International J.Math. Combin. Vol.4(2018), 38-44

Characteristic Properties of the Indicatrix Under a Kropina Change of Finsler Metric

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Abstract: The theory of $\beta-$change in Finsler geometry was first introduced by C. Shibata in [13]. In this paper, we study the behaviour of Indicatrices under a special $\beta-$change, known as Kropina change of Finsler metric.

Key Words: Indicatrix, $\beta-$change, Kropina change, curvature tensor, conformal flatness.

AMS(2010): 53B40, 53C60.

§1. Introduction

The notion of $\beta-$change in Finsler spaces was introduced by C. Shibata in [13]. Since then so many results have been obtained using this theory. In [1], S. H. Abed generalized the theory of $\beta-$change and introduced a new change, called conformal $\beta-$change. In differential geometry, the theory of indicatrices has been very interesting topic for geometers from all over the world for both pure mathematical and applied reasons. The theory of indicatries and its properties have been studied by so many authors (([7], [10], $\cdots$, [14]) In the present paper we study the behavior of the indicatrices given by a particular $\beta-$change, known as Kropina change.

This paper is organized as follows:

In the second section, we discuss the basic definitions and examples of some special Finsler spaces. In Section 3, we consider the Indicatrices given by a $\beta-$change, called Kropina change and study its properties in detail. The terminologies and notations are referred to Matsumoto's monograph [11] in this paper.

§2. Preliminaries

Let M be an $n-$ dimensional smooth manifold, T_xM, the tangent space at $x \in M$, and TM the tangent bundle, the disjoint union of tangent spaces, i.e.,

$$TM := \bigsqcup_{x\in M} T_xM.$$

[1]Received April 7, 2018, Accepted November 22, 2018.

The elements of TM are denoted by (x, y), where $x = (x^i) \in M$ and $y \in T_xM$, called supporting element. The slit tangent bundle TM_0 is defined as $TM \setminus \{0\}$.

A Finsler metric on a smooth manifold M is a function $F : TM \longrightarrow [0, \infty)$ satisfying the following properties:

(1) F is smooth on TM_0,

(2) F is positively $1-$homogeneous on the fibers of tangent bundle TM and

(3) the hession of F^2 with elements $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positively defined on TM.

A smooth manifold M equipped with the Finsler metric F is called Finsler manifold and the corresponding space, denoted by $F^n = (M, F)$ is called a Finsler space. F is called fundamental function and g_{ij} is called fundamental metric tensor of the Finsler space F^n. The normalized supporting element ℓ_i, angular metric tensor h_{ij}, and the metric tensor g_{ij} of F^n are defined respectively as:

$$\ell_i = \frac{\partial F}{\partial y^i}, \quad h_{ij} = \frac{\partial^2 F}{\partial y^i \partial y^j} \ \ \& \ g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}. \tag{2.1}$$

Finsler metrics were introduced in order to generalize the Riemannian ones in the sense that metric should not depend only on the point, but also on the direction. In Finsler geometry, (α, β) metrics, introduced in [12], form a very important and rich class of Finsler metrics which can be expressed in the form $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a $1-$form and ϕ is a positive smooth function on the domain of definition. The notable (α, β) metrics are Randers metric, Kropina metric, generalized Kropina metric, Z. Shen's square metric and Matsumoto metric. If $\phi(s) = 1 + s$, we get $F = \alpha + \beta$, called Randers metric. In particular, when $\phi(s) = \frac{1}{s}$, we get $F = \frac{\alpha^2}{\beta}$, called Kropina metric. Kropina metrics were induced by V. K. Kropina [8]. Kropina metrics seem to be among the simplest non-trivial Finsler metrics with many interesting applications in physics, electron optics with a magnatic field etc.([2], [3], [6]). Now we give some definitions and results that have been used in the next section.

Definition 2.1 *A Finsler space $F^n = (M, F)(n > 2)$ is called $P2-$like, if there exist a covariant vector field P_i such that the hv curvature tensor P_{hijk} of F^n can be written in the form*

$$P_{hijk} = P_h C_{ijk} - P_i C_{hjk}.$$

Let the Finsler space $F^n(n > 2)$ is $P2-$like. Then we have the result following.

Theorem 2.1([9]) *For a $P2-$like Finsler space $F^n = (M, F)(n > 2)$, the hv curvature tensor P_{hijk} vanishes, or the $v-$ curvature tensor S_{hijk} of F^n vanishes.*

Definition 2.2 *A Finsler space $F^n = (M, F)(n > 3)$ is called $R3-$like, if the third curvature tensor R_{hijk} of Cartan is expressible in the form $R_{hijk} = g_{hj}L_{ik} + g_{ik}L_{hj} - g_{hk}L_{ij} - g_{ij}L_{hk}$, where $L_{ik} = \frac{1}{n-2}\left(R_{ik} - \frac{r}{2}g_{ik}\right)$, $R_{hj} = R^{m}_{h\ jm}$ and $r = \frac{1}{n-1}R^m_m$.*

For the $(v)hv-$torsion tensor P_{hij} and the $(h)hv-$ torsion tensor C_{hij}, we define

$$^*P_{hij} = P_{hij} - \lambda C_{hij},$$

where the scalar λ is homogeneous of degree one with respect to y^i and is given by $\dfrac{P_iC^i}{C_jC^j}$ for $C_j \neq 0$.

Definition 2.3 *A Finsler space $F^n = (M, F)(n > 2)$ is called a $^*P-$Finsler space, if the torsion tensor $^*P_{hij} = 0$.*

Definition 2.4 *A Finsler space $F^n = (M, F)$ is called a Landsberg space, if the $(v)hv-$torsion tensor $P_{hij} = 0$.*

Definition 2.5([5]) *A non-Riemannian Finsler space $F^n = (M, F)(n > 4)$ is called $S4-$like, if the $v-$curvature tensor S_{hijk} is written in the form*

$$L^2 S_{hijk} = h_{hj}M_{ik} + h_{ik}M_{hj} - h_{hk}M_{ij} - h_{ij}M_{hk},$$

where M_{ij} is symmetric and indicatory tensor given by $M_{ij} = \dfrac{1}{n-3}\left[S_{ij} - \dfrac{Sh_{ij}}{2(n-2)}\right]$.

Theorem 2.2([15]) *Let $F^n = (M, F)(n > 4)$ be a $R3-$like (non-Landsberg) $^*P-$Finsler space. Then F^n is $S4-$like.*

Theorem 2.3([15]) *An $R3-$like Landsberg space $F^n = (M, F)(n > 3)$ is a Finsler space satisfying $S_{hijk} = 0$, or a Riemannian space of constant curvature.*

After some calculation, we find the following result.

Theorem 2.4([15]) *If a Finsler space $F^n = (M, F)(n > 4)$ is $S4-$like, then the Finsler space $\bar{F}^n = (M, \bar{F})$, obtained from F^n by a Kropina change, is also $S4-$like.*

§3. Indicatrices Given by a Kropina Change

Let $F^n = (M, F)$ be a Finsler space. For any $x \in M$, the tangent space T_xM is regarded as an $n-$ dimensional Riemannian space with the fundamental tensor $g_{ij}(x, y)$, where $x = (x^i)$ is fixed. In terms of the Cartan connection $C\Gamma$ of F^n, components C^i_{jk} of the $(h)hv-$torsion tensor are christoffel symbols of T_xM and the $v-$curvature tensor S^i_{hjk} is the Riemannian curvature tensor of T_xM. The indicatrix I_x at a point x is a hypersurface of the Riemannian space T_xM which is defined by the equation $F(x, y) = 1$, where x is fixed. Consequently, I_x is regarded as an $(n-1)-$dimensional Riemannian space.

Now, we consider a special $\beta-$change, called Kropina change, defined by

$$\bar{F} = \frac{F^2}{\beta} = f(F, \beta), \tag{3.1}$$

where $\beta = b_i(x)y^i$ is a non-zero $1-$form on M.

Differentiation of (3.1) with respect t0 F and β gives us the following relations:

$$\begin{aligned} f_1 &= \frac{\partial \bar{F}}{\partial F} = \frac{2F}{\beta},\ f_2 = \frac{\partial \bar{F}}{\partial \beta} = -\frac{F^2}{\beta^2}, \\ f_{11} &= \frac{\partial^2 \bar{F}}{\partial F^2} = \frac{2}{\beta},\ f_{22} = \frac{\partial^2 \bar{F}}{\partial \beta^2} = \frac{2F^2}{\beta^3},\ f_{12} = \frac{\partial^2 \bar{F}}{\partial \beta \partial F} = -\frac{2F}{\beta^2} \end{aligned} \tag{3.2}$$

$$\bar{F} = f_1 + f_2\beta = \frac{F^2}{\beta},\ Ff_{12} + \beta f_{22} = 0,\ Ff_{11} + \beta f_{12} = 0. \tag{3.3}$$

$$p = ff_1/F = \frac{2F^2}{\beta^2},\ q = ff_2 = -\frac{F^4}{\beta^3},\ q_{0} = ff_{22} = \frac{2F^4}{\beta^4}. \tag{3.4}$$

Further, $\bar{\ell}_i = \bar{F}_{y^i}$ gives

$$\bar{\ell}_i = f_1\ell_i + f_2 b_i = -\frac{F^2}{\beta^2}\left(b_i - \frac{2\beta}{F^2}y_i\right) \tag{3.5}$$

$$\bar{h}_{ij} = \bar{F}\dot{\partial}_i\dot{\partial}_j\bar{F} \ \text{ gives}$$

$$\bar{h}_{ij} = ph_{ij} + q_{0}m_i m_j = \frac{2F^2}{\beta^2}h_{ij} + \frac{2F^4}{\beta^4}m_i m_j, \quad m_i = b_i - \frac{\beta}{F^2}y_i. \tag{3.6}$$

Furthermore, we find

$$\begin{aligned} p_{0} = q_{0} + {f_2}^2 &= \frac{3F^4}{\beta^4},\ q_{-1} = ff_{12}/F = -\frac{2F^2}{\beta^3},\ p_{-1} = q_{-1} + pf_2/f = -\frac{4F^2}{\beta^3}, \\ q_{-2} &= \frac{f\left(f_{11} - f_1/F\right)}{F^2} = 0,\ p_{-2} = q_{-2} + p^2/f^2 = \frac{4}{\beta^2}. \end{aligned} \tag{3.7}$$

Notice that $\bar{g}_{ij} = \frac{1}{2}\left(\bar{F}^2\right)_{y^i y^j}$ gives

$$\begin{aligned} \bar{g}_{ij} &= pg_{ij} + p_{0}b_i b_j + p_{-1}\left(b_i y_j + b_j y_i\right) + p_{-2}y_i y_j \\ &= \frac{2F^2}{\beta^2}g_{ij} + \frac{3F^4}{\beta^4}b_i b_j - \frac{4F^2}{\beta^3}\left(b_i y_j + b_j y_i\right) + \frac{4}{\beta^2}y_i y_j. \end{aligned} \tag{3.8}$$

By the Kropina change $F_{ij} = \dfrac{h_{ij}}{F}$ is invariant under certain conditions, where $h_{ij} = g_{ij} - \ell_i\ell_j$ is the angular metric tensor.

From now on, we shall call a tensor which is invariant under the Kropina change a K-invariant tensor. For the v-curvature tensor S_{hijk}, putting

$$LS^*_{hijk} = S_{hijk} + \frac{1}{n-3}\,\mathfrak{U}_{jk}\ \left\{h_{ij}S_{hk} + h_{hk}S_{ij} - Sh_{ij}h_{hk}/\left(n-2\right)\right\}, \tag{3.9}$$

we find that S^*_{hijk} is K-invariant under certain restrictions, where we use the notation $\mathfrak{U}_{jk}$ to denote the interchange of indices j, k and subtraction.

For a $S4-$like Finsler space, we have the following result.

Theorem 3.1([14]) *Let $F^n = (M,F)(n > 4)$ be a $S4-$like Finsler space. Then the indicatrix I_x is conformally flat.*

Also, we can easily prove the result following.

Theorem 3.2) *A non-Riemannian Finsler space $F^n = (M,F)(n > 4)$ is $S4-$like if and only if the K-invariant tensor S^*_{hijk} vanishes.*

From equation (3.1), Theorems 2.1, 2.4, 3.1 and 3.2, we find the following result.

Theorem 3.3 *For a $P2-$like Finsler space $F^n = (M,F)(n > 4)$, the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a Kropina change is conformally flat provided that $P_{hijk} \neq 0$.*

From Theorems 2.2, 2.4 and 3.1, we immediately find the following theorem.

Theorem 3.4 *Let $F^n = (M,F)(n > 4)$, be a $R3-$ like (non-Landsberg) $^*P-$ Finsler space. Then the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a Kropina change, is conformally flat.*

From equation (3.1), Theorems 2.3, 2.4, 3.1 and 3.2, we immediately find the next result.

Theorem 3.5 *Let $F^n = (M,F)(n > 4)$, be an $R3-$ like Landsberg space. If F^n is not a Riemannian space of constant curvature, then the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a Kropina change, is conformally flat.*

Theorem 3.6([4]) *Let $F^n = (M,F)(n > 2)$, be a $^*P-$ Finsler space. If the $hv-$curvature tensor P_{hijk} is symmetric in $j\&k$, then $P_{hijk} = 0$, or the $v-$curvature tensor $S_{hijk} = 0$.*

Therefore, by, equation (3.1) and Theorems 2.1, 2.43.13.23.6 we immediately get the following conclusion.

Theorem 3.7 *Let $F^n = (M,F)(n > 2)$, be a $^*P-$ Finsler space. If the $hv-$curvature tensor P_{hijk} is symmetric in $j\&k$, then the indicatrix $\bar{I}_x$ of $\bar{F}^n$, obtained from F^n by a Kropina change, is conformally flat provided that $P_{hijk} \neq 0$.*

According to the $\beta-$ change of a Finsler metric, the $v-$curvature tensor S^i_{hjk} changes as follows ([13]):

$$\bar{S}^i_{hjk} = S^i_{hjk} + \mathfrak{U}_{jk}\ \left(C^i_{mk}V^m_{hj} - C^m_{hk}V^i_{mj} - V^i_{mk}V^m_{hj}\right), \quad V^h_{ij} = C^h_{ij} - \bar{C}^h_{ij}. \tag{3.10}$$

In case of Kropina change, from (3.2), we get a conclusion following.

Theorem 3.8 *Let $S^i_{hjk} = \mathfrak{U}_{jk}\ \left(C^m_{hk}V^i_{mj} + V^i_{mk}V^m_{hj} - C^i_{mk}V^m_{hj}\right)$. Then we get $\bar{S}^i_{hjk} = 0$, where*

$$\begin{aligned} V^h_{ij} &= Q^h\left(p\ C_{imj}b^m - p_{_{-1}}m_im_j\right) \\ &\quad -\frac{1}{2}\left(\frac{m^h}{p} - \nu\ Q^h\right)\left(p_{02}\ m_im_j + p_{_{-1}}h_{ij}\right) - \frac{p_{_{-1}}}{2p}\left(h^h_im_j + h^h_jm_i\right) \end{aligned}$$

and

$$Q^h = s_0 b^h + s_{-1} y^h,\ s_0 = \frac{\beta^2}{2b^2\ F^2},\ s_{-1} = -\frac{\beta^3}{b^2\ F^4},\ p = \frac{2F^2}{\beta^2},$$
$$p_{-1} = -\frac{4F^2}{\beta^3},\ \nu = b^2 - \frac{\beta^2}{F^2},\ p_{02} = \frac{\partial p_0}{\partial \beta} = -\frac{12F^4}{\beta^5}. \tag{3.11}$$

In [6], we have known the following result.

Theorem 3.9 *Let $F^n = (M, F)(n > 2)$, be a Finsler space. Then its $v-$curvature tensor S_{hijk} vanishes at a point x, if and only if the indicatrix I_x is of constant curvature* 1.

By Theorems (3.8) and (3.9), we get

Theorem 3.10 *Let $S^i_{hjk} = \mathfrak{U}_{jk}\left(C^m_{hk}V^i_{mj} + V^i_{mk}V^m_{hj} - C^i_{mk}V^m_{hj}\right)$. Then the indicarix $\bar{I}_x$ of $\bar{F}^n$, obtained from $F^n(n > 2)$ by a Kropina change, is of constant curvature* 1.

Acknowledgement

The author is very much thankful to the Central University of Punjab, Bathinda for providing finansial assistance to this research work via the RSM Grant (CUPB/CC/17/369).

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