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Different Domination Energies in Graphs

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Abstract: Representing a set of vertices in a graph means of a matrix was introduced by E. Sampath Kumar. Let $G(V, E)$ be a graph and $S \subseteq V$ be a set of vertices. We can represent the set S by means of a matrix as follows, in the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if, $v_i \in S$. In this paper we study the special case of set S being dominating set and corresponding domination energy of some class of graphs.

Key Words: Adjacency matrix, Smarandachely k-dominating set, domination number, eigenvalues, energy of graph.

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§1. Introduction

A set $D \subseteq V$ of G is said to be a Smarandachely k-dominating set if each vertex of G is dominated by at least k vertices of S and the Smarandachely k-domination number $\gamma_k(G)$ of G is the minimum cardinality of a Smarandachely k-dominating set of G. Particularly, if k =1, such a set is called a dominating set of G and the Smarandachely 1-domination number of G is called the domination number of G and denoted by $\gamma(G)$ in general.

The concept of graph energy arose in theoretical chemistry where certain numerical quantities like the heat of formation of a hydrocarbon are related to total π electron energy that can be calculated as the energy of corresponding molecular graph. The molecular graph is a representation of the molecular structure of a hydrocarbon whose vertices are the position of carbon atoms and two vertices are adjacent if there is a bond connecting them.

Eigen values and eigenvectors provide insight into the geometry of the associated linear transformation. The energy of a graph is the sum of the absolute values of the Eigen values of its adjacency matrix. From the pioneering work of Coulson [1] there exists a continuous interest towards the general mathematical properties of the total π electron energy ε as calculated within the framework of the Huckel Molecular Orbital (HMO) model. These efforts enabled one to get an insight into the dependence of ε on molecular structure. The properties of $\varepsilon(G)$ are discussed in detail in [2,3,4,5].

The importance of Eigen values is not only used in theoretical chemistry but also in analyzing structures. Car designers analyze Eigen values in order to damp out the noise to reduce

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the vibration of the car due to music. Eigen values can be used to test for cracks or deformities in a solid. Oil companies frequently use Eigen value analysis to explore land for oil. Eigen values are also used to discover new and better designs for the future [6].

Representation of a set of vertices in a graph by means of a matrix was first introduced by E. Sampath Kumar [7]. Let $G(V,E)$be a graph and $S \subseteq V$ be a set of vertices. We can represent the set S by means of a matrix as follows:

In the adjacency matrix $A(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the adjacency matrix can be taken as the matrix of the set S denoted by $A_S(G)$. The energy $E(G)$ obtained from the matrix $A_S(G)$ is called the set energy denoted by $E_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix as domination matrix denoted by $A_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the domination matrix $A_\gamma(G)$ is defined as domination energy denoted by $E_\gamma(G)$.

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The domination matrix of G is defined to be the square matrix $A_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the domination matrix denoted by $\kappa_1, \kappa_2, \kappa_3, \cdots, \kappa_n$ are said to be the A_γ Eigen values of G. Since the A_γ matrix is symmetric, its Eigen values are real and can be ordered $\kappa_1 \geqslant \kappa_2 \geqslant \kappa_3 \geqslant \cdots \geqslant \kappa_n$. Therefore, the domination energy

$$E_\gamma = E_\gamma(G) = \sum_{i=1}^{n} |\kappa_i|. \tag{1}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy [2].

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \tag{2}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots, \geqslant \lambda_n$ are the Eigen values of the adjacency matrix $A(G)$. Recall that in the last few years, the graph energy $E(G)$ and domination energy [20,21] or covering energy [8] has been extensively studied in mathematics [8-13] and mathematic-chemical literature [14-24].

Definition 1.1(Minimal domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The domination energy $E_\gamma(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{\gamma-\min}(G)$.*

Definition 1.2(Maximal domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The domination energy $E_\gamma(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{\gamma-\max}(G)$.*

Similarly to domination energy of graph G, *distance domination energy* can also be defined as follows:

Let the vertices of G be labeled as $v_1, v_2, v_3, \cdots, v_n$. The *distance matrix* of G is denoted by $D(G)$ is defined to be the square matrix $D(G) = [d_{ij}]$, where d_{ij} is the shortest distance between the vertex v_i and v_j in G. The Eigen values of the distance matrix denoted by $\mu_1, \mu_2, \mu_3, \cdots, \mu_n$ are said to be the D Eigen values of G. Since the $D(G)$ matrix is symmetric, its Eigen values are real and can be ordered $\mu_1 \geqslant \mu_2 \geqslant \mu_3 \geqslant \cdots \geqslant \mu_n$. Therefore, the distance energy

$$E_D = E_D(G) = \sum_{i=1}^{n} |\mu_i|. \tag{3}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy [2].

In the distance matrix $D(G)$ of G replace the a_{ii} element by 1 if and only if $v_i \in S$. The matrix thus obtained from the distance matrix can be considered as the *distance matrix of the set* S denoted by $D_S(G)$. The energy $E(G)$ obtained from the matrix $D_S(G)$ is called the *distance set energy* denoted by $D_S(G)$. In this paper we consider the set S as dominating set and the corresponding matrix is *distance domination matrix* denoted by $D_\gamma(G)$ of G. Thus the energy $E(G)$ obtained from the distance domination matrix $D_\gamma(G)$ is defined as *distance domination energy* denoted by $E_{D\gamma}(G)$.

The distance domination matrix of G is defined to be the square matrix $D_\gamma(G)$ corresponding to the dominating set of G. The Eigen values of the distance domination matrix denoted by $\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n$ are said to be the D_γ Eigen values of G. Since the $D_\gamma(G)$ matrix is symmetric, its D-Eigen values are real and can be ordered $\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3 \geqslant \cdots \geqslant \sigma_n$. Therefore, the distance domination energy

$$E_{D\gamma} = E_{D\gamma}(G) = \sum_{i=1}^{n} |\sigma_i|. \tag{4}$$

This equation has been chosen so as to be fully analogous to the definition of graph energy [2].

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|, \tag{5}$$

where $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots, \geqslant \lambda_n$ are the Eigen values of the adjacency matrix $A(G)$.

Definition 1.3(Minimal distance domination energy) *A dominating set D in G is a minimal dominating set if no proper subset of D is a dominating set. The distance domination energy $E_{D\gamma}(G)$ obtained for a minimal dominating set is called the minimal domination energy denoted by $E_{D\gamma-\min}(G)$.*

Definition 1.4(Maximal distance domination energy) *A dominating set D in G is a maximal dominating set if D contains all the vertices of G. The distance domination energy $E_{D\gamma}(G)$ obtained for a maximal dominating set is called the maximal domination energy denoted by $E_{D\gamma-\max}(G)$.*

§2. Different Energies of Graph with $\gamma(G) = 1$

In this section, we characterize graphs with respect to the unique domination set and hence find their different domination energies.

Remark 2.1 For the complete graph K_n the matrices $A(G) = D(G)$ and $A_\gamma(G) = D_\gamma(G)$.

Hence, the energy of complete graph K_n is given by $2(n-1)$, i.e., $E(K_n) = E_D(K_n) = 2(n-1)$.

Theorem 2.1 *Let $G = K_n$. Then,*

$$E_{\gamma-\min}(K_n) = E_{D\gamma-\min}(K_n) = \sqrt{n^2 - 2n + 5} + (n-2), n \geqslant 3.$$

Proof Calculation enables one to find the characteristic polynomial of K_n for $n \geqslant 3$ directly. Label the vertices of K_n as $v_1, v_2, v_3, \cdots, v_n$ such that v_1 is the dominating set. The domination matrix and the distance domination matrix are same. Hence, in the domination matrix or distance domination matrix $a_{11} = 1$ and $a_{ii} = 0, i \neq 1$.

The characteristic polynomial of domination matrix and the distance domination matrix is given by $\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \cdots + q_{n-1}\kappa + q_n = 0$ and $\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0$ respectively.

The domination matrix and the characteristic polynomial of K_3 are given by

$$A_\gamma(G) = D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^3 - \kappa^2 - 3\kappa - 1 = (\kappa+1)\left(\kappa^2 - 2\kappa - 1\right)$.

The domination matrix and the characteristic polynomial of K_4 are given by

$$A_\gamma(G) = D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^4 - \kappa^3 - 6\kappa^2 - 5\kappa - 1 = (\kappa+1)^2\left(\kappa^2 - 3\kappa - 1\right)$.

The domination matrix and the characteristic polynomial of K_5 are given by

$$A_\gamma(G) = D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^5 - \kappa^4 - 10\kappa^3 - 14\kappa^2 - 7\kappa - 1 = (\kappa+1)^3\left(\kappa^2 - 4\kappa - 1\right)$.

Therefore, the characteristic polynomial of K_n using domination matrix is

$$(\kappa+1)^{n-2}\left(\kappa^2 - (n-1)\kappa - 1\right) = 0.$$

Solving the equation we get

$$(\kappa+1)^{n-2}=0 \text{ or } \left(\kappa^2-(n-1)\kappa-1\right)=0.$$

$$\kappa=-1,-1,-1,\cdots,-1(n-2) \text{ times}$$

$$\kappa^2-(n-1)\kappa-1=0$$

Therefore,

$$\kappa=\frac{n-1\pm\sqrt{(n-1)^2-4(1)(-1)}}{2}=\frac{n-1\pm\sqrt{n^2-2n+5}}{2},$$

where $n\geqslant 3$. Hence the roots are

$$\kappa_1=\frac{n-1+\sqrt{n^2-2n+5}}{2},\ \kappa_2=-\left(\frac{\sqrt{n^2-2n+5}-(n-1)}{2}\right)$$

and

$$\begin{aligned}E_{\gamma-\min}(K_n)&=\sum_{i=1}^{n}|\kappa_i|\\&=\frac{n-1+\sqrt{n^2-2n+5}+\sqrt{n^2-2n+5}-(n-1)}{2}+n-2,\\E_{\gamma-\min}(K_n)&=E_{D\gamma-\min}(K_n)=\sqrt{n^2-2n+5}+(n-2).\end{aligned}$$

Hence, we get the proof. □

Remark 2.2 All four types of energies of a complete graph can be compared as follows:

$$\begin{aligned}E(K_n)&=E_D(K_n)=2(n-1)\geqslant E_{\gamma-\min}(K_n)\\&=E_{D\gamma-\min}(K_n)=\sqrt{n^2-2n+5}+(n-2).\end{aligned}$$

Remark 2.3 Energy of a star graph $K_{1,n-1}$ is given by $2\sqrt{n-1}$.

Theorem 2.2([21]) *Let $G=K_{1,n-1}$, $n\geqslant 3$. Then,*

$$E_{\gamma-\min}\left(K_{1,n-1}\right)=\sqrt{4n-3}.$$

Remark 2.4 $E\left(K_{1,n-1}\right)=2\sqrt{n-1}\leqslant E_{\gamma-\min}\left(K_{1,n-1}\right)=\sqrt{4n-3}$.

Theorem 2.3 *Let $G=K_{1,n-1}$, $n\geqslant 3$. Then,*

$$E_D\left(K_{1,n-1}\right)_= 2n-4+\sqrt{n^2-3n+3}.$$

Proof Calculation enables one to find the characteristic polynomial of $K_{1,n-1}$ for $n\geqslant 3$ directly. Label the vertices of $K_{1,n-1}$ as $v_1,v_2,v_3,\cdots,v_n$. The characteristic polynomial of distance matrix $D(G)$ is given by

$$\mu^n + q_1\mu^{n-1} + q_2\mu^{n-2} + \cdots + q_{n-1}\mu + q_n = 0.$$

The distance matrix and the characteristic polynomial of $K_{1,2}$ are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

and $\mu^3 - 6\mu - 4 = (\mu + 2)\left(\mu^2 - 2\mu - 2\right).$

The distance matrix and the characteristic polynomial of $K_{1,3}$ are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

and $\mu^4 - 15\mu^2 - 28\mu - 12 = (\mu + 2)^2\left(\mu^2 - 4\mu - 3\right).$

The distance matrix and the characteristic polynomial of $K_{1,4}$ are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and $\mu^5 - 28\mu^3 - 88\mu^2 - 96\mu - 32 = (\mu + 2)^3\left(\mu^2 - 6\mu - 4\right).$

The distance matrix and the characteristic polynomial of $K_{1,5}$ are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 \end{bmatrix}$$

and $\mu^6 - 45\mu^4 - 200\mu^3 - 360\mu^2 - 288\mu - 80 = (\mu + 2)^4\left(\mu^2 - 8\mu - 5\right).$

Therefore the characteristic polynomial of $K_{1,n-1}$ using distance matrix is

$$(\mu + 2)^{n-2}\left(\mu^2 - (2n - 4)\mu - (n - 1)\right).$$

Solving the equation we get

$$(\mu+2)^{n-2}=0 \quad \text{or} \quad \mu^2-(2n-4)\mu-(n-1)=0,$$

$$\mu=-2,-2,-2,\cdots,-2(n-2)(\text{times}) \text{ or } \mu^2-(2n-4)\mu-(n-1)=0.$$

Therefore,

$$\mu=\frac{(2n-4)\pm\sqrt{(2n-4)^2-4(-(n-1))}}{2}=\frac{(2n-4)\pm\sqrt{4\left(n^2-3n+3\right)}}{2}$$

where $n\geqslant 3$. Hence the roots are

$$\mu_1=\frac{(n-4)+\sqrt{n^2-3n+3}}{2} \quad \text{and} \quad \mu_2=-\left(\frac{\sqrt{n^2-3n+3}-(n-4)}{2}\right).$$

$$\begin{aligned} E_D\left(K_{1,n-1}\right) &= \sum_{i=1}^{n}|\mu_i| \\ &= \frac{2\sqrt{n^2-3n+3}}{2}+2(n-2) \\ &= 2n-4+\sqrt{n^2-3n+3}. \end{aligned}$$

Hence, we get the proof. □

Theorem 2.4 *Let $G=K_{1,n-1}$, $n\geqslant 3$. Then,*

$$E_{D\gamma}\left(K_{1,n-1}\right)_= 4n-7.$$

Proof Calculation enables one to find the characteristic polynomial of $K_{1,n-1}$ for $n\geqslant 3$ directly. Label the vertices of $K_{1,n-1}$ as $v_1,v_2,v_3,\cdots,v_n$.

The characteristic polynomial of distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n+q_1\sigma^{n-1}+q_2\sigma^{n-2}+\cdots+q_{n-1}\sigma+q_n=0.$$

The distance domination matrix and the characteristic polynomial of $K_{1,2}$ are given by

$$D_\gamma(G)=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 2\\ 1 & 2 & 0\end{bmatrix}$$

and $\sigma^3-\sigma^2-6\sigma=(\sigma+2)\left(\sigma^2-3\sigma+0\right)$.

The distance domination matrix and the characteristic polynomial of $K_{1,3}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 1 & 2 & 2 & 0 \end{bmatrix}$$

and $\sigma^4 - \sigma^3 - 15\sigma^2 - 16\sigma + 4 = (\sigma + 2)^2 \left(\sigma^2 - 5\sigma + 1\right)$.

The distance domination matrix and the characteristic polynomial of $K_{1,4}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and $\sigma^5 - \sigma^4 - 28\sigma^3 - 64\sigma^2 - 32\sigma + 16 = (\sigma + 2)^3 \left(\sigma^2 - 7\sigma + 2\right)$.

The distance domination matrix and the characteristic polynomial of $K_{1,5}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 2 & 2 \\ 1 & 2 & 0 & 2 & 2 & 2 \\ 1 & 2 & 2 & 0 & 2 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 \\ 1 & 2 & 2 & 2 & 0 & 2 \end{bmatrix}$$

and $\sigma^6 - \sigma^5 - 45\sigma^4 - 160\sigma^3 - 200\sigma^2 - 48\sigma + 48 = (\sigma + 2)^4 \left(\sigma^2 - 9\sigma + 3\right)$.

Therefore the characteristic polynomial of $K_{1,n-1}$ using distance domination matrix is

$$(\sigma + 2)^{n-2} \left(\sigma^2 - (2n - 3)\sigma + (n - 3)\right) = 0.$$

Solving the equation we get

$$(\sigma + 2)^{n-2} = 0 \text{ or } \sigma^2 - (2n - 3)\sigma + (n - 3) = 0.$$

Whence, $\sigma = -2, -2, -2, \cdots, -2$ ($(n-2)$ times) or $\sigma^2 - (2n-3)\sigma + (n-3) = 0$. Therefore,

$$\sigma = \frac{(2n-3) \pm \sqrt{(2n-3)^2 - 4((n-3))}}{2} = \frac{(2n-3) \pm \sqrt{4n^2 - 16n + 21}}{2},$$

where $n \geqslant 3$, i.e., the roots are

$$\sigma_1 = \frac{(2n-3)+\sqrt{4n^2-16n+21}}{2},$$
$$\sigma_2 = \frac{(2n-3)-\sqrt{4n^2-16n+21}}{2}$$

and

$$E_{D\gamma}(K_{1,n-1}) = \sum_{i=1}^{n}|\sigma_i| = (2n-3)+2(n-2) = 4n-7.$$

Hence, we get the proof. □

§3. Domination Energies for the Graph with $\gamma(G) = 2$

During the study of chemical graphs and its Weiner number, the Yugoslavian chemist Ivan Gutman introduced the concept of Thorn graphs. This idea was further extended to the broader concept of generalized thorny graphs by Danail Bonchev and Douglas J Klein of USA. This class of graphs gain importance in Spectral theory as it represents the structural formula of aliphatic and aromatic hydrocarbons9[.

Theorem 3.1 *Let $G = P_{2,t}$, $n = 2t$. Then,*

$$E(P_{2,t}) = 2\sqrt{4t-3}.$$

Proof Calculation enables one to find the characteristic polynomial of $G = P_{2,t}$ for $n = 2t$ directly. For $t = 1$, $P_{2,1}$ is a path with 2 vertices, $t = 2$, $P_{2,2}$ is a path with 4 vertices.

The adjacency matrix and the characteristic polynomial of $P_{2,3}$ are given by

$$A(G) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

and $\lambda^6 - 5\lambda^4 + 4\lambda^2 = \lambda^2(\lambda^2-\lambda-2)(\lambda^2+\lambda-2)$.

The adjacency matrix and the characteristic polynomial of $P_{2,4}$ are given by

$$A(G) = \begin{bmatrix} 0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 1&1&1&0&1&0&0&0\\ 0&0&0&1&0&1&1&1\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0 \end{bmatrix}$$

and $\lambda^8 - 7\lambda^6 + 9\lambda^4 = \lambda^4(\lambda^2 - \lambda - 3)(\lambda^2 + \lambda - 3)$.

The adjacency matrix and the characteristic polynomial of $P_{2,5}$ are given by

$$A(G) = \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0\\ 1&1&1&1&0&1&0&0&0&0\\ 0&0&0&0&1&0&1&1&1&1\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0 \end{bmatrix}$$

and $\lambda^{10} - 9\lambda^8 + 16\lambda^6 = \lambda^6(\lambda^2 - \lambda - 4)(\lambda^2 + \lambda - 4)$.

The adjacency matrix and the characteristic polynomial of $P_{2,6}$ are given by

$$A(G) = \begin{bmatrix} 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 1&1&1&1&1&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&1&1&1&1&1\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0 \end{bmatrix}$$

and $\lambda^{12}-11\lambda^{10}+25\lambda^{8}=\lambda^{8}(\lambda^{2}-\lambda-5)(\lambda^{2}+\lambda-5)$.

Therefore the characteristic polynomial of $P_{2,t}$ using adjacency matrix is

$$\lambda^{2t-4}(\lambda^{2}-\lambda-(t-1))(\lambda^{2}+\lambda-(t-1)).$$

Solving the equation we get

$$\lambda^{2t-4}=0,\ \lambda^{2}-\lambda-(t-1)=0 \text{ or } \lambda^{2}+\lambda-(t-1)=0,$$

i.e.,

$$\lambda=0,0,0,\cdots,0((2t-4)\text{ times}),\ \lambda^{2}-\lambda-(t-1)=0.$$

Therefore,

$$\lambda=\frac{1\pm\sqrt{1+4t-4}}{2}=1\pm\sqrt{4t-3},$$

where $t\geqslant 3$. Hence the roots are

$$\lambda_1=1+\sqrt{4t-3} \text{ and } \lambda_2=-\left(\sqrt{4t-3}-1\right)$$

and

$$E=\sum_{i=1}^{n}|\lambda_i|=\frac{1+\sqrt{4t-3}+\sqrt{4t-3}-1}{2}=\sqrt{4t-3}.$$

Similarly, solving the equation $\lambda^{2}+\lambda-(t-1)=0$ we get that

$$E=\sqrt{4t-3}.$$

Whence,

$$E\left(P_{2,t}\right)=2\sqrt{4t-3}.$$

Hence, we get the proof. □

Theorem 3.2([21]) *Let $G=P_{2,t}$, $n=2t$. Then,*

$$E_{\gamma-\min}\left(P_{2,t}\right)=2\sqrt{t-1}+2\sqrt{t}.$$

Theorem 3.3 *Let $G=P_{2,t}$, $n=2t$. Then,*

$$E_{D}\left(P_{2,t}\right)_{=}\sqrt{25t^{2}-28t+20}+(5t-6).$$

Proof Calculation enables one to find the characteristic polynomial of $P_{2,t}$ for $n=2t$ directly. For $t=1$, $P_{2,1}$ is a path with 2 vertices, $t=2$, $P_{2,2}$ is a path with 4 vertices.

The characteristic polynomial of $P_{2,t}$ using distance matrix $D(G)$ is given by

$$\mu^{n}+q_1\mu^{n-1}+q_2\mu^{n-2}+\cdots+q_{n-1}\mu+q_n=0.$$

The distance matrix and the characteristic polynomial of $P_{2,3}$ are given by

$$D(G)=\begin{bmatrix}0&2&1&2&3&3\\2&0&1&2&3&3\\1&1&0&1&2&2\\2&2&1&0&1&1\\3&3&2&1&0&2\\3&3&2&1&2&0\end{bmatrix}$$

and

$$\mu^6-65\mu^4-296\mu^3-504\mu^2-352\mu-80=(\mu+2)^2\left(\mu^2-9\mu-10\right)\left(\mu^2+5\mu+2\right).$$

The distance matrix and the characteristic polynomial of $P_{2,4}$ are given by

$$D(G)=\begin{bmatrix}0&2&2&1&2&3&3&3\\2&0&2&1&2&3&3&3\\2&2&0&1&2&3&3&3\\1&1&1&0&1&2&2&2\\2&2&2&1&0&1&1&1\\3&3&3&2&1&0&2&2\\3&3&3&2&1&2&0&2\\3&3&3&2&1&2&2&0\end{bmatrix}$$

and

$$\begin{aligned}&\mu^8-136\mu^6-1040\mu^5-3468\mu^4-6112\mu^3-5792\mu^2-2688\mu-448\\&=(\mu+2)^4\left(\mu^2-14\mu-14\right)\left(\mu^2+6\mu+2\right).\end{aligned}$$

The distance matrix and the characteristic polynomial of $P_{2,5}$ are given by

$$D(G)=\begin{bmatrix}0&2&2&2&1&2&3&3&3&3\\2&0&2&2&1&2&3&3&3&3\\2&2&0&2&1&2&3&3&3&3\\2&2&2&0&1&2&3&3&3&3\\1&1&1&1&0&1&2&2&2&2\\2&2&2&2&1&0&1&1&1&1\\3&3&3&3&2&1&0&2&2&2\\3&3&3&3&2&1&2&0&2&2\\3&3&3&3&2&1&2&2&0&2\\3&3&3&3&2&1&2&2&2&0\end{bmatrix}$$

and

$$\mu^{10} - 233\mu^8 - 2512\mu^7 - 12624\mu^6 - 36800\mu^5 - 66400\mu^4 - 74496\mu^3$$
$$-49664\mu^2 - 17408\mu - 2304 = (\mu+2)^6\left(\mu^2 - 19\mu - 18\right)\left(\mu^2 + 7\mu + 2\right).$$

Therefore, the characteristic polynomial of $P_{2,t}$ using distance matrix is

$$(\mu+2)^{2t-4}\left(\mu^2 - (5t-6)\mu - (4t-2)\right)\left(\mu^2 + (t+2)\mu + 2\right),$$

i.e.,

$$(\mu+2)^{2t-4} = 0,\ \mu^2 - (5t-6)\mu - (4t-2),\ \text{or}\ \mu^2 + (t+2)\mu + 2\left(\mu+2\right)^{2t-4} = 0.$$

Solving the equation $(\mu+2)^{2t-4}$ we get $\mu = -2, -2, -2, \cdots, -2((2t-4)$ times. Similarly, Solving the equation $\mu^2 - (5t-6)\mu - (4t-2)$ we get

$$\mu = \frac{(5t-6) \pm \sqrt{(5t-6)^2 + 4(4t-2)}}{2}$$

, and the equation $+(t+2)\mu + 2$ we get

$$\mu = \frac{-(t+2) \pm \sqrt{(t+2)^2 - 8}}{2}.$$

Therefore,

$$\begin{aligned} E_D\left(P_{2,t}\right) = \sum_{i=1}^{n} |\mu_i| &= \sqrt{25t^2 - 28t + 20} + (t+2) + (4t-8) \\ &= \sqrt{25t^2 - 28t + 20} + (5t-6). \end{aligned}$$

Hence, we get the proof. □

Theorem 3.4 *Let $G = P_{2,t}$, $n = 2t$. Then,*

$$E_{D\gamma}\left(P_{2,t}\right) = \begin{cases} \sqrt{25t^2 - 54t + 45} + \sqrt{t^2 + 6t - 3} + (4t-8) & t = 3, 4 \\ (5t-5) + \sqrt{t^2 + 6t - 3} + (4t-8) & t > 5 \end{cases}$$

and for $t = 5$,

$$E_{D\gamma}\left(P_{2,t}\right) = \frac{(5t-5) + \sqrt{25t^2 + 54t + 45}}{2} + \sqrt{t^2 + 6t - 3} + (4t-8).$$

Proof Calculation enables one to find the characteristic polynomial of $P_{2,t}$ for $n = 2t$ directly. For $t = 1$, $P_{2,1}$ is a path with 2 vertices, $t = 2$, $P_{2,2}$ is a path with 4 vertices.

The characteristic polynomial of $P_{2,t}$ using distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0.$$

The distance domination matrix and the characteristic polynomial of $P_{2,3}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 1 & 2 & 3 & 3 \\ 2 & 0 & 1 & 2 & 3 & 3 \\ 1 & 1 & 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 1 & 0 & 2 \\ 3 & 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

and $\sigma^6 - 2\sigma^5 - 64\sigma^4 - 188\sigma^3 - 124\sigma^2 + 64\sigma + 16 = (\sigma+2)^2\left(\sigma^2 - 10\sigma - 2\right)\left(\sigma^2 + 4\sigma - 2\right)$.

The distance domination matrix and the characteristic polynomial of $P_{2,4}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 2 & 1 & 2 & 3 & 3 & 3 \\ 2 & 0 & 2 & 1 & 2 & 3 & 3 & 3 \\ 2 & 2 & 0 & 1 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 2 & 1 & 0 & 2 & 2 \\ 3 & 3 & 3 & 2 & 1 & 2 & 0 & 2 \\ 3 & 3 & 3 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\begin{aligned} &\sigma^8 - 2\sigma^7 - 135\sigma^6 - 800\sigma^5 - 1877\sigma^4 - 1704\sigma^3 + 88\sigma^2 + 736\sigma + 48 \\ &= (\sigma+2)^4\left(\sigma^2 - 15\sigma - 1\right)\left(\sigma^2 + 5\sigma - 3\right). \end{aligned}$$

The distance domination matrix and the characteristic polynomial of $P_{2,5}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 0 & 2 & 2 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 0 & 2 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 0 & 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 2 & 1 & 0 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 2 & 0 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 2 & 2 & 0 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\sigma^{10}-2\sigma^{9}-232\sigma^{8}-2088\sigma^{7}-8480\sigma^{6}-18208\sigma^{5}-19584\sigma^{4}$$
$$-5504\sigma^{3}+7424\sigma^{2}+5120\sigma=(\sigma+2)^{6}\left(\sigma^{2}-20\sigma-0\right)\left(\sigma^{2}+6\sigma-4\right).$$

The distance domination matrix and the characteristic polynomial of $P_{2,6}$ are given by

$$D_\gamma(G)=\begin{bmatrix}0&2&2&2&2&1&2&3&3&3&3&3\\2&0&2&2&2&1&2&3&3&3&3&3\\2&2&0&2&2&1&2&3&3&3&3&3\\2&2&2&0&2&1&2&3&3&3&3&3\\2&2&2&2&0&1&2&3&3&3&3&3\\1&1&1&1&1&1&1&2&2&2&2&2\\2&2&2&2&2&1&1&1&1&1&1&1\\3&3&3&3&3&2&1&0&2&2&2&2\\3&3&3&3&3&2&1&2&0&2&2&2\\3&3&3&3&3&2&1&2&2&0&2&2\\3&3&3&3&3&2&1&2&2&2&0&2\\3&3&3&3&3&2&1&2&2&2&2&0\end{bmatrix}$$

and

$$\sigma^{12}-2\sigma^{11}-355\sigma^{10}-4300\sigma^{9}-24885\sigma^{8}-83856\sigma^{7}-172368\sigma^{6}-206400\sigma^{5}-108000\sigma^{4}$$
$$+39680\sigma^{3}+80384\sigma^{2}+28672\sigma-1280=(\sigma+2)^{8}\left(\sigma^{2}-25\sigma+1\right)\left(\sigma^{2}+7\sigma-5\right).$$

Therefore, the characteristic polynomial of $P_{2,t}$ using distance domination matrix is

$$(\sigma+2)^{2t-4}\left(\sigma^{2}-(5t-5)\sigma+(t-5)\right)\left(\sigma^{2}+(t+1)\sigma-(t-1)\right),$$

i.e.,

$$(\sigma+2)^{2t-4},\ \sigma^{2}-(5t-5)\sigma+(t-5)\text{ or }\sigma^{2}+(t+1)\sigma-(t-1).$$

Solving the equation $(\sigma+2)^{2t-4}=0$ we get $\sigma=-2,-2,-2,\cdots,-2$ ($(2t-4)$ times). Similarly, solving the equation $\sigma^2-(5t-5)\sigma+(t-5)$ we get

$$\sigma=\frac{(5t-5)\pm\sqrt{(5t-5)^2-4(t-5)}}{2}$$

and the equation $\sigma^2+(t+1)\sigma-(t-1)$ implies

$$\sigma=\frac{(t+1)\pm\sqrt{(t+2)^2+4(t-1)}}{2}.$$

Therefore,

$$E_{D\gamma}\left(P_{2,t}\right)=\sum_{i=1}^{n}|\sigma_i|=\begin{cases}\sqrt{25t^2-54t+45}+\sqrt{t^2+6t-3}+(4t-8), & t=3,4\\(5t-5)+\sqrt{t^2+6t-3}+(4t-8), & t>5.\end{cases}$$

and for $t=5$,

$$E_{D\gamma}\left(P_{2,t}\right)=\frac{(5t-5)+\sqrt{25t^2+54t+45}}{2}+\sqrt{t^2+6t-3}+(4t-8).\qquad\square$$

Theorem 3.5 *Let* $G=P_{3,t}$, $n=2t+1$. *Then,*

$$E\left(P_{3,t}\right)=2\sqrt{t-1}+2\sqrt{t+1}.$$

Proof Calculation enables one to find the characteristic polynomial of $P_{3,t}$ for $n=2t+1$ directly. For $t=1$, $P_{3,1}$ is a path with 3 vertices, $t=2$, $P_{3,2}$ is a path with 5 vertices.

The adjacency matrix and the characteristic polynomial of $P_{3,3}$ are given by

$$A(G)=\begin{bmatrix}0&0&1&0&0&0&0\\0&0&1&0&0&0&0\\1&1&0&1&0&0&0\\0&0&1&0&1&0&0\\0&0&0&1&0&1&1\\0&0&0&0&1&0&0\\0&0&0&0&1&0&0\end{bmatrix}$$

and $\lambda^7-6\lambda^5+8\lambda^3=\lambda^3(\lambda^2-2)(\lambda^2-4)$.

The adjacency matrix and the characteristic polynomial of $P_{3,4}$ are given by

$$A(G)=\begin{bmatrix}0&0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\1&1&1&0&1&0&0&0&0\\0&0&0&1&0&1&0&0&0\\0&0&0&0&1&0&1&1&1\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0&0\end{bmatrix}$$

and $\lambda^9-8\lambda^7+15\lambda^5=\lambda^5(\lambda^2-3)(\lambda^2-5)$.

The adjacency matrix and the characteristic polynomial of $P_{3,5}$ are given by

$$A(G) = \begin{bmatrix} 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0 \\ 1&1&1&1&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&1&1&1&1 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0 \end{bmatrix}$$

and $\lambda^{11} - 10\lambda^9 + 24\lambda^7 = \lambda^7(\lambda^2 - 4)(\lambda^2 - 6)$.

Therefore the characteristic polynomial of $P_{3,t}$ using adjacency matrix is

$$\lambda^{2t-3}(\lambda^2 - (t-1))(\lambda^2 - (t+1)).$$

Solving the equation we get

$$E\left(P_{3,t}\right) = 2\sqrt{t-1} + 2\sqrt{t+1}.$$

Hence, we get the proof. □

Theorem 3.6([21]0) *Let $G = P_{3,t}$, $n = 2t+1$. Then,*

$$E_{\gamma-\min}\left(P_{3,t}\right) = \sqrt{4t-3} + \sqrt{4t+5}.$$

Theorem 3.7 *Let $G = P_{3,t}$, $n = 2t+1$ Then, the characteristic polynomial of $P_{3,t}$ using distance matrix of G is*

$$(\mu+2)^{2t-4}\left(\mu^2 + (2t+2)\mu + 4\right)\left(\mu^3 - (6t-6)\mu^2 - (12t-6)\mu - 4t\right) = 0.$$

Proof Calculation enables one to find the characteristic polynomial of $P_{3,t}$ for $n = 2t+1$ directly. For $t = 1$, $P_{3,1}$ is a path with 3 vertices, $t = 2$, $P_{3,2}$ is a path with 5 vertices.

The characteristic polynomial of distance matrix $D(G)$ is given by

$$\mu^n + q_1\mu^{n-1} + q_2\mu^{n-2} + \cdots + q_{n-1}\mu + q_n = 0.$$

The distance matrix and the characteristic polynomial of $P_{3,3}$ are given by

$$D(G)=\begin{bmatrix}0&2&1&2&3&4&4\\2&0&1&2&3&4&4\\1&1&0&1&2&3&3\\2&2&1&0&1&2&2\\3&3&2&1&0&1&1\\4&4&3&2&1&0&2\\4&4&3&2&1&2&0\end{bmatrix}$$

and

$$\begin{aligned}&\mu^7-134\mu^5-804\mu^4-1904\mu^3-2112\mu^2-1056\mu-192\\&=(\mu+2)^2\left(\mu^2+8\mu+4\right)\left(\mu^3-12\mu^2-30\mu-12\right).\end{aligned}$$

The distance matrix and the characteristic polynomial of $P_{3,4}$ are given by

$$D(G)=\begin{bmatrix}0&2&2&1&2&3&4&4&4\\2&0&2&1&2&3&4&4&4\\2&2&0&1&2&3&4&4&4\\1&1&1&0&1&2&3&3&3\\2&2&2&1&0&1&2&2&2\\3&3&3&2&1&0&1&1&1\\4&4&4&3&2&1&0&2&2\\4&4&4&3&2&1&2&0&2\\4&4&4&3&2&1&2&2&0\end{bmatrix}$$

and

$$\begin{aligned}&\mu^9-258\mu^7-2412\mu^6-9864\mu^5-21984\mu^4-28128\mu^3-20160\mu^2\\&-7296\mu-1024=(\mu+2)^4\left(\mu^2+10\mu+4\right)\left(\mu^3-18\mu^2-42\mu-16\right).\end{aligned}$$

The distance matrix and the characteristic polynomial of $P_{3,5}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 0 & 2 & 2 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 0 & 2 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 2 & 0 & 2 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 2 & 2 & 0 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\mu^{11} - 422\mu^9 - 5380\mu^8 - 31584\mu^7 - 108160\mu^6 - 233920\mu^5 - 326784\mu^4 - 290560\mu^3$$
$$-155648\mu^2 - 44544\mu - 5120 = (\mu+2)^6\left(\mu^2 + 12\mu + 4\right)\left(\mu^3 - 24\mu^2 - 54\mu - 20\right).$$

Therefore, the characteristic polynomial of $P_{3,t}$ using distance matrix is

$$(\mu+2)^{2t-4}\left(\mu^2 + (2t+2)\mu + 4\right)\left(\mu^3 - (6t-6)\mu^2 - (12t-6)\mu - 4t\right) = 0. \qquad \square$$

Theorem 3.8 *Let $G = P_{3,t}$, $n = 2t+1$ Then, the characteristic polynomial of $P_{2,t}$ using distance domination matrix of G, is given by*

$$(\sigma+2)^{2t-4}\left(\sigma^2 + (2t+1)\sigma - (2t-4)\right)\left(\sigma^3 - (6t-5)\sigma^2 - (6t+2)\sigma + (4t+8)\right) = 0.$$

Proof Calculation enables one to find the characteristic polynomial of $P_{3,t}$ for $n = 2t+1$ directly. For $t = 1$, $P_{3,1}$ is a path with 3 vertices, $t = 2$, $P_{3,2}$ is a path with 5 vertices.
The characteristic polynomial of distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0.$$

The distance domination matrix and the characteristic polynomial of $P_{3,3}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 1 & 2 & 3 & 4 & 4 \\ 2 & 0 & 1 & 2 & 3 & 4 & 4 \\ 1 & 1 & 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 1 & 0 & 1 & 2 & 2 \\ 3 & 3 & 2 & 1 & 1 & 1 & 1 \\ 4 & 4 & 3 & 2 & 1 & 0 & 2 \\ 4 & 4 & 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

and

$$\sigma^7 - 2\sigma^6 - 133\sigma^5 - 586\sigma^4 - 824\sigma^3 - 176\sigma^2 + 240\sigma - 32$$
$$= (\sigma + 2)^2 \left(\sigma^2 + 7\sigma - 2\right) \left(\sigma^3 - 13\sigma^2 - 20\sigma + 4\right).$$

The distance domination matrix and the characteristic polynomial of $P_{3,4}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 2 & 1 & 2 & 3 & 4 & 4 & 4 \\ 2 & 0 & 2 & 1 & 2 & 3 & 4 & 4 & 4 \\ 2 & 2 & 0 & 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 \\ 2 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 \\ 3 & 3 & 3 & 2 & 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 3 & 2 & 1 & 0 & 2 & 2 \\ 4 & 4 & 4 & 3 & 2 & 1 & 2 & 0 & 2 \\ 4 & 4 & 4 & 3 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\sigma^9 - 2\sigma^8 - 257\sigma^7 - 1966\sigma^6 - 6152\sigma^5 - 8816\sigma^4 - 4048\sigma^3 + 2464\sigma^2 + 1792\sigma - 512$$
$$= (\sigma + 2)^4 \left(\sigma^2 + 9\sigma - 4\right) \left(\sigma^3 - 19\sigma^2 - 26\sigma + 18\right).$$

The distance domination matrix and the characteristic polynomial of $P_{3,5}$ are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 0 & 2 & 2 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 0 & 2 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 3 & 3 & 3 \\ 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 2 & 0 & 2 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 2 & 2 & 0 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\sigma^{11}-2\sigma^{10}-421\sigma^9-4626\sigma^8-22736\sigma^7-60832\sigma^6-89568\sigma^5-59072\sigma^4+9728\sigma^3$$
$$+32768\sigma^2+6912\sigma-4608=(\sigma+2)^6\left(\sigma^2+11\sigma-6\right)\left(\sigma^3-25\sigma^2-32\sigma+12\right).$$

Therefore the characteristic polynomial of $P_{2,t}$ using distance domination matrix of G is

$$(\sigma+2)^{2t-4}\left(\sigma^2+(2t+1)\sigma-(2t-4)\right)\left(\sigma^3-(6t-5)\sigma^2-(6t+2)\sigma+(4t+8)\right)=0. \quad \Box$$

Theorem 3.9 *Let $G=P_{4,t}$, $n=2t+2$. Then, the characteristic polynomial using adjacency matrix of G is given by*

$$\lambda^{2t-4}(\lambda^3-\lambda^2-t\lambda+(t-1))(\lambda^3+\lambda^2-t\lambda-(t-1)).$$

Proof Calculation enables one to find the characteristic polynomial of $P_{4,t}$ for $n=2t+2$ directly. For $t=1$, $P_{4,1}$ is a path with 4 vertices, $t=2$, $P_{4,2}$ is a path with 6 vertices.

The adjacency matrix and the characteristic polynomial of $P_{4,3}$ are given by

$$A(G)=\begin{bmatrix}0&0&1&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&1&0&1&0&0&0&0\\0&0&1&0&1&0&0&0\\0&0&0&1&0&1&0&0\\0&0&0&0&1&0&1&1\\0&0&0&0&0&1&0&0\\0&0&0&0&0&1&0&0\end{bmatrix}$$

and $\lambda^8-7\lambda^6+13\lambda^4-4\lambda^2=\lambda^2(\lambda^3-\lambda^2-3\lambda+2)(\lambda^3+\lambda^2-3\lambda-2)$.

The adjacency matrix and the characteristic polynomial of $P_{4,4}$ are given by

$$A(G)=\begin{bmatrix}0&0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0\\1&1&1&0&1&0&0&0&0&0\\0&0&0&1&0&1&0&0&0&0\\0&0&0&0&1&0&1&0&0&0\\0&0&0&0&0&1&0&1&1&1\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0&0\\0&0&0&0&0&0&1&0&0&0\end{bmatrix}$$

and $\lambda^{10} - 9\lambda^8 + 22\lambda^6 - 9\lambda^4 = \lambda^4(\lambda^3 - \lambda^2 - 4\lambda + 3)(\lambda^3 + \lambda^2 - 4\lambda - 3)$.

The adjacency matrix and the characteristic polynomial of $P_{4,5}$ are given by

$$A_\gamma(G) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\lambda^{12} - 11\lambda^{10} + 33\lambda^8 - 16\lambda^6 = \lambda^6(\lambda^3 - \lambda^2 - 5\lambda + 4)(\lambda^3 + \lambda^2 - 5\lambda - 4).$$

Therefore, the characteristic polynomial of $P_{4,t}$ using adjacency matrix of G is

$$\lambda^{2t-4}(\lambda^3 - \lambda^2 - t\lambda + (t-1))(\lambda^3 + \lambda^2 - t\lambda - (t-1)).$$

Hence, we get the proof. □

Theorem 3.10([21]) *Let $G = P_{4,t}$, $n = 2t + 2$. Then, the characteristic polynomial using domination matrix of G is given by*

$$\kappa^{2t-4}(\kappa^3 - (t+1)\kappa - (t-1))(\kappa^3 - 2\kappa^2 - (t-1)\kappa + (t-1)).$$

Theorem 3.11 *Let $G = P_{4,t}$, $n = 2t + 2$. Then, the characteristic polynomial using distance matrix of G is given by*

$$(\mu+2)^{2t-4}(\mu^3 - (7t-5)\mu^2 - (22t-8)\mu - (8t+4))(\mu^3 + (3t+2)\mu^2 + (2t+8)\mu + 4).$$

Proof Calculation enables one to find the characteristic polynomial of $P_{4,t}$ for $n = 2t + 2$ directly. For $t = 1$, $P_{4,1}$ is a path with 4 vertices, $t = 2$, $P_{4,2}$ is a path with 6 vertices.

The distance matrix and the characteristic polynomial of $P_{4,3}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 1 & 2 & 3 & 4 & 5 & 5 \\ 2 & 0 & 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 1 & 0 & 1 & 2 & 3 & 4 & 4 \\ 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 \\ 3 & 3 & 2 & 1 & 0 & 1 & 2 & 2 \\ 4 & 4 & 3 & 2 & 1 & 0 & 1 & 1 \\ 5 & 5 & 4 & 3 & 2 & 1 & 0 & 2 \\ 5 & 5 & 4 & 3 & 2 & 1 & 2 & 0 \end{bmatrix}$$

and $\mu^8 - 248\mu^6 - 1904\mu^5 - 5932\mu^4 - 9248\mu^3 - 7456\mu^2 - 2944\mu - 448 = (\mu+2)^2\,(\mu^3 - 16\mu^2 - 58\mu - 28)(\mu^3 + 12\mu^2 + 14\mu + 4)$.

The distance matrix and the characteristic polynomial of $P_{4,4}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\ 2 & 0 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\ 2 & 2 & 0 & 1 & 2 & 3 & 4 & 5 & 5 & 5 \\ 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 4 & 4 \\ 2 & 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 & 3 \\ 3 & 3 & 3 & 2 & 1 & 0 & 1 & 2 & 2 & 2 \\ 4 & 4 & 4 & 3 & 2 & 1 & 0 & 1 & 1 & 1 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 0 & 2 \\ 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 2 & 0 \end{bmatrix}$$

and $\mu^{10} - 449\mu^8 - 5032\mu^7 - 24768\mu^6 - 67808\mu^5 - 110944\mu^4 - 109440\mu^3 - 62720\mu^2 - 18944\mu - 2304 = (\mu+2)^4\,(\mu^3 - 23\mu^2 - 80\mu - 36)(\mu^3 + 15\mu^2 + 16\mu + 4)$.

The distance matrix and the characteristic polynomial of $P_{4,5}$ are given by

$$D(G) = \begin{bmatrix} 0 & 2 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 2 & 0 & 2 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 2 & 2 & 0 & 2 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 2 & 2 & 2 & 0 & 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 1 & 1 & 1 & 1 & 0 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 & 1 & 0 & 1 & 2 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 2 & 1 & 0 & 1 & 2 & 2 & 2 & 2 \\ 4 & 4 & 4 & 4 & 3 & 2 & 1 & 0 & 1 & 1 & 1 & 1 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 0 & 2 & 2 & 2 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 0 & 2 & 2 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 2 & 0 & 2 \\ 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 2 & 2 & 2 & 0 \end{bmatrix}$$

and

$$\begin{aligned}&\mu^{12}-708\mu^{10}-10464\mu^{9}-70860\mu^{8}-281664\mu^{7}-718016\mu^{6}-1214208\mu^{5}\\&\quad-1365888\mu^{4}-998400\mu^{3}-448512\mu^{2}-110592\mu-11264\\&=(\mu+2)^{6}\,(\mu^{3}-30\mu^{2}-102\mu-44)(\mu^{3}+18\mu^{2}+18\mu+4).\end{aligned}$$

Therefore the characteristic polynomial of $P_{4,t}$ using distance matrix of G is

$$\begin{aligned}&(\mu+2)^{2t-4}\,(\mu^{3}-(7t-5)\mu^{2}-(22t-8)\mu-(8t+4))\\&\times(\mu^{3}+(3t+2)\mu^{2}+(2t+8)\mu+4).\end{aligned}$$

Hence, we get the proof. □

Theorem 3.12 *Let $G=P_{4,t}$, $n=2t+2$. Then, the characteristic polynomial using distance domination matrix of G is given by*

$$\begin{aligned}&(\sigma+2)^{2t-4}\,(\sigma^{3}-(7t-4)\sigma^{2}-(5t)\sigma+(10t-20))\\&\times(\sigma^{3}+(3t+2)\sigma^{2}+(8-t)\sigma+4).\end{aligned}$$

Proof Calculation enables one to find the characteristic polynomial of $P_{4,t}$ for $n=2t+2$ directly. For $t=1$, $P_{4,1}$ is a path with 4 vertices, $t=2$, $P_{4,2}$ is a path with 6 vertices.

The distance domination matrix and the characteristic polynomial of $P_{4,3}$ are given by

$$D_\gamma(G)=\begin{bmatrix}0&2&1&2&3&4&5&5\\2&0&1&2&3&4&5&5\\1&1&1&1&2&3&4&4\\2&2&1&0&1&2&3&3\\3&3&2&1&0&1&2&2\\4&4&3&2&1&1&1&1\\5&5&4&3&2&1&0&2\\5&5&4&3&2&1&2&0\end{bmatrix}$$

and

$$\begin{aligned}&\sigma^{8}-2\sigma^{7}-247\sigma^{6}-1504\sigma^{5}-3277\sigma^{4}-2472\sigma^{3}+216\sigma^{2}+480\sigma-80\\&=(\sigma+2)^{2}\,(\sigma^{3}-17\sigma^{2}-45\sigma+10)(\sigma^{3}+11\sigma^{2}+5\sigma-2).\end{aligned}$$

The distance domination matrix and the characteristic polynomial of $P_{4,4}$ are given by

$$D_\gamma(G)=\begin{bmatrix}0&2&2&1&2&3&4&5&5&5\\2&0&2&1&2&3&4&5&5&5\\2&2&0&1&2&3&4&5&5&5\\1&1&1&1&1&2&3&4&4&4\\2&2&2&1&0&1&2&3&3&3\\3&3&3&2&1&0&1&2&2&2\\4&4&4&3&2&1&1&1&1&1\\5&5&5&4&3&2&1&0&2&2\\5&5&5&4&3&2&1&2&0&2\\5&5&5&4&3&2&1&2&2&0\end{bmatrix}$$

and

$$\sigma^{10}-2\sigma^9-448\sigma^8-4264\sigma^7-16936\sigma^6-33376\sigma^5-29968\sigma^4-3328\sigma^3+10496\sigma^2$$
$$+2560\sigma-1280=(\sigma+2)^4(\sigma^3-24\sigma^2-60\sigma+20)(\sigma^3+14\sigma^2+4\sigma-4).$$

The distance domination matrix and the characteristic polynomial of $P_{4,5}$ are given by

$$D_\gamma(G)=\begin{bmatrix}0&2&2&2&1&2&3&4&5&5&5&5\\2&0&2&2&1&2&3&4&5&5&5&5\\2&2&0&2&1&2&3&4&5&5&5&5\\2&2&2&0&1&2&3&4&5&5&5&5\\1&1&1&1&1&1&2&3&4&4&4&4\\2&2&2&2&1&0&1&2&3&3&3&3\\3&3&3&3&2&1&0&1&2&2&2&2\\4&4&4&4&3&2&1&1&1&1&1&1\\5&5&5&5&4&3&2&1&0&2&2&2\\5&5&5&5&4&3&2&1&2&0&2&2\\5&5&5&5&4&3&2&1&2&2&0&2\\5&5&5&5&4&3&2&1&2&2&2&0\end{bmatrix}$$

and

$$\sigma^{12}-2\sigma^{11}-707\sigma^{10}-9212\sigma^9-53597\sigma^8-173456\sigma^7-326864\sigma^6-332864\sigma^5-107744\sigma^4$$
$$+105216\sigma^3+90624\sigma^2-11520=(\sigma+2)^6(\sigma^3-31\sigma^2-75\sigma+30)(\sigma^3+17\sigma^2+3\sigma-6).$$

Therefore the characteristic polynomial of $P_{4,t}$ using distance domination matrix of G is

$$(\sigma+2)^{2t-4}(\sigma^3-(7t-4)\sigma^2-(5t)\sigma+(10t-20))(\sigma^3+(3t+2)\sigma^2+(8-t)\sigma+4).$$

Hence, we get the proof. □

§4. Generalized Characteristic Polynomial Can Not Be Obtained

It is not easy to find the generalized characteristic polynomial with respect to domination energies for all class of graphs, as the problem of finding the characteristic polynomial for an arbitrary matrix is still open. Here we illustrate that for paths, cycles and wheel graphs finding the generalized characteristic polynomial is not possible. Hence for this kind of graphs the absolute energies cannot be found. Therefore only the upper and lower bound can be obtained.

Theorem 4.1 *Let $G = P_n$, $n \geqslant 3$. Then the exact $E(P_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of P_n for $n \geqslant 3$ directly. Label the vertices of P_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of adjacency matrix $A(G)$ is given by

$$\lambda^n + q_1\lambda^{n-1} + q_2\lambda^{n-2} + \cdots + q_{n-1}\lambda + q_n = 0.$$

The adjacency matrix and the characteristic polynomial of P_3 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^3 - 2\lambda = \lambda(\lambda^2 - 1)$.

The adjacency matrix and the characteristic polynomial of P_4 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^4 - 3\lambda^2 + 1 = (\lambda^2 - \lambda - 1)(\lambda^2 + \lambda - 1)$.

The adjacency matrix and the characteristic polynomial of P_5 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^5 - 4\lambda^3 + 3\lambda = \lambda\,(\lambda - 1)\,(\lambda + 1)(\lambda^2 - 3)$.

The adjacency matrix and the characteristic polynomial of P_6 are given by

$$A(G)=\begin{bmatrix}0&1&0&0&0&0\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\0&0&0&0&1&0\end{bmatrix}$$

and

$$\lambda^6-5\lambda^4+6\lambda^2-1=(\lambda^3-\lambda^2-2\lambda+1)(\lambda^3+\lambda^2-2\lambda-1).$$

Hence, we get the proof. □

Theorem 4.2 *Let $G=P_n$, $n\geqslant 3$. Then the exact $E_\gamma(P_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of P_n for $n\geqslant 3$ directly. Label the vertices of P_n as $v_1,v_2,v_3,\cdots,v_n$.

The characteristic polynomial of domination matrix $A_\gamma(G)$ is given by $\kappa^n+q_1\kappa^{n-1}+q_2\kappa^{n-2}+\cdots+q_{n-1}\kappa+q_n=0$.

The domination matrix and the characteristic polynomial of P_3 are given by

$$A_\gamma(G)=\begin{bmatrix}0&1&0\\1&1&1\\0&1&0\end{bmatrix}$$

and $\kappa^3-\kappa^2-2\kappa=\kappa(\kappa+1)(\kappa-2)$.

The domination matrix and the characteristic polynomial of P_4 are given by

$$A_\gamma(G)=\begin{bmatrix}0&1&0&0\\1&1&1&0\\0&1&0&1\\0&0&1&1\end{bmatrix},\quad A_\gamma(G)=\begin{bmatrix}0&1&0&0\\1&1&1&0\\0&1&1&1\\0&0&1&0\end{bmatrix}\quad\text{or}\quad A_\gamma(G)=\begin{bmatrix}1&1&0&0\\1&0&1&0\\0&1&0&1\\0&0&1&1\end{bmatrix}$$

whose polynomial are respectively

$$\kappa^4-2\kappa^3-2\kappa^2+3\kappa+1,$$
$$\kappa^4-2\kappa^3-2\kappa^2+2\kappa+1=(\kappa-1)(\kappa+1)(\kappa^2-2\kappa-1),$$
$$\kappa^4-2\kappa^3-2\kappa^2+4\kappa=\kappa(\kappa-2)(\kappa^2-2).$$

The domination matrix and the characteristic polynomial of P_5 are given by

$$A_\gamma(G)=\begin{bmatrix}1&1&0&0&0\\1&0&1&0&0\\0&1&0&1&0\\0&0&1&1&1\\0&0&0&1&0\end{bmatrix} \text{ or } A_\gamma(G)=\begin{bmatrix}0&1&0&0&0\\1&1&1&0&0\\0&1&0&1&0\\0&0&1&1&1\\0&0&0&1&0\end{bmatrix}$$

whose polynomial are respectively

$$\begin{aligned}\kappa^5-2\kappa^4-3\kappa^3+5\kappa^2+2\kappa-1 &= (\kappa^2-\kappa-1)(\kappa^3-\kappa^2-3\kappa+1)\\ \kappa^5-2\kappa^4-3\kappa^3+4\kappa^2+3\kappa &= \kappa(\kappa^2-\kappa-3)(\kappa^2-\kappa-1).\end{aligned}$$

The domination matrix and the characteristic polynomial of P_6 are given by

$$A_\gamma(G)=\begin{bmatrix}0&1&0&0&0&0\\1&1&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&1&1\\0&0&0&0&1&0\end{bmatrix}$$

and

$$\kappa^6-2\kappa^5-4\kappa^4+6\kappa^3+5\kappa^2-2\kappa-1=(\kappa^3-3\kappa-1)(\kappa^3-2\kappa^2-\kappa+1).$$

Hence, we get the proof. □

Theorem 4.3 *Let $G=P_n$, $n\geqslant 3$. Then the exact $E_D(P_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of P_n for $n\geqslant 3$ directly. Label the vertices of P_n as $v_1,v_2,v_3,\cdots,v_n$.

The characteristic polynomial of P_n using distance matrix $D(G)$ is given by $\mu^n+q_1\mu^{n-1}+q_2\mu^{n-2}+\cdots+q_{n-1}\mu+q_n=0$.

The distance matrix and the characteristic polynomial of P_3 are given by

$$D(G)=\begin{bmatrix}0&1&2\\1&0&1\\2&1&0\end{bmatrix}$$

and $\mu^3-6\mu-4=(\mu+2)\left(\mu^2-2\mu-2\right)$.

The distance matrix and the characteristic polynomial of P_4 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^4 - 20\mu^2 - 32\mu - 12 = \left(\mu^2 - 4\mu - 6\right)\left(\mu^2 + 4\mu + 2\right).$

The distance matrix and the characteristic polynomial of P_5 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^5 - 50\mu^3 - 140\mu^2 - 120\mu - 32 = \left(\mu^2 + 6\mu + 4\right)\left(\mu^3 - 6\mu^2 - 18\mu - 8\right).$

The distance matrix and the characteristic polynomial of P_6 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 4 & 3 & 2 & 1 & 0 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^6 - 105\mu^4 - 448\mu^3 - 648\mu^2 - 384\mu - 80 = (\mu + 1)\left(\mu^2 + 8\mu + 4\right)\left(\mu^3 - 9\mu^2 - 36\mu - 20\right).$
Hence, we get the proof. $\square$

Theorem 4.4 *Let $G = P_n$, $n \geqslant 3$. Then the exact $E_{D\gamma}(P_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of P_n for $n \geqslant 3$ directly. Label the vertices of P_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of P_n using distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0.$$

The distance domination matrix and the characteristic polynomial of P_3 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^3 - \sigma^2 - 6\sigma = \sigma(\sigma+2)(\sigma-3)$.

The distance domination matrix and the characteristic polynomial of P_4 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}, \quad D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \quad \text{or} \quad D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 1 \end{bmatrix}$$

and

$$\begin{aligned} \sigma^4 - 2\sigma^3 - 19\sigma^2 - 12\sigma &= \left(\sigma^2 - 5\sigma - 3\right)\left(\sigma^2 + 3\sigma - 1\right), \\ \sigma^4 - 2\sigma^3 - 19\sigma^2 - 4\sigma + 3 &= \sigma(\sigma+3)\left(\sigma^2 - 5\sigma - 4\right), \\ \sigma^4 - 2\sigma^3 - 19\sigma^2 - 20\sigma - 5 &= \left(\sigma^2 - 5\sigma - 5\right)\left(\sigma^2 + 3\sigma + 1\right). \end{aligned}$$

The distance domination matrix and the characteristic polynomial of P_5 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix} \quad \text{or} \quad D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} &\sigma^5 - 2\sigma^4 - 49\sigma^3 - 70\sigma^2 = \sigma^2(\sigma+5)\left(\sigma^2 - 7\sigma - 14\right), \\ &\sigma^5 - 2\sigma^4 - 49\sigma^3 - 85\sigma^2 - 30\sigma. \end{aligned}$$

The distance domination matrix and the characteristic polynomial of P_6 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 4 & 3 & 2 & 1 & 1 & 1 \\ 5 & 4 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and

$$\sigma^6 - 2\sigma^5 - 104\sigma^4 - 300\sigma^3 - 180\sigma^2 = \sigma^2\left(\sigma^2 - 10\sigma - 30\right)\left(\sigma^2 + 8\sigma + 6\right).$$

Hence, we get the proof. □

Theorem 4.5 *Let $G = C_n$, $n \geqslant 3$. Then the exact $E(C_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of C_n for $n \geqslant 3$ directly. Label the vertices of C_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of adjacency matrix $A(G)$ is given by

$$\lambda^n + q_1\lambda^{n-1} + q_2\lambda^{n-2} + \cdots + q_{n-1}\lambda + q_n = 0.$$

The adjacency matrix and the characteristic polynomial of C_3 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\lambda^3 - 3\lambda - 2 = (\lambda - 2)(\lambda + 1)^2$.

The adjacency matrix and the characteristic polynomial of C_4 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^4 - 4\lambda^2 = \lambda^2(\lambda - 2)(\lambda + 2)$.

The adjacency matrix and the characteristic polynomial of C_5 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^5 - 5\lambda^3 + 5\lambda - 2 = (\lambda - 2)\,(\lambda^2 + \lambda - 1)^2$.

The adjacency matrix and the characteristic polynomial of C_6 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^6 - 6\lambda^4 + 9\lambda^2 - 4 = (\lambda - 2)\,(\lambda - 1)^2(\lambda + 1)^2(\lambda + 2)$.

The adjacency matrix and the characteristic polynomial of C_7 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^7 - 7\lambda^5 + 14\lambda^3 - 7\lambda - 2 = (\lambda - 2)(\lambda^3 + \lambda^2 - 2\lambda - 1)^2$. Hence, we get the proof. □

Theorem 4.6 *Let $G = C_n$, $n \geqslant 3$. Then the exact $E_\gamma(C_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of C_n for $n \geqslant 3$ directly. Label the vertices of C_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of domination matrix $A_\gamma(G)$ is given by

$$\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \cdots + q_{n-1}\kappa + q_n = 0.$$

The domination matrix and the characteristic polynomial of C_3 are given by

$$A_\gamma(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^3 - \kappa^2 - 3\kappa - 1 = (\kappa + 1)(\kappa^2 - 2\kappa - 1)$.

The domination matrix and the characteristic polynomial of C_4 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \text{or} \quad A_\gamma(G) = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\kappa^4 - 2\kappa^3 - 3\kappa^2 + 4\kappa = \kappa(\kappa-1)(\kappa^2-\kappa-4) \quad \text{or} \quad \kappa^4 - 2\kappa^3 - 3\kappa^2 + 4\kappa - 1.$$

The domination matrix and the characteristic polynomial of C_5 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^5 - 2\kappa^4 - 4\kappa^3 + 6\kappa^2 + 4\kappa - 4 = (\kappa^2-2)(\kappa^3 - 2\kappa^2 - 2\kappa + 2)$.

The domination matrix and the characteristic polynomial of C_6 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^6 - 2\kappa^5 - 5\kappa^4 + 8\kappa^3 + 7\kappa^2 - 6\kappa - 3 = (\kappa-1)(\kappa+1)\left(\kappa^2-3\right)(\kappa^2-2\kappa-1)$.

The domination matrix and the characteristic polynomial of C_7 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^7 - 3\kappa^6 - 4\kappa^5 + 14\kappa^4 + 5\kappa^3 - 17\kappa^2 - 3\kappa + 1 = (\kappa^3 - 3\kappa - 1)(\kappa^4 - 3\kappa^3 - \kappa^2 + 6\kappa - 1)$. Hence, we get the proof. □

Theorem 4.7 *Let $G = C_n$, $n \geqslant 3$. Then the exact $E_D(C_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of C_n for $n \geqslant 3$ directly. Label the vertices of C_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of P_n using distance matrix $D(G)$ is given by

$$\mu^n + q_1\mu^{n-1} + q_2\mu^{n-2} + \cdots + q_{n-1}\mu + q_n = 0.$$

The distance matrix and the characteristic polynomial of C_3 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\mu^3 - 3\mu - 2 = (\mu - 2)(\mu + 1)^2$.

The distance matrix and the characteristic polynomial of C_4 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^4 - 12\mu^2 - 16\mu = \mu\left(\mu - 4\right)\left(\mu + 2\right)^2$.

The distance matrix and the characteristic polynomial of C_5 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^5 - 25\mu^3 - 60\mu^2 - 35\mu - 6 = (\mu - 6)\left(\mu^2 + 3\mu + 1\right)^2$.

The distance matrix and the characteristic polynomial of C_6 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 3 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^6 - 56\mu^4 - 203\mu^3 - 190\mu^2 - 72\mu = \mu\,(\mu+4)\,(\mu-9)\left(\mu^3+5\mu^2+5\mu+2\right).$

The distance matrix and the characteristic polynomial of C_7 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^7 - 98\mu^5 - 490\mu^4 - 707\mu^3 - 434\mu^2 - 119\mu - 12 = (\mu - 12)\left(\mu^3 + 6\mu^2 + 5\mu + 1\right)^2$. Hence, we get the proof. □

Theorem 4.8 *Let $G = C_n$, $n \geqslant 3$. Then the exact $E_{D\gamma}(C_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of C_n for $n \geqslant 3$ directly. Label the vertices of C_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of P_n using distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0.$$

The distance domination matrix and the characteristic polynomial of C_3 are given by

$$D_\gamma(G) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and $\sigma^3 - \sigma^2 - 3\sigma - 1 = (\sigma + 1)(\sigma^2 - 2\sigma - 1)$.

The distance domination matrix and the characteristic polynomial of C_4 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^4 - 2\sigma^3 - 11\sigma^2 - 4\sigma + 4 = (\sigma+1)(\sigma+2)\left(\sigma^2 - 5\sigma + 2\right)$.

The distance domination matrix and the characteristic polynomial of C_5 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^5 - 2\sigma^4 - 24\sigma^3 - 30\sigma^2 + 4\sigma = \sigma(\sigma+2)\left(\sigma^3 - 4\sigma^2 - 16\sigma + 2\right)$.

The distance domination matrix and the characteristic polynomial of C_6 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 3 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and

$$\begin{aligned} &\sigma^6 - 2\sigma^5 - 55\sigma^4 - 129\sigma^3 - 12\sigma^2 + 38\sigma + 24 \\ &= (\sigma+4)\left(\sigma^2 - 10\sigma + 6\right)\left(\sigma^3 + 4\sigma^2 + 3\sigma + 1\right). \end{aligned}$$

The distance matrix and the characteristic polynomial of C_7 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 3 & 2 \\ 2 & 1 & 1 & 1 & 2 & 3 & 3 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 3 & 3 & 2 & 1 & 1 & 1 & 2 \\ 2 & 3 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

and

$$\sigma^7 - 3\sigma^6 - 95\sigma^5 - 281\sigma^4 - 10\sigma^3 + 60\sigma^2 + 8\sigma$$
$$= \sigma\left(\mu^2 + 5\sigma + 2\right)\left(\mu^4 - 8\mu^3 - 57\mu^2 + 20\mu + 4\right).$$

Hence, we get the proof. □

Theorem 4.9 *Let $G = W_n$, $n \geqslant 3$. Then the exact $E(W_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of W_n for $n \geqslant 3$ directly. Label the vertices of W_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of adjacency matrix $A(G)$ is given by

$$\lambda^n + q_1\lambda^{n-1} + q_2\lambda^{n-2} + \cdots + q_{n-1}\lambda + q_n = 0.$$

The adjacency matrix and the characteristic polynomial of W_4 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\lambda^4 - 6\lambda^2 - 8\lambda - 3 = (\lambda - 3)(\lambda + 1)^3$.

The adjacency matrix and the characteristic polynomial of W_5 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^5 - 8\lambda^3 - 8\lambda^2 = \lambda^2(\lambda + 2)(\lambda^2 - 2\lambda - 4)$.

The adjacency matrix and the characteristic polynomial of W_6 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\lambda^6 - 10\lambda^4 - 10\lambda^3 + 10\lambda^2 + 8\lambda - 5 = \left(\lambda^2 - 2\lambda - 5\right)\left(\lambda^2 + \lambda - 1\right)^2$.

The adjacency matrix and the characteristic polynomial of W_7 are given by

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and

$$\lambda^7 - 12\lambda^5 - 12\lambda^4 + 21\lambda^3 + 24\lambda^2 - 10\lambda - 12 = (\lambda - 1)^2(\lambda + 1)^2(\lambda + 2)(\lambda^2 - 2\lambda - 6).$$

Hence, we get the proof. □

Theorem 4.10 *Let $G = W_n$, $n \geqslant 3$. Then the exact $E_\gamma(W_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of W_n for $n \geqslant 3$ directly. Label the vertices of W_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of domination matrix $A_\gamma(G)$ is given by

$$\kappa^n + q_1\kappa^{n-1} + q_2\kappa^{n-2} + \cdots + q_{n-1}\kappa + q_n = 0.$$

The domination matrix and the characteristic polynomial of W_4 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\kappa^4 - \kappa^3 - 6\kappa^2 - 5\kappa - 1 = (\kappa + 1)^2(\kappa^2 - 3\kappa - 1)$.

The domination matrix and the characteristic polynomial of W_5 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^5 - \kappa^4 - 8\kappa^3 - 4\kappa^2 = \kappa^2(\kappa + 2)(\kappa^2 - 3\kappa - 2)$.

The domination matrix and the characteristic polynomial of W_6 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^6 - \kappa^5 - 10\kappa^4 - 5\kappa^3 + 10\kappa^2 + 3\kappa - 3 = (\kappa^2 - 3\kappa - 3)(\kappa^2 + \kappa - 1)^2$.

The domination matrix and the characteristic polynomial of W_7 are given by

$$A_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and $\kappa^7 - \kappa^6 - 12\kappa^5 - 6\kappa^4 + 21\kappa^3 + 15\kappa^2 - 10\kappa - 8 = (\kappa - 1)^2(\kappa + 1)^3(\kappa + 2)(\kappa + 4)$. Hence, we get the proof. □

Theorem 4.11 *Let $G = W_n$, $n \geqslant 3$. Then the exact $E_D(W_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of W_n for $n \geqslant 3$ directly. Label the vertices of W_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of W_n using distance matrix $D(G)$ is given by

$$\mu^n + q_1\mu^{n-1} + q_2\mu^{n-2} + \cdots + q_{n-1}\mu + q_n = 0.$$

The distance matrix and the characteristic polynomial of W_4 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\mu^4 - 6\mu^2 - \mu - 3 = (\mu - 3)(\mu + 1)^3$.

The distance matrix and the characteristic polynomial of W_5 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^5 - 16\mu^3 - 32\mu^2 - 16\mu = \mu\left(\mu+2\right)^2\left(\mu^2 - 4\mu - 4\right).$

The distance matrix and the characteristic polynomial of W_6 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^6 - 30\mu^4 - 90\mu^3 - 90\mu^2 - 36\mu - 5 = \left(\mu^2 - 6\mu - 5\right)\left(\mu^2 + 3\mu + 1\right)^2.$

The distance matrix and the characteristic polynomial of W_7 are given by

$$D(G) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\mu^7 - 48\mu^5 - 200\mu^4 - 315\mu^3 - 216\mu^2 - 54\mu = \mu\left(\mu+1\right)^2\left(\mu+3\right)^2\left(\mu^2 - 8\mu - 6\right)$. Hence, we get the proof. $\square$

Theorem 4.12 *Let $G = W_n$, $n \geqslant 3$. Then the exact $E_{D\gamma}(W_n)$ cannot be calculated as characteristic polynomial cannot be generalized.*

Proof Calculation does not enable one to find the characteristic polynomial of W_n for $n \geqslant 3$ directly. Label the vertices of W_n as $v_1, v_2, v_3, \cdots, v_n$.

The characteristic polynomial of W_n using distance domination matrix $D_\gamma(G)$ is given by

$$\sigma^n + q_1\sigma^{n-1} + q_2\sigma^{n-2} + \cdots + q_{n-1}\sigma + q_n = 0.$$

The distance domination matrix and the characteristic polynomial of W_4 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

and $\sigma^4 - \sigma^3 - 6\sigma^2 - 5\sigma - 1 = (\sigma+1)^2\left(\sigma^2 - 3\sigma - 1\right)$.

The distance domination matrix and the characteristic polynomial of W_5 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^5 - \sigma^4 - 16\sigma^3 - 20\sigma^2 = \sigma^2(\sigma - 5)(\sigma + 2)^2$.

The distance domination matrix and the characteristic polynomial of W_6 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^6 - \sigma^5 - 30\sigma^4 - 65\sigma^3 - 30\sigma^2 - \sigma + 1 = \left(\sigma^2 - 7\sigma + 1\right)\left(\sigma^2 + 3\sigma + 1\right)^2$.

The distance matrix and the characteristic polynomial of W_7 are given by

$$D_\gamma(G) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 0 & 1 & 2 & 2 \\ 1 & 2 & 2 & 1 & 0 & 1 & 2 \\ 1 & 2 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & 2 & 2 & 1 & 0 \end{bmatrix}$$

and $\sigma^7 - \sigma^6 - 48\sigma^5 - 158\sigma^4 - 163\sigma^3 - 33\sigma^2 + 18\sigma = \sigma(\sigma+1)^2(\sigma+3)^2\left(\mu^2 - 9\mu + 2\right)$. Hence, we get the proof. $\square$

§5. Open Problems

Problem 5.1 *Finding the characteristic polynomial for an arbitrary graph.*

Problem 5.2 *Find upper and lower bound for various kinds of energies with respect to different parameters of graph.*

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