 *Let $S^i_{hjk} = \mathfrak{U}_{jk}\ \left(C^m_{hk}V^i_{mj} + V^i_{mk}V^m_{hj} - C^i_{mk}V^m_{hj}\right)$. Then we get $\bar{S}^i_{hjk} = 0$, where*

$$\begin{aligned} V^h_{ij} &= Q^h\left(p\ C_{imj}b^m - p_{-1}m_im_j\right) \\ &\quad -\frac{1}{2}\left(\frac{m^h}{p} - \nu\ Q^h\right)\left(p_{02}\ m_im_j + p_{-1}h_{ij}\right) - \frac{p_{-1}}{2p}\left(h^h_im_j + h^h_jm_i\right) \end{aligned}$$

and

$$
\begin{aligned}
Q^h &= s_0 b^h + s_{-1} y^h, \ s_0 = \frac{\beta^2}{2b^2 \ F^2}, \ s_{-1} = -\frac{\beta^3}{b^2 \ F^4}, \ p = \frac{2F^2}{\beta^2}, \\
p_{-1} &= -\frac{4F^2}{\beta^3}, \ \nu = b^2 - \frac{\beta^2}{F^2}, \ p_{02} = \frac{\partial p_0}{\partial \beta} = -\frac{12F^4}{\beta^5}.
\end{aligned} \tag{3.11}
$$

In [6], we have known the following result.

Theorem 3.9 *Let $F^n = (M, F)(n > 2)$, be a Finsler space. Then its $v-$curvature tensor S_{hijk} vanishes at a point x, if and only if the indicatrix I_x is of constant curvature* 1.

By Theorems (3.8) and (3.9), we get

Theorem 3.10 *Let $S^i_{hjk} = \mathfrak{U}_{jk}\left(C^m_{hk} V^i_{mj} + V^i_{mk} V^m_{hj} - C^i_{mk} V^m_{hj}\right)$. Then the indicarix $\bar{I}_x$ of $\bar{F}^n$, obtained from $F^n(n > 2)$ by a Kropina change, is of constant curvature* 1.

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Neighbourly Pseudo Irregular Fuzzy Graphs

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Abstract: In this paper, neighbourly pseudo irregular fuzzy graphs and neighbourly pseudo totally irregular fuzzy graphs are defined. Comparative study between neighbourly pseudo irregular fuzzy graph and neighbourly pseudo totally irregular fuzzy graph is done. A necessary and sufficient conditions under which they are equivalent are provided. Also, few properties of neighbourly pseudo irregular fuzzy graphs and neighbourly pseudo totally irregular fuzzy graphs are discussed.

Key Words: 2-degree, pseudo degree of a vertex in a graph, neighbourly pseudo irregular fuzzy graph, neighbourly pseudo totally irregular fuzzy graph.

AMS(2010): 05C12, 03E72, 05C72.

§1. Introduction

In this paper, we consider only finite, simple, connected graphs. We denote the vertex set and the edge set of a graph G by $V(G)$ and $E(G)$ respectively. The degree of a vertex v is the number of edges incident at v, and it is denoted by $d(v)$. A graph G is regular if all its vertices have the same degree. The 2-degree of v is the sum of the degrees of the vertices adjacent to v and it is denoted by $t(v)$. A pseudo degree of a vertex v is denoted by $d_a(v)$ and defined as $\dfrac{t(v)}{d^*_G(v)}$, where $d^*_G(v)$ is the number of edges incident at v.

A graph is called pseudo-regular if every vertex of G has equal pseudo (average) degree [3]. The notion of fuzzy sets was introduced by Zadeh as a way of representing uncertainly and vagueness [18]. The first definition of fuzzy graph was introduced by Haufmann in 1973. In 1975, A. Rosenfeld introduced the concept of fuzzy graphs [8]. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas. Irregular fuzzy graphs plays a central role in combinatorics and theoretical computer science.

§2. Review of Literature

Nagoorgani and Radha introduced the concept of degree, total degree, regular fuzzy graphs in

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2008 [7]. Nagoorgani and Latha introduced the concept of irregular fuzzy graphs, neighbourly irregular fuzzy graphs and highly irregular fuzzy graphs in 2008 [6]. Mathew, Sunitha and Anjali introduced some connectivity concepts in bipolar fuzzy graphs [16]. Akram and Dudek introduced the notions of regular bipolar fuzzy graphs [1] and also introduced intuitionistic fuzzy graphs [2]. Samanta and Pal introduced the concept of irregular bipolar fuzzy graphs in [14].

N.R.S. Maheswari and C. Sekar introduced (2,k)-regular fuzzy graphs and totally (2,k)-regular fuzzy graphs [9]. N.R.S. Maheswari and C. Sekar introduced m-neighbourly irregular fuzzy graphs [13]. N.R.S. Maheswari and C. Sekar introduced neighbourly edge irregular fuzzy graphs [10]. N.R.S. Maheswariand C. Sekar introduced neighbourly edge irregular bipolar fuzzy graphs [11]. Pal and Hossein introduced irregular interval-valued fuzzy graphs [17]. Sunitha and Mathew discussed about growth of fuzzy graph theory [15]. N.R.S. Maheswari and C. Sekar introduced pseudo degree and total pseudo degree in fuzzy graphs and pseudo regular fuzzy graphs and discussed some of its properties [12]. These motivate us to introduce neighbourly pseudo irregular fuzzy graphs, and neighbourly pseudo totally irregular fuzzy graphs discussed some of its properties.

§3. Preliminaries

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G denoted by $V(G)$ and $E(G)$ respectively [2].

Definition 3.1([5]) *A fuzzy graph $G:(\sigma,\mu)$ is a pair of functions (σ,μ), where $\sigma:V\rightarrow[0,1]$ is a fuzzy subset of a non-empty set V and $\mu:V\times V\rightarrow[0,1]$ is a symmetric fuzzy relation on σ such that for all u,v in V, the relation $\mu(uv)\leq\sigma(u)\Lambda\sigma(v)$ is satisfied. A fuzzy graph G is called complete fuzzy graph if the relation $\mu(uv)=\sigma(u)\Lambda\sigma(v)$ is satisfied.*

Definition 3.2([4]) *Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. The degree of a vertex u in G is denoted by $d(u)$ and is defined as $d(u)=\sum\mu(uv)$, for all $uv\in E$.*

Definition 3.3([6]) *Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. The total degree of a vertex u in G is denoted by $td(uv)$ and is defined as $td(uv)=d(u)+\sigma(u)$ for all $u\in V$.*

Definition 3.4([1]) *Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. Then G is said to be an irregular fuzzy graph, if there is a vertex which is adjacent to vertices with distinct degrees.*

Definition 3.5 *Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$.Then G is said to be a totally irregular fuzzy graph if there is vertex which is adjacent to vertices with distinct degrees.*

Definition 3.6 *let $G:(\sigma,\mu)$be a fuzzy graph on $G^*(V,E)$. Then G is said to be a neighbourly irregular fuzzy graph if every two adjacent vertices of G have distinct degree.*

Definition 3.7 *Let $G:(\sigma,\mu)$ be a fuzzy graph on $G^*(V,E)$. Then G is said to be a neighbourly total irregular fuzzy graph if every two adjacent vertices have distinct total degrees.*

Definition 3.8 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. The 2-degree of a vertex v is defined as the sum of degrees of vertices incident at v and it is denoted by $t(v)$.*

Definition 3.9 *A pseudo degree of a vertex v is denoted by $d_a(v)$ and defined as $\frac{t(v)}{d^*_G(v)}$, where $d^*_G(v)$ is the number of edges incident at v.*

Definition 3.10 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. The pseudo total degree of a vertex v in G is denoted by $td_a(v)$ and is defined as $td_a(v) = d_a(v) + \sigma(v)$ for all $v \in V$.*

§4. Neighbourly Pseudo Irregular Fuzzy Graphs

Definition 4.1 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be a neighbourly pseudo irregular fuzzy graph if every two adjacent vertices of G have distinct pseudo degree.*

Example 4.2 Consider a graph on $G^*(V, E)$.

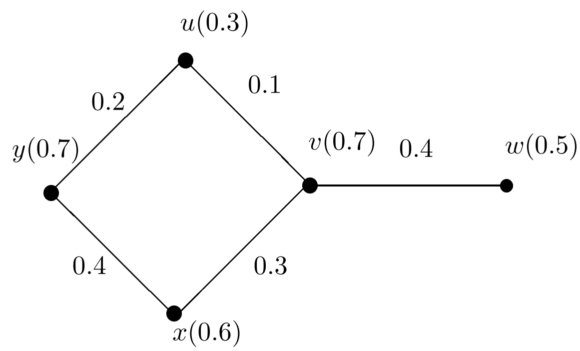

Figure 1

From Figure 1, $d_G(u) = 0.3$, $d_G(v) = 0.8$, $d_G(w) = 0.4$, $d_G(x) = 0.7$, $d_G(y) = 0.6$. Also, $d_a(u) = 0.7$, $d_a(v) = 0.46$, $d_a(w) = 0.8$, $d_a(x) = 0.7$, $d_a(y) = 0.5$. Here, pseudo degrees of all pair of adjacent vertices are distinct. Hence G is neighbourly pseudo irregular fuzzy graph.

Definition 4.3 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. Then G is said to be a neighbourly pseudo totally irregular fuzzy graph if every two adjacent vertices of G have distinct total pseudo degree.*

Example 4.4 Consider a graph on $G^*(V, E)$.

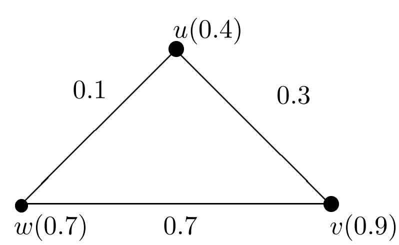

Figure 2

From Figure 2, $d_G(u) = 0.4$, $d_G(v) = 1.0$, $d_G(w) = 0.8$. Here, $d_G*(u) = 2$ for all u in G.

Also, $d_a(u) = 0.9$, $d_a(v) = 0.6$, $d_a(w) = 0.7$, $td_a(u) = 1.3$, $td_a(v) = 1.5$, $td_a(w) = 1.4$. Here, total pseudo degrees of all pair of adjacent vertices are distinct. Hence G is neighbourly pseudo totally irregular fuzzy graph.

Remark 4.5 A neighbourly pseudo irregular fuzzy graph need not be a neighbourly pseudo totally irregular fuzzy graph.

Example 4.6 Consider a graph on $G^*(V, E)$.

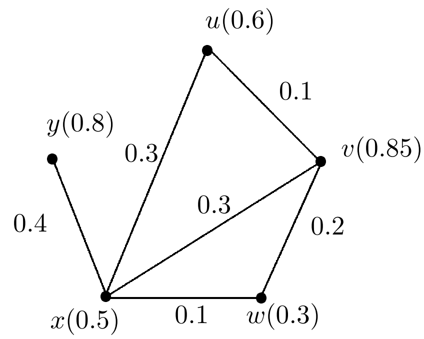

Figure 3

From the above figure, $d_G(u) = 0.4$, $d_G(v) = 0.6$, $d_G(w) = 0.3$, $d_G(x) = 1.1$, $d_G(y) = 0.4$. Also, $d_a(u) = 0.85$, $d_a(v) = 0.6$, $d_a(w) = 0.85$,$d_a(x) = 0.425$, $d_a(y) = 1.1$, $td_a(u) = 1.45$, $td_a(v) = 1.45$, $td_a(w) = 1.15$, $td_a(x) = 0.925$, $td_a(y) = 1.9$. Here, pseudo degrees of all pair of adjacent vertices are distinct. Hence G is neighbourly pseudo irregular fuzzy graph. But u and v are the adjacent vertices having same total pseudo degree. Hence G is not a neighbourly pseudo totally irregular fuzzy graph.

Remark 4.6 A neighbourly pseudo totally irregular fuzzy graph need not be a neighbourly pseudo irregular fuzzy graph.

Example 4.7 Consider a graph on $G^*(V, E)$.

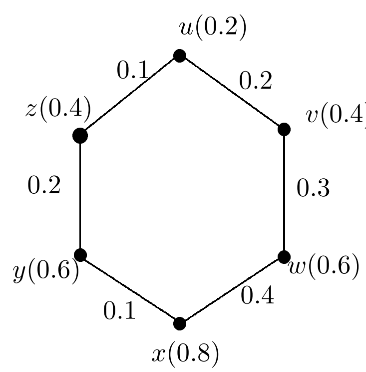

Figure 4

Here, $d_a(u) = 0.4$, $d_a(v) = 0.5$, $d_a(w) = 0.5$,$d_a(x) = 0.5$, $d_a(y) = 0.4$, $d_a(z) = 0.3$, $td_a(u) = 0.6$, $td_a(v) = 0.9$, $td_a(w) = 1.1$, $td_a(x) = 1.3$, $td_a(y) = 1.0$, $td_a(z) = 0.7$. Here, total pseudo degrees of all pair of adjacent vertices are distinct. Hence G is neighbourly pseudo totally irregular fuzzy graph. But the pairs v and w, w and x are the adjacent vertices having same pseudo degree. Hence G is not a neighbourly pseudo irregular fuzzy graph.

Theorem 4.9 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. If σ is a constant function then the following are equivalent.*

(i) G is neighbourly pseudo irregular fuzzy graph;

(ii) G is neighbourly pseudo totally irregular fuzzy graph.

Proof Assume that σ is a constant function. Let $\sigma(u) = c$ for all $u \in V$. Suppose G is a neighbourly pseudo irregular fuzzy graph. Then every two pair of adjacent vertices have distinct pseudo degrees. Let u_1 and u_2 be two adjacent vertices with pseudo degrees k_1 and k_2 respectively. Then $k_1 \neq k_2$. Suppose G is not a neighbourly pseudo totally irregular fuzzy graph. Then at least two adjacent vertices have same total pseudo degree. Suppose $td_a(u_1) = td_a(u_2) \implies k_1 + c = k_2 + c \implies k_1 = k_2$, which is a contradiction. Hence G is a neighbourly pseudo totally irregular fuzzy graph. Then $(i) \implies (ii)$ proved

Now, Suppose G is a neighbourly pseudo totally irregular fuzzy graph. Then every pair of adjacent vertices have distinct total pseudo degrees. Let u_1 and u_2 be two adjacent vertices with pseudo degrees k_1 and k_2 respectively. Now, $td_a(u_1) \neq td_a(u_2) \implies k_1 + c \neq k_2 + c \implies k_1 \neq k_2$. Thus every pair of adjacent vertices have distinct average degrees. Hence G is a neighbourly pseudo irregular fuzzy graph. Thus $(ii) \implies (i)$ proved. □

Remark 4.10 The converse of the above theorem need not be true.

Example 4.11 Consider a graph on $G^*(V, E)$.

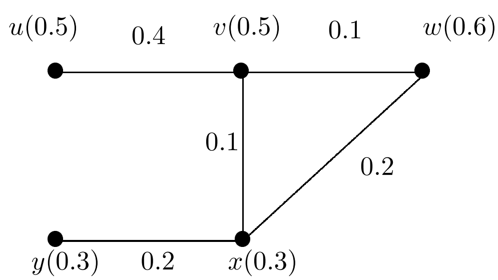

Figure 5

From the figure 5, $d_a(u) = 0.6$, $d_a(v) = 0.6$, $d_a(w) = 0.55$,$d_a(x) = 0.366$, $d_a(y) = 0.5$, $td_a(u) = 1.1$, $td_a(v) = 0.9$, $td_a(w) = 1.15$, $td_a(x) = 0.666$, $td_a(y) = 0.8$. Hence G is neighbourly pseudo irregular fuzzy graph and neighbourly pseudo totally irregular fuzzy graph. But σ is not a constant function.

Remark 4.12 Pseudo irregular fuzzy graph need not be a neighbourly pseudo irregular fuzzy graph.

Example 4.13 Consider a graph on $G^*(V, E)$.

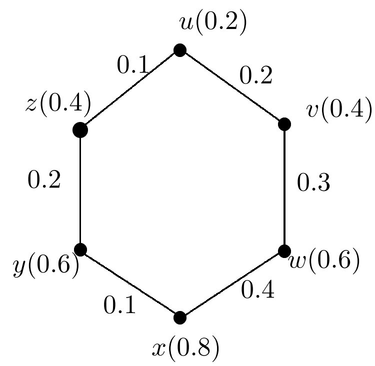

Figure 6

Here, $d_a(u) = 0.4$, $d_a(v) = 0.5$, $d_a(w) = 0.5$,$d_a(x) = 0.5$, $d_a(y) = 0.4$, $d_a(z) = 0.3$. Here But the pairs v & w and w & $x$are the adjacent vertices having same pseudo degree. Hence G is not a neighbourly pseudo irregular fuzzy graph. But G is pseudo irregular fuzzy graph, since the vertex u is adjacent to vertices v and z with distinct pseudo degrees

Theorem 4.14 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. If the pseudo degrees of all vertices of G are distinct, then G is neighbourly pseudo irregular fuzzy graph.*

Proof Assume that the pseudo degrees of all vertices of G are distinct. Then every pair of adjacent vertices have distinct pseudo degree and hence G is neighbourly pseudo irregular fuzzy graph. □

Theorem 4.15 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$. If the pseudo degrees of all vertices of G are distinct and σ is constant, then G is neighbourly pseudo totally irregular fuzzy graph.*

Proof Assume that the pseudo degrees of all vertices of G are distinct. Then by theorem G is neighbourly pseudo irregular fuzzy graph. Since σ is constant, by theorem, G is neighbourly pseudo totally irregular fuzzy graph. □

Theorem 4.16 *If $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$, a cycle of length n and μ is a constant function then G is not a neighbourly pseudo irregular fuzzy graph.*

Proof Assume that μ is a constant function, say $\mu(u_iu_j) = c$, $i \neq j$ for all $u_iu_j \in E$. Then $d_a(u_i) = 2c$ for all $u_i \in V$. Thus $d_a(u_i)$ is constant for all $u_i \in V$. Hence G is not a neighbourly pseudo irregular fuzzy graph. □

Theorem 4.17 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$, a cycle of length n. If μ is a constant and σ is distinct, then G is neighbourly pseudo totally irregular fuzzy graph.*

Proof Assume that μ is a constant and σ is distinct. (i.e.) $\mu(u_iu_j) = c$, $i \neq j$ for all $u_iu_j \in E$ and $\sigma(u_i) = k_i$ for all $u_i \in V$. Thus $k_1 \neq k_2 \neq k_3 \neq \cdots \neq k_n$. Then $d_a(u_i) = 2c$ for all $u_i \in V$. Now $td_a(u_i) = d_a(u_i) + \sigma(u_i) = 2c + k_i$, for $i = 1, 2, 3, \cdots, n$. Hence G is a neighbourly pseudo totally irregular fuzzy graph. □

Theorem 4.18 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$, an even cycle of length n and σ*

is distinct. If alternate edges have the same membership values, then G is neighbourly pseudo totally irregular fuzzy graph.

Proof Assume that alternate edges takes the same membership values and $\sigma(u_i) = k_i$, for $i = 1, 2, \cdots, n$ and $k_1 \neq k_2 \neq \cdots \neq k_n$. Let $e_1, e_2, \cdots, e_n$ be the edges of G. Since the alternate edges have the same membership values,

$$\mu(e_i) = \begin{cases} c_1 \text{if } i \text{ is odd,} \\ c_2 \text{ if } i \text{ iseven,} \end{cases}$$

$$\begin{aligned} d_a(u_i) &= c_1 + c_2,\ i = 1, 2, \cdots, n, \\ d_a(u_i) &= \text{constant}, \\ td_a(u_i) &= d_a(u_i) + \sigma(u_i), \\ &= d_a(u_i) + k_i,\ i = 1, 2, \cdots, n \text{ and } k_1 \neq k_2 \neq ... \neq k_n. \end{aligned}$$

So, every pair of adjacent vertices have distinct total pseudo degree. Hence G is neighbourly pseudo totally irregular fuzzy graph. □

Remarks 4.19 The above theorem does not hold for neighbourly pseudo irregular fuzzy graph.

Example 4.20 Consider a graph on $G^*(V, E)$.

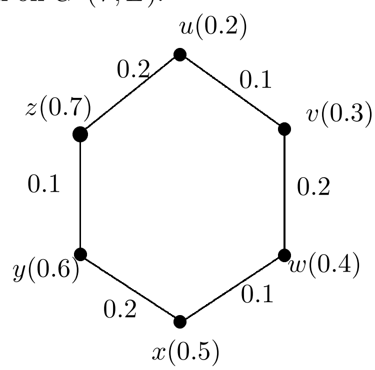

Figure 7

Here, $d_a(u) = 0.3$, $d_a(v) = 0.3$, $d_a(w) = 0.3$,$d_a(x) = 0.3$, $d_a(y) = 0.3$, $d_a(z) = 0.3$. Here $\sigma(u)$ is distinct. But G is not a neighbourly pseudo irregular fuzzy graph, since there is no pair of adjacent vertices having distinct pseudo degree.

Theorem 4.21 *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$, a cycle of length n and $n \geq 5$. If the membership values of the edges are $c_1, c_2, c_3, \cdots, c_n$ such that $c_1 < c_2 < c_3 < \cdots < c_n$. Then G is neighbourly pseudo irregular fuzzy graph.*

Proof Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^*(V, E)$, a cycle of length n and $n \geq 5$. Let $c_1, c_2, c_3, \cdots, c_n$ be the edges of the cycle C_n in that order. Let the membership values of the edges $e_1, e_2, e_3, \cdots, e_n$ be $c_1, c_2, c_3, \cdots, c_n$ such that $c_1 < c_2 < c_3 < \cdots < c_n$.

$$\text{Now, } d(v_i) = \begin{cases} c_n + c_1 \text{ if } i = 1 \\ c_{i-1} + c_i \text{ if } i = 2, 3, 4, \cdots, n \end{cases}$$

$$\implies d_a(v_i) = \begin{cases} \frac{d(v_2)+d(v_n)}{2} \text{ if } i = 1 \\ \frac{d(v_{i-1})+d(v_{i+1})}{2} \text{ if } i = 2, 3, \cdots, n-1 \\ \frac{d(v_{n-1})+d(v_1)}{2} \text{ if } i = n \end{cases}$$

$$\implies d_a(v_i) = \begin{cases} \frac{c_2+c_3+c_{n-1}+c_n}{2} \text{ if } i = 1 \\ \frac{c_n+c_1+c_2+c_3}{2} \text{ if } i = 2 \\ \frac{c_{i-2}+c_{i-1}+c_i+c_{i+1}}{2} \text{ if } i = 3, \cdots, n-1 \\ \frac{c_1+c_n+c_{n-1}+c_{n-2}}{2} \text{ if} i = n. \end{cases}$$

Also, since $c_1 < c_2 < c_3 < \cdots < c_n$, we have every pair of adjacent vertices have distinct pseudo degree. Hence the graph G is neighbourly pseudo irregular fuzzy graph. □

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Spectra of a New Join in Duplication Graph

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Abstract: The *duplication graph* $D_G(G)$ of a graph G is obtained by inserting new vertices corresponding to each vertex of G and making the vertex adjacent to the neighborhood of the corresponding vertex of G and deleting the edges of G. Let G_1 and G_2 be two graph with vertex sets $V(G_1)$ and $V(G_2)$ respectively. The D_G*-vertex join* of G_1 and G_2 is denoted by $G_1 \sqcup G_2$ and it is the graph obtained from $D_G(G_1)$ and G_2 by joining every vertex of $V(G_1)$ to every vertex of $V(G_2)$. The DG-*add vertex join* of G_1 and G_2 is denoted by $G_1 \bowtie G_2$ and is the graph obtained from $D_G(G_1)$ and G_2 by joining every additional vertex of $D_G(G_1)$ to every vertex of $V(G_2)$. In this paper we determine the A-spectra and L-spectra of the two new joins of graphs G_1 and G_2 when G_1 is a regular graph and G_2 is an arbitrary graph. As an application we give the number of spanning tree, the Kirchhoff index and Laplace energy like invariant of the new join. Also we obtain some infinite family of new class of integral graphs.

Key Words: Spectrum, cospectral graphs, Join of graphs, spanning tree, Kirchhoff index, Laplace-energy like invariant.

AMS(2010): 05C50.

§1. Introduction

All graphs described in this paper are simple and undirected. Let G be a graph with vertex set $V(G_1) = \{v_1, v_2, \cdots v_n\}$. The adjacency matrix of G, denoted by $A(G) = (a_{ij})_{n\times n}$ is an $n \times n$ symmetric matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Let d_i be the degree of the vertex v_i in G and $D(G) = diag(d_1, d_2, \cdots d_n)$ be the diagonal matrix of G. The Laplacian matrix is defined as $L(G) = D(G) - A(G)$. The characteristic polynomial of $A(G)$ is defined as $f_G(A : x) = det(xI_n - A)$, where I_n is the identity matrix of order n. The roots of the characteristic equation of $A(G)$ are called the *eigenvalues* of G. It is

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denoted by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. It is called the *A - Spectrum* of G. The eigen values of $L(G)$ is denoted by $0 = \mu_1(G) \leq \mu_2(G), \cdots \leq \mu_n(G)$ and it is called the *L - Spectrum* of G. Since $A(G)$ and $L(G)$ are real and symmetric, their eigen values are all real numbers. A graph is *A - integral*, if the A - spectrum consists only of integers [4,14]. Two graphs are said to be *A - Cospectral* if they have the same A - spectrum.

The characteristic polynomial and spectra of graphs help to investigate some properties of graphs such as energy [8,16], number of spanning trees [18, 9,1], the Kirchhoff index [2, 5, 11], Laplace energy like invariants [7] etc.

The first result on Laplacian matrix, which was discovered by Kirchhoff, appeared in a paper published in the year 1847 is related to electrical network. There exists a vast literature that studies the Laplacian eigen values and their relationship with various properties of graphs [12,13]. Most of the studies of the Laplacian eigen values has naturally concentrated on external non trivial eigen values. Gutman et al. [16] discovered the connection between photoelectron spectra of standard hydrocarbons and the Laplacian eigen values of the underlying molecular graphs.

In a recent paper Reji Kumar and Renny P. Varghese [18] introduced subdivision graph vertex join of two given graphs and studies its spectral properties. They also studied [19] the spectral properties of some classes of hypergraphs.

In the next section we define DG - vertex join and DG - add vertex join of two graphs and discuss some important results, which are found essential to prove the results given in the subsequent sections. In the third section we find the A - spectrum and the L - spectrum of the new join and prove some related results. As an application, we find the number of spanning trees, Kirchhoff index and Laplacian - energy like invariant. Fourth section contains a discussion on some infinite family of integral graphs.

§2. Preliminaries

In a paper published in 1973 on duplicate graphs, which appeared in the *Journal of Indian Mathematical Society*, Sampathkumar [10] defined duplicate graphs. Let G be a graph with vertex set V(G) $= \{v_1, v_2, \cdots, v_n\}$. Take another set $U = \{u_1, u_2, \cdots, u_n\}$. Make u_i adjacent to all the vertices in $N(v_i)$, the neighbourhood set of v_i, in G for each i and remove all edges of G. The resulting graph is called the *duplication graph* of G and is denoted by $D(G)$. The following result tells us an easy way to find the determinant of a bigger matrix using the determinant of relatively smaller matrices.

Proposition 2.1 *Let M_1, M_2, M_3, M_4 be respectively $p \times p, p \times q, q \times p, q \times q$ matrix with M_1 and M_4 are invertible then*

$$det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = det(M_1)det(M_4 - M_3M_1^{-1}M_2)$$
$$= det(M_4)det(M_1 - M_2M_4^{-1}M_3),$$

where $M_4 - M_3M_1^{-1}M_2$ and $M_1 - M_2M_4^{-1}M_3$ are called the Schur complements of M_1 and M_4 respectively.

Let G be a graph on n vertices, with the adjacency matrix A. The characteristic matrix $xI - A$ of A has determinant $det(xI - A) = f_G(A : x) \neq 0$, so is invertible. The A - *coronal* ([6]), $\Gamma_A(x)$ of G is defined to be the sum of the entries of the matrix $(xI - A)^{-1}$. This can be calculated as

$$\Gamma_A(x) = \mathbf{1}_n^T(xI - A)^{-1}\mathbf{1}_n.$$

The A - coronal of some classes of graphs are given here.

Lemma 2.2([6]) *Let G be r - regular on n vertices. Then*

$$\Gamma_A(x) = \frac{n}{x - r}.$$

Since for any graph G with n vertices, each row sum of the Laplacian matrix $L(G)$ is equal to 0, we have $\Gamma_L(x) = \frac{n}{x}$.

Lemma 2.3([6]) *Let G be the bipartite graph K_{pq}, where $p + q = n$. Then*

$$\Gamma_A(x) = \frac{nx + 2pq}{x^2 - pq}.$$

The following results on an $n \times n$ real matrix is useful in this context.

Proposition 2.4([15]) *Let A be an $n \times n$ real matrix, and $J_{s\times t}$ denote the $s \times t$ matrix with all entries equal to one. Then*

$$det(A + \alpha J_n \times n) = det(A) + \alpha \mathbf{1}_n^T adj(A)\mathbf{1}^n.$$

Here α is a real number and $adj(A)$ is the adjugate matrix of A.

Corollary2.5([15]) *Let A be an $n \times n$ real matrix. Then*

$$det(xI_n - A - \alpha J_{n\times n}) = (1 - \alpha\Gamma_A(x))\ det(xI_n - A).$$

Next we proceed to define the DG - vertex join and the DG - advertex join of two graphs.

Definition 2.6 *Let G_1 be a graph on n_1 vertices and m_1 edges. G_2 be an arbitrary graph on n_2 vertices The $DG -$ vertex join of G_1 and G_2 is denoted by $G_1 \sqcup G_2$ and is the graph obtained from $D(G_1)$ and G_2 by joining every vertex of $V(G_1)$ to every vertex of $V(G_2)$. Where $D(G_1)$ is the duplication graph of G_1.*

In Figure 1 an example of DG - vertex join of the graphs C_4 and K_2 is given.

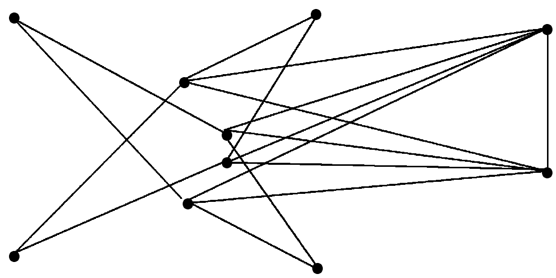

Figure 1 $C_4 \sqcup K_2$

Definition 2.7 *The $DG-$ addvertex join of G_1 and G_2 is denoted by $G_1 \bowtie G_2$ and is the graph obtained from $D(G_1)$ and G_2 by joining the additional vertices of $D(G_1)$ corresponding to the vertices of G_1 with every vertex of $V(G_2)$.*

In Figure 2 an example of DG - advertex join of the graphs C_4 and K_2 is given.

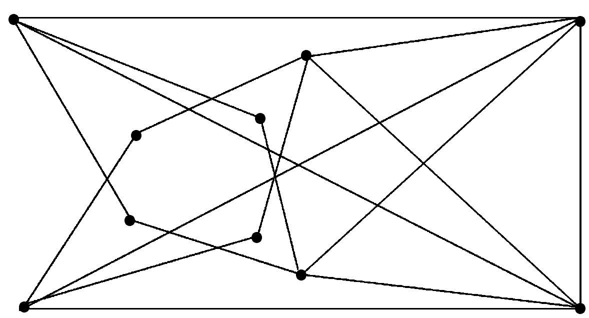

Figure 2 $C_4 \bowtie K_2$

§3. Spectrum of $G_1 \sqcup G_2$ for Some Classes of Graphs G_1 and G_2

In this section we study the spectrum of DG - vertex join of some classes of graphs G_1 and G_2. We prove the following results in this connection.

Theorem 3.1 *Let G_1 be an r_1 - regular graph on n_1 vertices and m_1 edges. G_2 be an arbitrary graph on n_2 vertices. Then, the Characteristic polynomial of $G_1 \sqcup G_2$ is*

$$f_{G_1 \sqcup G_2}(A:x) = (x^2 - n_1 x \Gamma_{A_2}(x) - r_1^2) \prod_{i=2}^{n_2}(x - \lambda_i(G_2)) \prod_{i=2}^{n_1}(x^2 - \lambda_i(G_1)^2).$$

Proof The adjacency matrix of $G_1 \sqcup G_2$ is

$$A = \begin{bmatrix} 0 & A_1 & J_{n_1\times n_2} \\ A_1 & 0_{n_1} & 0_{n_1\times n_2} \\ J_{n_2\times n_1} & 0_{n_2\times n_1} & A_2 \end{bmatrix}$$

where A_1 and A_2 are the adjacency matrix of G_1 and G_2 respectively and J is a matrix with each entries 1.

The characteristic polynomial of $G_1 \sqcup G_2$ is

$$\begin{aligned} f_{G_1\sqcup G_2}(A:x) &= \begin{vmatrix} xI_{n_1} & -A_1 & -J \\ -A_1 & xI_{n_1} & 0 \\ -J & 0 & xI_{n_2}-A_2 \end{vmatrix} \\ &= det(xI_{n_2} - A_2)\ det\ S, \end{aligned}$$

where

$$\begin{aligned} S &= \begin{pmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n_1} \end{pmatrix} - \begin{pmatrix} -J_{n_1\times n_2} \\ 0 \end{pmatrix} (xI_{n_2} - A_2)^{-1} \begin{pmatrix} -J_{n_2\times n_1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} xI_{n_1} & -A_1 \\ -A_1 & xI_{n1} \end{pmatrix} - \begin{pmatrix} \Gamma_{A_2}(x)J_{n_1\times n_1} & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} xI - \Gamma_{A_2}(x)J_{n_1\times n_1} & -A_1 \\ -A_1 & xI \end{pmatrix} \end{aligned}$$

Whence,

$$\begin{aligned} det\ S &= det(xI)\ det\left((xI - \Gamma_{A_2}(x)J - \frac{A_1^2}{x}\right) \\ &= x^{n_1}\ det\left(xI - \Gamma_{A_2}(x)J - \frac{A_1^2}{x})\right) \\ &= x^{n_1}\ det\left(xI - \frac{A_1^2}{x} - \Gamma_{A_2}(x)J\right) \\ &= x^{n_1}\ det\left(xI - \frac{A_1^2}{x}\right)\left(1 - \Gamma_{A_2}(x)\Gamma_{\frac{A_1^2}{x}}(x)\right), \end{aligned}$$

Notice that G_1 is r_1 - regular and the row sum of A_1^2 is r_1^2. We get

$$\Gamma_{\frac{A_1^2}{x}} = \frac{n_1}{x - \frac{r_1^2}{x}} = \frac{n_1 x}{x^2 - r_1^2}$$

and

$$\begin{aligned} det\ S &= x^{n_1} det\left(xI - \frac{A_1^2}{x})\right)\left(1 - \frac{n_1 x}{x^2 - r_1^2}\Gamma_{A_2}(x)\right) \\ &= det(x^2 I - A^2)\left(\frac{x^2 - r_1^2 - n_1 x \Gamma_{A_2}(x)}{x^2 - r_1^2}\right). \end{aligned}$$

Hence

$$det(xI - A) = (x^2 - n_1 x \Gamma_{A_2}(x) - r_1^2)\prod_{i=1}^{n_2}(x - \lambda_i(G_2))\prod_{i=2}^{n_1}(x^2 - \lambda_i(G_1)^2). \qquad \Box$$

Corollary 3.2 *Let G_1 be an r_1 - regular graph on n_1 vertices, G_2 be r_2 - regular graph on n_2 vertices. Then the $A-$ Spectrum of $G_1 \sqcup G_2$ consists of*

(*i*) $\lambda_i(G_2)$ *, for $i = 2, 3, \cdots, n_2$;*

(*ii*) $\pm\lambda_i(G_1)$ *, for $i = 2, 3, \cdots, n_1$;*

(*iii*) *Three roots of the equation*

$$x^3 - r_2 x^2 - (n_1 n_2 + r_1^2)x + r_1^2 r_2.$$

Proof If G_2 is r_2 - regular then

$$\Gamma_{A_2}(x) = \frac{n_2}{x - r_2}.$$

We get

$$\begin{aligned} det(xI - A) &= (x^3 - r_2 x^2 - (n_1 n_2 + r_1^2)x + r_1^2 r_2) \\ &\quad \times \prod_{i=2}^{n_2}(x - \lambda_i(G_2))\prod_{i=2}^{n_1}(x^2 - \lambda_i(G_1)^2). \qquad \Box \end{aligned}$$

Corollary 3.3 *Let G_1 be an r_1 - regular graph on n_1 vertices, $A-$ Spectrum of $G_1 \sqcup \overline{K_n}$ consists of*

(*i*) *0, repeats n_2 times;*

(*ii*) $\pm\lambda_i(G_1)$ *, for $i = 2, 3, \cdots, n_1$;*

(*iii*) $\pm\sqrt{n_1 n_2 + r_1^2}$.

Corollary 3.4 *Let G_1 be an r_1 - regular graph on n_1 vertices. $A-$ Spectrum of $G_1 \sqcup K_{pq}$ consists of*

(*i*) *0, repeats $p + q - 2$ times;*

(*ii*) $\pm\lambda_i(G_1)$ *, for $i = 2, 3, \cdots, n_1$;*

(*iii*) *Four roots of the equation*

$$x^4 - (pq + r_1^2 + n_1 p + n_1 q)x^2 - 2pqn_1 x + r_1^2 pq.$$

3.1 Laplacian Spectrum of $G_1 \sqcup G_2$ for Some Classes of Graphs G_1 and G_2

Theorem 3.5 *Let G_1 be an r_1 - regular graph on n_1 vertices and m_1 edges. G_2 be an arbitrary graph on n_2 vertices. then,*

$$\begin{aligned} f_{G_1 \sqcup G_2}(L:x) &= x(x^2-(n_1+n_2+2r_1)x+r_1(2n_1+n_2)) \\ &\quad \times \prod_{i=2}^{n_2}(x-n_1-\mu_i(G_2)) \prod_{i=2}^{n_1}(x^2-(2r_1+n_2)x+n_2r_1+r_1^2-\lambda_i(G_1)^2). \end{aligned}$$

Proof The Laplace adjacency matrix of $G_1 \sqcup G_2$ is

$$L = \begin{bmatrix} (r_1+n_2)I & -A_1 & J_{n_1\times n_2} \\ -A_1 & r_1 I & 0_{n_1\times n_2} \\ -J_{n_2\times n_1} & 0_{n_1\times n_1} & n_1 I_{n_2} + L_2 \end{bmatrix}$$

where L_2 is the Laplacian adjacency matrix of G_2

The Laplacian characteristic polynomial of $G_1 \sqcup G_2$ is

$$f_{G_1 \sqcup G_2}(L:x) = \begin{vmatrix} (x-r_1-n_2)I_{n_1} & A_1 & J \\ A_1 & (x-r_1)I_{n_1} & 0 \\ J & 0 & (x-n_1)I_{n_2}-L_2 \end{vmatrix}.$$

Using proposition 2.2 we get

$$f_{G_1 \sqcup G_2}(L:x) = det((x-n_1)I_{n_2}-L_2)\ detS,$$

where

$$\begin{aligned} S &= \begin{pmatrix} (x-r_1-n_2)I_{n_1} & A_1 \\ A_1 & (x-r_1)I_{n_1} \end{pmatrix} - \begin{pmatrix} J \\ 0 \end{pmatrix} ((x-n_1)I_{n_1}-L_2)^{-1} \begin{pmatrix} J & 0 \end{pmatrix} \\ &= \begin{pmatrix} (x-r_1-n_2)I & A_1 \\ A_1 & (x-r_1)I \end{pmatrix} - \begin{pmatrix} \Gamma_{L_2}(x-n_1)J_{n_1\times n_1} & 0 \\ 0 & o \end{pmatrix} \\ &= \begin{pmatrix} (x-r_1-n_2)I - \Gamma_{L_2}(x-n_1)J & A_1 \\ A_1 & (x-r_1)I \end{pmatrix} \end{aligned}$$

Therefore,

$$det\ S = (x-r_1)^{n_1} det\left((x-r_1-n_2)I - \Gamma_{L_2}(x-n_1)J - \frac{A_1^2}{x-r_1}\right).$$

By Corollary 2.7

$$\begin{aligned} det\ S &= (x-r_1)^{n_1} det\left((x-r_1-n_2)I - \frac{A_1^2}{x-r_1}\right) \\ &\quad \times \left(1-\Gamma_{L_2}(x-n_1)\Gamma_{\frac{A_1^2}{x-r_1}}(x-r_1-n_2)\right) \\ &= det\left((x-r_1-n_2)(x-r_1)I - A^2\right)\left(1-\Gamma_{L_2}(x-n_1)\Gamma_{\frac{A_1^2}{x-r_1}}(x-r_1-n_2)\right). \end{aligned}$$

Since G_1 is r_1 regular graph, the row sum of $\frac{A_1^2}{x-r_1}$ is $\frac{r_1^2}{x-r_1}$. Therefore,

$$\begin{aligned} \Gamma_{\frac{A_1^2}{x-r1}}(x-r_1-n_2) &= \frac{n_1(x-r_1)}{x^2-(2r_1+n_2)x+n_2r_1}, \\ 1-\Gamma_{L_2}(x-n_1)\Gamma_{\frac{A_1^2}{x-r_1}}(x-r_1-n_2) &= \frac{x(x^2-(n_1+n_2+2r_1)x+r_1(2n_1+n_2))}{(x-n_1)(x^2-(2r_1+n_2)x+n_2r_1)}. \end{aligned}$$

Hence

$$\begin{aligned} f_{G_1\sqcup G_2}(L:x) &= x(x^2-(n_1+n_2+2r_1)x+r_1(2n_1+n_2)) \\ &\quad \times \prod_{i=2}^{n_2}(x-n_1-\mu_i(G_2))\prod_{i=2}^{n_1}(x^2-(2r_1+n_2)x+n_2r_1+r_1^2-\lambda_i(G_1)^2). \quad \square \end{aligned}$$

Let $t(G)$ denote the number of spanning tree of the graph G, the total number of distinct spanning subgraphs of G that are trees. The number of spanning trees of the graph describe the network which is one of the natural characteristics of its reliability. If G is a connected graph with n vertices and the Laplacian spectrum $0=\mu_1(G)\le\mu_2(G),\cdots\le,\mu_n(G)$ then ([17])

$$t(G)=\frac{\mu_2(G)\mu_3(G)\cdots\mu_n(G)}{n}$$

Corollary 3.6 *Let G_1 be an r_1 - regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices. Then*

$$t(G_1\sqcup G_2)=\frac{r_1(2n_1+n_2)\prod_{i=2}^{n_1}(n_1+\mu_i(G_2))\prod_{i=2}^{n_2}(r_1^2+n_2r_1-\lambda_i^2(G_1))}{2n_1+n_2}.$$

Proof By Theorem 3.5 the roots of $f_{G_1\sqcup G_2}(L:x)$ are as follows:

(i) 0;

(ii) $n_1+\mu_i(G_2)$ for $i=2,3,\cdots,n_2$;

(iii) Two roots say x_1 and x_2 of the equation $x^2-(n_1+n_2+2r_1)x+r_1(2n_1+n_2)$;

(iv) Two roots say x_{i1} and x_{i2} of the equation $x^2-(2r_1+n_2)x+n_2r_1+r_1^2-\lambda_i(G_1)^2$ for $i=2,3,\cdots,n_2$.

For Case (iii), $x_1x_2=r_1(2n_1+n_2)$, and for Case (iv), $x_{i1}x_{i2}=n_2r_1+r_1^2-\lambda_i(G_1)^2$,

$i = 2, 3, \cdots, n_2$. Then, we get that

$$t(G_1 \sqcup G_2) = \frac{r_1(2n_1+n_2)\prod_{i=2}^{n_1}(n_1+\mu_i(G_2))\prod_{i=2}^{n_2}(r_1^2+n_2r_1-\lambda_i^2(G_1))}{2n_1+n_2}. \qquad \square$$

Another Laplacian spectrum based on graph invariant was defined by Liu and Liu [3] called the Laplacian - energy - like invariant. The Laplacian - energy - like invariant(LEL) of a graph G of n vertices is defined as

$$LEL(G) = \sum_{i=2}^{n}\sqrt{\mu_i}.$$

Corollary 3.7 *Let G_1 be an r_1 - regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices. Then Laplace - energy - like invariant*

$$\begin{aligned} LEL &= \left(n_1+n_2+2r_1+2\sqrt{r_1(2n_1+n_2)}\right)^{1/2} + \sum_{i=2}^{n_2}\left(n_1+\mu_i(G_1)^2\right)^{1/2} \\ &\quad + \sum_{i=2}^{n_1}\left(\frac{2r_1+n_2+\sqrt{r_1^2+n_2r_1-\lambda_i(G_1)^2}}{r_1^2+n_2r_1-\lambda_i(G_1)^2}\right)^{1/2}. \end{aligned}$$

Proof Using Theorem 3.5 and Corollary 3.6 we have

$$\begin{aligned} \sqrt{x_1}+\sqrt{x_2} &= (x_1+x_2+2\sqrt{x_1x_2})^{1/2} \\ &= \left(n_1+n_2+2\sqrt{r_1(2n_1+n_2)}\right)^{1/2}, \\ \frac{1}{\sqrt{x_{i1}}}+\frac{1}{\sqrt{x_{i2}}} &= \frac{\sqrt{x_{i1}}+\sqrt{x_{i2}}}{2\sqrt{x_{i1}x_{i2}}} \\ &= \left(\frac{x_1+x_2+\sqrt{x_1x_2}}{x_{i1}x_{i2}}\right)^{1/2} \\ &= \left(\frac{2r_1+n_2+\sqrt{r_1^2+n_2r_1-\lambda_i(G_1)^2}}{r_1^2+n_2r_1-\lambda_i(G_1)^2}\right)^{1/2}. \end{aligned}$$

Hence the required result is obtained using the formula for LEL. $\square$

Klein [5] propounder of *resistance distance* defined electric resistance in network corresponding to the considered graph as the resistance distance between any two adjacent nodes is 1 ohm. The sum of the resistance distance between all pairs of the vertices of a graph is conceived as a new graph invariant. The electric resistance is calculated by means of the Kirchhoff laws called *kirchhoff index*.

Kirchhoff index of a connected graph G with $n(n \geq 2)$ vertices is defined as

$$Kf(G) = n\sum_{i=1}^{n-1}\frac{1}{\mu_i}$$

Corollary 3.8 *Let G_1 be an r_1 - regular graph on n_1 vertices. G_2 be an arbitrary graph on n_2*

vertices. Then

$$Kf(G_1 \sqcup G_2) = (2n_1 + n_2)\left[\frac{n_1 + n_2 + 2r_1}{r_1(2n_1 + n_2)} + \sum_{i=2}^{n_2}\frac{1}{n_1 + \mu_i(G_2)} + \sum_{i=2}^{n_1}\frac{2r_1 + n_2}{r_1^2 + n_2 r_1 - \lambda_i(G_1)^2}\right].$$

Proof Using Theorem 3.5, Corollary 3.7 and the formula for Kirchhoff index we obtain the required result. □

3.2 Spectra of DG - add Vertex Graph of Some Classes of Graphs

Next we discuss some spectral properties of the DG - add vertex graph of some classes of graphs.

Proposition 3.9 *Let G_1 be an r_1 - regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices. Then $G_1 \sqcup G_2$ and $G_1 \bowtie G_2$ are A - cospectral*

Proof Notice that the characteristic polynomials of $G_1 \sqcup G_2$ and $G_1 \bowtie G_2$ are same. Hence we get the result. □

Proposition 3.10 *Let G_1 be an r_1 - regular graph on n_1 vertices and G_2 be an arbitrary graph on n_2 vertices then $G_1 \sqcup G_2$ and $G_1 \bowtie G_2$ are L - cospectral.*

§4. Infinite Families of Integral Graphs

The following properties give a necessary and sufficient condition for DG - vertex join and DG - add vertex join of G_1 and G_2 to be integral.

Proposition 4.1 *Let G_1 be r_1 - regular graph on n_1 vertices and G_2 be r_2 - regular graph on n_2 vertices. $G_1 \sqcup G_2$ (respectively $G_1 \bowtie G_2$) is an integral graph if and only if G_1 and G_2 are integral graphs and the roots of $x^3 - r_2x^2 - (n_1n_2 + r_1^2)x + r_1^2r_2$ are integers.*

In particular if $G_2 = \overline{K_n}$ (totally disconnected) then $r_2 = 0$ then $G_1 \sqcup G_2$ (respectively $G_1 \bowtie G_2$) is integral iff G_1 is an integral graph and $n_1n_2 + r_1^2$ is a perfect square.

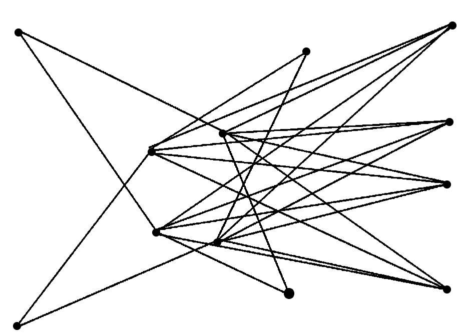

Figure 3 $K_4 \sqcup \overline{K_4}$ with spectrum $\{-5, -1^3, 0^4, 1^3, 5\}$

Proposition 4.2 *Let G_1 be r_1 - regular graph on n_1. $G_1 \sqcup K_{pq}$ (respectively $G_1 \bowtie K_{pq}$) is an integral graph if and only if G_1 is an integral graph and the roots of $x^4 - (pq + r_1^2 + n_1 p + n_1 q)x^2 - 2pqn_1 x + r_1^2 pq$ are integers.*

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The Gourava Index of Four Operations on Graphs

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Abstract: Molecular descriptor are major in the study of QSAR/QSPR. There are numerous importance of graph theory in the field of structural chemistry. In the present paper, we study the Gourava index of four operation on graphs.

Key Words: Gourava index, Zagreb index, graph operations.

AMS(2010): 5C05, 05C12.

§1. Introduction

Let ϑ denotes the collection entire graphs. A mapping $T : \vartheta \to R$ is called a topological index, if for every graph H isomorphic to G, $T(G) = T(H)$. In chemical graph theory, topological indices have several applications in isomer discrimination, QSAR/QSPR investigation, pharmaceutical drug design and many more [5]. There are few important class of topological indices that are extensively studied by a number of researchers. Out of these topological indices, the first and second Zagreb indices, first appeared in a topological structure for the total π -energy of conjugated molecules, were introduced by Gutman et.al., in [8].

The first and second Zagreb indices [3] of a molecular graph G are defined as

$$M_1(G) = \sum_{uv \in E(G)} [d(u) + d(v)].$$

and

$$M_2(G) = \sum_{uv \in E(G)} [d(u)d(v)].$$

Motivated by the definitions of the Zagreb indices and their wide applications, V. R. Kulli [10], introduced the first Gourava index of a molecular graph as follows.

The first Gourava index of a graph G is defined as

$$GO_1(G) = \sum_{uv \in E(G)} [d(u) + d(v) + d(u)d(v)].$$

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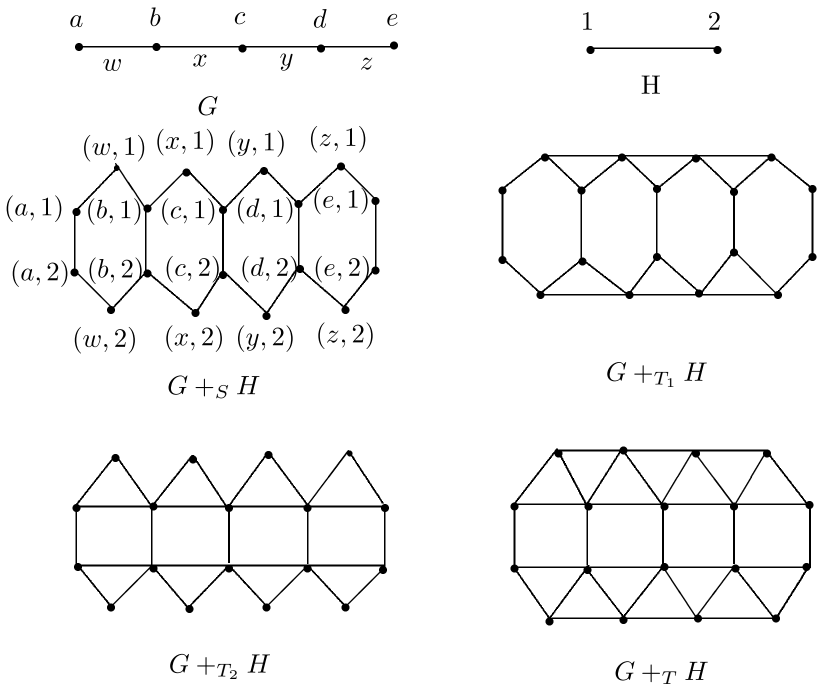

Figure 1: Graph G, H and $G +_F H$

The cartesian product is an important method to construct a ample graph and play vital role in the design and analysis the network. The cartesian product of two connected graphs G and H, which is denoted by $G\square H$, is a graph such that the set of vertices is $V(G)\square V(H)$ and two vertices (p_1, q_1) and (p_2, q_2) of $G\square H$ are adjacent if and only if $p_1 = p_2$ and q_1 is adjacent with q_2 in H otherwise $q_1 = q_2$ and p_1 is adjacent with p_2 in G. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$, there are four related graphs as follows:

For any connected graph G, define four operator graphs $S(G)$, $T(G)$, $Q(G) = T_1(G)$ and $R(G) = T_2(G)$ as follows:

• $S(G)$ is the graph obtained by inserting an additional vertex in each edge of G, i.e., replacing each edge of G by a path of length 2 ([1, 18]).

• The total graph $T(G)$ of a graph G is the graph whose vertex set $V \bigcup E$, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident ([14]).

• $Q(G)$ is the graph obtained by inserting a new vertex into each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G, by a new edge ([15]).

• $R(G)$ is the graph obtained by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge ([15]).

Suppose that G and H are two connected graphs. M. Eliasi, B. Taeri [6] introduced four new operations named as F-sum graphs, on these graphs that are based on S, T_2, T_1, T as follows.

Let F be one of the symbols S, T_2, T_1 or $, T$. The F-sum denoted by $G +_F H$ of graphs G and H, is a graph with the set of vertices $V(G +_F H) = (V(G) \bigcup E(G)) \times V(H)$ and (p_1, p_2)

$(q_1, q_2) \in E(G +_F H)$, if and only if $p_1 = p_2 \in V(G)$ and $q_1q_2 \in E(H)$ or $q_1 = q_2$ and $(p_1, p_2) \in E(F(G))$.

Throughout this paper, we consider only simple, connected, finite and undirected graphs. For a graph G, the order and the size of graph are denoted as n_G and e_G respectively.

In mathematical chemistry, graph operations act as a very essential role, viz., as some chemically interesting graphs can be derived from some simpler graphs by operations on graphs.

In [4], H. Deng et al. computed the first and second Zagreb indices for graph operations $S(G)$, $R(G)$, $Q(G)$ and $T(G)$. Here, we extend this study by investigate the Gourava index of four operation on graphs. Investigators need to study more details on calculating topological indices of graph operations can be refer [2, 7, 9, 11, 12, 16, 17, 19].

§2. The Gourava Index of F-Sum of Graphs

In this section, we discuss main results of Gourava index of F-sum of graphs.

Theorem 2.1 *Let G and H be two connected graphs. Then,*

$$GO_1(G +_s H) = n_H GO_1(G) + n_G GO_1(H) + e_H M_1(G) + 2e_G M_1(H) + 8n_H e_G + 12e_H e_G.$$

Proof From the definition of Gourava index,

$$\begin{aligned} GO_1(G +_s H) &= \sum_{(p_1,q_1)(p_2,q_2)\in E(G+_sH)} [d_{G+_sH}(p_1,q_1) + d_{G+_sH}(p_2,q_2) \\ &\quad + d_{G+_sH}(p_1,q_1)d_{G+_sH}(p_2,q_2)] \\ &= \sum_{p_1\in V(G)} \sum_{q_1q_2\in E(H)} [d_{G+_sH}(p_1,q_1) + d_{G+_sH}(p_1,q_2) \\ &\quad + d_{G+_sH}(p_1,q_1)d_{G+_sH}(p_1,q_2)] \\ &\quad + \sum_{q_1\in V(H)} \sum_{p_1p_2\in E(S(G))} [d_{G+_sH}(p_1,q_1) + d_{G+_sH}(p_1,q_2) \\ &\quad + d_{G+_sH}(p_1,q_1)d_{G+_sH}(p_1,q_2)] \\ &= I_1 + I_2, \end{aligned} \tag{1}$$

where I_1, I_2 are the sums of the above terms, in order.

For vertex $\forall p_1 \in V(G)$ and $q_1q_2 \in E(H)$ we get

$$\begin{aligned} I_1 &= \sum_{p_1\in V(G)} \sum_{q_1q_2\in E(H)} [d_G(p_1) + d_H(q_1) + d_G(p_1) + d_H(q_2) \\ &\quad + [d_G(p_1) + d_H(q_1)][d_G(p_1) + d_H(q_2)]] \\ &= \sum_{p_1\in V(G)} \sum_{q_1q_2\in E(H)} \big[2d_G(p_1) + d_H(q_1) + d_H(q_2) + d_G^2(p_1) + d_G(p_1)[d_H(q_1) + d_H(q_2)] \\ &\quad + d_H(q_1)d_H(q_2)] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{p_1\in V(G)} \big[2e_H d_G(p_1) + M_1(H) + e_H d_G^2(p_1) + d_G(p_1)M_1(H) + M_2(H)\big] \\
&= 4e_He_G + n_G GO_1(H) + e_H M_1(G) + 2e_G M_1(H).
\end{aligned}
$$

For edge $\forall p_1p_2 \in E(S(G)$, where the vertex $p_1 \in V(G)$, $p_2 \in V(S(G)) - V(G)$ and $q_1 \in V(H)$, since $|E(S(G)| = 2|E(G)|$,

$$
\begin{aligned}
I_2 &= \sum_{q_1\in V(H)} \sum_{p_1p_2\in E(S(G))} \big[d_{S(G)}(p_1) + d_H(q_1) + d_{S(G)}(p_2) \\
&\quad +[d_{S(G)}(p_1) + d_H(q_1)]d_{S(G)}(p_2)\big] \\
&= \sum_{q_1\in V(H)} [GO_1(S(G)) + 2e_G d_H(q_1) + 2e_G d_H(q_1)] \\
&= n_H GO_1(S(G)) + 8e_He_G
\end{aligned}
$$

We know that, $M_1S(G) = M_1(G) + 4e_G$ and $M_2S(G) = M_2(G) + 4e_G$. Therefore,

$$GO_1(S(G)) = GO_1(G) + 8e_G \quad \text{and} \quad I_2 = n_H GO_1(G) + 8n_He_G + 8e_He_G.$$

Substituting I_1 and I_2 in (1) we get required result

$$GO_1(G +_s H) = n_H GO_1(G) + n_G GO_1(H) + e_H M_1(G) + 2e_G M_1(H) + 8n_He_G + 12e_He_G. \quad \square$$

Theorem 2.2 *Let G and H be two connected graphs. Then,*

$$
\begin{aligned}
GO_1(G +_{T_1} H) &= n_G GO_1(H) + 5e_H M_1(G) + 3e_G M_1(H) + 2n_H M_1(G) + 2e_G n_H M_1(G) \\
&\quad +10e_He_G + n_H \sum_{\substack{u_iu_j\in E(G),\\ u_ju_k\in E(G)}} [d_G(u_i)[1 + d_G(u_k)] + d_G(u_k)[1 + d_G(u_j)] \\
&\quad +d_G(u_j)[d_G(u_i) + d_G(u_j)]]
\end{aligned}
$$

Proof Consider

$$
\begin{aligned}
GO_1(G +_{T_1} H) &= \sum_{(p_1,q_1)(p_2,q_2)\in E(G+_{T_1}H)} \big[d_{G+_{T_1}H}(p_1,q_1) + d_{G+_{T_1}H}(p_2,q_2) \\
&\quad +d_{G+_{T_1}H}(p_1,q_1)d_{G+_{T_1}H}(p_2,q_2)\big] \\
&= \sum_{p_1\in V(G)} \sum_{q_1q_2\in E(H)} \big[d_{G+_{T_1}H}(p_1,q_1) + d_{G+_{T_1}H}(p_1,q_2) \\
&\quad +d_{G+_{T_1}H}(p_1,q_1)d_{G+_{T_1}H}(p_1,q_2)\big] \\
&\quad + \sum_{q_1\in V(H)} \sum_{p_1p_2\in E(T_1(G))} \big[d_{G+_{T_1}H}(p_1,q_1) + d_{G+_{T_1}H}(p_2,q_1) \\
&\quad +d_{G+_{T_1}H}(p_1,q_1)d_{G+_{T_1}H}(p_2,q_1)\big].
\end{aligned}
$$

The edge set $E(T_1(G))$ split in to $E(S(G))$ and $E(L(G))$. Let $E(T_1(G)) = \alpha_1$, $V(G) = \beta$,

$V(T_1(G)) - V(G) = \gamma_1$. Then,

$$\begin{aligned} GO_1(G +_{T_1} H) &= \sum_{p_1 \in V(G)} \sum_{q_1 q_2 \in E(H)} \big[d_{G+_{T_1}H}(p_1, q_1) + d_{G+_{T_1}H}(p_1, q_2) \\ &\quad + d_{G+_{T_1}H}(p_1, q_1) d_{G+_{T_1}H}(p_1, q_2)\big] \\ &\quad + \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1 \in \beta, \\ p_2 \in \gamma_1}} \big[d_{G+_{T_1}H}(p_1, q_1) + d_{G+_{T_1}H}(p_2, q_1) \\ &\quad + d_{G+_{T_1}H}(p_1, q_1) d_{G+_{T_1}H}(p_2, q_1)\big] \\ &\quad + \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1, p_2 \in \gamma_1}} \big[d_{G+_{T_1}H}(p_1, q_1) + d_{G+_{T_1}H}(p_2, q_1) \\ &\quad + d_{G+_{T_1}H}(p_1, q_1) d_{G+_{T_1}H}(p_2, q_1)\big] \\ &= J_1 + J_2 + J_3, \end{aligned} \tag{2}$$

where J_1, J_2, J_3 are the sums of the above terms, in order

$$\begin{aligned} J_1 &= \sum_{p_1 \in V(G)} \sum_{q_1 q_2 \in E(H)} \big[2d_{T_1(G)}(p_1) + d_H(q_1) + d_H(q_2) \\ &\quad + [d_{T_1(G)}(p_1) + d_H(q_1)][d_{T_1(G)}(p_1) + d_H(q_2)]\big] \\ &= \sum_{p_1 \in V(G)} \sum_{q_1 q_2 \in E(H)} \Big[2d_{T_1(G)}(p_1) + d_H(q_1) + d_H(q_2) + d^2_{T_1(G)}(p_1) + d_{T_1(G)}(p_1) d_H(q_2) \\ &\quad + d_H(q_1) d_H(q_2) + d_{T_1(G)}(p_1) d_H(q_1)\big] \\ &= \sum_{p_1 \in V(G)} \big[2e_H d_G(p_1) + GO_1(H) + e_H d^2_G(p_1) + d_G(p_1) d_H(q_2) + d_G(p_1) d_H(q_1)\big] \\ &= n_G GO_1(H) + e_H M_1(G) + e_G M_1(H) + 2e_H e_G. \end{aligned}$$

$$\begin{aligned} J_2 &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1 \in \beta, \\ p_2 \in \gamma_1}} \big[[d_{T_1(G)}(p_1) + 2d_H(q_1) + d_{T_1(G)}(p_2)] \\ &\quad + [d_{T_1(G)}(p_1) + d_H(q_1)][d_{T_1(G)}(p_2) + d_H(q_1)]\big] \\ &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1 \in \beta, \\ p_2 \in \gamma_1}} \big[[d_G(p_1) + 2d_H(q_1) + d_{T_1(G)}(p_2)] \\ &\quad + [d_G(p_1) + d_H(q_1)][d_{T_1(G)}(p_2) + d_H(q_1)]\big] \\ &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1 p_2 \in \alpha_1, \\ p_1 \in \beta, \\ p_2 \in \gamma_1}} \big[d_G(p_1) + 2d_H(q_1) + d_{T_1(G)}(p_2) + d_G(p_1) d_{T_1(G)}(p_2) \\ &\quad + d_G(p_1) d_H(q_1) + d_H(q_1) d_{T_1(G)}(p_2) + d^2_H(q_1)\big] \end{aligned}$$

$$= \sum_{q_1\in V(H)}\sum_{p_1\in V(G)}\left[d_G(p_1)[d_G(p_1)+2d_H(q_1)+d_G(p_1)d_H(q_1)+d_H^2(q_1)]\right]$$
$$+\sum_{q_1\in V(H)}\sum_{\substack{p_1p_2\in\alpha_1,\\ p_1\in\beta,\\ p_2\in\gamma_1}}\left[d_{T_1(G)}(p_2)+d_G(p_1)d_{T_1(G)}(p_2)+d_H(q_1)d_{T_1(G)}(p_2)\right].$$

We observe that for $p_2\in V(T_1(G))-V(G)$, $d_{T_1(G)}(p_2)=d_G(w_i)+d_G(w_j)$, where $p_2=w_iw_j\in E(G)$. Hence,

$$\begin{aligned}J_2 &= n_HM_1(G)+8e_He_G+2e_HM_1(G)+2e_GM_1(H)\\ &\quad+\sum_{q_1\in V(H)}\sum_{w_iw_j\in E(G)}[d_G(w_i)+d_G(w_j)+d_G(p_1)[d_G(w_i)+d_G(w_j)]\\ &\quad+d_H(q_1)[d_G(w_i)+d_G(w_j)]]\\ &= 2n_HM_1(G)+8e_He_G+4e_HM_1(G)+2e_GM_1(H)+2e_Gn_HM_1(G).\end{aligned}$$

$$\begin{aligned}J_3 &= \sum_{q_1\in V(H)}\sum_{p_1p_2\in\alpha_1,p_1,p_2\in\gamma_1}\left[[d_{T_1(G)}(p_1)+d_{T_1(G)}(p_2)]+[d_{T_1(G)}(p_1)d_{T_1(G)}(p_2)]\right]\\ &= n_H\sum_{\substack{u_iu_j\in E(G),\\ u_ju_k\in E(G)}}[[d_G(u_i)+d_G(u_j)+d_G(u_j)+d_G(u_k)]\\ &\quad+[d_G(u_i)+d_G(u_j)][d_G(u_j)+d_G(u_k)]]\\ &= n_H\sum_{\substack{u_iu_j\in E(G),\\ u_ju_k\in E(G)}}[d_G(u_i)[1+d_G(u_k)]+d_G(u_k)[1+d_G(u_j)]+d_G(u_j)[d_G(u_i)+d_G(u_j)]].\end{aligned}$$

Adding J_1, J_2, J_3 in (2) we get desired result. □

Theorem 2.3 *Let G and H be two connected graphs. Then,*

$$\begin{aligned}GO_1(G+_{T_2}H) &= 4n_HGO_1(G)+GO_1(H)+8e_HM_1(G)+5e_GM_1(H)+6n_HM_1(G)\\ &\quad+4n_HM_2(G)+24e_He_G+4n_He_G\end{aligned}$$

Proof We know that,

$$\begin{aligned}GO_1(G+_{T_2}H) &= \sum_{(p_1,q_1)(p_2,q_2)\in E(G+_{T_2}H)}\left[d_{G+_{T_2}H}(p_1,q_1)+d_{G+_{T_2}H}(p_2,q_2)\right.\\ &\quad\left.+d_{G+_{T_2}H}(p_1,q_1)d_{G+_{T_2}H}(p_1,q_2)\right]\\ &= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)}\left[d_{G+_{T_2}H}(p_1,q_1)+d_{G+_{T_2}H}(p_1,q_2)\right.\\ &\quad\left.+d_{G+_{T_2}H}(p_1,q_1)d_{G+_{T_2}H}(p_1,q_2)\right]\\ &\quad+\sum_{q_1\in V(H)}\sum_{p_1p_2\in E(T_1(G))}\left[d_{G+_{T_2}H}(p_1,q_1)+d_{G+_{T_2}H}(p_2,q_1)\right.\\ &\quad\left.+d_{G+_{T_2}H}(p_1,q_1)d_{G+_{T_2}H}(p_2,q_1)\right]\end{aligned}$$

$$= K_1 + K_2, \quad (3)$$

where K_1 and K_1 are the sums of the above terms, in order

$$\begin{aligned}
K_1 &= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)}\Big[2d_{T_2(G)}(p_1)+d_H(q_1)+d_H(q_2)\\
&\quad +d^2_{T_2(G)}(p_1)+d_{T_2(G)}(p_1)[d_H(q_1)+d_H(q_2)]+d_H(q_1)d_H(q_2)\Big]\\
&= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)}\big[4d_{(G)}(p_1)+d_H(q_1)+d_H(q_2)\\
&\quad +4d^2_G(p_1)+2d_{(G)}(p_1)[d_H(q_1)+d_H(q_2)]+d_H(q_1)d_H(q_2)\big]\\
&= \sum_{p_1\in V(G)}\big[4e_Hd_G(p_1)+GO_1(H)+4e_Hd^2_G(p_1)+2d_G(p_1)M_1(H)\big]\\
&= 8e_He_G+GO_1(H)+4e_HM_1(G)+4e_GM_1(H) \quad (3a)
\end{aligned}$$

for edge $\forall p_1p_2 \in E(T_2(G))$ and vertex $q_1 \in V(H)$. Here we denote $E(T_2(G)) = \alpha_2$, $V(G) = \beta$, $V(T_2(G)) - V(G) = \gamma_2$.

$$\begin{aligned}
K_2 &= \sum_{q_1\in V(H)}\sum_{p_1p_2\in E(T_2(G))}\big[d_{G+_{T_2}H}(p_1,q_1)+d_{G+_{T_2}H}(p_2,q_1)\\
&\quad +d_{G+_{T_2}H}(p_1,q_1)d_{G+_{T_2}H}(p_2,q_1)\big]\\
&\quad +\sum_{q_1\in V(H)}\sum_{\substack{p_1p_2\in\alpha_2,\\ p_1\in\beta,\\ p_2\in\gamma_2}}\big[d_{G+_{T_2}H}(p_1,q_1)+d_{G+_{T_2}H}(p_2,q_1)+d_{G+_{T_2}H}(p_1,q_1)d_{G+_{T_2}H}(p_2,q_1)\big]\\
&= K_3+K_4 \quad (3b).
\end{aligned}$$

for $\forall q_1 \in V(H)$ and edge $p_1p_2 \in E(T_2(G))$ if and only if $p_1p_2 \in E(G)$.

$$\begin{aligned}
K_3 &= \sum_{q_1\in V(H)}\sum_{p_1p_2\in E(G)}\Big[d_{G+_{T_2(G)}H}(p_1,q_1)+d_{G+_{T_2(G)}H}(p_2,q_1)\\
&\quad +d_{G+_{T_2(G)}H}(p_1,q_1)d_{G+_{T_2(G)}H}(p_2,q_1)\Big]\\
&= \sum_{q_1\in V(H)}\sum_{p_1p_2\in E(G)}\big[d_{T_2(G)}(p_1)+d_H(q_1)+d_{T_2(G)}(p_2)+d_H(q_1)\\
&\quad +[d_{T_2(G)}(p_1)+d_H(q_1)][d_{T_2(G)}(p_2)+d_H(q_1)]\big]\\
&= \sum_{q_1\in V(H)}\sum_{p_1p_2\in E(G)}[2d_G(p_1)+2d_H(q_1)+2d_G(p_2)+4d_G(p_1)d_G(p_2)\\
&\quad +2d_G(p_1)d_H(q_1)+2d_H(q_1)d_G(p_2)+d^2_H(q_1)\big]\\
&= 4n_HGO_1(G)+4e_HM_1(G)+e_GM_1(H)+4n_HM_2(G)+4e_He_G.
\end{aligned}$$

Since we have $d_{T_2(G)}(p_1) = 2d_G(P_1)$ for each vertex $p_1 \in V(G)$ and $d_{T_2}(p_2) = 2$ for each vertex $p_2 \in V(T_2(G)) - V(G)$,

$$\begin{aligned} K_4 &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1p_2 \in \alpha_2, \\ p_1 \in \beta, \\ p_2 \in \gamma_2}} \big[d_{T_2(G)}(p_1) + d_H(q_1) + d_{T_2(G)}(p_2) \\ &\quad + [d_{T_2(G)}(p_1) + d_H(q_1)] d_{T_2(G)}(p_2)\big] \\ &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1p_2 \in \alpha_2, \\ p_1 \in \beta, \\ p_2 \in \gamma_2}} \big[d_{T_2(G)}(p_1) + d_H(q_1) + d_{T_2(G)}(p_2) \\ &\quad + d_{T_2(G)}(p_1) d_{T_2(G)}(p_2) + d_H(q_1) d_{T_2(G)}(p_2)\big] \\ &= \sum_{q_1 \in V(H)} \sum_{\substack{p_1p_2 \in \alpha_2, \\ p_1 \in \beta, \\ p_2 \in \gamma_2}} [6d_G(p_1) + 3d_H(q_1) + 2] \\ &= \sum_{q_1 \in V(H)} \sum_{p_1 \in V(G)} d_G(p_1) \, [6d_G(p_1) + 3d_H(q_1) + 2] \\ &= 6n_H M_1(G) + 12e_G e_H + 4n_H e_G. \end{aligned}$$

Adding K_3 and K_4 and substitute in (3b) we get

$$4n_H GO_1(G) + 16e_H e_G + 6n_H M_1(G) + 4e_H M_1(G) + e_G M_1(H) + 4n_H M_2(G) + 4n_H e_G. \quad (3c)$$

Substitute ($3a$) and (3c) in (3) we get desired results.

$$\begin{aligned} GO_1(G +_{T_2} H) &= 4n_H GO_1(G) + GO_1(H) + 8e_H M_1(G) + 5e_G M_1(H) + 6n_H M_1(G) \\ &\quad + 4n_H M_2(G) + 24e_H e_G + 4n_H e_G. \end{aligned}$$

This completes the proof. □

Theorem 2.4 *Let G and H be two connected graphs. Then,*

$$\begin{aligned} GO_1(G +_T H) &= 4n_H GO_1(G) + n_G GO_1(H) + 12e_H M_1(G) + 6e_G M_1(H) \\ &\quad + 2n_H M_1(G) + e_G M_2(H) + 8e_G M_1(G) + 20e_H e_G \\ &\quad + n_H \sum_{\substack{q_i q_j \in E(G), \\ q_j q_k \in E(G)}} [d_G(q_i) + 2d_G(q_j) + d_G(q_k) \\ &\quad + [d_G(q_i) + d_G(q_j)][d_G(q_j) + d_G(q_k)]] \end{aligned}$$

Proof Let

$$\begin{aligned} GO_1(G +_T H) &= \sum_{(p_1,q_1)(p_2,q_2) \in E(G +_T H)} [d_{G+_T H}(p_1, q_1) + d_{G+_T H}(p_2, q_2) \\ &\quad + d_{G+_T H}(p_1, q_1) d_{G+_T H}(p_2, q_2)] \end{aligned}$$

$$\begin{aligned}
&= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)}[d_{G+_TH}(p_1,q_1)+d_{G+_TH}(p_1,q_2)\\
&\quad +d_{G+_TH}(p_1,q_1)d_{G+_TH}(p_1,q_2)]\\
&\quad +\sum_{q_1\in V(H)}\sum_{p_1p_2\in E(T(G))}[d_{G+_TH}(p_1,q_1)+d_{G+_TH}(p_2,q_1)\\
&\quad +d_{G+_TH}(p_1,q_1)d_{G+_TH}(p_2,q_1)]\,.
\end{aligned}$$

Note that $E(T(G))=E(G)\bigcup E(S(G))\bigcup E(L(G))$. We get that

$$\begin{aligned}
&GO_1(G+_TH)\\
&=\sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)}[d_{G+_TH}(p_1,q_1)+d_{G+_TH}(p_1,q_2)+d_{G+_TH}(p_1,q_1)d_{G+_TH}(p_1,q_2)]\\
&+\sum_{q_1\in V(H)}\sum_{\substack{(p_1p_2)\in E(T(G)),\\(p_1,p_2)\in V(G)}}[d_{G+_TH}(p_1,q_1)+d_{G+_TH}(p_2,q_1)+d_{G+_TH}(p_1,q_1)d_{G+_TH}(p_2,q_1)]\\
&+\sum_{q_1\in V(H)}\sum_{\substack{(p_1p_2)\in\alpha_3,\\p_1\in\beta,\\p_2\in\gamma_3}}[d_{G+_TH}(p_1,q_1)+d_{G+_TH}(p_2,q_1)+d_{G+_TH}(p_1,q_1)d_{G+_TH}(p_2,q_1)]\\
&+\sum_{q_1\in V(H)}\sum_{\substack{(p_1p_2)\in\alpha_3,\\(p_1,p_2)\in\gamma_3}}[d_{G+_TH}(p_1,q_1)+d_{G+_TH}(p_2,q_1)+d_{G+_TH}(p_1,q_1)d_{G+_TH}(p_2,q_1)]\\
&=L_1+L_2+L_3+L_4, \qquad (4)
\end{aligned}$$

where L_1,L_2,L_3,L_4 are the sums of the above terms, in order

$$\begin{aligned}
L_1 &= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)}\big[2d_{T(G)}(p_1)+d_H(q_1)+d_H(q_2)\\
&\quad +[d_{T(G)}(p_1)+d_H(q_1)][d_{T(G)}(p_1)d_H(q_2)]\big]\\
&= \sum_{p_1\in V(G)}\sum_{q_1q_2\in E(H)}\big[4d_G(p_1)+d_H(q_1)+d_H(q_2)+4d_G^2(p_1)+2d_G(p_1)d_H(q_1)\big]\\
&\quad +2d_G(p_1)d_H(q_2)+d_H(q_1)d_H(q_2)\\
&= n_GGO_1(H)+4e_HM_1(G)+4e_GM_1(H)+8e_Ge_H.
\end{aligned}$$

$$\begin{aligned}
L_2 &= \sum_{q_1\in V(H)}\sum_{p_1p_2\in\alpha_3,p_1,p_2\in\beta}\big[d_{T(G)}(p_1)+2d_H(q_1)+d_{T(G)}(p_2)\\
&\quad +[d_{T(G)}(p_1)+d_H(q_1)][d_{T(G)}(p_2)d_H(q_1)]\big]\\
&= \sum_{q_1\in V(H)}\sum_{p_1p_2\in E(G)}\big[2d_G(p_1)+2d_G(p_2)+2d_H(q_1)+d_H^2(q_1)+2d_G(p_2)d_H(q_1)\\
&\quad +4d_G(p_1)d_G(p_2)+2d_G(p_1)d_H(q_1)]\\
&= 2n_HGO_1(G)+4e_HM_1(G)+e_GM_2(H)+2n_HM_2(G)+4e_Ge_H.
\end{aligned}$$

$$\begin{aligned}
L_3 &= \sum_{q_1\in V(H)} \sum_{\substack{p_1p_2\in\alpha_3,\\ p_1\in\beta,\\ p_2\in\gamma_3}} \big[d_{T(G)}(p_1)+d_{T(G)}(p_2)+2d_H(q_1) \\
&\quad +[d_{T(G)}(p_1)+d_H(q_1)][d_{T(G)}(p_2)d_H(q_1)]\big] \\
&= \sum_{q_1\in V(H)} \sum_{(p_1\in V(G)} \big[d_G(p_1)2d_G(p_1)+d_H(q_1)+d_H(q_1)+d_G(p_1)d_H(q_1)+d_H^2(q_1)\big] \\
&\quad + \sum_{q_1\in V(H)} \sum_{\substack{p_1p_2\in\alpha_3,\\ p_1\in\beta,\\ p_2\in\gamma_3}} \big[d_{T(G)}(p_2)+2d_G(p_1)d_{T(G)}(p_2)+d_H(q_1)d_{T(G)}(p_2)\big].
\end{aligned}$$

Note that $p_2 \in V(T(G)) - V(G)$, $d_{T(G)}(p_2) = d_G(p) + d_G(q)$ where $p_2 = pq \in E(G)$, we further get that

$$\begin{aligned}
L_3 &= 2n_HM_1(G)+4e_HM_1(G)+2e_GM_1(H)+8e_He_G \\
&\quad + \sum_{q_1\in V(H)} \sum_{\substack{p_1\in\beta,\\ p_2\in\gamma_3}} [(d_G(p)+d_G(q))+2d_G(p_1)(d_G(p)+d_G(q))+d_H(q_1)(d_G(p)+d_G(q))] \\
&= 2n_HM_1(G)+4e_HM_1(G)+2e_GM_1(H)+2n_HM_1(G)+8e_GM_1(G)+4e_HM_1(G) \\
&= 4n_HM_1(G)+4e_HM_1(G)+8e_GM_1(G)+2e_GM_1(H)+8e_He_G.
\end{aligned}$$

$$\begin{aligned}
L_4 &= \sum_{q_1\in V(H)} \sum_{(p_1,p_2)\in\gamma_3} [d_{G+_TH}(p_1,q_1)+d_{G+_TH}(p_2,q_1)+d_{G+_TH}(p_1,q_1)d_{G+_TH}(p_2,q_1)] \\
&= \sum_{q_1\in V(H)} \sum_{\substack{p_1,\\ p_2\in\gamma_3}} \big[d_{T(G)}(p_1)+d_{T(G)}(p_2)+d_{T(G)}(p_1)d_{T(G)}(p_2)\big] \\
&= n_H \sum_{q_iq_j\in E(G), q_jq_k\in E(G)} [(d_G(q_i)+d_G(q_j))+(d_G(q_j)+d_G(q_k)) \\
&\quad +[d_G(q_i)+d_G(q_j)][d_G(q_j)+d_G(q_k)]]
\end{aligned}$$

Adding L_1, L_2, L_3, L_4 in (4) we get required result. □

§3. Conclusion

In this paper, we obtain explicit expression for the Gourava index of four operation on graphs in terms of first Zagreb and second Zagreb index.

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Strongly 2-Multiplicative Graphs

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Abstract: Since the year 2000 a number of authors have studied strongly multiplicative graphs. In this vein we introduce the concept of strongly k-multiplicative graph and prove that certain class of graphs such as paths, binary tree, cycle etc. are strongly 2-multiplicative.

Key Words: Strongly 2-multiplicative, graph labelling, paths, star, fan graph, binary tree, comb graph, triangular snake, ladder.

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§1. Introduction

A graph G consists of a nonempty set $V = V(G)$ of points called vertices and another set $E = E(G)$ whose elements are called edges where each edge is identified with an unordered pair of vertices in V. Each pair $e = (u, v)$ in E of points of V is an edge of G and is said to be incident with u and v. In this case u and v are said to be adjacent to each other. The number of vertices in G is called the order of G.

We begin with some basic definitions and notations [7], [12], [6].

Definition 1.1 *A walk of a graph G is a finite, alternative sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \cdots, v_{n-1}, e_n, v_n$, beginning with v_0 and ending with v_n such that each edge e_i is incident with v_{i-1} and v_i. The number of edges is called the length of the walk. A walk is called a path if all its vertices (and thus necessarily all the edges) are distinct. A path on n vertices is denoted by P_n.*

Definition 1.2 *A walk in a graph is closed if its initial and terminal vertices are identical. A closed walk is called a cycle. A cycle on $n(\geq 3)$ vertices is denoted by C_n.*

Definition 1.3 *A graph G is said to be complete if every pair of its distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n.*

Definition 1.4 *A bigraph or bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. (V_1, V_2) is a bipartition of G.*

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A complete bipartite graph is a bipartite graph with bipartition (V_1, V_2) such that every vertex of V_1 joined to all the vertices of V_2. If V_1 contains m points and V_2 contains n points then the complete bipartite graph is denoted by $K_{m,n}$. A star $K_{1,n}$ is a complete bipartite graph.

Definition 1.5 *A graph is acyclic if it has no cycles. A tree is a connected acyclic graph.*

Definition 1.6 *The wheel $W_n(n \geq 4)$ is the graph obtained from the join of K_1 and C_{n-1}.*

Definition 1.7 *A fan $F_n(n \geq 2)$ is the graph obtained from the join of the path P_n and K_1.*

Definition 1.8 *A ladder L_n is a graph with vertex set $V(L_n) = \{v_i : 1 \leq i \leq 2n\}$ and edge set $E(L_n) = \{v_{2i}v_{2i+2}, v_{2i-1}v_{2i+1} : 1 \leq i \leq n-1\} \cup \{v_{2i-1}v_{2i} : 1 \leq i \leq n\}$.*

Definition 1.9 *A triangular ladder is a graph T_n, whose vertex set is $V(T_n) = \{v_i : 1 \leq i \leq 2n\}$ and whose edge set is $E(T_n) = E(L_n) \bigcup \{v_{2i}v_{2i+1} : 1 \leq i \leq n-1\}$.*

Definition 1.10 *A complete n-ary tree is a tree in which every internal vertex is of degree $n+1$, the root vertex is of degree n and the pendent vertices are of degree 1 and have the same depth.*

Definition 1.11 *A chord of a cycle C_n is an edge joining two non-adjacent vertices of the cycle C_n.*

Definition 1.12 *The graph obtained by joining a single pendent edge to each vertex of a path is called a comb.*

Definition 1.13 *Duplication of a vertex v by a new edge $e = uw$ in a graph G produces a new graph $G^{'}$ such that $N(u) = \{v, w\}$ and $N(w) = \{u, v\}$.*

Definition 1.14 *Duplication of an edge $e = uv$ by a new vertex w in a graph G produces a new graph $G^{'}$ such that $N(w) = \{u, v\}$.*

Definition 1.15 *A triangular snake is a graph obtained from the duplication of each edge of a path by a new vertex.*

Definition 1.16 *The windmill graph $K_n^m, (n > 3)$ consists of m copies of K_n with a vertex in common.*

Consider a graph G of order n. Let P_1 and P_2 be two paths in G with the same vertex set V. Then we say that P_1 and P_2 are path homotopic with respect to V. We denote this by $P_1 \simeq_V P_2$. One can easily prove that this relation is an equivalence relation. Let $\mathcal{P}$ be the path homotopy class consisting of those paths which are path homotopic to the path P with a given vertex set and let $\mathcal{A}$ denote the set of all distinct path homotopy classes in G.

Definition 1.17 *A graph G of order n is said to be strongly k-multiplicative if there is an injective mapping $f : V(G) \to \{1, 2, \cdots, n\}$ such that the induced mapping $h : \mathcal{A} \to \mathbb{Z}^+$ defined by $h(\mathcal{P}) = \prod_{i=1}^{k+1} f(v_{j_i})$, where $j_1, j_2, \cdots, j_{k+1} \in \{1, 2, \cdots, n\}$, $k+1 \leq n$ and $\mathcal{P}$ is the path* *homotopy class of paths having the vertex set* $\{v_{j_1}, v_{j_2}, \cdots, v_{j_{k+1}}\}$*, is injective.*

In particular, if k=2 we call G, strongly 2- multiplicative and if k=1, then we call G, strongly 1- multiplicative or simply strongly multiplicative.

In 2001, L. W. Beineke and S. M. Hegde [5] have introduced the concept of strongly multiplicative graphs. Since then many authors including C. Adiga, H. N. Ramaswamy and D. D. Somashekara [2],[3], [4], M. A. Seoud and A. Zid [9], B. D. Acharya, Germina and Ajitha [1], S. K. Vaidya and K. K. Kanani [10], [11] and M. Muthusamy, K. C. Raajasekar and J. Basker Babujee [8] have also studied and contributed to the concept of strongly multiplicative graphs. For more details one may refer the survey article "*A dynamic survey of graph labeling*" by J. A. Gallian [6].

In the next section we prove our main results.

§2. Main Results

We first note that for a graph to be strongly 2-multiplicative, it has to have at least 3 vertices.

Theorem 2.1 *The path P_n is strongly* 2*-multiplicative.*

Proof Consider a path P_n of length $n-1$. We label the vertices as follows: $v_i = i$ for all i. Then $\mathcal{A}$ consists of $n-2$ distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots, \mathcal{P}_{n-2}$, where $\mathcal{P}_i$ is the path homotopy class of paths having the vertex set $\{v_i, v_{i+1}, v_{i+2}\}$, for $1 \leqslant i \leq n-2$. Then $h(\mathcal{P}_i)$=$(i)(i+1)(i+2)$, for $1 \leq i \leq n-2$. Since $i(i+1)(i+2) < (i+1)(i+2)(i+3)$, for $1 \leq i \leq n-3$, it follows that $h(\mathcal{P}_i) < h(\mathcal{P}_{i+1})$, for $1 \leq i \leq n-3$. Hence h is injective and P_n is strongly 2-multiplicative. □

Theorem 2.2 *Every cycle C_n, is strongly* 2*-multiplicative.*

Proof Consider a cycle C_n=$(v_1, v_2, v_3, \cdots, v_n, v_1)$ of order n and let p be the largest prime less than n. We label the vertices as follows: v_i=i, for $1 \leq i \leq p-1$, $v_i = i+1$, for $p \leq i \leq n-1$ and $v_n = p$. If $n = 3$, then $\mathcal{A}$ consists of only one path homotopy class and is trivially strongly 2-multiplicative. If $n > 3$, then $\mathcal{A}$ consists of n distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots, \mathcal{P}_n$, where $\mathcal{P}_i$ is the path homotopy classes of paths having the vertex sets $\{v_i, v_{i+1}, v_{i+2}\}$, for $1 \leq i \leq n-2$, $\mathcal{P}_{n-1}$ is the path homotopy class of paths having the vertex set $\{v_{n-1}, v_n, v_1\}$ and $\mathcal{P}_n$ is the path homotopy class of paths having the vertex set $\{v_n, v_1, v_2\}$. Then $h(\mathcal{P}_i) = (i)(i+1)(i+2)$, for $1 \leq i \leq p-3$, $h(\mathcal{P}_{p-2}) = (p-2)(p-1)(p+1)$, $h(\mathcal{P}_{p-1}) = (p-1)(p+1)(p+2)$, $h(\mathcal{P}_i) = (i+1)(i+2)(i+3)$, for $p \leq i \leq n-3$, $h(\mathcal{P}_{n-2}) = (n-1)(n)(p)$ or $h(\mathcal{P}_{n-2}) = (n-2)(n)(p)$, if p is the immediate predecessor of n, $h(\mathcal{P}_{n-1}) = n \cdot p \cdot 1$ and $h(\mathcal{P}_n) = p \cdot 1 \cdot 2$. Then from the definition of h it follows that $h(\mathcal{P}_i) < h(\mathcal{P}_{i+1})$, $1 \leq i \leq n-3$ and $h(\mathcal{P}_n) < h(\mathcal{P}_{n-1}) < h(\mathcal{P}_{n-2})$, also $h(\mathcal{P}_i) \neq h(\mathcal{P}_j)$, $n-2 \leq j \leq n$ and $1 \leq i \leq n-3$. Since $h(\mathcal{P}_j)$ is divisible by p, where as $h(\mathcal{P}_i)$ is not, h is injective and the graph C_n is strongly 2-multiplicative. □

Theorem 2.3 *Every cycle with one chord is strongly 2-multiplicative.*

Proof First, consider a cycle C_4 with vertices v_1, v_2, v_3, v_4. Let the chord be $e = v_1v_3$. We label the vertices as follows: $v_1 = 1$, $v_2 = 4$, $v_3 = 2$ and $v_4 = 3$. Then $\mathcal{A}$ consists of 4 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$, corresponding to the path homotopy classes of paths having the vertex sets $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_1\}$ and $\{v_4, v_1, v_2\}$ respectively. Then $h(\mathcal{P}_1) = 8$, $h(\mathcal{P}_2) = 24$, $h(\mathcal{P}_3) = 6$, $h(\mathcal{P}_4) = 12$. Clearly h is injective and C_4 with one chord is strongly 2-multiplicative.

Second, consider a cycle C_5 with vertices v_1, v_2, v_3, v_4, v_5. Let the chord be $e = v_1v_3$. We label the vertices as follows: $v_1 = 1$, $v_2 = 4$, $v_3 = 2$, $v_4 = 5$ and $v_5 = 3$. Then $\mathcal{A}$ consists of 7 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, $\mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6$ and $\mathcal{P}_7$, corresponding to path homotopy classes of paths having the vertex sets $\{v_1, v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_1\}$, $\{v_5, v_1, v_2\}$, $\{v_5, v_1, v_3\}$ and $\{v_4, v_3, v_1\}$ respectively. Then $h(\mathcal{P}_1)$=8, $h(\mathcal{P}_2) = 40$, $h(\mathcal{P}_3) = 30$, $h(\mathcal{P}_4) = 15$, $h(\mathcal{P}_5) = 12$, $h(\mathcal{P}_6) = 6$ and $h(\mathcal{P}_7) = 10$. Clearly h is injective and C_5 with one chord is strongly 2-multiplicative.

Finally, let $n > 5$. Consider a cycle $C_n = (v_1, v_2, v_3, \cdots, v_n, v_1)$ of order n and let p_1 and p_2 be the two consecutive primes such that $0 < p_2 < p_1 < n$ and that p_1 is the largest. Let $e = v_1v_{p_2}$ be the chord of the cycle C_n. We label the vertices as follows: $v_i = i$, for $1 \leq i \leq p_1-1$, $v_i = i+1$, for $p_1 \leq i \leq n-1$ and v_n=p_1. Then $\mathcal{A}$ consists of $n+4$ ($n+2$, in case $n = 6$ and $n = 7$) distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots, \mathcal{P}_n$, $\mathcal{P}_{n+1}$, $\mathcal{P}_{n+2}$, $\mathcal{P}_{n+3}$, $\mathcal{P}_{n+4}$, where $\mathcal{P}_i$ is the path homotopy class of paths having the vertex sets $\{v_i, v_{i+1}, v_{i+2}\}$, for $1 \leq i \leq n-2$ and $\mathcal{P}_{n-1}$, $\mathcal{P}_n$, $\mathcal{P}_{n+1}$, $\mathcal{P}_{n+2}$, $\mathcal{P}_{n+3}$ and $\mathcal{P}_{n+4}$ are the path homotopy classes of paths having the vertex set $\{v_{n-1}, v_n, v_1\}$, $\{v_n, v_1, v_2\}$, $\{v_n, v_1, v_{p_2}\}$, $\{v_{p_2+1}, v_{p_2}, v_1\}$, $\{v_2, v_1, v_{p_2}\}$ and $\{v_{p_2-1}, v_{p_2}, v_1\}$ respectively. Then $h(\mathcal{P}_i)$=$(i)(i+1)(i+2)$, for $1 \leq i \leq p_1-3$, $h(\mathcal{P}_{p_1-2}) = (p_1-2)(p_1-1)(p_1+1)$, $h(\mathcal{P}_{p_1-1}) = (p_1-1)(p_1+1)(p_1+2)$, $h(\mathcal{P}_i) = (i+1)(i+2)(i+3)$, for $p_1 \leq i \leq n-3$, $h(\mathcal{P}_{n-2}) = (n-1)(n)(p_1)$ or $h(\mathcal{P}_{n-2}) = (n-2)(n)(p_1)$, if p_1 is the immediate predecessor of n, $h(\mathcal{P}_{n-1})$=$n.p_1.1$, $h(\mathcal{P}_n)$=$p_1.1.2$, $h(\mathcal{P}_{n+1})$=$p_1.1.p_2$, $h(\mathcal{P}_{n+2})$=$(p_2+1).p_2.1$ $h(\mathcal{P}_{n+3})$=$2.1.p_2$ and $h(\mathcal{P}_{n+4})$=$(p_2-1).p_2.1$. Then from the definition of h it follows that $h(\mathcal{P}_i)$<$h(\mathcal{P}_{i+1})$, for 1≤i≤p_2-3 and p_2+1≤i≤$n-3$ and $h(\mathcal{P}_n)$<$h(\mathcal{P}_{n-1})$< $h(\mathcal{P}_{n-2})$, also $h(\mathcal{P}_i)$≠ $h(\mathcal{P}_j)$, 1≤i≤p_2-3, p_2+1≤i≤$n-3$ and $n-2$≤j≤n. Since $h(\mathcal{P}_j)$ is divisible by p_1, where as $h(\mathcal{P}_i)$ is not. $h(\mathcal{P}_{n+3}) < h(\mathcal{P}_{n+4}) < h(\mathcal{P}_{n+2}) < h(\mathcal{P}_{n+1}) < h(\mathcal{P}_{p_2-2}) < h(\mathcal{P}_{p_2-1}) < h(\mathcal{P}_{p_2})$ and these are not equal to $h(\mathcal{P}_i)$ and $h(\mathcal{P}_j)$, where 1≤i≤p_2-3, p_2+1≤i≤$n-3$ and $n-2$≤j≤n, since these are divisible by p_2 whereas $h(\mathcal{P}_i)$ and $h(\mathcal{P}_j)$ are not. Hence h is injective and $C_n, n > 5$ with one chord is strongly 2-multiplicative. □

Remark 2.4 (1) In general, a cycle C_n=$(v_1, v_2, v_3, \cdots, v_n, v_1)$ with one chord joining any two non adjacent vertices, can be shown to be strongly 2-multiplicative.

(2) A cycle with twin chords can be shown to be strongly 2-multiplicative.

Theorem 2.5 *The graph obtained by duplication of an arbitrary vertex of a cycle by a new edge is strongly 2-multiplicative.*

Proof Consider a cycle C_n=$(v_1, v_2, v_3, \cdots, v_n, v_1)$. We duplicate the vertex v_n by an edge e with end vertices v_{n+1} and v_{n+2}. Let the graph so obtained be G. Then $\mid V(G) \mid= n+2$ and $\mid E(G) \mid= n+3$. Let p be the largest prime less than n. We label the vertices as follows: v_i=i, for $1 \leq i \leq p-1$ and for $n < i \leq n+2$, $v_i = i+1$, for $p \leq i \leq n-1$ and $v_n = p$. If $n = 3$, then $\mathcal{A}$ consists of 6 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$, corresponding to the path homotopy classes of paths having the vertex sets $\{v_1, v_2, v_3\}$, $\{v_3, v_4, v_5\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_3, v_5\}$, $\{v_1, v_3, v_4\}$ and $\{v_1, v_3, v_5\}$ respectively. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 40$, $h(\mathcal{P}_3) = 24$, $h(\mathcal{P}_4) = 30$, $h(\mathcal{P}_5) = 8$, $h(\mathcal{P}_4) = 10$. If $n > 3$, then $\mathcal{A}$ consists of $n+5$ distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots, \mathcal{P}_n$, $\mathcal{P}_{n+1}$, $\mathcal{P}_{n+2}$, $\mathcal{P}_{n+3}$, $\mathcal{P}_{n+4}$, $\mathcal{P}_{n+5}$, where $\mathcal{P}_i$ is the path homotopy class of paths having the vertex sets $\{v_i, v_{i+1}, v_{i+2}\}$, for $1 \leq i \leq n-2$ and $\mathcal{P}_{n-1}$, $\mathcal{P}_n$, $\mathcal{P}_{n+1}$, $\mathcal{P}_{n+2}$, $\mathcal{P}_{n+3}$, $\mathcal{P}_{n+4}$ and $\mathcal{P}_{n+5}$ are the path homotopy classes of paths having the vertex sets $\{v_{n-1}, v_n, v_1\}$, $\{v_n, v_1, v_2\}$, $\{v_n, v_{n+1}, v_{n+2}\}$, $\{v_{n+1}, v_n, v_{n-1}\}$, $\{v_{n+1}, v_n, v_1\}$, $\{v_{n+2}, v_n, v_{n-1}\}$ and $\{v_{n+2}, v_n, v_1\}$ respectively. Then $h(\mathcal{P}_i) = (i)(i+1)(i+2)$, for $1 \leq i \leq p-3$, $h(\mathcal{P}_{p-2}) = (p-2)(p-1)(p+1)$, $h(\mathcal{P}_{p-1}) = (p-1)(p+1)(p+2)$, $h(\mathcal{P}_i) = (i+1)(i+2)(i+3)$, for $p \leq i \leq n-3$, $h(\mathcal{P}_{n-2}) = (n-1)(n)(p)$ or $h(\mathcal{P}_{n-2}) = (n-2)(n)(p)$, if p is the immediate predecessor of n, $h(\mathcal{P}_{n-1}) = n \cdot p \cdot 1$, $h(\mathcal{P}_n) = p \cdot 1 \cdot 2$, $h(\mathcal{P}_{n+1}) = p \cdot (n+1) \cdot (n+2)$, $h(\mathcal{P}_{n+2}) = n \cdot p \cdot (n+1)$, $h(\mathcal{P}_{n+3}) = (n+1) \cdot p \cdot 1$, $h(\mathcal{P}_{n+4}) = (n+2) \cdot p \cdot n$ and $h(\mathcal{P}_{n+5}) = (n+2) \cdot p \cdot 1$. Then from the definition of h it follows that $h(\mathcal{P}_i) < h(\mathcal{P}_{i+1})$, $1 \leq i \leq n-3$ and $h(\mathcal{P}_n) < h(\mathcal{P}_{n-1}) < h(\mathcal{P}_{n+3}) < h(\mathcal{P}_{n+5}) < h(\mathcal{P}_{n-2}) < h(\mathcal{P}_{n+2}) < h(\mathcal{P}_{n+4}) < h(\mathcal{P}_{n+1})$ and these not equal to $h(\mathcal{P}_k)$ where $1 \leq k \leq n-3$, since these are divisible by p whereas $h(\mathcal{P}_k)$ is not. Hence h is injective and the graph obtained by duplication of an arbitrary vertex of a cycle by a new edge is strongly 2-multiplicative. □

Remark 2.6 If we duplicate an edge in a cycle of an order n by a new vertex, then we obtain a cycle of order $n+1$ with one chord. Hence by Theorem 2.3 the graph obtained by duplication of an arbitrary edge of cycle by a new vertex is strongly 2-multiplicative.

Theorem 2.7 *The comb graph is strongly 2-multiplicative*

Proof Consider the comb graph G of order $2n (n \geq 2)$ with vertex set $G = \{v_1, v_2, v_3, \cdots, v_{2n}\}$ as shown below.

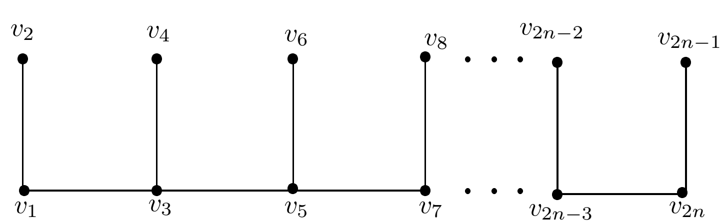

Figure 1

Then $\mathcal{A}$ consists of $3n-4$ distinct path homotopy classes $\mathcal{P}_{2i-1,2i+1,2i+3}$, $\mathcal{P}_{2i-1,2i,2i+1}$, $\mathcal{P}_{2i-1,2i+1,2i+2}$, corresponding to path homotopy classes of paths having vertex sets $\{v_{2i-1}, v_{2i+1}, v_{2i+3}\}$, $\{v_{2i-1}, v_{2i}, v_{2i+1}\}$ and $\{v_{2i-1}, v_{2i+1}, v_{2i+2}\}$ respectively, for $1 \leq i \leq n-2$ and path homotopy classes $\mathcal{P}_{2n-3,2n-2,2n-1}$, $\mathcal{P}_{2n-3,2n-1,2n}$ corresponding to path homotopy classes of paths having the vertex sets $\{v_{2n-3}, v_{2n-2}, v_{2n-1}\}$ and $\{v_{2n-3}, v_{2n-1}, v_{2n}\}$ respectively. We label the vertices as follows: $v_i = i$, for all i. Then $h(\mathcal{P}_{i,j,k}) = i \cdot j \cdot k$. Since $(2i-1) \cdot (2i) \cdot (2i+1) < (2i-1) \cdot (2i+1) \cdot (2i+2) < (2i-1) \cdot$

$(2i+1)\cdot(2i+3)$, $(2i-1)\cdot(2i+1)\cdot(2i+3) < (2i+1)\cdot(2i+2)\cdot(2i+3)$, for $1 \le i \le n-2$ and $(2i-1)\cdot(2i)\cdot(2i+1) < (2i-1)\cdot(2i+1)\cdot(2i+2)$ for $i = n-1$, it follows that $h(\mathcal{P}_{1,2,3}) < h(\mathcal{P}_{1,3,4}) < \cdots < h(\mathcal{P}_{2n-3,2n-1,2n})$. Therefore h is injective and the comb graph is strongly 2-multiplicative. □

Theorem 2.8 *The triangular snake graph is strongly 2-multiplicative.*

Proof Consider the triangular snake graph $T_n (n \ge 2)$ with vertex set $V(T_n) = \{v_1, v_2, v_3, \cdots, v_{2n-1}\}$ as shown below.

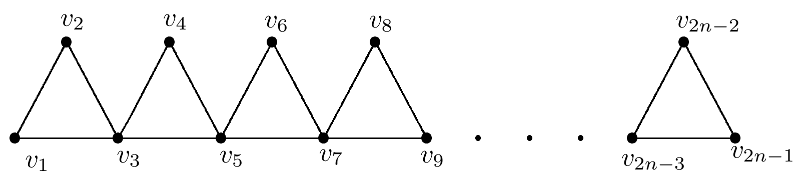

Figure 2

Then $\mathcal{A}$ consists of $5n-9$ distinct path homotopy classes

$$\mathcal{P}_{2i-1,2i,2i+1},\ \mathcal{P}_{2i-1,2i+1,2i+2},\ \mathcal{P}_{2i-1,2i+1,2i+3},\ \mathcal{P}_{2i,2i+1,2i+2},\ \mathcal{P}_{2i,2i+1,2i+3}$$

corresponding to path homotopy classes of paths having vertex sets

$$\{v_{2i-1}, v_{2i}, v_{2i+1}\},\ \{v_{2i-1}, v_{2i+1}, v_{2i+2}\},\ \{v_{2i-1}, v_{2i+1}, v_{2i+3}\},\ \{v_{2i}, v_{2i+1}, v_{2i+2}\}$$

and $\{v_{2i}, v_{2i+1}, v_{2i+3}\}$ respectively, for $1 \le i \le n-2$ and path homotopy classes $\mathcal{P}_{2n-3,2n-2,2n-1}$ corresponding to path homotopy class of paths having the vertex set $\{v_{2n-3}, v_{2n-2}, v_{2n-1}\}$. We label the vertices as follows: $v_i = i$, for all i. Then $h(\mathcal{P}_{i,j,k}) = i\cdot j\cdot k$. Since $(2i-1)\cdot(2i)\cdot(2i+1) < (2i-1)\cdot(2i+1)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+3) < (2i)\cdot(2i+1)\cdot(2i+2) < (2i)\cdot(2i+1)\cdot(2i+3)$, $(2i)\cdot(2i+1)\cdot(2i+3) < (2i+1)\cdot(2i+2)\cdot(2i+3)$, for $1 \le i \le n-2$ and $(2n-3)\cdot(2n-1)\cdot(2n-4) < (2n-3)\cdot(2n-2)\cdot(2n-1)$ it follows that $h(\mathcal{P}_{1,2,3}) < h(\mathcal{P}_{1,3,4}) < \ldots < h(\mathcal{P}_{2n-3,2n-2,2n-1})$. Therefore h is injective and the triangular snake graph is strongly 2-multiplicative. □

Theorem 2.9 *The ladder graph L_n is strongly 2-multiplicative.*

Proof Consider the ladder graph L_n with vertex set $V(L_n) = \{v_1, v_2, v_3, \cdots, v_{2n}\}$ as shown below.

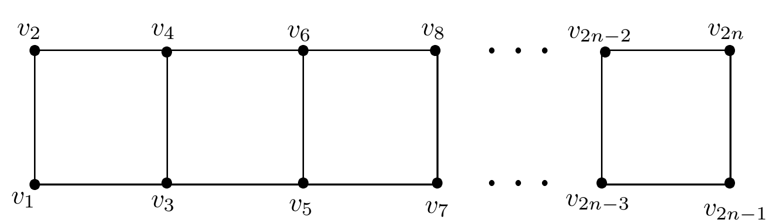

Figure 3

Then $\mathcal{A}$ consists of $6n-8$ distinct path homotopy classes

$\mathcal{P}_{2i,2i-1,2i+1}$, $\mathcal{P}_{2i-1,2i,2i+2}$, $\mathcal{P}_{2i-1,2i+1,2i+2}$, $\mathcal{P}_{2i-1,2i+1,2i+3}$, $\mathcal{P}_{2i,2i+2,2i+4}$, $\mathcal{P}_{2i,2i+2,2i+1}$, corresponding to path homotopy classes of paths having vertex sets $\{v_{2i}, v_{2i-1}, v_{2i+1}\}$, $\{v_{2i-1}, v_{2i}, v_{2i+2}\}$, $\{v_{2i-1}, v_{2i+1}, v_{2i+2}\}$, $\{v_{2i-1}, v_{2i+1}, v_{2i+3}\}$, $\{v_{2i}, v_{2i+2}, v_{2i+4}\}$ and $\{v_{2i}, v_{2i+2}, v_{2i+1}\}$ respectively, for $1 \leq i \leq n-2$ and path homotopy classes $\mathcal{P}_{2n-2,2n-3,2n-1}$, $\mathcal{P}_{2n-3,2n-2,2n}$, $\mathcal{P}_{2n-3,2n-1,2n}$, $\mathcal{P}_{2n-2,2n,2n-1}$ corresponding to path homotopy classes of paths having the vertex sets $\{v_{2n-2}, v_{2n-3}, v_{2n-1}\}$, $\{v_{2n-3}, v_{2n-2}, v_{2n}\}$, $\{v_{2n-3}, v_{2n-1}, v_{2n}\}$ and $\{v_{2n-2}, v_{2n}, v_{2n-1}\}$ respectively. We label the vertices as follows: v_i=i, for all i. Then $h(\mathcal{P}_{i,j,k}) = i \cdot j \cdot k$. Since $(2i)\cdot(2i-1)\cdot(2i+1) < (2i-1)\cdot(2i)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+3) < (2i)\cdot(2i+2)\cdot(2i+1) < (2i)\cdot(2i+2)\cdot(2i+4)$, $(2i)\cdot(2i+2)\cdot(2i+4) < (2i+2)\cdot(2i+1)\cdot(2i+3)$, for $1 \leq i \leq n-2$ and $(2i)\cdot(2i-1)\cdot(2i+1) < (2i-1)\cdot(2i)\cdot(2i+2) < (2i-1)\cdot(2i+1)\cdot(2i+2) < (2i)\cdot(2i+2)\cdot(2i+1)$ for $i = n-1$ it follows that $h(\mathcal{P}_{2,1,3}) < h(\mathcal{P}_{1,2,4}) < \cdots < h(\mathcal{P}_{2n-2,2n,2n-1})$. Therefore h is injective and the graph L_n is strongly 2-multiplicative. □

Theorem 2.10 *The binary tree is strongly 2-multiplicative.*

Proof Consider the binary tree G consisting of $2^{n+1}-1$ vertices with n levels. We label the vertices, using breadth-first search method as follows $v_i = i$, for $1 \leq i \leq 2^{n+1}-1$ as shown in the figure.

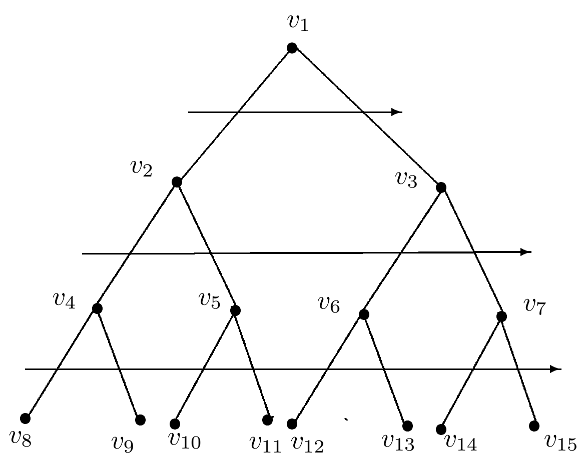

Figure 4

If $n = 1$ then the tree becomes a path with 3 vertices and is trivially strongly 2-multiplicative. So, let $n > 1$. Then for each m, consisting of the edges of level $m-1$ and of the level m, $1 < m \leq n-1$, there are 5.2^{m-2} distinct path homotopy classes consisting of 2^{m-2} bunches of 5 path homotopy classes $\mathcal{P}_{m,r,1}$, $\mathcal{P}_{m,r,2}$, $\mathcal{P}_{m,r,3}$, $\mathcal{P}_{m,r,4}$, $\mathcal{P}_{m,r,5}$ corresponding to path homotopy classes of paths having vertex sets

$$\{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)}, v_{2(2^{m-2}+r-1)+1}\},\ \{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)}, v_{4(2^{m-2}+r-1)}\},$$
$$\{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)}, v_{4(2^{m-2}+r-1)+1}\},\ \{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)+1}, v_{4(2^{m-2}+r-1)+2}\}$$

and $\{v_{2^{m-2}+r-1}, v_{2(2^{m-2}+r-1)+1}, v_{4(2^{m-2}+r-1)+3}\}$ respectively, where $1 \leq r \leq 2^{m-2}$ and if $m = n$, in addition to $5 \cdot 2^{m-2}$ distinct path homotopy classes described above we have 2^{n-1} distinct path homotopy classes $\mathcal{P}_{n+1,r,1}$ corresponding to the paths having vertex sets

$\{v_{2^{n-1}+r-1}, v_{2(2^{n-1}+r-1)}, v_{2(2^{n-1}+r-1)+1}\}$, where $1 \leq r \leq 2^{n-1}$. Then $h(\mathcal{P}_{m,r,1}) = (2^{m-2}+r-1)\cdot(2(2^{m-2}+r-1))\cdot(2(2^{m-2}+r-1)+1)$, $h(\mathcal{P}_{m,r,2}) = (2^{m-2}+r-1)\cdot(2(2^{m-2}+r-1))\cdot(4(2^{m-2}+r-1))$, $h(\mathcal{P}_{m,r,3}) = (2^{m-2}+r-1)\cdot(2(2^{m-2}+r-1))\cdot(4(2^{m-2}+r-1)+1)$, $h(\mathcal{P}_{m,r,4}) = (2^{m-2}+r-1)\cdot(2(2^{m-2}+r-1)+1)\cdot(4(2^{m-2}+r-1)+2)$, $h(\mathcal{P}_{m,r,5}) = (2^{m-2}+r-1)\cdot(2(2^{m-2}+r-1)+1)\cdot(4(2^{m-2}+r-1)+3)$, for $1 < m \leq n$, $1 \leq r \leq 2^{m-2}$ and $h(\mathcal{P}_{n+1,r,1}) = (2^{n-1}+r-1)\cdot(2(2^{n-1}+r-1))\cdot(2(2^{n-1}+r-1)+1)$, for $1 \leq r \leq 2^{n-1}$. Then to show h is injective, consider the following cases:

Case 1. Let $k = 2^{m-2}+r-1$. Then $h(\mathcal{P}_{m,r,2}) = k\cdot 2k\cdot 4k$, $h(\mathcal{P}_{m,r,3}) = k\cdot 2k\cdot(4k+1)$, $h(\mathcal{P}_{m,r,4}) = k\cdot(2k+1)\cdot(4k+2)$ and $h(\mathcal{P}_{m,r,5}) = k\cdot(2k+1)\cdot(4k+3)$. Since $2k < 2k+1$ and $4k < 4k+1 < 4k+2 < 4k+3$, we have $k\cdot 2k\cdot 4k < k\cdot 2k\cdot(4k+1) < k\cdot(2k+1)\cdot(4k+2) < k\cdot(2k+1)\cdot(4k+3)$. Hence $h(\mathcal{P}_{m,r,2}) < h(\mathcal{P}_{m,r,3}) < h(\mathcal{P}_{m,r,4}) < h(\mathcal{P}_{m,r,5})$.

Case 2. Let $k = 2^{m-1}-1$. Then $h(\mathcal{P}_{m,2^{m-2},5}) = k\cdot(2k+1)\cdot(4k+3)$, $h(\mathcal{P}_{m+1,1,2}) = (k+1)\cdot(2(k+1))\cdot(4(k+1))$. Since $k < k+1$, $2k+1 < 2k+2$ and $4k+3 < 4k+4$, we have $k\cdot 2k+1\cdot 4k+3 < k+1\cdot 2k+2\cdot 4k+4$. Hence $h(\mathcal{P}_{m,2^{m-2},5}) < h(\mathcal{P}_{m+1,1,2})$.

Case 3. Let $k = 2^{m-2}+r-1$. Then $h(\mathcal{P}_{m,r,5}) = k\cdot(2k+1)\cdot(4k+3)$, $h(\mathcal{P}_{m,r+1,1}) = (k+1)\cdot(2(k+1))\cdot(4(k+1))$. Since $k < k+1$, $2k+1 < 2k+2$ and $4k+3 < 4k+4$, we have $k\cdot 2k+1\cdot 4k+3 < k+1\cdot 2k+2\cdot 4k+4$. Hence $h(\mathcal{P}_{m,r,5}) < h(\mathcal{P}_{m,r+1,2})$, for $1 \leq r \leq 2^{m-2}-1$.

Case 4. Since $r-1 < r$, we have $(2^{m-2}+r-1)\cdot(2(2^{m-2}+r-1))\cdot(2(2^{m-2}+r-1)+1) < ((2^{m-2})+r)\cdot(2(2^{m-2}+r))\cdot(2(2^{m-2}+r)+1)$, which is same as $h(\mathcal{P}_{m,r,1}) < h(\mathcal{P}_{m,r+1,1})$, for $1 \leq r \leq 2^{m-2}-1$.

Case 5. Let $k = 2^{m-1}-1$. Then $h(\mathcal{P}_{m,2^{m-2},1}) = k\cdot 2k\cdot(2k+1)$, $h(\mathcal{P}_{m+1,1,1}) = (k+1)\cdot(2(k+1))\cdot(2(k+1)+1)$. Since $k < k+1$, $2k < 2k+2$ and $2k+1 < 2k+3$, we have $k\cdot 2k\cdot(2k+1) < (k+1)\cdot(2k+2)\cdot(2k+3)$. Hence $h(\mathcal{P}_{m,2^{m-2},1}) < h(\mathcal{P}_{m+1,1,1})$.

Case 6. For given m and r, we have $h(\mathcal{P}_{m,r,1}) = (2^{m-2}+r-1)\cdot(2(2^{m-2}+r-1))\cdot(2(2^{m-2}+r-1)+1)$ in which one of the three factors differs from the three factors of $h(\mathcal{P}_{s,t,i})$ for $s < m$, $1 \leq t \leq 2^{s-1}$, $1 \leq i \leq 5$ and for $s = m$, $1 \leq t < r \leq 2^{m-2}$, $1 \leq i \leq 5$. Hence $h(\mathcal{P}_{m,r,1}) \neq h(\mathcal{P}_{s,t,i})$ for $s < m$, $1 \leq t \leq 2^{s-1}$, $1 \leq i \leq 5$ and for $s = m$, $1 \leq t < r \leq 2^{m-2}$, $1 \leq i \leq 5$.

Thus, by Cases (1)-(6) it follows that h is injective and G is strongly 2-multiplicative. □

Theorem 2.11 *The complete graph K_n is strongly 2-multiplicative if and only if $3 \leq n \leq 5$.*

Proof First, consider K_3 with vertices v_1, v_2 and v_3. Then there is only one path homotopy class and is trivially strongly 2-multiplicative.

Second, consider K_4 with vertices v_1, v_2, v_3 and v_4. Then there are four distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$, corresponding to paths having vertex sets $\{v_1, v_2, v_3\}$, $\{v_1, v_2, v_4\}$, $\{v_1, v_3, v_4\}$ and $\{v_2, v_3, v_4\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 4$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 12$, $h(\mathcal{P}_4) = 24$. Clearly h is injective and K_4 is strongly 2-multiplicative.

Third, consider K_5 with vertices v_1, v_2, v_3, v_4 and v_5. Then there are ten distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$ and $\mathcal{P}_{10}$, corresponding to paths having vertex sets $\{v_1, v_2, v_3\}$, $\{v_1, v_3, v_4\}$, $\{v_1, v_4, v_5\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_4, v_5\}$, $\{v_2, v_5, v_1\}$, $\{v_3, v_4, v_5\}$, $\{v_3, v_5, v_1\}$, $\{v_4, v_1, v_2\}$ and $\{v_5, v_2, v_3\}$ respectively. We label the vertices as follows : v_i=i, $1 \leq i \leq 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 20$, $h(\mathcal{P}_4) = 24$, $h(\mathcal{P}_5) = 40$, $h(\mathcal{P}_6) = 10$, $h(\mathcal{P}_7) = 60$, $h(\mathcal{P}_8) = 15$, $h(\mathcal{P}_9) = 8$ and $h(\mathcal{P}_{10}) = 30$. Clearly h is injective and K_5 is strongly 2-multiplicative.

Finally, consider a complete graph K_n, where $n \geq 6$. Clearly corresponding to each triangle, one can always find a path homotopy class of paths of length 2 having the vertex set, the vertices of triangle. In any labelling of the vertices, we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ where $\mathcal{P}$ consisting of paths having the vertices labelled $1, 3$ and 4 and $\mathcal{P}'$ consisting of paths having the vertices labelled $1, 2$ and 6. Clearly $\mathcal{P} \neq \mathcal{P}'$, but $h(\mathcal{P}) = 12 = h(\mathcal{P}')$. Hence for $n \geq 6$, K_n is not strongly 2-multiplicative. □

Theorem 2.12 *The star graph S_n is strongly 2-multiplicative if and only if* $3 \leq n \leq 7$.

Proof First, consider S_3 with vertices v_1,v_2 and v_3. Here v_2,v_3 are pendent vertices. Then there is only one path homotopy class and is trivially strongly 2-multiplicative.

Second, consider S_4 with vertices v_1,v_2,v_3 and v_4. Here v_2, v_3, v_4 are pendent vertices. Then there are three distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$ and $\{v_3, v_1, v_4\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 4$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 12$. Clearly h is injective and S_4 is strongly 2-multiplicative.

Third, consider S_5 with vertices v_1, v_2, v_3, v_4 and v_5. Here v_2, v_3, v_4, v_5 are pendent vertices. Then there are six distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$ and $\mathcal{P}_6$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$ and $\{v_4, v_1, v_5\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 15$, $h(\mathcal{P}_6) = 20$. Clearly h is injective and S_5 is strongly 2-multiplicative.

Fourth, consider S_6 with vertices v_1, v_2, v_3, v_4, v_5 and v_6. Here v_2, v_3, v_4, v_5, v_6 are pendent vertices. Then there are ten distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$ and $\mathcal{P}_{10}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$ and $\{v_5, v_1, v_6\}$ respectively. We label the vertices as follows: $v_1 = 2$, $v_2 = 1$, $v_i = i$, $3 \leq i \leq 6$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 24$, $h(\mathcal{P}_6) = 30$, $h(\mathcal{P}_7) = 36$, $h(\mathcal{P}_8) = 40$, $h(\mathcal{P}_9) = 48$, $h(\mathcal{P}_{10}) = 60$. Clearly h is injective and S_6 is strongly 2-multiplicative.

Fifth, consider S_7 with vertices $v_1, v_2, v_3, v_4, v_5, v_6$ and v_7. Here $v_2, v_3, v_4, v_5, v_6, v_7$ are pendent vertices. Then there are fifteen distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{14}$ and $\mathcal{P}_{15}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_2, v_1, v_7\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_3, v_1, v_7\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_4, v_1, v_7\}$, $\{v_5, v_1, v_6\}$, $\{v_5, v_1, v_7\}$ and $\{v_6, v_1, v_7\}$ respectively. We label the vertices as follows : v_1=2, $v_2 = 1$, v_i=i, $3 \leq i \leq 7$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 14$, $h(\mathcal{P}_6) = 24$, $h(\mathcal{P}_7) = 30$, $h(\mathcal{P}_8) = 36$, $h(\mathcal{P}_9) = 42$, $h(\mathcal{P}_{10}) = 40$, $h(\mathcal{P}_{11}) = 48$, $h(\mathcal{P}_{12}) = 56$, $h(\mathcal{P}_{13}) = 60$, $h(\mathcal{P}_{14}) = 70$,

$h(\mathcal{P}_{15}) = 84$. Clearly h is injective and S_7 is strongly 2-multiplicative.

Finally, consider a star graph S_n, where $n \geq 8$. In any labeling of the vertices we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$ but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 8$, S_n is not strongly 2-multiplicative. □

Theorem 2.13 *The fan graph F_n is strongly 2-multiplicative if and only if $3 \leq n \leq 6$.*

Proof First, consider $F_2 = K_1 + P_2$. Let the vertex of K_1 be v_1 and the vertices of P_2 be v_2 and v_3. Then there is only one path homotopy class and is trivially strongly 2-multiplicative.

Second, consider $F_3 = K_1 + P_3$. Let the vertex of K_1 be v_1 and the vertices of P_3 be v_2, v_3 and v_4. Then there are four distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2$,$\mathcal{P}_3$ and $\mathcal{P}_4$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_3, v_1, v_4\}$ and $\{v_2, v_3, v_4\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 4$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 12$, $h(\mathcal{P}_4) = 24$. Clearly h is injective and F_3 is strongly 2-multiplicative.

Third, consider $F_4 = K_1 + P_4$. Let the vertex of K_1 be v_1 and the vertices of P_4 be v_2, v_3, v_4 and v_5. Then there are eight distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4, \mathcal{P}_5, \mathcal{P}_6, \mathcal{P}_7$ and $\mathcal{P}_8$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_4, v_1, v_5\}$, $\{v_2, v_3, v_4\}$ and $\{v_3, v_4, v_5\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 15$, $h(\mathcal{P}_6) = 20$, $h(\mathcal{P}_7) = 24$, $h(\mathcal{P}_8) = 60$. Clearly h is injective and F_4 is strongly 2-multiplicative.

Fourth, consider $F_5 = K_1 + P_5$. Let the vertex of K_1 be v_1 and the vertices of P_5 be v_2, v_3, v_4,v_5 and v_6. Then there are thirteen distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$, $\mathcal{P}_4, \mathcal{P}_5$, $\mathcal{P}_6, \mathcal{P}_7, \mathcal{P}_8, \mathcal{P}_9, \mathcal{P}_{10}, \mathcal{P}_{11}, \mathcal{P}_{12}$ and $\mathcal{P}_{13}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_5, v_1, v_6\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$ and $\{v_4, v_5, v_6\}$ respectively. We label the vertices as follows: $v_1 = 3$, $v_i = i - 1$, $i = 2, 3$, $v_i = i$, $4 \leq i \leq 6$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 15$, $h(\mathcal{P}_4) = 18$, $h(\mathcal{P}_5) = 24$, $h(\mathcal{P}_6) = 30$, $h(\mathcal{P}_7) = 36$, $h(\mathcal{P}_8) = 60$, $h(\mathcal{P}_9) = 72$, $h(\mathcal{P}_{10}) = 90$, $h(\mathcal{P}_{11}) = 8$, $h(\mathcal{P}_{12}) = 40$, $h(\mathcal{P}_{13}) = 120$. Clearly h is injective and F_5 is strongly 2-multiplicative.

Fifth, consider $F_6 = K_1 + P_6$. Let the vertex of K_1 be v_1 and the vertices of P_6 be v_2, v_3, v_4, v_5, v_6 and v_7. Then there are nineteen distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{14}$, $\mathcal{P}_{15}$, $\mathcal{P}_{16}$, $\mathcal{P}_{17}$, $\mathcal{P}_{18}$ and $\mathcal{P}_{19}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_2, v_1, v_7\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_3, v_1, v_7\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_4, v_1, v_7\}$, $\{v_5, v_1, v_6\}$, $\{v_5, v_1, v_7\}$, $\{v_6, v_1, v_7\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$ and $\{v_5, v_6, v_7\}$ respectively. We label the vertices as follows : $v_1 = 3$, $v_i = i - 1$, $i = 2, 3$, $v_i = i$, $4 \leq i \leq 7$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 15$, $h(\mathcal{P}_4) = 18$, $h(\mathcal{P}_5) = 21$, $h(\mathcal{P}_6) = 24$, $h(\mathcal{P}_7) = 30$, $h(\mathcal{P}_8) = 36$, $h(\mathcal{P}_9) = 42$, $h(\mathcal{P}_{10}) = 60$, $h(\mathcal{P}_{11}) = 72$, $h(\mathcal{P}_{12}) = 84$, $h(\mathcal{P}_{13}) = 90$, $h(\mathcal{P}_{14}) = 105$, $h(\mathcal{P}_{15}) = 126$,$h(\mathcal{P}_{16}) = 8$, $h(\mathcal{P}_{17}) = 40$, $h(\mathcal{P}_{18}) = 120$, $h(\mathcal{P}_{19}) = 210$. Clearly h is injective and F_6 is strongly 2-multiplicative.

Finally, consider a fan graph F_n, where $n \geq 7$. In any labeling of the vertices we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$ but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 7$, F_n is not strongly 2-multiplicative. □

Theorem 2.14 *The wheel graph W_n is strongly 2-multiplicative if and only if $4 \leq n \leq 7$.*

Proof First, consider $W_4 = K_1 + C_3$. Let the vertex of K_1 be v_1 and the vertices of C_3 be v_2, v_3 and v_4. Then there are four distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_3, v_1, v_4\}$ and $\{v_2, v_3, v_4\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 4$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 12$, $h(\mathcal{P}_4) = 24$. Clearly h is injective and W_4 is strongly 2-multiplicative.

Second, consider $W_5 = K_1 + C_4$. Let the vertex of K_1 be v_1 and the vertices of C_4 be v_2, v_3, v_4 and v_5. Then there are ten distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$ and $\mathcal{P}_{10}$ corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_4, v_1, v_5\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_2\}$ and $\{v_5, v_2, v_3\}$ respectively. We label the vertices as follows : $v_i = i$, $1 \leq i \leq 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 15$, $h(\mathcal{P}_6) = 20$, $h(\mathcal{P}_7) = 24$, $h(\mathcal{P}_8) = 60$, $h(\mathcal{P}_9) = 40$, $h(\mathcal{P}_{10}) = 30$. Clearly h is injective and W_5 is strongly 2-multiplicative.

Third, consider $W_6 = K_1 + C_5$. Let the vertex of K_1 be v_1 and the vertices of C_5 be v_2, v_3, v_4, v_5 and v_6. Then there are fifteen distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{14}$ and $\mathcal{P}_{15}$ corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_5, v_1, v_6\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$, $\{v_5, v_6, v_2\}$ and $\{v_6, v_2, v_3\}$ respectively. We label the vertices as follows: $v_1 = 2$, $v_2 = 1$, $v_3 = 3$, $v_4 = 6$, $v_5 = 4$, $v_6 = 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 8$, $h(\mathcal{P}_4) = 10$, $h(\mathcal{P}_5) = 36$, $h(\mathcal{P}_6) = 24$, $h(\mathcal{P}_7) = 30$, $h(\mathcal{P}_8) = 48$, $h(\mathcal{P}_9) = 60$, $h(\mathcal{P}_{10}) = 40$, $h(\mathcal{P}_{11}) = 18$, $h(\mathcal{P}_{12}) = 72$, $h(\mathcal{P}_{13}) = 120$, $h(\mathcal{P}_{14}) = 20$, $h(\mathcal{P}_{15}) = 15$. Clearly h is injective and W_6 is strongly 2-multiplicative.

Fourth, consider $W_7 = K_1 + C_6$. Let the vertex of K_1 be v_1 and the vertices of C_6 be v_2, v_3, v_4, v_5, v_6 and v_7. Then there are twenty one distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$, $\mathcal{P}_5$, $\mathcal{P}_6$, $\mathcal{P}_7$, $\mathcal{P}_8$, $\mathcal{P}_9$, $\mathcal{P}_{10}$, $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{14}$, $\mathcal{P}_{15}$, $\mathcal{P}_{16}$, $\mathcal{P}_{17}$, $\mathcal{P}_{18}$, $\mathcal{P}_{19}$, $\mathcal{P}_{20}$ and $\mathcal{P}_{21}$, corresponding to paths having the vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_2, v_1, v_7\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_3, v_1, v_7\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_4, v_1, v_7\}$, $\{v_5, v_1, v_6\}$, $\{v_5, v_1, v_7\}$, $\{v_6, v_1, v_7\}$, $\{v_2, v_3, v_4\}$, $\{v_3, v_4, v_5\}$, $\{v_4, v_5, v_6\}$, $\{v_5, v_6, v_7\}$, $\{v_6, v_7, v_2\}$ and $\{v_7, v_2, v_3\}$ respectively. We label the vertices as follows : $v_1 = 2$, $v_2 = 1$, $v_i = i$, for $i = 3, 7$, $v_4 = 6$, $v_5 = 4$, $v_6 = 5$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 12$, $h(\mathcal{P}_3) = 8$, $h(\mathcal{P}_4) = 10$, $h(\mathcal{P}_5) = 14$, $h(\mathcal{P}_6) = 36$, $h(\mathcal{P}_7) = 24$, $h(\mathcal{P}_8) = 30$, $h(\mathcal{P}_9) = 42$, $h(\mathcal{P}_{10}) = 48$, $h(\mathcal{P}_{11}) = 60$, $h(\mathcal{P}_{12}) = 84$, $h(\mathcal{P}_{13}) = 40$, $h(\mathcal{P}_{14}) = 56$, $h(\mathcal{P}_{15}) = 70$, $h(\mathcal{P}_{16}) = 18$, $h(\mathcal{P}_{17}) = 72$, $h(\mathcal{P}_{18}) = 120$, $h(\mathcal{P}_{19}) = 140$, $h(\mathcal{P}_{20}) = 35$, $h(\mathcal{P}_{21}) = 21$. Clearly h is injective and W_7 is strongly 2-multiplicative.

Finally, consider a wheel graph W_n, where $n \geq 8$. In any labeling of the vertices we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$ but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 8$, W_n is not strongly 2-multiplicative. □

Theorem 2.15 *The complete bipartite graph $K_{2,n}$ is strongly 2-multiplicative if and only if $2 \leq n \leq 3$.*

Proof First, consider complete bipartite graph $K_{2,2}$. Let $A = \{v_1, v_2\}$ and $B = \{v_3, v_4\}$ be the two partitions of vertex set of $K_{2,2}$. Then $\mathcal{A}$ consists of 4 distinct path homotopy classes

$\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ and $\mathcal{P}_4$, corresponding to paths having vertex sets $\{v_3, v_1, v_4\}$, $\{v_3, v_2, v_4\}$, $\{v_1, v_3, v_2\}$ and $\{v_1, v_4, v_2\}$ respectively. We label the vertices as follows: $v_i = i$, for all $i \in \{1, 2, 3, 4\}$. Then $h(\mathcal{P}_1) = 12$, $h(\mathcal{P}_2) = 24$, $h(\mathcal{P}_3) = 6$, $h(\mathcal{P}_4) = 8$. Clearly h is injective $K_{2,2}$ is strongly 2-multiplicative.

Second, consider complete bipartite graph $K_{2,3}$. Let $A = \{v_1, v_2\}$ and $B =\{v_3, v_4, v_5\}$ be the two partitions of vertex set of $K_{2,3}$. Then $\mathcal{A}$ consists of 9 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ $\mathcal{P}_4$,$\mathcal{P}_5, \mathcal{P}_6$,$\mathcal{P}_7, \mathcal{P}_8$ and $\mathcal{P}_9$, corresponding to paths having vertex sets $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_2, v_4\}$, $\{v_3, v_2, v_5\}$, $\{v_2, v_5, v_1\}$, $\{v_2, v_4, v_1\}$, $\{v_4, v_2, v_5\}$, $\{v_1, v_4, v_5\}$ and $\{v_2, v_3, v_1\}$ respectively. We label the vertices as follows: $v_i = i$, for all $i \in \{1, 2, 3, 4, 5\}$. Then $h(\mathcal{P}_1) = 12$, $h(\mathcal{P}_2) = 15$, $h(\mathcal{P}_3) = 24$, $h(\mathcal{P}_4) = 30$, $h(\mathcal{P}_5) = 10$, $h(\mathcal{P}_6) = 8$, $h(\mathcal{P}_7) = 40$, $h(\mathcal{P}_8) = 20$, $h(\mathcal{P}_9) = 6$. Clearly h is injective $K_{2,3}$ is strongly 2-multiplicative.

Finally, consider a complete bipartite graph $K_{2,4}$. In any labelling of the vertices we get as the value of $h(\mathcal{P})$ one of 12,24 and 30. Since $12 = 1 \cdot 3 \cdot 4 = 1 \cdot 6 \cdot 2$, $24 = 1 \cdot 6 \cdot 4 = 3 \cdot 4 \cdot 2$ and $30 = 3 \cdot 2 \cdot 5 = 1 \cdot 6 \cdot 5$, we get two distinct path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ with $h(\mathcal{P})$=$h(\mathcal{P}')$. Hence $K_{2,4}$ is not strongly 2-multiplicative. Like this one can show that, a complete bipartite graph $K_{2,n}$, for $n > 4$ is not strongly 2-multiplicative. □

Theorem 2.16 *The graph $P_2 + P_n$ is strongly 2-multiplicative if and only if $n \leq 3$.*

Proof First, consider the graph $P_2 + P_2$. This is same as K_4, which is strongly 2-multiplicative by Theorem 2.11.

Second, consider the graph $P_2 + P_3$. Then $\mathcal{A}$ consists of 10 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ $\mathcal{P}_4$,$\mathcal{P}_5, \mathcal{P}_6$,$\mathcal{P}_7, \mathcal{P}_8$, $\mathcal{P}_9$ and $\mathcal{P}_{10}$ corresponding to paths having vertex sets $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_2, v_4\}$, $\{v_3, v_2, v_5\}$, $\{v_2, v_5, v_1\}$, $\{v_2, v_4, v_1\}$, $\{v_4, v_2, v_5\}$, $\{v_1, v_4, v_5\}$, $\{v_2, v_3, v_1\}$ and $\{v_3, v_4, v_5\}$ respectively. We label the vertices as follows: $v_i = i$, for all $i \in \{1, 2, 3, 4, 5\}$. Then $h(\mathcal{P}_1) = 12$, $h(\mathcal{P}_2) = 15$, $h(\mathcal{P}_3) = 24$, $h(\mathcal{P}_4) = 30$, $h(\mathcal{P}_5) = 10$, $h(\mathcal{P}_6) = 8$, $h(\mathcal{P}_7) = 40$, $h(\mathcal{P}_8) = 20$, $h(\mathcal{P}_9) = 6$, $h(\mathcal{P}_{10}) = 60$. Clearly h is injective and $P_2 + P_3$ is strongly 2-multiplicative.

Finally, consider graph $P_2 + P_n$ where $n \geq 4$. In any labeling of the vertices we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$ but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 4$, $P_2 + P_n$ is not strongly 2-multiplicative. □

Theorem 2.17 *The peterson graph is strongly 2-multiplicative.*

Proof Consider a peterson graph with vertices $v_1, v_2, v_3, v_4, \cdots, v_{10}$.

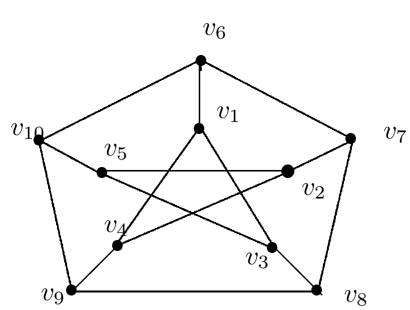

Figure 5

Then $\mathcal{A}$ consists of 21 distinct path homotopy classes $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \cdots, \mathcal{P}_{21}$, corresponding to paths having vertex sets $\{v_4, v_1, v_3\}$, $\{v_5, v_2, v_4\}$, $\{v_5, v_3, v_1\}$, $\{v_1, v_4, v_2\}$, $\{v_2, v_5, v_3\}$, $\{v_6, v_7, v_8\}$, $\{v_7, v_8, v_9\}$, $\{v_8, v_9, v_{10}\}$, $\{v_9, v_{10}, v_6\}$, $\{v_9, v_{10}, v_4\}$, $\{v_{10}, v_6, v_7\}$, $\{v_6, v_1, v_4\}$, $\{v_6, v_1, v_3\}$, $\{v_7, v_2, v_5\}$, $\{v_7, v_2, v_3\}$, $\{v_8, v_3, v_1\}$, $\{v_8, v_3, v_5\}$, $\{v_9, v_4, v_2\}$, $\{v_9, v_4, v_1\}$, $\{v_{10}, v_5, v_3\}$ and $\{v_{10}, v_5, v_2\}$ respectively. We label the vertices as follows: $v_i = i$, for $1 \leq i \leq 7$, $v_8 = 9$, $v_9 = 8$, $v_{10} = 10$. Then $h(\mathcal{P}_1) = 12$, $h(\mathcal{P}_2) = 40$, $h(\mathcal{P}_3) = 15$, $h(\mathcal{P}_4) = 8$, $h(\mathcal{P}_5) = 30$, $h(\mathcal{P}_6) = 378$, $h(\mathcal{P}_7) = 504$, $h(\mathcal{P}_8) = 720$, $h(\mathcal{P}_9) = 480$, $h(\mathcal{P}_{10}) = 320$, $h(\mathcal{P}_{11}) = 420$, $h(\mathcal{P}_{12}) = 24$, $h(\mathcal{P}_{13}) = 18$, $h(\mathcal{P}_{14}) = 70$, $h(\mathcal{P}_{15}) = 42$, $h(\mathcal{P}_{16}) = 27$, $h(\mathcal{P}_{17}) = 135$, $h(\mathcal{P}_{18}) = 64$, $h(\mathcal{P}_{19}) = 32$, $h(\mathcal{P}_{20}) = 150$, $h(\mathcal{P}_{21}) = 100$. Clearly h is injective peterson graph is strongly 2-multiplicative. □

Theorem 2.18 *The windmill K_n^m is strongly 2-multiplicative if and only if $m \leq 3, n \leq 3$.*

Proof First, if $m = 2$, then the proof follows from the proof of Theorem 2.5, with $n = 3$.

Second, consider the K_3^3 with vertices v_1, v_2, v_3, v_4, v_5, v_6 and v_7 such that v_1 be the common vertex as shown in the figure.

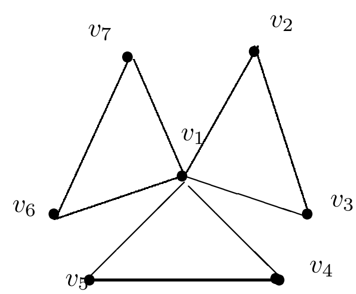

Figure 6

Then $\mathcal{A}$ consists of 15 distinct path homotopy classes $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4, \ldots$, $\mathcal{P}_{15}$ corresponding to paths having vertex sets $\{v_2, v_1, v_3\}$, $\{v_2, v_1, v_4\}$, $\{v_2, v_1, v_5\}$, $\{v_2, v_1, v_6\}$, $\{v_2, v_1, v_7\}$, $\{v_3, v_1, v_4\}$, $\{v_3, v_1, v_5\}$, $\{v_3, v_1, v_6\}$, $\{v_3, v_1, v_7\}$, $\{v_4, v_1, v_5\}$, $\{v_4, v_1, v_6\}$, $\{v_4, v_1, v_7\}$, $\{v_5, v_1, v_6\}$, $\{v_5, v_1, v_7\}$ and $\{v_6, v_1, v_7\}$ respectively. We label the vertices as follows: $v_1 = 2$, $v_2 = 1$ and $v_i = i$ for all i for $3 \leq i \leq 7$. Then $h(\mathcal{P}_1) = 6$, $h(\mathcal{P}_2) = 8$, $h(\mathcal{P}_3) = 10$, $h(\mathcal{P}_4) = 12$, $h(\mathcal{P}_5) = 14$, $h(\mathcal{P}_6) = 24$, $h(\mathcal{P}_7) = 30$, $h(\mathcal{P}_8) = 36$, $h(\mathcal{P}_9) = 42$, $h(\mathcal{P}_{10}) = 40$, $h(\mathcal{P}_{11}) = 48$, $h(\mathcal{P}_{12}) = 56$, $h(\mathcal{P}_{13}) = 60$, $h(\mathcal{P}_{14}) = 70$, $h(\mathcal{P}_{15}) = 84$. Clearly h is injective K_3^3 is strongly 2-multiplicative.

Finally, consider a windmill K_n^m for $m \geq 3$, $n > 3$. In any labelling of the vertices, we can find two path homotopy classes $\mathcal{P}$ and $\mathcal{P}'$ such that $\mathcal{P} \neq \mathcal{P}'$, but $h(\mathcal{P}) = h(\mathcal{P}')$. Hence for $n \geq 3$,$m \geq 3$, K_n^m is not strongly 2-multiplicative. □

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A Characterization of Directed Pathos Line Digraph of an Arborescence

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Abstract: For an arborescence A_r, *a directed pathos line digraph* $Q = DPL(A_r)$ has vertex set $V(Q) = A(A_r) \cup P(A_r)$, where $A(A_r)$ is the arc set and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b; Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j. The purpose of this note is to characterize $DPL(A_r)$, i.e., when is a digraph a directed pathos line digraph of an arborescence A_r and is A_r reconstructible from $DPL(A_r)$?

Key Words: Line digraph, complete bipartite subdigraph, directed pathos vertex.

AMS(2010): 05C20.

§1. Introduction

Notations and definitions not introduced here can be found in [2]. There are many digraph operators (or digraph valued functions) with which one can construct a new digraph from a given digraph, such as the line digraph, the total digraph, and their generalizations. One such a digraph operator is called *a directed pathos line digraph* of an arborescence.

The concept of *pathos* of a graph G was introduced by Harary [3] as a collection of minimum number of edge disjoint open paths whose union is G. The path number of a graph G is the number of paths in any pathos. The path number of a tree T equals k, where $2k$ is the number of odd degree vertices of T.

For a tree T with vertex set $V(T) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(T) = \{e_1, e_2, \cdots, e_{n-1}\}$, the authors in [4] gave the following definition. A *pathos line graph of* T, written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T have a vertex in common or the edge lies on the corresponding path of the pathos.

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The order and size of $PL(T)$ are $n+k-1$ and $\frac{1}{2}\sum_{i=1}^{n} d_i^2$, respectively, where k is the path number and d_i is the degree of vertices of T. The characterization of graphs whose $PL(T)$ are planar, outerplanar, and maximal outerplanar were presented. A necessary and sufficient condition for $PL(T)$ to be Eulerian was given. They also showed that for any tree T, $PL(T)$ is not minimally nonouterplanar.

See Figure.1 for an example of a tree T and its pathos line graph $PL(T)$.

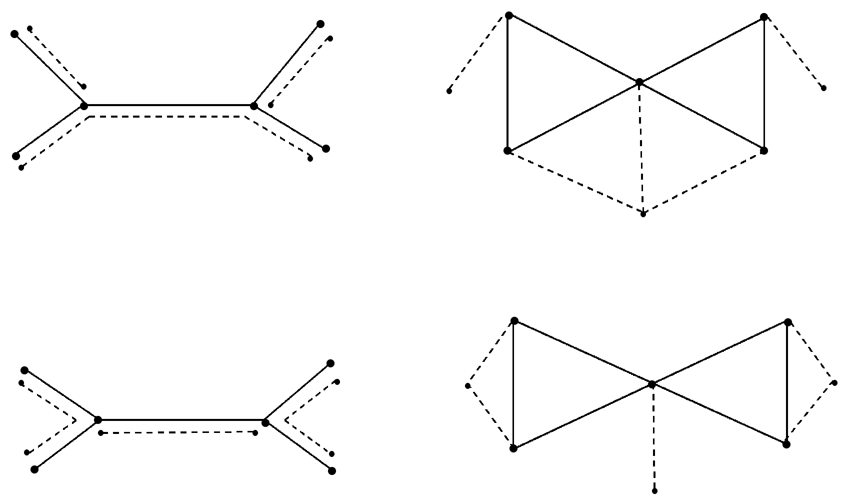

Figure 1

A *directed graph* (or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pair of distinct vertices called *arcs*. Here $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. For an arc (u,v) or uv of D the first vertex u is its *tail* and the second vertex v is its *head*. An *arborescence* is a directed graph in which, for a vertex u called the *root* and any other vertex v, there is exactly one directed path from u to v. We shall use A_r to denote an arborescence. A vertex with an in-degree (out-degree) zero is called a *source* (*sink*).

M. Aigner [1] defines the *line digraph* of a digraph as follows. Let D be a digraph with n vertices $v_1, v_2, \cdots, v_n$ and m arcs, and $L(D)$ its associated *line digraph* with $n^{'}$ vertices and $m^{'}$ arcs. We immediately have $n^{'} = m$ and $m^{'} = \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i)$. Furthermore, the in-degree and out-degree of a vertex $v^{'} = (v_i, v_j)$ in $L(D)$ are $d^-(v^{'}) = d^-(v_i)$ and $d^+(v^{'}) = d^+(v_j)$, respectively. A digraph D is said to be a line digraph if it is isomorphic to the line digraph of a certain digraph H.

The authors in [5] extended the definition of a pathos line graph of a tree to an arborescence by introducing the concept of directed pathos line digraph of an arborescence and studied some of the characterization problems such planarity, outer planarity, etc. It is the object of this paper to discuss the problem of reconstructing an arborescence from its directed pathos line digraph.

§2. Definition of $DPL(A_r)$

Definition 2.1 *If a directed path $\vec{P}_n$ starts at one vertex and ends at a different vertex, then $\vec{P}_n$ is called an open directed path.*

Definition 2.2 *The directed pathos of an arborescence A_r is defined as a collection of minimum number of arc disjoint open directed paths whose union is A_r.*

Definition 2.3 *The directed path number k' of A_r is the number of directed paths in any directed pathos of A_r and is equal to the number of sinks in A_r, i.e., k' = number of sinks in A_r.*

Definition 2.4 *A directed pathos vertex is a vertex corresponding to a directed path of the directed pathos of A_r.*

Definition 2.5 *For an arborescence A_r, a directed pathos line digraph $Q = DPL(A_r)$ has vertex set $V(Q) = A(A_r) \cup P(A_r)$, where $A(A_r)$ is the arc set and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b; Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j.*

Note that the directed path number k' of an arborescence A_r is minimum only when the out-degree of the root of A_r is one. Therefore, unless otherwise specified, the out-degree of the root of every arborescence is one. Finally, we assume that the direction of the directed pathos is along the direction of the arcs in A_r.

See Figure.2 for an example of an arborescence A_r and its directed pathos line digraph $DPL(A_r)$.

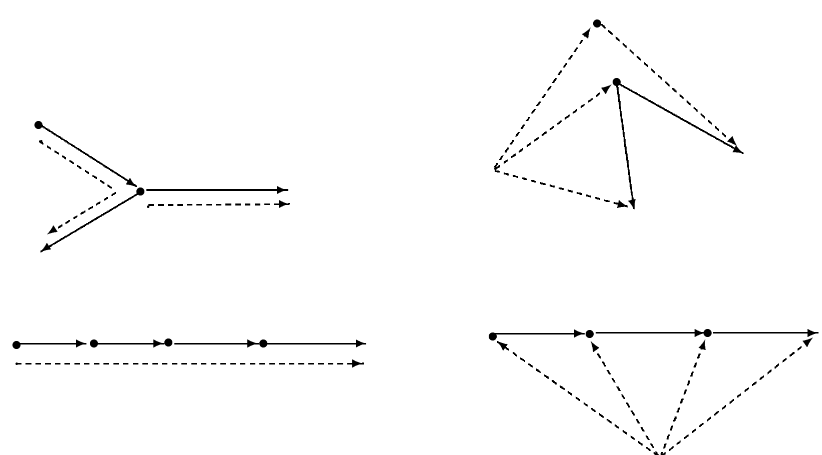

Figure 2

§3. A Criterion for Directed Pathos Line Digraphs

The main objective is to determine a necessary and sufficient condition that a digraph be a directed pathos line digraph.

A *complete bipartite digraph* is a directed graph D whose vertices can be partitioned into nonempty disjoint sets A and B such that each vertex of A has exactly one arc directed towards each vertex of B and such that D contains no other arc.

Theorem 3.1 *A digraph $A_r^{'}$ is a directed pathos line digraph of an arborescence A_r if and only if $V(A_r^{'}) = A(A_r) \cup P(A_r)$ and arc sets :*
(i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of A_r, respectively;
(ii) $\cup_{k=1}^{r} \cup_{j=1}^{r} P_k \times Z_j$ such that $P_k \times Z_j = \phi$ for $k \neq j$, where Z_j is the set of arcs on which P_k lies in A_r;
(iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} P_k \times Z_j^{'}$ such that $P_k \times Z_j^{'} = \phi$ for $k \neq j$, where $Z_j^{'}$ is the set of directed paths whose heads are reachable from the tail of P_k through a common vertex in A_r.

Proof Suppose that A_r is an arborescence with vertex set $V(A_r) = \{v_1, v_2, \cdots, v_n\}$ and a directed pathos set $P(A_r) = \{P_1, P_2, \cdots, P_r\}$. We consider the following three cases.

Case 1. Let v be a vertex of A_r with $d^{-}(v) = \alpha$ and $d^{+}(v) = \beta$. Then α arcs incident into v and the β arcs incident out of v give rise to a complete bipartite subdigraph with α tails and β heads and $\alpha \cdot \beta$ arcs joining each tail with each head.

Case 2. Let P_i be a directed path which lies on $\alpha^{'}$ arcs in A_r. Then $\alpha^{'}$ arcs give rise to a complete bipartite subdigraph with a single tail (i.e., P_i) and $\alpha^{'}$ heads and $\alpha^{'}$ arcs joining P_i with each head.

Case 3. Let P_i be a directed path, and let $\beta^{'}$ be the number of directed paths whose heads are reachable from the tail of P_i through a common vertex in A_r. Then $\beta^{'}$ arcs give rise to a complete bipartite subdigraph with a single tail (i.e., P_i) and $\beta^{'}$ heads and $\beta^{'}$ arcs joining P_i with each head.

Hence by all the above cases, $DPL(A_r)$ is decomposed into mutually arc-disjoint complete bipartite subdigraphs with vertex set $A(A_r) \cup P(A_r)$ and arc sets:

(i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of A_r, respectively;

(ii) $\cup_{k=1}^{r} \cup_{j=1}^{r} P_k \times Z_j$ such that $P_k \times Z_j = \phi$ for $k \neq j$, where Z_j is the set of arcs on which P_k lies in A_r;

(iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} P_k \times Z_j^{'}$ such that $P_k \times Z_j^{'} = \phi$ for $k \neq j$, where $Z_j^{'}$ is the set of directed paths whose heads are reachable from the tail of P_k through a common vertex in A_r.

Conversely, let $A_r^{'}$ be a digraph of the type described above. Let $t_1, t_2, \ldots, t_l$ be the vertices corresponding to complete bipartite subdigraphs $A_{r1}, A_{r2}, \cdots, A_{rl}$ of *Case* **1**, respectively, and let $t^1, t^2, \ldots, t^r$ be the vertices corresponding to complete bipartite subdigraphs $P_1^{'}, P_2^{'}, \cdots, P_r^{'}$ of *Case* **2**, respectively. Finally, let t_0 be a vertex chosen arbitrarily.

For each vertex v of the complete bipartite subdigraphs $A_{r1}, A_{r2}, \cdots, A_{rl}$, we draw an arc

a_v as follows.

(i) If $d^+(v) > 0$ and $d^-(v) = 0$, then $a_v := (t_0, t_i)$, where i is the base (or index) of A_{ri} such that $v \in Y_i$.

(ii) If $d^+(v) > 0$ and $d^-(v) > 0$, then $a_v := (t_i, t_j)$, where i and j are the indices of A_{ri} and A_{rj} such that $v \in X_j \cap Y_i$.

(c) If $d^+(v) = 0$ and $d^-(v) = 1$, then $a_v := (t_j, t^n)$, $n = 1, 2, \cdots, r$, where j is the base of A_{rj} such that $v \in X_j$.

Note that in (t_j, t^n) no matter what the value of j is, n varies from 1 to r such that the number of arcs of the form (t_j, t^n) is exactly r.

We now mark the directed pathos as follows. It is easy to observe that the directed path number $k^{'}$ equals the number of subdigraphs of Case 2. Let $\psi_1, \psi_2, \cdots, \psi_r$ be the number of heads of subdigraphs $P_1^{'}, P_2^{'}, \cdots, P_r^{'}$, respectively. Suppose we mark the directed path P_1. For this we choose any ψ_1 number of arcs and mark P_1 on ψ_1 arcs such that the direction of P_1 must be along the direction of ψ_1 arcs. Similarly, we choose ψ_2 number of arcs and mark P_2 on ψ_2 arcs. This process is repeated until all directed paths are marked. The digraph A_r with directed paths thus constructed apparently has $A_r^{'}$ as directed pathos line digraph. □

Given a directed pathos line digraph Q, the proof of the sufficiency of the Theorem above shows how to find an arborescence A_r such that $DPL(A_r) = Q$. This obviously raises the question of whether Q determines A_r uniquely. Although the answer to this in general is no, the extent to which A_r is determined is given as follows. One can easily check that using reconstruction procedure of the sufficiency of the Theorem above, any arborescence (without directed pathos) is uniquely reconstructed from its directed pathos line digraph. Since the pattern of directed pathos for an arborescence is not unique, there is freedom in marking the directed pathos for an arborescence in different ways. This clearly shows that if the directed path number is one, any arborescence with directed pathos is uniquely reconstructed from its directed pathos line digraph. It is known that the directed path is a special case of an arborescence. Since the directed path number of a directed path of order n $(n \geq 2)$ is exactly one, it follows that a directed path is uniquely reconstructed from its directed pathos line digraph.

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Independent Open Irredundant Colorings of Graphs

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Abstract: A vertex $v \in V - S$ is an external private neighbor of u with respect to S if v is adjacent to u but no other vertex in S. A set $S \subseteq V$ is *open irredundant* if every vertex in S has an external private neighbor with respect to S. A set S is called an *independent open irredundant set* or *ioir*-set if S is an independent set and every vertex in S has an external private neighbor with respect to S. An *independent open irredundant coloring* of a graph G is a partition of $V(G)$ into *independent open irredundant* sets. In this paper, we introduce the study of *independent open irredundant colorings* of graphs.

Key Words: Independence, irredundance, open irredundant coloring, independent open irredundant coloring, Smarandachely k-independent open irredundant set.

AMS(2010): 05C15, 05C69.

§1. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2].

Domination is a well studied concept in graph theory. For an excellent treatment of fundamentals of domination we refer to the book by Haynes et al. [6]. Several advanced topics in domination are given in the book edited by Haynes et al. [7].

The neighbourhood of a vertex $x \in V(G)$ in the graph G is denoted by $N(x)$ and the closed neighbourhood $\{x\} \cup N(x)$ by $N[x]$. If X is a subset of $V(G)$, then $N[X] = \bigcup_{x \in X} N[x]$ and the subgraph induced by X is denoted by $G[X]$.

In 1999, Cockayne [3] introduced the study of a large class of generalized irredundant sets in graphs. Each type of a generalized irredundant set $S \subset V$ is defined by the types of private neighbors (i.e self, internal or external) that each vertex in the set must have. A subset S of V in a graph G is said to be *independent* if no two vertices in S are adjacent. Let $u \in S$. A vertex $v \in V - S$ is an external private neighbor of u with respect to S if v is adjacent to u but no other vertex in S. A vertex $u \in S$ is its own private neighbor if it is not adjacent to any vertex in S. A set S is called *irredundant* if every vertex in S is either its own private neighbor or has an external private neighbor, with respect to S. A set S is called an *independent open*

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irredundant set or *ioir*-set if S is an independent set and every vertex in S has an external private neighbor.

Generally, a set S is called a *Smarandachely k-independent open irredundant set* if there is a subset $V_0 \subset V$ with $|V_0| = k$ such that S is an independent set and every vertex in S has an external private neighbor in V_0. Clearly, if $V_0 = V$, a Smarandachely $|G|$-independent open irredundant set is nothing else but an *ioir*-set.

In [3], Cockayne identifies 12 types of generalised irredundant sets the properties of which are hereditary. Perhaps the most interesting of these are the *ioir*-sets. One can therefore define $ioir(G)$ to equal the minimum size of a maximal *ioir*-set and $IOIR(G)$ to equal the maximum size of an *ioir*-set. These generalized irredundant sets are also studied by Finbow in [5] and Cockayne and Finbow in [4].

If a collection of edges between two sets of vertices, say A and B, define a bijection between A and B, then we call such a perfect matching a bijective matching.

A proper k-coloring of a graph G is a partition $\pi = \{V_1, V_2, \cdots, V_k\}$ of V into k non-empty independent sets. The chromatic number $\chi(G)$ equals the minimum integer k for which G has a k-coloring. More generally given a property P concerning subsets of V, a P-coloring is a partition $\pi = \{V_1, V_2, \cdots, V_k\}$ of V into sets, such that each V_i has the property P. If the property P is independence, the P-coloring is the usual coloring and if the property P is domination, the corresponding P-coloring gives the concept of domatic partition. Haynes et al. [8] introduced the concept of irredundant colorings and open irredundant colorings of graphs. Arumugam et al. [1] initiate a study of open irredundant colorings and obtain some results on irredundant colorings and open irredundant colorings. Motivated by the work on [1,8], we initiate a study of *independent open irredundant colorings*. An *independent open irredundant coloring* of a graph G is a partition of V into nonempty independent open irredundant sets. The *independent open irratic number* is the minimum order of an independent open irredundant coloring of G, and it is denoted by $\chi_{ioir}(G)$. In section 2, we obtain some results on independent open irredundant colorings. A study of harmonious, achromatic coloring on middle graph, central graph, total graph, line graph of various classes of graphs can be found in [10, 11, 12, 13]. In Section 3, we investigate the *independent open irratic number* for the middle graph, central graph, total graph, line graph of double star graph families.

We need the following theorems.

Theorem 1.1([6]) *If a graph G has no isolated vertices, then G has a minimum dominating set which is also open irredundant.*

Theorem 1.2([8]) *For any graph G, $n/IR(G) \leq \chi_{ir}(G) \leq n - IR(G) + 1$.*

Observation 1.3([1]) Since any *oir*-coloring of G is an *ir*-coloring of G, it follows that $\chi_{ir}(G) \leq \chi_{oir}(G)$.

Theorem 1.4([8]) *For any graph G, $\chi_{oir}(G) = 2$ if and only if $V(G)$ can be partitioned into two subsets V_1 and V_2 such that there exists a bijective matching between V_1 and V_2.*

Throughout, we assume that G is a graph without isolated vertices.

§2. Independent Open Irredundant Colorings

Observation 2.1 Since any *ioir*-coloring of G is an *oir*-coloring and χ-coloring of G, it follows that $\chi_{ir}(G) \leq \chi_{oir}(G) \leq \chi_{ioir}(G)$ and $\chi_{ir}(G) \leq \chi(G) \leq \chi_{ioir}(G)$.

Observation 2.1 Since $V(G)$ is not an *ioir*-set of G, it follows that $2 \leq \chi_{ioir}(G) \leq n$.

Theorem 2.3 *For any graph G, $\chi_{ioir}(G) = 2$ if and only if $V(G)$ can be partitioned into two independent subsets V_1 and V_2 such that there exists a bijective matching between V_1 and V_2.*

Proof The proof follows from Theorem 1.4. □

Theorem 2.4 *Let G be a graph of order n. Then $\chi_{ioir}(G) = n$ if and only if for any independent set $S \subset V$, there exists $v, w \in S$ such that $N(v) \subseteq N(w)$ or $N(w) \subseteq N(v)$.*

Proof Assume that $\chi_{ioir}(G) = n$. Suppose there is an independent set $S \subset V$ such that $N(v) \nsubseteq N(w)$ and $N(w) \nsubseteq N(v)$ $\forall v, w \in S$. Then there exists a vertex $z_1 \in N(v)$ such that z_1 is not adjacent to w and there exists a vertex $z_2 \in N(w)$ such that z_2 is not adjacent to v. Hence $\{v, w\}$ is an *ioir*-set and $IOIR(G) \geq 2$. Therefore $\chi_{ioir}(G) \leq n - 1$ which is a contradiction. The converse is obvious. □

Observation 2.5 For any complete graph K_n and complete bipartite graph $K_{m,n}$, we have $\chi_{ioir}(K_n) = n$ and $\chi_{ioir}(K_{m,n}) = m + n$.

Observation 2.6 For any tree T, $\chi_{ioir}(T) = n$ if and only if T is a star.

Theorem 2.7 *For the path $P_n = (v_1, v_2, \cdots, v_n)$, we have $\chi_{ioir}(P_n) = 3$.*

Proof Let $V_1 = \{v_1, v_4, v_7, v_{10}, \cdots\}$, $V_2 = \{v_2, v_5, v_8, v_{11}, \cdots\}$, $V_3 = \{v_3, v_6, v_9, v_{12}, \cdots\}$.

Clearly $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ into independent open irredundant sets. Hence $\chi_{ioir}(P_n) \leq 3$. By Theorem 2.3, $\chi_{ioir}(P_n) \geq 3$ and so $\chi_{ioir}(P_n) = 3$. □

Theorem 2.8 *For the cycle $C_n = (v_1, v_2, \cdots, v_n)$, we have*

$$\chi_{ioir}(C_n) = \begin{cases} 4 & \text{if } n = 4 \text{ or } n = 7 \\ 3 & \text{otherwise} \end{cases}$$

Proof We can easily observe that $\chi_{ioir}(C_4) = 4$. We now prove that $\chi_{ioir}(C_n) = 3$ for $n \neq 4$ or 7. By Theorem 2.3, $\chi_{ioir}(C_n) \geq 3$. Now we consider three cases.

Case 1. $n \equiv 0(mod 3)$.

Let $V_1 = \{v_1, v_4, v_7, v_{10}, \cdots, v_{n-2}\}$, $V_2 = \{v_2, v_5, v_8, v_{11}, \cdots, v_{n-1}\}$ and $V_3 = \{v_3, v_6, v_9, v_{12}, \cdots, v_n\}$. Clearly $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ into independent open irredundant sets since any three consecutive vertices in the cycle receives distinct colors. Hence $\chi_{ioir}(C_n) \leq 3$.

Case 2. $n \equiv 1(mod 3)$.

Let $V_1 = \{v_1, v_3, v_6, v_8, v_{11}, v_{14}, v_{17}, \cdots, v_{l-3}, v_l, v_{l+3}, \cdots, v_{n-2}\}$, $V_2 = \{v_2, v_4, v_7, v_9, v_{12},$

$v_{15}, v_{18}, \cdots, v_{l-3}, v_l, v_{l+3}, \cdots, v_{n-1}\}$, $V_3 = \{v_5, v_{10}, v_{13}, v_{16}, v_{19}, \cdots, v_{l-3}, v_l, v_{l+3}, \cdots, v_n\}$. We now prove that $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ into independent open irredundant sets. Clearly the sets V_i, $i = 1, 2, 3$ are independent. Hence it is enough to prove that every vertex in the set V_i has an external private neighbour with respect to V_i, $i = 1, 2, 3$. Note that v_1, v_5, v_6 are the external private neighbors of v_2, v_4, v_7 respectively and v_n, v_4, v_7 and v_{10} are the external private neighbors of v_1, v_3, v_8 and v_9 respectively. All other remaining vertices v_i have external private neighbor v_{i-1}.

Case 3. $n \equiv 2(mod 3)$.

Let $V_1 = \{v_1, v_4, v_7, v_{10}, \cdots, v_{n-1}\}$, $V_2 = \{v_2, v_5, v_8, v_{11}, \cdots, v_n\}$ and $V_3 = \{v_3, v_6, v_9, v_{12}, \cdots, v_{n-2}\}$. Since v_2, v_{n-1}, v_{n-2} are the external private neighbors of v_1, v_n, v_{n-1} respectively and remaining vertices v_i have external private neighbor v_{i+1}, $\{V_1, V_2, V_3\}$ is a partition of $V(G)$ into independent open irredundant sets. Hence $\chi_{ioir}(C_n) \leq 3$. Now we prove that $\chi_{ioir}(C_7) = 4$. Since any independent open irredundant set of C_7 has at most two vertices, minimum four colors are required to color the vertices of C_7. Let $V_1 = \{v_1, v_3\}$, $V_2 = \{v_2, v_6\}$, $V_3 = \{v_3, v_5\}$ and $V_4 = \{v_7\}$. Clearly $\{V_1, V_2, V_3, V_4\}$ is an *ioir*-coloring of C_7. Hence $\chi_{ioir}(C_7) = 4$. □

Proposition 2.9 *For any graph G, $n/IOIR(G) \leq \chi_{ioir}(G) \leq n - IOIR(G) + 1$, where $IOIR(G)$ is the upper independent open irredundance number of G.*

Proof Let $\chi_{ioir}(G) = k$. Let $\{V_1, V_2, \cdots, V_k\}$ be an *ioir*-coloring of G. Since $|V_i| \leq IOIR(G)$, it follows that $n = \sum_{i=1}^{k} |V_i| \leq k.IOIR(G)$. Hence $n/IOIR(G) \leq \chi_{ioir}(G)$. Now, let S be an independent open irredundant set of G with $|S| = IOIR(G)$. Then $\{S\} \cup \{\{v\} : v \in V - S\}$ is an *ioir*-coloring of G. Hence $\chi_{ioir}(G) \leq n - IOIR(G) + 1$. □

Theorem 2.10 *Let G be a connected graph with $\delta = 1$ and let r denote the maximum number of leaves adjacent to a support vertex v of G. Then $\chi_{ioir}(G) \geq r + 2$.*

Proof Let $v_1, v_2, \cdots, v_r$ be the leaves adjacent to v. Since any independent open irredundant set in G contains at most one of the leaves v_i, the result follows. □

Observation 2.11 Let $T \neq K_{1,n}$ be any tree and let r denote the maximum number of leaves adjacent to a support vertex v of T. Then $\chi_{ioir}(T) \geq r + 2$.

§3. IOIR-Coloring on Double Star Graph Families

In this section we investigate the independent open irratic number for the central graph, middle graph, total graph, line graph of star graph $K_{1,n}$ and double star graph $K_{1,n,n}$.

The central graph $C(G)$ of a graph G is formed by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i)x, y are in $E(G)$ and x, y are adjacent in G. (ii)x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

The total graph of G has vertex set $V(G) \cup E(G)$, and edges joining all elements of this vertex set which are adjacent or incident in G.

The line graph of G denoted by $L(G)$ is the graph with vertices are the edges of G with two vertices of $L(G)$ adjacent whenever the corresponding edges of G are adjacent.

A star is a complete bipartite graph $K_{1,m}$ with $m \geq 2$, and the unique vertex v of this star of degree m is called the center.

Double star $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendant edge of the existing n pendant vertices. It has $2n+1$ vertices and $2n$ edges. Let $V(K_{1,n,n}) = \{v\} \cup \{v_1, v_2, \cdots, v_n\} \cup \{u_1, u_2, \cdots, u_n\}$ and $E(K_{1,n,n}) = \{e_1, e_2, \cdots, e_n\} \cup \{s_1, s_2, \cdots, s_n\}$.

Proposition 3.1 *For any star graph $K_{1,n}$, we have*

(i) $\chi_{ioir}(M(K_{1,n})) = n+2$;
(ii) $\chi_{ioir}(C(K_{1,n})) = n+1$;
(iii) $\chi_{ioir}(T(K_{1,n})) = n+2$;
(iv) $\chi_{ioir}(L(K_{1,n})) = n$.

Proof (i) By the definition of middle graph, each edge vv_i in $K_{1,n}$ is subdivided by the vertex e_i in $M(K_{1,n})$ and the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $n+1$ in $M(K_{1,n})$. i.e $V(M(K_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Hence $n+1$ distinct colors are required to color the vertices $v, e_1, e_2, \cdots, e_n$. Note that e_i is the only external private neighbour of v_i with respect to any set $S \subseteq V$. Therefore we assign the color which is different from the already assigned colors to v_i. Hence $\chi_{ioir}(M(K_{1,n})) \geq n+2$. Assign *ioir*-coloring as follows: For $1 \leq i \leq n$, assign the color c_i for e_i and assign the color c_{n+1} to v. For $1 \leq i \leq n$, assign the color c_{n+2} to all the vertices $v_1, v_2, \cdots, v_n$. Hence $\chi_{ioir}(M(K_{1,n})) \leq n+2$.

(ii) By the definition of central graph, each edge vv_i in $K_{1,n}$ is subdivided by the vertex e_i in $C(K_{1,n})$ and the vertices $v_1, v_2, \cdots, v_n$ induce a clique of order n in $C(K_{1,n})$. i.e $V(C(K_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$. Since v_i $(1 \leq i \leq n)$ induce a clique of order n, we have $\chi_{ioir}(C(K_{1,n})) \geq n$. We now prove that $\chi_{ioir}(C(K_{1,n})) \geq n+1$. Suppose $\chi_{ioir}(C(K_{1,n})) = n$. Let V_i be the set of vertices which are colored with c_i, $i = 1$ to n. Let we assign the color c_i to v_i $(1 \leq i \leq n)$ and assign the color c_1 to v. Therefore the vertices $e_1, e_2, \cdots, e_n$ are colored by $c_2, c_3, \cdots, c_{n-1}, c_n$ in some arrangement. Hence at least two of the vertices e_i and e_j are colored with the same color c_m. Clearly any vertex adjacent to vertices e_i and e_j is also joined to vertex of color c_m. It follows that there is no external private neighbour for the vertices e_i and e_j with respect to V_m. This is a contradiction. Hence $\chi_{ioir}(C(K_{1,n})) \geq n+1$. Assign *ioir*-coloring as follows: For $1 \leq i \leq n$, assign the color c_i for v_i and assign the color c_{n+1} for each e_i. Finally we assign the color c_1 to v. Hence $\chi_{ioir}(C(K_{1,n})) \leq n+1$.

(iii) By the definition of total graph, we have $V(T(K_{1,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\}$, in which the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $n+1$. Clearly $\chi_{ioir}(T(K_{1,n})) \geq n+1$. Let we assign the color c_i to e_i $(1 \leq i \leq n)$ and assign the color c_{n+1} to v. Since e_i and v are the external private neighbors of v_i with respect to V_i and V_{n+1}, we need one more color to v_i. Hence $\chi_{ioir}(T(K_{1,n})) \geq n+2$. Assign *ioir*-coloring as follows: For $1 \leq i \leq n$, assign the color c_i for e_i and assign the color c_{n+1} to v. Finally we assign the color c_{n+2} to each v_i. Hence $\chi_{ioir}(T(K_{1,n})) \leq n+2$.

(iv) Since $L(K_{1,n}) \cong K_n$, $\chi_{ioir}(L(K_{1,n})) = n$. □

Proposition 3.2 *For any double star graph $K_{1,n,n}$, we have*

$$\chi_{ioir}(M(K_{1,n,n})) = \begin{cases} n+1 & \forall n \geq 3 \\ 4 & n = 2 \end{cases}$$

Proof Clearly we observe that $\chi_{ioir}(M(K_{1,2,2})) = 4$. By the definition of middle graph, each edge vv_i and v_iu_i $(1 \leq i \leq n)$ in $K_{1,n,n}$ are subdivided by the vertices e_i and s_i in $M(K_{1,n,n})$ and the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $n+1$ (say K_{n+1}) in $M(K_{1,n,n})$. i.e $V(M(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. Clearly $\chi_{ioir}(M(K_{1,n,n})) \geq n+1$. Assign *ioir*-coloring as follows: For $1 \leq i \leq n$, assign the color c_i for e_i and assign the color c_{n+1} to v. For $1 \leq i \leq n$, assign two distinct colors c_l and c_m other than c_{n+1} and c_i to the vertices v_i and s_i. Finally, assign the color c_{n+1} to each $u_i(1 \leq i \leq n)$. Let V_i be the set of vertices which are colored with c_i, $i = 1$ to $n+1$. Note that v is the external private neighbor of all the vertices e_i with respect to V_i, $1 \leq i \leq n$ and e_i's are the external private neighbors of v with respect to V_{n+1}. For $1 \leq i \leq n$, s_i is the external private neighbor of u_i and v_i with respect to V_{n+1} and V_l. Finally, v_i is the external private neighbor of s_i with respect to V_m. Hence $\chi_{ioir}(M(K_{1,n,n})) \leq n+1$. □

Proposition 3.3 *For any double star graph $K_{1,n,n}$, we have $\chi_{ioir}(C(K_{1,n,n})) = n+2$.*

Proof By the definition of central graph, each edge vv_i and v_iu_i $(1 \leq i \leq n)$ in $K_{1,n,n}$ are subdivided by the vertices e_i and s_i in $C(K_{1,n,n})$. The vertices $v, u_1, u_2, \cdots, u_n$ induce a clique of order $n+1$ (say K_{n+1}) and the vertices $v_i(1 \leq i \leq n)$ induce a clique of order n in $C(K_{1,n,n})$. i.e $V(C(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq n\}$. Clearly $\chi_{ioir}(C(K_{1,n,n})) \geq n+1$. We now prove that $\chi_{ioir}(C(K_{1,n,n})) \geq n+2$. Suppose $\chi_{ioir}(C(K_{1,n,n})) = n+1$. Since v, u_i $(1 \leq i \leq n)$ induce a clique of order $n+1$, let us assign the color c_{n+1} to v and assign the color c_i to $u_i(1 \leq i \leq n)$. Since e_i has degree 2 and v is adjacent to the vertex of color c_i $\forall i$, v_i is the only external private neighbour of e_i. But v_i is adjacent to the vertex of color c_j, $\forall j \neq i$. Therefore e_i must be colored only with c_i and v_i must be colored only with c_{n+1}. Since v_i $(1 \leq i \leq n)$ induce a clique of order n, v_l, it leads to a contradiction. Hence $\chi_{ioir}(C(K_{1,n,n})) \geq n+2$. Consider the colors $c_1, c_2, \cdots, c_{n+2}$. Assign *ioir*-coloring as follows: Assign the colour c_{n+1} to v and assign the color c_i to u_i, where $1 \leq i \leq n$. Assign the color c_{n+1} to all the vertices $s_1, s_2, \cdots, s_n$ and assign the color c_{n+2} to all the vertices $e_1, e_2, \cdots, e_n$. Finally, we assign the color c_i to v_i for $1 \leq i \leq n$. Let V_i be the set of vertices which are colored with c_i, $i = 1$ to $n+2$. For $1 \leq i \leq n$, e_i is the external private neighbor of v with respect to V_{n+1} and v_i is the external private neighbor of e_i with respect to V_{n+2}. For $1 \leq i \leq n$, e_i is the external private neighbor of v_i with respect to V_i and v_i is the external private neighbor of s_i with respect to V_{n+1}. Finally, v is the external private neighbor of all the vertices u_i with respect to V_i. Hence $\chi_{ioir}(C(K_{1,n,n})) \leq n+2$. □

Proposition 3.4 *For any double star graph $K_{1,n,n}$, we have $\chi_{ioir}(T(K_{1,n,n})) = n+1$.*

Proof By the definition of total graph, we have $V(T(K_{1,n,n})) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup$

$\{u_i : 1 \le i \le n\} \cup \{e_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le n\}$ in which the vertices $v, e_1, e_2, \cdots, e_n$ induce a clique of order $n+1$. Clearly $\chi_{ioir}(T(K_{1,n,n})) \ge n+1$. Consider the colors $c_1, c_2, \cdots, c_{n+1}$. Assign *ioir*-coloring as follows: Assign the color c_{n+1} to v and assign the colour c_i to e_i, where $1 \le i \le n$. For $1 \le i \le n$, assign two distinct colors other than c_{n+1} and c_i to the vertices v_i and s_i. Finally, assign the color c_{n+1} to each $u_i (1 \le i \le n)$. Hence $\chi_{ioir}(T(K_{1,n,n})) \le n+1$. □

Proposition 3.5 *For any double star graph $K_{1,n,n}$, we have $\chi_{ioir}(L(K_{1,n,n})) = n+1$.*

Proof By the definition of line graph, each edge of $K_{1,n,n}$ taken to be as vertex in $(L(K_{1,n,n}))$. The vertices $e_1, e_2, \cdots, e_n$ induce a clique of order n and the vertices $s_1, s_2, \cdots, s_n$ are all pendant in $(L(K_{1,n,n}))$. i.e $V(L(K_{1,n,n})) = \{e_i : 1 \le i \le n\} \cup \{s_i : 1 \le i \le n\}$. From Theorem 2.10, we have $\chi_{ioir}(L(K_{1,n,n})) \ge n+1$. Assign *ioir*-coloring as follows: Assign the color c_{n+1} to all the vertices s_i, where $1 \le i \le n$ and assign the color c_i to e_i, where $1 \le i \le n$. Hence $\chi_{ioir}(L(K_{1,n,n})) \le n+1$. □

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Different Domination Energies in Graphs

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Abstract: Representing a set of vertices in a graph means of a matrix was introduced by E. Sampath Kumar. Let $G(V,E)$ be a graph and $S \subseteq V$ be a set of vertices. We can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if, $v_i \in S$. In this paper we study the special case of set S being dominating set and corresponding domination energy of some class of graphs.

Key Words: Adjacency matrix, Smarandachely k-dominating set, domination number, eigenvalues, energy of graph.

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§1. Introduction

A set $D \subseteq V$ of G is said to be a Smarandachely k-dominating set if each vertex of G is dominated by at least k vertices of S and the Smarandachely k-domination number $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if k =1, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the domination number of G and denoted by $\gamma(G)$ in general.

The concept of graph energy arose in theoretical chemistry where certain numerical quantities like the heat of formation of a hydrocarbon are related to total π electron energy that can be calculated as the energy of corresponding molecular graph. The molecular graph is a representation of the molecular structure of a hydrocarbon whose vertices are the position of carbon atoms and two vertices are adjacent if there is a bond connecting them.

Eigen values and eigenvectors provide insight into the geometry of the associated linear transformation. The energy of a graph is the sum of the absolute values of the Eigen values of its adjacency matrix. From the pioneering work of Coulson [1] there exists a continuous interest towards the general mathematical properties of the total π electron energy ε as calculated within the framework of the Huckel Molecular Orbital (HMO) model. These efforts enabled one to get an insight into the dependence of ε on molecular structure. The properties of $\varepsilon(G)$ are discussed in detail in [2,3,4,5].

The importance of Eigen values is not only used in theoretical chemistry but also in analyzing structures. Car designers analyze Eigen values in order to damp out the noise to reduce

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the vibration of the car due to music. Eigen values can be used to test for cracks or deformities in a solid. Oil companies frequently use Eigen value analysis to explore land for oil. Eigen values are also used to discover new and better designs for the future [6].

Representation of a set of vertices in a graph by means of a matrix was first introduced by E. Sampath Kumar [7]. Let $G(V, E)$be a graph and $S \subseteq V$ be a set of vertices. We can represent the set S by means of a matrix as follows:

In the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S denoted by $A_S(G)$. The energy $E(G)$ obtained from the matrix $A_S(G)$ is called the set energy denoted by $E_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix as domination matrix denoted by $A_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the domination matrix $A_\gamma(G)$ is defined as domination energy denoted by $E_\gamma(G)$.

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The domination matrix of G is defined to be the square matrix $A_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the domination matrix denoted by $\kappa_1, \kappa_2, \kappa_3, \cdots, \kappa_n$ are said to be the A_γ Eigen values of G. Since the A_γ matrix is symmetric, its Eigen values are real and can be ordered $\kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \cdots \geqslant \kappa_n$. Therefore, the domination energy

$$E_\gamma = E_\gamma(G) = \sum_{i=1}^{n} |\kappa_i|. \tag{1}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy [2].

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \tag{2}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots, \geqslant \lambda_n$ are the Eigen values of the adjacency matrix $A(G)$. Recall that in the last few years, the graph energy $E(G)$ and domination energy [20,21] or covering energy [8] has been extensively studied in mathematics [8-13] and mathematic-chemical literature [14-24].

Definition 1.1(Minimal domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The domination energy $E_\gamma(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{\gamma-\min}(G)$.*

Definition 1.2(Maximal domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The domination energy $E_\gamma(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{\gamma-\max}(G)$.*

Similarly to domination energy of graph G, *distance domination energy* can also be defined as follows:

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The *distance matrix* of G is denoted by $D(G)$ is defined to be the square matrix $D(G) = [d_{ij}]$, where d_{ij} is the shortest distance between the vertex v_i and v_j in G. The Eigen values of the distance matrix denoted by $\mu_1, \mu_2, \mu_3, \cdots, \mu_n$ are said to be the D Eigen values of G. Since the $D(G)$ matrix is symmetric, its Eigen values are real and can be ordered $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots \geqslant \mu_n$. Therefore, the distance energy

$$E_D = E_D(G) = \sum_{i=1}^{n} |\mu_i|. \tag{3}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy [2].

In the distance matrix $D(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the distance matrix can be considered as the *distance matrix of the set* S denoted by $D_S(G)$. The energy $E(G)$ obtained from the matrix $D_S(G)$ is called the *distance set energy* denoted by $D_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix is *distance domination matrix* denoted by $D_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the distance domination matrix $D_\gamma(G)$ is defined as *distance domination energy* denoted by $E_{D\gamma}(G)$.

The distance domination matrix of G is defined to be the square matrix $D_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the distance domination matrix denoted by $\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n$ are said to be the D_γ Eigen values of G. Since the $D_\gamma(G)$ matrix is symmetric, its D-Eigen values are real and can be ordered $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant \cdots \geqslant \sigma_n$. Therefore, the distance domination energy

$$E_{D\gamma} = E_{D\gamma}(G) = \sum_{i=1}^{n} |\sigma_i|. \tag{4}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy [2].

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \tag{5}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots, \geqslant \lambda_n$ are the Eigen values of the adjacency matrix $A(G)$.

Definition 1.3(Minimal distance domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The distance domination energy $E_{D\gamma}(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{D\gamma-\min}(G)$.*

Definition 1.4(Maximal distance domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The distance domination energy $E_{D\gamma}(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{D\gamma-\max}(G)$.*

§2. Different Energies of Graph with $\gamma(G) = 1$

In this section, we characterize graphs with respect to the unique domination set and hence find their different domination energies.

Remark 2.1 For the complete graph K_n the matrices $A(G) = D(G)$ and $A_\gamma(G) = D_\gamma(G)$.

Hence, the energy of complete graph K_n is given by $2(n-1)$, i.e., $E(K_n) = E_D(K_n) = 2(n-1)$.

Theorem 2.1 *Let $G = K_n$. Then,*

$$E_{\gamma-\min}(K_n) = E_{D\gamma-\min}(K_n) = \sqrt{n^2 - 2n + 5} + (n-2), n \geqslant 3.$$

Proof Calculation enables one to find the characteristic polynomial of K_n for $n \geqslant 3$ directly. Label the vertices of K_n as $v_1, v_2, v_3, \cdots, v_n$ such that v_1 is the dominating set. The domination matrix and the distance domination matrix are same. Hence, in the domination matrix or distance domination matrix $a_{11} = 1$ and $a_{ii} = 0, i \neq 1$.

The characteristic polynomial of domination matrix and the distance domination matrix is given by $\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \cdots + q_{n-1}\kappa + q_n = 0$ and $\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0$ respectively.

The domination matrix and the characteristic polynomial of K_3 are given by

$$A_\gamma(G) = D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^3 - \kappa^2 - 3\kappa - 1 = (\kappa+1)\left(\kappa^2 - 2\kappa - 1\right)$.

The domination matrix and the characteristic polynomial of K_4 are given by

$$A_\gamma(G) = D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^4 - \kappa^3 - 6\kappa^2 - 5\kappa - 1 = (\kappa+1)^2\left(\kappa^2 - 3\kappa - 1\right)$.

The domination matrix and the characteristic polynomial of K_5 are given by

$$A_\gamma(G) = D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^5 - \kappa^4 - 10\kappa^3 - 14\kappa^2 - 7\kappa - 1 = (\kappa+1)^3\left(\kappa^2 - 4\kappa - 1\right)$.

Therefore, the characteristic polynomial of K_n using domination matrix is

$$(\kappa+1)^{n-2}\left(\kappa^2 - (n-1)\kappa - 1\right) = 0.$$

Solving the equation we get

$$(\kappa+1)^{n-2}=0 \text{ or } \left(\kappa^2-(n-1)\kappa-1\right)=0.$$

$$\kappa=-1,-1,-1,\cdots,-1(n-2) \text{ times}$$

$$\kappa^2-(n-1)\kappa-1=0$$

Therefore,

$$\kappa=\frac{n-1\pm\sqrt{(n-1)^2-4(1)(-1)}}{2}=\frac{n-1\pm\sqrt{n^2-2n+5}}{2},$$

where $n\geqslant 3$. Hence the roots are

$$\kappa_1=\frac{n-1+\sqrt{n^2-2n+5}}{2},\ \kappa_2=-\left(\frac{\sqrt{n^2-2n+5}-(n-1)}{2}\right)$$

and

$$\begin{aligned}E_{\gamma-\min}(K_n)&=\sum_{i=1}^{n}|\kappa_i|\\&=\frac{n-1+\sqrt{n^2-2n+5}+\sqrt{n^2-2n+5}-(n-1)}{2}+n-2,\\E_{\gamma-\min}(K_n)&=E_{D\gamma-\min}(K_n)=\sqrt{n^2-2n+5}+(n-2).\end{aligned}$$

Hence, we get the proof. □

Remark 2.2 All four types of energies of a complete graph can be compared as follows:

$$\begin{aligned}E(K_n)&=E_D(K_n)=2(n-1)\geqslant E_{\gamma-\min}(K_n)\\&=E_{D\gamma-\min}(K_n)=\sqrt{n^2-2n+5}+(n-2).\end{aligned}$$

Remark 2.3 Energy of a star graph $K_{1,n-1}$ is given by $2\sqrt{n-1}$.

Theorem 2.2([21]) *Let $G=K_{1,n-1}$, $n\geqslant 3$. Then,*

$$E_{\gamma-\min}\left(K_{1,n-1}\right)=\sqrt{4n-3}.$$

Remark 2.4 $E\left(K_{1,n-1}\right)=2\sqrt{n-1}\leqslant E_{\gamma-\min}\left(K_{1,n-1}\right)=\sqrt{4n-3}$.

Theorem 2.3 *Let $G=K_{1,n-1}$, $n\geqslant 3$. Then,*

$$E_D\left(K_{1,n-1}\right)_= 2n-4+\sqrt{n^2-3n+3}.$$

Proof Calculation enables one to find the characteristic polynomial of $K_{1,n-1}$ for $n\geqslant 3$ directly. Label the vertices of $K_{1,n-1}$ as $v_1,v_2,v_3,\cdots,v_n$. The characteristic polynomial of distance matrix $D(G)$ is given by

$$\mu^n + q_1\mu^{n-1} + q_2\mu^{n-2} + \cdots + q_{n-1}\mu + q_n = 0.$$

The distance matrix and the characteristic polynomial of $K_{1,2}$ are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

and $\mu^3 - 6\mu - 4 = (\mu + 2)\left(\mu^2 - 2\mu - 2\right).$

The distance matrix and the characteristic polynomial of $K_{1,3}$ are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

and $\mu^4 - 15\mu^2 - 28\mu - 12 = (\mu + 2)^2\left(\mu^2 - 4\mu - 3\right).$

The distance matrix and the characteristic polynomial of $K_{1,4}$ are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and $\mu^5 - 28\mu^3 - 88\mu^2 - 96\mu - 32 = (\mu + 2)^3\left(\mu^2 - 6\mu - 4\right).$

The distance matrix and the characteristic polynomial of $K_{1,5}$ are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 \end{bmatrix}$$

and $\mu^6 - 45\mu^4 - 200\mu^3 - 360\mu^2 - 288\mu - 80 = (\mu + 2)^4\left(\mu^2 - 8\mu - 5\right).$

Therefore the characteristic polynomial of $K_{1,n-1}$ using distance matrix is

$$(\mu + 2)^{n-2}\left(\mu^2 - (2n - 4)\mu - (n - 1)\right).$$

Solving the equation we get

$$(\mu+2)^{n-2}=0 \ \text{ or } \ \mu^2-(2n-4)\mu-(n-1)=0,$$

$$\mu=-2,-2,-2,\cdots,-2(n-2)(\text{times}) \text{ or } \mu^2-(2n-4)\mu-(n-1)=0.$$

Therefore,

$$\mu=\frac{(2n-4)\pm\sqrt{(2n-4)^2-4(-(n-1))}}{2}=\frac{(2n-4)\pm\sqrt{4\left(n^2-3n+3\right)}}{2}$$

where $n\geqslant 3$. Hence the roots are

$$\mu_1=\frac{(n-4)+\sqrt{n^2-3n+3}}{2} \ \text{ and } \ \mu_2=-\left(\frac{\sqrt{n^2-3n+3}-(n-4)}{2}\right).$$

$$\begin{aligned} E_D\left(K_{1,n-1}\right) &= \sum_{i=1}^{n}|\mu_i| \\ &= \frac{2\sqrt{n^2-3n+3}}{2}+2(n-2) \\ &= 2n-4+\sqrt{n^2-3n+3}. \end{aligned}$$

Hence, we get the proof. □

Theorem 2.4 *Let $G=K_{1,n-1}$, $n\geqslant 3$. Then,*

$$E_{D\gamma}\left(K_{1,n-1}\right)_= 4n-7.$$

Proof Calculation enables one to find the characteristic polynomial of $K_{1,n-1}$ for $n\geqslant 3$ directly. Label the vertices of $K_{1,n-1}$ as $v_1,v_2,v_3,\cdots,v_n$.

The characteristic polynomial of distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n+q_1\sigma^{n-1}+q_2\sigma^{n-2}+\cdots+q_{n-1}\sigma+q_n=0.$$

The distance domination matrix and the characteristic polynomial of $K_{1,2}$ are given by

$$D_\gamma(G)=\begin{bmatrix}1&1&1\\1&0&2\\1&2&0\end{bmatrix}$$

and $\sigma^3-\sigma^2-6\sigma=(\sigma+2)\left(\sigma^2-3\sigma+0\right)$.

The distance domination matrix and the characteristic polynomial of $K_{1,3}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

and $\sigma^4 - \sigma^3 - 15\sigma^2 - 16\sigma + 4 = (\sigma+2)^2\left(\sigma^2 - 5\sigma + 1\right)$.

The distance domination matrix and the characteristic polynomial of $K_{1,4}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and $\sigma^5 - \sigma^4 - 28\sigma^3 - 64\sigma^2 - 32\sigma + 16 = (\sigma+2)^3\left(\sigma^2 - 7\sigma + 2\right)$.

The distance domination matrix and the characteristic polynomial of $K_{1,5}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 \end{bmatrix}$$

and $\sigma^6 - \sigma^5 - 45\sigma^4 - 160\sigma^3 - 200\sigma^2 - 48\sigma + 48 = (\sigma+2)^4\left(\sigma^2 - 9\sigma + 3\right)$.

Therefore the characteristic polynomial of $K_{1,n-1}$ using distance domination matrix is

$$(\sigma+2)^{n-2}\left(\sigma^2 - (2n-3)\sigma + (n-3)\right) = 0.$$

Solving the equation we get

$$(\sigma+2)^{n-2} = 0 \text{ or } \sigma^2 - (2n-3)\sigma + (n-3) = 0.$$

Whence, $\sigma = -2, -2, -2, \cdots, -2$ ($(n-2)$ times) or $\sigma^2 - (2n-3)\sigma + (n-3) = 0$. Therefore,

$$\sigma = \frac{(2n-3) \pm \sqrt{(2n-3)^2 - 4((n-3))}}{2} = \frac{(2n-3) \pm \sqrt{4n^2 - 16n + 21}}{2},$$

where $n \geqslant 3$, i.e., the roots are

$$\sigma_1 = \frac{(2n-3)+\sqrt{4n^2-16n+21}}{2},$$
$$\sigma_2 = \frac{(2n-3)-\sqrt{4n^2-16n+21}}{2}$$

and

$$E_{D\gamma}(K_{1,n-1}) = \sum_{i=1}^{n} |\sigma_i| = (2n-3)+2(n-2) = 4n-7.$$

Hence, we get the proof. □

§3. Domination Energies for the Graph with $\gamma(G) = 2$

During the study of chemical graphs and its Weiner number, the Yugoslavian chemist Ivan Gutman introduced the concept of Thorn graphs. This idea was further extended to the broader concept of generalized thorny graphs by Danail Bonchev and Douglas J Klein of USA. This class of graphs gain importance in Spectral theory as it represents the structural formula of aliphatic and aromatic hydrocarbons9[.

Theorem 3.1 *Let $G = P_{2,t}$, $n = 2t$. Then,*

$$E(P_{2,t}) = 2\sqrt{4t-3}.$$

Proof Calculation enables one to find the characteristic polynomial of $G = P_{2,t}$ for $n = 2t$ directly. For $t = 1$, $P_{2,1}$ is a path with 2 vertices, $t = 2$, $P_{2,2}$ is a path with 4 vertices.

The adjacency matrix and the characteristic polynomial of $P_{2,3}$ are given by

$$A(G) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and $\lambda^6 - 5\lambda^4 + 4\lambda^2 = \lambda^2(\lambda^2 - \lambda - 2)(\lambda^2 + \lambda - 2)$.

The adjacency matrix and the characteristic polynomial of $P_{2,4}$ are given by

$$A(G) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

and $\lambda^8 - 7\lambda^6 + 9\lambda^4 = \lambda^4(\lambda^2 - \lambda - 3)(\lambda^2 + \lambda - 3)$.

The adjacency matrix and the characteristic polynomial of $P_{2,5}$ are given by

$$A(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\lambda^{10} - 9\lambda^8 + 16\lambda^6 = \lambda^6(\lambda^2 - \lambda - 4)(\lambda^2 + \lambda - 4)$.

The adjacency matrix and the characteristic polynomial of $P_{2,6}$ are given by

$$A(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\lambda^{12}-11\lambda^{10}+25\lambda^{8}=\lambda^{8}(\lambda^{2}-\lambda-5)(\lambda^{2}+\lambda-5)$.

Therefore the characteristic polynomial of $P_{2,t}$ using adjacency matrix is

$$\lambda^{2t-4}(\lambda^{2}-\lambda-(t-1))(\lambda^{2}+\lambda-(t-1)).$$

Solving the equation we get

$$\lambda^{2t-4}=0,\ \lambda^{2}-\lambda-(t-1)=0 \text{ or } \lambda^{2}+\lambda-(t-1)=0,$$

i.e.,

$$\lambda=0,0,0,\cdots,0((2t-4)\text{ times}),\ \lambda^{2}-\lambda-(t-1)=0.$$

Therefore,

$$\lambda=\frac{1\pm\sqrt{1+4t-4}}{2}=1\pm\sqrt{4t-3},$$

where $t\geqslant 3$. Hence the roots are

$$\lambda_1=1+\sqrt{4t-3} \text{ and } \lambda_2=-\left(\sqrt{4t-3}-1\right)$$

and

$$E=\sum_{i=1}^{n}|\lambda_i|=\frac{1+\sqrt{4t-3}+\sqrt{4t-3}-1}{2}=\sqrt{4t-3}.$$

Similarly, solving the equation $\lambda^{2}+\lambda-(t-1)=0$ we get that

$$E=\sqrt{4t-3}.$$

Whence,

$$E\left(P_{2,t}\right)=2\sqrt{4t-3}.$$

Hence, we get the proof. □

Theorem 3.2([21]) *Let $G=P_{2,t}$, $n=2t$. Then,*

$$E_{\gamma-\min}\left(P_{2,t}\right)=2\sqrt{t-1}+2\sqrt{t}.$$

Theorem 3.3 *Let $G=P_{2,t}$, $n=2t$. Then,*

$$E_D\left(P_{2,t}\right)_{=}\sqrt{25t^{2}-28t+20}+(5t-6).$$

Proof Calculation enables one to find the characteristic polynomial of $P_{2,t}$ for $n=2t$ directly. For $t=1$, $P_{2,1}$ is a path with 2 vertices, $t=2$, $P_{2,2}$ is a path with 4 vertices.

The characteristic polynomial of $P_{2,t}$ using distance matrix $D(G)$ is given by

$$\mu^{n}+q_1\mu^{n-1}+q_2\mu^{n-2}+\cdots+q_{n-1}\mu+q_n=0.$$

The distance matrix and the characteristic polynomial of $P_{2,3}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 1 & 2 & 3 & 3 \\ 2 & 0 & 1 & 2 & 3 & 3 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 2 & 2 & 1 & 0 & 1 & 1 \\ 3 & 3 & 2 & 1 & 0 & 2 \\ 3 & 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

and

$$\mu^6 - 65\mu^4 - 296\mu^3 - 504\mu^2 - 352\mu - 80 = (\mu+2)^2\left(\mu^2 - 9\mu - 10\right)\left(\mu^2 + 5\mu + 2\right).$$

The distance matrix and the characteristic polynomial of $P_{2,4}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 2 & 1 & 2 & 3 & 3 & 3 \\ 2 & 0 & 2 & 1 & 2 & 3 & 3 & 3 \\ 2 & 2 & 0 & 1 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 0 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 1 & 0 & 2 & 2 \\ 3 & 3 & 3 & 2 & 1 & 2 & 0 & 2 \\ 3 & 3 & 3 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\begin{aligned} &\mu^8 - 136\mu^6 - 1040\mu^5 - 3468\mu^4 - 6112\mu^3 - 5792\mu^2 - 2688\mu - 448 \\ &= (\mu+2)^4\left(\mu^2 - 14\mu - 14\right)\left(\mu^2 + 6\mu + 2\right). \end{aligned}$$

The distance matrix and the characteristic polynomial of $P_{2,5}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 0 & 2 & 2 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 0 & 2 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 0 & 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 & 0 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 2 & 0 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 2 & 2 & 0 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\mu^{10} - 233\mu^8 - 2512\mu^7 - 12624\mu^6 - 36800\mu^5 - 66400\mu^4 - 74496\mu^3$$
$$-49664\mu^2 - 17408\mu - 2304 = (\mu+2)^6\left(\mu^2 - 19\mu - 18\right)\left(\mu^2 + 7\mu + 2\right).$$

Therefore, the characteristic polynomial of $P_{2,t}$ using distance matrix is

$$(\mu+2)^{2t-4}\left(\mu^2 - (5t-6)\mu - (4t-2)\right)\left(\mu^2 + (t+2)\mu + 2\right),$$

i.e.,

$$(\mu+2)^{2t-4} = 0,\ \mu^2 - (5t-6)\mu - (4t-2),\ \text{or}\ \mu^2 + (t+2)\mu + 2\,(\mu+2)^{2t-4} = 0.$$

Solving the equation $(\mu+2)^{2t-4}$ we get $\mu = -2, -2, -2, \cdots, -2((2t-4)$ times. Similarly, Solving the equation $\mu^2 - (5t-6)\mu - (4t-2)$ we get

$$\mu = \frac{(5t-6) \pm \sqrt{(5t-6)^2 + 4(4t-2)}}{2}$$

, and the equation $+(t+2)\mu + 2$ we get

$$\mu = \frac{-(t+2) \pm \sqrt{(t+2)^2 - 8}}{2}.$$

Therefore,

$$E_D\left(P_{2,t}\right) = \sum_{i=1}^{n} |\mu_i| = \sqrt{25t^2 - 28t + 20} + (t+2) + (4t-8)$$
$$= \sqrt{25t^2 - 28t + 20} + (5t-6).$$

Hence, we get the proof. □

Theorem 3.4 *Let $G = P_{2,t}$, $n = 2t$. Then,*

$$E_{D\gamma}\left(P_{2,t}\right) = \begin{cases} \sqrt{25t^2 - 54t + 45} + \sqrt{t^2 + 6t - 3} + (4t-8) & t = 3, 4 \\ (5t-5) + \sqrt{t^2 + 6t - 3} + (4t-8) & t > 5 \end{cases}$$

and for $t = 5$,

$$E_{D\gamma}\left(P_{2,t}\right) = \frac{(5t-5) + \sqrt{25t^2 + 54t + 45}}{2} + \sqrt{t^2 + 6t - 3} + (4t-8).$$

Proof Calculation enables one to find the characteristic polynomial of $P_{2,t}$ for $n = 2t$ directly. For $t = 1$, $P_{2,1}$ is a path with 2 vertices, $t = 2$, $P_{2,2}$ is a path with 4 vertices.

The characteristic polynomial of $P_{2,t}$ using distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0.$$

The distance domination matrix and the characteristic polynomial of $P_{2,3}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 1 & 2 & 3 & 3 \\ 2 & 0 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 1 & 0 & 2 \\ 3 & 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

and $\sigma^6 - 2\sigma^5 - 64\sigma^4 - 188\sigma^3 - 124\sigma^2 + 64\sigma + 16 = (\sigma+2)^2\left(\sigma^2 - 10\sigma - 2\right)\left(\sigma^2 + 4\sigma - 2\right).$

The distance domination matrix and the characteristic polynomial of $P_{2,4}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 2 & 1 & 2 & 3 & 3 & 3 \\ 2 & 0 & 2 & 1 & 2 & 3 & 3 & 3 \\ 2 & 2 & 0 & 1 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 1 & 0 & 2 & 2 \\ 3 & 3 & 3 & 2 & 1 & 2 & 0 & 2 \\ 3 & 3 & 3 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\sigma^8 - 2\sigma^7 - 135\sigma^6 - 800\sigma^5 - 1877\sigma^4 - 1704\sigma^3 + 88\sigma^2 + 736\sigma + 48$$
$$= (\sigma+2)^4\left(\sigma^2 - 15\sigma - 1\right)\left(\sigma^2 + 5\sigma - 3\right).$$

The distance domination matrix and the characteristic polynomial of $P_{2,5}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 0 & 2 & 2 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 0 & 2 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 0 & 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 & 0 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 2 & 0 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 2 & 2 & 0 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\sigma^{10} - 2\sigma^9 - 232\sigma^8 - 2088\sigma^7 - 8480\sigma^6 - 18208\sigma^5 - 19584\sigma^4$$
$$-5504\sigma^3 + 7424\sigma^2 + 5120\sigma = (\sigma + 2)^6 \left(\sigma^2 - 20\sigma - 0\right)\left(\sigma^2 + 6\sigma - 4\right).$$

The distance domination matrix and the characteristic polynomial of $P_{2,6}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 1 & 2 & 3 & 3 & 3 & 3 & 3 \\ 2 & 0 & 2 & 2 & 2 & 1 & 2 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 0 & 2 & 2 & 1 & 2 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 0 & 2 & 1 & 2 & 3 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 3 & 2 & 1 & 0 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 2 & 1 & 2 & 0 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 2 & 1 & 2 & 2 & 0 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 & 2 & 1 & 2 & 2 & 2 & 0 & 2 \\ 3 & 3 & 3 & 3 & 3 & 2 & 1 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\sigma^{12} - 2\sigma^{11} - 355\sigma^{10} - 4300\sigma^9 - 24885\sigma^8 - 83856\sigma^7 - 172368\sigma^6 - 206400\sigma^5 - 108000\sigma^4$$
$$+39680\sigma^3 + 80384\sigma^2 + 28672\sigma - 1280 = (\sigma + 2)^8 \left(\sigma^2 - 25\sigma + 1\right)\left(\sigma^2 + 7\sigma - 5\right).$$

Therefore, the characteristic polynomial of $P_{2,t}$ using distance domination matrix is

$$(\sigma + 2)^{2t-4} \left(\sigma^2 - (5t - 5)\sigma + (t - 5)\right)\left(\sigma^2 + (t + 1)\sigma - (t - 1)\right),$$

i.e.,

$$(\sigma + 2)^{2t-4}, \ \sigma^2 - (5t - 5)\sigma + (t - 5) \text{ or } \sigma^2 + (t + 1)\sigma - (t - 1).$$

Solving the equation $(\sigma + 2)^{2t-4} = 0$ we get $\sigma = -2, -2, -2, \cdots, -2$ ($(2t - 4)$ times). Similarly, solving the equation $\sigma^2 - (5t - 5)\sigma + (t - 5)$ we get

$$\sigma = \frac{(5t - 5) \pm \sqrt{(5t - 5)^2 - 4(t - 5)}}{2}$$

and the equation $\sigma^2 + (t + 1)\sigma - (t - 1)$ implies

$$\sigma = \frac{(t + 1) \pm \sqrt{(t + 2)^2 + 4(t - 1)}}{2}.$$

Therefore,

$$E_{D\gamma}\left(P_{2,t}\right)=\sum_{i=1}^{n}|\sigma_i|=\begin{cases}\sqrt{25t^2-54t+45}+\sqrt{t^2+6t-3}+(4t-8), & t=3,4\\ (5t-5)+\sqrt{t^2+6t-3}+(4t-8), & t>5.\end{cases}$$

and for $t=5$,

$$E_{D\gamma}\left(P_{2,t}\right)=\frac{(5t-5)+\sqrt{25t^2+54t+45}}{2}+\sqrt{t^2+6t-3}+(4t-8).\qquad\square$$

Theorem 3.5 *Let $G=P_{3,t}$, $n=2t+1$. Then,*

$$E\left(P_{3,t}\right)=2\sqrt{t-1}+2\sqrt{t+1}.$$

Proof Calculation enables one to find the characteristic polynomial of $P_{3,t}$ for $n=2t+1$ directly. For $t=1$, $P_{3,1}$ is a path with 3 vertices, $t=2$, $P_{3,2}$ is a path with 5 vertices.

The adjacency matrix and the characteristic polynomial of $P_{3,3}$ are given by

$$A(G)=\begin{bmatrix}0&0&1&0&0&0&0\\0&0&1&0&0&0&0\\1&1&0&1&0&0&0\\0&0&1&0&1&0&0\\0&0&0&1&0&1&1\\0&0&0&0&1&0&0\\0&0&0&0&1&0&0\end{bmatrix}$$

and $\lambda^7-6\lambda^5+8\lambda^3=\lambda^3(\lambda^2-2)(\lambda^2-4)$.

The adjacency matrix and the characteristic polynomial of $P_{3,4}$ are given by

$$A(G)=\begin{bmatrix}0&0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\1&1&1&0&1&0&0&0&0\\0&0&0&1&0&1&0&0&0\\0&0&0&0&1&0&1&1&1\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0\end{bmatrix}$$

and $\lambda^9-8\lambda^7+15\lambda^5=\lambda^5(\lambda^2-3)(\lambda^2-5)$.

The adjacency matrix and the characteristic polynomial of $P_{3,5}$ are given by

$$A(G)=\begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 1&1&1&1&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&1&1&1&1\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0 \end{bmatrix}$$

and $\lambda^{11}-10\lambda^9+24\lambda^7=\lambda^7(\lambda^2-4)(\lambda^2-6)$.

Therefore the characteristic polynomial of $P_{3,t}$ using adjacency matrix is

$$\lambda^{2t-3}(\lambda^2-(t-1))(\lambda^2-(t+1)).$$

Solving the equation we get

$$E\left(P_{3,t}\right)=2\sqrt{t-1}+2\sqrt{t+1}.$$

Hence, we get the proof. □

Theorem 3.6([21]0) *Let $G=P_{3,t}$, $n=2t+1$. Then,*

$$E_{\gamma-\min}\left(P_{3,t}\right)=\sqrt{4t-3}+\sqrt{4t+5}.$$

Theorem 3.7 *Let $G=P_{3,t}$, $n=2t+1$ Then, the characteristic polynomial of $P_{3,t}$ using distance matrix of G is*

$$(\mu+2)^{2t-4}\left(\mu^2+(2t+2)\mu+4\right)\left(\mu^3-(6t-6)\mu^2-(12t-6)\mu-4t\right)=0.$$

Proof Calculation enables one to find the characteristic polynomial of $P_{3,t}$ for $n=2t+1$ directly. For $t=1$, $P_{3,1}$ is a path with 3 vertices, $t=2$, $P_{3,2}$ is a path with 5 vertices.

The characteristic polynomial of distance matrix $D(G)$ is given by

$$\mu^n+q_1\mu^{n-1}+q_2\mu^{n-2}+\cdots+q_{n-1}\mu+q_n=0.$$

The distance matrix and the characteristic polynomial of $P_{3,3}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 1 & 2 & 3 & 4 & 4 \\ 2 & 0 & 1 & 2 & 3 & 4 & 4 \\ 1 & 1 & 0 & 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 0 & 1 & 2 & 2 \\ 3 & 3 & 2 & 1 & 0 & 1 & 1 \\ 4 & 4 & 3 & 2 & 1 & 0 & 2 \\ 4 & 4 & 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

and

$$\mu^7 - 134\mu^5 - 804\mu^4 - 1904\mu^3 - 2112\mu^2 - 1056\mu - 192$$
$$= (\mu+2)^2\left(\mu^2+8\mu+4\right)\left(\mu^3 - 12\mu^2 - 30\mu - 12\right).$$

The distance matrix and the characteristic polynomial of $P_{3,4}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 2 & 1 & 2 & 3 & 4 & 4 & 4 \\ 2 & 0 & 2 & 1 & 2 & 3 & 4 & 4 & 4 \\ 2 & 2 & 0 & 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 1 & 1 & 0 & 1 & 2 & 3 & 3 & 3 \\ 2 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & 2 & 1 & 0 & 1 & 1 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 & 0 & 2 & 2 \\ 4 & 4 & 4 & 3 & 2 & 1 & 2 & 0 & 2 \\ 4 & 4 & 4 & 3 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\mu^9 - 258\mu^7 - 2412\mu^6 - 9864\mu^5 - 21984\mu^4 - 28128\mu^3 - 20160\mu^2$$
$$-7296\mu - 1024 = (\mu+2)^4\left(\mu^2+10\mu+4\right)\left(\mu^3 - 18\mu^2 - 42\mu - 16\right).$$

The distance matrix and the characteristic polynomial of $P_{3,5}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 0 & 2 & 2 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 0 & 2 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 2 & 0 & 2 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 2 & 2 & 0 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\mu^{11} - 422\mu^9 - 5380\mu^8 - 31584\mu^7 - 108160\mu^6 - 233920\mu^5 - 326784\mu^4 - 290560\mu^3$$
$$-155648\mu^2 - 44544\mu - 5120 = (\mu + 2)^6 \left(\mu^2 + 12\mu + 4\right) \left(\mu^3 - 24\mu^2 - 54\mu - 20\right).$$

Therefore, the characteristic polynomial of $P_{3,t}$ using distance matrix is

$$(\mu + 2)^{2t-4} \left(\mu^2 + (2t + 2)\mu + 4\right) \left(\mu^3 - (6t - 6)\mu^2 - (12t - 6)\mu - 4t\right) = 0. \qquad \square$$

Theorem 3.8 *Let $G = P_{3,t}$, $n = 2t + 1$ Then, the characteristic polynomial of $P_{2,t}$ using distance domination matrix of G, is given by*

$$(\sigma + 2)^{2t-4} \left(\sigma^2 + (2t + 1)\sigma - (2t - 4)\right) \left(\sigma^3 - (6t - 5)\sigma^2 - (6t + 2)\sigma + (4t + 8)\right) = 0.$$

Proof Calculation enables one to find the characteristic polynomial of $P_{3,t}$ for $n = 2t + 1$ directly. For $t = 1$, $P_{3,1}$ is a path with 3 vertices, $t = 2$, $P_{3,2}$ is a path with 5 vertices. The characteristic polynomial of distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0.$$

The distance domination matrix and the characteristic polynomial of $P_{3,3}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 1 & 2 & 3 & 4 & 4 \\ 2 & 0 & 1 & 2 & 3 & 4 & 4 \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 0 & 1 & 2 & 2 \\ 3 & 3 & 2 & 1 & 1 & 1 & 1 \\ 4 & 4 & 3 & 2 & 1 & 0 & 2 \\ 4 & 4 & 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

and

$$\begin{aligned}&\sigma^7-2\sigma^6-133\sigma^5-586\sigma^4-824\sigma^3-176\sigma^2+240\sigma-32\\&=(\sigma+2)^2\left(\sigma^2+7\sigma-2\right)\left(\sigma^3-13\sigma^2-20\sigma+4\right).\end{aligned}$$

The distance domination matrix and the characteristic polynomial of $P_{3,4}$ are given by

$$D_\gamma(G)=\begin{bmatrix}0&2&2&1&2&3&4&4&4\\2&0&2&1&2&3&4&4&4\\2&2&0&1&2&3&4&4&4\\1&1&1&1&1&2&3&3&3\\2&2&2&1&0&1&2&2&2\\3&3&3&2&1&1&1&1&1\\4&4&4&3&2&1&0&2&2\\4&4&4&3&2&1&2&0&2\\4&4&4&3&2&1&2&2&0\end{bmatrix}$$

and

$$\begin{aligned}&\sigma^9-2\sigma^8-257\sigma^7-1966\sigma^6-6152\sigma^5-8816\sigma^4-4048\sigma^3+2464\sigma^2+1792\sigma-512\\&=(\sigma+2)^4\left(\sigma^2+9\sigma-4\right)\left(\sigma^3-19\sigma^2-26\sigma+18\right).\end{aligned}$$

The distance domination matrix and the characteristic polynomial of $P_{3,5}$ are given by

$$D_\gamma(G)=\begin{bmatrix}0&2&2&2&1&2&3&4&4&4&4\\2&0&2&2&1&2&3&4&4&4&4\\2&2&0&2&1&2&3&4&4&4&4\\2&2&2&0&1&2&3&4&4&4&4\\1&1&1&1&1&1&2&3&3&3&3\\2&2&2&2&1&0&1&2&2&2&2\\3&3&3&3&2&1&1&1&1&1&1\\4&4&4&4&3&2&1&0&2&2&2\\4&4&4&4&3&2&1&2&0&2&2\\4&4&4&4&3&2&1&2&2&0&2\\4&4&4&4&3&2&1&2&2&2&0\end{bmatrix}$$

and

$$\sigma^{11}-2\sigma^{10}-421\sigma^9-4626\sigma^8-22736\sigma^7-60832\sigma^6-89568\sigma^5-59072\sigma^4+9728\sigma^3$$
$$+32768\sigma^2+6912\sigma-4608=(\sigma+2)^6\left(\sigma^2+11\sigma-6\right)\left(\sigma^3-25\sigma^2-32\sigma+12\right).$$

Therefore the characteristic polynomial of $P_{2,t}$ using distance domination matrix of G is

$$(\sigma+2)^{2t-4}\left(\sigma^2+(2t+1)\sigma-(2t-4)\right)\left(\sigma^3-(6t-5)\sigma^2-(6t+2)\sigma+(4t+8)\right)=0. \qquad \square$$

Theorem 3.9 *Let $G=P_{4,t}$, $n=2t+2$. Then, the characteristic polynomial using adjacency matrix of G is given by*

$$\lambda^{2t-4}(\lambda^3-\lambda^2-t\lambda+(t-1))(\lambda^3+\lambda^2-t\lambda-(t-1)).$$

Proof Calculation enables one to find the characteristic polynomial of $P_{4,t}$ for $n=2t+2$ directly. For $t=1$, $P_{4,1}$ is a path with 4 vertices, $t=2$, $P_{4,2}$ is a path with 6 vertices.

The adjacency matrix and the characteristic polynomial of $P_{4,3}$ are given by

$$A(G)=\begin{bmatrix}0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&1&0&0&0&0\\0&0&1&0&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&0&0&1&0&1&1\\0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\end{bmatrix}$$

and $\lambda^8-7\lambda^6+13\lambda^4-4\lambda^2=\lambda^2(\lambda^3-\lambda^2-3\lambda+2)(\lambda^3+\lambda^2-3\lambda-2)$.

The adjacency matrix and the characteristic polynomial of $P_{4,4}$ are given by

$$A(G)=\begin{bmatrix}0&0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\1&1&1&0&1&0&0&0&0&0\\0&0&0&1&0&1&0&0&0&0\\0&0&0&0&1&0&1&0&0&0\\0&0&0&0&0&1&0&1&1&1\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0&0\end{bmatrix}$$

and $\lambda^{10} - 9\lambda^8 + 22\lambda^6 - 9\lambda^4 = \lambda^4(\lambda^3 - \lambda^2 - 4\lambda + 3)(\lambda^3 + \lambda^2 - 4\lambda - 3)$.

The adjacency matrix and the characteristic polynomial of $P_{4,5}$ are given by

$$A_\gamma(G) = \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 1&1&1&1&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&1&1&1&1 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \end{bmatrix}$$

and

$$\lambda^{12} - 11\lambda^{10} + 33\lambda^8 - 16\lambda^6 = \lambda^6(\lambda^3 - \lambda^2 - 5\lambda + 4)(\lambda^3 + \lambda^2 - 5\lambda - 4).$$

Therefore, the characteristic polynomial of $P_{4,t}$ using adjacency matrix of G is

$$\lambda^{2t-4}(\lambda^3 - \lambda^2 - t\lambda + (t-1))(\lambda^3 + \lambda^2 - t\lambda - (t-1)).$$

Hence, we get the proof. □

Theorem 3.10([21]) *Let $G = P_{4,t}$, $n = 2t + 2$. Then, the characteristic polynomial using domination matrix of G is given by*

$$\kappa^{2t-4}(\kappa^3 - (t+1)\kappa - (t-1))(\kappa^3 - 2\kappa^2 - (t-1)\kappa + (t-1)).$$

Theorem 3.11 *Let $G = P_{4,t}$, $n = 2t + 2$. Then, the characteristic polynomial using distance matrix of G is given by*

$$(\mu+2)^{2t-4}(\mu^3 - (7t-5)\mu^2 - (22t-8)\mu - (8t+4))(\mu^3 + (3t+2)\mu^2 + (2t+8)\mu + 4).$$

Proof Calculation enables one to find the characteristic polynomial of $P_{4,t}$ for $n = 2t + 2$ directly. For $t = 1$, $P_{4,1}$ is a path with 4 vertices, $t = 2$, $P_{4,2}$ is a path with 6 vertices.

The distance matrix and the characteristic polynomial of $P_{4,3}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 1 & 2 & 3 & 4 & 5 & 5 \\ 2 & 0 & 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 1 & 0 & 1 & 2 & 3 & 4 & 4 \\ 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 \\ 3 & 3 & 2 & 1 & 0 & 1 & 2 & 2 \\ 4 & 4 & 3 & 2 & 1 & 0 & 1 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 & 0 & 2 \\ 5 & 5 & 4 & 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

and $\mu^8 - 248\mu^6 - 1904\mu^5 - 5932\mu^4 - 9248\mu^3 - 7456\mu^2 - 2944\mu - 448 = (\mu+2)^2\,(\mu^3 - 16\mu^2 - 58\mu - 28)(\mu^3 + 12\mu^2 + 14\mu + 4)$.

The distance matrix and the characteristic polynomial of $P_{4,4}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\ 2 & 0 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\ 2 & 2 & 0 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\ 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 4 & 4 \\ 2 & 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 & 3 \\ 3 & 3 & 3 & 2 & 1 & 0 & 1 & 2 & 2 & 2 \\ 4 & 4 & 4 & 3 & 2 & 1 & 0 & 1 & 1 & 1 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 0 & 2 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

and $\mu^{10} - 449\mu^8 - 5032\mu^7 - 24768\mu^6 - 67808\mu^5 - 110944\mu^4 - 109440\mu^3 - 62720\mu^2 - 18944\mu - 2304 = (\mu+2)^4\,(\mu^3 - 23\mu^2 - 80\mu - 36)(\mu^3 + 15\mu^2 + 16\mu + 4)$.

The distance matrix and the characteristic polynomial of $P_{4,5}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 2 & 0 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 2 & 2 & 0 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 2 & 2 & 2 & 0 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 0 & 2 & 2 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 2 & 0 & 2 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\mu^{12}-708\mu^{10}-10464\mu^{9}-70860\mu^{8}-281664\mu^{7}-718016\mu^{6}-1214208\mu^{5}$$
$$-1365888\mu^{4}-998400\mu^{3}-448512\mu^{2}-110592\mu-11264$$
$$=(\mu+2)^{6}(\mu^{3}-30\mu^{2}-102\mu-44)(\mu^{3}+18\mu^{2}+18\mu+4).$$

Therefore the characteristic polynomial of $P_{4,t}$ using distance matrix of G is

$$(\mu+2)^{2t-4}(\mu^{3}-(7t-5)\mu^{2}-(22t-8)\mu-(8t+4))$$
$$\times(\mu^{3}+(3t+2)\mu^{2}+(2t+8)\mu+4).$$

Hence, we get the proof. □

Theorem 3.12 *Let $G=P_{4,t}$, $n=2t+2$. Then, the characteristic polynomial using distance domination matrix of G is given by*

$$(\sigma+2)^{2t-4}(\sigma^{3}-(7t-4)\sigma^{2}-(5t)\sigma+(10t-20))$$
$$\times(\sigma^{3}+(3t+2)\sigma^{2}+(8-t)\sigma+4).$$

Proof Calculation enables one to find the characteristic polynomial of $P_{4,t}$ for $n=2t+2$ directly. For $t=1$, $P_{4,1}$ is a path with 4 vertices, $t=2$, $P_{4,2}$ is a path with 6 vertices.

The distance domination matrix and the characteristic polynomial of $P_{4,3}$ are given by

$$D_{\gamma}(G)=\begin{bmatrix}0&2&1&2&3&4&5&5\\2&0&1&2&3&4&5&5\\1&1&1&1&2&3&4&4\\2&2&1&0&1&2&3&3\\3&3&2&1&0&1&2&2\\4&4&3&2&1&1&1&1\\5&5&4&3&2&1&0&2\\5&5&4&3&2&1&2&0\end{bmatrix}$$

and

$$\sigma^{8}-2\sigma^{7}-247\sigma^{6}-1504\sigma^{5}-3277\sigma^{4}-2472\sigma^{3}+216\sigma^{2}+480\sigma-80$$
$$=(\sigma+2)^{2}(\sigma^{3}-17\sigma^{2}-45\sigma+10)(\sigma^{3}+11\sigma^{2}+5\sigma-2).$$

The distance domination matrix and the characteristic polynomial of $P_{4,4}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\ 2 & 0 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\ 2 & 2 & 0 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\ 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 4 & 4 \\ 2 & 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 & 3 \\ 3 & 3 & 3 & 2 & 1 & 0 & 1 & 2 & 2 & 2 \\ 4 & 4 & 4 & 3 & 2 & 1 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 0 & 2 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\sigma^{10} - 2\sigma^9 - 448\sigma^8 - 4264\sigma^7 - 16936\sigma^6 - 33376\sigma^5 - 29968\sigma^4 - 3328\sigma^3 + 10496\sigma^2$$
$$+2560\sigma - 1280 = (\sigma+2)^4\,(\sigma^3 - 24\sigma^2 - 60\sigma + 20)(\sigma^3 + 14\sigma^2 + 4\sigma - 4).$$

The distance domination matrix and the characteristic polynomial of $P_{4,5}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 2 & 0 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 2 & 2 & 0 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 2 & 2 & 2 & 0 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 0 & 2 & 2 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 2 & 0 & 2 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\sigma^{12} - 2\sigma^{11} - 707\sigma^{10} - 9212\sigma^9 - 53597\sigma^8 - 173456\sigma^7 - 326864\sigma^6 - 332864\sigma^5 - 107744\sigma^4$$
$$+105216\sigma^3 + 90624\sigma^2 - 11520 = (\sigma+2)^6\,(\sigma^3 - 31\sigma^2 - 75\sigma + 30)(\sigma^3 + 17\sigma^2 + 3\sigma - 6).$$

Therefore the characteristic polynomial of $P_{4,t}$ using distance domination matrix of G is

$$(\sigma+2)^{2t-4}\,(\sigma^3 - (7t-4)\sigma^2 - (5t)\sigma + (10t-20))(\sigma^3 + (3t+2)\sigma^2 + (8-t)\sigma + 4).$$

Hence, we get the proof. □

§4. Generalized Characteristic Polynomial Can Not Be Obtained

It is not easy to find the generalized characteristic polynomial with respect to domination energies for all class of graphs, as the problem of finding the characteristic polynomial for an arbitrary matrix is still open. Here we illustrate that for paths, cycles and wheel graphs finding the generalized characteristic polynomial is not possible. Hence for this kind of graphs the absolute energies cannot be found. Therefore only the upper and lower bound can be obtained.

Theorem 4.1 *Let $G = P_n$, $n \geqslant 3$. Then the exact $E(P_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of P_n for $n \geqslant 3$ directly. Label the vertices of P_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of adjacency matrix $A(G)$ is given by

$$\lambda^n + q_1\lambda^{n-1} + q_2\lambda^{n-2} + \cdots + q_{n-1}\lambda + q_n = 0.$$

The adjacency matrix and the characteristic polynomial of P_3 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^3 - 2\lambda = \lambda(\lambda^2 - 1)$.

The adjacency matrix and the characteristic polynomial of P_4 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^4 - 3\lambda^2 + 1 = (\lambda^2 - \lambda - 1)(\lambda^2 + \lambda - 1)$.

The adjacency matrix and the characteristic polynomial of P_5 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^5 - 4\lambda^3 + 3\lambda = \lambda\,(\lambda - 1)\,(\lambda + 1)(\lambda^2 - 3)$.

The adjacency matrix and the characteristic polynomial of P_6 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\lambda^6 - 5\lambda^4 + 6\lambda^2 - 1 = (\lambda^3 - \lambda^2 - 2\lambda + 1)(\lambda^3 + \lambda^2 - 2\lambda - 1).$$

Hence, we get the proof. □

Theorem 4.2 *Let $G = P_n$, $n \geqslant 3$. Then the exact $E_\gamma(P_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of P_n for $n \geqslant 3$ directly. Label the vertices of P_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of domination matrix $A_\gamma(G)$ is given by $\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \cdots + q_{n-1}\kappa + q_n = 0$.

The domination matrix and the characteristic polynomial of P_3 are given by

$$A_\gamma(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^3 - \kappa^2 - 2\kappa = \kappa(\kappa + 1)(\kappa - 2)$.

The domination matrix and the characteristic polynomial of P_4 are given by

$$A_\gamma(G) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad A_\gamma(G) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

whose polynomial are respectively

$$\kappa^4 - 2\kappa^3 - 2\kappa^2 + 3\kappa + 1,$$
$$\kappa^4 - 2\kappa^3 - 2\kappa^2 + 2\kappa + 1 = (\kappa - 1)(\kappa + 1)(\kappa^2 - 2\kappa - 1),$$
$$\kappa^4 - 2\kappa^3 - 2\kappa^2 + 4\kappa = \kappa(\kappa - 2)(\kappa^2 - 2).$$

The domination matrix and the characteristic polynomial of P_5 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad A_\gamma(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

whose polynomial are respectively

$$\begin{aligned} \kappa^5 - 2\kappa^4 - 3\kappa^3 + 5\kappa^2 + 2\kappa - 1 &= (\kappa^2 - \kappa - 1)(\kappa^3 - \kappa^2 - 3\kappa + 1) \\ \kappa^5 - 2\kappa^4 - 3\kappa^3 + 4\kappa^2 + 3\kappa &= \kappa(\kappa^2 - \kappa - 3)(\kappa^2 - \kappa - 1). \end{aligned}$$

The domination matrix and the characteristic polynomial of P_6 are given by

$$A_\gamma(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\kappa^6 - 2\kappa^5 - 4\kappa^4 + 6\kappa^3 + 5\kappa^2 - 2\kappa - 1 = (\kappa^3 - 3\kappa - 1)(\kappa^3 - 2\kappa^2 - \kappa + 1).$$

Hence, we get the proof. □

Theorem 4.3 *Let $G = P_n$, $n \geqslant 3$. Then the exact $E_D(P_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of P_n for $n \geqslant 3$ directly. Label the vertices of P_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of P_n using distance matrix $D(G)$ is given by $\mu^n + q_1\mu^{n-1} + q_2\mu^{n-2} + \cdots + q_{n-1}\mu + q_n = 0$.

The distance matrix and the characteristic polynomial of P_3 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

and $\mu^3 - 6\mu - 4 = (\mu + 2)\left(\mu^2 - 2\mu - 2\right)$.

The distance matrix and the characteristic polynomial of P_4 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^4 - 20\mu^2 - 32\mu - 12 = \left(\mu^2 - 4\mu - 6\right)\left(\mu^2 + 4\mu + 2\right)$.

The distance matrix and the characteristic polynomial of P_5 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^5 - 50\mu^3 - 140\mu^2 - 120\mu - 32 = \left(\mu^2 + 6\mu + 4\right)\left(\mu^3 - 6\mu^2 - 18\mu - 8\right)$.

The distance matrix and the characteristic polynomial of P_6 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 4 & 3 & 2 & 1 & 0 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^6 - 105\mu^4 - 448\mu^3 - 648\mu^2 - 384\mu - 80 = (\mu + 1)\left(\mu^2 + 8\mu + 4\right)\left(\mu^3 - 9\mu^2 - 36\mu - 20\right)$.
Hence, we get the proof. □

Theorem 4.4 *Let $G = P_n$, $n \geqslant 3$. Then the exact $E_{D\gamma}(P_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of P_n for $n \geqslant 3$ directly. Label the vertices of P_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of P_n using distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0.$$

The distance domination matrix and the characteristic polynomial of P_3 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^3 - \sigma^2 - 6\sigma = \sigma(\sigma+2)(\sigma-3)$.

The distance domination matrix and the characteristic polynomial of P_4 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}, \quad D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad \text{or} \quad D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \sigma^4 - 2\sigma^3 - 19\sigma^2 - 12\sigma &= \left(\sigma^2 - 5\sigma - 3\right)\left(\sigma^2 + 3\sigma - 1\right), \\ \sigma^4 - 2\sigma^3 - 19\sigma^2 - 4\sigma + 3 &= \sigma(\sigma+3)\left(\sigma^2 - 5\sigma - 4\right), \\ \sigma^4 - 2\sigma^3 - 19\sigma^2 - 20\sigma - 5 &= \left(\sigma^2 - 5\sigma - 5\right)\left(\sigma^2 + 3\sigma + 1\right). \end{aligned}$$

The distance domination matrix and the characteristic polynomial of P_5 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix} \quad \text{or} \quad D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} &\sigma^5 - 2\sigma^4 - 49\sigma^3 - 70\sigma^2 = \sigma^2(\sigma+5)\left(\sigma^2 - 7\sigma - 14\right), \\ &\sigma^5 - 2\sigma^4 - 49\sigma^3 - 85\sigma^2 - 30\sigma. \end{aligned}$$

The distance domination matrix and the characteristic polynomial of P_6 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and

$$\sigma^6 - 2\sigma^5 - 104\sigma^4 - 300\sigma^3 - 180\sigma^2 = \sigma^2\left(\sigma^2 - 10\sigma - 30\right)\left(\sigma^2 + 8\sigma + 6\right).$$

Hence, we get the proof. □

Theorem 4.5 *Let $G = C_n$, $n \geqslant 3$. Then the exact $E(C_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of C_n for $n \geqslant 3$ directly. Label the vertices of C_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of adjacency matrix $A(G)$ is given by

$$\lambda^n + q_1\lambda^{n-1} + q_2\lambda^{n-2} + \cdots + q_{n-1}\lambda + q_n = 0.$$

The adjacency matrix and the characteristic polynomial of C_3 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\lambda^3 - 3\lambda - 2 = (\lambda - 2)(\lambda + 1)^2$.

The adjacency matrix and the characteristic polynomial of C_4 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^4 - 4\lambda^2 = \lambda^2(\lambda - 2)(\lambda + 2)$.

The adjacency matrix and the characteristic polynomial of C_5 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^5 - 5\lambda^3 + 5\lambda - 2 = (\lambda - 2)\left(\lambda^2 + \lambda - 1\right)^2$.

The adjacency matrix and the characteristic polynomial of C_6 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^6 - 6\lambda^4 + 9\lambda^2 - 4 = (\lambda - 2)\,(\lambda - 1)^2(\lambda + 1)^2(\lambda + 2)$.

The adjacency matrix and the characteristic polynomial of C_7 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^7 - 7\lambda^5 + 14\lambda^3 - 7\lambda - 2 = (\lambda - 2)(\lambda^3 + \lambda^2 - 2\lambda - 1)^2$. Hence, we get the proof. $\square$

Theorem 4.6 *Let $G = C_n$, $n \geqslant 3$. Then the exact $E_\gamma(C_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of C_n for $n \geqslant 3$ directly. Label the vertices of C_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of domination matrix $A_\gamma(G)$ is given by

$$\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \cdots + q_{n-1}\kappa + q_n = 0.$$

The domination matrix and the characteristic polynomial of C_3 are given by

$$A_\gamma(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^3 - \kappa^2 - 3\kappa - 1 = (\kappa + 1)(\kappa^2 - 2\kappa - 1)$.

The domination matrix and the characteristic polynomial of C_4 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad A_\gamma(G) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\kappa^4 - 2\kappa^3 - 3\kappa^2 + 4\kappa = \kappa(\kappa - 1)(\kappa^2 - \kappa - 4) \quad \text{or} \quad \kappa^4 - 2\kappa^3 - 3\kappa^2 + 4\kappa - 1.$$

The domination matrix and the characteristic polynomial of C_5 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^5 - 2\kappa^4 - 4\kappa^3 + 6\kappa^2 + 4\kappa - 4 = (\kappa^2 - 2)(\kappa^3 - 2\kappa^2 - 2\kappa + 2)$.

The domination matrix and the characteristic polynomial of C_6 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^6 - 2\kappa^5 - 5\kappa^4 + 8\kappa^3 + 7\kappa^2 - 6\kappa - 3 = (\kappa - 1)(\kappa + 1)\left(\kappa^2 - 3\right)(\kappa^2 - 2\kappa - 1)$.

The domination matrix and the characteristic polynomial of C_7 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^7 - 3\kappa^6 - 4\kappa^5 + 14\kappa^4 + 5\kappa^3 - 17\kappa^2 - 3\kappa + 1 = (\kappa^3 - 3\kappa - 1)(\kappa^4 - 3\kappa^3 - \kappa^2 + 6\kappa - 1)$. Hence, we get the proof. □

Theorem 4.7 *Let $G = C_n$, $n \geqslant 3$. Then the exact $E_D(C_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of C_n for $n \geqslant 3$ directly. Label the vertices of C_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of P_n using distance matrix $D(G)$ is given by

$$\mu^n + q_1\mu^{n-1} + q_2\mu^{n-2} + \cdots + q_{n-1}\mu + q_n = 0.$$

The distance matrix and the characteristic polynomial of C_3 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\mu^3 - 3\mu - 2 = (\mu - 2)(\mu + 1)^2$.

The distance matrix and the characteristic polynomial of C_4 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^4 - 12\mu^2 - 16\mu = \mu\left(\mu - 4\right)\left(\mu + 2\right)^2$.

The distance matrix and the characteristic polynomial of C_5 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^5 - 25\mu^3 - 60\mu^2 - 35\mu - 6 = \left(\mu - 6\right)\left(\mu^2 + 3\mu + 1\right)^2$.

The distance matrix and the characteristic polynomial of C_6 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 3 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^6 - 56\mu^4 - 203\mu^3 - 190\mu^2 - 72\mu = \mu\left(\mu + 4\right)\left(\mu - 9\right)\left(\mu^3 + 5\mu^2 + 5\mu + 2\right).$

The distance matrix and the characteristic polynomial of C_7 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^7 - 98\mu^5 - 490\mu^4 - 707\mu^3 - 434\mu^2 - 119\mu - 12 = \left(\mu - 12\right)\left(\mu^3 + 6\mu^2 + 5\mu + 1\right)^2$. Hence, we get the proof. □

Theorem 4.8 *Let $G = C_n$, $n \geqslant 3$. Then the exact $E_{D\gamma}(C_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of C_n for $n \geqslant 3$ directly. Label the vertices of C_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of P_n using distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0.$$

The distance domination matrix and the characteristic polynomial of C_3 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\sigma^3 - \sigma^2 - 3\sigma - 1 = (\sigma + 1)(\sigma^2 - 2\sigma - 1)$.

The distance domination matrix and the characteristic polynomial of C_4 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^4 - 2\sigma^3 - 11\sigma^2 - 4\sigma + 4 = (\sigma+1)(\sigma+2)\left(\sigma^2 - 5\sigma + 2\right)$.

The distance domination matrix and the characteristic polynomial of C_5 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^5 - 2\sigma^4 - 24\sigma^3 - 30\sigma^2 + 4\sigma = \sigma(\sigma+2)\left(\sigma^3 - 4\sigma^2 - 16\sigma + 2\right)$.

The distance domination matrix and the characteristic polynomial of C_6 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 3 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} &\sigma^6 - 2\sigma^5 - 55\sigma^4 - 129\sigma^3 - 12\sigma^2 + 38\sigma + 24 \\ &= (\sigma+4)\left(\sigma^2 - 10\sigma + 6\right)\left(\sigma^3 + 4\sigma^2 + 3\sigma + 1\right). \end{aligned}$$

The distance matrix and the characteristic polynomial of C_7 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 3 & 2 \\ 2 & 1 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and

$$\begin{aligned}&\sigma^7-3\sigma^6-95\sigma^5-281\sigma^4-10\sigma^3+60\sigma^2+8\sigma\\&=\sigma\left(\mu^2+5\sigma+2\right)\left(\mu^4-8\mu^3-57\mu^2+20\mu+4\right).\end{aligned}$$

Hence, we get the proof. □

Theorem 4.9 *Let $G=W_n$, $n\geqslant 3$. Then the exact $E(W_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of W_n for $n\geqslant 3$ directly. Label the vertices of W_n as $v_1,v_2,v_3,\cdots,v_n$.

The characteristic polynomial of adjacency matrix $A(G)$ is given by

$$\lambda^n+q_1\lambda^{n-1}+q_2\lambda^{n-2}+\cdots+q_{n-1}\lambda+q_n=0.$$

The adjacency matrix and the characteristic polynomial of W_4 are given by

$$A(G)=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}$$

and $\lambda^4-6\lambda^2-8\lambda-3=(\lambda-3)(\lambda+1)^3$.

The adjacency matrix and the characteristic polynomial of W_5 are given by

$$A(G)=\begin{bmatrix}0&1&1&1&1\\1&0&1&0&1\\1&1&0&1&0\\1&0&1&0&1\\1&1&0&1&0\end{bmatrix}$$

and $\lambda^5-8\lambda^3-8\lambda^2=\lambda^2\,(\lambda+2)\,(\lambda^2-2\lambda-4)$.

The adjacency matrix and the characteristic polynomial of W_6 are given by

$$A(G)=\begin{bmatrix}0&1&1&1&1&1\\1&0&1&0&0&1\\1&1&0&1&0&0\\1&0&1&0&1&0\\1&0&0&1&0&1\\1&1&0&0&1&0\end{bmatrix}$$

and $\lambda^6-10\lambda^4-10\lambda^3+10\lambda^2+8\lambda-5=\left(\lambda^2-2\lambda-5\right)\left(\lambda^2+\lambda-1\right)^2$.

The adjacency matrix and the characteristic polynomial of W_7 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\lambda^7 - 12\lambda^5 - 12\lambda^4 + 21\lambda^3 + 24\lambda^2 - 10\lambda - 12 = (\lambda-1)^2(\lambda+1)^2(\lambda+2)(\lambda^2-2\lambda-6).$$

Hence, we get the proof. □

Theorem 4.10 *Let $G = W_n$, $n \geqslant 3$. Then the exact $E_\gamma(W_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of W_n for $n \geqslant 3$ directly. Label the vertices of W_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of domination matrix $A_\gamma(G)$ is given by

$$\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \cdots + q_{n-1}\kappa + q_n = 0.$$

The domination matrix and the characteristic polynomial of W_4 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^4 - \kappa^3 - 6\kappa^2 - 5\kappa - 1 = (\kappa+1)^2(\kappa^2 - 3\kappa - 1)$.

The domination matrix and the characteristic polynomial of W_5 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^5 - \kappa^4 - 8\kappa^3 - 4\kappa^2 = \kappa^2(\kappa+2)(\kappa^2 - 3\kappa - 2)$.

The domination matrix and the characteristic polynomial of W_6 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^6 - \kappa^5 - 10\kappa^4 - 5\kappa^3 + 10\kappa^2 + 3\kappa - 3 = (\kappa^2 - 3\kappa - 3)(\kappa^2 + \kappa - 1)^2$.

The domination matrix and the characteristic polynomial of W_7 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^7 - \kappa^6 - 12\kappa^5 - 6\kappa^4 + 21\kappa^3 + 15\kappa^2 - 10\kappa - 8 = (\kappa - 1)^2(\kappa + 1)^3(\kappa + 2)(\kappa + 4)$. Hence, we get the proof. □

Theorem 4.11 *Let $G = W_n$, $n \geqslant 3$. Then the exact $E_D(W_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of W_n for $n \geqslant 3$ directly. Label the vertices of W_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of W_n using distance matrix $D(G)$ is given by

$$\mu^n + q_1\mu^{n-1} + q_2\mu^{n-2} + \cdots + q_{n-1}\mu + q_n = 0.$$

The distance matrix and the characteristic polynomial of W_4 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\mu^4 - 6\mu^2 - \mu - 3 = (\mu - 3)(\mu + 1)^3$.

The distance matrix and the characteristic polynomial of W_5 are given by

$$D(G)=\begin{bmatrix}0&1&1&1&1\\1&0&1&2&1\\1&1&0&1&2\\1&2&1&0&1\\1&1&2&1&0\end{bmatrix}$$

and $\mu^5-16\mu^3-32\mu^2-16\mu=\mu\left(\mu+2\right)^2\left(\mu^2-4\mu-4\right).$

The distance matrix and the characteristic polynomial of W_6 are given by

$$D(G)=\begin{bmatrix}0&1&1&1&1&1\\1&0&1&2&2&1\\1&1&0&1&2&2\\1&2&1&0&1&2\\1&2&2&1&0&1\\1&1&2&2&1&0\end{bmatrix}$$

and $\mu^6-30\mu^4-90\mu^3-90\mu^2-36\mu-5=\left(\mu^2-6\mu-5\right)\left(\mu^2+3\mu+1\right)^2.$

The distance matrix and the characteristic polynomial of W_7 are given by

$$D(G)=\begin{bmatrix}0&1&1&1&1&1&1\\1&0&1&2&2&2&1\\1&1&0&1&2&2&2\\1&2&1&0&1&2&2\\1&2&2&1&0&1&2\\1&2&2&2&1&0&1\\1&1&2&2&2&1&0\end{bmatrix}$$

and $\mu^7-48\mu^5-200\mu^4-315\mu^3-216\mu^2-54\mu=\mu\left(\mu+1\right)^2\left(\mu+3\right)^2\left(\mu^2-8\mu-6\right)$. Hence, we get the proof. □

Theorem 4.12 *Let $G=W_n$, $n\geqslant 3$. Then the exact $E_{D\gamma}(W_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of W_n for $n\geqslant 3$ directly. Label the vertices of W_n as $v_1,v_2,v_3,\cdots,v_n$.

The characteristic polynomial of W_n using distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n+q_1\sigma^{n-1}+q_2\sigma^{n-2}+\cdots+q_{n-1}\sigma+q_n=0.$$

The distance domination matrix and the characteristic polynomial of W_4 are given by

$$D_{\gamma}(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\sigma^4 - \sigma^3 - 6\sigma^2 - 5\sigma - 1 = (\sigma+1)^2\left(\sigma^2 - 3\sigma - 1\right).$

The distance domination matrix and the characteristic polynomial of W_5 are given by

$$D_{\gamma}(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^5 - \sigma^4 - 16\sigma^3 - 20\sigma^2 = \sigma^2(\sigma-5)(\sigma+2)^2.$

The distance domination matrix and the characteristic polynomial of W_6 are given by

$$D_{\gamma}(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^6 - \sigma^5 - 30\sigma^4 - 65\sigma^3 - 30\sigma^2 - \sigma + 1 = \left(\sigma^2 - 7\sigma + 1\right)\left(\sigma^2 + 3\sigma + 1\right)^2.$

The distance matrix and the characteristic polynomial of W_7 are given by

$$D_{\gamma}(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^7 - \sigma^6 - 48\sigma^5 - 158\sigma^4 - 163\sigma^3 - 33\sigma^2 + 18\sigma = \sigma(\sigma+1)^2(\sigma+3)^2\left(\mu^2 - 9\mu + 2\right).$ Hence, we get the proof. □

§5. Open Problems

Problem 5.1 *Finding the characteristic polynomial for an arbitrary graph.*

Problem 5.2 *Find upper and lower bound for various kinds of energies with respect to different parameters of graph.*

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Product Cordial Labeling of Extensions of Barbell Graph

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Abstract: A barbell graph $B(r, n)$ is a graph consists of path P_n joining two complete graphs K_r. This paper deals with study of the product cordial labeling of graphs that are obtained by applying various graph operations on barbell graph.

Key Words: Barbell graph, product cordial labeling, Smarandachely product cordial labeling, duplication, switching, degree splitting.

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§1. Introduction

All the graphs considered in this paper are finite, simple, connected and undirected. Through out this work, $|X|$ denotes the cardinality of the set X. By order and size of a graph we means the cardinality of vertex set and the cardinality of edge set respectively. For various graph theoretic notations and terminology we follow [1].

A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices(or edges) then the labeling is called vertex labeling(or edge labeling). A mapping $f : V(G) \to \{0, 1\}$ is called *binary vertex labeling* of a graph $G = (V(G), E(G))$. Also the number of vertices(or edges) having label i under the map f are denoted by $v_f(i)$(or $e_f(i)$) and the set of all vertices adjacent to v are denoted by $N(v)$.

A *product cordial labeling* of a graph $G = (V(G), E(G))$ is a function f from $V(G)$ to $\{0, 1\}$ such that if each edge uv is assigned the label $f(u)f(v)$, the number $v_f(0)$ of vertices labeled with 0 and the number $v_f(1)$ of vertices labeled with 1 differ by at most 1, and the number $e_f(0)$ of edges labeled with 0 and the number $e_f(1)$ of edges labeled with 1 differ by at most 1. A graph with a product cordial labeling is called a *product cordial graph*. Opposed to the product cordial labeling, a *Smarandachely product cordial labeling* on G is such a labeling $f: V(G) \to \{0, 1\}$ with induced labeling $f(u)f(v)$ on edge $uv \in E(G)$ that $|v_f(0) - v_f(1)| \geq 2$ or $|e_f(0) - e_f(1)| \geq 2$.

The product cordial labeling was introduced by Sundaram et. al. [3], [4]. They proved that many graphs are product cordial: trees; unicyclic graphs of odd order; triangular snakes;

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dragons; helms; path and cycle related graphs. They also proved that a graph having p vertices and q edges is product cordial, then $q \leq \frac{(p-1)(p+1)}{4} + 1$. For further results on product cordial labeling we refer to the dynamic survey of graph labeling by Gallian [2].

A *barbell graph* consists of a path graph of order n connecting two complete graphs of order $r \geq 3$ each and it is denoted by $B(r, n)$. S K Vaidya and Chirag Barasara [5] proved that if G and G' are the graphs such that their orders or sizes differ at most by 1, then the new graph obtained by joining G and G' by a path P_k of $k \in \mathbb{N}$ length is product cordial. This result along with the definition of barbell graph shows that barbell graph is product cordial. In this paper we study the product cordial labeling of graphs that are obtained by performing certain operations on barbell graph. We first define these operations.

Definition 1.1 *The duplication of a vertex v of graph G produces a new graph G' by adding a new vertex v' such that $N(v') = N(v)$. In other words a vertex v' is said to be duplication of v if all the vertices which are adjacent to v in G are also adjacent to v' in G'.*

Definition 1.2 *The duplication of vertex v_k by a new edge $e = v'_k v''_k$ in a graph G produce a new graph G' such that $N(v'_k) = \{v_k, v''_k\}$ and $N(v''_k) = \{v_k, v'_k\}$.*

Definition 1.3 *The duplication of an edge $e = uv$ by a new vertex w in a graph G produce a new graph G' such that $N(w) = \{u, v\}$.*

Definition 1.4 *The duplication of an edge $e = uv$ of a graph G produce a new graph G' by adding an edge $e' = u'v'$ such that $N(u') = \{N(u) \cup \{v'\}\} \setminus \{v\}$ and $N(v') = \{N(v) \cup \{u'\}\} \setminus \{u\}$.*

Definition 1.5 *A vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G, removing all the edges incident to v and adding edges joining v to every other vertex which are not adjacent to v in G.*

Definition 1.6 *Let $G = (V(G), E(G))$ be a graph with $V(G) = S_1 \cup S_2 \cup \cdots \cup S_t \cup T$ where each S_i is a set of vertices having at least two vertices and having the same degree and $T = V(G) \setminus \cup S_i$. Then the degree splitting graph of G is a graph obtained from G by adding vertices $w_1, w_2, \cdots w_t$ and joining w_i to each vertex of $S_i (1 \leq i \leq t)$.*

In the present work we proved that graphs obtained from barbell graph $B(r, n)$ by duplicating all vertices by edges and duplicating all edges by vertices in path joining complete graphs are product cordial for all r and n. We also show that a graph obtained by switching a vertex of path in barbell graph B(r; n) admits product cordial labeling for all r and n. We also derive partial results for the product cordial labeling of graphs that are obtained from barbell graph $B(r, n)$ by duplicating vertex by vertex and edge by edge in the path joining complete graphs. Further we show that for certain values of r and n the degree splitting graph of barbell graph as well as degree splitting graph of path in barbell graph are product cordial.

§2. Main Results

Theorem 2.1 *A barbell graph $B(r, n)$ with duplication of edges of path joining complete graphs by vertices, is product cordial for all possible values of r and n.*

Proof In a barbell graph $G = B(r, n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from Barbell graph by taking duplication of edges of path by vertices and also $v'_1, v'_2, \cdots, v'_{n-1}$ be vertices of duplication of path edges $v_1v_2,\ v_2v_3, \cdots, v_{n-1}v_n$ respectively. Then $|V(G')| = 2r+2n-1$ and $|E(G')| = r(r-1)+3n-1$. We define $f : V(G') \to \{0, 1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r$$

$$f(u'_i) = 0; 1 \leqslant i \leqslant r$$

$$f(v_j) = \begin{cases} 1, & 1 \le j \le \left\lceil \frac{n}{2} \right\rceil; \\ 0, & \left\lceil \frac{n}{2} \right\rceil + 1 \le j \le n. \end{cases}$$

$$f(v'_j) = \begin{cases} 1, & 1 \le j \le \left\lceil \frac{n}{2} \right\rceil; \\ 0, & \left\lceil \frac{n}{2} \right\rceil + 1 \le j \le n-1. \end{cases}$$

According to above definition of f, we have $v_f(0)+1 = r+n = v_f(1)$. Thus $|v_f(0)-v_f(1)| \le 1$. For the edges labeled with 0 and 1 consider the following cases.

Case 1. n is odd.

In this case we have $e_f(0) = \frac{r(r-1)}{2} + \frac{3n-1}{2} = e_f(1)$. So, $|e_f(0) - e_f(1)| \le 1$.

Case 2. n is even

In this case we have $e_f(0) = \frac{r(r-1)}{2} + \frac{3n-2}{2} = e_f(1) + 1$. Hence, $|e_f(0) - e_f(1)| \le 1$.

Thus G' has product cordial labeling. □

Example 2.1 A barbell graph $B(5, 4)$ with duplication of edges of path joining complete graphs by vertices and its product cordial labeling is shown in Figure 1.

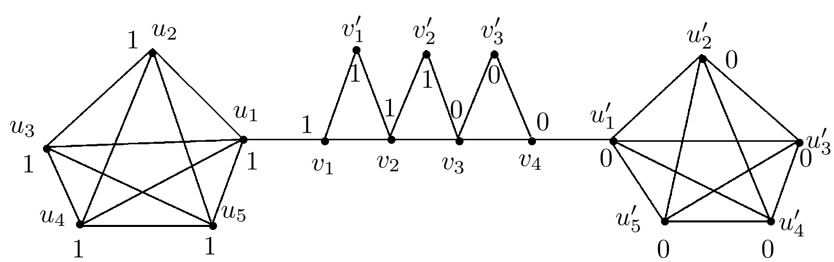

Figure 1 Barbell graph $B(5, 4)$ with duplication of edges of path by vertices

Theorem 2.2 *A barbell graph $B(r, n)$ with duplication of vertices of path joining complete graphs by edges, is product cordial for all r and n.*

Proof In a barbell graph $G = B(r, n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from barbell graph by taking duplication of vertices of path by edges and also $v_1'v_2'$, $v_2'v_3'$, $\cdots$, $v_{2n-1}'v_{2n}'$ be edges of duplication of path vertices $v_1, v_2, \cdots, v_n$. Then $|V(G')| = 2r + 3n$ and $|E(G')| = r(r-1) + 4n + 1$. We define $f : V(G') \to \{0, 1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r$$
$$f(u_i') = 0; 1 \leqslant i \leqslant r$$
$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \lceil \frac{n}{2} \rceil; \\ 0, & \lceil \frac{n}{2} \rceil + 1 \leq j \leq n. \end{cases}$$
$$f(v_j') = \begin{cases} 1, & 1 \leq j \leq n; \\ 0, & n+1 \leq j \leq 2n. \end{cases}$$

According to above definition of f, we have $e_f(0) = \frac{r(r-1)}{2} + 2n + 1 = e_f(1) + 1$. Thus $|e_f(0) - e_f(1)| \leq 1$. For the vertices labeled with 0 and 1 consider the following cases.

Case 1. n is odd

In this case we have $v_f(0) = r + \frac{3n-1}{2} = v_f(1) + 1$. So $|v_f(0) - v_f(1)| \leq 1$.

Case 2. n is even

In this case we have $v_f(0) = r + \frac{3n}{2} = v_f(1)$. Thus $|v_f(0) - v_f(1)| \leq 1$.

And hence G' is product cordial. □

Example 2.2 A barbell graph $B(5, 6)$ with duplication of vertices of path joining complete graphs by edges and its product cordial labeling is shown in Figure 2.

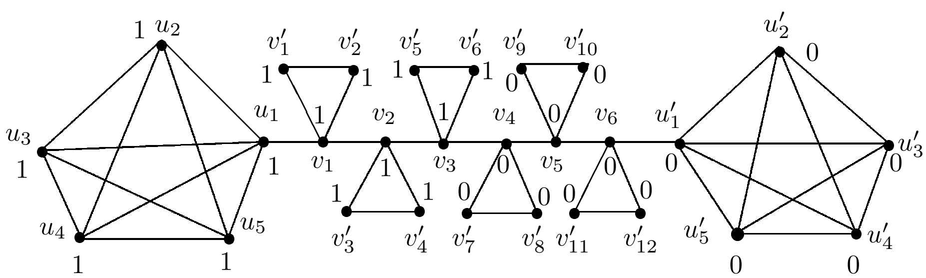

Figure 2 Barbell graph $B(5, 6)$ with duplication of vertices by edges

Theorem 2.3 *A barbell graph $B(r, n)$ with switching of a vertex of path joining complete graphs is product cordial for all possible values of r and n.*

Proof Let G be a barbell graph and let $u_1, u_2, \cdots, u_r$ and $u_1', u_2', \cdots, u_r'$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from G by switching vertex v of path. Here for v we have two choices either v is end vertex of path or internal vertex of path.

Case 1. v is end vertex say v_1.

In this case we have $|V(G')| = 2r + n$ and $|E(G')| = r(r-1) + 2n - 3$. Define $f : V(G') \rightarrow \{0, 1\}$ as

$$f(u_i) = 0; 1 \leqslant i \leqslant r,$$
$$f(u_i') = 1; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & j = 1, n, n-1, \cdots, \lceil \frac{n}{2} \rceil + 2; \\ 0, & j = 2, 3, \cdots, \rceil \frac{n}{2} \rceil + 1. \end{cases}$$

Subcase 1.1 is n odd.

In this case we have $e_f(1) = \frac{r(r-1)}{2} + n - 1 = e_f(0) + 1$ and $v_f(1) = r + \frac{n+1}{2} = v_f(0) + 1$.

Subcase 1.2 n is even.

In this case we have $e_f(1) + 1 = \frac{r(r-1)}{2} + n - 1 = e_f(0)$ and $v_f(1) = r + \frac{n}{2} = v_f(0)$.

Thus from both the sub cases we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 2. v is internal vertex say v_2.

In this case we have $|V(G')| = 2r + n$ and $|E(G')| = r(r-1) + 2n - 4$. Define $f : V(G') \rightarrow \{0, 1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u_i') = 0; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & j = 2, n, n-1, \cdots, \rceil \frac{n}{2} \rceil + 2; \\ 0, & j = 1, 3, 4, \cdots, \rceil \frac{n}{2} \rceil + 1. \end{cases}$$

Then we have $e_f(1) = \frac{r(r-1)}{2} + n - 2 = e_f(0)$ and $v_f(1) = r + \frac{n+1}{2} = v_f(0) + 1$. Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Thus G' is product cordial graph. □

Example 2.3 Consider a barbell graph $B(6, 6)$ with switching of end vertex of path joining complete graphs. Then it is product cordial and its labeling is as shown in Figure 3.

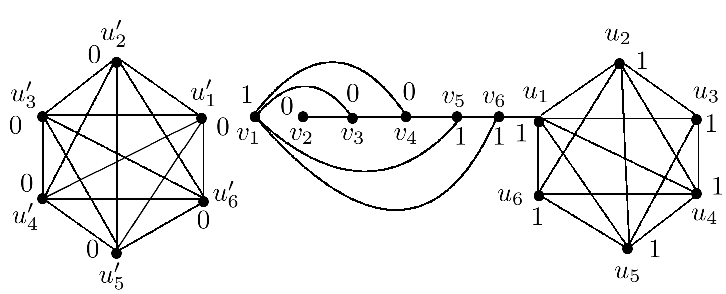

Figure 3 Barbell graph $B(6, 6)$ with switching of end vertex of path

Theorem 2.4 *A barbell graph with duplication of vertices of path joining complete graphs by vertices is product cordial for the following choices of r and n:*

(1) $r \geqslant 3$ *and* $n = 4$;

(2) $r \geq 5$ *and* $n \geq 6$.

Proof In a barbell graph $G = B(r, n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from barbell graph by taking duplication of edges of path by vertices and also $v'_1, v'_2, \cdots, v'_n$ be vertices of duplication of path edges $v_1,\ v_2, \cdots, v_n$ respectively. Then $|V(G')| = 2r + 2n$ and $|E(G')| = r(r-1) + 3n + 1$.

Case 1. $r \geqslant 3$ and $n = 4$.

We consider the following sub cases for r to define the function on $V(G')$.

Subcase 1.1 $r = 3$.

We define $f : V(G') \to \{0, 1\}$ as

$$f(u_i) = 0; 1 \leqslant i \leqslant 3,$$
$$f(u'_i) = 0; 1 \leqslant i \leqslant 3,$$
$$f(v_j) = \begin{cases} 1, & 1 \leqslant i \leqslant 3; \\ 0, & i = 4, \end{cases}$$
$$f(v'_j) = 1; 1 \leqslant j \leqslant 4.$$

Subcase 1.2 $r \geqslant 4$.

We define $f : V(G') \to \{0, 1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = \begin{cases} 1, & 1 \leqslant i \leqslant 4; \\ 0, & 5 \leqslant i \leqslant r, \end{cases}$$
$$f(v_j) = 0; 1 \leqslant j \leqslant 4,$$
$$f(v'_j) = 0; 1 \leqslant j \leqslant 4.$$

According to above definitions of f in different sub cases, we have $v_f(0) = r + 4 = v_f(1)$ and $e_f(0) = \frac{r(r-1)}{2} + 7 = e_f(1) + 1$. So, Thus $|v_f(0) - v_f(1)| \leq 1$. $|e_f(0) - e_f(1)| \leq 1$.

Case 2. $r \geq 5$ and $n \geq 6$.

We consider the following sub cases for r to define the function on $V(G')$.

Subcase 2.1 $n = 6$.

We define $f : V(G') \rightarrow \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = \begin{cases} 1, & 1 \leqslant i \leqslant 5; \\ 0, & 6 \leqslant i \leqslant r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & j = 2; \\ 0, & 3 \leq j \leq n, \end{cases}$$
$$f(v'_j) = 0; 3 \leq j \leq n.$$

Subcase 2.2 $n = 7$.

We define $f : V(G') \rightarrow \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = \begin{cases} 1, & 1 \leqslant i \leqslant 5; \\ 0, & 6 \leqslant i \leqslant r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & j = 2; \\ 0, & 3 \leq j \leq n, \end{cases}$$
$$f(v'_j) = 0; 3 \leq j \leq n.$$

Subcase 2.3 $n \geq 8$.

We define $f : V(G') \rightarrow \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = \begin{cases} 1, & 1 \leqslant i \leqslant 5; \\ 0, & 6 \leqslant i \leqslant r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & 2 \leq j \leq \lceil \frac{n}{2} \rceil - 1; \\ 0, & j = 1, \ \rceil \frac{n}{2} \rceil \leq j \leq n, \end{cases}$$
$$f(v'_j) = \begin{cases} 1, & 3 \leq j \leq \lceil \frac{n}{2} \rceil - 1; \\ 0, & j = 1, 2, \ \lceil \frac{n}{2} \rceil \leq j \leq n. \end{cases}$$

According to above definitions of f in different subcases, we have $v_f(0) = r + n = v_f(1)$. Thus $|v_f(0) - v_f(1)| \leq 1$. For the number of edges labeled with 0 and 1 consider the following cases.

Case 1. n is odd.

In this case we have $e_f(0) = \frac{r(r-1)}{2} + \frac{3n+1}{2} = e_f(1)$. So, $|e_f(0) - e_f(1)| \leq 1$.

Case 2. n is even.

In this case we have $e_f(0) = \frac{r(r-1)}{2} + \frac{3n}{2} + 1 = e_f(1) + 1$. Hence, $|e_f(0) - e_f(1)| \leq 1$. Thus G' has product cordial labeling. □

Example 2.4 A barbell graph $B(5,6)$ with duplication of vertices of path joining complete graphs by vertices is product cordial and its product cordial labeling is shown in Figure 4.

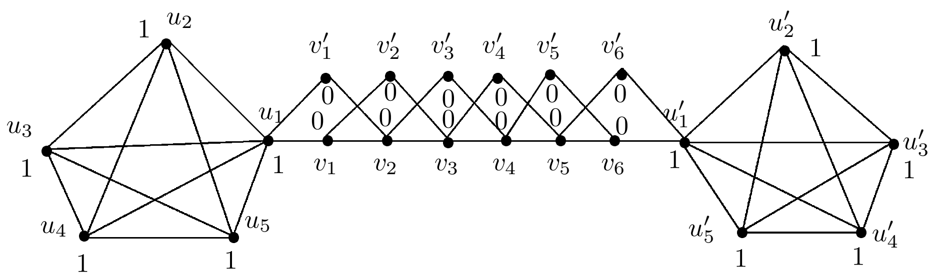

Figure 4 Barbell graph $B(5,6)$ with duplication of vertices by vertices

Theorem 2.5 *A barbell graph $B(r,n)$ with duplication of edges of path joining complete graphs by edges is product cordial for*

(1) $r \geq 4$ *and* $n = 4$*;*

(2) $r \geq 4$ *and n is odd with* $n \geq 5$.

Proof In a barbell graph $G = B(r,n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from barbell graph by taking duplication of edges of path by edges and also $v'_1v'_2$, $v'_2v'_3, \cdots, v'_{2n-3}v'_{2n-2}$ be edges of duplication of path vertices $v_1v_2, v_2v_3, \cdots, v_{n-1}v_n$ respectively. Then $|V(G)| = 2r + 3n - 2$ and $|E(G)| = r(r-1) + 4n - 2$.

Case 1. $r \geq 4$ and $n = 4$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$

$$f(u'_i) = \begin{cases} 1, & 1 \leqslant i \leqslant 4; \\ 0, & 5 \leqslant i \leqslant r, \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1; \\ 0, & 2 \leq j \leq 4, \end{cases}$$

$$f(v'_j) = 0; 1 \leqslant j \leqslant 6.$$

According to above definitions of f in different subcases, we have $v_f(0) = r + 5 = v_f(1)$ and $e_f(0) = \frac{r(r-1)}{2} + 7 = e_f(1)$. So, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 2. $r \geq 4$ and odd $n \geq 5$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$

$$f(u'_i) = \begin{cases} 1, & 1 \leqslant i \leqslant 4; \\ 0, & 5 \leqslant i \leqslant r, \end{cases}$$

$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \frac{n+1}{2}; \\ 0, & j = 1,\ \frac{n+3}{2} \leq j \leq n, \end{cases}$$

$$f(v'_j) = \begin{cases} 1, & 3 \leq j \leq n-5; \\ 0, & n-4 \leq j \leq 2n-2. \end{cases}$$

According to above definitions of f in different subcases, we have $v_f(0) = r + 3\left(\frac{n-1}{2}\right) = v_f(1)$ and $e_f(0) = \frac{r(r-1)}{2} + 2n - 1 = e_f(1)$. So, $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Thus G' has product cordial labeling. □

Example 2.5 A barbell graph $B(5,5)$ with duplication of edges of path joining complete graphs by edges is product cordial and its product cordial labeling is shown in Figure 5.

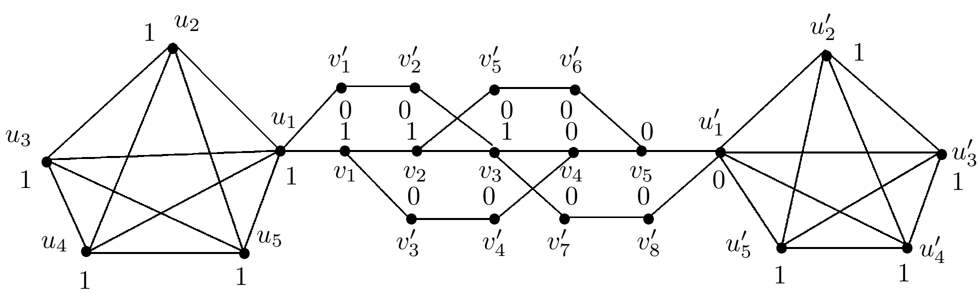

Figure 5 Barbell graph $B(5,5)$ with duplication of edges by edges

Theorem 2.6 *A degree splitting graph of barbell graph $B(r,n)$ is product cordial for $r = 3$ and n is odd.*

Proof For a barbell graph $G = B(r,n)$, let $u_1, u_2, \cdots, u_r$ and $u'_1, u'_2, \cdots, u'_r$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1.

Let G' be degree splitting graph of G and w_1, w_2 be inserting vertices with the properties. $N(w_1) = \{v \in V(G) : d(v) = r\}$, $N(w_2) = \{v \in V(G) : d(v) = 2\}$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u_i') = 0; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & 1 \le j \le \lceil \frac{n}{2} \rceil; \\ 0, & \lceil \frac{n}{2} \rceil + 1 \le j \le n, \end{cases}$$
$$f(w_1) = 0,$$
$$f(w_2) = 1.$$

Then we have $e_f(1) = 6 + n = e_f(0) - 1$ and $v_f(1) - 1 = 4 + \frac{n}{2} - \frac{1}{2} = v_f(0)$.

Hence in this case we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. Thus G' is product cordial graph. □

Example 2.6 Consider the degree splitting graph of $B(3,5)$.Then it is product cordial and its product cordial labeling is shown in Figure 6.

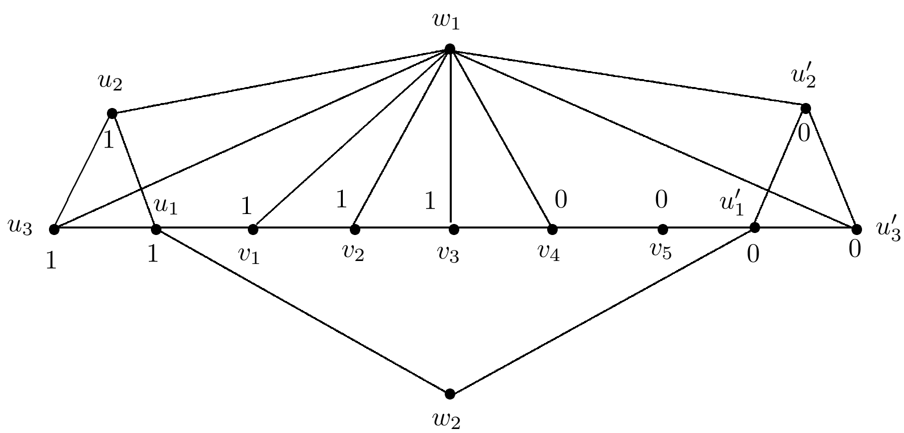

Figure 6 Degree splitting graphs of $B(3,5)$

Theorem 2.7 *A graph obtained by taking degree splitting graph of path joining complete graphs in barbell graph $B(r,n)$ is product cordial for*

(1) $r \ge 3$ *and n is even;*

(2) $r = 3$ *and n is odd with* $n \ne 1$*;*

(3) $r = 4$ *and n is odd with* $n \ne 1, 3, 5$*;*

(4) $r \ge 5$ *and n is odd with* $n \ne 1, 3, 5, 7, 13$.

Proof For a barbell graph $G = B(r,n)$, let $u_1, u_2, \cdots, u_r$ and $u_1', u_2', \cdots, u_r'$ be vertices of complete graphs and $v_1, v_2, \cdots, v_n$ be vertices of path joining complete graphs where v_1 is adjacent to u_1. Let G' be graph obtained from $B(r,n)$ by taking degree splitting graph of path joining complete graphs and v' be the inserting vertex. Then we have $|V(G)| = 2r + n + 1$ and $|E(G)| = r(r-1) + 2n + 1$.

Case 1. $r \geq 3$ and n is even.

We define $f : V(G') \rightarrow \{0, 1\}$ as

$$f(u_i) = 0; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \lceil \frac{n}{2} \rceil; \\ 0, & \lceil \frac{n}{2} \rceil + 1 \leq j \leq n, \end{cases}$$
$$f(v') = 1.$$

Then we have $e_f(1) = \frac{r(r-1)}{2} + n = e_f(0) + 1$ and $v_f(1) - 1 = r + \frac{n}{2} = v_f(0)$. Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus G' is product cordial graph in this case.

Case 2. $r = 3$ and n is odd with $n \neq 1$.

Subcase 2.1 $n = 3$.

We define $f : V(G') \rightarrow \{0, 1\}$ as

$$f(u_i) = \begin{cases} 1, & i = 1; \\ 0, & 2 \leq i \leq r, \end{cases}$$
$$f(u'_i) = 0; 1 \leqslant i \leqslant r;$$
$$f(v_j) = 1; 1 \leq j \leq n,$$
$$f(v') = 1.$$

Subcase 2.2 $n \geq 5$.

We define $f : V(G') \rightarrow \{0, 1\}$ as

$$f(u_i) = f(u'_i) = 0; 1 \leqslant i \leqslant r,$$
$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \frac{n+5}{2}; \\ 0, & \frac{n+7}{2} \leq j \leq n, \end{cases}$$
$$f(v') = 1.$$

According to the above definitions of f in different sub cases, we have $e_f(1) - 1 = n + 3 = e_f(0)$ and $v_f(1) = \frac{n+7}{2} = v_f(0)$. Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 3. $r = 4$ and n is odd with $n \neq 1, 3, 5$.

We define $f : V(G') \rightarrow \{0,1\}$ as

$$f(u_i) = f(u'_i) = 0; 1 \leqslant i \leqslant rm$$

$$f(v_j) = \begin{cases} 1, & 1 \leq j \leq \frac{n+7}{2}; \\ 0, & \frac{n+9}{2} \leq j \leq nm, \end{cases}$$

$$f(v') = 1.$$

Then we have $e_f(1) - 1 = n + 6 = e_f(0)$ and $v_f(1) = \frac{n+9}{2} = v_f(0)$. Hence in this case we have $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$.

Case 4. $r \geq 5$ and n is odd with $n \neq 1, 3, 5, 7, 13$.

Subcase 4.1 $n = 9$.

We define $f : V(G') \rightarrow \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$

$$f(u'_i) = \begin{cases} 1, & 1 \leq i \leq 5; \\ 0, & 6 \leq i \leq r, \end{cases}$$

$$f(v_j) = 0; 1 \leqslant j \leqslant n,$$

$$f(v') = 0.$$

Subcase 4.2 $n = 11$.

We define $f : V(G') \rightarrow \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$

$$f(u'_i) = \begin{cases} 1, & 1 \leq i \leq 5; \\ 0, & 6 \leq i \leq r, \end{cases}$$

$$f(v_j) = \begin{cases} 1, & j = 1; \\ 0, & 2 \leq j \leq n, \end{cases}$$

$$f(v') = 0.$$

Subcase 4.3 $n = 15$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = \begin{cases} 1, & 1 \le i \le 6; \\ 0, & 7 \le i \le r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & j = 2, 4; \\ 0, & \text{otherwise}, \end{cases}$$
$$f(v') = 0.$$

Subcase 4.4 $n \ge 17$.

We define $f : V(G') \to \{0,1\}$ as

$$f(u_i) = 1; 1 \leqslant i \leqslant r,$$
$$f(u'_i) = \begin{cases} 1, & 1 \le i \le 6; \\ 0, & 7 \le i \le r, \end{cases}$$
$$f(v_j) = \begin{cases} 1, & j = 1, 3 \le j \le \frac{n-11}{2}; \\ 0, & j = 2, \frac{n-9}{2} \leqslant j \leqslant n, \end{cases}$$
$$f(v') = 1.$$

According to the above definitions of f in different sub cases, we have $e_f(1) - 1 = \frac{r(r-1)}{2} + n = e_f(0)$ and $v_f(1) = r + \frac{n+1}{2} = v_f(0)$. Hence in this case we have $|v_f(0) - v_f(1)| \le 1$ and $|e_f(0) - e_f(1)| \le 1$. □

Example 2.7 Consider the degree splitting graphs of path joining complete graphs in barbell graphs $B(3,5), B(4,4)$ and $B(7,9)$. Then they are product cordials and their labeling are as shown in Figures 7, 8 and 9.

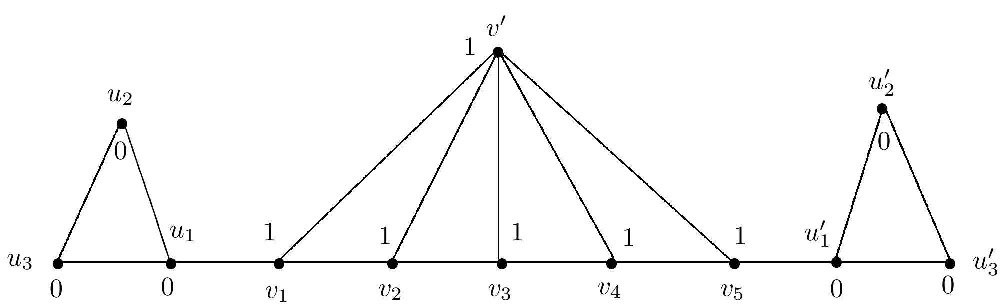

Figure 7 Degree splitting graph of path joining complete graphs in barbell graph $B(3,5)$

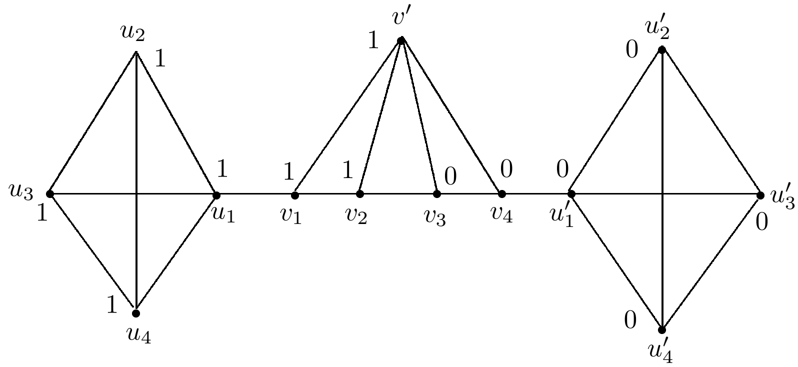

Figure 8 Degree splitting graph of path joining complete graphs in barbell graph $B(4, 4)$

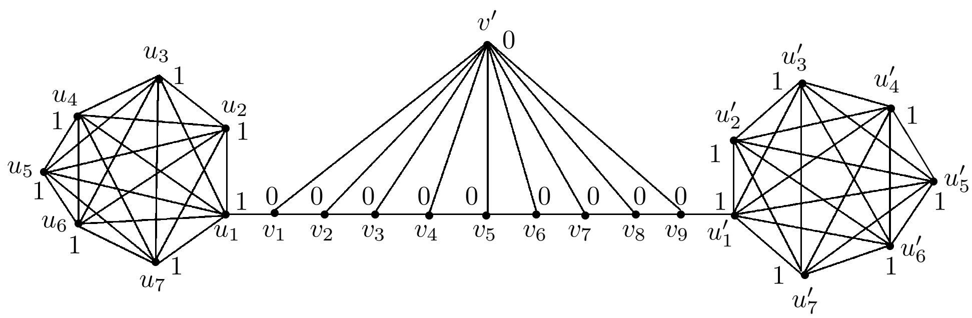

Figure 9 Degree splitting graph of path joining complete graphs in barbell graph $B(7, 9)$

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New mathematical methods and concepts, often more important than itself to follow in solving mathametical problems.

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Research papers

[6]Linfan Mao, Mathematics on non-mathematics - A combinatorial contribution, *International J.Math. Combin.*, Vol.3(2014), 1-34.
[9]Kavita Srivastava, On singular H-closed extensions, *Proc. Amer. Math. Soc.* (to appear).

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