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A Characterization of Directed Pathos Line Digraph of an Arborescence

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Abstract: For an arborescence A_r, *a directed pathos line digraph* $Q = DPL(A_r)$ has vertex set $V(Q) = A(A_r) \cup P(A_r)$, where $A(A_r)$ is the arc set and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b; Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j. The purpose of this note is to characterize $DPL(A_r)$, i.e., when is a digraph a directed pathos line digraph of an arborescence A_r and is A_r reconstructible from $DPL(A_r)$?

Key Words: Line digraph, complete bipartite subdigraph, directed pathos vertex.

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§1. Introduction

Notations and definitions not introduced here can be found in [2]. There are many digraph operators (or digraph valued functions) with which one can construct a new digraph from a given digraph, such as the line digraph, the total digraph, and their generalizations. One such a digraph operator is called *a directed pathos line digraph* of an arborescence.

The concept of *pathos* of a graph G was introduced by Harary [3] as a collection of minimum number of edge disjoint open paths whose union is G. The path number of a graph G is the number of paths in any pathos. The path number of a tree T equals k, where $2k$ is the number of odd degree vertices of T.

For a tree T with vertex set $V(T) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(T) = \{e_1, e_2, \cdots, e_{n-1}\}$, the authors in [4] gave the following definition. A *pathos line graph of* T, written $PL(T)$, is a graph whose vertices are the edges and paths of a pathos of T, with two vertices of $PL(T)$ adjacent whenever the corresponding edges of T have a vertex in common or the edge lies on the corresponding path of the pathos.

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The order and size of $PL(T)$ are $n+k-1$ and $\frac{1}{2}\sum_{i=1}^{n} d_i^2$, respectively, where k is the path number and d_i is the degree of vertices of T. The characterization of graphs whose $PL(T)$ are planar, outerplanar, and maximal outerplanar were presented. A necessary and sufficient condition for $PL(T)$ to be Eulerian was given. They also showed that for any tree T, $PL(T)$ is not minimally nonouterplanar.

See Figure.1 for an example of a tree T and its pathos line graph $PL(T)$.

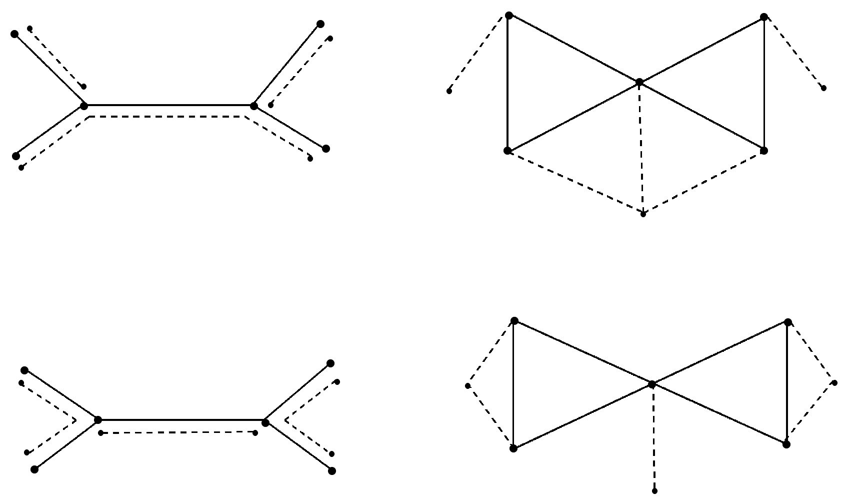

Figure 1

A *directed graph* (or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pair of distinct vertices called *arcs*. Here $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. For an arc (u,v) or uv of D the first vertex u is its *tail* and the second vertex v is its *head*. An *arborescence* is a directed graph in which, for a vertex u called the *root* and any other vertex v, there is exactly one directed path from u to v. We shall use A_r to denote an arborescence. A vertex with an in-degree (out-degree) zero is called a *source* (*sink*).

M. Aigner [1] defines the *line digraph* of a digraph as follows. Let D be a digraph with n vertices $v_1, v_2, \cdots, v_n$ and m arcs, and $L(D)$ its associated *line digraph* with $n^{'}$ vertices and $m^{'}$ arcs. We immediately have $n^{'}=m$ and $m^{'}=\sum_{i=1}^{n} d^-(v_i)\cdot d^+(v_i)$. Furthermore, the in-degree and out-degree of a vertex $v^{'}=(v_i,v_j)$ in $L(D)$ are $d^-(v^{'})=d^-(v_i)$ and $d^+(v^{'})=d^+(v_j)$, respectively. A digraph D is said to be a line digraph if it is isomorphic to the line digraph of a certain digraph H.

The authors in [5] extended the definition of a pathos line graph of a tree to an arborescence by introducing the concept of directed pathos line digraph of an arborescence and studied some of the characterization problems such planarity, outer planarity, etc. It is the object of this paper to discuss the problem of reconstructing an arborescence from its directed pathos line digraph.

§2. Definition of $DPL(A_r)$

Definition 2.1 *If a directed path $\vec{P}_n$ starts at one vertex and ends at a different vertex, then $\vec{P}_n$ is called an open directed path.*

Definition 2.2 *The directed pathos of an arborescence A_r is defined as a collection of minimum number of arc disjoint open directed paths whose union is A_r.*

Definition 2.3 *The directed path number k' of A_r is the number of directed paths in any directed pathos of A_r and is equal to the number of sinks in A_r, i.e., k' = number of sinks in A_r.*

Definition 2.4 *A directed pathos vertex is a vertex corresponding to a directed path of the directed pathos of A_r.*

Definition 2.5 *For an arborescence A_r, a directed pathos line digraph $Q = DPL(A_r)$ has vertex set $V(Q) = A(A_r) \cup P(A_r)$, where $A(A_r)$ is the arc set and $P(A_r)$ is a directed pathos set of A_r. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(A_r)$ and the head of a coincides with the tail of b; Pa such that $a \in A(A_r)$ and $P \in P(A_r)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(A_r)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j.*

Note that the directed path number k' of an arborescence A_r is minimum only when the out-degree of the root of A_r is one. Therefore, unless otherwise specified, the out-degree of the root of every arborescence is one. Finally, we assume that the direction of the directed pathos is along the direction of the arcs in A_r.

See Figure.2 for an example of an arborescence A_r and its directed pathos line digraph $DPL(A_r)$.

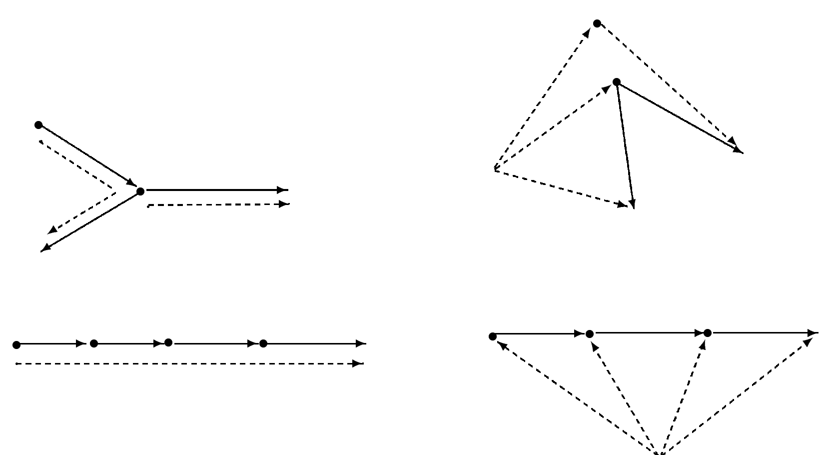

Figure 2

§3. A Criterion for Directed Pathos Line Digraphs

The main objective is to determine a necessary and sufficient condition that a digraph be a directed pathos line digraph.

A *complete bipartite digraph* is a directed graph D whose vertices can be partitioned into nonempty disjoint sets A and B such that each vertex of A has exactly one arc directed towards each vertex of B and such that D contains no other arc.

Theorem 3.1 *A digraph $A_r^{'}$ is a directed pathos line digraph of an arborescence A_r if and only if $V(A_r^{'}) = A(A_r) \cup P(A_r)$ and arc sets :*
(i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of A_r, respectively;
(ii) $\cup_{k=1}^{r} \cup_{j=1}^{r} P_k \times Z_j$ such that $P_k \times Z_j = \phi$ for $k \neq j$, where Z_j is the set of arcs on which P_k lies in A_r;
(iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} P_k \times Z_j^{'}$ such that $P_k \times Z_j^{'} = \phi$ for $k \neq j$, where $Z_j^{'}$ is the set of directed paths whose heads are reachable from the tail of P_k through a common vertex in A_r.

Proof Suppose that A_r is an arborescence with vertex set $V(A_r) = \{v_1, v_2, \cdots, v_n\}$ and a directed pathos set $P(A_r) = \{P_1, P_2, \cdots, P_r\}$. We consider the following three cases.

Case 1. Let v be a vertex of A_r with $d^-(v) = \alpha$ and $d^+(v) = \beta$. Then α arcs incident into v and the β arcs incident out of v give rise to a complete bipartite subdigraph with α tails and β heads and $\alpha \cdot \beta$ arcs joining each tail with each head.

Case 2. Let P_i be a directed path which lies on $\alpha^{'}$ arcs in A_r. Then $\alpha^{'}$ arcs give rise to a complete bipartite subdigraph with a single tail (i.e., P_i) and $\alpha^{'}$ heads and $\alpha^{'}$ arcs joining P_i with each head.

Case 3. Let P_i be a directed path, and let $\beta^{'}$ be the number of directed paths whose heads are reachable from the tail of P_i through a common vertex in A_r. Then $\beta^{'}$ arcs give rise to a complete bipartite subdigraph with a single tail (i.e., P_i) and $\beta^{'}$ heads and $\beta^{'}$ arcs joining P_i with each head.

Hence by all the above cases, $DPL(A_r)$ is decomposed into mutually arc-disjoint complete bipartite subdigraphs with vertex set $A(A_r) \cup P(A_r)$ and arc sets:

(i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of A_r, respectively;

(ii) $\cup_{k=1}^{r} \cup_{j=1}^{r} P_k \times Z_j$ such that $P_k \times Z_j = \phi$ for $k \neq j$, where Z_j is the set of arcs on which P_k lies in A_r;

(iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} P_k \times Z_j^{'}$ such that $P_k \times Z_j^{'} = \phi$ for $k \neq j$, where $Z_j^{'}$ is the set of directed paths whose heads are reachable from the tail of P_k through a common vertex in A_r.

Conversely, let $A_r^{'}$ be a digraph of the type described above. Let $t_1, t_2, \ldots, t_l$ be the vertices corresponding to complete bipartite subdigraphs $A_{r1}, A_{r2}, \cdots, A_{rl}$ of *Case* **1**, respectively, and let $t^1, t^2, \ldots, t^r$ be the vertices corresponding to complete bipartite subdigraphs $P_1^{'}, P_2^{'}, \cdots, P_r^{'}$ of *Case* **2**, respectively. Finally, let t_0 be a vertex chosen arbitrarily.

For each vertex v of the complete bipartite subdigraphs $A_{r1}, A_{r2}, \cdots, A_{rl}$, we draw an arc

a_v as follows.

(*i*) If $d^+(v) > 0$ and $d^-(v) = 0$, then $a_v := (t_0, t_i)$, where i is the base (or index) of A_{ri} such that $v \in Y_i$.

(*ii*) If $d^+(v) > 0$ and $d^-(v) > 0$, then $a_v := (t_i, t_j)$, where i and j are the indices of A_{ri} and A_{rj} such that $v \in X_j \cap Y_i$.

(c) If $d^+(v) = 0$ and $d^-(v) = 1$, then $a_v := (t_j, t^n)$, $n = 1, 2, \cdots, r$, where j is the base of A_{rj} such that $v \in X_j$.

Note that in (t_j, t^n) no matter what the value of j is, n varies from 1 to r such that the number of arcs of the form (t_j, t^n) is exactly r.

We now mark the directed pathos as follows. It is easy to observe that the directed path number $k^{'}$ equals the number of subdigraphs of Case 2. Let $\psi_1, \psi_2, \cdots, \psi_r$ be the number of heads of subdigraphs $P_1^{'}, P_2^{'}, \cdots, P_r^{'}$, respectively. Suppose we mark the directed path P_1. For this we choose any ψ_1 number of arcs and mark P_1 on ψ_1 arcs such that the direction of P_1 must be along the direction of ψ_1 arcs. Similarly, we choose ψ_2 number of arcs and mark P_2 on ψ_2 arcs. This process is repeated until all directed paths are marked. The digraph A_r with directed paths thus constructed apparently has $A_r^{'}$ as directed pathos line digraph. □

Given a directed pathos line digraph Q, the proof of the sufficiency of the Theorem above shows how to find an arborescence A_r such that $DPL(A_r) = Q$. This obviously raises the question of whether Q determines A_r uniquely. Although the answer to this in general is no, the extent to which A_r is determined is given as follows. One can easily check that using reconstruction procedure of the sufficiency of the Theorem above, any arborescence (without directed pathos) is uniquely reconstructed from its directed pathos line digraph. Since the pattern of directed pathos for an arborescence is not unique, there is freedom in marking the directed pathos for an arborescence in different ways. This clearly shows that if the directed path number is one, any arborescence with directed pathos is uniquely reconstructed from its directed pathos line digraph. It is known that the directed path is a special case of an arborescence. Since the directed path number of a directed path of order n $(n \geq 2)$ is exactly one, it follows that a directed path is uniquely reconstructed from its directed pathos line digraph.

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