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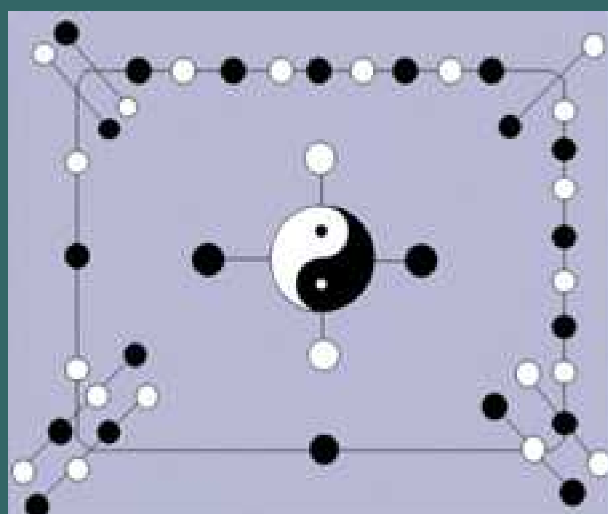

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The greatest lesson in life is to know that even fools are right sometimes.

By Winston Churchill, a British statesman.

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Direct Product of Multigroups and Its Generalization

P.A. Ejegwa

(Department of Mathematics/Statistics/Computer Science, University of Agriculture, P.M.B. 2373, Makurdi, Nigeria)

A. M. Ibrahim

(Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria)

E-mail: ejegwa.augustine@uam.edu.ng, amibrahim@abu.edu.ng

Abstract: This paper proposes the concept of direct product of multigroups and its generalization. Some results are obtained with reference to root sets and cuts of multigroups. We prove that the direct product of multigroups is a multigroup. Finally, we introduce the notion of homomorphism and explore some of its properties in the context of direct product of multigroups and its generalization.

Key Words: Multisets, multigroups, direct product of multigroups.

AMS(2010): 03E72, 06D72, 11E57, 19A22.

§1. Introduction

In set theory, repetition of objects are not allowed in a collection. This perspective rendered set almost irrelevant because many real life problems admit repetition. To remedy the handicap in the idea of sets, the concept of multiset was introduced in [10] as a generalization of set wherein objects repeat in a collection. Multiset is very promising in mathematics, computer science, website design, etc. See [14, 15] for details.

Since algebraic structures like groupoids, semigroups, monoids and groups were built from the idea of sets, it is then natural to introduce the algebraic notions of multiset. In [12], the term *multigroup* was proposed as a generalization of group in analogous to some non-classical groups such as fuzzy groups [13], intuitionistic fuzzy groups [3], etc. Although the term *multigroup* was earlier used in [4, 11] as an extension of group theory, it is only the idea of multigroup in [12] that captures multiset and relates to other non-classical groups. In fact, every multigroup is a multiset but the converse is not necessarily true and the concept of classical groups is a specialize multigroup with a unit count [5].

In furtherance of the study of multigroups, some properties of multigroups and the analogous of isomorphism theorems were presented in [2]. Subsequently, in [1], the idea of order of an element with respect to multigroup and some of its related properties were discussed. A complete account on the concept of multigroups from different algebraic perspectives was outlined in [8]. The notions of upper and lower cuts of multigroups were proposed and some of

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their algebraic properties were explicated in [5]. In continuation to the study of homomorphism in multigroup setting (cf. [2, 12]), some homomorphic properties of multigroups were explored in [6]. In [9], the notion of multigroup actions on multiset was proposed and some results were established. An extensive work on normal submultigroups and comultisets of a multigroup were presented in [7].

In this paper, we explicate the notion of direct product of multigroups and its generalization. Some homomorphic properties of direct product of multigroups are also presented. This paper is organized as follows; in Section 2, some preliminary definitions and results are presented to be used in the sequel. Section 3 introduces the concept of direct product between two multigroups and Section 4 considers the case of direct product of k^{th} multigroups. Meanwhile, Section 5 contains some homomorphic properties of direct product of multigroups.

§2. Preliminaries

Definition 2.1([14]) *Let $X = \{x_1, x_2, \cdots, x_n, \cdots\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$ denoted the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$.*

The set of all multisets over X is denoted by $MS(X)$.

Definition 2.2([15]) *Let $A, B \in MS(X)$, A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x)$ for $\forall x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. A multiset is called the parent in relation to its submultiset.*

Definition 2.3([12]) *Let X be a group. A multiset G is called a multigroup of X if it satisfies the following conditions:*

(i) $C_G(xy) \geq C_G(x) \wedge C_G(y) \forall x, y \in X$;
(ii) $C_G(x^{-1}) = C_G(x) \forall x \in X$,

where C_G denotes count function of G from X into a natural number $\mathbb{N}$ and $\wedge$ denotes minimum, respectively.

By implication, a multiset G is called a multigroup of a group X if

$$C_G(xy^{-1}) \geq C_G(x) \wedge C_G(y), \quad \forall x, y \in X.$$

It follows immediately from the definition that,

$$C_G(e) \geq C_G(x), \quad \forall x \in X,$$

where e is the identity element of X.

The count of an element in G is the number of occurrence of the element in G. While the order of G is the sum of the count of each of the elements in G, and is given by

$$|G| = \sum_{i=1}^{n} C_G(x_i), \quad \forall x_i \in X.$$

We denote the set of all multigroups of X by $MG(X)$.

Definition 2.4([5]) *Let $A \in MG(X)$. A nonempty submultiset B of A is called a submultigroup of A denoted by $B \sqsubseteq A$ if B form a multigroup. A submultigroup B of A is a proper submultigroup denoted by $B \sqsubset A$, if $B \sqsubseteq A$ and $A \neq B$.*

Definition 2.5([5]) *Let $A \in MG(X)$. Then the sets $A_{[n]}$ and $A_{(n)}$ defined as*

(i) $A_{[n]} = \{x \in X \mid C_A(x) \geq n, n \in \mathbb{N}\}$ and
(ii) $A_{(n)} = \{x \in X \mid C_A(x) > n, n \in \mathbb{N}\}$

are called strong upper cut and weak upper cut of A.

Definition 2.6([5]) *Let $A \in MG(X)$. Then the sets $A^{[n]}$ and $A^{(n)}$ defined as*

(i) $A^{[n]} = \{x \in X \mid C_A(x) \leq n, n \in \mathbb{N}\}$ and
(ii) $A^{(n)} = \{x \in X \mid C_A(x) < n, n \in \mathbb{N}\}$

are called strong lower cut and weak lower cut of A.

Definition 2.7([12]) *Let $A \in MG(X)$. Then the sets A_* and A^* are defined as*

(i) $A_* = \{x \in X \mid C_A(x) > 0\}$ and
(ii) $A^* = \{x \in X \mid C_A(x) = C_A(e)\}$, where e is the identity element of X.

Proposition 2.8([12]) *Let $A \in MG(X)$. Then A_* and A^* are subgroups of X.*

Theorem 2.9([5]) *Let $A \in MG(X)$. Then $A_{[n]}$ is a subgroup of X $\forall\, n \leq C_A(e)$ and $A^{[n]}$ is a subgroup of X $\forall\, n \geq C_A(e)$, where e is the identity element of X and $n \in \mathbb{N}$.*

Definition 2.10([7]) *Let $A, B \in MG(X)$ such that $A \subseteq B$. Then A is called a normal submultigroup of B if for all $x, y \in X$, it satisfies $C_A(xyx^{-1}) \geq C_A(y)$.*

Proposition 2.11([7]) *Let $A, B \in MG(X)$. Then the following statements are equivalent:*

(i) A is a normal submultigroup of B;
(ii) $C_A(xyx^{-1}) = C_A(y) \forall x, y \in X$;
(iii) $C_A(xy) = C_A(yx) \forall x, y \in X$.

Definition 2.12([7]) *Two multigroups A and B of X are conjugate to each other if for all $x, y \in X$, $C_A(x) = C_B(yxy^{-1})$ and $C_B(y) = C_A(xyx^{-1})$.*

Definition 2.13([6]) *Let X and Y be groups and let $f : X \to Y$ be a homomorphism. Suppose A and B are multigroups of X and Y, respectively. Then f induces a homomorphism from A to B which satisfies* (*i*) $C_A(f^{-1}(y_1y_2)) \geq C_A(f^{-1}(y_1)) \wedge C_A(f^{-1}(y_2)) \, \forall y_1, y_2 \in Y$;

(*ii*) $C_B(f(x_1x_2)) \geq C_B(f(x_1)) \wedge C_B(f(x_2)) \, \forall x_1, x_2 \in X$,

where

(*i*) *the image of A under f, denoted by* $f(A)$*, is a multiset of* Y *defined by*

$$C_{f(A)}(y) = \begin{cases} \bigvee_{x \in f^{-1}(y)} C_A(x), & f^{-1}(y) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for each $y \in Y$ *and*

(*ii*) *the inverse image of B under f, denoted by* $f^{-1}(B)$*, is a multiset of* X *defined by*

$$C_{f^{-1}(B)}(x) = C_B(f(x)) \, \forall x \in X.$$

Proposition 2.14([12]) *Let X and Y be groups and* $f : X \to Y$ *be a homomorphism. If* $A \in MG(X)$*, then* $f(A) \in MG(Y)$.

Corollary 2.15([12]) *Let X and Y be groups and* $f : X \to Y$ *be a homomorphism. If* $B \in MG(Y)$*, then* $f^{-1}(B) \in MG(X)$.

§3. Direct Product of Multigroups

Given two groups X and Y, the direct product, $X \times Y$ is the Cartesian product of ordered pair (x, y) such that $x \in X$ and $y \in Y$, and the group operation is component-wise, so

$$(x_1, y_1) \times (x_2, y_2) = (x_1x_2, y_1y_2).$$

The resulting algebraic structure satisfies the axioms for a group. Since the ordered pair (x, y) such that $x \in X$ and $y \in Y$ is an element of $X \times Y$, we simply write $(x, y) \in X \times Y$. In this section, we discuss the notion of direct product of two multigroups defined over X and Y, respectively.

Definition 3.1 *Let X and Y be groups,* $A \in MG(X)$ *and* $B \in MG(Y)$*, respectively. The direct product of A and B depicted by* $A \times B$ *is a function*

$$C_{A \times B} : X \times Y \to \mathbb{N}$$

defined by

$$C_{A \times B}((x, y)) = C_A(x) \wedge C_B(y) \forall x \in X, \forall y \in Y.$$

Example 3.2 Let $X = \{e, a\}$ be a group, where $a^2 = e$ and $Y = \{e', x, y, z\}$ be a Klein 4-group, where $x^2 = y^2 = z^2 = e'$. Then

$$A = [e^5, a]$$

and

$$B = [(e')^6, x^4, y^5, z^4]$$

are multigroups of X and Y by Definition 2.3. Now

$$X \times Y = \{(e, e'), (e, x), (e, y), (e, z), (a, e'), (a, x), (a, y), (a, z)\}$$

is a group such that

$$(e, x)^2 = (e, y)^2 = (e, z)^2 = (a, e')^2 = (a, x)^2 = (a, y)^2 = (a, z)^2 = (e, e')$$

is the identity element of $X \times Y$. Then using Definition 3.1,

$$A \times B = [(e, e')^5, (e, x)^4, (e, y)^5, (e, z)^4, (a, e'), (a, x), (a, y), (a, z)]$$

is a multigroup of $X \times Y$ satisfying the conditions in Definition 2.3.

Example 3.3 Let X and Y be groups as in Example 3.2. Let

$$A = [e^5, a^4]$$

and

$$B = [(e')^7, x^9, y^6, z^5]$$

be multisets of X and Y, respectively. Then

$$A \times B = [(e, e')^5, (e, x)^5, (e, y)^5, (e, z)^5, (a, e')^4, (a, x)^4, (a, y)^4, (a, z)^4].$$

By Definition 2.3, it follows that $A \times B$ is a multigroup of $X \times Y$ although B is not a multigroup of Y while A is a multigroup of X.

From the notion of direct product in multigroup context, we observe that

$$|A \times B| < |A||B|$$

unlike in classical group where $|X \times Y| = |X||Y|$.

Theorem 3.4 *Let $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then for all $n \in \mathbb{N}$, $(A \times B)_{[n]} = A_{[n]} \times B_{[n]}$.*

Proof Let $(x, y) \in (A \times B)_{[n]}$. Using Definition 2.5, we have

$$C_{A \times B}((x, y)) = (C_A(x) \wedge C_B(y)) \geq n.$$

This implies that $C_A(x) \geq n$ and $C_B(y) \geq n$, then $x \in A_{[n]}$ and $y \in B_{[n]}$. Thus,

$$(x, y) \in A_{[n]} \times B_{[n]}.$$

Also, let $(x,y) \in A_{[n]} \times B_{[n]}$. Then $C_A(x) \geq n$ and $C_B(y) \geq n$. That is,

$$(C_A(x) \wedge C_B(y)) \geq n.$$

This yields us $(x,y) \in (A \times B)_{[n]}$. Therefore, $(A \times B)_{[n]} = A_{[n]} \times B_{[n]}$ $\forall n \in \mathbb{N}$. □

Corollary 3.5 *Let $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then for all $n \in \mathbb{N}$, $(A \times B)^{[n]} = A^{[n]} \times B^{[n]}$.*

Proof Straightforward from Theorem 3.4. □

Corollary 3.6 *Let $A \in MG(X)$ and $B \in MG(Y)$, respectively. Then*

(i) $(A \times B)_* = A_* \times B_*$;
(ii) $(A \times B)^* = A^* \times B^*$.

Proof Straightforward from Theorem 3.4. □

Theorem 3.7 *Let A and B be multigroups of X and Y, respectively, then $A \times B$ is a multigroup of $X \times Y$.*

Proof Let $(x,y) \in X \times Y$ and let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. We have

$$\begin{aligned} C_{A\times B}(xy) &= C_{A\times B}((x_1,x_2)(y_1,y_2)) \\ &= C_{A\times B}((x_1y_1, x_2y_2)) \\ &= C_A(x_1y_1) \wedge C_B(x_2y_2) \\ &\geq \wedge(C_A(x_1) \wedge C_A(y_1), C_B(x_2) \wedge C_B(y_2)) \\ &= \wedge(C_A(x_1) \wedge C_B(x_2), C_A(y_1) \wedge C_B(y_2)) \\ &= C_{A\times B}((x_1,x_2)) \wedge C_{A\times B}((y_1,y_2)) \\ &= C_{A\times B}(x) \wedge C_{A\times B}(y). \end{aligned}$$

Also,

$$\begin{aligned} C_{A\times B}(x^{-1}) &= C_{A\times B}((x_1,x_2)^{-1}) = C_{A\times B}((x_1^{-1}, x_2^{-1})) \\ &= C_A(x_1^{-1}) \wedge C_B(x_2^{-1}) = C_A(x_1) \wedge C_B(x_2) \\ &= C_{A\times B}((x_1,x_2)) = C_{A\times B}(x). \end{aligned}$$

Hence, $A \times B \in MG(X \times Y)$. □

Corollary 3.8 *Let $A_1, B_1 \in MG(X_1)$ and $A_2, B_2 \in MG(X_2)$, respectively such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$. If A_1 and A_2 are normal submultigroups of B_1 and B_2, then $A_1 \times A_2$ is a normal submultigroup of $B_1 \times B_2$.*

Proof By Theorem 3.7, $A_1 \times A_2$ is a multigroup of $X_1 \times X_2$. Also, $B_1 \times B_2$ is a multigroup of $X_1 \times X_2$. We show that $A_1 \times A_2$ is a normal submultigroup of $B_1 \times B_2$. Let $(x,y) \in X_1 \times X_2$

such that $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then we get

$$\begin{aligned} C_{A_1 \times A_2}(xy) &= C_{A_1 \times A_2}((x_1, x_2)(y_1, y_2)) \\ &= C_{A_1 \times A_2}((x_1 y_1, x_2 y_2)) \\ &= C_{A_1}(x_1 y_1) \wedge C_{A_2}(x_2 y_2) \\ &= C_{A_1}(y_1 x_1) \wedge C_{A_2}(y_2 x_2) \\ &= C_{A_1 \times A_2}((y_1 x_1, y_2 x_2)) \\ &= C_{A_1 \times A_2}((y_1, y_2)(x_1, x_2)) \\ &= C_{A_1 \times A_2}(yx). \end{aligned}$$

Hence $A_1 \times A_2$ is a normal submultigroup of $B_1 \times B_2$ by Proposition 2.11. □

Theorem 3.9 *Let A and B be multigroups of X and Y, respectively. Then*

(i) $(A \times B)_*$ *is a subgroup of* $X \times Y$*;*
(ii) $(A \times B)^*$ *is a subgroup of* $X \times Y$*;*
(iii) $(A \times B)_{[n]}, n \in \mathbb{N}$ *is a subgroup of* $X \times Y$, $\forall\, n \leq C_{A \times B}(e, e')$*;*
(iv) $(A \times B)^{[n]}, n \in \mathbb{N}$ *is a subgroup of* $X \times Y$, $\forall\, n \geq C_{A \times B}(e, e')$.

Proof Combining Proposition 2.8, Theorem 2.9 and Theorem 3.7, the results follow. □

Corollary 3.10 *Let $A, C \in MG(X)$ such that $A \subseteq C$ and $B, D \in MG(Y)$ such that $B \subseteq D$, respectively. If A and B are normal, then*

(i) $(A \times B)_*$ *is a normal subgroup of* $(C \times D)_*$*;*
(ii) $(A \times B)^*$ *is a normal subgroup of* $(C \times D)^*$*;*
(iii) $(A \times B)_{[n]}, n \in \mathbb{N}$ *is a normal subgroup of* $(C \times D)_{[n]}$, $\forall\, n \leq C_{A \times B}(e, e')$*;*
(iv) $(A \times B)^{[n]}, n \in \mathbb{N}$ *is a normal subgroup of* $(C \times D)^{[n]}$, $\forall\, n \geq C_{A \times B}(e, e')$.

Proof Combining Proposition 2.8, Theorem 2.9, Theorem 3.7 and Corollary 3.8, the results follow. □

Proposition 3.11 *Let $A \in MG(X)$, $B \in MG(Y)$ and $A \times B \in MG(X \times Y)$. Then $\forall (x, y) \in X \times Y$, we have*

(i) $C_{A \times B}((x^{-1}, y^{-1})) = C_{A \times B}((x, y))$*;*
(ii) $C_{A \times B}((e, e')) \geq C_{A \times B}((x, y))$*;*
(iii) $C_{A \times B}((x, y)^n) \geq C_{A \times B}((x, y))$, *where e and e' are the identity elements of X and Y, respectively and $n \in \mathbb{N}$.*

Proof For $x \in X$, $y \in Y$ and $(x, y) \in X \times Y$, we get

(i) $C_{A \times B}((x^{-1}, y^{-1})) = C_A(x^{-1}) \wedge C_B(y^{-1}) = C_A(x) \wedge C_B(y) = C_{A \times B}((x, y))$.
Clearly, $C_{A \times B}((x^{-1}, y^{-1})) = C_{A \times B}((x, y))\ \forall (x, y) \in X \times Y$.

(*ii*)

$$\begin{aligned} C_{A\times B}((e,e')) &= C_{A\times B}((x,y)(x^{-1},y^{-1})) \\ &\geq C_{A\times B}((x,y)) \wedge C_{A\times B}((x^{-1},y^{-1})) \\ &= C_{A\times B}((x,y)) \wedge C_{A\times B}((x,y)) \\ &= C_{A\times B}((x,y))\ \forall (x,y) \in X \times Y. \end{aligned}$$

Hence, $C_{A\times B}((e,e')) \geq C_{A\times B}((x,y))$.

(*iii*)

$$\begin{aligned} C_{A\times B}((x,y)^n) &= C_{A\times B}((x^n,y^n)) \\ &= C_{A\times B}((x^{n-1},y^{n-1})(x,y)) \\ &\geq C_{A\times B}((x^{n-1},y^{n-1})) \wedge C_{A\times B}((x,y)) \\ &\geq C_{A\times B}((x^{n-2},y^{n-2})) \wedge C_{A\times B}((x,y)) \wedge C_{A\times B}((x,y)) \\ &\geq C_{A\times B}((x,y)) \wedge C_{A\times B}((x,y)) \wedge ... \wedge C_{A\times B}((x,y)) \\ &= C_{A\times B}((x,y)), \end{aligned}$$

which implies that $C_{A\times B}((x,y)^n) = C_{A\times B}((x^n,y^n)) \geq C_{A\times B}((x,y))\ \forall (x,y) \in X \times Y$. □

Theorem 3.12 *Let A and B be multisets of groups X and Y, respectively. Suppose that e and e' are the identity elements of X and Y, respectively. If $A \times B$ is a multigroup of $X \times Y$, then at least one of the following statements hold.*

(i) $C_B(e') \geq C_A(x)\ \forall x \in X$;

(ii) $C_A(e) \geq C_B(y)\ \forall y \in Y$.

Proof Let $A \times B \in MG(X \times Y)$. By contrapositive, suppose that none of the statements holds. Then suppose we can find a in X and b in Y such that

$$C_A(a) > C_B(e') \text{ and } C_B(b) > C_A(e).$$

From these we have

$$\begin{aligned} C_{A\times B}((a,b)) &= C_A(a) \wedge C_B(b) \\ &> C_A(e) \wedge C_B(e') \\ &= C_{A\times B}((e,e')). \end{aligned}$$

Thus, $A \times B$ is not a multigroup of $X \times Y$ by Proposition 3.11. Hence, either $C_B(e') \geq C_A(x)\ \forall x \in X$ or $C_A(e) \geq C_B(y)\ \forall y \in Y$. This completes the proof. □

Theorem 3.13 *Let A and B be multisets of groups X and Y, respectively, such that $C_A(x) \leq C_B(e')\ \forall x \in X$, e' being the identity element of Y. If $A \times B$ is a multigroup of $X \times Y$, then A is a multigroup of X.*

Proof Let $A \times B$ be a multigroup of $X \times Y$ and $x, y \in X$. Then $(x, e'), (y, e') \in X \times Y$. Now, using the property $C_A(x) \leq C_B(e')\,\forall x \in X$, we get

$$\begin{aligned} C_A(xy) &= C_A(xy) \wedge C_B(e'e') \\ &= C_{A\times B}((x,e')(y,e')) \\ &\geq C_{A\times B}((x,e')) \wedge C_{A\times B}((y,e')) \\ &= \wedge(C_A(x) \wedge C_B(e'), C_A(y) \wedge C_B(e')) \\ &= C_A(x) \wedge C_A(y). \end{aligned}$$

Also,

$$\begin{aligned} C_A(x^{-1}) &= C_A(x^{-1}) \wedge C_B(e'^{-1}) = C_{A\times B}((x^{-1}, e'^{-1})) \\ &= C_{A\times B}((x,e')^{-1}) = C_{A\times B}((x,e')) \\ &= C_A(x) \wedge C_B(e') = C_A(x). \end{aligned}$$

Hence, A is a multigroup of X. This completes the proof. □

Theorem 3.14 *Let A and B be multisets of groups X and Y, respectively, such that $C_B(x) \leq C_A(e)\,\forall x \in Y$, e being the identity element of X. If $A \times B$ is a multigroup of $X \times Y$, then B is a multigroup of Y.*

Proof Similar to Theorem 3.13. □

Corollary 3.15 *Let A and B be multisets of groups X and Y, respectively. If $A \times B$ is a multigroup of $X \times Y$, then either A is a multigroup of X or B is a multigroup of Y.*

Proof Combining Theorems 3.12 − 3.14, the result follows. □

Theorem 3.16 *If A and C are conjugate multigroups of a group X, and B and D are conjugate multigroups of a group Y. Then $A \times B \in MG(X \times Y)$ is a conjugate of $C \times D \in MG(X \times Y)$.*

Proof Since A and C are conjugate, it implies that for $g_1 \in X$, we have

$$C_A(x) = C_C(g_1^{-1} x g_1)\,\forall x \in X.$$

Also, since B and D are conjugate, for $g_2 \in Y$, we get

$$C_B(y) = C_D(g_2^{-1} y g_2)\,\forall y \in Y.$$

Now,

$$\begin{aligned} C_{A\times B}((x,y)) = C_A(x) \wedge C_B(y) &= C_C(g_1^{-1}xg_1) \wedge C_D(g_2^{-1}yg_2) \\ &= C_{C\times D}((g_1^{-1}xg_1),(g_2^{-1}yg_2)) \\ &= C_{C\times D}((g_1^{-1},g_2^{-1})(x,y)(g_1,g_2)) \\ &= C_{C\times D}((g_1,g_2)^{-1}(x,y)(g_1,g_2)). \end{aligned}$$

Hence, $C_{A\times B}((x,y)) = C_{C\times D}((g_1,g_2)^{-1}(x,y)(g_1,g_2))$. This completes the proof. □

§4. Generalized Direct Product of Multigroups

In this section, we defined direct product of k^{th} multigroups and obtain some results which generalized the results in Section 3.

Definition 4.1 *Let $A_1, A_2, \cdots, A_k$ be multigroups of $X_1, X_2, \cdots, X_k$, respectively. Then the direct product of $A_1, A_2, \cdots, A_k$ is a function*

$$C_{A_1\times A_2\times\cdots\times A_k} : X_1 \times X_2 \times \cdots \times X_k \to \mathbb{N}$$

defined by

$$C_{A_1\times A_2\times\cdots\times A_k}(x) = C_{A_1}(x_1) \wedge C_{A_2}(x_2) \wedge \cdots \wedge C_{A_{k-1}}(x_{k-1}) \wedge C_{A_k}(x_k)$$

where $x = (x_1, x_2, \cdots, x_{k-1}, x_k)$, $\forall x_1 \in X_1, \forall x_2 \in X_2, \cdots, \forall x_k \in X_k$. If we denote $A_1, A_2, \cdots, A_k$ by $A_i, (i \in I), X_1, X_2, \cdots, X_k$ by $X_i, (i \in I), A_1\times A_2\times\cdots\times A_k$ by $\prod_{i=1}^k A_i$ and $X_1\times X_2\times\cdots\times X_k$ by $\prod_{i=1}^k X_i$. Then the direct product of A_i is a function

$$C_{\prod_{i=1}^k A_i} : \prod_{i=1}^k X_i \to \mathbb{N}$$

defined by

$$C_{\prod_{i=1}^k A_i}((x_i)_{i\in I}) = \wedge_{i\in I} C_{A_i}((x_i))\ \forall x_i \in X_i,\ I = 1, \cdots, k.$$

Unless otherwise specified, it is assumed that X_i is a group with identity e_i for all $i \in I$, $X = \prod_{i\in I}^k X_i$, and so $e = (e_i)_{i\in I}$.

Theorem 4.2 *Let $A_1, A_2, \cdots, A_k$ be multisets of the sets $X_1, X_2, \cdots, X_k$, respectively and let $n \in \mathbb{N}$. Then*

$$(A_1 \times A_2 \times \cdots \times A_k)_{[n]} = A_{1[n]} \times A_{2[n]} \times \cdots \times A_{k[n]}.$$

Proof Let $(x_1, x_2, \cdots, x_k) \in (A_1 \times A_2 \times \cdots \times A_k)_{[n]}$. From Definition 2.5, we have

$$C_{A_1\times A_2\times\cdots\times A_k}((x_1, x_2, \cdots, x_k)) = (C_{A_1}(x_1) \wedge C_{A_2}(x_2) \wedge \cdots \wedge C_{A_k}(x_k)) \geq n.$$

This implies that $C_{A_1}(x_1) \geq n$, $C_{A_2}(x_2) \geq n, \cdots, C_{A_k}(x_k) \geq n$ and $x_1 \in A_{1[n]}$, $x_2 \in A_{2[n]}, \cdots, x_k \in A_{k[n]}$. Thus, $(x_1, x_2, \cdots, x_k) \in A_{1[n]} \times A_{2[n]} \times \cdots \times A_{k[n]}$.

Again, let $(x_1, x_2, \cdots, x_k) \in A_{1[n]} \times A_{2[n]} \times \cdots \times A_{k[n]}$. Then $x_i \in A_{i[n]}$, for $i = 1, 2, \cdots, k$, $C_{A_1}(x_1) \geq n$, $C_{A_2}(x_2) \geq n, \cdots, C_{A_k}(x_k) \geq n$. That is,

$$(C_{A_1}(x_1) \wedge C_{A_2}(x_2) \wedge \cdots \wedge C_{A_k}(x_k)) \geq n.$$

Implies that

$$(x_1, x_2, \cdots, x_k) \in (A_1 \times A_2 \times ... \times A_k)_{[n]}.$$

Hence, $(A_1 \times A_2 \times \cdots \times A_k)_{[n]} = A_{1[n]} \times A_{2[n]} \times \cdots \times A_{k[n]}$. □

Corollary 4.3 *Let $A_1, A_2, \cdots, A_k$ be multisets of the sets $X_1, X_2, \cdots, X_k$, respectively and let $n \in \mathbb{N}$. Then*

(i) $(A_1 \times A_2 \times \cdots \times A_k)^{[n]} = A_1^{[n]} \times A_2^{[n]} \times \cdots \times A_k^{[n]}$;
(ii) $(A_1 \times A_2 \times \cdots \times A_k)^* = A_1^* \times A_2^* \times \cdots \times A_k^*$;
(iii) $(A_1 \times A_2 \times \cdots \times A_k)_* = A_{1*} \times A_{2*} \times \cdots \times A_{k*}$.

Proof Straightforward from Theorem 4.2. □

Theorem 4.4 *Let $A_1, A_2, \cdots, A_k$ be multigroups of the groups $X_1, X_2, \cdots, X_k$, respectively. Then $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$.*

Proof Let $(x_1, x_2, \cdots, x_k), (y_1, y_2, \cdots, y_k) \in X_1 \times X_2 \times \cdots \times X_k$. We get

$$\begin{aligned}
& C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k)(y_1, \cdots, y_k)) \\
&= C_{A_1 \times \cdots \times A_k}((x_1 y_1, \cdots, x_k y_k)) \\
&= C_{A_1}(x_1 y_1) \wedge \cdots \wedge C_{A_k}(x_k y_k) \\
&\geq (C_{A_1}(x_1) \wedge C_{A_1}(y_1)) \wedge \cdots \wedge (C_{A_k}(x_k) \wedge C_{A_k}(y_k)) \\
&= \wedge(\wedge(C_{A_1}(x_1), C_{A_1}(y_1)), \cdots, \wedge(C_{A_k}(x_k), C_{A_k}(y_k)) \\
&= \wedge(\wedge(C_{A_1}(x_1), \cdots, C_{A_k}(x_k)), \wedge(C_{A_1}(y_1), \cdots, C_{A_k}(y_k))) \\
&= C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k)) \wedge C_{A_1 \times \cdots \times A_k}((y_1, \cdots, y_k)).
\end{aligned}$$

Also,

$$\begin{aligned}
C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k)^{-1}) &= C_{A_1 \times \cdots \times A_k}((x_1^{-1}, \cdots, x_k^{-1})) \\
&= C_{A_1}(x_1^{-1}) \wedge \cdots \wedge C_{A_k}(x_k^{-1}) \\
&= C_{A_1}(x_1) \wedge \cdots \wedge C_{A_k}(x_k) \\
&= C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k))
\end{aligned}$$

Hence, $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$. □

Corollary 4.5 *Let $A_1, A_2, \cdots, A_k$ and $B_1, B_2, \cdots, B_k$ be multigroups of $X_1, X_2, \cdots, X_k$, re-*

spectively, such that $A_1, A_2, \cdots, A_k \subseteq B_1, B_2, \cdots, B_k$. *If* $A_1, A_2, \cdots, A_k$ *are normal submultigroups of* $B_1, B_2, \cdots, B_k$, *then* $A_1 \times A_2 \times \cdots \times A_k$ *is a normal submultigroup of* $B_1 \times B_2 \times \cdots \times B_k$.

Proof By Theorem 4.4, $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1, X_2, \cdots, X_k$. Also, $B_1 \times B_2 \times \cdots \times B_k$ is a multigroup of $X_1, X_2, \cdots, X_k$.

Let $(x_1, x_2, \cdots, x_k), (y_1, y_2, \cdots, y_k) \in X_1 \times X_2 \times \cdots \times X_k$. Then we get

$$\begin{aligned} C_{A_1 \times \cdots \times A_k}((x_1, \cdots, x_k)(y_1, \cdots, y_k)) &= C_{A_1 \times \cdots \times A_k}((x_1 y_1, \cdots, x_k y_k)) \\ &= C_{A_1}(x_1 y_1) \wedge \cdots \wedge C_{A_k}(x_k y_k) \\ &= C_{A_1}(y_1 x_1) \wedge \cdots \wedge C_{A_k}(y_k x_k) \\ &= C_{A_1 \times \cdots \times A_k}((y_1 x_1, \cdots, y_k x_k)) \\ &= C_{A_1 \times \cdots \times A_k}((y_1, \cdots, y_k)(x_1, \cdots, x_k)). \end{aligned}$$

Thus, $A_1 \times \cdots \times A_k$ is a normal submultigroup of $B_1 \times \cdots \times B_k$ by Proposition 2.11. □

Theorem 4.6 *If* $A_1, A_2, \cdots, A_k$ *are multigroups of* $X_1, X_2, \cdots, X_k$, *respectively, then*

(i) $(A_1 \times A_2 \times \cdots \times A_k)_*$ *is a subgroup of* $X_1 \times X_2 \times \cdots \times X_k$;

(ii) $(A_1 \times A_2 \times \cdots \times A_k)^*$ *is a subgroup of* $X_1 \times X_2 \times \cdots \times X_k$;

(iii) $(A_1 \times A_2 \times \cdots \times A_k)_{[n]}, n \in \mathbb{N}$ *is a subgroup of* $X_1 \times X_2 \times \cdots \times X_k$, $\forall\, n \leq C_{A_1}(e_1) \wedge C_{A_2}(e_2) \wedge \cdots \wedge C_{A_k}(e_k)$;

(iv) $(A_1 \times A_2 \times \cdots \times A_k)^{[n]}, n \in \mathbb{N}$ *is a subgroup of* $X_1 \times X_2 \times \cdots \times X_k$, $\forall\, n \geq C_{A_1}(e_1) \wedge C_{A_2}(e_2) \wedge \cdots \wedge C_{A_k}(e_k)$.

Proof Combining Proposition 2.8, Theorem 2.9 and Theorem 4.4, the results follow. □

Corollary 4.7 *Let* $A_1, A_2, \cdots, A_k$ *and* $B_1, B_2, \cdots, B_k$ *be multigroups of* $X_1, X_2, \cdots, X_k$ *such that* $A_1, A_2, \cdots, A_k \subseteq B_1, B_2, \cdots, B_k$. *If* $A_1, A_2, \cdots, A_k$ *are normal submultigroups of* $B_1, B_2, \cdots, B_k$, *then*

(i) $(A_1 \times A_2 \times \cdots \times A_k)_*$ *is a normal subgroup of* $(B_1 \times B_2 \times \cdots \times B_k)_*$;

(ii) $(A_1 \times A_2 \times \cdots \times A_k)^*$ *is a normal subgroup of* $(B_1 \times B_2 \times \cdots \times B_k)^*$;

(iii) $(A_1 \times A_2 \times \cdots \times A_k)_{[n]}, n \in \mathbb{N}$ *is a normal subgroup of* $(B_1 \times B_2 \times \cdots \times B_k)_{[n]}$, $\forall\, n \leq C_{A_1}(e_1) \wedge C_{A_2}(e_2) \wedge \cdots \wedge C_{A_k}(e_k)$;

(iv) $(A_1 \times A_2 \times \cdots \times A_k)^{[n]}, n \in \mathbb{N}$ *is a normal subgroup of* $(B_1 \times B_2 \times \cdots \times B_k)^{[n]}$, $\forall\, n \geq C_{A_1}(e_1) \wedge C_{A_2}(e_2) \wedge \cdots \wedge C_{A_k}(e_k)$.

Proof Combining Proposition 2.8, Theorem 2.9, Theorem 4.4 and Corollary 4.5, the results follow. □

Theorem 4.8 *Let* $A_1, A_2, \cdots, A_k$ *and* $B_1, B_2, \cdots, B_k$ *be multigroups of groups* $X_1, X_2, \cdots, X_k$, *respectively. If* $A_1, A_2, \cdots, A_k$ *are conjugate to* $B_1, B_2, \cdots, B_k$, *then the multigroup* $A_1 \times A_2 \times \cdots \times A_k$ *of* $X_1 \times X_2 \times \cdots \times X_k$ *is conjugate to the multigroup* $B_1 \times B_2 \times \cdots \times B_k$ *of* $X_1 \times X_2 \times \cdots \times X_k$.

Proof By Definition 2.12, if multigroup A_i of X_i conjugates to multigroup B_i of X_i, then exist $x_i \in X_i$ such that for all $y_i \in X_i$,

$$C_{A_i}(y_i) = C_{B_i}(x_i^{-1} y_i x_i), i = 1, 2, \cdots, k.$$

Then we have

$$\begin{aligned} C_{A_1 \times \cdots \times A_k}((y_1, \cdots, y_k)) &= C_{A_1}(y_1) \wedge \cdots \wedge C_{A_k}(y_k) \\ &= C_{B_1}(x_1^{-1} y_1 x_1) \wedge \cdots \wedge C_{B_k}(x_k^{-1} y_k x_k) \\ &= C_{B_1 \times \cdots \times B_k}((x_1^{-1} y_1 x_1, \cdots, x_k^{-1} y_k x_k)). \end{aligned}$$

This completes the proof. □

Theorem 4.9 *Let $A_1, A_2, \cdots, A_k$ be multisets of the groups $X_1, X_2, \cdots, X_k$, respectively. Suppose that $e_1, e_2, \cdots, e_k$ are identities elements of $X_1, X_2, \cdots, X_k$, respectively. If $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$, then for at least one $i = 1, 2, \cdots, k$, the statement*

$$C_{A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, e_2, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \geq C_{A_i}((x_i)), \quad \forall x_i \in X_i$$

holds.

Proof Let $A_1 \times A_2 \times \cdots \times A_k$ be a multigroup of $X_1 \times X_2 \times \cdots \times X_k$. By contraposition, suppose that for none of $i = 1, 2, \cdots, k$, the statement holds. Then we can find $(a_1, a_2, \cdots, a_k) \in X_1 \times X_2 \times \cdots \times X_k$, respectively, such that

$$C_{A_i}((a_i)) > C_{A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, e_2, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)).$$

Then we have

$$\begin{aligned} C_{A_1 \times \cdots \times A_k}((a_1, \cdots, a_k)) &= C_{A_1}(a_1) \wedge \cdots \wedge C_{A_k}(a_k) \\ &> C_{A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \\ &= C_{A_1}(e_1) \wedge \cdots \wedge C_{A_{i-1}}(e_{i-1}) \wedge C_{A_{i+1}}(e_{i+1}) \wedge \cdots \wedge C_{A_k}(e_k) \\ &= C_{A_1}(e_1) \wedge \cdots \wedge C_{A_k}(e_k) \\ &= C_{A_1 \times \cdots \times A_k}((e_1, ..., e_k)). \end{aligned}$$

So, $A_1 \times A_2 \times ... \times A_k$ is not a multigroup of $X_1 \times X_2 \times \cdots \times X_k$. Hence, for at least one $i = 1, 2, \cdots, k$, the inequality

$$C_{A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \geq C_{A_i}((x_i))$$

is satisfied for all $x_i \in X_i$. □

Theorem 4.10 *Let $A_1, A_2, \cdots, A_k$ be multisets of the groups $X_1, X_2, \cdots, X_k$, respectively, such that*

$$C_{A_i}((x_i)) \leq C_{A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k}((e_1, e_2, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k))$$

$\forall x_i \in X_i$, e_i being the identity element of X_i. If $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$, then A_i is a multigroup of X_i.

Proof Let $A_1 \times A_2 \times \cdots \times A_k$ be a multigroup of $X_1 \times X_2 \times \cdots \times X_k$ and $x_i, y_i \in X_i$. Then

$$(e_1, \cdots, e_{i-1}, x_i, e_{i+1}, \cdots, e_k), (e_1, \cdots, e_{i-1}, y_i, e_{i+1}, \cdots, e_k) \in X_1 \times X_2 \times \cdots \times X_k.$$

Now, using the given inequality, we have

$$\begin{aligned}
C_{A_i}((x_i y_i)) &= C_{A_i}((x_i y_i)) \wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k) \\
&\quad (e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \\
&= C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((e_1, \cdots, x_i, \cdots, e_k)(e_1, \cdots, y_i, \cdots, e_k)) \\
&\geq C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((e_1, \cdots, x_i, \cdots, e_k)) \wedge C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((e_1, \cdots, y_i, \cdots, e_k)) \\
&= \wedge(C_{A_i}((x_i)) \wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)), C_{A_i}((y_i)) \\
&\quad \wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \\
&= C_{A_i}((x_i)) \wedge C_{A_i}((y_i)).
\end{aligned}$$

Also,

$$\begin{aligned}
C_{A_i}((x_i^{-1})) &= C_{A_i}((x_i^{-1})) \wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((e_1^{-1}, \cdots, e_{i-1}^{-1}, e_{i+1}^{-1}, \cdots, e_k^{-1})) \\
&= C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((e_1^{-1}, \cdots, x_i^{-1}, \cdots, e_k^{-1})) \\
&= C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((e_1, \cdots, x_i, \cdots, e_k)^{-1}) \\
&= C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((e_1, \cdots, x_i, \cdots, e_k)) \\
&= C_{A_i}((x_i)) \wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((e_1, \cdots, e_{i-1}, e_{i+1}, \cdots, e_k)) \\
&= C_{A_i}((x_i)).
\end{aligned}$$

Hence, $A_i \in MG(X_i)$. □

Theorem 4.11 *Let $A_1, A_2, \cdots, A_k$ be multisets of the groups $X_1, X_2, \cdots, X_k$, respectively, such that*

$$C_{A_1\times A_2\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k)) \leq C_{A_i}((e_i))$$

for $\forall(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) \in X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_k$, e_i being the identity element of X_i. If $A_1 \times A_2 \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_k$, then $A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k$ is a multigroup of $X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_k$.

Proof Let $A_1 \times A_2 \times \cdots \times A_k$ be a multigroup of $X_1 \times X_2 \times \cdots \times X_k$ and $(x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k), (y_1, y_2, \cdots, y_{i-1}, y_{i+1}, \cdots, y_k) \in X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_k$. Then

$$(x_1, \cdots, x_{i-1}, e_i, x_{i+1}, \cdots, x_k), (y_1, \cdots, y_{i-1}, e_i, y_{i+1}, \cdots, y_k) \in X_i.$$

Using the given inequality, we arrive at

$$\begin{aligned}
&C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)(y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_k))\\
&= C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)(y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_k))\\
&\quad \wedge C_{A_i}((e_i)) = C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1,\cdots,e_i,\cdots,x_k)(y_1,\cdots,e_i,\cdots,y_k))\\
&\geq C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1,\cdots,e_i,\cdots,x_k)) \wedge C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((y_1,\cdots,e_i,\cdots,y_k))\\
&= \wedge(C_{A_i}((e_i)) \wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)), C_{A_i}((e_i))\\
&\quad \wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((y_1,\cdots,y_{i-1},y_{i+1},\cdots,y_k))) = C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}\\
&\quad ((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)) \wedge C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((y_1,y_2,\cdots,y_{i-1},y_{i+1},\cdots,y_k)).
\end{aligned}$$

Again,

$$\begin{aligned}
&C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1^{-1},\cdots,x_{i-1}^{-1},x_{i+1}^{-1},\cdots,x_k^{-1}))\\
&= C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1^{-1},\cdots,x_{i-1}^{-1},x_{i+1}^{-1},\cdots,x_k^{-1})) \wedge C_{A_i}((e_i^{-1}))\\
&= C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1^{-1},\cdots,e_i^{-1},\cdots,x_k^{-1}))\\
&= C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1,\cdots,e_i,\cdots,x_k)^{-1}) = C_{A_1\times\cdots\times A_i\times\cdots\times A_k}((x_1,\cdots,e_i,\cdots,x_k))\\
&= C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)) \wedge C_{A_i}((e_i))\\
&= C_{A_1\times\cdots\times A_{i-1}\times A_{i+1}\times\cdots\times A_k}((x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_k)).
\end{aligned}$$

Hence, $A_1 \times A_2 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_k$ is the multigroup of $X_1 \times X_2 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_k$. □

§5. Homomorphism of Direct Product of Multigroups

In this section, we present some homomorphic properties of direct product of multigroups. This is an extension of the notion of homomorphism in multigroup setting (cf. [6, 12]) to direct product of multigroups.

Definition 5.1 *Let $W \times X$ and $Y \times Z$ be groups and let $f : W \times X \to Y \times Z$ be a homomorphism. Suppose $A \times B \in MS(W \times X)$ and $C \times D \in MS(Y \times Z)$, respectively. Then*

(i) the image of $A \times B$ under f, denoted by $f(A \times B)$, is a multiset of $Y \times Z$ defined by

$$C_{f(A\times B)}((y,z)) = \begin{cases} \bigvee_{(w,x)\in f^{-1}((y,z))} C_{A\times B}((w,x)), & f^{-1}((y,z)) \neq \emptyset \\ 0, & \textit{otherwise}, \end{cases}$$

for each $(y,z) \in Y \times Z$;

(ii) the inverse image of $C \times D$ under f, denoted by $f^{-1}(C \times D)$, is a multiset of $W \times X$ defined by

$$C_{f^{-1}(C\times D)}((w,x)) = C_{C\times D}(f((w,x)))\ \forall (w,x) \in W \times X.$$

Theorem 5.2 *Let W, X, Y, Z be groups, $A \in MS(W), B \in MS(X), C \in MS(Y)$ and $D \in MS(Z)$. If $f : W \times X \to Y \times Z$ is a homomorphism, then*

(*i*) $f(A \times B) \subseteq f(A) \times f(B)$;
(*ii*) $f^{-1}(C \times D) = f^{-1}(C) \times f^{-1}(D)$.

Proof (*i*) Let $(w, x) \in W \times X$. Suppose $\exists\,(y, z) \in Y \times Z$ such that

$$f((w, x)) = (f(w), f(x)) = (y, z).$$

Then we get

$$\begin{aligned} C_{f(A\times B)}((y,z)) &= C_{A\times B}(f^{-1}((y,z))) \\ &= C_{A\times B}((f^{-1}(y), f^{-1}(z))) \\ &= C_A(f^{-1}(y)) \wedge C_B(f^{-1}(z)) \\ &= C_{f(A)}(y) \wedge C_{f(B)}(z) \\ &= C_{f(A)\times f(B)}((y,z)) \end{aligned}$$

Hence, we conclude that, $f(A \times B) \subseteq f(A) \times f(B)$.

(*ii*) For $(w, x) \in W \times X$, we have

$$\begin{aligned} C_{f^{-1}(C\times D)}((w,x)) &= C_{C\times D}(f((w,x))) \\ &= C_{C\times D}((f(w), f(x))) \\ &= C_C(f(w)) \wedge C_D(f(x)) \\ &= C_{f^{-1}(C)}(w) \wedge C_{f^{-1}(D)}(x) \\ &= C_{f^{-1}(C)\times f^{-1}(D)}((w,x)). \end{aligned}$$

Hence, $f^{-1}(C \times D) \subseteq f^{-1}(C) \times f^{-1}(D)$.

Similarly,

$$\begin{aligned} C_{f^{-1}(C)\times f^{-1}(D)}((w,x)) &= C_{f^{-1}(C)}(w) \wedge C_{f^{-1}(D)}(x) \\ &= C_C(f(w)) \wedge C_D(f(x)) \\ &= C_{C\times D}((f(w), f(x))) \\ &= C_{C\times D}(f((w,x))) \\ &= C_{f^{-1}(C\times D)}((w,x)). \end{aligned}$$

Again, $f^{-1}(C) \times f^{-1}(D) \subseteq f^{-1}(C \times D)$. Therefore, the result follows. □

Theorem 5.3 *Let $f : W \times X \to Y \times Z$ be an isomorphism, A, B, C and D be multigroups of W, X, Y and Z, respectively. Then the following statements hold:*

(*i*) $f(A \times B) \in MG(Y \times Z)$;
(*ii*) $f^{-1}(C) \times f^{-1}(D) \in MG(W \times X)$.

Proof (i) Since $A \in MG(W)$ and $B \in MG(X)$, then $A \times B \in MG(W \times X)$ by Theorem 3.7. From Proposition 2.14 and Definition 5.1, it follows that, $f(A \times B) \in MG(Y \times Z)$.

(ii) Combining Corollary 2.15, Theorem 3.7, Definition 5.1 and Theorem 5.2, the result follows. □

Corollary 5.4 *Let X and Y be groups, $A \in MG(X)$ and $B \in MG(Y)$. If*

$$f : X \times X \to Y \times Y$$

be homomorphism, then

(i) $f(A \times A) \in MG(Y \times Y)$;
(ii) $f^{-1}(B \times B) \in MG(X \times X)$.

Proof Straightforward from Theorem 5.3. □

Proposition 5.5 *Let $X_1, X_2, \cdots, X_k$ and $Y_1, Y_2, \cdots, Y_k$ be groups, and*

$$f : X_1 \times X_2 \times \cdots \times X_k \to Y_1 \times Y_2 \times \cdots \times Y_k$$

be homomorphism. If $A_1 \times A_2 \times \cdots \times A_k \in MG(X_1 \times X_2 \times \cdots \times X_k)$ and $B_1 \times B_2 \times \cdots \times B_k \in MG(Y_1 \times Y_2 \times \cdots \times Y_k)$, then

(i) $f(A_1 \times A_2 \times \cdots \times A_k) \in MG(Y_1 \times Y_2 \times \cdots \times Y_k)$;
(ii) $f^{-1}(B_1 \times B_2 \times \cdots \times B_k) \in MG(X_1 \times X_2 \times \cdots \times X_k)$.

Proof Straightforward from Corollary 5.4. □

§6. Conclusions

The concept of direct product in groups setting has been extended to multigroups. We lucidly exemplified direct product of multigroups and deduced several results. The notion of generalized direct product of multigroups was also introduced in the case of finitely k^{th} multigroups. Finally, homomorphism and some of its properties were proposed in the context of direct product of multigroups.

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Hilbert Flow Spaces with Operators over Topological Graphs

Linfan MAO

1. Chinese Academy of Mathematics and System Science, Beijing 100190, P.R.China

2. Academy of Mathematical Combinatorics & Applications (AMCA), Colorado, USA

E-mail: maolinfan@163.com

Abstract: A complex system $\mathscr{S}$ consists m components, maybe inconsistence with $m \geq 2$, such as those of biological systems or generally, interaction systems and usually, a system with contradictions, which implies that there are no a mathematical subfield applicable. Then, *how can we hold on its global and local behaviors or reality*? All of us know that there always exists universal connections between things in the world, i.e., a topological graph $\overrightarrow{G}$ underlying components in $\mathscr{S}$. We can thereby establish mathematics over graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots$ by viewing labeling graphs $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots$ to be globally mathematical elements, not only game objects or combinatorial structures, which can be applied to characterize dynamic behaviors of the system $\mathscr{S}$ on time t. Formally, a continuity flow $\overrightarrow{G}^L$ is a topological graph $\overrightarrow{G}$ associated with a mapping $L : (v, u) \to L(v, u)$, 2 end-operators $A_{vu}^+ : L(v, u) \to L^{A_{vu}^+}(v, u)$ and $A_{uv}^+ : L(u, v) \to L^{A_{uv}^+}(u, v)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$ with $L(v, u) = -L(u, v)$ and $A_{vu}^+(-L(v, u)) = -L^{A_{vu}^+}(v, u)$ for $\forall (v, u) \in E\left(\overrightarrow{G}\right)$ holding with continuity equations

$$\sum_{u \in N_G(v)} L^{A_{vu}^+}(v, u) = L(v), \qquad \forall v \in V\left(\overrightarrow{G}\right).$$

The main purpose of this paper is to extend Banach or Hilbert spaces to Banach or Hilbert continuity flow spaces over topological graphs $\left\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots\right\}$ and establish differentials on continuity flows for characterizing their globally change rate. A few well-known results such as those of Taylor formula, L'Hospital's rule on limitation are generalized to continuity flows, and algebraic or differential flow equations are discussed in this paper. All of these results form the elementary differential theory on continuity flows, which contributes mathematical combinatorics and can be used to characterizing the behavior of complex systems, particularly, the synchronization.

Key Words: Complex system, Smarandache multispace, continuity flow, Banach space, Hilbert space, differential, Taylor formula, L'Hospital's rule, mathematical combinatorics.

AMS(2010): 34A26, 35A08, 46B25, 92B05, 05C10, 05C21, 34D43, 51D20.

§1. Introduction

A *Banach* or *Hilbert space* is respectively a linear space $\mathscr{A}$ over a field $\mathbb{R}$ or $\mathbb{C}$ equipped with a complete norm $\|\cdot\|$ or inner product $\langle \cdot , \cdot \rangle$, i.e., for every Cauchy sequence $\{x_n\}$ in $\mathscr{A}$, there

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exists an element x in $\mathscr{A}$ such that

$$\lim_{n\to\infty} \|x_n - x\|_{\mathscr{A}} = 0 \quad \text{or} \quad \lim_{n\to\infty} \langle x_n - x, x_n - x\rangle_{\mathscr{A}} = 0$$

and a topological graph $\varphi(G)$ is an embedding of a graph G with vertex set $V(G)$, edge set $E(G)$ in a space $\mathscr{S}$, i.e., there is a $1-1$ continuous mapping $\varphi: G \to \varphi(G) \subset \mathscr{S}$ with $\varphi(p) \neq \varphi(q)$ if $p \neq q$ for $\forall p, q \in G$, i.e., edges of G only intersect at vertices in $\mathscr{S}$, an embedding of a topological space to another space. A well-known result on embedding of graphs without loops and multiple edges in $\mathbb{R}^n$ concluded that *there always exists an embedding of G that all edges are straight segments in $\mathbb{R}^n$ for $n > 3$* ([22]) such as those shown in Fig.1.

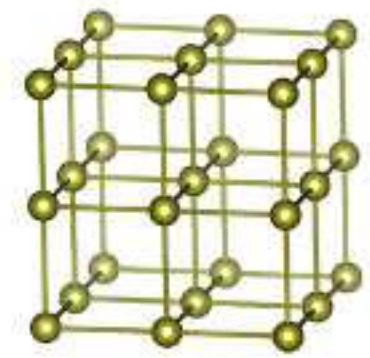

Fig.1

As we known, the purpose of science is hold on the reality of things in the world. However, the reality of a thing $\mathscr{T}$ is complex and there are no a mathematical subfield applicable unless a system maybe with contradictions in general. *Is such a contradictory system meaningless to human beings*? Certain not because all of these contradictions are the result of human beings, not the nature of things themselves, particularly on those of contradictory systems in mathematics. Thus, holding on the reality of things motivates one to turn contradictory systems to compatible one by a combinatorial notion and establish an envelope theory on mathematics, i.e., mathematical combinatorics ([9]-[13]). Then, *Can we globally characterize the behavior of a system or a population with elements≥ 2, which maybe contradictory or compatible*? The answer is certainly YES by *continuity flows*, which needs one to establish an envelope mathematical theory over topological graphs, i.e., views labeling graphs G^L to be mathematical elements ([19]), not only a game object or a combinatorial structure with labels in the following sense.

Definition 1.1 *A continuity flow $\left(\overrightarrow{G}; L, A\right)$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L: v \to L(v)$, $(v,u) \to L(v,u)$, 2 end-operators A^+_{vu} : $L(v,u) \to L^{A^+_{vu}}(v,u)$ and $A^+_{uv}: L(u,v) \to L^{A^+_{uv}}(u,v)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$*

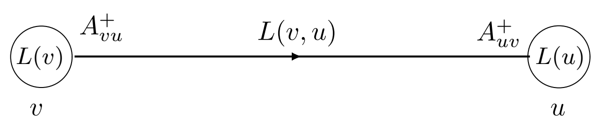

Fig.2

with $L(v,u) = -L(u,v)$ *and* $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ *for* $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ *holding with continuity equation*

$$\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u) = L(v) \quad for \quad \forall v \in V\left(\overrightarrow{G}\right)$$

such as those shown for vertex v *in Fig.*3 *following*

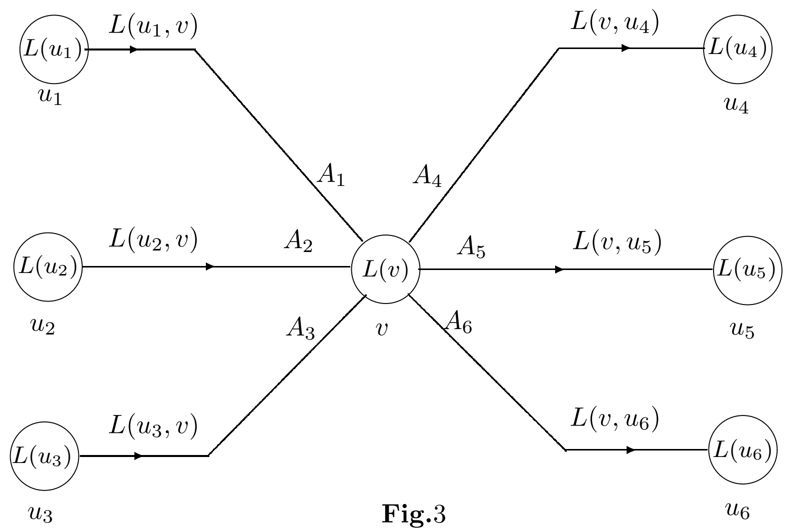

Fig.3

with a continuity equation

$$L^{A_1}(v,u_1) + L^{A_2}(v,u_2) + L^{A_3}(v,u_3) - L^{A_4}(v,u_4) - L^{A_5}(v,u_5) - L^{A_6}(v,u_6) = L(v),$$

where $L(v)$ *is the surplus flow on vertex* v.

Particularly, if $L(v) = \dot{x}_v$ *or constants* $\mathbf{v}_v, v \in V\left(\overrightarrow{G}\right)$, *the continuity flow* $\left(\overrightarrow{G}; L, A\right)$ *is respectively said to be a complex flow or an action* A *flow, and* $\overrightarrow{G}$*-flow if* $A = \mathbf{1}_{\mathscr{V}}$, *where* $\dot{x}_v = dx_v/dt$, x_v *is a variable on vertex* v *and* $\mathbf{v}$ *is an element in* $\mathscr{B}$ *for* $\forall v \in E\left(\overrightarrow{G}\right)$.

Clearly, an action flow is an equilibrium state of a continuity flow $\left(\overrightarrow{G}; L, A\right)$. We have shown that Banach or Hilbert space can be extended over topological graphs ([14],[17]), which can be applied to understanding the reality of things in [15]-[16], and we also shown that complex flows can be applied to hold on the global stability of biological n-system with $n \geq 3$ in [19]. For further discussing continuity flows, we need conceptions following.

Definition 1.2 *Let* $\mathscr{B}_1, \mathscr{B}_2$ *be Banach spaces over a field* $\mathbb{F}$ *with norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$, *respectively. An operator* $\mathbf{T} : \mathscr{B}_1 \to \mathscr{B}_2$ *is linear if*

$$\mathbf{T}(\lambda \mathbf{v}_1 + \mu \mathbf{v}_2) = \lambda \mathbf{T}(\mathbf{v}_1) + \mu \mathbf{T}(\mathbf{v}_2)$$

for $\lambda, \mu \in \mathbb{F}$, *and* $\mathbf{T}$ *is said to be continuous at a vector* $\mathbf{v}_0$ *if there always exist such a number*

$\delta(\varepsilon)$ for $\forall\epsilon>0$ that

$$\left\|\mathbf{T}\left(\mathbf{v}\right)-\mathbf{T}\left(\mathbf{v}_0\right)\right\|_2<\varepsilon$$

if $\left\|\mathbf{v}-\mathbf{v}_0\right\|_1<\delta(\varepsilon)$ for $\forall\mathbf{v},\mathbf{v}_0,\mathbf{v}_1,\mathbf{v}_2\in\mathscr{B}_1$.

Definition 1.3 *Let $\mathscr{B}_1,\mathscr{B}_2$ be Banach spaces over a field $\mathbb{F}$ with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. An operator $\mathbf{T}:\mathscr{B}_1\to\mathscr{B}_2$ is bounded if there is a constant $M>0$ such that*

$$\left\|\mathbf{T}\left(\mathbf{v}\right)\right\|_2\le M\left\|\mathbf{v}\right\|_1, \quad i.e., \quad \frac{\|\mathbf{T}(\mathbf{v})\|_2}{\|\mathbf{v}\|_1}\le M$$

for $\forall\mathbf{v}\in\mathscr{B}$ and furthermore, $\mathbf{T}$ is said to be a contractor if

$$\left\|\mathbf{T}\left(\mathbf{v}_1\right)-\mathbf{T}\left(\mathbf{v}_2\right)\right\|\le c\left\|\mathbf{v}_1-\mathbf{v}_2)\right\|$$

for $\forall\mathbf{v}_1,\mathbf{v}_2\in\mathscr{B}$ with $c\in[0,1)$.

We only discuss the case that all end-operators A_{vu}^{+},A_{uv}^{+} are both linear and continuous. In this case, the result following on linear operators of Banach space is useful.

Theorem 1.4 *Let $\mathscr{B}_1,\mathscr{B}_2$ be Banach spaces over a field $\mathbb{F}$ with norms $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Then, a linear operator $\mathbf{T}:\mathscr{B}_1\to\mathscr{B}_2$ is continuous if and only if it is bounded, or equivalently,*

$$\|\mathbf{T}\|:=\sup_{\mathbf{0}\neq\mathbf{v}\in\mathscr{B}_1}\frac{\|\mathbf{T}(\mathbf{v})\|_2}{\|\mathbf{v}\|_1}<+\infty.$$

Let $\left\{\overrightarrow{G}_1,\overrightarrow{G}_2,\cdots\right\}$ be a graph family. The main purpose of this paper is to extend Banach or Hilbert spaces to Banach or Hilbert continuity flow spaces over topological graphs $\left\{\overrightarrow{G}_1,\overrightarrow{G}_2,\cdots\right\}$ and establish differentials on continuity flows, which enables one to characterize their globally change rate constraint on the combinatorial structure. A few well-known results such as those of Taylor formula, L'Hospital's rule on limitation are generalized to continuity flows, and algebraic or differential flow equations are discussed in this paper. All of these results form the elementary differential theory on continuity flows, which contributes mathematical combinatorics and can be used to characterizing the behavior of complex systems, particularly, the synchronization.

For terminologies and notations not defined in this paper, we follow references [1] for mechanics, [4] for functionals and linear operators, [22] for topology, [8] combinatorial geometry, [6]-[7],[25] for Smarandache systems, Smarandache geometries and Smaarandache multispaces and [2], [20] for biological mathematics.

§2. Banach and Hilbert Flow Spaces

2.1 Linear Spaces over Graphs

Let $\overrightarrow{G}_1,\overrightarrow{G}_2,\cdots,\overrightarrow{G}_n$ be oriented graphs embedded in topological space $\mathscr{S}$ with $\overrightarrow{\mathscr{G}}=\bigcup\limits_{i=1}^{n}\overrightarrow{G}_i$, i.e., $\overrightarrow{G}_i$ is a subgraph of $\overrightarrow{\mathscr{G}}$ for integers $1 \leq i \leq n$. In this case, these is naturally an embedding $\iota : \overrightarrow{G}_i \to \overrightarrow{\mathscr{G}}$.

Let $\mathscr{V}$ be a linear space over a field $\mathscr{F}$. A vector labeling $L : \overrightarrow{G} \to \mathscr{V}$ is a mapping with $L(v), L(e) \in \mathscr{V}$ for $\forall v \in V(\overrightarrow{G}), e \in E(\overrightarrow{G})$. Define

$$\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1+L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{L_2} \tag{2.1}$$

and

$$\lambda \cdot \overrightarrow{G}^L = \overrightarrow{G}^{\lambda \cdot L} \tag{2.2}$$

for $\forall \lambda \in \mathscr{F}$. Clearly, if , and $\overrightarrow{G}^L, \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}$ are continuity flows with linear end-operators A_{vu}^+ and A_{uv}^+ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}$ and $\lambda \cdot \overrightarrow{G}^L$ are continuity flows also. If we consider each continuity flow $\overrightarrow{G}_i^L$ a continuity subflow of $\overrightarrow{\mathscr{G}}^{\widehat{L}}$, where $\widehat{L} : \overrightarrow{G}_i = L(\overrightarrow{G}_i)$ but $\widehat{L} : \overrightarrow{\mathscr{G}} \setminus \overrightarrow{G}_i \to \mathbf{0}$ for integers $1 \leq i \leq n$, and define $\mathbf{O} : \overrightarrow{\mathscr{G}} \to \mathbf{0}$, then all continuity flows, particularly, all complex flows, or all action flows on oriented graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ naturally form a linear space, denoted by $\left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}; +, \cdot\right)$ over a field $\mathscr{F}$ under operations (2.1) and (2.2) because it holds with:

(1) A field $\mathscr{F}$ of scalars;

(2) A set $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ of objects, called continuity flows;

(3) An operation "+", called continuity flow addition, which associates with each pair of continuity flows $\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}$ in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ a continuity flows $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}$ in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$, called the sum of $\overrightarrow{G}_1^{L_1}$ and $\overrightarrow{G}_2^{L_2}$, in such a way that

(a) Addition is commutative, $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} = \overrightarrow{G}_2^{L_2} + \overrightarrow{G}_1^{L_1}$ because of

$$\begin{aligned}
\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} &= \left(\overrightarrow{G}_1 - \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1+L_2} \bigcup \left(\overrightarrow{G}_2 - \overrightarrow{G}_1\right)^{L_2} \\
&= \left(\overrightarrow{G}_2 - \overrightarrow{G}_1\right)^{L_2} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_2+L_1} \bigcup \left(\overrightarrow{G}_1 - \overrightarrow{G}_2\right)^{L_1} \\
&= \overrightarrow{G}_2^{L_2} + \overrightarrow{G}_1^{L_1};
\end{aligned}$$

(b) Addition is associative, $\left(\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right) + \overrightarrow{G}_3^{L_3} = \overrightarrow{G}_1^{L_1} + \left(\overrightarrow{G}_2^{L_2} + \overrightarrow{G}_3^{L_3}\right)$ because if we let

$$L_{ijk}^+(x) = \begin{cases}
L_i(x), & \text{if } x \in \overrightarrow{G}_i \setminus \left(\overrightarrow{G}_j \bigcup \overrightarrow{G}_k\right) \\
L_j(x), & \text{if } x \in \overrightarrow{G}_j \setminus \left(\overrightarrow{G}_i \bigcup \overrightarrow{G}_k\right) \\
L_k(x), & \text{if } x \in \overrightarrow{G}_k \setminus \left(\overrightarrow{G}_i \bigcup \overrightarrow{G}_j\right) \\
L_{ij}^+(x), & \text{if } x \in \left(\overrightarrow{G}_i \bigcap \overrightarrow{G}_j\right) \setminus \overrightarrow{G}_k \\
L_{ik}^+(x), & \text{if } x \in \left(\overrightarrow{G}_i \bigcap \overrightarrow{G}_k\right) \setminus \overrightarrow{G}_j \\
L_{jk}^+(x), & \text{if } x \in \left(\overrightarrow{G}_j \bigcap \overrightarrow{G}_k\right) \setminus \overrightarrow{G}_i \\
L_i(x) + L_j(x) + L_k(x) & \text{if } x \in \overrightarrow{G}_i \bigcap \overrightarrow{G}_j \bigcap \overrightarrow{G}_k
\end{cases} \tag{2.3}$$

and

$$L_{ij}^{+}(x) = \begin{cases} L_i(x), & \text{if } x \in \overrightarrow{G}_i \setminus \overrightarrow{G}_j \\ L_j(x), & \text{if } x \in \overrightarrow{G}_j \setminus \overrightarrow{G}_i \\ L_i(x) + L_j(x), & \text{if } x \in \overrightarrow{G}_i \bigcap \overrightarrow{G}_j \end{cases} \tag{2.4}$$

for integers $1 \le i, j, k \le n$, then

$$\begin{aligned} \left(\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right) + \overrightarrow{G}_3^{L_3} &= \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)^{L_{12}^{+}} + \overrightarrow{G}_3^{L_3} = \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2 \bigcup \overrightarrow{G}_3\right)^{L_{123}^{+}} \\ &= \overrightarrow{G}_1^{L_1} + \left(\overrightarrow{G}_2 \bigcup \overrightarrow{G}_3\right)^{L_{23}^{+}} = \overrightarrow{G}_1^{L_1} + \left(\overrightarrow{G}_2^{L_2} + \overrightarrow{G}_3^{L_3}\right); \end{aligned}$$

(c) There is a unique continuity flow $\mathbf{O}$ on $\overrightarrow{\mathscr{G}}$ hold with $\mathbf{O}(v,u) = \mathbf{0}$ for $\forall (v,u) \in E\left(\overrightarrow{\mathscr{G}}\right)$ and $V\left(\overrightarrow{\mathscr{G}}\right)$ in $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$, called zero such that $\overrightarrow{G}^L + \mathbf{O} = \overrightarrow{G}^L$ for $\overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$;

(d) For each continuity flow $\overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$ there is a unique continuity flow $\overrightarrow{G}^{-L}$ such that $\overrightarrow{G}^L + \overrightarrow{G}^{-L} = \mathbf{O}$;

(4) An operation "$\cdot$, called scalar multiplication, which associates with each scalar k in F and a continuity flow $\overrightarrow{G}^L$ in $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$ a continuity flow $k \cdot \overrightarrow{G}^L$ in $\mathscr{V}$, called the product of k with $\overrightarrow{G}^L$, in such a way that

(a) $1 \cdot \overrightarrow{G}^L = \overrightarrow{G}^L$ for every $\overrightarrow{G}^L$ in $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$;

(b) $(k_1 k_2) \cdot \overrightarrow{G}^L = k_1(k_2 \cdot \overrightarrow{G}^L)$;

(c) $k \cdot (\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}) = k \cdot \overrightarrow{G}_1^{L_1} + k \cdot \overrightarrow{G}_2^{L_2}$;

(d) $(k_1 + k_2) \cdot \overrightarrow{G}^L = k_1 \cdot \overrightarrow{G}^L + k_2 \cdot \overrightarrow{G}^L$.

Usually, we abbreviate $\left(\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}; +, \cdot\right)$ to $\left\langle \overrightarrow{G}_i, 1 \le i \le n \right\rangle^{\mathscr{V}}$ if these operations $+$ and $\cdot$ are clear in the context.

By operation (1.1), $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} \ne \overrightarrow{G}_1^{L_1}$ if and only if $\overrightarrow{G}_1 \not\preceq \overrightarrow{G}_2$ with $L_1 : \overrightarrow{G}_1 \setminus \overrightarrow{G}_2 \not\to \mathbf{0}$ and $\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2} \ne \overrightarrow{G}_2^{L_2}$ if and only if $\overrightarrow{G}_2 \not\preceq \overrightarrow{G}_1$ with $L_2 : \overrightarrow{G}_2 \setminus \overrightarrow{G}_1 \not\to \mathbf{0}$, which allows us to introduce the conception of linear irreducible. Generally, a continuity flow family $\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\}$ is *linear irreducible* if for any integer i,

$$\overrightarrow{G}_i \not\preceq \bigcup_{l \ne i} \overrightarrow{G}_l \quad \text{with} \quad L_i : \overrightarrow{G}_i \setminus \bigcup_{l \ne i} \overrightarrow{G}_l \not\to \mathbf{0}, \tag{2.5}$$

where $1 \le i \le n$. We know the following result on linear generated sets.

Theorem 2.1 *Let $\mathscr{V}$ be a linear space over a field $\mathscr{F}$ and let $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\right\}$ be an linear irreducible family, $L_i : \overrightarrow{G}_i \to \mathscr{V}$ for integers $1 \le i \le n$ with linear operators A_{vu}^{+}, A_{uv}^{+} for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. Then, $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\right\}$ is an independent generated set of*

$\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$, *called basis, i.e.,*

$$\dim \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}} = n.$$

Proof By definition, $\overrightarrow{G}_i^{L_i}, 1 \leq i \leq n$ is a linear generated of $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ with $L_i : \overrightarrow{G}_i \to \mathscr{V}$, i.e.,

$$\dim \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}} \leq n.$$

We only need to show that $\overrightarrow{G}_i^{L_i}, 1 \leq i \leq n$ is linear independent, i.e.,

$$\dim \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}} \geq n,$$

which implies that if there are n scalars $c_1, c_2, \cdots, c_n$ holding with

$$c_1 \overrightarrow{G}_1^{L_1} + c_2 \overrightarrow{G}_2^{L_2} + \cdots + c_n \overrightarrow{G}_n^{L_n} = \mathbf{O},$$

then $c_1 = c_2 = \cdots = c_n = 0$. Notice that $\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n\}$ is linear irreducible. We are easily know $\overrightarrow{G_i} \setminus \bigcup\limits_{l \neq i} \overrightarrow{G}_l \neq \emptyset$ and find an element $x \in E(\overrightarrow{G_i} \setminus \bigcup\limits_{l \neq i} \overrightarrow{G}_l)$ such that $c_i L_i(x) = \mathbf{0}$ for integer $i, 1 \leq i \leq n$. However, $L_i(x) \neq \mathbf{0}$ by (1.5). We get that $c_i = 0$ for integers $1 \leq i \leq n$. □

A subspace of $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ is called an A_0-*flow space* if its elements are all continuity flows $\overrightarrow{G}^L$ with $L(v)$, $v \in V\left(\overrightarrow{G}\right)$ are constant $\mathbf{v}$. The result following is an immediately conclusion of Theorem 2.1.

Theorem 2.2 *Let $\overrightarrow{G}, \overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ be oriented graphs embedded in a space $\mathscr{S}$ and $\mathscr{V}$ a linear space over a field $\mathscr{F}$. If $\overrightarrow{G}^{\mathbf{v}}, \overrightarrow{G}_1^{\mathbf{v}_1}, \overrightarrow{G}_2^{\mathbf{v}_2}, \cdots, \overrightarrow{G}_n^{\mathbf{v}_n}$ are continuity flows with $\mathbf{v}(v) = \mathbf{v}, \mathbf{v}_i(v) = \mathbf{v}_i \in \mathscr{V}$ for $\forall v \in V\left(\overrightarrow{G}\right)$, $1 \leq i \leq n$, then*

(1) $\left\langle \overrightarrow{G}^{\mathbf{v}} \right\rangle$ *is an A_0-flow space;*

(2) $\left\langle \overrightarrow{G}_1^{\mathbf{v}_1}, \overrightarrow{G}_2^{\mathbf{v}_2}, \cdots, \overrightarrow{G}_n^{\mathbf{v}_n} \right\rangle$ *is an A_0-flow space if and only if $\overrightarrow{G}_1 = \overrightarrow{G}_2 = \cdots = \overrightarrow{G}_n$ or* $\mathbf{v}_1 = \mathbf{v}_2 = \cdots = \mathbf{v}_n = \mathbf{0}$.

Proof By definition, $\overrightarrow{G}_1^{\mathbf{v}_1} + \overrightarrow{G}_2^{\mathbf{v}_2}$ and $\lambda \overrightarrow{G}^{\mathbf{v}}$ are A_0-flows if and only if $\overrightarrow{G}_1 = \overrightarrow{G}_1$ or $\mathbf{v}_1 = \mathbf{v}_2 = \mathbf{0}$ by definition. We therefore know this result. □

2.2 Commutative Rings over Graphs

Furthermore, if $\mathscr{V}$ is a commutative ring $(\mathscr{R}; +, \cdot)$, we can extend it over oriented graph family $\{\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n\}$ by introducing operation $+$ with (2.1) and operation $\cdot$ following:

$$\overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{L_1 \cdot L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{L_2}, \tag{2.6}$$

where $L_1 \cdot L_2 : \ x \to L_1(x) \cdot L_2(x)$, and particularly, the scalar product for $\mathbb{R}^n, n \geq 2$ for $x \in \overrightarrow{G}_1 \bigcap \overrightarrow{G}_2$.

As we shown in Subsection 2.1, $\left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{R}}; +\right)$ is an Abelian group. We show $\left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{R}}; +, \cdot\right)$ is a commutative semigroup also.

In fact, define

$$L_{ij}^{\times}(x) = \begin{cases} L_i(x), & \text{if } x \in \overrightarrow{G}_i \setminus \overrightarrow{G}_j \\ L_j(x), & \text{if } x \in \overrightarrow{G}_j \setminus \overrightarrow{G}_i \\ L_i(x) \cdot L_j(x), & \text{if } x \in \overrightarrow{G}_i \bigcap \overrightarrow{G}_j \end{cases}$$

Then, we are easily known that $\overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_2^{L_2} = \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)^{L_{12}^{\times}} = \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)^{L_{21}^{\times}} = \overrightarrow{G}_2^{L_2} \cdot \overrightarrow{G}_1^{L_1}$ for $\forall \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{R}}; \cdot\right)$ by definition (2.6), i.e., it is commutative.

Let

$$L_{ijk}^{\times}(x) = \begin{cases} L_i(x), & \text{if } x \in \overrightarrow{G}_i \setminus \left(\overrightarrow{G}_j \bigcup \overrightarrow{G}_k\right) \\ L_j(x), & \text{if } x \in \overrightarrow{G}_j \setminus \left(\overrightarrow{G}_i \bigcup \overrightarrow{G}_k\right) \\ L_k(x), & \text{if } x \in \overrightarrow{G}_k \setminus \left(\overrightarrow{G}_i \bigcup \overrightarrow{G}_j\right) \\ L_{ij}(x), & \text{if } x \in \left(\overrightarrow{G}_i \bigcap \overrightarrow{G}_j\right) \setminus \overrightarrow{G}_k \\ L_{ik}(x), & \text{if } x \in \left(\overrightarrow{G}_i \bigcap \overrightarrow{G}_k\right) \setminus \overrightarrow{G}_j \\ L_{jk}(x), & \text{if } x \in \left(\overrightarrow{G}_j \bigcap \overrightarrow{G}_k\right) \setminus \overrightarrow{G}_i \\ L_i(x) \cdot L_j(x) \cdot L_k(x) & \text{if } x \in \overrightarrow{G}_i \bigcap \overrightarrow{G}_j \bigcap \overrightarrow{G}_k \end{cases}$$

Then,

$$\begin{aligned} \left(\overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_2^{L_2}\right) \cdot \overrightarrow{G}_3^{L_3} &= \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)^{L_{12}^{\times}} \cdot \overrightarrow{G}_3^{L_3} = \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2 \bigcup \overrightarrow{G}_3\right)^{L_{123}^{\times}} \\ &= \overrightarrow{G}_1^{L_1} \cdot \left(\overrightarrow{G}_2 \bigcup \overrightarrow{G}_3\right)^{L_{23}^{\times}} = \overrightarrow{G}_1^{L_1} \cdot \left(\overrightarrow{G}_2^{L_2} \cdot \overrightarrow{G}_3^{L_3}\right). \end{aligned}$$

Thus,

$$\left(\overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_2^{L_2}\right) \cdot \overrightarrow{G}_3^{L_3} = \overrightarrow{G}_1^{L_1} \cdot \left(\overrightarrow{G}_2^{L_2} \cdot \overrightarrow{G}_3^{L_3}\right)$$

for $\forall \overrightarrow{G}^L, \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left(\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{R}}; \cdot\right)$, which implies that it is a semigroup.

We are also need to verify the distributive laws, i.e.,

$$\overrightarrow{G}_3^{L_3} \cdot \left(\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right) = \overrightarrow{G}_3^{L_3} \cdot \overrightarrow{G}_1^{L_1} + \overrightarrow{G}_3^{L_3} \cdot \overrightarrow{G}_2^{L_2} \tag{2.7}$$

and

$$\left(\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right) \cdot \overrightarrow{G}_3^{L_3} = \overrightarrow{G}_1^{L_1} \cdot \overrightarrow{G}_3^{L_3} + \overrightarrow{G}_2^{L_2} \cdot \overrightarrow{G}_3^{L_3} \tag{2.8}$$

for $\forall\overrightarrow{G}_3, \overrightarrow{G}_1, \overrightarrow{G}_2 \in \left(\left\langle\overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{R}}; +, \cdot\right)$. Clearly,

$$\begin{aligned}\overrightarrow{G}_3^{L_3} \cdot \left(\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right) &= \overrightarrow{G}_3^{L_3} \cdot \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)^{L_{12}^{+}} = \left(\overrightarrow{G}_3\left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)\right)^{L_{3(21)}^{\times}} \\ &= \left(\overrightarrow{G}_3 \bigcup \overrightarrow{G}_1\right)^{L_{31}^{\times}} \bigcup \left(\overrightarrow{G}_3 \bigcup \overrightarrow{G}_2\right)^{L_{32}^{\times}} = \overrightarrow{G}_3^{L_3} \cdot \overrightarrow{G}_1^{L_1} + \overrightarrow{G}_3^{L_3} \cdot \overrightarrow{G}_2^{L_2},\end{aligned}$$

which is the (2.7). The proof for (2.8) is similar. Thus, we get the following result.

Theorem 2.3 *Let $(\mathscr{R}; +, \cdot)$ be a commutative ring and let $\left\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\right\}$ be a linear irreducible family, $L_i : \overrightarrow{G}_i \to \mathscr{R}$ for integers $1 \leq i \leq n$ with linear operators A_{vu}^{+}, A_{uv}^{+} for $\forall(v, u) \in E\left(\overrightarrow{G}\right)$. Then, $\left(\left\langle\overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{R}}; +, \cdot\right)$ is a commutative ring.*

2.3 Banach or Hilbert Flow Spaces

Let $\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\}$ be a basis of $\left\langle\overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}}$, where $\mathscr{V}$ is a Banach space with a norm $\|\cdot\|$. For $\forall\overrightarrow{G}^L \in \left\langle\overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}}$, define

$$\left\|\overrightarrow{G}^L\right\| = \sum_{e \in E\left(\overrightarrow{G}\right)} \|L(e)\|. \tag{2.9}$$

Then, for $\forall\overrightarrow{G}, \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left\langle\overrightarrow{G}_i, 1 \leq i \leq n\right\rangle^{\mathscr{V}}$ we are easily know that

(1) $\left\|\overrightarrow{G}^L\right\| \geq 0$ and $\left\|\overrightarrow{G}^L\right\| = 0$ if and only if $\overrightarrow{G}^L = \mathbf{O}$;

(2) $\left\|\overrightarrow{G}^{\xi L}\right\| = \xi\left\|\overrightarrow{G}^L\right\|$ for any scalar ξ;

(3) $\left\|\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right\| \leq \left\|\overrightarrow{G}_1^{L_1}\right\| + \left\|\overrightarrow{G}_2^{L_2}\right\|$ because of

$$\begin{aligned}\left\|\overrightarrow{G}_1^{L_1} + \overrightarrow{G}_2^{L_2}\right\| &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)} \|L_1(e)\| \\ &\quad + \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)} \|L_1(e) + L_2(e)\| + \sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)} \|L_2(e)\| \\ &\leq \left(\sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)} \|L_1(e)\| + \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)} \|L_1(e)\|\right) \\ &\quad + \left(\sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)} \|L_2(e)\| + \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)} \|L_2(e)\|\right) = \left\|\overrightarrow{G}_1^{L_1}\right\| + \left\|\overrightarrow{G}_2^{L_2}\right\|.\end{aligned}$$

for $\|L_1(e) + L_2(e)\| \leq \|L_1(e)\| + \|L_2(e)\|$ in Banach space $\mathscr{V}$. Therefore, $\|\cdot\|$ is also a norm on $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$.

Furthermore, if $\mathscr{V}$ is a Hilbert space with an inner product $\langle\cdot,\cdot\rangle$, for $\forall \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$, define

$$\begin{aligned}\left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \right\rangle &= \sum_{e\in E\left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)} \langle L_1(e), L_1(e)\rangle \\ &+ \sum_{e\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)} \langle L_1(e), L_2(e)\rangle + \sum_{e\in E\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)} \langle L_2(e), L_2(e)\rangle . \qquad (2.10)\end{aligned}$$

Then we are easily know also that

(1) For $\forall \overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$,

$$\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle = \sum_{e\in E\left(\overrightarrow{G}\right)} \langle L(e), L(e)\rangle \geq 0$$

and $\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle = 0$ if and only if $\overrightarrow{G}^L = \mathbf{O}$.

(2) For $\forall \overrightarrow{G}^{L_1}, \overrightarrow{G}^{L_2} \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$,

$$\left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \right\rangle = \overline{\left\langle \overrightarrow{G}_2^{L_2}, \overrightarrow{G}_1^{L_1} \right\rangle}$$

because of

$$\begin{aligned}\left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \right\rangle &= \sum_{e\in E\left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)} \langle L_1(e), L_1(e)\rangle + \sum_{e\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)} \langle L_1(e), L_2(e)\rangle \\ &+ \sum_{e\in E\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)} \langle L_2(e), L_2(e)\rangle \\ &= \sum_{e\in E\left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)} \overline{\langle L_1(e), L_1(e)\rangle} + \sum_{e\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)} \overline{\langle L_2(e), L_1(e)\rangle} \\ &+ \sum_{e\in E\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)} \overline{\langle L_2(e), L_2(e)\rangle} = \overline{\left\langle \overrightarrow{G}_2^{L_2}, \overrightarrow{G}_1^{L_1} \right\rangle}\end{aligned}$$

for $\langle L_1(e), L_2(e)\rangle = \overline{\langle L_2(e), L_1(e)\rangle}$ in Hilbert space $\mathscr{V}$.

(3) For $\overrightarrow{G}^L, \overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2} \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ and $\lambda, \mu \in \mathscr{F}$, there is

$$\left\langle \lambda\overrightarrow{G}_1^{L_1} + \mu\overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle = \lambda\left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}^L \right\rangle + \mu\left\langle \overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle$$

because of

$$\left\langle \lambda\overrightarrow{G}_1^{L_1} + \mu\overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle = \left\langle \overrightarrow{G}_1^{\lambda L_1} + \overrightarrow{G}_2^{\mu L_2}, \overrightarrow{G}^L \right\rangle$$
$$= \left\langle \left(\overrightarrow{G}_1 \setminus \overrightarrow{G}_2\right)^{\lambda L_1} \bigcup \left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}_2\right)^{\lambda L_1 + \mu L_2} \bigcup \left(\overrightarrow{G}_2 \setminus \overrightarrow{G}_1\right)^{\mu L_2}, \overrightarrow{G}^L \right\rangle.$$

Define $L_{1_\lambda 2_\mu} : \overrightarrow{G}_1 \bigcup \overrightarrow{G}_2 \to \mathscr{V}$ by

$$L_{1_\lambda 2_\mu}(x) = \begin{cases} \lambda L_1(x), & \text{if } x \in \overrightarrow{G}_1 \setminus \overrightarrow{G}_2 \\ \mu L_2(x), & \text{if } x \in \overrightarrow{G}_2 \setminus \overrightarrow{G}_1 \\ \lambda L_1(x) + \mu L_2(x), & \text{if } x \in \overrightarrow{G}_2 \bigcap \overrightarrow{G}_1 \end{cases}$$

Then, we know that

$$\begin{aligned} \left\langle \lambda\overrightarrow{G}_1^{L_1} + \mu\overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle &= \sum_{e \in E\left(\left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right) \setminus \overrightarrow{G}\right)} \left\langle L_{1_\lambda 2_\mu}(e), L_{1_\lambda 2_\mu}(e) \right\rangle \\ &+ \sum_{e \in E\left(\left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right) \bigcap \overrightarrow{G}\right)} \left\langle L_{1_\lambda 2_\mu}(e), L(e) \right\rangle \\ &+ \sum_{e \in E\left(\overrightarrow{G} \setminus \left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right)\right)} \langle L(e), L(e) \rangle . \end{aligned}$$

and

$$\begin{aligned} &\lambda \left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}^L \right\rangle + \mu \left\langle \overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L \right\rangle \\ &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)} \langle \lambda L_1(e), \lambda L_1(e) \rangle + \sum_{e \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)} \langle \lambda L_1(e), L(e) \rangle \\ &\quad + \sum_{e \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)} \langle L(e), L(e) \rangle + \sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}\right)} \langle \mu L_2(e), \mu L_2(e) \rangle \\ &\quad + \sum_{e \in E\left(\overrightarrow{G}_2 \bigcap \overrightarrow{G}\right)} \langle \mu L_2(e), L(e) \rangle + \sum_{e \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_2\right)} \langle L(e), L(e) \rangle . \end{aligned}$$

Notice that

$$\begin{aligned} &\sum_{e \in E\left(\left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right) \setminus \overrightarrow{G}\right)} \left\langle L_{1_\lambda 2_\mu}(e), L_{1_\lambda 2_\mu}(e) \right\rangle \\ &= \sum_{e \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)} \langle \lambda L_1(e), \lambda L_1(e) \rangle + \sum_{e \in E\left(\overrightarrow{G}_2 \setminus \overrightarrow{G}\right)} \langle \mu L_2(e), \mu L_2(e) \rangle \\ &\quad + \sum_{e \in E\left(\left(\overrightarrow{G}_1 \bigcup \overrightarrow{G}_2\right) \bigcap \overrightarrow{G}\right)} \left\langle L_{1_\lambda 2_\mu}(e), L(e) \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \sum_{e\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}\right)} \langle\lambda L_1(e), L(e)\rangle + \sum_{e\in E\left(\overrightarrow{G}_2\bigcap\overrightarrow{G}\right)} \langle\mu L_2(e), L(e)\rangle \\
&\quad + \sum_{e\in E\left(\overrightarrow{G}\setminus\overrightarrow{G}_2\right)} \langle L(e), L(e)\rangle \\
&= \sum_{e\in E\left(\overrightarrow{G}\setminus\overrightarrow{G}_1\right)} \langle L(e), L(e)\rangle + \sum_{e\in E\left(\overrightarrow{G}\setminus\overrightarrow{G}_2\right)} \langle L(e), L(e)\rangle .
\end{aligned}
$$

We therefore know that

$$
\left\langle \lambda\overrightarrow{G}_1^{L_1} + \mu\overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L\right\rangle = \lambda\left\langle \overrightarrow{G}_1^{L_1}, \overrightarrow{G}^L\right\rangle + \mu\left\langle \overrightarrow{G}_2^{L_2}, \overrightarrow{G}^L\right\rangle .
$$

Thus, $\overrightarrow{G}^{\mathscr{V}}$ is an inner space

If $\{\overrightarrow{G}_1^{L_1}, \overrightarrow{G}_2^{L_2}, \cdots, \overrightarrow{G}_n^{L_n}\}$ is a basis of space $\left\langle\overrightarrow{G}_i, 1\le i\le n\right\rangle^{\mathscr{V}}$, we are easily find the exact formula on L by $L_1.L_2,\cdots,L_n$. In fact, let

$$
\overrightarrow{G}^L = \lambda_1\overrightarrow{G}_1^{L_1} + \lambda_2\overrightarrow{G}_2^{L_2} + \cdots + \lambda_n\overrightarrow{G}_n^{L_n},
$$

where $(\lambda_1, \lambda_2, \cdots, \lambda_n) \neq (0, 0, \cdots, 0)$, and let

$$
\widehat{L} : \left(\bigcap_{l=1}^{i}\overrightarrow{G}_{k_l}\right) \setminus \left(\bigcup_{s\neq k_l,\cdots,k_i}\overrightarrow{G}_s\right) \to \sum_{l=1}^{i}\lambda_{k_l}L_{k_l}
$$

for integers $1\le i\le n$. Then, we are easily knowing that $\widehat{L}$ is nothing else but the labeling L on $\overrightarrow{G}$ by operation (2.1), a generation of (2.3) and (2.4) with

$$
\left\|\overrightarrow{G}^L\right\| = \sum_{i=1}^{n}\sum_{e\in E\left(\overrightarrow{G}_i\right)}\left\|\sum_{l=1}^{i}\lambda_{k_l}L_{k_l}(e)\right\|, \tag{2.11}
$$

$$
\left\langle\overrightarrow{G}^L, \overrightarrow{G'}^{L'}\right\rangle = \sum_{i=1}^{n}\sum_{e\in E\left(\overrightarrow{G}_i\right)}\left\langle\sum_{l=1}^{i}\lambda_{k_l}L^1_{k_l}(e), \sum_{s=1}^{i}\lambda'_{k_s}L^2_{k_s}\right\rangle, \tag{2.12}
$$

where $\overrightarrow{G'}^{L'} = \lambda'_1\overrightarrow{G}_1^{L_1} + \lambda'_2\overrightarrow{G}_2^{L_2} + \cdots + \lambda'_n\overrightarrow{G}_n^{L_n}$ and $\overrightarrow{G}_i = \left(\bigcap_{l=1}^{i}\overrightarrow{G}_{k_l}\right) \setminus \left(\bigcup_{s\neq k_l,\cdots,k_i}\overrightarrow{G}_s\right)$.

We therefore extend the Banach or Hilbert space $\mathscr{V}$ over graphs $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ following.

Theorem 2.4 *Let $\overrightarrow{G}_1, \overrightarrow{G}_2, \cdots, \overrightarrow{G}_n$ be oriented graphs embedded in a space $\mathscr{S}$ and $\mathscr{V}$ a Banach space over a field $\mathscr{F}$. Then $\left\langle\overrightarrow{G}_i, 1\le i\le n\right\rangle^{\mathscr{V}}$ with linear operators A^+_{vu}, A^+_{uv} for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ is a Banach space, and furthermore, if $\mathscr{V}$ is a Hilbert space, $\left\langle\overrightarrow{G}_i, 1\le i\le n\right\rangle^{\mathscr{V}}$ is a Hilbert space too.*

Proof We have shown, $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ is a linear normed space or inner space if $\mathscr{V}$ is a linear normed space or inner space, and for the later, let

$$\left\| \overrightarrow{G}^L \right\| = \sqrt{\left\langle \overrightarrow{G}^L, \overrightarrow{G}^L \right\rangle}$$

for $\overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$. Then $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ is a normed space and furthermore, it is a Hilbert space if it is complete. Thus, we are only need to show that any Cauchy sequence is converges in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$.

In fact, let $\left\{ \overrightarrow{H}_n^{L_n} \right\}$ be a Cauchy sequence in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$, i.e., for any number $\varepsilon > 0$, there always exists an integer $N(\varepsilon)$ such that

$$\left\| \overrightarrow{H}_n^{L_n} - \overrightarrow{H}_m^{L_m} \right\| < \varepsilon$$

if $n, m \geq N(\varepsilon)$. Let $\mathscr{G}^{\mathscr{V}}$ be the continuity flow space on $\overrightarrow{\mathscr{G}} = \bigcup\limits_{i=1}^{n} \overrightarrow{G}_i$. We embed each $\overrightarrow{H}_n^{L_n}$ to a $\overrightarrow{\mathscr{G}}^{\widehat{L}} \in \overrightarrow{\mathscr{G}}^{\mathscr{V}}$ by letting

$$\widehat{L}_n(e) = \begin{cases} L_n(e), & \text{if } \ e \in E\left(H_n\right) \\ \mathbf{0}, & \text{if } \ e \in E\left(\overrightarrow{\mathscr{G}} \setminus \overrightarrow{H}_n\right). \end{cases}$$

Then

$$\begin{aligned} \left\| \overrightarrow{\mathscr{G}}^{\widehat{L}_n} - \overrightarrow{\mathscr{G}}^{\widehat{L}_m} \right\| &= \sum_{e \in E\left(\overrightarrow{G}_n \setminus \overrightarrow{G}_m\right)} \|L_n(e)\| + \sum_{e \in E\left(\overrightarrow{G}_n \bigcap \overrightarrow{G}_m\right)} \|L_n(e) - L_m(e)\| \\ &+ \sum_{e \in E\left(\overrightarrow{G}_m \setminus \overrightarrow{G}_n\right)} \|-L_m(e)\| = \left\| \overrightarrow{H}_n^{L_n} - \overrightarrow{H}_m^{L_m} \right\| \leq \varepsilon. \end{aligned}$$

Thus, $\left\{ \overrightarrow{\mathscr{G}}^{\widehat{L}_n} \right\}$ is a Cauchy sequence also in $\overrightarrow{\mathscr{G}}^{\mathscr{V}}$. By definition,

$$\left\| \widehat{L}_n(e) - \widehat{L}_m(e) \right\| \leq \left\| \overrightarrow{\mathscr{G}}^{L_n} - \overrightarrow{\mathscr{G}}^{L_m} \right\| < \varepsilon,$$

i.e., $\{L_n(e)\}$ is a Cauchy sequence for $\forall e \in E\left(\overrightarrow{\mathscr{G}}\right)$, which is converges on in $\mathscr{V}$ by definition.

Let

$$\widehat{L}(e) = \lim_{n \to \infty} \widehat{L}_n(e)$$

for $\forall e \in E\left(\overrightarrow{\mathscr{G}}\right)$. Then it is clear that $\lim\limits_{n \to \infty} \overrightarrow{\mathscr{G}}^{\widehat{L}_n} = \overrightarrow{\mathscr{G}}^{\widehat{L}}$, which implies that $\{\overrightarrow{\mathscr{G}}^{\widehat{L}_n}\}$, i.e., $\left\{ \overrightarrow{H}_n^{L_n} \right\}$ is converges to $\overrightarrow{\mathscr{G}}^{\widehat{L}} \in \overrightarrow{\mathscr{G}}^{\mathscr{V}}$, an element in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ because of $\widehat{L}(e) \in \mathscr{V}$ for $\forall e \in E\left(\overrightarrow{\mathscr{G}}\right)$ and $\overrightarrow{\mathscr{G}} = \bigcup\limits_{i=1}^{n} \overrightarrow{G}_i$. $\square$

§3. Differential on Continuity Flows

3.1 Continuity Flow Expansion

Theorem 2.4 enables one to establish differentials and generalizes results in classical calculus in space $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$. Let L be kth differentiable to t on a domain $\mathscr{D} \subset \mathbb{R}$, where $k \geq 1$. Define

$$\frac{d\overrightarrow{G}^L}{dt} = \overrightarrow{G}^{\frac{dL}{dt}} \quad \text{and} \quad \int_0^t \overrightarrow{G}^L dt = \overrightarrow{G}^{\int_0^t Ldt}.$$

Then, we are easily to generalize Taylor formula in $\left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathscr{V}}$ following.

Theorem 3.1(Taylor) *Let $\overrightarrow{G}^L \in \left\langle \overrightarrow{G}_i, 1 \leq i \leq n \right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ and there exist kth order derivative of L to t on a domain $\mathscr{D} \subset \mathbb{R}$, where $k \geq 1$. If A^+_{vu}, A^+_{uv} are linear for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$, then*

$$\overrightarrow{G}^L = \overrightarrow{G}^{L(t_0)} + \frac{t-t_0}{1!}\overrightarrow{G}^{L'(t_0)} + \cdots + \frac{(t-t_0)^k}{k!}\overrightarrow{G}^{L^{(k)}(t_0)} + o\left((t-t_0)^{-k}\overrightarrow{G}\right), \qquad (3.1)$$

for $\forall t_0 \in \mathscr{D}$, where $o\left((t-t_0)^{-k}\overrightarrow{G}\right)$ denotes such an infinitesimal term $\widehat{L}$ of L that

$$\lim_{t\to t_0} \frac{\widehat{L}(v,u)}{(t-t_0)^k} = 0 \quad for \quad \forall (v,u) \in E\left(\overrightarrow{G}\right).$$

Particularly, if $L(v,u) = f(t)c_{vu}$, where c_{vu} is a constant, denoted by $f(t)\overrightarrow{G}^{L_C}$ with L_C : $(v,u) \to c_{vu}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and

$$f(t) = f(t_0) + \frac{(t-t_0)}{1!}f'(t_0) + \frac{(t-t_0)^2}{2!}f''(t_0) + \cdots + \frac{(t-t_0)^k}{k!}f^{(k)}(t_0) + o\left((t-t_0)^k\right),$$

then

$$f(t)\overrightarrow{G}^{L_C} = f(t) \cdot \overrightarrow{G}^{L_C}.$$

Proof Notice that $L(v,u)$ has kth order derivative to t on $\mathscr{D}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. By applying Taylor formula on t_0, we know that

$$L(v,u) = L(v,u)(t_0) + \frac{L'(v,u)(t_0)}{1!}(t-t_0) + \cdots + \frac{L^{(k)(v,u)(t_0)}}{k!} + o\left((t-t_0)^k\right)$$

if $t \to t_0$, where $o\left((t-t_0)^k\right)$ is an infinitesimal term $\widehat{L}(v,u)$ of $L(v,u)$ hold with

$$\lim_{t\to t_0} \frac{\widehat{L}(v,u)}{(t-t_0)^t} = 0$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. By operations (2.1) and (2.2),

$$\overrightarrow{G}^{L_1}+\overrightarrow{G}^{L_2}=\overrightarrow{G}^{L_1+L_2}\quad\text{and}\quad\lambda\overrightarrow{G}^{L}=\overrightarrow{G}^{\lambda\overrightarrow{L}}$$

because A^+_{vu}, A^+_{uv} are linear for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. We therefore get

$$\overrightarrow{G}^{L}=\overrightarrow{G}^{L(t_0)}+\frac{(t-t_0)}{1!}\overrightarrow{G}^{L'(t_0)}+\cdots+\frac{(t-t_0)^k}{k!}\overrightarrow{G}^{L^{(k)}(t_0)}+o\left((t-t_0)^{-k}\,\overrightarrow{G}\right)$$

for $t_0\in\mathscr{D}$, where $o\left((t-t_0)^{-k}\,\overrightarrow{G}\right)$ is an infinitesimal term $\widehat{L}$ of L, i.e.,

$$\lim_{t\to t_0}\frac{\widehat{L}(v,u)}{(t-t_0)^t}=0$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. Calculation also shows that

$$\begin{aligned}
f(t)\overrightarrow{G}^{L_C(v,u)} &= \overrightarrow{G}^{f(t)L_C(v,u)}=\overrightarrow{G}^{\left(f(t_0)+\frac{(t-t_0)}{1!}f'(t_0)\cdots+\frac{(t-t_0)^k}{k!}f^{(k)}(t_0)+o\left((t-t_0)^k\right)\right)c_{vu}}\\
&= f(t_0)c_{vu}\overrightarrow{G}+\frac{f'(t_0)(t-t_0)}{1!}c_{vu}\overrightarrow{G}+\frac{f''(t_0)(t-t_0)^2}{2!}c_{vu}\overrightarrow{G}\\
&\quad+\cdots+\frac{f^{(k)}(t_0)\,(t-t_0)^k}{k!}c_{vu}\overrightarrow{G}+o\left((t-t_0)^k\right)\overrightarrow{G}\\
&= \left(f(t_0)+\frac{(t-t_0)}{1!}f'(t_0)\cdots+\frac{(t-t_0)^k}{k!}f^{(k)}(t_0)+o\left((t-t_0)^k\right)\right)c_{vu}\overrightarrow{G}\\
&= f(t)c_{vu}\overrightarrow{G}=f(t)\cdot\overrightarrow{G}^{L_C(v,u)},
\end{aligned}$$

i.e.,

$$f(t)\overrightarrow{G}^{L_C}=f(t)\cdot\overrightarrow{G}^{L_C}.$$

This completes the proof. □

Taylor expansion formula for continuity flow $\overrightarrow{G}^{L}$ enables one to find interesting results on $\overrightarrow{G}^{L}$ such as those of the following.

Theorem 3.2 *Let $f(t)$ be a k differentiable function to t on a domain $\mathscr{D}\subset\mathbb{R}$ with $0\in\mathscr{D}$ and $f(0\overrightarrow{G})=f(0)\overrightarrow{G}$. If A^+_{vu}, A^+_{uv} are linear for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, then*

$$f(t)\overrightarrow{G}=f\left(t\overrightarrow{G}\right).\tag{3.2}$$

Proof Let $t_0=0$ in the Taylor formula. We know that

$$f(t)=f(0)+\frac{f'(0)}{1!}t+\frac{f''(0)}{2!}t^2+\cdots+\frac{f^{(k)}(0)}{k!}t^k+o\left(t^k\right).$$

Notice that

$$\begin{aligned} f(t)\overrightarrow{G} &= \left(f(0)+\frac{f'(0)}{1!}t+\frac{f''(0)}{2!}t^2+\cdots+\frac{f^{(k)}(0)}{k!}t^k+o\left(t^k\right)\right)\overrightarrow{G} \\ &= \overrightarrow{G}^{f(0)+\frac{f'(0)}{1!}t+\frac{f''(0)}{2!}t^2+\cdots+\frac{f^{(k)}(0)}{k!}t^k+o\left(t^k\right)} \\ &= f(0)\overrightarrow{G}+\frac{f'(0)t}{1!}\overrightarrow{G}+\cdots+\frac{f^{(k)}(0)t^k}{k!}\overrightarrow{G}+o\left(t^k\right)\overrightarrow{G} \end{aligned}$$

and by definition,

$$\begin{aligned} f\left(t\overrightarrow{G}\right) &= f\left(0\overrightarrow{G}\right)+\frac{f'(0)}{1!}\left(t\overrightarrow{G}\right)+\frac{f''(0)}{2!}\left(t\overrightarrow{G}\right)^2 \\ &\quad +\cdots+\frac{f^{(k)}(0)}{k!}\left(t\overrightarrow{G}\right)^k+o\left(\left(t\overrightarrow{G}\right)^k\right) \\ &= f\left(0\overrightarrow{G}\right)+\frac{f'(0)}{1!}t\overrightarrow{G}+\frac{f''(0)}{2!}t^2\overrightarrow{G}+\cdots+\frac{f^{(k)}(0)}{k!}t^k\overrightarrow{G}+o\left(t^k\right)\overrightarrow{G} \end{aligned}$$

because of $\left(t\overrightarrow{G}\right)^i=\overrightarrow{G}^{t^i}=t^i\overrightarrow{G}$ for any integer $1\leq i\leq k$. Notice that $f(0\overrightarrow{G})=f(0)\overrightarrow{G}$. We therefore get that

$$f(t)\overrightarrow{G}=f\left(t\overrightarrow{G}\right).$$

□

Theorem 3.2 enables one easily getting Taylor expansion formulas by $f\left(t\overrightarrow{G}\right)$. For example, let $f(t)=e^t$. Then

$$e^t\overrightarrow{G}=e^{t\overrightarrow{G}} \tag{3.3}$$

by Theorem 3.5. Notice that $(e^t)'=e^t$ and $e^0=1$. We know that

$$e^{t\overrightarrow{G}}=e^t\overrightarrow{G}=\overrightarrow{G}+\frac{t}{1!}\overrightarrow{G}+\frac{t^2}{2!}\overrightarrow{G}+\cdots+\frac{t^k}{k!}\overrightarrow{G}+\cdots \tag{3.4}$$

and

$$e^{t\overrightarrow{G}}\cdot e^{s\overrightarrow{G}}=\overrightarrow{G}^{e^t}\cdot\overrightarrow{G}^{e^s}=\overrightarrow{G}^{e^t\cdot e^s}=\overrightarrow{G}^{e^{t+s}}=e^{(t+s)\overrightarrow{G}}, \tag{3.5}$$

where t and s are variables, and similarly, for a real number α if $|t|<1$,

$$\left(\overrightarrow{G}+t\overrightarrow{G}\right)^\alpha=\overrightarrow{G}+\frac{\alpha t}{1!}\overrightarrow{G}+\cdots+\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)t^n}{n!}\overrightarrow{G}+\cdots \tag{3.6}$$

3.2 Limitation

Definition 3.3 *Let $\overrightarrow{G}^L,\overrightarrow{G}_1^{L_1}\in\left\langle\overrightarrow{G}_i,1\leq i\leq n\right\rangle^{\mathscr{V}}$ with L,L_1 dependent on a variable $t\in[a,b]\subset(-\infty,+\infty)$ and linear continuous end-operators A_{vu}^+ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$. For $t_0\in[a,b]$ and any number $\varepsilon>0$, if there is always a number $\delta(\varepsilon)$ such that if $|t-t_0|\leq\delta(\varepsilon)$ then $\left\|\overrightarrow{G}_1^{L_1}-\overrightarrow{G}^L\right\|<\varepsilon$, then, $\overrightarrow{G}_1^{L_1}$ is said to be converged to $\overrightarrow{G}^L$ as $t\to t_0$, denoted by $\lim\limits_{t\to t_0}\overrightarrow{G}_1^{L_1}=\overrightarrow{G}^L$. Particularly, if $\overrightarrow{G}^L$ is a continuity flow with a constant $L(v)$ for $\forall v\in V\left(\overrightarrow{G}\right)$ and $t_0=+\infty$, $\overrightarrow{G}_1^{L_1}$ is said to be $\overrightarrow{G}$-synchronized.*

Applying Theorem 1.4, we know that there are positive constants $c_{vu} \in \mathbb{R}$ such that $\|A_{vu}^{+}\| \leq c_{vu}^{+}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$.

By definition, it is clear that

$$\begin{aligned}
&\left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^{L}\right\| \\
&= \left\|\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)^{L_1}\right\| + \left\|\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)^{L_1-L}\right\| + \left\|\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)^{-L}\right\| \\
&= \sum_{u\in N_{G_1\setminus G}(v)} \left\|L_1^{A'^{+}_{vu}}(v,u)\right\| + \sum_{u\in N_{G_1\bigcap G}(v)} \left\|\left(L_1^{A'^{+}_{vu}} - L_{vu}^{A^{+}_{vu}}\right)(v,u)\right\| + \sum_{u\in N_{G\setminus G_1}(v)} \left\|-L^{A^{+}_{vu}}(v,u)\right\| \\
&\leq \sum_{u\in N_{G_1\setminus G}(v)} c_{vu}^{+}\|L_1(v,u)\| + \sum_{u\in N_{G_1\bigcap G}(v)} c_{vu}^{+}\|\left(L_1 - L\right)(v,u)\| + \sum_{u\in N_{G\setminus G_1}(v)} c_{vu}^{+}\| - L(v,u)\|.
\end{aligned}$$

and $\|L(v,u)\| \geq 0$ for $(v,u) \in E\left(\overrightarrow{G}\right)$ and $E\left(\overrightarrow{G}_1\right)$. Let

$$c_{G_1G}^{\max} = \left\{\max_{(v,u)\in E(G_1)} c_{vu}^{+}, \max_{(v,u)\in E(G_1)} c_{vu}^{+}\right\}.$$

If $\left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^{L}\right\| < \varepsilon$, we easily get that $\|L_1(v,u)\| < c_{G_1G}^{\max}\varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)$, $\|(L_1 - L)(v,u)\| < c_{G_1G}^{\max}\varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)$ and $\| - L(v,u)\| < c_{G_1G}^{\max}\varepsilon$ for $(v,u) \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)$.

Conversely, if $\|L_1(v,u)\| < \varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)$, $\|(L_1 - L)(v,u)\| < \varepsilon$ for $(v,u) \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)$ and $\| - L(v,u)\| < \varepsilon$ for $(v,u) \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)$, we easily know that

$$\begin{aligned}
\left\|\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^{L}\right\| \quad &= \sum_{u\in N_{G_1\setminus G}(v)} \left\|L_1^{A'^{+}_{vu}}(v,u)\right\| + \sum_{u\in N_{G_1\bigcap G}(v)} \left\|\left(L_1^{A'^{+}_{vu}} - L_{vu}^{A^{+}_{vu}}\right)(v,u)\right\| \\
&+ \sum_{u\in N_{G\setminus G_1}(v)} \left\|-L^{A^{+}_{vu}}(v,u)\right\| \\
&\leq \sum_{u\in N_{G_1\setminus G}(v)} c_{vu}^{+}\|L_1(v,u)\| + \sum_{u\in N_{G_1\bigcap G}(v)} c_{vu}^{+}\|\left(L_1 - L\right)(v,u)\| \\
&+ \sum_{u\in N_{G\setminus G_1}(v)} c_{vu}^{+}\| - L(v,u)\| \\
&< \left|\overrightarrow{G}_1 \setminus \overrightarrow{G}\right| c_{G_1G}^{\max}\varepsilon + \left|\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right| c_{G_1G}^{\max}\varepsilon + \left|\overrightarrow{G} \setminus \overrightarrow{G}_1\right| c_{G_1G}^{\max}\varepsilon = \left|\overrightarrow{G}_1 \bigcup \overrightarrow{G}\right| c_{G_1G}^{\max}\varepsilon.
\end{aligned}$$

Thus, we get an equivalent condition for $\lim\limits_{t\to t_0} \overrightarrow{G}_1^{L_1} = \overrightarrow{G}^{L}$ following.

Theorem 3.4 $\lim\limits_{t\to t_0} \overrightarrow{G}_1^{L_1} = \overrightarrow{G}^{L}$ *if and only if for any number* $\varepsilon > 0$ *there is always a number* $\delta(\varepsilon)$ *such that if* $|t - t_0| \leq \delta(\varepsilon)$ *then* $\|L_1(v,u)\| < \varepsilon$ *for* $(v,u) \in E\left(\overrightarrow{G}_1 \setminus \overrightarrow{G}\right)$, $\|(L_1 - L)(v,u)\| < \varepsilon$ *for* $(v,u) \in E\left(\overrightarrow{G}_1 \bigcap \overrightarrow{G}\right)$ *and* $\| - L(v,u)\| < \varepsilon$ *for* $(v,u) \in E\left(\overrightarrow{G} \setminus \overrightarrow{G}_1\right)$,*i.e.,* $\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^{L}$ *is an infinitesimal or* $\lim\limits_{t\to t_0}\left(\overrightarrow{G}_1^{L_1} - \overrightarrow{G}^{L}\right) = \mathbf{O}$.

If $\lim_{t\to t_0}\overrightarrow{G}^L$, $\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}$ and $\lim_{t\to t_0}\overrightarrow{G_2}^{L_2}$ exist, the formulas following are clearly true by definition:

$$\lim_{t\to t_0}\left(\overrightarrow{G_1}^{L_1}+\overrightarrow{G_2}^{L_2}\right)=\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}+\lim_{t\to t_0}\overrightarrow{G_2}^{L_2},$$
$$\lim_{t\to t_0}\left(\overrightarrow{G_1}^{L_1}\cdot\overrightarrow{G_2}^{L_2}\right)=\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}\cdot\lim_{t\to t_0}\overrightarrow{G_2}^{L_2},$$
$$\lim_{t\to t_0}\left(\overrightarrow{G}^{L}\cdot\left(\overrightarrow{G_1}^{L_1}+\overrightarrow{G_2}^{L_2}\right)\right)=\lim_{t\to t_0}\overrightarrow{G}^{L}\cdot\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}+\lim_{t\to t_0}\overrightarrow{G}^{L}\cdot\lim_{t\to t_0}\overrightarrow{G_2}^{L_2},$$
$$\lim_{t\to t_0}\left(\left(\overrightarrow{G_1}^{L_1}+\overrightarrow{G_2}^{L_2}\right)\cdot\overrightarrow{G}^{L}\right)=\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}\cdot\lim_{t\to t_0}\overrightarrow{G}^{L}+\lim_{t\to t_0}\overrightarrow{G_2}^{L_2}\cdot\lim_{t\to t_0}\overrightarrow{G}^{L}$$

and furthermore, if $\lim_{t\to t_0}\overrightarrow{G_2}^{L_2}\neq\mathbf{O}$, then

$$\lim_{t\to t_0}\left(\frac{\overrightarrow{G_1}^{L_1}}{\overrightarrow{G_2}^{L_2}}\right)=\lim_{t\to t_0}\left(\overrightarrow{G_1}^{L_1}\cdot\overrightarrow{G_2}^{L_2^{-1}}\right)=\frac{\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}}{\lim_{t\to t_0}\overrightarrow{G_2}^{L_2}}.$$

Theorem 3.5(L'Hospital's rule) *If $\lim_{t\to t_0}\overrightarrow{G_1}^{L_1}=\mathbf{O}$, $\lim_{t\to t_0}\overrightarrow{G_2}^{L_2}=\mathbf{O}$ and L_1,L_2 are differentiable respect to t with $\lim_{t\to t_0}L_1'(v,u)=0$ for $(v,u)\in E\left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)$, $\lim_{t\to t_0}L_2'(v,u)\neq 0$ for $(v,u)\in E\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)$ and $\lim_{t\to t_0}L_2'(v,u)=0$ for $(v,u)\in E\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)$, then,*

$$\lim_{t\to t_0}\left(\frac{\overrightarrow{G_1}^{L_1}}{\overrightarrow{G_2}^{L_2}}\right)=\frac{\lim_{t\to t_0}\overrightarrow{G_1}^{L_1'}}{\lim_{t\to t_0}\overrightarrow{G_2}^{L_2'}}.$$

Proof By definition, we know that

$$\begin{aligned}
\lim_{t\to t_0}\left(\frac{\overrightarrow{G_1}^{L_1}}{\overrightarrow{G_2}^{L_2}}\right) &= \lim_{t\to t_0}\left(\overrightarrow{G_1}^{L_1}\cdot\overrightarrow{G_2}^{L_2^{-1}}\right)\\
&= \lim_{t\to t_0}\left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)^{L_1}\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{L_1\cdot L_2^{-1}}\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)^{L_2}\\
&= \lim_{t\to t_0}\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{L_1\cdot L_2^{-1}}=\lim_{t\to t_0}\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{\frac{L_1}{L_2^{-1}}}\\
&= \left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{\lim_{t\to t_0}\frac{L_1}{L_2^{-1}}}=\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{\frac{\lim_{t\to t_0}{L'}_1}{\lim_{t\to t_0}{L'}_2^{-1}}}\\
&= \left(\overrightarrow{G}_1\setminus\overrightarrow{G}_2\right)^{\lim_{t\to t_0}L_1'}\left(\overrightarrow{G}_1\bigcap\overrightarrow{G}_2\right)^{\lim_{t\to t_0}{L'}_1\cdot\lim_{t\to t_0}{L'}_2^{-1}}\left(\overrightarrow{G}_2\setminus\overrightarrow{G}_1\right)^{\lim_{t\to t_0}L_2'}\\
&= \overrightarrow{G}_1^{\lim_{t\to t_0}L_1'}\cdot\overrightarrow{G}_2^{\lim_{t\to t_0}{L'}_2^{-1}}=\frac{\lim_{t\to t_0}\overrightarrow{G}_1^{L_1'}}{\lim_{t\to t_0}\overrightarrow{G}_2^{L_2'}}.
\end{aligned}$$

This completes the proof. □

Corollary 3.6 *If* $\lim\limits_{t\to t_0}\overrightarrow{G}^{L_1}=\mathbf{O}$, $\lim\limits_{t\to t_0}\overrightarrow{G}^{L_2}=\mathbf{O}$ *and* L_1, L_2 *are differentiable respect to* t *with* $\lim\limits_{t\to t_0}L_2'(v,u)\neq 0$ *for* $(v,u)\in E\left(\overrightarrow{G}\right)$, *then*

$$\lim_{t\to t_0}\left(\frac{\overrightarrow{G}^{L_1}}{\overrightarrow{G}^{L_2}}\right)=\frac{\lim\limits_{t\to t_0}\overrightarrow{G}^{L_1'}}{\lim\limits_{t\to t_0}\overrightarrow{G}^{L_2'}}.$$

Generally, by Taylor formula

$$\overrightarrow{G}^{L}=\overrightarrow{G}^{L(t_0)}+\frac{t-t_0}{1!}\overrightarrow{G}^{L'(t_0)}+\cdots+\frac{(t-t_0)^k}{k!}\overrightarrow{G}^{L^{(k)}(t_0)}+o\left((t-t_0)^{-k}\overrightarrow{G}\right),$$

if $L_1(t_0)=L_1'(t_0)=\cdots=L_1^{(k-1)}(t_0)=0$ and $L_2(t_0)=L_2'(t_0)=\cdots=L_2^{(k-1)}(t_0)=0$ but $L_2^{(k)}(t_0)\neq 0$, then

$$\begin{aligned}\overrightarrow{G}_1^{L_1} &= \frac{(t-t_0)^k}{k!}\overrightarrow{G}_1^{L_1^{(k)}(t_0)}+o\left((t-t_0)^{-k}\overrightarrow{G}_1\right),\\ \overrightarrow{G}_2^{L_2} &= \frac{(t-t_0)^k}{k!}\overrightarrow{G}_2^{L_2^{(k)}(t_0)}+o\left((t-t_0)^{-k}\overrightarrow{G}_2\right).\end{aligned}$$

We are easily know the following result.

Theorem 3.7 *If* $\lim\limits_{t\to t_0}\overrightarrow{G_1}^{L_1}=\mathbf{O}$, $\lim\limits_{t\to t_0}\overrightarrow{G_2}^{L_2}=\mathbf{O}$ *and* $L_1(t_0)=L_1'(t_0)=\cdots=L_1^{(k-1)}(t_0)=0$ *and* $L_2(t_0)=L_2'(t_0)=\cdots=L_2^{(k-1)}(t_0)=0$ *but* $L_2^{(k)}(t_0)\neq 0$, *then*

$$\lim_{t\to t_0}\frac{\overrightarrow{G_1}^{L_1}}{\overrightarrow{G}_2^{L_2}}=\frac{\lim\limits_{t\to t_0}\overrightarrow{G}_1^{L_1^{(k)}(t_0)}}{\lim\limits_{t\to t_0}\overrightarrow{G}_2^{L_2^{(k)}(t_0)}}.$$

Example 3.8 Let $\overrightarrow{G}_1=\overrightarrow{G}_2=\overrightarrow{C}_n$, $A^{+}_{v_iv_{i+1}}=1$, $A^{+}_{v_iv_{i-1}}=2$ and

$$f_i=\frac{f_1+\left(2^{i-1}-1\right)F(\overline{x})}{2^{i-1}}+\frac{n!}{(2n+1)e^t}$$

for integers $1\leq i\leq n$ in Fig.4.

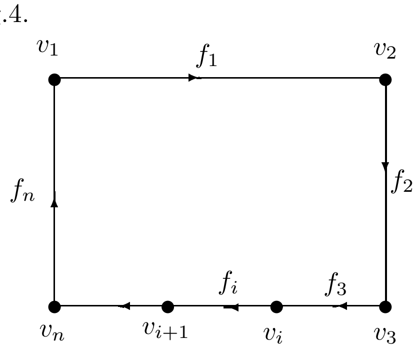

Fig.4

Calculation shows That

$$\begin{aligned} L(v_i) &= 2f_{i+1} - f_i = 2 \times \frac{f_1 + \left(2^i - 1\right) F(\overline{x})}{2^i} - \frac{f_1 + \left(2^{i-1} - 1\right) F(\overline{x})}{2^{i-1}} \\ &= F(\overline{x}) + \frac{n!}{(2n+1)e^t}. \end{aligned}$$

Calculation shows that $\lim\limits_{t\to\infty} L(v_i) = F(\overline{x})$, i.e., $\lim\limits_{t\to\infty} \overrightarrow{C}_n^L = \overrightarrow{C}_n^{\widehat{L}}$, where, $\widehat{L}(v_i) = F(\overline{x})$ for integers $1 \leq i \leq n$, i.e., $\overrightarrow{C}_n^L$ is $\overrightarrow{G}$-synchronized.

§4. Continuity Flow Equations

A continuity flow $\overrightarrow{G}^L$ is in fact an operator $L : \overrightarrow{G} \to \mathscr{B}$ determined by $L(v,u) \in \mathscr{B}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$. Generally, let

$$[L]_{m\times n} = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ L_{m1} & L_{m2} & \cdots & L_{mn} \end{pmatrix}$$

with $L_{ij} : \overrightarrow{G} \to \mathscr{B}$ for $1 \leq i \leq m, 1 \leq j \leq n$, called operator matrix. Particularly, if for integers $1 \leq i \leq m, 1 \leq j \leq n$, $L_{ij} : \overrightarrow{G} \to \mathbb{R}$, we can also determine its rank as the usual, labeled the edge (v,u) by $\mathrm{Rank}[L]_{m\times n}$ for $\forall (v,u) \in E\left(\overrightarrow{G}\right)$ and get a labeled graph $\overrightarrow{G}^{\mathrm{Rank}[L]}$. Then we get a result following.

Theorem 4.1 *A linear continuity flow equations*

$$\begin{cases} x_1\overrightarrow{G}^{L_{11}} + x_2\overrightarrow{G}^{L_{12}} + \cdots + x_n\overrightarrow{G}^{L_{n1}} = \overrightarrow{G}^{L_1} \\ x_1\overrightarrow{G}^{L_{21}} + x_2\overrightarrow{G}^{L_{22}} + \cdots + x_n\overrightarrow{G}^{L_{2n}} = \overrightarrow{G}^{L_2} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1\overrightarrow{G}^{L_{n1}} + x_2\overrightarrow{G}^{L_{n2}} + \cdots + x_n\overrightarrow{G}^{L_{nn}} = \overrightarrow{G}^{L_n} \end{cases} \tag{4.1}$$

is solvable if and only if

$$\overrightarrow{G}^{\mathrm{Rank}[L]} = \overrightarrow{G}^{\mathrm{Rank}\left[\overline{L}\right]}, \tag{4.2}$$

where

$$[L] = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix} \quad \textit{and} \quad \left[\overline{L}\right] = \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} & L_1 \\ L_{21} & L_{22} & \cdots & L_{2n} & L_2 \\ \cdots & \cdots & \cdots & \cdots & \\ L_{n1} & L_{n2} & \cdots & L_{nn} & L_n \end{pmatrix}.$$

Proof Clearly, if (4.1) is solvable, then for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, the linear equations

$$\begin{cases} x_1L_{11}(v,u)+x_2L_{12}(v,u)+\cdots+x_nL_{n1}(v,u0=L_1(v,u)\\ x_1L_{21}(v,u)+x_2L_{22}(v,u)+\cdots+x_nL_{21}(v,u0=L_2(v,u)\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_1L_{n1}(v,u)+x_2L_{n2}(v,u)+\cdots+x_nL_{nn}(v,u0=L_n(v,u)\end{cases}$$

is solvable. By linear algebra, there must be

$$\text{Rank}\begin{pmatrix} L_{11}(v,u) & L_{12}(v,u) & \cdots & L_{1n}(v,u)\\ L_{21}(v,u) & L_{22}(v,u) & \cdots & L_{2n}(v,u)\\ \cdots & \cdots & \cdots & \cdots\\ L_{n1}(v,u) & L_{n2}(v,u) & \cdots & L_{nn}(v,u)\end{pmatrix}=$$

$$\text{Rank}\begin{pmatrix} L_{11}(v,u) & L_{12}(v,u) & \cdots & L_{1n}(v,u) & L_1(v,u)\\ L_{21}(v,u) & L_{22}(v,u) & \cdots & L_{2n}(v,u) & L_2(v,u)\\ \cdots & \cdots & \cdots & \cdots & \\ L_{n1}(v,u) & L_{n2}(v,u) & \cdots & L_{nn}(v,u) & L_n(v,u)\end{pmatrix},$$

which implies that

$$\overrightarrow{G}^{\text{Rank}[L]}=\overrightarrow{G}^{\text{Rank}\left[\overline{L}\right]}.$$

Conversely, if the (4.2) is hold, then for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, the linear equations

$$\begin{cases} x_1L_{11}(v,u)+x_2L_{12}(v,u)+\cdots+x_nL_{n1}(v,u0=L_1(v,u)\\ x_1L_{21}(v,u)+x_2L_{22}(v,u)+\cdots+x_nL_{21}(v,u0=L_2(v,u)\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ x_1L_{n1}(v,u)+x_2L_{n2}(v,u)+\cdots+x_nL_{nn}(v,u0=L_n(v,u)\end{cases}$$

is solvable, i.e., the equations (4.1) is solvable. □

Theorem 4.2 *A continuity flow equation*

$$\lambda^s\overrightarrow{G}^{L_s}+\lambda^{s-1}\overrightarrow{G}^{L_{s-1}}+\cdots+\overrightarrow{G}^{L_0}=\mathbf{O} \tag{4.3}$$

always has solutions $\overrightarrow{G}^{L_\lambda}$ with $L_\lambda:(v,u)\in E\left(\overrightarrow{G}\right)\to\{\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_s^{vu}\}$, where $\lambda_i^{vu}, 1\leq i\leq s$ are roots of the equation

$$\alpha_s^{vu}\lambda^s+\alpha_{s-1}^{vu}\lambda^{s-1}+\cdots+\alpha_0^{vu}=0 \tag{4.4}$$

with $L_i:(v,u)\to\alpha_i^{v,u}$, $\alpha_s^{vu}\neq 0$ a constant for $(v,u)\in E\left(\overrightarrow{G}\right)$ and $1\leq i\leq s$.

For $(v,u)\in E\left(\overrightarrow{G}\right)$, if n^{vu} is the maximum number i with $L_i(v,u)\neq 0$, then there are

$\prod\limits_{(v,u)\in E\left(\overrightarrow{G}\right)} n^{vu}$ *solutions* $\overrightarrow{G}^{L_\lambda}$, *and particularly, if* $L_s(v,u)\neq 0$ *for* $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, *there are* $s^{\left|E\left(\overrightarrow{G}\right)\right|}$ *solutions* $\overrightarrow{G}^{L_\lambda}$ *of equation* (4.3).

Proof By the fundamental theorem of algebra, we know there are s roots $\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_s^{vu}$ for the equation (4.3). Whence, $L_\lambda\overrightarrow{G}$ is a solution of equation (4.2) because of

$$\begin{aligned}&\left(\lambda\overrightarrow{G}\right)^s\cdot\overrightarrow{G}^{L_s}+\left(\lambda\overrightarrow{G}\right)^{s-1}\cdot\overrightarrow{G}^{L_{s-1}}+\cdots+\left(\lambda\overrightarrow{G}\right)^0\cdot\overrightarrow{G}^{L_0}\\&=\overrightarrow{G}^{\lambda^sL_s}+\overrightarrow{G}^{\lambda^{s-1}L_{s-1}}+\cdots+\overrightarrow{G}^{\lambda^0L_0}=\overrightarrow{G}^{\lambda^sL_s+\lambda^{s-1}L_{s-1}+\cdots+L_0}\end{aligned}$$

and

$$\lambda^sL_s+\lambda^{s-1}L_{s-1}+\cdots+L_0:\ (v,u)\to\alpha_s^{vu}\lambda^s+\alpha_{s-1}^{vu}\lambda^{s-1}+\cdots+\alpha_0^{vu}=0,$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, i.e.,

$$\left(\lambda\overrightarrow{G}\right)^s\cdot\overrightarrow{G}^{L_s}+\left(\lambda\overrightarrow{G}\right)^{s-1}\cdot\overrightarrow{G}^{L_{s-1}}+\cdots+\left(\lambda\overrightarrow{G}\right)^0\cdot\overrightarrow{G}^{L_0}=0\overrightarrow{G}=\mathbf{O}.$$

Count the number of different L_λ for $(v,u)\in E\left(\overrightarrow{G}\right)$. It is nothing else but just n^{vu}. Therefore, the number of solutions of equation (4.3) is $\prod\limits_{(v,u)\in E\left(\overrightarrow{G}\right)} n^{vu}$. $\square$

Theorem 4.3 *A continuity flow equation*

$$\frac{d\overrightarrow{G}^L}{dt}=\overrightarrow{G}^{L_\alpha}\cdot\overrightarrow{G}^L \tag{4.5}$$

with initial values $\left.\overrightarrow{G}^L\right|_{t=0}=\overrightarrow{G}^{L_\beta}$ *always has a solution*

$$\overrightarrow{G}^L=\overrightarrow{G}^{L_\beta}\cdot\left(e^{tL_\alpha}\overrightarrow{G}\right),$$

where $L_\alpha:(v,u)\to\alpha_{vu}$, $L_\beta:(v,u)\to\beta_{vu}$ *are constants for* $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Proof A calculation shows that

$$\overrightarrow{G}^{\frac{dL}{dt}}=\frac{d\overrightarrow{G}^L}{dt}=\overrightarrow{G}^{L_\alpha}\cdot\overrightarrow{G}^L=\overrightarrow{G}^{L_\alpha\cdot L},$$

which implies that

$$\frac{dL}{dt}=\alpha_{vu}L \tag{4.6}$$

for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Solving equation (4.6) enables one knowing that $L(v,u)=\beta_{vu}e^{t\alpha_{vu}}$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Whence, the solution of (4.5) is

$$\overrightarrow{G}^{L}=\overrightarrow{G}^{L_\beta e^{tL_\alpha}}=\overrightarrow{G}^{L_\beta}\cdot\left(e^{tL_\alpha}\overrightarrow{G}\right)$$

and conversely, by Theorem 3.2,

$$\begin{aligned}\frac{d\overrightarrow{G}^{L_\beta e^{tL_\alpha}}}{dt} &= \overrightarrow{G}^{\frac{d\left(L_\beta e^{tL_\alpha}\right)}{dt}}=\overrightarrow{G}^{L_\alpha L_\beta e^{tL_\alpha}}\\ &= \overrightarrow{G}^{L_\alpha}\cdot\overrightarrow{G}^{L_\beta e^{tL_\alpha}},\end{aligned}$$

i.e.,

$$\frac{d\overrightarrow{G}^{L}}{dt}=\overrightarrow{G}^{L_\alpha}\cdot\overrightarrow{G}^{L}$$

if $\overrightarrow{G}^{L}=\overrightarrow{G}^{L_\beta}\cdot\left(e^{tL_\alpha}\overrightarrow{G}\right)$. This completes the proof. □

Theorem 4.3 can be generalized to the case of $L:(v,u)\to\mathbb{R}^n, n\geq 2$ for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Theorem 4.4 *A complex flow equation*

$$\frac{d\overrightarrow{G}^{L}}{dt}=\overrightarrow{G}^{L_\alpha}\cdot\overrightarrow{G}^{L} \tag{4.7}$$

with initial values $\left.\overrightarrow{G}^{L}\right|_{t=0}=\overrightarrow{G}^{L_\beta}$ *always has a solution*

$$\overrightarrow{G}^{L}=\overrightarrow{G}^{L_\beta}\cdot\left(e^{tL_\alpha}\overrightarrow{G}\right),$$

where $L_\alpha:(v,u)\to\left(\alpha_{vu}^1,\alpha_{vu}^2,\cdots,\alpha_{vu}^n\right)$, $L_\beta:(v,u)\to\left(\beta_{vu}^1,\beta_{vu}^2,\cdots,\beta_{vu}^n\right)$ *with constants* α_{vu}^i, β_{vu}^i, $1\leq i\leq n$ *for* $\forall(v,u)\in E\left(\overrightarrow{G}\right)$.

Theorem 4.5 *A complex flow equation*

$$\overrightarrow{G}^{L_{\alpha_n}}\cdot\frac{d^n\overrightarrow{G}^{L}}{dt^n}+\overrightarrow{G}^{L_{\alpha_{n-1}}}\cdot\frac{d^{n-1}\overrightarrow{G}^{L}}{dt^{n-1}}+\cdots+\overrightarrow{G}^{L_{\alpha_0}}\cdot\overrightarrow{G}^{L}=\mathbf{O} \tag{4.8}$$

with $L_{\alpha_i}:(v,u)\to\alpha_i^{vu}$ *constants for* $\forall(v,u)\in E\left(\overrightarrow{G}\right)$ *and integers* $0\leq i\leq n$ *always has a general solution* $\overrightarrow{G}^{L_\lambda}$ *with*

$$L_\lambda:(v,u)\to\left\{\ 0,\ \sum_{i=1}^{s}h_i(t)e^{\lambda_i^{vu}t}\ \right\}$$

for $(v,u)\in E\left(\overrightarrow{G}\right)$, *where* $h_{m_i}(t)$ *is a polynomial of degree* $\leq m_i-1$ *on* t, $m_1+m_2+\cdots+m_s=n$ *and* $\lambda_1^{vu},\lambda_2^{vu},\cdots,\lambda_s^{vu}$ *are the distinct roots of characteristic equation*

$$\alpha_n^{vu}\lambda^n+\alpha_{n-1}^{vu}\lambda^{n-1}+\cdots+\alpha_0^{vu}=0$$

with $\alpha_n^{vu} \neq 0$ *for* $(v,u) \in E\left(\overrightarrow{G}\right)$.

Proof Clearly, the equation (4.8) on an edge $(v,u) \in E\left(\overrightarrow{G}\right)$ is

$$\alpha_n^{vu}\frac{d^nL(v,u)}{dt^n} + \alpha_{n-1}^{vu}\frac{d^{n-1}L(v,u)}{dt^{n-1}} + \cdots + \alpha_0 = 0. \tag{4.9}$$

As usual, assuming the solution of (4.6) has the form $\overrightarrow{G}^L = e^{\lambda t}\overrightarrow{G}$. Calculation shows that

$$\begin{aligned}
\frac{d\overrightarrow{G}^L}{dt} &= \lambda e^{\lambda t}\overrightarrow{G} = \lambda\overrightarrow{G},\\
\frac{d^2\overrightarrow{G}^L}{dt^2} &= \lambda^2 e^{\lambda t}\overrightarrow{G} = \lambda^2\overrightarrow{G},\\
\cdots\cdots &\cdots \cdots\cdots\cdots\cdots\cdots,\\
\frac{d^n\overrightarrow{G}^L}{dt^n} &= \lambda^n e^{\lambda t}\overrightarrow{G} = \lambda^n\overrightarrow{G}.
\end{aligned}$$

Substituting these calculation results into (4.8), we get that

$$\left(\lambda^n\overrightarrow{G}^{L_{\alpha_n}} + \lambda^{n-1}\overrightarrow{G}^{L_{\alpha_{n-1}}} + \cdots + \overrightarrow{G}^{L_{\alpha_0}}\right)\cdot\overrightarrow{G}^L = \mathbf{O},$$

i.e.,

$$\overrightarrow{G}^{\left(\lambda^n\cdot L_{\alpha_n} + \lambda^{n-1}\cdot L_{\alpha_{n-1}} + \cdots + L_{\alpha_0}\right)\cdot L} = \mathbf{O},$$

which implies that for $\forall(v,u) \in E\left(\overrightarrow{G}\right)$,

$$\lambda^n\alpha_n^{vu} + \lambda^{n-1}\alpha_{n-1}^{vu} + \cdots + \alpha_0 = 0 \tag{4.10}$$

or

$$L(v,u) = 0.$$

Let $\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_s^{vu}$ be the distinct roots with respective multiplicities $m_1^{vu}, m_2^{vu}, \cdots, m_s^{vu}$ of equation (4.8). We know the general solution of (4.9) is

$$L(v,u) = \sum_{i=1}^{s} h_i(t)e^{\lambda_i^{vu}t}$$

with $h_{m_i}(t)$ a polynomial of degree$\leq m_i - 1$ on t by the theory of ordinary differential equations. Therefore, the general solution of (4.8) is $\overrightarrow{G}^{L_\lambda}$ with

$$L_\lambda : (v,u) \to \left\{\ 0,\ \sum_{i=1}^{s} h_i(t)e^{\lambda_i^{vu}t}\ \right\}$$

for $(v,u) \in E\left(\overrightarrow{G}\right)$. $\square$

§5. Complex Flow with Continuity Flows

The difference of a complex flow $\overrightarrow{G}^L$ with that of a continuity flow $\overrightarrow{G}^L$ is the labeling L on a vertex is $L(v)=\dot{x}_v$ or x_v. Notice that

$$\frac{d}{dt}\left(\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u)\right)=\sum_{u\in N_G(v)}\frac{d}{dt}L^{A^+_{vu}}(v,u)$$

for $\forall v\in V\left(\overrightarrow{G}\right)$. There must be relations between complex flows $\overrightarrow{G}^L$ and continuity flows $\overrightarrow{G}^L$. We get a general result following.

Theorem 5.1 *If end-operators A^+_{vu}, A^+_{uv} are linear with $\left[\int_0^t, A^+_{vu}\right]=\left[\int_0^t, A^+_{uv}\right]=\mathbf{0}$ and $\left[\frac{d}{dt}, A^+_{vu}\right]=\left[\frac{d}{dt}, A^+_{uv}\right]=\mathbf{0}$ for $\forall (v,u)\in E\left(\overrightarrow{G}\right)$, and particularly, $A^+_{vu}=\mathbf{1}_{\mathscr{V}}$, then $\overrightarrow{G}^L\in\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ is a continuity flow with a constant $L(v)$ for $\forall v\in V\left(\overrightarrow{G}\right)$ if and only if $\int_0^t\overrightarrow{G}^L dt$ is such a continuity flow with a constant one each vertex v, $v\in V\left(\overrightarrow{G}\right)$.*

Proof Notice that if $A^+_{vu}=\mathbf{1}_{\mathscr{V}}$, there always is $\left[\int_0^t, A^+_{vu}\right]=\mathbf{0}$ and $\left[\frac{d}{dt}, A^+_{vu}\right]=\mathbf{0}$, and by definition, we know that

$$\begin{aligned}\left[\int_0^t, A^+_{vu}\right]=\mathbf{0} \quad &\Leftrightarrow\quad \int_0^t\circ A^+_{vu}=A^+_{vu}\circ\int_0^t,\\ \left[\frac{d}{dt}, A^+_{vu}\right]=\mathbf{0}\quad &\Leftrightarrow\quad \frac{d}{dt}\circ A^+_{vu}=A^+_{vu}\circ\frac{d}{dt}.\end{aligned}$$

If $\overrightarrow{G}^L$ is a continuity flow with a constant $L(v)$ for $\forall v\in V\left(\overrightarrow{G}\right)$, i.e.,

$$\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u)=\mathbf{v}\ \text{ for }\ \forall v\in V\left(\overrightarrow{G}\right),$$

we are easily know that

$$\begin{aligned}\int_0^t\left(\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u)\right)dt &= \sum_{u\in N_G(v)}\left(\int_0^t\circ A^+_{vu}\right)L(v,u)dt=\sum_{u\in N_G(v)}\left(A^+_{vu}\circ\int_0^t\right)L(v,u)dt\\ &= \sum_{u\in N_G(v)} A^+_{vu}\left(\int_0^t L(v,u)dt\right)=\int_0^t\mathbf{v}dt\end{aligned}$$

for $\forall v\in V\left(\overrightarrow{G}\right)$ with a constant vector $\int_0^t\mathbf{v}dt$, i.e., $\int_0^t\overrightarrow{G}^L dt$ is a continuity flow with a constant flow on each vertex v, $v\in V\left(\overrightarrow{G}\right)$.

Conversely, if $\int_0^t\overrightarrow{G}^L dt$ is a continuity flow with a constant flow on each vertex v, $v\in$

$V\left(\overrightarrow{G}\right)$, i.e.,

$$\sum_{u\in N_G(v)} A^+_{vu}\circ\int_0^t L(v,u)dt = \mathbf{v} \quad \text{for} \quad \forall v\in V\left(\overrightarrow{G}\right),$$

then

$$\overrightarrow{G}^L = \frac{d\left(\int_0^t \overrightarrow{G}^L dt\right)}{dt}$$

is such a continuity flow with a constant flow on vertices in $\overrightarrow{G}$ because of

$$\begin{aligned}\frac{d\left(\sum\limits_{u\in N_G(v)} L^{A^+_{vu}}(v,u)\right)}{dt} &= \sum_{u\in N_G(v)} \frac{d}{dt}\circ A^+_{vu}\circ\int_0^t L(v,u)dt \\ &= \sum_{u\in N_G(v)} A^+_{vu}\circ\frac{d}{dt}\circ\int_0^t L(v,u)dt = \sum_{u\in N_G(v)} L(v,u)^{A^+_{vu}} = \frac{d\mathbf{v}}{dt}\end{aligned}$$

with a constant flow $\frac{d\mathbf{v}}{dt}$ on vertex v, $v\in V\left(\overrightarrow{G}\right)$. This completes the proof. □

If all end-operators A^+_{vu} and A^+_{uv} are constant for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, the conditions $\left[\int_0^t, A^+_{vu}\right] = \left[\int_0^t, A^+_{uv}\right] = \mathbf{0}$ and $\left[\frac{d}{dt}, A^+_{vu}\right] = \left[\frac{d}{dt}, A^+_{uv}\right] = \mathbf{0}$ are clearly true. We immediately get a conclusion by Theorem 5.1 following.

Corollary 5.2 *For $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, if end-operators A^+_{vu} and A^+_{uv} are constant c_{vu}, c_{uv} for $\forall(v,u)\in E\left(\overrightarrow{G}\right)$, then $\overrightarrow{G}^L\in\left\langle\overrightarrow{G}_i, 1\leq i\leq n\right\rangle^{\mathbb{R}\times\mathbb{R}^n}$ is a continuity flow with a constant $L(v)$ for $\forall v\in V\left(\overrightarrow{G}\right)$ if and only if $\int_0^t \overrightarrow{G}^L dt$ is such a continuity flow with a constant flow on each vertex v, $v\in V\left(\overrightarrow{G}\right)$.*

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$\beta-$Change of Finsler Metric by h-Vector and Imbedding Classes of Their Tangent Spaces

O.P.Pandey and H.S.Shukla

(Department of Mathematics & Statistics, DDU Gorakhpur University,Gorakhpur, India)

E-mail: oppandey1988@gmail.com, profhsshuklagkp@rediffmail.com

Abstract: We have considered the $\beta-$change of Finsler metric L given by $\overline{L} = f(L, \beta)$ where f is any positively homogeneous function of degree one in L and β. Here $\beta = b_i(x, y)y^i$, in which b_i are components of a covariant h-vector in Finsler space F^n with metric L. We have obtained that due to this change of Finsler metric, the imbedding class of their tangent Riemannian space is increased at the most by two.

Key Words: $\beta-$Change of Finsler metric, h-vector, imbedding class.

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§1. Introduction

Let (M^n, L) be an n-dimensional Finsler space on a differentiable manifold M^n, equipped with the fundamental function $L(x, y)$. In 1971, Matsumoto [2] introduced the transformation of Finsler metric given by

$$\overline{L}(x, y) = L(x, y) + \beta(x, y), \tag{1.1}$$

$$\overline{L}^2(x, y) = L^2(x, y) + \beta^2(x, y), \tag{1.2}$$

where $\beta(x, y) = b_i(x)y^i$ is a one-form on M^n. He has proved the following.

Theorem A. *Let $(M^n, \overline{L})$ be a locally Minkowskian n-space obtained from a locally Minkowskian n-space (M^n, L) by the change (1.1). If the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L) is of imbedding class r, then the tangent Riemannian n-space $(M^n_x, \overline{g}_x)$ to $(M^n, \overline{L})$ is of imbedding class at most $r + 2$.*

Theorem B. *Let $(M^n, \overline{L})$ be a locally Minkowskian n-space obtained from a locally Minkowskian n-space (M^n, L) by the change (1.2). If the tangent Riemannian n-space (M^n_x, g_x) to (M^n, L) is of imbedding class r, then the tangent Riemannian n-space $(M^n_x, \overline{g}_x)$ to $(M^n, \overline{L})$ is of imbedding class at most $r + 1$.*

Theorem B is included in theorem A due to the phrase "*at most*".

In [6] Singh, Prasad and Kumari Bindu have proved that the theorem A is valid for Kropina

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change of Finsler metric given by

$$\overline{L}(x,y) = \frac{L^2(x,y)}{\beta(x,y)}.$$

In [3], Prasad, Shukla and Pandey have proved that the theorem A is also valid for exponential change of Finsler metric given by

$$\overline{L}(x,y) = Le^{\beta/L}.$$

Recently Prasad and Kumari Bindu [5] have proved that the theorem A is valid for $\beta-$change [7] given by

$$\overline{L}(x,y) = f(L,\beta),$$

where f is any positively homogeneous function of degree one in L, β and β is one-form.

In all these works it has been assumed that $b_i(x)$ in β is a function of positional coordinate only.

The concept of $h-$vector has been introduced by H.Izumi. The covariant vector field $b_i(x,y)$ is said to be $h-$vector if $\frac{\partial b_i}{\partial y^j}$ is proportional to angular metric tensor.

In 1990, Prasad, Shukla and Singh [4] have proved that the theorem A is valid for the transformation (1.1) in which b_i in β is $h-$vector.

All the above $\beta-$changes of Finsler metric encourage the authors to check whether the theorem A is valid for any change of Finsler metric by $h-$vector.

In this paper we have proved that the theorem A is valid for the $\beta-$change of Finsler metric given by

$$\overline{L}(x,y) = f(L,\beta), \tag{1.3}$$

where f is positively homogeneous function of degree one in L, β and

$$\beta(x,y) = b_i(x,y)y^i. \tag{1.4}$$

Here $b_i(x,y)$ are components of a covariant $h-$vector satisfying

$$\frac{\partial b_i}{\partial y^j} = \rho h_{ij}, \tag{1.5}$$

where ρ is any scalar function of x, y and h_{ij} are components of angular metric tensor. The homogeneity of f gives

$$Lf_1 + \beta f_2 = f, \tag{1.6}$$

where the subscripts 1 and 2 denote the partial derivatives with respect to L and β respectively.

Differentiating (1.6) with respect to L and β respectively, we get

$$Lf_{11} + \beta f_{12} = 0 \quad \text{and} \quad Lf_{12} + \beta f_{22} = 0.$$

Hence, we have

$$\frac{f_{11}}{\beta^2} = -\frac{f_{12}}{\beta L} = \frac{f_{22}}{L^2}$$

which gives

$$f_{11} = \beta^2\omega, \quad f_{22} = L^2\omega, \quad f_{12} = -\beta L\omega, \tag{1.7}$$

where Weierstrass function ω is positively homogeneous function of degree -3 in L and β. Therefore

$$L\omega_1 + \beta\omega_2 + 3\omega = 0. \tag{1.8}$$

Again ω_1 and ω_2 are positively homogeneous function of degree - 4 in L and β, so

$$L\omega_{11} + \beta\omega_{12} + 4\omega_1 = 0 \quad \text{and} \quad L\omega_{21} + \beta\omega_{22} + 4\omega_2 = 0. \tag{1.9}$$

Throughout the paper we frequently use equation (1.6) to (1.9) without quoting them.

§2. An $h-$Vector

Let $b_i(x, y)$ be components of a covariant vector in the Finsler space (M^n, L). It is called an $h-$vector if there exists a scalar function ρ such that

$$\frac{\partial b_i}{\partial y^j} = \rho h_{ij}, \tag{2.1}$$

where h_{ij} are components of angular metric tensor given by

$$h_{ij} = g_{ij} - l_i l_j = L\frac{\partial^2 L}{\partial y^i\, \partial y^j}.$$

Differentiating (2.1) with respect to y^k, we get

$$\dot{\partial}_j\dot{\partial}_k b_i = (\dot{\partial}_k\rho)h_{ij} + \rho L^{-1}\{L^2\dot{\partial}_i\dot{\partial}_j\dot{\partial}_k L + h_{ij}l_k\},$$

where $\dot{\partial}_i$ stands for $\frac{\partial}{\partial y^i}$.

The skew-symmetric part of the above equation in j and k gives

$$(\dot{\partial}_k\rho + \rho L^{-1}l_k)h_{ij} - (\dot{\partial}_j\rho + \rho L^{-1}l_j)h_{ik} = 0.$$

Contracting this equation by g^{ij}, we get

$$(n-2)[\dot{\partial}_k\rho + \rho L^{-1}l_k] = 0,$$

which for $n > 2$, gives

$$\dot{\partial}_k\rho = -\frac{\rho}{L}l_k, \tag{2.2}$$

where we have used the fact that ρ is positively homogeneous function of degree -1 in y^i, i.e.,

$\frac{\partial\rho}{\partial y^j}y^j = -\rho.$

We shall frequently use equation (2.2) without quoting it in the next article.

§3. Fundamental Quantities of $(M^n, \overline{L})$

To find the relation between fundamental quantities of (M^n, L) and $(M^n, \overline{L})$, we use the following results

$$\dot{\partial}_i\beta = b_i, \quad \dot{\partial}_i L = l_i, \quad \dot{\partial}_j l_i = L^{-1}h_{ij}. \tag{3.1}$$

The successive differentiation of (1.3) with respect to y^i and y^j give

$$\overline{l}_i = f_1 l_i + f_2 b_i, \tag{3.2}$$

$$\overline{h}_{ij} = \frac{fp}{L}h_{ij} + fL^2 w m_i m_j, \tag{3.3}$$

where

$$p = f_1 + Lf_2\rho, \quad m_i = b_i - \frac{\beta}{L}l_i.$$

The quantities corresponding to $(M^n, \overline{L})$ will be denoted by putting bar on the top of those quantities.

From (3.2) and (3.3) we get the following relations between metric tensors of (M^n, L) and $(M^n, \overline{L})$

$$\begin{aligned} \overline{g}_{ij} &= \frac{fp}{L}g_{ij} - L^{-1}\{\beta(f_1f_2 - f\beta L\omega) + L\rho f f_2\}l_i l_j \\ &\quad +(fL^2\omega + f_2^2)b_i b_j + (f_1 f_2 - f\beta L\omega)(l_i b_j + l_j b_i). \end{aligned} \tag{3.4}$$

The contravariant components of the metric tensor of $(M^n, \overline{L})$ will be obtained from (3.4) as follows:

$$\overline{g}^{ij} = \frac{L}{fp}g^{ij} + \frac{Lv}{f^3pt}l^i l^j - \frac{L^4\omega}{fpt}b^i b^j - \frac{L^2u}{f^2pt}(l^i b^j + l^j b^i), \tag{3.5}$$

where, we put $b^i = g^{ij}b_j,\ l^i = g^{ij}l_j,\ b^2 = g^{ij}b_i b_j$ and

$$\begin{aligned} u &= f_1 f_2 - f\beta L\omega + L\rho f_2^2, \\ v &= (f_1f_2 - f\beta L\omega)(f\beta + \triangle f_2 L^2) + L\rho f_2\{f(f + L^2\rho f_2) \\ &\quad +L^2\triangle(f_2^2 + fL^2\omega)\} \end{aligned}$$

and

$$t = f_1 + L^3\omega\triangle + Lf_2\rho, \qquad \triangle = b^2 - \frac{\beta^2}{L^2}. \tag{3.6}$$

Putting $q = f_1 f_2 - f\beta L\omega + L\rho(f_2^2 + fL^2\omega)$, $s = 3f_2\omega + f\omega_2$, we find that

$$\begin{aligned}
&\text{(a)} \quad \dot{\partial}_i f = \frac{f}{L} l_i + f_2 m_i \\
&\text{(b)} \quad \dot{\partial}_i f_1 = -\beta L\omega m_i \\
&\text{(c)} \quad \dot{\partial}_i f_2 = L^2\omega m_i \\
&\text{(d)} \quad \dot{\partial}_i p = -L\omega(\beta - \rho L^2) m_i \\
&\text{(e)} \quad \dot{\partial}_i \omega = -\frac{3\omega}{L} l_i + \omega_2 m_i \\
&\text{(f)} \quad \dot{\partial}_i b^2 = -2C_{..i} + 2\rho m_i \\
&\text{(g)} \quad \dot{\partial}_i \triangle = -2C_{..i} - \frac{2}{L^2}(\beta - \rho L^2) m_i,
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&\text{(a)} \quad \dot{\partial}_i q = -(\beta - \rho L^2) s L m_i \\
&\text{(b)} \quad \dot{\partial}_i t = -2L^3\omega C_{..i} + [L^3 \triangle \omega_2 - 3(\beta - \rho L^2) L\omega] m_i \\
&\text{(c)} \quad \dot{\partial}_i s = -\frac{3s}{L} l_i + (4 f_2 \omega_2 + 3\omega^2 L^2 + f\omega_{22}) m_i
\end{aligned} \tag{3.8}$$

where "." denotes the contraction with b^i, viz. $C_{..i} = C_{jki} b^j b^k$.

Differentiating (3.4) with respect to y^k and using (d that

$$m_i l^i = 0, \quad m_i m^i = \triangle = m_i b^i, \quad h_{ij} m^j = h_{ij} b^j = m_i, \tag{3.10}$$

where $m^i = g^{ij} m_j = b^i - \frac{\beta}{L} l^i$.

To find $\overline{C}^i_{jk} = \overline{g}^{ih} \overline{C}_{jhk}$ we use (3.5), (3.9), (3.10) and get

$$\begin{aligned}
\overline{C}^i_{jk} \quad = \quad & C^i_{jk} + \frac{q}{2fp}(h_{jk} m^i + h^i_j m_k + h^i_k m_j) + \frac{sL^3}{2fp} m_j m_k m^i - \frac{L}{ft} C_{.jk} n^i \\
& - \frac{Lq\triangle}{2f^2pt} h_{jk} n^i - \frac{2Lq + L^4 \triangle s}{2f^2pt} m_j m_k n^i,
\end{aligned} \tag{3.11}$$

where $n^i = fL^2\omega b^i + u l^i$.

Corresponding to the vectors with components n^i and m^i, we have the following:

$$C_{ijk} m^i = C_{.jk}, \quad C_{ijk} n^i = fL^2\omega C_{.jk}, \quad m_i n^i = fL^2\omega\triangle. \tag{3.12}$$

To find the v-curvature tensor of $(M^n, \overline{L})$ with respect to Cartan's connection, we use the following:

$$C^h_{ij} h_{hk} = C_{ijk}, \quad h^i_k h^k_j = h^i_j, \quad h_{ij} n^i = fL^2\omega m_j. \tag{3.13}$$

The v-curvature tensors $\overline{S}_{hijk}$ of $(M^n, \overline{L})$ is defined as

$$\overline{S}_{hijk} = \overline{C}^r_{hk} C_{hjr} - \overline{C}^r_{hj} \overline{C}_{ikr}. \tag{3.14}$$

From (3.9)–(3.14), we get the following relation between v-curvature tensors of (M^n, L)

and $(M^n, \overline{L})$:

$$\overline{S}_{hijk} = \frac{fp}{L} S_{hijk} + d_{hj}d_{ik} - d_{hk}d_{ij} + E_{hk}E_{ij} - E_{hj}E_{ik}, \tag{3.15}$$

where

$$d_{ij} = PC_{.ij} - Qh_{ij} + Rm_i m_j, \tag{3.16}$$

$$E_{ij} = Sh_{ij} + Tm_i m_j, \tag{3.17}$$

$$P = L\left(\frac{fp\omega}{t}\right)^{1/2}, \quad Q = \frac{pq}{2L^2\sqrt{fp\omega t}}, \quad R = \frac{L(2\omega q - sp)}{2\sqrt{f\omega pt}},$$

$$S = \frac{q}{2L^2\sqrt{f\omega}}, \quad T = \frac{L(sp - \omega q)}{2p\sqrt{f\omega}}.$$

§4. Imbedding Class Numbers

The tangent vector space M^n_x to M^n at every point x is considered as the Riemannian n-space (M^n_x, g_x) with the Riemannian metric $g_x = g_{ij}(x,y)dy^i dy^j$. Then the components of the Cartan's tensor are the Christoffel symbols associated with g_x:

$$C^i_{jk} = \frac{1}{2} g^{ih}(\dot{\partial}_k g_{jh} + \dot{\partial}_j g_{hk} - \dot{\partial}_h g_{jk}).$$

Thus C^i_{jk} defines the components of the Riemannian connection on M^n_x and v-covariant derivative, say

$$X_i|_j = \dot{\partial}_j X_i - X_h C^h_{ij}$$

is the covariant derivative of covariant vector X_i with respect to Riemannian connection C^i_{jk} on M^n_x. It is observed that the v-curvature tensor S_{hijk} of (M^n, L) is the Riemannian Christoffel curvature tensor of the Riemannian space (M^n, g_x) at a point x. The space (M^n, g_x) equipped with such a Riemannian connection is called the tangent Riemannian n-space [2].

It is well known [1] that any Riemannian n-space V^n can be imbedded isometrically in a Euclidean space of dimension $\frac{n(n+1)}{2}$. If $n+r$ is the lowest dimension of the Euclidean space in which V^n is imbedded isometrically, then the integer r is called the imbedding class number of V^n. The fundamental theorem of isometric imbedding ([1] page 190) is that the tangent Riemannian n-space (M^n_x, g_x) is locally imbedded isometrically in a Euclidean $(n+r)-$space if and only if there exist $r-$number $\epsilon_P = \pm 1$, $r-$symmetric tensors $H_{(P)ij}$ and $\frac{r(r-1)}{2}$ covariant vector fields $H_{(P,Q)i} = -H_{(Q,P)i}$; $P, Q = 1, 2, \cdots, r$, satisfying the Gauss equations

$$S_{hijk} = \sum_P \epsilon_P \{H_{(P)hj}H_{(P)ik} - H_{(P)ij}H_{(P)hk}\}, \tag{4.1}$$

The Codazzi equations

$$H_{(P)ij}|_k - H_{(P)ik}|_j = \sum_Q \epsilon_Q \{H_{(Q)ij}H_{(Q,P)k} - H_{(Q)ik}H_{(Q,P)j}\}, \tag{4.2}$$

and the Ricci-Kühne equations

$$\begin{aligned} H_{(P,Q)i}|_j - H_{(P,Q)j}|_i \quad &+ \quad \sum_R \epsilon_R \{H_{(R,P)i}H_{(R,Q)j} - H_{(R,P)j}H_{(R,Q)i}\} \\ &+ \quad g^{hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\} = 0. \end{aligned} \tag{4.3}$$

The numbers $\epsilon_P = \pm 1$ are the indicators of unit normal vector N_P to M^n and $H_{(P)ij}$ are the second fundamental tensors of M^n with respect to the normals N_P. Thus if g_x is assumed to be positive definite, there exists a Cartesian coordinate system (z^i, u^p) of the enveloping Euclidean space E^{n+r} such that ds^2 in E^{n+r} is expressed as

$$ds^2 = \sum_i (dz^i)^2 + \sum_p \epsilon_p (du^p)^2.$$

§5. Proof of Theorem A

In order to prove the theorem A, we put

$$\begin{aligned} &\text{(a)} \quad \overline{H}_{(P)ij} = \sqrt{\frac{fp}{L}}\, H_{(P)ij}, \qquad \overline{\epsilon}_P = \epsilon_P, \quad P = 1, 2, \cdots, r \\ &\text{(b)} \quad \overline{H}_{(r+1)ij} = d_{ij}, \qquad \overline{\epsilon}_{r+1} = 1 \\ &\text{(c)} \quad \overline{H}_{(r+2)ij} = E_{ij}, \qquad \overline{\epsilon}_{r+2} = -1. \end{aligned} \tag{5.1}$$

Then it follows from (3.15) and (4.1) that

$$\overline{S}_{hijk} = \sum_{\lambda=1}^{r+2} \overline{\epsilon}_\lambda \{\overline{H}_{(\lambda)hj}\overline{H}_{(\lambda)ik} - \overline{H}_{(\lambda)hk}\overline{H}_{(\lambda)ij}\},$$

which is the Gauss equation of $(M_x^n, \overline{g}_x)$.

Moreover, to verify Codazzi and Ricci Kühne equation of $(M_x^n, \overline{g}_x)$, we put

$$\begin{aligned} &\text{(a)} \quad \overline{H}_{(P,Q)i} = -\overline{H}_{(Q,P)i} = H_{(P,Q)i}, \quad P, Q = 1, 2, , \cdots, r \\ &\text{(b)} \quad \overline{H}_{(P,r+1)i} = -\overline{H}_{(r+1,P)i} = \frac{L\sqrt{L\omega}}{\sqrt{t}} H_{(P).i}, \quad P = 1, 2, \cdots, r \\ &\text{(c)} \quad \overline{H}_{(P,r+2)i} = -\overline{H}_{(r+2,P)i} = 0, \quad P = 1, 2, \cdots, r. \\ &\text{(d)} \quad \overline{H}_{(r+1,r+2)i} = -\overline{H}_{(r+2,r+1)i} = \frac{sp - 2q\omega}{2f\omega\sqrt{pt}} m_i. \end{aligned} \tag{5.2}$$

The Codazzi equations of $(M_x^n, \overline{g}_x)$ consists of the following three equations:

$$\begin{aligned} \text{(a)} \quad \overline{H}_{(P)ij}\|_k - \overline{H}_{(P)ik}\|_j = &\sum_Q \overline{\epsilon}_Q \{\overline{H}_{(Q)ij}\overline{H}_{(Q,P)k} - \overline{H}_{(Q)ik}\overline{H}_{(Q,P)j}\} \\ &+ \overline{\epsilon}_{r+1}\{\overline{H}_{(r+1)ij}\overline{H}_{(r+1,P)k} - \overline{H}_{(r+1)ik}\overline{H}_{(r+1,P)j}\} \\ &+ \overline{\epsilon}_{r+2}\{\overline{H}_{(r+2)ij}\overline{H}_{(r+2,P)k} - \overline{H}_{(r+2)ik}\overline{H}_{(r+2,P)k}\} \end{aligned} \tag{5.3}$$

$$\text{(b)}\quad \overline{H}_{(r+1)ij}\|_k - \overline{H}_{(r+1)ik}\|_j = \sum_Q \overline{\epsilon}_Q\{\overline{H}_{(Q)ij}\overline{H}_{(Q,r+1)k} - \overline{H}_{(Q)ik}\overline{H}_{(Q,r+1)j}\}$$
$$+\overline{\epsilon}_{r+2}\{\overline{H}_{(r+2)ij}\overline{H}_{(r+2,r+1)k} - \overline{H}_{(r+2)ik}\overline{H}_{(r+2,r+1)j}\},$$

$$\text{(c)}\quad \overline{H}_{(r+2)ij}\|_k - \overline{H}_{(r+2)ik}\|_j = \sum_Q \overline{\epsilon}_Q\{\overline{H}_{(Q)ij}\overline{H}_{(Q,r+2)k} - \overline{H}_{(Q)ik}\overline{H}_{(Q,r+2)j}\}$$
$$+\overline{\epsilon}_{r+1}\{\overline{H}_{(r+1)ij}\overline{H}_{(r+1,r+2)k} - \overline{H}_{(r+1)ik}\overline{H}_{(r+1,r+2)j}\}.$$

where $\|_i$ denotes v-covariant derivative in $(M^n, \overline{L})$, i.e. covariant derivative in tangent Riemannian n-space $(M^n_x, \overline{g}_x)$ with respect to its Christoffel symbols $\overline{C}^i_{jk}$. Thus

$$X_i\|_j = \dot{\partial}_j X_i - X_h\overline{C}^h_{ij}.$$

To prove these equations we note that for any symmetric tensor X_{ij} satisfying $X_{ij}l^i = X_{ij}l^j = 0$, we have from (3.11),

$$\begin{aligned} X_{ij}\|_k - X_{ik}\|_j &= X_{ij}|_k - X_{ik}|_j - \frac{q}{2ft}(h_{ik}X_{.j} - h_{ij}X_{.k}) \\ &+\frac{L^3\omega}{t}(C_{.ik}X_{.j} - C_{.ij}X_{.k}) - \frac{q}{2fp}(X_{ij}m_k - X_{ik}m_j) \\ &+\frac{L^3(2q\omega - sp)}{2fpt}(X_{.j}m_k - X_{.k}m_j)\, m_i. \qquad (5.4) \end{aligned}$$

Also if X is any scalar function, then $X\|_j = X|_j = \dot{\partial}_j X$.

Verification of (5.3)(a) In view of (5.1) and (5.2), equation (5.3)a is equivalent to

$$\left(\sqrt{\frac{fp}{L}}H_{(P)ij}\right)\Big\|_k - \left(\sqrt{\frac{fp}{L}}H_{(P)ik}\right)\Big\|_j$$
$$= \sqrt{\frac{fp}{L}}.\sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q,P)k} - H_{(Q)ik}H_{(Q,P)j}\} - \frac{L\sqrt{L\omega}}{\sqrt{t}}\{d_{ij}H_{(P).k} - d_{ik}H_{(P).j}\}. \qquad (5.5)$$

Since $\left(\sqrt{\frac{fp}{L}}\right)\Big\|_k = \dot{\partial}_k\left(\sqrt{\frac{fp}{L}}\right) = \frac{q}{2\sqrt{fLp}}\, m_k$, applying formula (5.4) for $H_{(P)ij}$, we get

$$\begin{aligned} &\left(\sqrt{\frac{fp}{L}}H_{(P)ij}\right)\Big\|_k - \left(\sqrt{\frac{fp}{L}}H_{(P)ik}\right)\Big\|_j = \sqrt{\frac{fp}{L}}\{H_{(P)ij}|_k - H_{(P)ik}|_j\} \\ &-\frac{q}{2ft}\sqrt{\frac{fp}{L}}\{h_{ik}H_{(P).j} - h_{ij}H_{(P).k}\} + \frac{L^3\omega}{t}\sqrt{\frac{fp}{L}}\{C_{.ik}H_{(p).j} - C_{.ij}H_{(p).k}\} \\ &+\frac{L^2\sqrt{L}(2q\omega - sp)}{2t\sqrt{fp}}\{H_{(P).j}m_k - H_{(P).k}m_j\}m_i. \end{aligned} \qquad (5.6)$$

Substituting the values of $\left(\sqrt{\frac{fp}{L}}H_{(P)ij}\right)\Big\|_k - \left(\sqrt{\frac{fp}{L}}H_{(P)ik}\right)\Big\|_j$ from (5.6) and the values of d_{ij} from (3.16) in (5.5) we find that equation (5.5) is identically satisfied due to equation

(4.2).

Verification of (5.3)(b) In view of (5.1) and (5.2), equation (5.3)b is equivalent to

$$\begin{aligned} d_{ij}\|_k - d_{ik}\|_j = L\sqrt{\frac{f\omega p}{t}} \sum_Q \epsilon_Q\{H_{(Q)ij}H_{(Q).k} - H_{(Q)ik}H_{(Q).j}\} \\ + \frac{sp-2q\omega}{2f\omega\sqrt{pt}}\{E_{ij}m_k - E_{ik}m_j\}. \end{aligned} \tag{5.7}$$

To verify (5.7), we note that

$$C_{.ij}|_k - C_{.ik}|_j = -b^h S_{hijk} \tag{5.8}$$

$$h_{ij}|_k - h_{ik}|_j = L^{-1}(h_{ij}l_k - h_{ik}l_j), \tag{5.9}$$

$$m_i|_k = -C_{.ik} - \left(\frac{\beta}{L^2} - \rho\right) h_{ik} - \frac{1}{L} l_i m_k. \tag{5.10}$$

$$\dot\partial_k(f\omega p) = -2L^{-1} f\omega p l_k + (q\omega + fp\omega_2)m_k. \tag{5.11}$$

Contracting (3.16) with b^i and using (3.10), we find that

$$d_{.j} = L\sqrt{\frac{f\omega p}{t}} C_{..j} + \frac{q(2L^3\omega\triangle - p) - L^3\triangle sp}{2L^2\sqrt{f\omega pt}} m_j. \tag{5.12}$$

Applying formula (5.4) for d_{ij} and substituting the values of $d_{.j}$ from (5.12) and d_{ij} from (3.16), we get

$$\begin{aligned} d_{ij}\|_k - d_{ik}\|_j = \; & d_{ij}|_k - d_{ik}|_j - \frac{Lq\sqrt{f\omega p}}{2ft^{3/2}}(h_{ik}C_{..j} - h_{ij}C_{..k}) \\ & + \frac{L^4\omega(2q\omega - sp)}{2\sqrt{f\omega p}.t^{3/2}}(C_{..j}m_k - C_{..k}m_j)m_i \\ & + \frac{L^4\omega\sqrt{f\omega p}}{t^{3/2}}(C_{.ik}C_{..j} - C_{.ij}C_{..k}) \\ & + \frac{L^4\omega\triangle(3q\omega - sp)}{2\sqrt{f\omega p}.t^{3/2}}(C_{.ik}m_j - C_{.ij}m_k) \\ & - \frac{Lq\triangle(3q\omega - sp)}{4f\sqrt{f\omega p}.t^{3/2}}(h_{ik}m_j - h_{ij}m_k). \end{aligned} \tag{5.13}$$

From (3.16), we obtain

$$\begin{aligned} d_{ij}|_k - d_{ik}|_j = \; & P(C_{.ij}|_k - C_{.ik}|_j) - Q(h_{ij}|_k - h_{ik}|_j) \\ & + R(m_i|_k m_j + m_j|_k m_i - m_i|_j m_k - m_k|_j m_i) \\ & + (\dot\partial_k P)C_{.ij} - (\dot\partial_j P)C_{.ik} - (\dot\partial_k Q)h_{ij} + (\dot\partial_j Q)h_{ik}) \\ & + (\dot\partial_k R)m_i m_j - (\dot\partial_j R)m_i m_k). \end{aligned} \tag{5.14}$$

Since,

$$\begin{aligned}
\dot{\partial}_k P &= \frac{L^4\omega\sqrt{f\omega p}}{t^{3/2}}C_{..k} + \left[\frac{Lfp\{p\omega_2 + 3L\omega^2(\beta-\rho L^2)\}}{2\sqrt{f\omega p}.t^{3/2}}\right. \\
&\quad \left. + \frac{Lq\omega}{2\sqrt{f\omega pt}}\right] m_k, \\
\dot{\partial}_k Q &= \frac{Lpq\omega}{2\sqrt{f\omega p}.t^{3/2}}C_{..k} - \frac{pq}{2L^3\sqrt{f\omega pt}}l_k \\
&\quad - \frac{(\beta-\rho L^2)(q\omega+sp)}{2L\sqrt{f\omega pt}}m_k - \frac{pq(q\omega+fp\omega_2)}{4L^2(f\omega p)^{3/2}\sqrt{t}}m_k \\
&\quad + \frac{pq\{3\omega(\beta-\rho L^2) - L^2\triangle\omega_2\}}{4L\sqrt{f\omega p}\,t^{3/2}}m_k
\end{aligned} \tag{5.15}$$

and

$$\dot{\partial}_k R = \frac{L^4\omega(2q\omega-sp)}{2\sqrt{f\omega p}.t^{3/2}}C_{..k} - \frac{2q\omega-sp}{2\sqrt{f\omega pt}}l_k + \text{term containing } m_k,$$

where we have used the equations (3.6), (3.7) and (3.8).

From equations (5.8)–(5.15), we have

$$\begin{aligned}
d_{ij}|_k - d_{ik}|_j &= L\sqrt{\frac{f\omega p}{t}}(-b^h S_{hijk}) \\
&\quad + \frac{L^4\omega\triangle(3q\omega-sp)}{2\sqrt{f\omega p}.t^{3/2}}(C_{.ij}m_k - C_{.ik}m_j) \\
&\quad + \frac{L^4\omega\sqrt{f\omega p}}{t^{3/2}}(C_{.ij}C_{..k} - C_{.ik}C_{..j}) \\
&\quad + \frac{L\omega pq}{2\sqrt{f\omega p}.t^{3/2}}(h_{ik}C_{..j} - h_{ij}C_{..k}) \\
&\quad + \frac{pq[q\omega t + f(L^3\omega\triangle + t)\{3L\omega^2(\beta-\rho L^2) + p\omega_2\}]}{4L^2(f\omega pt)^{3/2}} \times \\
&\quad (h_{ij}m_k - h_{ik}m_j) + \frac{L^4\omega(2q\omega-sp)}{2\sqrt{f\omega p}.t^{3/2}}(C_{..k}m_j - C_{..j}m_k)\,m_i.
\end{aligned} \tag{5.16}$$

Substituting the value of $d_{ij}|_k - d_{ik}|_j$ from (5.16) in (5.13), then value of $d_{ij}\|_k - d_{ik}\|_j$ thus obtained in (5.7), and using equations (4.1) and (3.17), it follows that equation (5.7) holds identically.

Verification of (5.3)(c) In view of (5.1) and (5.2), equation (5.3)c is equivalent to

$$E_{ij}\|_k - E_{ik}\|_j = \frac{sp-2q\omega}{2f\omega\sqrt{pt}}(d_{ij}m_k - d_{ik}m_j). \tag{5.17}$$

Contracting (3.17) by b^i and using equation (3.10), we find that

$$E_{.j} = \frac{pq + L^3\triangle(sp-q\omega)}{2L^2p\sqrt{f\omega}}\,m_j. \tag{5.18}$$

Applying formula (5.4) for E_{ij} and substituting the value of $E_{.j}$ from (5.18) and the value of E_{ij} from (3.17), we get

$$\begin{aligned} E_{ij}\|_k - E_{ik}\|_j \ &= E_{ij}|_k - E_{ik}|_j + \frac{qL\triangle(sp-2q\omega)}{4fpt\sqrt{f\omega}}(h_{ij}m_k - h_{ik}m_j) \\ &+ \frac{L\omega\{pq + L^3\triangle(sp-q\omega)\}}{2pt\sqrt{f\omega}}(C_{.ik}m_j - C_{.ij}m_k). \end{aligned} \tag{5.19}$$

From (3.17), we get

$$\begin{aligned} E_{ij}|_k - E_{ik}|_j \ &= S(h_{ij}|_k - h_{ik}|_j) + T\{m_i|_k m_j + m_j|_k m_i \\ &-m_i|_j m_k - m_k|_j m_i\} + (\dot\partial_k S)h_{ij} \\ &-(\dot\partial_j S)h_{ik} + (\dot\partial_k T)m_i m_j - (\dot\partial_j T)m_i m_k. \end{aligned} \tag{5.20}$$

Now,

$$(\dot\partial_k S) = -\frac{q}{2L^3\sqrt{f\omega}}l_k - \left[\frac{(\beta-\rho L^2)s}{2L\sqrt{f\omega}} + \frac{q(f\omega_2 + f_2\omega)}{4L^2(f\omega)^{3/2}}\right]m_k \tag{5.21}$$

and

$$(\dot\partial_k T) = -\frac{sp-q\omega}{2p\sqrt{f\omega}}l_k + \text{term containing } m_k,$$

where we have used the equations (3.7) and (3.8).

From equation (5.9)–(5.11), (5.20) and (5.21), we get

$$\begin{aligned} E_{ij}|_k - E_{ik}|_j \ &= \frac{L(sp-q\omega)}{2p\sqrt{f\omega}}(C_{.ij}m_k - C_{.ik}m_j) \\ &- \frac{q(sp-2q\omega)}{4L^2p(f\omega)^{3/2}}(h_{ij}m_k - h_{ik}m_j). \end{aligned} \tag{5.22}$$

Substituting the value of $E_{ij}|_k - E_{ik}|_j$ from (5.22) in (5.19), then the value of $E_{ij}\|_k - E_{ik}\|_j$ thus obtained in (5.17), and then using (3.16) in the right-hand side of (5.17), we find that the equation (5.17) holds identically.

This completes the proof of Codazzi equations of $(M_x^n, \overline{g}_x)$. The Ricci Kühne equations of $(M_x^n, \overline{g}_x)$ consist of the following four equations

$$\begin{aligned} \text{(a)}\quad & \overline{H}_{(P,Q)i}\|_j - \overline{H}_{(P,Q)j}\|_i + \sum_R \overline{\epsilon}_R\{\overline{H}_{(R,P)i}\overline{H}_{(R,Q)j} \\ & -\overline{H}_{(R,P)j}\overline{H}_{(R,Q)i}\} + \overline{\epsilon}_{r+1}\{\overline{H}_{(r+1,P)i}\overline{H}_{(r+1,Q)j} \\ & -\overline{H}_{(r+1,P)j}\overline{H}_{(r+1,Q)i}\} + \overline{\epsilon}_{r+2}\{\overline{H}_{(r+2,P)i}\overline{H}_{(r+2,Q)j} \\ & -\overline{H}_{(r+2,P)j}\overline{H}_{(r+2,Q)i}\} + \overline{g}^{hk}\{\overline{H}_{(P)hi}\overline{H}_{(Q)kj} \\ & -\overline{H}_{(P)hj}\overline{H}_{(Q)ki}\} = 0, \qquad P,Q = 1,2,\cdots,r \end{aligned} \tag{5.23}$$

$$\begin{aligned}
\text{(b)}\quad & \overline{H}_{(P,r+1)i}\|_j - \overline{H}_{(P,r+1)j}\|_i + \sum_R \overline{\epsilon}_R\{\overline{H}_{(R,P)i}\overline{H}_{(R,r+1)j} - \overline{H}_{(R,P)j}\overline{H}_{(R,r+1)i}\} \\
& + \overline{\epsilon}_{r+2}\{\overline{H}_{(r+2,P)i}\overline{H}_{(r+2,r+1)j} - \overline{H}_{(r+2,P)j}\overline{H}_{(r+2,r+1)i}\} \\
& + \overline{g}^{hk}\{\overline{H}_{(P)hi}\overline{H}_{(r+1)kj} - \overline{H}_{(P)hj}\overline{H}_{(r+1)ki}\} = 0, \quad P = 1,2,\cdots,r \\
\text{(c)}\quad & \overline{H}_{(P,r+2)i}\|_j - \overline{H}_{(P,r+2)j}\|_i + \sum_R \overline{\epsilon}_R\{\overline{H}_{(R,P)i}\overline{H}_{(R,r+2)j} - \overline{H}_{(R,P)j}\overline{H}_{(R,r+2)i}\} \\
& + \overline{\epsilon}_{r+1}\{\overline{H}_{(r+1,P)i}\overline{H}_{(r+1,r+2)j} - \overline{H}_{(r+1,P)j}\overline{H}_{(r+1,r+2)i}\} \\
& + \overline{g}^{hk}\{\overline{H}_{(P)hi}\overline{H}_{(r+2)kj} - \overline{H}_{(P)hj}\overline{H}_{(r+2)ki}\} = 0, \quad P = 1,2,\cdots,r
\end{aligned}$$

$$\begin{aligned}
\text{(d)}\quad & \overline{H}_{(r+1,r+2)i}\|_j - \overline{H}_{(r+1,r+2)j}\|_i + \sum_R \overline{\epsilon}_R\{\overline{H}_{(R,r+1)i}\overline{H}_{(R,r+2)j} - \overline{H}_{(R,r+1)j} \\
& \times\overline{H}_{(R,r+2)i}\} + \overline{g}^{hk}\{\overline{H}_{(r+1)hi}\overline{H}_{(r+2)kj} - \overline{H}_{(r+1)hj}\overline{H}_{(r+2)ki}\} = 0.
\end{aligned}$$

Verification of (5.23)(a) In view of (5.1) and (5.2), equation (5.23)a is equivalent to

$$\begin{aligned}
& H_{(P,Q)i}\|_j - H_{(P,Q)j}\|_i + \sum_R \epsilon_R\{H_{(R,P)i}H_{(R,Q)j} - H_{(R,P)j}H_{(R,Q)i}\} \\
& + \frac{L^3\omega}{t}\{H_{(P).i}H_{(Q).j} - H_{(P).j}H_{(Q).i}\} + \overline{g}^{hk}\{H_{(P)hi}H_{(Q)kj} \\
& - H_{(P)hj}H_{(Q)ki}\}\frac{fp}{L} = 0. \qquad P,Q = 1,2,\ldots,r. \qquad (5.24)
\end{aligned}$$

Since $H_{(P)ij}l^i = 0 = H_{(P)ji}l^i$, from (3.5), we get

$$\begin{aligned}
& \overline{g}^{hk}\{H_{(P)hi}H_{(Q)kj} - H_{(P)hj}H_{(Q)ki}\}\frac{fp}{L} = g^{hk}\{H_{(P)hi}H_{(Q)kj} \\
& - H_{(P)hj}H_{(Q)ki}\} - \frac{L^3\omega}{t}\{H_{(P).i}H_{(Q).j} - H_{(P).j}H_{(Q).i}\}.
\end{aligned}$$

Also, we have $H_{(P,Q)i}\|_j - H_{(P,Q)j}\|_i = H_{(P,Q)i}|_j - H_{(P,Q)j}|_i$. Hence equation (5.24) is satisfied identically by virtue of (4.3).

Verification of (5.23)(b) In view of (5.1) and (5.2), equation (5.23)b is equivalent to

$$\begin{aligned}
& \left(\frac{L\sqrt{L\omega}}{\sqrt{t}}H_{(P).i}\right)\Big\|_j - \left(\frac{L\sqrt{L\omega}}{\sqrt{t}}H_{(P).j}\right)\Big\|_i \\
& + \frac{L\sqrt{L\omega}}{\sqrt{t}}\sum_R \epsilon_R\{H_{(R,P)i}H_{(R).j} - H_{(R,P)j}H_{(R).i}\} \\
& + \overline{g}^{hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{\frac{fp}{L}} = 0. \quad P,Q = 1,2,\cdots,r. \qquad (5.25)
\end{aligned}$$

Since $b^h|_j = g^{hk}C_{.jk}$, $H_{(P)hi}l^i = 0$, we have

$$\begin{aligned}
H_{(P).i}\|_j - H_{(P).j}\|_i = H_{(P).i}|_j - H_{(P).j}|_i = \{H_{(P)hi}|_j - H_{(P)hj}|_i\}b^h \\
- g^{hk}\{H_{(P)hi}C_{.kj} - H_{(P)hj}C_{.ki}\} \qquad (5.26)
\end{aligned}$$

$$\left(\frac{L\sqrt{L\omega}}{\sqrt{t}}\right)\Big\|_j = \dot{\partial}_j\left(\frac{L\sqrt{L\omega}}{\sqrt{t}}\right)$$

$$= \frac{L^4\omega\sqrt{L\omega}}{t^{3/2}}C_{..j} + \frac{L\sqrt{L\omega}}{2\omega t^{3/2}}\{p\omega_2 + 3L\omega^2(\beta - \rho L^2)\}\, m_j \tag{5.27}$$

and

$$\overline{g}^{hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{\frac{fp}{L}} = \sqrt{\frac{L}{fp}}g^{hk}\times$$

$$\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\} - \frac{L^3\omega\sqrt{L}}{t\sqrt{fp}}\{H_{(P).i}d_{.j} - H_{(P).j}d_{.i}\}. \tag{5.28}$$

After using (3.16) and (5.12) the equation (5.28) may be written as

$$\overline{g}^{hk}\{H_{(P)hi}d_{kj} - H_{(P)hj}d_{ki}\}\sqrt{\frac{fp}{L}} = \frac{L\sqrt{L\omega}}{\sqrt{t}}g^{hk}\times$$

$$\{H_{(P)hi}C_{.kj} - H_{(P)hj}C_{.ki}\} - \frac{L^4\omega\sqrt{L\omega}}{t^{3/2}}\{H_{(P).i}C_{..j} - H_{(P).j}C_{..i}\}$$

$$-\frac{L\sqrt{L\omega}}{2\omega t^{3/2}}[p\omega_2 + 3L\omega^2(\beta - \rho L^2)]\{H_{(P).i}m_j - H_{(P).j}m_i\}. \tag{5.29}$$

From (4.2), (5.26)–(5.29) it follows that equation (5.25) holds identically.

Verification of (5.23)(c) In view of (5.1) and (5.2), equation (5.23)c is equivalent to

$$\frac{L\sqrt{L\omega}(2q\omega - sp)}{2f\omega t\sqrt{p}}\{H_{(P).i}m_j - H_{(P).j}m_i\}$$

$$+\overline{g}^{hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\}\sqrt{\frac{fp}{L}} = 0, \tag{5.30}$$

Since $E_{kj}l^k = 0 = E_{jk}l^k$, from (3.5), we find that the value of $\overline{g}^{hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\}$ is

$$\sqrt{\frac{L}{fp}}.g^{hk}\{H_{(P)hi}E_{kj} - H_{(P)hj}E_{ki}\} - \frac{L^3\omega\sqrt{L}}{t\sqrt{fp}}\{H_{(P).i}E_{.j} - H_{(P).j}E_{.i}\},$$

which, in view of (3.17) and (5.18), is equal to

$$-\frac{L\sqrt{L\omega}(2q\omega - sp)}{2f\omega t\sqrt{p}}\{H_{(P).i}m_j - H_{(P).j}m_i\}.$$

Hence equation (5.30) is satisfied identically.

Verification of (5.23)(d) In view of (5.1) and (5.2), equation (5.23)d is equivalent to

$$(Nm_i)\|_j - (Nm_j)\|_i + \overline{g}^{hk}(d_{hi}E_{kj} - d_{hj}E_{ki}) = 0, \tag{5.31}$$

where $N = \frac{sp-2q\omega}{2f\omega\sqrt{pt}}$.

Since $d_{hi}l^h = 0$, $E_{kj}l^k = 0$, from (3.5), we find that the value of $\overline{g}^{hk}\{d_{hi}E_{kj} - d_{hj}E_{ki}\}$ is

$$\frac{L}{fp}g^{hk}\{d_{hi}E_{kj} - d_{hj}E_{ki}\} - \frac{L^4\omega}{fpt}\{d_{.i}E_{.j} - d_{.j}E_{.i}\},$$

which, in view of (3.16), (3.17), (5.12) and (5.18), is equal to

$$-\frac{L^3(2q\omega - sp)}{2f\sqrt{p}.t^{3/2}}\{C_{..i}m_j - C_{..j}m_i\}.$$

Also,

$$(Nm_i)\|_j - (Nm_j)\|_i = N(m_i\|_j - m_j\|_i) + (\dot{\partial}_j N)m_i - (\dot{\partial}_i N)m_j.$$

Since $m_i\|_j - m_j\|_i = m_i|_j - m_j|_i = L^{-1}(l_j m_i - l_i m_j)$ and

$$\dot{\partial}_j N = -\frac{2q\omega - sp}{2Lf\omega\sqrt{pt}}l_j + \frac{L^3(sp - 2q\omega)}{2f\sqrt{p}.t^{3/2}}C_{..j},$$

we have

$$(Nm_i)\|_j - (Nm_j)\|_i = \frac{L^3(sp - 2q\omega)}{2f\sqrt{p}.t^{3/2}}(C_{..j}m_i - C_{..i}m_j). \tag{5.32}$$

Hence equation (5.31) is satisfied identically. Therefore Ricci Kühne equations of $(M_x^n, \overline{g}_x)$ given in (5.23) are satisfied.

Hence the Theorem A given in introduction is satisfied for the $\beta-$change (1.3) of Finsler metric given by $h-$vector. □

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A Note on Hyperstructres and Some Applications

B.O.Onasanya

(Department of Mathematics, University of Ibadan, Ibadan, Oyo State, Nigeria)

E-mail: babtu2001@yahoo.com

Abstract: In classical group theory, two elements composed yield another element. This theory, definitely, has limitations in its use in the study of atomic reactions and reproduction in organisms where two elements composed can yield more than one. In this paper, we partly give a review of some properties of hyperstructures with some examples in chemical sciences. On the other hand, we also construct some examples of hyperstructures in genotype, extending the works of Davvaz (2007) to blood genotype. This is to motivate new and collaborative researches in the use of hyperstructures in these related fields.

Key Words: Genotype as a hyperstructure, hypergroup, offspring.

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§1. Introduction

The theory of *hyperstructures* began in 1934 by F. Marty. In his presentation at the 8th congress of Scandinavian Mathematicians, he illustrated the definition of hypergroup and some applications, giving some of its uses in the study of groups and some functions. It is a kind of generalization of the concept of abstract group and an extension of well-known group theory and as well leading to new areas of study.

The study of hypergroups now spans to the investigation and studying of subhypergroups, relations defined on hyperstructures, cyclic hypergroups, canonical hypergroups, P-hypergroups, hyperrings, hyperlattices, hyperfields, hypermodules and H_ν-structures but to mention a few.

A very close concept to this is that of HX Group which was developed by Li [11] in 1985. There have been various studies linking HX Groups to hyperstructures. In the late 20th century, the theory experienced more development in the applications to other mathematical theories such as character theory of finite groups, combinatorics and relation theory. Researchers like P. Corsini, B. Davvaz, T. Vougiouklis, V. Leoreanu, but to mention a few, have done very extensive studies in the theory of hyperstructures and their uses.

§2. Definitions and Examples of Hyperstrutures

Definition 2.1 *Let H be a non empty set. The operation $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$ is called a*

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hyperoperation and $(H, \circ)$ *is called a hypergroupoid, where* $\mathcal{P}^*(H)$ *is the collection of all non empty subsets of* H. *In this case, for* $A, B \subseteq H$, $A \circ B = \cup\{a \circ b | a \in A, b \in B\}$.

Remark 2.1 A hyperstructure is a set on which a hyperoperation is defined. Some major kinds of hyperstructures are hypergroups, HX groups, H_ν groups, hyperrings and so on.

Definition 2.2 *A hypergroupoid* $(H, \circ)$ *is called a semihypergroup if*

$$(a \circ b) \circ c = a \circ (b \circ c) \quad \forall a, b, c \in H \quad (Associativity)$$

.

Definition 2.3 *A hypergroupoid* $(H, \circ)$ *is called a quasihypergroup if*

$$a \circ H = H = H \circ a \quad \forall a \in H \quad (Reproduction\ \ Axiom).$$

Definition 2.4 *A hypergroupoid* $(H, \circ)$ *is called a hypergroup if it is both a semihypergroup and quasihypergroup.*

Example 2.1 (1) For any group G, if the hyperoperation is defined on the cosets, it generally yields a hypergroup.

(2) If we partition $H = \{1, -1, i, -i\}$ by $K^* = \{\{1, -1\}, \{i, -i\}\}$, then $(H/K^*, \circ)$ is a hypergroup.

(3)([8]) Let $(G, +) = (\mathrm{Z}, +)$ be an abelian group with an equivalence relation ρ partitioning G into $\overline{x} = \{x, -x\}$. Then, if $\overline{x} \circ \overline{y} = \{\overline{x+y}, \overline{x-y}\}$ $\forall \overline{x}, \overline{y} \in G/\rho, (G/\rho, \circ)$ is a hypergroup.

Definition 2.5 *A hypergroupoid* $(H, \circ)$ *is called a* H_ν *group if it satisfies*

(1) $(a \circ b) \circ c \cap a \circ (b \circ c) \neq \emptyset \quad \forall a, b, c \in H \quad (Weak\ \ Associativity);$

(2) $a \circ H = H = H \circ a \quad \forall a \in H \quad (Reproduction\ \ Axiom).$

Remark 2.2 An H_ν group may not be a hypergroup. A subset $K \subseteq H$ is called a subhypergroup if $(K, \circ)$ is also a hypergroup. A hypergroup $(H, \circ)$ is said to have an identity e if $\forall a \in H \quad a \in e \circ a \cap a \circ e \neq \emptyset$.

Example 2.2 Davvaz [8] has given an example of a H_ν group as the chemical reaction

$$A_2 + B_2 \ \underset{\longleftrightarrow}{energy}\ 2AB$$

in which A° and B° are the fragments of A_2, B_2, AB and $\mathcal{H} = \{A^\circ, B^\circ, A_2, B_2, AB\}$.

Definition 2.6 *Let* G *be a group and* $\circ : G \times G \longrightarrow \mathcal{P}^*(G)$ *a hyperoperation. Let* $\mathcal{C} \subseteq \mathcal{P}^*(G)$ *and* $A, B \in \mathcal{C}$. *If* $\mathcal{C}$, *under the product* $A \circ B = \cup\{a \circ b | a \in A, b \in B\}$, *is a group, then* $(\mathcal{C}, \circ)$ *is a* HX *group on* G *with unit element* $E \subseteq \mathcal{C}$ *such that* $E \circ A = A = A \circ E\ \forall A \in \mathcal{C}$.

It is important to study HX group separately because some hypergroups exist but are not

HX groups. An example is $(\{\{0\},(0,+\infty),(-\infty,0)\},+)$; it a hypergroup but not a HX group. Note that if the unit element E of the quotient group of G by E is a normal subgroup of G, then the quotient group is a HX group.

Definition 2.7 *If for the identity element $e \in G$ we have $e \in E$, then $(\mathcal{C}, \circ)$ is a regular HX group on G.*

Theorem 2.1([10]) *If $\mathcal{C}$ is a HX group on G, then $\forall A, B \in \mathcal{C}$*

(1) $|A| = |B|$;
(2) $A \cap B \neq \emptyset \Longrightarrow |A \cap B| = |E|$.

Remark 2.3 Corsini [4] has shown that a HX group, also referred to as *Chinese Hyperstructure* is a H_ν Group and that, under some condition, is a hypergroup. But, trivially, a hypergroup is a H_ν Group since only that associativity was relaxed in a hypergroup to obtain a H_ν Group. Besides, Onasanya [12] has shown that no additional condition is needed by a *Chinese Hyperstructure*, that is a HX group, to become a hypergroup.

§3. Applications and Occurrences of Hyperstrutures in Biological and Chemical Sciences

The chain reactions that occur between hydrogen and halogens, say iodine (I), give interesting examples of hyperstructures [8]. This can be seen in Table 1. Many properties of these reactions can be seen from the study of hyperstructures.

Table 1. Reaction of Hydrogen with Iodine

$+$	H^o	I^o	H_2	I_2	HI
H^o	H^o, H_2	H^o, I^o, HI	H^o, H_2	H^o, I^o, HI, I_2	H^o, I^o, H_2, HI
I^o	I^o, H^o, HI	I^o, I_2	I^o, H^o, HI, H_2	I^o, I_2	H^o, I^o, HI, I_2
H_2	H^o, H_2	H^o, I^o, HI, H_2	H^o, H_2	H^o, I^o, HI, H_2, I_2	H^o, I^o, H_2, HI
I_2	I^o, H^o, I_2, HI	I^o, I_2	H^o, I^o, HI, H_2, I_2	I^o, I_2	H^o, I^o, HI, I_2
HI	H^o, I^o, H_2, HI	H^o, I^o, HI, I_2	H^o, I^o, HI, H_2	H^o, I^o, HI, I_2	H^o, I^o, HI, I_2, H_2

Let $G = \{H^o, I^o, H_2, I_2, HI\}$ so that $(G, \circ)$ is such that $\forall A, B \in G$, we have that $A \circ B$ are the possible product(s) representing the reaction between A and B. Then, $(G, \circ)$ is a H_v-group. The subsets $G_1 = \{H^o, H_2\}$ and $G_2 = \{I^o, I_2\}$ are the only H_v-subgroups of $(G, \circ)$ and indeed they are trivial hypergroups.

Davvaz [6] has the following examples: Dismutation is a kind of chemical reaction. Comproportionation is a kind of dismutation in which two different reactants of the same element having different oxidation numbers combine to form a new product with intermediate oxidation number. An example is the reaction

$$Sn + Sn^{4+} \to 2Sn^{2+}.$$

In this reaction, letting $\mathcal{G} = \{Sn, Sn^{2+}, Sn^{4+}\}$, the following table shows all possible occurrences.

Table 2. Dismutation Reaction of Tin

$\circ$	Sn	Sn^{2+}	Sn^{4+}
Sn	Sn	Sn, Sn^{2+}	Sn^{2+}
Sn^{2+}	Sn, Sn^{2+}	Sn^{2+}	Sn^{2+}, Sn^{4+}
Sn^{4+}	Sn^{2+}	Sn^{2+}, Sn^{4+}	Sn^{4+}

While it is agreeable that $(\mathcal{G}, \circ)$ is weak associative as claimed by [6], we say further that it is a H_ν group. Also, while $(\{Sn, Sn^{2+}\}, \circ)$ is agreed to be a hypergroup, we say that $(\{Sn^{2+}, Sn^{4+}\}, \circ)$ is not just a H_ν semigroup as claimed by [6] but a H_ν group.

Furthermore, Cu(0), Cu(I), Cu(II) and Cu(III) are the four oxidation states of copper. Its different species can react with themselves (without energy) as defined below

(1) $Cu^{3+} + Cu^{+} \mapsto Cu^{2+}$;

(2) $Cu^{3+} + Cu \mapsto Cu^{2+} + Cu^{+}$.

Table 3. Redox (Oxidation-Reduction) reaction of Copper

$\circ$	Cu	Cu^{+}	Cu^{2+}	Cu^{3+}
Cu	Cu	Cu, Cu^{+}	Cu, Cu^{2+}	Cu^{+}, Cu^{2+}
Cu^{+}	Cu, Cu^{+}	Cu^{+}	Cu^{+}, Cu^{2+}	Cu^{2+}
Cu^{2+}	Cu, Cu^{2+}	Cu^{+}, Cu^{2+}	Cu^{2+}	Cu^{2+}, Cu^{3+}
Cu^{3+}	Cu^{+}, Cu^{2+}	Cu^{2+}	Cu^{2+}, Cu^{3+}	Cu^{3+}

Let $G = \{Cu, Cu^{+}, Cu^{2+}, Cu^{3+}\}$. Then $(G, \circ)$ is weak associative and

$$Cu^{+} \circ X = X \circ Cu^{+} \neq X$$

so that $(G, \circ)$ is an H_v semigroup. $\{Cu, Cu^{+}\}$, $\{Cu, Cu^{2+}\}$ $\{Cu^{+}, Cu^{2+}\}$ and $\{Cu^{2+}, Cu^{3+}\}$ are hypergroups with respect to $\circ$. From Table 4 we also have that $(\{Cu, Cu^{+}, Cu^{2+}\}, \circ)$ is a hypergroup.

Table 4. Another Redox reaction of Cu

$\circ$	Cu	Cu^{+}	Cu^{2+}
Cu	Cu	Cu, Cu^{+}	Cu, Cu^{2+}
Cu^{+}	Cu, Cu^{+}	Cu^{+}	Cu^{+}, Cu^{2+}
Cu^{2+}	Cu, Cu^{2+}	Cu^{+}, Cu^{2+}	Cu^{2+}

It should be noted that $\{Cu, Cu^{+}\}$, $\{Cu, Cu^{2+}\}$ and $\{Cu^{+}, Cu^{2+}\}$ are subhypergroups of $(\{Cu, Cu^{+}, Cu^{2+}\}, \circ)$.

§4. Identities of Hyperstructures

Definition 4.1([8]) *The set $I_p = \{e \in H | \exists x \in H$ such that $x \in x \circ e \cup e \circ x\}$ is refered to as partial identities of H.*

Definition 4.2([3]) *An element $e \in H$ is called the right (analogously the left) identity of H if $x \in x \circ e (x \in e \circ x)\ \forall x \in H$. It is called an identity of H if it is both right and left identity.*

Definition 4.3([3]) *A hypergroup H is semi regular if each $x \in H$ has at least one right and one left identity.*

It can be seen that every right or left identity of H is in I_p.

4.1 Blood Genotype as a Hyperstructure

Let $G = \{AA, AS, SS\}$ and the hyperoperation $\oplus$ denote mating. The blood genotype is a kind of hyperstructure.

Table 5. Genotype Table [12]

$\oplus$	AA	AS	SS
AA	$\{AA\}$	$\{AA, AS\}$	$\{AS\}$
AS	$\{AA, AS\}$	$\{AA, AS, SS\}$	$\{AS, SS\}$
SS	$\{AS\}$	$\{AS, SS\}$	$\{SS\}$

In Table 5, $\{AA\} \oplus G \neq G \neq G \oplus \{AA\}$; the *reproduction axiom* is not satisfied. Also, it is weak associative. It is a H_ν semigroup.

Note that a lot has been discussed on the occurrence of hyperstructure algebra in inheritance [7]. For most of the monohybrid and dihybrid crossing of the pea plant, they are hypergroups in the second generation. Take for instance, *the monohybrid Crossing in the Pea Plant* , the parents has the RR(Round) and rr(Wrinkled) genes. The first generation has Rr(Round). The second generation has RR(Round), Rr(Round) and rr(Wrinkled). Now consider the set $G = \{R, W\}$; R for Round and W for Wrinkled. Crossing this generation under the operation $\oplus$ for mating, [7] already established it is a hypergroup.

In the following section, a little more information about their properties would be given and an extension to cases which are hypergroups in earlier generations are made.

§5. Main Results

5.1 Hyperstructures in Group Theory

The following example is a construction of an HX group which is also a hypergroup and a H_ν Group by Remark 2.1.

Example 5.1 Let us partition $(\mathrm{Z}_{10}, +)$ by $\rho = \{\{0, 5\}, \{1, 6\}, \{2, 7\}, \{3, 8\}, \{4, 9\}\}$. Then we can see that $E = \{0, 5\}$ is a normal subgroup of $(\mathrm{Z}_{10}, +)$ and that $E^2 = E$. $(\mathrm{Z}_{10}/\rho, \circ)$ is also a regular HX group since $0 \in E$.

We give some further clarifications on Table 5, that this is a H_ν cyclic semigroup, with generator $\{AS\}$. It has no H_ν subsemigroups. The set of partial identites I_p of $(G, \oplus)$ is G itself by Definition 4.1, and the identity (which is both right and left identity) of G is $\{AS\}$ by Definition 4.2. Then, $(G, \oplus)$ is also a semi regular hypergroupoid by Defnition 4.3. Note that if the parents' genotype are $\{AA, AS\}$ or $\{AA, SS\}$ or $\{AS, SS\}$, the first generations of each of these are H_ν semigroups. These can be seen in the tables below.

Table 6. Parents with the genotype AA and AS

$\oplus$	AA	AS
AA	$\{AA\}$	$\{AA, AS\}$
AS	$\{AA, AS\}$	$\{AA, AS, SS\}$

The first generation $H_1 = \{AA, AS, SS\}$ is a H_ν semigroup under $\oplus$.

Table 7. Parents with the genotype AA and SS

$\oplus$	AA	SS
AA	$\{AA\}$	$\{AS\}$
SS	$\{AS\}$	$\{SS\}$

The first generation $H_2 = \{AA, AS, SS\}$ is a H_ν semigroup under $\oplus$.

Table 8. Parents with the genotype AA and SS

$\oplus$	AS	SS
AS	$\{AA, AS, SS\}$	$\{AS, SS\}$
SS	$\{AS, SS\}$	$\{SS\}$

The first generation $H_3 = \{AA, AS, SS\}$ is a H_ν semigroup under $\oplus$.

It is established in this work that the case of crossing between organism which have lethal genes (i.e. the genes that cause the death of the carrier at homozygous condition), such as the crossing of mice parents with traits Yellow(Yy) and Grey(yy), is a semi regular hypergroup at all generations, including the parents' generation. However, the parents with traits Yellow(Yy) and Yellow(Yy) have their first generation and the generations of all other offsprings to be semi regular hypergyoups. These are summarized in the tables below.

Table 9. Parents with the genotype Yellow(Yy) and Grey(yy)

$\oplus$	Yy	yy
Yy	$\{Yy, yy\}$	$\{Yy, yy\}$
yy	$\{Yy, yy\}$	$\{yy\}$

They produce the offspring Yy and yy like themselves in the first generation in the ratio 2:3. Let $G = \{Yy, yy\}$, $(G, \oplus)$ is a semi regular hypergroup.

Table 10. Parents with the genotype Yellow(Yy) and Yellow(Yy)

$\oplus$	Yy	Yy
Yy	$\{Yy, yy\}$	$\{Yy, yy\}$
Yy	$\{Yy, yy\}$	$\{yy\}$

They produce the offspring Yy and yy in the first generation in the ratio 2:1 but is not a hypergroupoid for the occurrence of yy. But crossing this first generation produces the result of Table 9, showing that the first generation with $\oplus$ is a hypergroup. This same result is obtained for all other generations in this crossing henceforth.

It is important to note that the monohybrid and dihybrid mating of pea plant considered in [7] are not just hypergroups but semi regular hypergroups. The particular case mentioned above has a right and a left identity $I = \{W\}$.

§6. Conclusions

The following is just to make some conclusions. Far reaching ones can be made from the in-depth studies and applications of the theory of hyperstructures. The algebraic properties of these hyperstructures can be used to gain insight into what happens in the biological situations and chemical reactions which they have modelled. For instance, the *weak associativity*, in case it is a case of H_ν group, of some of the chemical reactions suggests that, given reactants A, B, and C, one must be careful in the order of mixture as you may not always have the same product when $A + B$ is done before adding C as in when $B + C$ is done before adding A. In other words, $A+(B+C)$ does not always equal $(A+B)+C$. Moreover, the *strong associativity*, in the case of hypergroup, indicates that same products are obtained in both orders.

From blood the genotype table of $G = \{AA, AS, SS\}$, reproduction axiom is not satisfied with the element $\{SS\}$, meaning that if marriages are only contracted between any member of the group and someone with $\{SS\}$ genotype, all offsprings shall be carriers of sickle cell in all subsequent generations. Besides, its *weak associativity* property indicates that if there were to be marriages between individuals with genotypes A, B, and C so that those with the genotypes A and B marry and produce offsprings which now marry those with genotype C, then some of the offsprings of this marriage will always have the same genotype as some of the offsprings of those with genotype A marrying the offsprings produced by the marriages of people with the genotypes B and C.

If the operation $\oplus$ denotes cross breeding, it should also be noted that genetic crossing (in terms of genotype or phenotype) is not always, at the parents level, a hyperstructure. This is because in the collection of all traits $\mathcal{P}^*(T)$ of *Parents*, there sometimes will be *trait* A and *trait* B which combine to form a *trait* C but such that $C \notin \mathcal{P}^*(T)$. An example is in the *incomplete dominance* reported when Mendel crossed the four O' clock plant (Mirabilis jalapa) which produced an intermediate flower colour (Pink) from parents having Red and White colours. Not even at any generation will it be a hyperstructure as long as there is incomplete dominance. Hence, the theory of hyperstructures should not be applied in this case.

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A Class of Lie-admissible Algebras

Qiuhui Mo, Xiangui Zhao and Qingnian Pan

(Department of Mathematics, Huizhou University Huizhou 516007, P. R. China)

E-mail: scnuhuashimomo@126.com, xiangui.zhao@foxmail.com, pqn@hzu.edu.cn

Abstract: In this paper, we study nonassociative algebras which satisfy the following identities: $(xy)z = (yx)z, x(yz) = x(zy)$. These algebras are Lie-admissible algebras i.e., they become Lie algebras under the commutator $[f, g] = fg - gf$. We obtain a nonassociative Gröbner-Shirshov basis for the free algebra $LA(X)$ with a generating set X of the above variety. As an application, we get a monomial basis for $LA(X)$. We also give a characterization of the elements of $S(X)$ among the elements of $LA(X)$, where $S(X)$ is the Lie subalgebra, generated by X, of $LA(X)$.

Key Words: Nonassociative algebra, Lie admissible algebra, Gröbner-Shirshov basis.

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§1. Introduction

In 1948, A. A. Albert introduced a new family of (nonassociative) algebras whose commutator algebras are Lie algebras [1]. These algebras are called Lie-admissible algebras, and they arise naturally in various areas of mathematics and mathematical physics such as differential geometry of affine connections on Lie groups. Examples include associative algebras, pre-Lie algebras and so on.

Let $k\langle X\rangle$ be the free associative algebra generated by X. It is well known that the Lie subalgebra, generated X, of $k\langle X\rangle$ is a free Lie algebra (see for example [6]). Friedrichs [15] has given a characterization of Lie elements among the set of noncommutative polynomials. A proof of characterization theorem was also given by Magnus [18], who refers to other proofs by P. M. Cohn and D. Finkelstein. Later, two short proofs of the characterization theorem were given by R. C. Lyndon [17] and A. I. Shirshov [21], respectively.

Pre-Lie algebras arise in many areas of mathematics and physics. As was pointed out by D. Burde [8], these algebras first appeared in a paper by A. Cayley in 1896 (see [9]). Survey [8] contains detailed discussion of the origin, theory and applications of pre-Lie algebras in geometry and physics together with an extensive bibliography. Free pre-Lie algebras had already

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been studied as early as 1981 by Agrachev and Gamkrelidze [2]. They gave a construction of monomial bases for free pre-Lie algebras. Segal [20] in 1994 gave an explicit basis (called good words in [20]) for a free pre-Lie algebra and applied it for the PBW-type theorem for the universal pre-Lie enveloping algebra of a Lie algebra. Linear bases of free pre-Lie algebras were also studied in [3, 10, 11, 14, 25]. As a special case of Segal's latter result, the Lie subalgebra, generated by X, of the free pre-Lie algebra with generating set X is also free. Independently, this result was also proved by A. Dzhumadil'daev and C. Löfwall [14]. M. Markl [19] gave a simple characterization of Lie elements in free pre-Lie algebras as elements of the kernel of a map between spaces of trees.

Gröbner bases and Gröbner-Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [22, 24], free Lie algebras [23, 24] and implicitly free associative algebras [23, 24] (see also [4, 5, 12, 13]), by H. Hironaka [16] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [7] for ideals of the polynomial algebras.

In this paper, we study a class of Lie-admissible algebras. These algebras are nonassociative algebras which satisfy the following identities: $(xy)z = (yx)z, x(yz) = x(zy)$. Let $LA(X)$ be the free algebra with a generating set X of the above variety. We obtain a nonassociative Gröbner-Shirshov basis for the free algebra $LA(X)$. Using the Composition-Diamond lemma of nonassociative algebras, we get a monomial basis for $LA(X)$. Let $S(X)$ be the Lie subalgebra, generated by X, of $LA(X)$. We get a linear basis of $S(X)$. As a corollary, we show that $S(X)$ is not a free Lie algebra when the cardinality of X is greater than 1. We also give a characterization of the elements of $S(X)$ among the elements of $LA(X)$. For the completeness of this paper, we formulate the Composition-Diamond lemma for free nonassociative algebras in Section 2.

§2. Composition-Diamond Lemma for Nonassociative Algebras

Let X be a well ordered set. Each letter $x \in X$ is a nonassociative word of degree 1. Suppose that u and v are nonassociative words of degrees m and n respectively. Then uv is a nonassociative word of degree $m + n$. Denoted by $|uv|$ the degree of uv, by X^* the set of all associative words on X and by X^{**} the set of all nonassociative word on X. If $u = (p(v)q)$, where $p, q \in X^*, u, v \in X^{**}$, then v is called a subword of u. Denote u by $u|_v$, if this is the case.

The set X^{**} can be ordered by the following way: $u > v$ if either

(1) $|u| > |v|$; or

(2) $|u| = |v|$ and $u = u_1u_2$, $v = v_1v_2$, and either

(2a) $u_1 > v_1$; or

(2b) $u_1 = v_1$ and $u_2 > v_2$.

This ordering is called degree lexicographical ordering and used throughout this paper.

Let k be a field and $M(X)$ be the free nonassociative algebra over k, generated by X. Then each nonzero element $f \in M(X)$ can be presented as

$$f = \alpha\overline{f} + \sum_i \alpha_i u_i,$$

where $\overline{f} > u_i, \alpha, \alpha_i \in k, \alpha \neq 0,\ u_i \in X^{**}$. Then $\overline{f}$, α are called the leading term and leading coefficient of f respectively and f is called monic if $\alpha = 1$. Denote by $d(f)$ the degree of f, which is defined by $d(f) = |\bar{f}|$.

Let $S \subset M(X)$ be a set of monic polynomials, $s \in S$ and $u \in X^{**}$. We define S-word $(u)_s$ in a recursive way:

(i) $(s)_s = s$ is an S-word of s-length 1;

(ii) If $(u)_s$ is an S-word of s-length k and v is a nonassociative word of degree l, then

$$(u)_s v \ \text{ and } \ v(u)_s$$

are S-words of s-length $k + l$.

Note that for any S-word $(u)_s = (asb)$, where $a, b \in X^*$, we have $\overline{(asb)} = (a(\bar{s})b)$.

Let f, g be monic polynomials in $M(X)$. Suppose that there exist $a, b \in X^*$ such that $\bar{f} = (a(\bar{g})b)$. Then we define the composition of inclusion

$$(f, g)_{\bar{f}} = f - (agb).$$

The composition $(f, g)_{\bar{f}}$ is called trivial modulo $(S, \bar{f})$, if

$$(f, g)_{\bar{f}} = \sum_i \alpha_i(a_i s_i b_i)$$

where each $\alpha_i \in k,\ a_i, b_i \in X^*,\ s_i \in S,\ (a_i s_i b_i)$ an S-word and $(a_i(\bar{s_i})b_i) < \bar{f}$. If this is the case, then we write $(f, g)_{\bar{f}} \equiv 0 \text{ mod}(S, \bar{f})$. In general, for $p, q \in M(X)$ and $w \in X^{**}$, we write

$$p \equiv q \mod(S, w)$$

which means that $p - q = \sum \alpha_i(a_i s_i b_i)$, where each $\alpha_i \in k, a_i, b_i \in X^*,\ s_i \in S,\ (a_i s_i b_i)$ an S-word and $(a_i(\bar{s_i})b_i) < w$.

Definition 2.1([22,24]) *Let $S \subset M(X)$ be a nonempty set of monic polynomials and the ordering $>$ defined as before. Then S is called a Gröbner-Shirshov basis in $M(X)$ if any composition $(f, g)_{\bar{f}}$ with $f, g \in S$ is trivial modulo $(S, \bar{f})$, i.e., $(f, g)_{\bar{f}} \equiv 0$ mod$(S, \bar{f})$.*

Theorem 2.2([22,24]) (Composition-Diamond lemma for nonassociative algebras) *Let $S \subset M(X)$ be a nonempty set of monic polynomials, $Id(S)$ the ideal of $M(X)$ generated by S and the ordering $>$ on X^{**} defined as before. Then the following statements are equivalent:*

(i) S is a Gröbner-Shirshov basis in $M(X)$;

(ii) $f \in Id(S) \Rightarrow \bar{f} = (a(\bar{s})b)$ for some $s \in S$ and $a, b \in X^*$, where (asb) is an S-word;

(iii) $Irr(S) = \{u \in X^{**} | u \neq (a(\bar{s})b)\ a, b \in X^*,\ s \in S$ *and* (asb) *is an* S*-word*$\}$ *is a linear basis of the algebra* $M(X|\ S) = M(X)/Id(S)$.

§3. A Nonassociative Gröbner-Shirshov Basis for the Algebra $LA(X)$

Let $\mathcal{LA}$ be the variety of nonassociative algebras which satisfy the following identities: $(xy)z = (yx)z, x(yz) = x(zy)$. Let $LA(X)$ be the free algebra with a generating set X of the variety $\mathcal{LA}$. It's clear that the free algebra $LA(X)$ is isomorphic to $M(X|(uv)w - (vu)w, w(uv) - w(vu), u, v, w \in X^{**})$.

Theorem 3.1 *Let* $S = \{(uv)w - (vu)w, w(uv) - w(vu), u > v, u, v, w \in X^{**}\}$. *Then* S *is a Gröbner-Shirshov basis of the algebra* $M(X|\ (uv)w - (vu)w, w(uv) - w(uv), u, v, w \in X^{**})$.

Proof It is clear that $Id(S)$ is the same as the ideal generated by the set $\{(uv)w - (vu)w, w(uv) - w(vu), u, v, w \in X^{**}\}$ of $M(X)$. Let $f_{123} = (u_1u_2)u_3 - (u_2u_1)u_3, g_{123} = v_1(v_2v_3) - v_1(v_3v_2), u_1 > u_2, v_2 > v_3, u_i, v_i \in X^{**}, 1 \leq i \leq 3$. Clearly, $\overline{f_{123}} = (u_1u_2)u_3$ and $\overline{g_{123}} = v_1(v_2v_3)$. Then all possible compositions in S are the following:

(c_1) $(f_{123}, f_{456})_{(u_1|_{(u_4u_5)u_6}u_2)u_3}$;
(c_2) $(f_{123}, f_{456})_{(u_1u_2|_{(u_4u_5)u_6})u_3}$;
(c_3) $(f_{123}, f_{456})_{(u_1u_2)u_3|_{(u_4u_5)u_6}}$;
(c_4) $(f_{123}, f_{456})_{((u_4u_5)u_6)u_3}, u_1u_2 = (u_4u_5)u_6$;
(c_5) $(f_{123}, f_{456})_{(u_1u_2)u_3}, (u_1u_2)u_3 = (u_4u_5)u_6$;
(c_6) $(f_{123}, g_{123})_{(u_1|_{v_1(v_2v_3)}u_2)u_3}$;
(c_7) $(f_{123}, g_{123})_{(u_1u_2|_{v_1(v_2v_3)})u_3}$;
(c_8) $(f_{123}, g_{123})_{(u_1u_2)u_3|_{v_1(v_2v_3)}}$;
(c_9) $(f_{123}, g_{123})_{(v_1(v_2v_3))u_3}, u_1u_2 = v_1(v_2v_3)$;
(c_{10}) $(f_{123}, g_{123})_{(u_1u_2)(v_2v_3)}, u_1u_2 = v_1, u_3 = v_2v_3$;
(c_{11}) $(g_{123}, f_{123})_{v_1|_{(u_1u_2)u_3}(v_2v_3)}$;
(c_{12}) $(g_{123}, f_{123})_{v_1(v_2|_{(u_1u_2)u_3}v_3)}$;
(c_{13}) $(g_{123}, f_{123})_{v_1(v_2v_3|_{(u_1u_2)u_3})}$;
(c_{14}) $(g_{123}, f_{123})_{v_1((u_1u_2)u_3)}, v_2v_3 = (u_1u_2)u_3$;
(c_{15}) $(g_{123}, g_{456})_{v_1|_{v_4(v_5v_6)}(v_2v_3)}$;
(c_{16}) $(g_{123}, g_{456})_{v_1(v_2|_{v_4(v_5v_6)}v_3)}$;
(c_{17}) $(g_{123}, g_{456})_{v_1(v_2v_3|_{v_4(v_5v_6)})}$;
(c_{18}) $(g_{123}, g_{456})_{v_1(v_4(v_5v_6))}, v_2v_3 = v_4(v_5v_6)$;
(c_{19}) $(g_{123}, g_{456})_{v_1(v_2v_3)}, v_1(v_2v_3) = v_4(v_5v_6)$.

The above compositions in S all are trivial module S. Here, we only prove the following cases: (c_1), (c_4), (c_9), (c_{10}), (c_{14}), (c_{18}). The other cases can be proved similarly.

$$\begin{aligned}(f_{123}, f_{456})_{(u_1|_{(u_4u_5)u_6}u_2)u_3} \equiv&(u_2u_1|_{(u_4u_5)u_6})u_3 - (u_1'|_{(u_5u_4)u_6}u_2)u_3\\ \equiv&(u_2u_1'|_{(u_5u_4)u_6})u_3 - (u_1'|_{(u_5u_4)u_6}u_2)u_3 \equiv 0,\end{aligned}$$

$$\begin{aligned}(f_{123}, f_{456})_{((u_4u_5)u_6)u_3}, u_1u_2 = (u_4u_5)u_6 =&(u_6(u_4u_5))u_3 - ((u_5u_4)u_6)u_3\\ \equiv&(u_6(u_5u_4))u_3 - ((u_5u_4)u_6)u_3 \equiv 0,\end{aligned}$$

$$\begin{aligned}(f_{123}, g_{123})_{(v_1(v_2v_3))u_3}, u_1u_2 = v_1(v_2v_3) =&((v_2v_3)v_1)u_3 - (v_1(v_3v_2))u_3\\ \equiv&((v_3v_2)v_1)u_3 - (v_1(v_3v_2))u_3 \equiv 0,\end{aligned}$$

$$\begin{aligned}(f_{123}, g_{123})_{(u_1u_2)(v_2v_3)}, u_1u_2 = v_1, u_3 = v_2v_3 =&(u_2u_1)(v_2v_3) - (u_1u_2)(v_3v_2)\\ \equiv&(u_2u_1)(v_3v_2) - (u_2u_1)(v_3v_2) = 0,\end{aligned}$$

$$\begin{aligned}(g_{123}, f_{123})_{v_1((u_1u_2)u_3)}, v_2v_3 =&(u_1u_2)u_3 = v_1(u_3(u_1u_2)) - v_1((u_2u_1)u_3)\\ \equiv&v_1(u_3(u_2u_1)) - v_1((u_2u_1)u_3) \equiv 0,\end{aligned}$$

$$\begin{aligned}(g_{123}, g_{456})_{v_1(v_4(v_5v_6))}, v_2v_3 =&(v_4(v_5v_6)) = v_1((v_5v_6)v_4) - v_1(v_4(v_6v_5))\\ \equiv&v_1((v_6v_5)v_4) - v_1(v_4(v_6v_5)) \equiv 0.\end{aligned}$$

Therefore S is a Gröbner-Shirshov basis of the algebra $M(X|(uv)w - (vu)w, w(uv) - w(uv), u, v, w \in X^{**})$. □

Definition 3.2 *Each letter $x_i \in X$ is called a regular word of degree 1. Suppose that $u = vw$ is a nonassociative word of degree $m, m > 1$. Then $u = vw$ is called a regular word of degree m if it satisfies the following conditions:*

($S1$) *both v and w are regular words;*

($S2$) *if $v = v_1v_2$, then $v_1 \leq v_2$;*

($S3$) *if $w = w_1w_2$, then $w_1 \leq w_2$.*

Lemma 3.3 *Let $N(X)$ be the set of all regular words on X. Then $Irr(S) = N(X)$.*

Proof Suppose that $u \in Irr(S)$. If $|u| = 1$, then $u = x \in N(X)$. If $|u| > 1$ and $u = vw$, then by induction $v, w \in N(X)$. If $v = v_1v_2$, then $v_1 \leq v_2$, since $u \in Irr(S)$. If $w = w_1w_2$, then $w_1 \leq w_2$, since $u \in Irr(S)$. Therefore $u \in N(X)$.

Suppose that $u \in N(X)$. If $|u| = 1$, then $u = x \in Irr(S)$. If $u = vw$, then v, w are regular and by induction $v, w \in Irr(S)$. If $v = v_1v_2$, then $v_1 \leq v_2$, since $u \in N(X)$. If $w = w_1w_2$, then $w_1 \leq w_2$, since $u \in N(X)$. Therefore $u \in Irr(S)$. □

From Theorems 2.2, 3.1 and Lemma 3.3, the following result follows.

Theorem 3.4 *The set $N(X)$ of all regular words on X forms a linear basis of the free algebra $LA(X)$.*

§4. A Characterization Theorem

Let X be a well ordered set, $S(X)$ the Lie subalgebra, generated by X, of $LA(X)$ under the commutator $[f,g] = fg - gf$. Let $T = \{[x_i, x_j] | x_i > x_j, x_i, x_j \in X\}$ where $[x_i, x_j] = x_i x_j - x_j x_i$.

Lemma 5.1 *The set $X \bigcup T$ forms a linear basis of the Lie algebra $S(X)$.*

Proof Let $u \in X \bigcup T$. If $u = x_i$, then $\bar{u} = x_i$. If $u = [x_i, x_j], x_i > x_j$, then $u = x_i x_j - x_j x_i$ and thus $\bar{u} = x_i x_j$. Then we may conclude that if $u, v \in X \bigcup T$ and $u \neq v$, then $\bar{u} \neq \bar{v}$. Therefore the elements in $X \bigcup T$ are linear independent. Since $[[f,g],h] = (fg)h - (gf)h - h(fg) + h(gf) = 0 = -[h,[f,g]]$, then all the Lie words with degree ≥ 3 equal zero. Therefore, the set $X \bigcup T$ forms a linear basis of the Lie algebra $S(X)$. □

Corollary 5.2 *Let $|X| > 1$. Then the Lie subalgebra $S(X)$ of $LA(X)$ is not a free Lie algebra.*

Theorem 5.3 *An element $f(x_1, x_2, \cdots, x_s)$ of the algebra $LA(X)$ belongs to $S(X)$ if and only if $d(f) < 3$ and the relations $x_i x'_j = x'_j x_i, i, j = 1, 2, \cdots, n$ imply the equation $f(x_1 + x'_1, x_2 + x'_2, \cdots, x_s + x'_s) = f(x_1, x_2, \cdots, x_s) + f(x'_1, x'_2, \cdots, x'_s)$.*

Proof Suppose that an element $f(x_1, x_2, \cdots, x_s)$ of the algebra $LA(X)$ belongs to $S(X)$. From Lemma 4.1, it follows that $d(f) < 3$ and it suffices to prove that if $u(x_1, x_2, \cdots, x_s) \in X \bigcup T$, then the relations $x_i x'_j = x'_j x_i$ imply the equation $u(x_1 + x'_1, x_2 + x'_2, \cdots, x_s + x'_s) = u(x_1, x_2, \cdots, x_s) + u(x'_1, x'_2, \cdots, x'_s)$. This holds since $d(f) < 3$ and $[x'_i, x_j] = [x_j, x'_i] = 0$, $x'_i, x_j, 1 \leq i, j \leq s$.

Let d_1 be an element of the algebra $LA(X)$ that does not belong to $S(X)$. If $\bar{d_1} = x_i x_j$ where $x_i > x_j$, then let $d_2 = d_1 - [x_i, x_j]$. Clearly, d_2 is also an element of the algebra $LA(X)$ that does not belong to $S(X)$. Then after a finite number of steps of the above algorithm, we will obtain an element d_t whose leading term is u_t where $u_t = x_p x_q, x_p \leq x_q$. It's easy to see that in the expression

$$d_t(x_1 + x'_1, x_2 + x'_2, \cdots, x_s + x'_s) - d_t(x_1, x_2, \cdots, x_s) - d_t(x'_1, x'_2, \cdots, x'_s)$$

the element $x'_q x_p$ occurs with nonzero coefficient. □

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Intrinsic Geometry of the Special Equations in Galilean $3-$Space G_3

Handan Oztekin

(Firat University, Science Faculty, Department of Mathematics, Elazig, Turkey)

Sezin Aykurt Sepet

(Ahi Evran University, Art and Science Faculty, Department of Mathematics, Kirsehir, Turkey)

E-mail: saykurt@ahievran.edu.tr

Abstract: In this study, we investigate a general intrinsic geometry in 3-dimensional Galilean space G_3. Then, we obtain some special equations by using intrinsic derivatives of orthonormal triad in G_3.

Key Words: NLS Equation, Galilean Space.

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§1. Introduction

A Galilean space is a three dimensional complex projective space, where $\{w, f, I_1, I_2\}$ consists of a real plane w (the absolute plane), real line $f \subset w$ (the absolute line) and two complex conjugate points I_1, $I_2 \in f$ (the absolute points). We shall take as a real model of the space G_3, a real projective space P_3 with the absolute $\{w, f\}$ consisting of a real plane $w \subset G_3$ and a real line $f \subset w$ on which an elliptic involution ε has been defined. The Galilean scalar product between two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ is defined [3]

$$(a.b)_G = \begin{cases} a_1 b_1, & if\ a_1 \neq 0\ or\ b_1 \neq 0, \\ a_2 b_2 + a_3 b_3, & if\ \ a_1 = b_1 = 0. \end{cases}$$

and the Galilean vector product is defined

$$(a \wedge b)_G = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & if\ a_1 \neq 0\ or\ b_1 \neq 0, \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & if\ a_1 = b_1 = 0. \end{cases}$$

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Let $\alpha : I \to G_3,\ I \subset R$ be an unit speed curve in Galilean space G_3 parametrized by the invariant parameter $s \in I$ and given in the coordinate form
$\alpha(s) = (s, y(s), z(s))$. Then the curvature and the torsion of the curve α are given by

$$\kappa(s) = \|\alpha''(s)\|, \quad \tau(s) = \frac{1}{\kappa^2(s)} Det\left(\alpha'(s), \alpha''(s), \alpha'''(s)\right)$$

respectively. The Frenet frame $\{t, n, b\}$ of the curve α is given by

$$\begin{aligned} t(s) &= \alpha'(s) = (1, y'(s), z'(s)), \\ n(s) &= \frac{\alpha''(s)}{\|\alpha''(s)\|} = \frac{1}{\kappa(s)}(1, y''(s), z''(s)), \\ b(s) &= (t(s) \wedge n(s))_G = \frac{1}{\kappa(s)}(1, -z''(s), y''(s)), \end{aligned}$$

where $t(s)$, $n(s)$ and $b(s)$ are called the tangent vector, principal normal vector and binormal vector, respectively. The Frenet formulas for $\alpha(s)$ given by [3] are

$$\begin{bmatrix} t'(s) \\ n'(s) \\ b'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ 0 & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} t(s) \\ n(s) \\ b(s) \end{bmatrix}. \tag{1.1}$$

The binormal motion of curves in the Galilean 3-space is equivalent to the nonlinear Schrödinger equation (NLS$^-$) of repulsive type

$$iq_b + q_{ss} - \frac{1}{2}\left|\langle q, q\rangle\right|^2 \bar{q} = 0 \tag{1.2}$$

where

$$q = \kappa \exp\left(\int_0^s \sigma ds\right), \quad \sigma{=}\kappa\exp\left(\int_0^s r ds\right). \tag{1.3}$$

§2. Basic Properties of Intrinsic Geometry

Intrinsic geometry of the nonlinear Schrodinger equation was investigated in E^3 by Rogers and Schief. According to anholonomic coordinates, characterization of three dimensional vector field was introduced in E^3 by Vranceau [5], and then analyse Marris and Passman [3].

Let ϕ be a 3-dimensional vector field according to anholonomic coordinates in G_3. The $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$ is the tangent, principal normal and binormal directions to the vector lines of ϕ. Intrinsic derivatives of this orthonormal triad are given by following

$$\frac{\delta}{\delta s}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{2.1}$$

$$\frac{\delta}{\delta n}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \theta_{ns} & (\Omega_b+\tau) \\ -\theta_{ns} & 0 & -div\boldsymbol{b} \\ -(\Omega_b+\tau) & div\boldsymbol{b} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{2.2}$$

$$\frac{\delta}{\delta b}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & -(\Omega_n+\tau) & \theta_{bs} \\ (\Omega_n+\tau) & 0 & div\boldsymbol{n} \\ -\theta_{bs} & -div\boldsymbol{n} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}, \tag{2.3}$$

where $\frac{\delta}{\delta s}$, $\frac{\delta}{\delta n}$ and $\frac{\delta}{\delta b}$ are directional derivatives in the tangential, principal normal and binormal directions in G_3. Thus, the equation (2.1) show the usual Serret-Frenet relations, also (2.2) and (2.3) give the directional derivatives of the orthonormal triad $\{\boldsymbol{t},\boldsymbol{n},\boldsymbol{b}\}$ in the n- and b-directions, respectively. Accordingly,

$$grad = \boldsymbol{t}\frac{\delta}{\delta s}+\boldsymbol{n}\frac{\delta}{\delta n}+\boldsymbol{b}\frac{\delta}{\delta b}, \tag{2.4}$$

where θ_{bs} and θ_{ns} are the quantities originally introduced by Bjorgum in 1951 [2] via

$$\theta_{ns}=\boldsymbol{n}\cdot\frac{\delta\boldsymbol{t}}{\delta n}, \qquad \theta_{bs}=\boldsymbol{b}\cdot\frac{\delta\boldsymbol{t}}{\delta b}. \tag{2.5}$$

From the usual Serret Frenet relations in G_3, we obtain the following equations

$$div\boldsymbol{t}=(\boldsymbol{t}\frac{\delta}{\delta s}+\boldsymbol{n}\frac{\delta}{\delta n}+\boldsymbol{b}\frac{\delta}{\delta b})\boldsymbol{t}=\boldsymbol{t}(\kappa\boldsymbol{n})+\boldsymbol{n}\frac{\delta\boldsymbol{t}}{\delta n}+\boldsymbol{b}\frac{\delta\boldsymbol{t}}{\delta b}=\theta_{ns}+\theta_{bs}, \tag{2.6}$$

$$div\boldsymbol{n}=(\boldsymbol{t}\frac{\delta}{\delta s}+\boldsymbol{n}\frac{\delta}{\delta n}+\boldsymbol{b}\frac{\delta}{\delta b})\boldsymbol{n}=\boldsymbol{t}(\tau\boldsymbol{b})+\boldsymbol{n}\frac{\delta\boldsymbol{n}}{\delta n}+\boldsymbol{b}\frac{\delta\boldsymbol{n}}{\delta b}=\boldsymbol{b}\frac{\delta\boldsymbol{n}}{\delta b}, \tag{2.7}$$

$$div\boldsymbol{b}=(\boldsymbol{t}\frac{\delta}{\delta s}+\boldsymbol{n}\frac{\delta}{\delta n}+\boldsymbol{b}\frac{\delta}{\delta b})\boldsymbol{b}=\boldsymbol{t}(-\tau\boldsymbol{n})+\boldsymbol{n}\frac{\delta\boldsymbol{b}}{\delta n}+\boldsymbol{b}\frac{\delta\boldsymbol{b}}{\delta b}=\boldsymbol{n}\frac{\delta\boldsymbol{b}}{\delta n}. \tag{2.8}$$

Moreover, we get

$$\begin{aligned} curl\boldsymbol{t} &= \left(\boldsymbol{t}\times\frac{\delta}{\delta s}+\boldsymbol{n}\times\frac{\delta}{\delta n}+\boldsymbol{b}\times\frac{\delta}{\delta b}\right)\boldsymbol{t} \\ &= \boldsymbol{t}\times(\kappa\boldsymbol{n})+\boldsymbol{n}\times\frac{\delta\boldsymbol{t}}{\delta n}+\boldsymbol{b}\times\frac{\delta\boldsymbol{t}}{\delta b} \\ &= \left[\frac{\delta\boldsymbol{t}}{\delta n}\boldsymbol{b}-\frac{\delta\boldsymbol{t}}{\delta b}\boldsymbol{n}\right](1,0,0)+\kappa\boldsymbol{b} \end{aligned}$$

$$\Rightarrow curl\boldsymbol{t}=\Omega_s\,(1,0,0)+\kappa\boldsymbol{b}, \tag{2.9}$$

where

$$\Omega_s=\boldsymbol{t}\cdot curl\boldsymbol{t}=\boldsymbol{b}\cdot\frac{\delta\boldsymbol{t}}{\delta n}-\boldsymbol{n}\cdot\frac{\delta\boldsymbol{t}}{\delta b} \tag{2.10}$$

is defined the abnormality of the $\boldsymbol{t}$-field. Firstly, the relation (2.9) was obtained in E^3 by

Masotti. Also, we find

$$\begin{aligned} curl\boldsymbol{n} &= \left(\boldsymbol{t}\times\frac{\delta}{\delta s}+\boldsymbol{n}\times\frac{\delta}{\delta n}+\boldsymbol{b}\times\frac{\delta}{\delta b}\right)\boldsymbol{n} \\ &= \boldsymbol{t}\times(\tau\boldsymbol{b})+\boldsymbol{n}\times\frac{\delta\boldsymbol{n}}{\delta n}+\boldsymbol{b}\times\frac{\delta\boldsymbol{n}}{\delta b} \\ &= \left[\boldsymbol{t}\cdot\frac{\delta\boldsymbol{n}}{\delta b}-\tau\right]\boldsymbol{n}+\left(\boldsymbol{b}\frac{\delta\boldsymbol{n}}{\delta n}\right)(1,0,0)-\left(\boldsymbol{t}\frac{\delta\boldsymbol{n}}{\delta n}\right)\boldsymbol{b} \end{aligned}$$

$$\Rightarrow curl\boldsymbol{n}=-\left(div\boldsymbol{b}\right)(1,0,0)+\Omega_n\boldsymbol{n}+\theta_{ns}\boldsymbol{b}, \tag{2.11}$$

where

$$\Omega_n=\boldsymbol{n}\cdot curl\boldsymbol{n}=\boldsymbol{t}\cdot\frac{\delta\boldsymbol{n}}{\delta b}-\tau \tag{2.12}$$

is defined the abnormality of the $\boldsymbol{n}$-field and

$$\begin{aligned} curl\boldsymbol{b} &= \left(\boldsymbol{t}\times\frac{\delta}{\delta s}+\boldsymbol{n}\times\frac{\delta}{\delta n}+\boldsymbol{b}\times\frac{\delta}{\delta b}\right)\boldsymbol{b} \\ &= \boldsymbol{t}\times(-\tau\boldsymbol{n})+\boldsymbol{n}\times\left[\left(\boldsymbol{t}\frac{\delta\boldsymbol{b}}{\delta n}\right)\boldsymbol{t}\right]+\boldsymbol{b}\times\left[\left(\boldsymbol{t}\frac{\delta\boldsymbol{b}}{\delta b}\right)\boldsymbol{t}+\left(\boldsymbol{n}\frac{\delta\boldsymbol{b}}{\delta b}\right)\boldsymbol{n}\right] \\ &= -\left[\tau+\boldsymbol{t}\cdot\frac{\delta\boldsymbol{b}}{\delta n}\right]\boldsymbol{b}+\left(\boldsymbol{t}\frac{\delta\boldsymbol{b}}{\delta b}\right)\boldsymbol{n}+\left(\boldsymbol{b}\frac{\delta\boldsymbol{n}}{\delta b}\right)(1,0,0), \end{aligned}$$

$$\Rightarrow curl\boldsymbol{b}=\Omega_b\boldsymbol{b}-\theta_{bs}\boldsymbol{n}+\left(div\boldsymbol{n}\right)(1,0,0), \tag{2.13}$$

where

$$\Omega_b=\boldsymbol{b}\cdot curl\boldsymbol{b}=-\left(\tau+\boldsymbol{t}\cdot\frac{\delta\boldsymbol{b}}{\delta n}\right) \tag{2.14}$$

is defined the abnormality of the $\boldsymbol{b}$-field. By using the identity $curlgrad\varphi=0$, we have

$$\left(\frac{\delta^2\varphi}{\delta n\delta b}-\frac{\delta^2\varphi}{\delta b\delta n}\right)\boldsymbol{t}+\left(\frac{\delta^2\varphi}{\delta b\delta s}-\frac{\delta^2\varphi}{\delta s\delta b}\right)\boldsymbol{n}+\left(\frac{\delta^2\varphi}{\delta s\delta n}-\frac{\delta^2\varphi}{\delta n\delta s}\right)\boldsymbol{b}$$

$$+\frac{\delta\varphi}{\delta s}curl\boldsymbol{t}+\frac{\delta\varphi}{\delta n}curl\boldsymbol{n}+\frac{\delta\varphi}{\delta b}curl\boldsymbol{b}=0. \tag{2.15}$$

Substituting (2.9), (2.11) and (2.13) in (2.15), we find

$$\begin{aligned} \frac{\delta^2\phi}{\delta n\delta b}-\frac{\delta^2\phi}{\delta n\delta b} &= -\frac{\delta\phi}{\delta s}\Omega_s+\frac{\delta\phi}{\delta n}(div\boldsymbol{b})-\frac{\delta\phi}{\delta b}(div\boldsymbol{n}) \\ \frac{\delta^2\phi}{\delta b\delta s}-\frac{\delta^2\phi}{\delta s\delta b} &= -\frac{\delta\phi}{\delta n}\Omega_n+\frac{\delta\phi}{\delta b}\theta_{bs} \\ \frac{\delta^2\phi}{\delta s\delta n}-\frac{\delta^2\phi}{\delta n\delta s} &= -\frac{\delta\phi}{\delta s}\kappa-\frac{\delta\phi}{\delta n}\theta_{ns}-\frac{\delta\phi}{\delta b}\Omega_b. \end{aligned} \tag{2.16}$$

By using the linear system (2.1), (2.2) and (2.3) we can write the following nine relations in terms of the eight parameters κ, τ, Ω_s, Ω_n, $div\boldsymbol{n}$,$div\boldsymbol{b}$, θ_{ns} and θ_{bs}. But we take (2.20), (2.21) and (2.22) relations for this work.

$$\frac{\delta}{\delta b}\theta_{ns}+\frac{\delta}{\delta n}\left(\Omega_n+\tau\right)=\left(div\boldsymbol{n}\right)\left(\Omega_s-2\Omega_n-2\tau\right)+\left(\theta_{bs}-\theta_{ns}\right)div\boldsymbol{b}+\kappa\Omega_s, \tag{2.17}$$

$$\frac{\delta}{\delta b}\left(\Omega_n-\Omega_s+\tau\right)+\frac{\delta}{\delta n}\theta_{bs}=div\boldsymbol{n}\left(\theta_{ns}-\theta_{bs}\right)+div\boldsymbol{b}\left(\Omega_s-2\Omega_n-2\tau\right), \tag{2.18}$$

$$\begin{aligned}\frac{\delta}{\delta b}\left(div\boldsymbol{b}\right)+\frac{\delta}{\delta n}\left(div\boldsymbol{n}\right) &= \left(\tau+\Omega_n\right)\left(\tau+\Omega_n-\Omega_s\right)-\theta_{ns}\theta_{bs}-\tau\Omega_s\\ &\quad -\left(div\boldsymbol{b}\right)^2-\left(div\boldsymbol{n}\right)^2,\end{aligned} \tag{2.19}$$

$$\frac{\delta}{\delta s}\left(\tau+\Omega_n\right)+\frac{\delta\kappa}{\delta b}=-\Omega_n\theta_{ns}-\left(2\tau+\Omega_n\right)\theta_{bs}, \tag{2.20}$$

$$\frac{\delta}{\delta s}\theta_{bs}=-\theta_{bs}^2+\kappa div\boldsymbol{n}-\Omega_n\left(\tau+\Omega_n-\Omega_s\right)+\tau\left(\tau+\Omega_n\right), \tag{2.21}$$

$$\frac{\delta}{\delta s}\left(div\boldsymbol{n}\right)-\frac{\delta\tau}{\delta b}=-\Omega_n\left(div\boldsymbol{b}\right)-\theta_{bs}\left(\kappa+div\boldsymbol{n}\right), \tag{2.22}$$

$$\frac{\delta\kappa}{\delta n}-\frac{\delta}{\delta s}\theta_{ns}=\kappa^2+\theta_{ns}^2+\left(\tau+\Omega_n\right)\left(3\tau+\Omega_n\right)-\Omega_s\left(2\tau+\Omega_n\right), \tag{2.23}$$

$$\frac{\delta}{\delta s}\left(\tau+\Omega_n-\Omega_s\right)=-\theta_{ns}\left(\Omega_n-\Omega_s\right)+\theta_{bs}\left(-2\tau-\Omega_n+\Omega_s\right)+\kappa div\boldsymbol{b}, \tag{2.24}$$

$$\frac{\delta\tau}{\delta n}+\frac{\delta}{\delta s}\left(div\boldsymbol{b}\right)=-\kappa\left(\Omega_n-\Omega_s\right)-\theta_{ns}div\boldsymbol{b}+\left(div\boldsymbol{n}\right)\left(-2\tau+\Omega_n+\Omega_s\right). \tag{2.25}$$

§3. General Properties

The relation

$$\frac{\delta\boldsymbol{n}}{\delta n}=\kappa_n\boldsymbol{n}_n=-\theta_{ns}\boldsymbol{t}-\left(div\boldsymbol{b}\right)\boldsymbol{b} \tag{3.1}$$

gives that the unit normal to the n-lines and their curvatures are given, respectively, by

$$\boldsymbol{n}_n=\frac{-\theta_{ns}\boldsymbol{t}-\left(div\boldsymbol{b}\right)\boldsymbol{b}}{\left\|-\theta_{ns}-\left(div\boldsymbol{b}\right)\boldsymbol{b}\right\|}=\frac{-\theta_{ns}\boldsymbol{t}-\left(div\boldsymbol{b}\right)\boldsymbol{b}}{-\theta_{ns}}, \tag{3.2}$$

$$\kappa_n=-\theta_{ns}. \tag{3.3}$$

In addition, from the relation (2.11) can be written,

$$curl\boldsymbol{n}=\Omega_n\boldsymbol{n}+\kappa_n\boldsymbol{b}_n, \tag{3.4}$$

where

$$\boldsymbol{b}_n=\boldsymbol{n}\times\boldsymbol{n}_n=\frac{-\left(div\boldsymbol{b}\right)\left(1,0,0\right)+\theta_{ns}\boldsymbol{b}}{-\theta_{ns}} \tag{3.5}$$

gives the unit binormal to the n-lines. Similarly, the relation

$$\frac{\delta \boldsymbol{b}}{\delta b} = \kappa_b \boldsymbol{n}_b = -\theta_{bs} \boldsymbol{t} - (div\boldsymbol{n})\,\boldsymbol{n} \tag{3.6}$$

gives that the unit normal to the b-lines and their curvature are given, respectively, by

$$\boldsymbol{n}_b = \frac{\theta_{bs} \boldsymbol{t} + (div\boldsymbol{n})\,\boldsymbol{n}}{\theta_{bs}}, \tag{3.7}$$

$$\kappa_b = -\theta_{bs}. \tag{3.8}$$

Moreover, from the relation (2.13) we can be written as

$$curl\boldsymbol{b} = \Omega_b \boldsymbol{b} + \kappa_b \boldsymbol{b}_b, \tag{3.9}$$

where

$$\boldsymbol{b}_b = \boldsymbol{b} \times \boldsymbol{n}_b = \frac{\theta_{bs} \boldsymbol{n} - (div\boldsymbol{n})\,(1,0,0)}{\theta_{bs}} \tag{3.10}$$

is the unit binormal to the b-line. To determine the torsions of the n-lines and b-lines, we take the relations

$$\frac{\delta \boldsymbol{b}_n}{\delta n} = -\tau_n \boldsymbol{n}_n, \tag{3.11}$$

$$\frac{\delta \boldsymbol{b}_b}{\delta b} = -\tau_b \boldsymbol{n}_b, \tag{3.12}$$

respectively. Thus, from (3.11) we have

$$-\frac{\delta}{\delta n}\left(\ln|\kappa_n|\right)(div\boldsymbol{b}) - \frac{\delta}{\delta n}(div\boldsymbol{b}) - \theta_{ns}\left(\Omega_b + \tau\right) = \tau_n \theta_{ns}, \tag{3.13}$$

$$-\frac{\delta}{\delta n} \ln|\kappa_n|\,\theta_{ns} + \frac{\delta}{\delta n}\theta_{ns} = \tau_n\,(div\boldsymbol{b})\,. \tag{3.14}$$

Accordingly,

$$\tau_n = \begin{cases} -\left(\Omega_b + \tau\right) + \frac{div\boldsymbol{b}}{\theta_{ns}} \frac{\delta}{\delta n} \ln\left|\frac{\theta_{ns}}{div\boldsymbol{b}}\right| & \text{if } div\boldsymbol{b} \neq 0,\ \theta_{ns} \neq 0 \\ -\left(\Omega_b + \tau\right) & \text{if } div\boldsymbol{b} = 0,\ \theta_{ns} \neq 0 \\ & \text{or } \theta_{ns} = 0,\ div\boldsymbol{b} \neq 0. \end{cases} \tag{3.15}$$

Similarly, from (3.12) we have

$$-\frac{\delta}{\delta b}\left(\ln \kappa_b\right)(div\boldsymbol{n}) + \frac{\delta}{\delta b}(div\boldsymbol{n}) - \theta_{bs}\left(\Omega_n + \tau\right) = \tau_b \theta_{bs}, \tag{3.16}$$

$$\frac{\delta}{\delta b}\left(\ln \kappa_b\right)\theta_{bs} - \frac{\delta}{\delta b}\theta_{bs} = \tau_b\,(div\boldsymbol{n})\,. \tag{3.17}$$

Thus,

$$\tau_b = \begin{cases} -(\Omega_n+\tau) - \frac{(div\boldsymbol{n})}{\theta_{bs}}\frac{\delta}{\delta b}\ln\left|\frac{\theta_{bs}}{div\boldsymbol{n}}\right| & \text{if } div\boldsymbol{n}\neq 0,\ \theta_{bs}\neq 0, \\ (\Omega_n+\tau) & \text{if } div\boldsymbol{n}=0, \theta_{bs}\neq 0 \\ & \text{or } \theta_{bs}=0,\ div\boldsymbol{n}\neq 0. \end{cases} \tag{3.18}$$

Also, we obtain an important relation

$$\Omega_s - \tau = \frac{1}{2}\left(\Omega_s + \Omega_n + \Omega_b\right) \tag{3.19}$$

is obtained by combining the equations (2.10), (2.12) and (2.14). Ω_s, Ω_n and Ω_b are defined the total moments of the $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$ congruences, respectively.

In conclusion, we see that the relation (3.19) has cognate relations

$$\Omega_n - \tau_n = \frac{1}{2}\left(\Omega_n + \Omega_{n_n} + \Omega_{b_n}\right), \tag{3.20}$$

$$\Omega_b - \tau_b = \frac{1}{2}\left(\Omega_b + \Omega_{n_b} + \Omega_{b_b}\right), \tag{3.21}$$

where

$$\begin{aligned} \Omega_{n_n} &= \boldsymbol{n}_n \cdot curl\boldsymbol{n}_n, \quad \Omega_{\boldsymbol{b}_n} = \boldsymbol{b}_n \cdot curl\boldsymbol{b}_n, \\ \Omega_{n_b} &= \boldsymbol{n}_b \cdot curl\boldsymbol{n}_b, \quad \Omega_{b_b} = \boldsymbol{b}_b \cdot curl\boldsymbol{b}_b. \end{aligned} \tag{3.22}$$

§4. The Nonlinear Schrödinger Equation

In geometric restriction

$$\Omega_n = 0 \tag{4.1}$$

imposed. Here, our purpose is to obtain the nonlinear Schrodinger equation with such a restriction in G_3. The condition indicate the necessary and sufficient restriction for the existence of a normal congruence of Σ surfaces containing the s-lines and b-lines. If the s-lines and b-lines are taken as parametric curves on the member surfaces U = constant of the normal congruence, then the surface metric is given by [4]

$$I_U = ds^2 + g\left(s,b\right)db^2. \tag{4.2}$$

where $g_{11} = g(s,s), g_{12} = g(s,b), g_{22} = g(b,b)$, and

$$grad_U = \boldsymbol{t}\frac{\delta}{\delta s} + \boldsymbol{b}\frac{\delta}{\delta b} = \boldsymbol{t}\frac{\partial}{\partial s} + \frac{\boldsymbol{b}}{g^{1/2}}\frac{\partial}{\partial b}. \tag{4.3}$$

Therefore, from equation (2.1) and (2.3), we have

$$\frac{\partial}{\partial s}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} \tag{4.4}$$

$$g^{-1/2}\frac{\partial}{\partial b}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 0 & -(\Omega_n+\tau) & \theta_{bs} \\ (\Omega_n+\tau) & 0 & div\boldsymbol{n} \\ -\theta_{bs} & -div\boldsymbol{n} & 0 \end{bmatrix}\begin{bmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{bmatrix}. \tag{4.5}$$

Also, if r shows the position vector to the surface then (4.4) and (4.5) implies that

$$r_{bs} = \frac{\partial \boldsymbol{t}}{\partial b} = g^{1/2}\left[-\tau\boldsymbol{n} + \theta_{bs}\boldsymbol{b}\right] \tag{4.6}$$

and

$$r_{sb} = \frac{\partial}{\partial s}\left(g^{1/2}\boldsymbol{b}\right) = -g^{1/2}\tau\boldsymbol{n} + \frac{\partial g^{1/2}}{\partial s}\boldsymbol{b}. \tag{4.7}$$

Thus, we obtain

$$\theta_{bs} = \frac{1}{2}\frac{\partial \ln g}{\partial s}. \tag{4.8}$$

In the case $\Omega_n = 0$, the compatibility conditions equations (2.20)-(2.22) become the non-linear system

$$\frac{\partial \tau}{\partial s} + \frac{\partial \kappa}{\partial b} = -2\tau\theta_{bs}, \tag{4.9}$$

$$\frac{\partial}{\partial s}\theta_{bs} = -\theta_{bs}^2 + \kappa div\boldsymbol{n} + \tau^2, \tag{4.10}$$

$$\frac{\partial}{\partial s}(div\boldsymbol{n}) - \frac{\partial \tau}{\partial b} = -\theta_{bs}(\kappa + div\boldsymbol{n}). \tag{4.11}$$

The Gauss-Mainardi-Codazzi equations become with (4.8)

$$\frac{\partial}{\partial s}(g^{1/2}div\boldsymbol{n}) + \kappa\frac{\partial}{\partial s}(g^{1/2}) - \frac{\partial \tau}{\partial b} = 0, \tag{4.12}$$

$$\frac{\partial}{\partial s}(g\tau) + g^{1/2}\frac{\partial \kappa}{\partial b} = 0, \tag{4.13}$$

$$(g^{1/2})_{ss} = g^{1/2}(\kappa div\boldsymbol{n} + \tau^2). \tag{4.14}$$

With elimination of $div\boldsymbol{n}$ of between (4.12) and (4.14), we have

$$\frac{\partial \tau}{\partial b} = \frac{\partial}{\partial s}\left[\frac{\left(g^{1/2}\right)_{ss} - \tau^2 g^{1/2}}{\kappa}\right] + \kappa\frac{\partial}{\partial s}\left(g^{1/2}\right). \tag{4.15}$$

If we accept

$$g^{1/2} = \lambda\kappa,$$

where λ varies only in the direction normal congruence, then $\lambda b \to b$, thus the pair equations (4.13) and (4.15) reduces to

$$\kappa_b = 2\kappa_s\tau + \kappa\tau_s, \tag{4.16}$$

$$\tau_b = \left(\tau^2 - \frac{\kappa_{ss}}{\kappa} + \frac{\kappa^2}{2}\right)_s. \tag{4.17}$$

By using equations (4.16) and (4.17), we obtain

$$iq_b + q_{ss} - \frac{1}{2}\left|\langle q, q\rangle\right|^2 \bar{q} - \Phi\left(b\right) q = 0, \tag{4.18}$$

where $\Phi\left(b\right) = \left(\tau^2 - \frac{\kappa_{ss}}{\kappa} + \frac{\kappa^2}{2}\right)_{s=s_0}$. This is nonlinear Schrodinger equation of repulsive type.

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Some Lower and Upper Bounds on the Third ABC Co-index

Deepak S. Revankar[1], Priyanka S. Hande[2], Satish P. Hande[3] and Vijay Teli[3]

1. Department of Mathematics, KLE, Dr. M. S. S. C. E. T., Belagavi - 590008, India
2. Department of Mathematics, KLS, Gogte Institute of Technology, Belagavi - 590008, India
3. Department of Mathematics,KLS, Vishwanathrao Deshpande Rural, Institute of Technology, Haliyal - 581 329, India

E-mail: revankards@gmail.com, priyanka18hande@gmail.com, handesp1313@gmail.com, vijayteli22@gmail.com

Abstract: Graonac defined the second ABC index as

$$ABC_2(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{1}{n_i} + \frac{1}{n_j} - \frac{2}{n_in_j}}.$$

Dae Won Lee defined the third ABC index as

$$ABC_3(G) = \sum_{v_iv_j \in E(G)} \sqrt{\frac{1}{e_i} + \frac{1}{e_j} - \frac{2}{e_ie_j}}$$

and studied lower and upper bounds. In this paper, we defined a new index which is called third ABC Coindex and it is defined as

$$\overline{ABC_3(G)} = \sum_{v_iv_j \notin E(G)} \sqrt{\frac{1}{n_i} + \frac{1}{n_j} - \frac{2}{n_in_j}}$$

and we found some lower and upper bounds on $\overline{ABC_3(G)}$ index.

Key Words: Molecular graph, the third atom - bond connectivity (ABC_3) index, the third atom - bond connectivity co-index ($\overline{ABC_3}$).

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§1. Introduction

The topological indices plays vital role in chemistry, pharmacology etc [1]. Let $G = (V, E)$ be a simple connected graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E(G)$, with $|V(G)| = n$ and $|E(G)| = m$. Let $u, v \in V(G)$ then the distance between u and v is denoted by $d(u, v)$ and is defined as the length of the shortest path in G connecting u and v.

The eccentricity of a vertex $v_i \in V(G)$ is the largest distance between v_i and any other vertex v_j of G. The diameter $d(G)$ of G is the maximum eccentricity of G and radius $r(G)$ of G is the minimum eccentricity of G.

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The Zagreb indices have been introduced by Gutman and Trinajstic [2]-[5]. They are defined as,

$$M_1(G) = \sum_{v_i \in V(G)} {d_i}^2, \qquad M_2(G) = \sum_{v_i v_j \in V(G)} d_i d_j.$$

The Zagreb co-indices have been introduced by Doslic [6],

$$\overline{M_1(G)} = \sum_{v_i v_j \notin E(G)} (d_i^2 + {d_j}^2), \qquad \overline{M_2(G)} = \sum_{v_i v_j \notin E(G)} (d_i d_j).$$

Similarly Zagreb eccentricity indices are defined as

$$E_1(G) = \sum_{v_i \in V(G)} {e_i}^2, \qquad E_2(G) = \sum_{v_i v_j \in V(G)} e_i e_j.$$

Estrada et al. defined atom bond connectivity index [7-10]

$$ABC(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{1}{d_i} + \frac{1}{d_j} - \frac{2}{d_i d_j}}$$

and Graovac defined second ABC index as

$$ABC_2(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{1}{n_i} + \frac{1}{n_j} - \frac{2}{n_i n_j}},$$

which was given by replacing d_i, d_j to n_i, n_j where n_i is the number of vertices of G whose distance to the vertex v_i is smaller than the distance to the vertex v_j [11-14].

Dae and Wan Lee defined the third ABC index [16]

$$ABC_3(G) = \sum_{v_i v_j \in E(G)} \sqrt{\frac{1}{e_i} + \frac{1}{e_j} - \frac{2}{e_i e_j}}.$$

In this paper, we have defined the third ABC co - index; $\overline{ABC_3(G)}$ as

$$\overline{ABC_3(G)} = \sum_{v_i v_j \notin E(G)} \sqrt{\frac{1}{e_i} + \frac{1}{e_j} - \frac{2}{e_i e_j}}$$

found some lower and upper bounds on $\overline{ABC_3(G)}$.

§2. Lower and Upper Bounds on $\overline{ABC_3(G)}$ Index

Calculation shows clearly that

(i) $\overline{ABC_3(K_n)} = 0$;

(ii) $\overline{ABC_3(K_{1,n-1})} = \frac{1}{2}\binom{n}{2}$;

(*iii*) $\overline{ABC_3(C_{2n})} = 2(n-3)\sqrt{n-2}$;

(*iv*) $\overline{ABC_3(C_{2n+1})} = n(n-3)\sqrt{\frac{4n-12}{(n-1)^2}}$.

Theorem 2.1 *Let G be a simple connected graph. Then $\overline{ABC_3(G)} \geq \frac{1}{\sqrt{\overline{E_2(G)}}}$, where $\overline{E_2(G)}$ is the second zagreb eccentricity coindex.*

Proof Since $G \ncong K_n$, it is easy to see that for every $e = v_iv_j$ in $E(G)$, $e_i + e_j \geq 3$. By the definition of ABC_3 coindex

$$\begin{aligned} \overline{ABC_3(G)} &= \sum_{v_iv_j \notin E(G)} \sqrt{\frac{1}{e_i} + \frac{1}{e_j} - \frac{2}{e_ie_j}} \\ &\geq \sum_{v_iv_j \notin E(G)} \frac{1}{\sqrt{e_ie_j}} \geq \frac{1}{\sqrt{\sum\limits_{v_iv_j \notin E(G)} e_ie_j}} = \frac{1}{\sqrt{\overline{E_2(G)}}}. \end{aligned}$$ □

Theorem 2.2 *Let G be a connected graph with m edges, radius $r = r(G) \geq 2$, diameter $d = d(G)$. Then,*

$$\frac{\sqrt{2m}}{d}\sqrt{d-1} \leq \overline{ABC_3(G)} \leq \frac{\sqrt{2m}}{r}\sqrt{r-1}$$

with equality holds if and only if G is self-centered graph.

Proof For $2 \leq r \leq e_i,\ e_j \leq d$,

$$\begin{aligned} \frac{1}{e_i} + \frac{1}{e_j} - \frac{2}{e_ie_j} &\geq \frac{1}{e_i} + \frac{1}{e_j}(1 - \frac{2}{e_j}) \ \ (\text{as } e_j \leq d,\ 1 - \frac{2}{e_i} \geq 0 = \frac{1}{d} + \frac{1}{e_j}(1 - \frac{2}{d})) \\ &\geq \frac{1}{d} + \frac{1}{d}(1 - \frac{2}{d}) \ (\text{as } e_i \leq d \text{ and } (1 - \frac{2}{d}) \geq 0) \\ &\geq \frac{1}{d} + \frac{1}{d} - \frac{2}{d^2} \\ &\geq \frac{2}{d} - \frac{2}{d^2} \geq \frac{2}{d^2}(d-1) \end{aligned}$$

with equality holds if and only if $e_i = e_j = d$.

Similarly we can easily show that,

$$\frac{1}{e_i} + \frac{1}{e_j} - \frac{2}{e_ie_j} \leq \frac{2}{r^2}(r-1)$$

for $2 \leq r \leq e_i$, $e_j \leq d$ with equality holding if and only if $e_i = e_j = r$. □

The following lemma can be verified easily.

Lemma 2.1 *Let $(a_1, a_2, \cdots, a_n)$ be a positive n-tuple such that there exist positive numbers*

A, a satisfying $0 < a \leq a_i \leq A$. Then,

$$\frac{n\sum\limits_{i=1}^{n} {a_i}^2}{(\sum\limits_{i=1}^{n} a_i)^2} \leq \frac{1}{4}(\sqrt{A/a} + \sqrt{a/A})^2$$

with equality holds if and only if $a = A$ or $q = \dfrac{A/a}{(A/a)+1}n$ is an integer and q of numbers a_i coincide with a and the remaining $n-q$ of the a_i's coincide with $A(\neq a)$.

Theorem 2.3 *Let G be a simple connected graph with m edges, $r = r(G) \geq 2, d = d(G)$. Then,*

$$\overline{ABC_3(G)} = \sqrt{\frac{4m\sqrt{(r-1)(d-1)}}{rd(\frac{1}{r}\sqrt{r-1} + \frac{1}{d}\sqrt{d-1})^2\overline{E2(G)}}}.$$

Proof By Theorem 2.2 we know that

$$\frac{\sqrt{2}}{d}\sqrt{d-1} \leq \sqrt{\frac{1}{e_i} + \frac{1}{e_j} - \frac{2}{e_ie_j}} \leq \frac{\sqrt{2}}{r}\sqrt{r-1}, \quad v_iv_j \notin E(G). \tag{2.1}$$

Also by Lemma 2.3 we have

$$a \leq a_i \leq A. \tag{2.2}$$

Let

$$a = \frac{\sqrt{2}}{d}\sqrt{d-1} \ \text{ and } \ a_i = \sqrt{\frac{1}{e_i} + \frac{1}{e_j} - \frac{2}{e_ie_j}}, \quad v_iv_j \notin E(G)$$

and

$$A = \frac{\sqrt{2}}{r}\sqrt{r-1}.$$

in equations (2.1) and (2.2). We therefore know that

$$\frac{n\sum\limits_{i=1}^{n} {a_i}^2}{\left(\sum a_i\right)^2} \leq \frac{1}{4}\left(\sqrt{\frac{A}{a}} + \sqrt{\frac{a}{A}}\right)^2,$$

i.e.,

$$\frac{\left(\sum a_i\right)^2}{n\sum a_i^2} \geq 4\frac{1}{\left(\sqrt{\frac{A}{a}} + \sqrt{\frac{a}{A}}\right)^2},$$

which implies that

$$\left(\sum a_i\right)^2 \geq \frac{4n\sum a_i^2}{\left(\sqrt{\frac{A}{a}} + \sqrt{\frac{a}{A}}\right)^2} \geq \frac{4n\sum a_i^2}{\left[\frac{\sqrt{A}}{\sqrt{a}} + \frac{\sqrt{a}}{\sqrt{A}}\right]^2} \geq \frac{4n\sum a_i^2}{\left[\frac{A+a}{\sqrt{an}}\right]^2} \geq \frac{4n\sum a_i^2 aA}{[A+a]^2}$$

and

$$\left(\sum_{v_iv_j\in E(G)}\sqrt{\frac{1}{e_i}+\frac{1}{e_j}-\frac{2}{e_ie_j}}\right)^2 \geq \frac{4n\frac{\sqrt{2}}{d}\sqrt{d-1}\frac{\sqrt{2}}{r}\sqrt{r-1}}{\left[\frac{\sqrt{2}}{r}\sqrt{r-1}+\frac{\sqrt{2}}{d}\sqrt{r-1}\right]^2}\sum_{v_iv_j\notin E(G)}\left(\frac{1}{e_i}+\frac{1}{e_j}-\frac{2}{e_ie_j}\right)$$

$$\geq \frac{\frac{8n}{rd}\sqrt{d-1}\sqrt{r-1}}{\frac{2}{r}\left[\frac{\sqrt{r-1}}{r}+\frac{\sqrt{d-1}}{d}\right]^2}\sum_{v_iv_j\notin E(G)}\left(\frac{1}{e_i}+\frac{1}{e_j}-\frac{2}{e_ie_j}\right).$$

Therefore

$$\frac{8n\sqrt{(r-1)(d-1)}}{rd\left[\frac{\sqrt{r-1}}{r}+\frac{\sqrt{d-1}}{d}\right]^2}\sum\left(\frac{1}{e_i}+\frac{1}{e_j}-\frac{2}{e_ie_j}\right)=\frac{\frac{8n}{rd}\sqrt{(r-1)(d-1)}}{\frac{2}{r}\left[\frac{\sqrt{r-1}}{r}+\frac{\sqrt{d-1}}{d}\right]^2}\sum_{v_iv_j}\left(\frac{1}{e_i}+\frac{1}{e_j}-\frac{2}{e_ie_j}\right).$$

We know that

$$\sum_{v_iv_j\notin E(G)}\left(\frac{1}{e_i}+\frac{1}{e_j}-\frac{2}{e_ie_j}\right)\geq\frac{1}{\overline{E_2(G)}}$$

from Theorem 2.1. Thus,

$$\sum_{v_iv_j\notin E(G)}\left(\sqrt{\frac{1}{e_i}+\frac{1}{e_j}-\frac{2}{e_ie_j}}^{\,2}\right)\geq\frac{4m\times\sqrt{(r-1)(d-1)}}{rd\left(\frac{1}{r}\sqrt{r-1}+\frac{1}{d}\sqrt{d-1}\right)}\overline{E_2(G)},$$

$$\sum_{v_iv_j\notin E(G)}\sqrt{\left(\frac{1}{e_i}+\frac{1}{e_j}-\frac{2}{e_ie_j}\right)}\geq\sqrt{\frac{4m\sqrt{(r-1)(d-1)}}{rd\left(\frac{1}{r}\sqrt{r-1}+\frac{1}{d\sqrt{d-1}}\right)\overline{E_2(G)}}}.$$

□

Theorem 2.4 *Let G be a simple connected graph with n vertices and m edges. Then,*

$$\frac{1}{\sqrt{n^2m-n\overline{M_1(G)}+\overline{M_2(G)}}}\leq\overline{ABC_3(G)}\leq\frac{1}{\sqrt{2}}\sqrt{2nm^2-n\overline{M_1(G)}-2m^2}.$$

Proof From Theorem 2.1,

$$\sum_{v_iv_j\notin E(G)}\sqrt{\frac{1}{e_i}+\frac{1}{e_j}-\frac{2}{e_ie_j}}\geq\frac{1}{\sqrt{\sum\limits_{v_iv_j\notin E(G)}e_ie_j}}.$$

Since $e_i\leq(n-d_i)$, we know that

$$\frac{1}{\sqrt{\sum\limits_{v_iv_j\notin E(G)}e_ie_j}}\geq\frac{1}{\sqrt{\sum(n-d_i)(n-d_j)}}=\frac{1}{\sqrt{\sum(n^2-nd_i-nd_j+d_id_j)}}$$

$$=\frac{1}{\sqrt{mn^2-n\overline{M_1(G)}+\overline{M_2(G)}}}.$$

This completes the lower bound.

Now, since $G \ncong K_n$, $e_ie_j \geq 2$ for $v_iv_j \notin E(G)$, we get that

$$\sum_{v_iv_j \notin E(G)} \sqrt{\frac{1}{e_i} + \frac{1}{e_j} - \frac{2}{e_ie_j}} \leq \frac{1}{\sqrt{2}} \sum_{v_iv_j \notin E(G)} \sqrt{e_i + e_j - 2}.$$

By Cachy-Schwarz inequality, we also know that

$$\frac{1}{\sqrt{2}} \sum_{v_iv_j \notin E(G)} \sqrt{e_i + e_j - 2} \leq \frac{1}{\sqrt{2}} \sqrt{\sum_{v_iv_j \notin E(G)} 1 \sum_{v_iv_j \notin E(G)} (e_i + e_j - 2)}.$$

Since $e_i \leq n - d_i$ for $v_i \in V(G)$, we get that

$$\begin{aligned}
&\frac{1}{\sqrt{2}} \sqrt{\sum_{v_iv_j \notin E(G)} 1 \sum_{v_iv_j \notin E(G)} (e_i + e_j - 2)} \\
&\leq \frac{1}{\sqrt{2}} \sqrt{m \sum_{v_iv_j \notin E(G)} (n - d_i + n - d_j - 2)} \\
&\leq \frac{1}{\sqrt{2}} \sqrt{m \left[\sum_{v_iv_j \notin E(G)} 2n - \sum_{v_iv_j \notin E(G)} (d_i + d_j) - 2 \sum_{v_iv_j \notin E(G)} 1 \right]} \\
&= \frac{1}{\sqrt{2}} \sqrt{m \left[2nm - \overline{M_1(G)} - 2m \right]} \\
&= \frac{1}{\sqrt{2}} \sqrt{2m^2n - m\overline{M_1(G)} - 2m^2}. \qquad \square
\end{aligned}$$

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The k-Distance Degree Index of Corona, Neighborhood Corona Products and Join of Graphs

Ahmed M. Naji and Soner Nandappa D

(Department of Studies in Mathematics, University of Mysore, Manasagangotri, Mysuru - 570 006, India)

E-mail: ama.mohsen78@gmail.com, ndsoner@yahoo.co.in

Abstract: The k-distance degree index (N_k-index) of a graph G have been introduced in [11], and is defined as $N_k(G) = \sum_{k=1}^{diam(G)} \left(\sum_{v\in V(G)} d_k(v)\right) \cdot k$, where $d_k(v) = |N_k(v)| = |\{u \in V(G) : d(v,u) = k\}|$ is the k-distance degree of a vertex v in G, $d(u,v)$ is the distance between vertices u and v in G and $diam(G)$ is the diameter of G. In this paper, we extend the study of N_k-index of a graph for other graph operations. Exact formulas of the N_k-index for corona $G \circ H$ and neighborhood corona $G \star H$ products of connected graphs G and H are presented. An explicit formula for the splitting graph $S(G)$ of a graph G is computed. Also, the N_k-index formula of the join $G + H$ of two graphs G and H is presented. Finally, we generalize the N_k-index formula of the join for more than two graphs.

Key Words: Vertex degrees, distance in graphs, k-distance degree, Smarandachely k-distance degree, k-distance degree index, corona, neighborhood corona.

AMS(2010): 05C07, 05C12, 05C76, 05C31.

§1. Introduction

In this paper, we consider only simple graph $G = (V, E)$, i.e., finite, having no loops no multiple and directed edges. A graph G is said to be connected if there is a path between every pair of its vertices. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph G, respectively. The distance $d(u, v)$ between any two vertices u and v of G is the length of a minimum path connecting them. For a vertex $v \in V$ and a positive integer k, the open k-distance neighborhood of v in a graph G is $N_k(v/G) = \{u \in V(G) : d(u,v) = k\}$ and the closed k-neighborhood of v is $N_k[v/G] = N_k(v) \cup \{v\}$. The k-distance degree of a vertex v in G, denoted by $d_k(v/G)$ (or simply $d_k(v)$ if no misunderstanding) is defined as $d_k(v/G) = |N_k(v/G)|$, and generally, a Smarandachely k-distance degree $d_k(v/G : S)$ of v on vertex set $S \subset V(G)$ is $d_k(v/G) = |N_k(v/G : S)|$, where $N_k(v/G : S) = \{u \in V(G)\backslash S : d(u,v) = k\}$. Clearly, $d_k(v/G : \emptyset) = d_k(v/G)$ and $d_1(v/G) = d(v/G)$ for every $v \in V(G)$. A vertex of degree equals to zero in G is called an isolated vertex and a vertex of degree one is called a pendant vertex. The graph with just one vertex is referred to as trivial graph and denoted K_1. The complement

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$\overline{G}$ of a graph G is a graph with vertex set $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. A totally disconnected graph $\overline{K_n}$ is one in which no two vertices are adjacent (that is, one whose edge set is empty). If a graph G consists of $s \geq 2$ disjoint copies of a graph H, then we write $G = sH$. For a vertex v of G, the eccentricity $e(v) = \max\{d(v,u) : u \in V(G)\}$. The radius of G is $rad(G) = \min\{e(v) : v \in V(G)\}$ and the diameter of G is $diam(G) = \max\{e(v) : v \in V(G)\}$. For any terminology or notation not mention here, we refer the reader to the books [3, 5].

A topological index of a graph G is a numerical parameter mathematically derived from the graph structure. It is a graph invariant thus it does not depend on the labeling or pictorial representation of the graph and it is the graph invariant number calculated from a graph representing a molecule. The topological indices of molecular graphs are widely used for establishing correlations between the structure of a molecular compound and its physic-chemical properties or biological activity. The topological indices which are definable by a distance function $d(.,.)$ are called a distance-based topological index. All distance-based topological indices can be derived from the distance matrix or some closely related distance-based matrix, for more information on this matter see [2] and a survey paper [20] and the references therein.

There are many examples of such indices, especially those based on distances, which are applicable in chemistry and computer science. The Wiener index (1947), defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v)$$

is the first and most studied of the distance based topological indices [19]. The hyper-Wiener index,

$$WW(G) = \frac{1}{2} \sum_{\{u,v\} \subseteq V} (d(u,v) + d^2(u,v))$$

was introduced in (1993) by M. Randic [14]. The Harrary index

$$H(G) = \sum_{\{u,v\} \subseteq V} \frac{1}{d^2(u,v)}$$

was introduced in (1992) by Mihalic et al. [10]. In spite of this, the Harary index is nowadays defined [8, 12] as

$$H(G) = \sum_{\{u,v\} \subseteq V} \frac{1}{d(u,v)}.$$

The Schultz index

$$S(G) = \sum_{\{u,v\} \subseteq V} (d(u) + d(v))d(u,v)$$

was introduced in (1989) by H. P. Schultz [16]. A. Dobrynin et al. in (1994) also proposed the Schultz index and called it the degree distance index and denoted $DD(G)$ [1]. S. Klavzar and

I Gutman, motivated by Schultz index, introduced in (1997) the second kind of Schultz index

$$S^*(G) = \sum_{\{u,v\}\subseteq V} d(u)d(v)d(u,v)$$

called modified Schultz (or Gutman) index of G [9]. The eccentric connectivity index

$$\xi^c = \sum_{v\in V} d(v)e(v)$$

was proposed by Sharma et al. [17]. For more details and examples of distance-based topological indices, we refer the reader to [2, 20, 13, 6] and the references therein.

Recently, The authors in [11], have been introduced a new type of graph topological index, based on distance and degree, called k-distance degree of a graph, for positive integer number $k \geq 1$. Which, for simplicity of notion, referred as N_k-index, denoted by $N_k(G)$ and defined by

$$N_k(G) = \sum_{k=1}^{diam(G)} \left(\sum_{v\in V(G)} d_k(v) \right) \cdot k$$

where $d_k(v) = d_k(v/G)$ and $diam(G)$ is the diameter of G. They have obtained some basic properties and bounds for N_k-index of graphs and they have presented the exact formulas for the N_k-index of some well-known graphs. They also established the N_k-index formula for a cartesian product of two graphs and generalize this formula for more than two graphs. The k-distance degree index, $N_k(G)$, of a graph G is the first derivative of the k-distance neighborhood polynomial, $N_k(G,x)$, of a graph evaluated at $x = 1$,see ([18]).

The following are some fundamental results which will be required for many of our arguments in this paper and which are finding in [11].

Lemma 1.1 *For $n \geq 1$, $N_k(\overline{K_n}) = N_k(K_1) = 0$.*

Theorem 1.2 *For any connected graph G of order n with size m and $diam(G) = 2$, $N_k(G) = 2n(n-1) - 2m$.*

Theorem 1.3 *For any connected nontrivial graph G, $N_k(G)$ is an even integer number.*

In this paper, we extend our study of N_k-index of a graph for other graph operations. Namely, exact formulas of the N_k-index for corona $G \circ H$ and neighborhood corona $G \star H$ products of connected graphs G and H are presented. An explicit formula for the splitting graph $S(G)$ of a graph G is computed. Also, the N_k-index formula of the join $G + H$ of two graphs G and H is presented. Finally, we generalize the N_k-index formula of the join for more than two graphs.

§2. The N_k-Index of Corona Product of Graphs

The corona of two graphs was first introduced by Frucht and Harary in [4].

Definition 2.1 *Let G and H be two graphs on disjoint sets of n_1 and n_2 vertices, respectively. The corona $G \circ H$ of G and H is defined as the graph obtained by taking one copy of G and n_1 copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.*

It is clear from the definition of $G \circ H$ that

$$\begin{aligned} n &= |V(G \circ H)| = n_1 + n_1 n_2, \\ m &= |E(G \circ H)| = m_1 + n_1(n_2 + m_2) \end{aligned}$$

and

$$diam(G \circ H) = diam(G) + 2,$$

where m_1 and m_2 are the sizes of G and H, respectively. In the following results, H^j, for $1 \leq j \leq n_1$, denotes the copy of a graph H which joining to a vertex v_j of a graph G,i.e., $H^j = \{v_j\} \circ H$, $D = diam(G)$ and $d_k(v/G)$ denotes the degree of a vertex v in a graph G. Note that in general this operation is not commutative.

Theorem 2.2 *Let G and H be connected graphs of orders n_1 and n_2 and sizes m_1 and m_2, respectively. Then*

$$N_k(G \circ H) = \left(1 + 2n_2 + n_2^2\right) N_k(G) + 2n_1 n_2 \left(n_1 + n_1 n_2 - 1\right) - 2n_1 m_2.$$

Proof Let G and H be connected graphs of orders n_1 and n_2 and sizes m_1 and m_2, respectively and let $D = diam(G)$, $n = |V(G \circ H)|$ and $m = |E(G \circ H)|$. Then by the definition of $G \circ H$ and for every $1 \leq k \leq diam(G \circ H)$, we have the following cases.

Case 1. For every $v \in V(G)$,

$$d_k(v/G \circ H) = d_k(v/G) + n_2 d_{k-1}(v/G).$$

Case 2. For every $u \in H^j$, $1 \leq j \leq n_1$,

- $d_1(u/G \circ H^j) = 1 + d_1(u/H)$;
- $d_2(u/G \circ H^j) = d_1(v_j/G) + (n_2 - 1) - d_1(u/H)$;
- $d_k(u/G \circ H^j) = d_{k-1}(v_j/G) + n_2 d_{k-2}(v_j/G)$, for every $3 \leq k \leq D + 2$.

Since for every $v \in V(G \circ H)$ either $v \in V(G)$ or $v \in V(H^j)$, for some $1 \leq j \leq n_1$, it follows that for $1 \leq k \leq diam(G \circ H)$,

$$\sum_{v \in V(G \circ H)} d_k(v/G \circ H) = \sum_{v \in V(G)} d_k(v/G \circ H) + \sum_{j=1}^{n_1} \sum_{u \in V(H^j)} d_k(u/G \circ H^j).$$

Hence, by using the hypothesis above

$$
\begin{aligned}
N_k(G\circ H) &= \sum_{k=1}^{diam(G\circ H)} \Bigg[\sum_{v\in V(G\circ H)} d_k(v/G\circ H)\Bigg]k\\
&= \sum_{k=1}^{D+2} \Bigg[\sum_{v\in V(G)} d_k(v/G\circ H) + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)} d_k(u/G\circ H^j)\Bigg]k\\
&= \sum_{k=1}^{D+2} \Bigg[\sum_{v\in V(G)} \Big(d_k(v/G) + n_2 d_{k-1}(v/G)\Big)\Bigg].k + \sum_{k=1}^{D+2}\Bigg[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)} d_k(u/G\circ H^j)\Bigg]k\\
&= \sum_{k=1}^{D+2}\Big(\sum_{v\in V(G)} d_k(v/G)\Big).k + n_2\sum_{k=1}^{D+2}\Big(\sum_{v\in V(G)} d_{k-1}(v/G)\Big)k\\
&\quad + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(1+d_1(u/H^j)\Big) + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_1(v_j/G) + (n_2-1) - d(u/H^j)\Big)2\\
&\quad + \sum_{k=3}^{D+2}\Bigg[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_{k-1}(v_j/G) + n_2 d_{k-2}(v_j/G)\Big)\Bigg]k
\end{aligned}
$$

Set $x = x_1 + x_2$, where

$$
\begin{aligned}
x_1 &= \sum_{k=1}^{D+2}\Big(\sum_{v\in V(G)} d_k(v/G)\Big)k\\
&= \sum_{k=1}^{D}\Big(\sum_{v\in V(G)} d_k(v/G)\Big)k + \Big(\sum_{v\in V(G)} d_{D+1}(v/G)\Big)(D+1) + \Big(\sum_{v\in V(G)} d_{D+2}(v/G)\Big)(D+2)\\
&= \sum_{k=1}^{D}\Big(\sum_{v\in V(G)} d_k(v/G)\Big)k + 0 + 0 = N_k(G).
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= n_2\sum_{k=1}^{D+2}\Big(\sum_{v\in V(G)} d_{k-1}(v/G)\Big)k\\
&= n_2\Bigg[(\sum_{v\in V(G)} d_0(v/G))1 + (\sum_{v\in V(G)} d_1(v/G)).2 + \cdots + (\sum_{v\in V(G)} d_D(v/G))(D+1)\\
&\quad + (\sum_{v\in V(G)} d_{D+1}(v/G))(D+2)\Bigg] = n_2\Bigg[n_1 + \sum_{k=1}^{D}(\sum_{v\in V(G)} d_k(v/G))(k+1) + 0\Bigg]\\
&= n_2\Bigg[n_1 + \sum_{k=1}^{D}(\sum_{v\in V(G)} d_k(v/G))k + \sum_{k=1}^{D}(\sum_{v\in V(G)} d_k(v/G))1\Bigg]\\
&= n_2\Bigg[n_1 + N_k(G) + n_1(n_1-1)\Bigg].
\end{aligned}
$$

Thus, $x=(1+n_2)N_k(G)+n_1^2n_2$. Also, set $y=y_1+y_2+y_3$, where

$$\begin{aligned}
y_1 &= \sum_{j=1}^{n_1}\sum_{u\in V(H^j)}(1+d_1(u/H))1=n_1n_2+2n_1m_2,\\
y_2 &= \sum_{j=1}^{n_1}\sum_{u\in V(H^j)}(d_1(v_j/G)+(n_2-1)-d_1(u/H))2=2(2m_1n_2+n_1n_2(n_2-1)-2n_1m_2)
\end{aligned}$$

and

$$\begin{aligned}
y_3 &= \sum_{k=3}^{D+2}\left[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_{k-1}(v_j/G)+n_2d_{k-2}(v_j/G)\Big)\right]k\\
&= \sum_{k=3}^{D+2}\left[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_{k-1}(v_j/G)\Big)\right]k+n_2\sum_{k=3}^{D+2}\left[\sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big(d_{k-2}(v_j/G)\Big)\right]k\\
&= n_2\left[\sum_{k=3}^{D+2}\Big(\sum_{j=1}^{n_1}(d_{k-1}(v_j/G)\Big)k\right]+n_2^2\left[\sum_{k=3}^{D+2}\Big(\sum_{j=1}^{n_1}(d_{k-2}(v_j/G)\Big)k\right].
\end{aligned}$$

Now set $y_3=y_3'+y_3''$, where

$$\begin{aligned}
y_3' &= n_2\left[\sum_{k=3}^{D+2}(\sum_{j=1}^{n_1}(d_{k-1}(v_j/G))\right].k\\
&= n_2\left[(\sum_{v\in V(G)}d_2(v/G))3+(\sum_{v\in V(G)}d_2(v/G))4+\cdots+(\sum_{v\in V(G)}d_D(v/G))(D+1)+0\right]\\
&= n_2\left[\sum_{k=1}^{D}(\sum_{v\in V(G)}d_k(v/G))(k+1)-(\sum_{v\in V(G)}d_1(v/G))2\right]
\end{aligned}$$

$$\begin{aligned}
&= n_2\left[\sum_{k=1}^{D}(\sum_{v\in V(G)}d_k(v/G))k+\sum_{k=1}^{D}(\sum_{v\in V(G)}d_k(v/G))1-(\sum_{v\in V(G)}d_1(v/G))2\right]\\
&= n_2N_k(G)+n_1n_2(n_1-1)-4m_1n_2,
\end{aligned}$$

and similarly

$$\begin{aligned}
y_3'' &= n_2^2\left[\sum_{k=3}^{D+2}\Big(\sum_{j=1}^{n_1}(d_{k-2}(v_j/G)\Big)k\right]=n_2^2\left[\sum_{k=1}^{D}(\sum_{v\in V(G)}d_k(v/G))(k+2)\right]\\
&= n_2^2N_k(G)+2n_1n_2^2(n_1-1).
\end{aligned}$$

Thus, $y_3=(n_2^2+n_2)N_k(G)+n_1n_2(n_1-1)-4m_1n_2+2n_1n_2^2(n_1-1)$.

Accordingly,

$$y=(n_2^2+n_2)N_k(G)+2n_1^2n_2^2+n_1^2n_2-2n_1n_2-2n_1m_2$$

and

$$N_k(G \circ H) = x + y.$$

Therefore,

$$N_k(G \circ H) = (1 + 2n_2 + n_2^2)N_k(G) + 2n_1n_2(n_1n_2 + n_1 - 1) - 2n_1m_2. \qquad \square$$

Corollary 2.3 *Let G be a connected graph of order $n \geq 2$ and size $m \geq 1$. Then*

(1) $N_k(K_1 \circ G) = 2(n^2 - m)$;
(2) $N_k(G \circ K_1) = 4N_k(G) + 2n(2n - 1)$;
(3) $N_k(G \circ \overline{K_p}) = (1 + 2p + p^2)N_k(G) + 2pn(pn + n - 1)$, *where $\overline{K_p}$ is a totally disconnected graph with $p \geq 2$ vertices.*

§3. The N_k-Index of Neighborhood Corona Product of Graphs

The neighborhood corona was introduced in [7].

Definition 3.1 *Let G and H be connected graphs of orders n_1 and n_2, respectively. Then the neighborhood corona of G and H, denoted by $G \star H$, is the graph obtained by taking one copy of G and n_1 copies of H, and joining every neighbor of the i^{th} vertex of G to every vertex in the i^{th} copy of H.*

It is clear from the definition of $G \circ H$ that

- In general $G \star H$ is not commutative.
- When $H = K_1$, $G \star H = S(G)$ is the splitting graph defined in [**?**].
- When $G = K_1$, $G \star H = G \cup H$.
- $n = |V(G \star H)| = n_1 + n_1n_2$
- $diam(G \star H) = \begin{cases} 3, & \text{if } diam(G) \leq 3; \\ diam(G), & \text{if } diam(G) \geq 3; \end{cases}$

In the following results, H^j, for $1 \leq j \leq n_1$, denotes the j^{th} copy of a graph H which corresponding to a vertex v_j of a graph G, i.e., $H^j = \{v_j\} \star H$, $D = diam(G)$ and $d_k(v/G)$ denotes the degree of a vertex v in a graph G.

Theorem 3.2 *Let G and H be connected graphs of orders and sizes n_1, n_2, m_1 and m_2 respectively such that $diam(G) \geq 3$. Then*

$$N_k(G \star H) = (1 + 2n_2 + n_2^2)N_k(G) + 2n_2^2(n_1 + m_1) + 2n_1(n_2 - m_2).$$

Proof Let G and H be connected graphs of orders and sizes n_1, m_1, n_2 and m_2 respectively and let $\{v_1, v_2, \cdots, v_{n_1}\}$ and $\{u_1, u_2, \cdots, u_{n_2}\}$ be the vertex sets of G and H respectively. Then for every $w|inv(G \star H)$ either $w = v \in V(G)$ or $w = u \in V(H)$. Since, for every $v \in V(G)$,

$$\begin{aligned}
|N_1(v/G \star H)| &= |N_1(v/G)| + |V(H)||N_1(v/G)| \\
d_1(v/G \star H) &= d_1(v/G) + n_2 d_1(v/G) \\
&= (1 + n_2) d_1(v/G)
\end{aligned}$$

and for every $u \in V(H^j)$, $1 \le j \le n_1$

$$\begin{aligned}
|N_1(u/G \star H^j)| &= |N_1(u/H)| + |N_1(v_j/G)|, \\
d_1(u/G \star H^j) &= d_1(u/H) + d_1(v_j/G).
\end{aligned}$$

Thus, for ever $w \in V(G \star H)$

$$\begin{aligned}
\sum_{w \in V(G \star H)} d_1(w/G \star H) &= \sum_{v \in V(G)} d_1(v/G \star H) + \sum_{j=1}^{n_1} \sum_{u \in V(H^j)} d_1(u/G \star H^j) \\
&= \sum_{v \in V(G)} (1 + n_2) d_1(v/G) + \sum_{j=1}^{n_1} \sum_{u \in V(H^j)} (d_1(u/H^j) + d_1(v_j/G)) \\
&= (1 + n_2) \sum_{v \in V(G)} d_1(v/G) + \sum_{j=1}^{n_1} 2m_2 + n_2 \sum_{i=1}^{n_1} d_1(v_j/G) \\
&= (1 + 2n_2) \sum_{v \in V(G)} d_1(v/G) + 2n - 1m_2.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
|N_2(v_j/G \star H)| &= |N_2(v_j/G)| + |V(H^j)| + |V(H^j)||N_2(v_j/G)|, \\
d_2(v_j/G \star H) &= d_2(v_j/G) + n_2 + n_2 d_2(v/G) \\
&= (1 + n_2) d_2(v/G) + n_2
\end{aligned}$$

for every $v_j \in V(G)$, $1 \le j \le n_1$, and

$$\begin{aligned}
|N_2(u/G \star H^j)| &= (|V(H^j)| - 1) - |N_1(u/H^j)| + |\{v_j\}| \\
&\quad + |V(H^j)||N_2(v_j/G)| + |N_2(v_j/G)| \\
d_2(u/G \star H^j) &= (n_2 - 1) - d_1(u/H) + 1 + n_2 d_2(v_j/G) + d_2(v/G) \\
&= n_2 + d_1(u/H) + (1 + n_2) d_2(v_j/G)
\end{aligned}$$

for every $u \in H^j$, $1 \leq j \leq n_1$. Thus, for ever $w \in V(G \star H)$,

$$\begin{aligned}\sum_{w\in V(G\star H)} d_2(w/G\star H) &= \sum_{v\in V(G)} d_2(v/G\star H) + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)} d_2(u/G\star H^j)\\ &= \sum_{v\in V(G)} \Big[(1+n_2)d_1(v/G)+n_2\Big]\\ &\quad + \sum_{j=1}^{n_1}\sum_{u\in V(H^j)}\Big[n_2+d_1(u/H)+(1+n_2)d_1(v_j/G)\Big]\\ &= (1+n_2+n_2^2)\sum_{v\in V(G)} d_2(v/G) + n_1n_2^2 + n_1n_2 - 2n_1m_2.\end{aligned}$$

Also, for every $v \in V(G)$, $d_3(v/G\star H) = (1+n_2)d_3(v/G)$ and for every $u \in V(H^j)$,

$$d_3(u/G\star H^j) = n_2d_1(v_j/G) + (1+n_2)d_3(v_j/G).$$

Hence, For every $w \in V(G\star H)$,

$$\begin{aligned}d_3(w/G\star H) &= (1+n_2+n_2^2)\sum_{v\in V(G)} d_3(v/G)\\ &\quad + n_2^2\sum_{v\in V(G)} d_1(v/G).\end{aligned}$$

By continue in same process we get, for every $4 \leq k \leq diam(G\star H)$, that is, for every $v \in V(G)$,

$$d_k(v/G\star H) = (1+n_2)d_k(v/G)$$

and for every $u \in V(H^j)$,

$$d_k(u/G\star H^j) = (1+n+2)d_k(v_j/G),$$

and hence for every $w \in V(G\star H)$,

$$d_k(w/G\star H) = (1+2n_2+n_2^2)d_k(v/G).$$

Accordingly,

$$\begin{aligned}N_k(G\star H) &= \sum_{k=1}^{D}\Big(\sum_{w\in V(G\star H)} d_k(w/G\star H)\Big)\ k\\ &= \sum_{w\in V(G\star H)} d_1(w/G\star H))1 + \sum_{w\in V(G\star H)} d_2(w/G\star H))2+\cdots\\ &\quad + \sum_{w\in V(G\star H)} d_D(w/G\star H))\ D\end{aligned}$$

$$
\begin{aligned}
&= \Big[(1+2n_2)\sum_{v\in V(G)} d_1(v/G) + 2n_1m_2\Big]1 \\
&\quad + \Big[(1+2n_2+n_2^2)\sum_{v\in V(G)} d_2(v/G) + n_1n_2^2 + n_1n_2 \\
&\quad - 2n_1m_2\Big]\ 2 + \Big[(1+2n_2+n_2^2)\sum_{v\in V(G)} d_3(v/G) + n_2^2\sum_{v\in V(G)} d_1(v/G)\Big]\ 3 \\
&\quad + \Big[(1+2n_2+n_2^2)\sum_{v\in V(G)} d_4(v/G)\Big]\ 4 + \cdots + \Big[(1+2n_2+n_2^2)\sum_{v\in V(G)} d_D(v/G)\Big]\ D \\
&= (1+2n_2+n_2^2)\Big[\sum_{v\in V(G)} d_1(v/G)\ 1 + \sum_{v\in V(G)} d_2(v/G)\ 2 + \cdots + \sum_{v\in V(G)} d_D(v/G)\ D\Big] \\
&\quad + \Big[(-n_2^2\sum_{v\in V(G)} d_1(v/G) + 2n_1m_2)\ 1 + (n_1n_2^2 + n_1n_2 - 2n_1m_2)\ 2 \\
&\quad + (n_2^2\sum_{v\in V(G)} d_1(v/G))\ 3 \\
&= (1+2n_2+n_2^2)N_k(G) + 2n_2^2(n_1+m_1) + 2n_1(n_2-m_2). \qquad \square
\end{aligned}
$$

Corollary 3.3 *Let G be a connected graph of order $n \geq 2$ and size m and let $S(G)$ be the splitting graph of G. Then*

$$N_k(S(G)) = 4N_k(G) + 2(2n+m).$$

§4. The N_k-Index of Join of Graphs

Definition 4.1([5]) *Let G_1 and G_2 be two graphs with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$. Then the join $G_1 + G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv|u \in V(G_1)\&v \in V(G_2)\}$.*

Definition 4.2 *It is clear that, $G_1 + G_2$ is a connected graph, $n = |V(G_1 + G_2)| = |V(G_1)| + |V(G_2)|$, $m = |E(G_1 + G_2)| = |V(G_1)||V(G_2)| + |E(G_1)| + |E(G_2)|$ and $diam(G_1 + G_2) \leq 2$. Furthermore, $diam(G_1 + G_2) = 1$ if and only if G_1 and G_2 are complete graphs. We denote by $d_k(v/G)$ to the k-distance degree of a vertex v in a graph G.*

Theorem 4.2 *Let G and H be connected graphs of order n_1 and n_2 and size m_1 and m_2, respectively. Then*

$$N_k(G+H) = 4\binom{n_1+n_2}{2} - 2(n_1n_2 + m_1 + m_2).$$

Proof The proof is an immediately consequences of Theorem 1.2. $\square$

Since, For any connected graph G, $G + K_1 = K_1 + G = K_1 \circ G$ then the next result follows

Corollary 2.3.

Corollary 4.3 *For any connected graph G with n vertices and m edges,*

$$N_k(G + K_1) = 2(n^2 - m).$$

The join of more than two graphs is defined inductively as following,

$$G_1 + G_2 + \cdots + G_t = (G_1 + G_2 + \cdots + G_{t-1}) + G_t$$

for some positive integer number $t \geq 2$. We denote by $\sum_{i=1}^{t} G_i$ to $G_1 + G_2 + \cdots + G_t$. It is clear for this definition that

- $n = |V(\sum_{i=1}^{t} G_i)| = \sum_{i=1}^{t} |V(G_i)|$.
- $m = |E(\sum_{i=1}^{t} G_i)| = \sum_{i=1}^{t} |E(G_i)| + \sum_{i=2}^{t} |V(G_i)| \Big(\sum_{j=1}^{i-1} |V(G_j)| \Big)$.
- $diam(\sum_{i=1}^{t} G_i) \leq 2$.

Accordingly, we can generalize Theorem 4.2 by using Theorem 1.2 as following.

Theorem 4.4 *For some positive integer number $t \geq 2$, let $G_1, G_2, \cdots, G_t$ be connected graphs of orders $n_1, n_2, \cdots, n_t$ and sizes $m_1, m_2, \cdots, m_t$, respectively. Then*

$$N_k(\sum_{i=1}^{t} G_i) = 4\binom{\sum_{i=1}^{t} n_i}{2} - 2\left[\sum_{i=1}^{t} m_i + \sum_{i=2}^{t} n_i \Big(\sum_{j=1}^{i-1} n_j\Big)\right].$$

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On Terminal Hosoya Polynomial of Some Thorn Graphs

Harishchandra S.Ramane, Gouramma A.Gudodagi and Raju B.Jummannaver

(Department of Mathematics, Karnatak University, Dharwad - 580003, India)

E-mail: hsramane@yahoo.com, gouri.gudodagi@gmail.com, rajesh.rbj065@gmail.com

Abstract: The terminal Hosoya polynomial of a graph G is defined as $TH(G,\lambda) = \sum\limits_{k\geq 1} d_T(G,k)\lambda^k$ is the number of pairs of pendant vertices of G that are at distance k. In this paper we obtain the terminal Hosoya polynomial for caterpillers, thorn stars and thorn rings. These results generalizes the existing results.

Key Words: Terminal Hosoya polynomial, thorn graphs, thorn trees, thorn stars, thorn rings.

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§1. Introduction

Let G be a connected graph with a vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \cdots, e_m\}$, where $|V(G)| = n$ and $|E(G)| = m$. The *degree* of a vertex v in G is the number of edges incident to it and denoted by $deg_G(v)$. If $deg_G(v) = 1$, then v is called a *pendent vertex* or a *terminal vertex*. The *distance* between the vertices v_i and v_j in G is equal to the length of the shortest path joining them and is denoted by $d(v_i, v_j|G)$.

The *Wiener index* $W = W(G)$ of a graph G is defined as the sum of the distances between all pairs of vertices of G, that is

$$W = W(G) = \sum_{1\leq i<j\leq n} d(u_i, v_j|G).$$

This molecular structure descriptor was put forward by Harold Wiener [29] in 1947. Details on its chemical applications and mathematical properties can be found in [5, 12, 21, 28].

The Hosoya polynomial of a graph was introduced in Hosoya's seminal paper [16] in 1988 and received a lot of attention afterwards. The polynomial was later independently introduced and considered by Sagan et al. [22] under the name Wiener polynomial of a graph. Both names are still used for the polynomial but the term Hosoya polynomial is nowadays used by the majority of researchers. The main advantage of the Hosoya polynomial is that it contains a wealth of information about distance based graph invariants. For instance, knowing the Hosoya polynomial of a graph, it is straight forward to determine the Wiener index of a graph as the first derivative of the polynomial at the point $\lambda = 1$. Cash [2] noticed that the hyper-Wiener

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index can be obtained from the Hosoya polynomial in a similar simple manner.

Estrada et al. [6] studied the chemical applications of Hosoya polynomial. The *Hosoya polynomial* of a graph is a distance based polynomial introduced by Hosoya [15] in 1988 under the name Wiener polynomial. However today it is called the Hosoya polynomial [8, 11, 17, 18, 23, 27]. For a connected graph G, the *Hosoya polynomial* denoted by $H(G,\lambda)$ is defined as

$$H(G,\lambda) = \sum_{k\geq 1} d(G,k)\lambda^k = \sum_{1\leq i<j\leq n} \lambda^{d(v_i,v_j|G)}. \tag{1.1}$$

where $d(G,k)$ is the number of pairs of vertices of G that are at distance k and λ is the parameter.

The Hosoya polynomial has been obtained for trees, composite graphs, benzenoid graphs, tori, zig-zag open-ended nano-tubes, certain graph decorations, armchair open-ended nanotubes, zigzag polyhex nanotorus, nanotubes, pentachains, polyphenyl chains, the circumcoronene series, Fibonacci and Lucas cubes, Hanoi graphs, and so forth. These can be found in [4].

Recently the *terminal Wiener index* $TW(G)$ was put forward by Gutman et al. [10]. The *terminal Wiener index* $TW(G)$ of a connected graph G is defined as the sum of the distances between all pairs of its pendant vertices. Thus if $V_T(G) = v_1, v_2, \ldots, v_k$ is the number of pendant vertices of G, then

$$TW(G) = \sum_{1\leq i<j\leq k} d(v_i, v_j|G).$$

The recent work on terminal Wiener index can be found in [3, 9, 14, 20, 24]. In analogy of (1.1), the *terminal Hosoya polynomial* $TH(G,\lambda)$ was put forward by Narayankar et al. [19] and is defined as follows: if $v_1, v_2, \ldots, v_k$ are the pendant vertices of G, then

$$TH(G,\lambda) = \sum_{k\geq 1} d_T(G,k)\lambda^k = \sum_{1\leq i<j\leq n} \lambda^{d(v_i,v_j|G)},$$

where $d_T(G,k)$ is the number of pairs of pendant vertices of the graph G that are at distance k. It is easy to check that

$$TW(G) = \frac{d}{d\lambda}(TH(G,\lambda))|_{\lambda=1}.$$

In [19], the terminal Hosoya polynomial of thorn graph is obtained. In this paper we generalize the results obtained in [19].

§2. Terminal Hosoya Polynomial of Thorn Graphs

Definition 2.1 *Let G be a connected n-vertex graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$. The thorn graph $G_P = G(p_1, p_2, \cdots, p_n : k)$ is the graph obtained by attaching p_i paths of length k to the vertex v_i for $i = 1, 2, \cdots, n$ of a graph G. The p_i paths of length k attached to the vertex v_i will be called the thorns of v_i.*

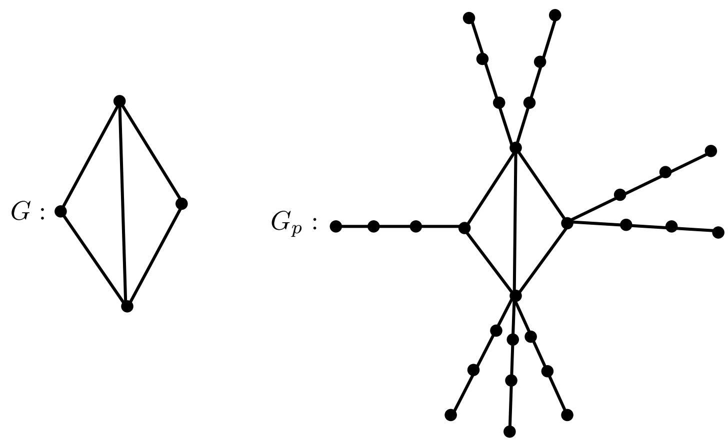

Fig. 1.

A thorn graph $Gp = G(2,1,3,2:3)$ obtained from G by attaching paths of length 3 is shown in Fig.1. Notice that the concept of thorny graph was introduced by Gutman [7] and eventually found a variety of applications [1, 25, 26, 27].

Theorem 2.2 *For a thorn graph $G_P = G(p_1, p_2, \ldots, p_n : k)$, the terminal Hosoya polynomial is*

$$TH(G_P, \lambda) = \sum_{i=1}^{n} \binom{p_i}{2} \lambda^{2k} + \sum_{1 \leq i < j \leq n} p_i p_j \lambda^{2k + d(v_i, v_j | G)}. \tag{2.1}$$

Proof Consider p_i path of length k attached to a vertex v_i, $i = 1, 2, \cdots, n$. Each of these are at distance $2k$. Thus for each v_i, there are $\binom{p_i}{2}$ pairs of vertices which are distance $2k$. This leads to the first term of (2.1).

For the second term of (2.1), consider p_i thorns $v_1^i, v_2^i, \cdots, v_{p_i}^i$ attached to the vertex v_i and p_j thorns $v_1^j, v_2^j, \cdots, v_{p_j}^j$ attached to the vertex v_j of G, $i \neq j$. In G_P,

$$d(v_m^i, v_l^j | G_P) = 2k + d(v_, v_j | G), \quad m = 1, 2, \cdots, p_i \text{ and } l = 1, 2, \cdots, p_j.$$

Since there are $p_i \times p_j$ pairs of paths of length k of such kind, their contribution to $TH(G_P, \lambda)$ is equal to $p_i p_j \lambda^{2k + d(v_i, v_j | G)}$, $i \neq j$. This leads to the second term of (2.1). $\square$

Corollary 2.3 *Let G be a connected graph with n vertices. If $p_i = p > 0$, $i = 1, 2, \cdots, n$. Then*

$$TH(G_P, \lambda) = \frac{np(p-1)}{2} \lambda^{2k} + p^2 \lambda^{2k} \sum_{1 \leq i < j \leq n} \lambda^{d(v, v_j | G)}. \tag{2.2}$$

Corollary 2.4 *Let G be a complete graph on n vertices. If $p_i = p > 0$, $i = 1, 2, \cdots, n$. Then*

$$TH(G_P, \lambda) = \frac{np(p-1)}{2} \lambda^{2k} + \frac{p^2 n(n-1)}{2} \lambda^{2k+1}.$$

Proof If G is a complete graph then $d(v_, v_j | G) = 1$ for all $v_i, v_j \in V(G)$, $i \neq j$. Therefore from (2.2)

$$\begin{aligned} TH(G_P,\lambda) &= \frac{np(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k}\sum_{1\le i<j\le n}\lambda \\ &= \frac{np(p-1)}{2}\lambda^{2k} + \frac{p^2 n(n-1)}{2}\lambda^{2k+1}. \end{aligned}$$

This completes the proof. □

Corollary 2.5 *Let G be a connected graph with n vertices and m edges. If $diam(G) \le 2$ and $p_i = p > 0$, $i = 1, 2, \ldots, n$. Then*

$$TH(G_P,\lambda) = \frac{np(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k+1}m + \left(\frac{n(n-1)}{2} - m\right)p^2\lambda^{2k+2}.$$

Proof Since $diam(G) \le 2$, there are m pairs of vertices at distance 1 and $\binom{n}{2} - m$ pairs of vertices are at distance 2 in G. Therefore from (2.2)

$$\begin{aligned} TH(G_P,\lambda) &= \frac{np(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k}\left[\sum_{m}\lambda + \sum_{\binom{n}{2}-m}\lambda^2\right] \\ &= \frac{n(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k}\left[m\lambda + \left(\frac{n(n-1)}{2} - m\right)\lambda^2\right] \\ &= \frac{np(p-1)}{2}\lambda^{2k} + p^2\lambda^{2k+1}m + \left(\frac{n(n-1)}{2} - m\right)p^2\lambda^{2k+2}. \end{aligned}$$

This completes the proof. □

Bonchev and Klein [1] proposed the terminology of thorn trees, where the parent graph is a tree. In a thorn tree if the parent graph is a path then it is a caterpiller [13].

Definition 2.6 *Let P_l be path on l vertices, $l \ge 3$ labeled as $u_1, u_2, \cdots, u_l$, where u_i is adjacent to u_{i+1}, $i = 1, 2, \cdots, (l-1)$. Let $T_P = T(p_1, p_2, \cdots, p_l : k)$ be a thorn tree obtained from P_l by attaching $p_i \ge 0$ path of length k to u_i, $i = 1, 2, \cdots, l$.*

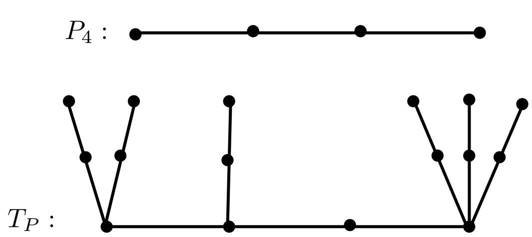

Fig. 2

A thorn graph $T_P = T(2, 1, 0, 3 : 2)$ obtained from T by attaching paths of length 2 is shown in Fig.2.

Theorem 2.7 *For a thorn tree $T_P = T(p_1, p_2, \cdots, p_l : k)$ of order $n \geq 3$, the terminal Hosoya polynomial is*

$$Th(T_P, \lambda) = a_1\lambda + a_2\lambda^2 + \ldots + a_{2k+1}\lambda^{2k+1},$$

where

$$\begin{aligned} a_1 &= 0 \\ a_{2k} &= \sum_{i=1}^{l} \binom{p_i}{2} \\ a_{2k+l-j} &= \sum_{i=1}^{j} p_i p_{i+l-j} \qquad j = 1, 2, \ldots, (l-1). \end{aligned}$$

Proof Notice that there is no pair of pendant vertices which are at distance 1 and there are $\binom{p_i}{2}$ pairs of pendant vertices of which are at distance $2k$ in T. Therefore $a_1 = 0$ and

$$a_{2k} = \sum_{i=1}^{l} \binom{p_i}{2}.$$

For a_k, $2 \leq k \leq l$, $d(u, v|T) = 2k + l - j$, where u and v are the vertices of T_P. There are $p_i \times p_{i+l-j}$ pairs of pendant vertices which are at distance $2k + l - j$, where $j = 1, 2, \cdots, n-1$. Therefore

$$a_{2k+l-j} = \sum_{i=1}^{j} p_i p_{i+l-j}. \qquad \square$$

Definition 2.8 *Let $S_n = K_{1,n-1}$ be the star on n-vertices and let $u_1, u_2, \cdots, u_{n-1}$ be the pendant vertices of the star S_n and u_n be the central vertex. Let $S_P = S(p_1, p_2, \cdots, p_{n-1} : k)$ be the thorn star obtained from S_n by attaching p_i paths of length k to the vertex u_i, $i = 1, 2, \cdots, (n-1)$ and $p_i \geq 0$.*

Theorem 2.9 *The terminal Hosoya polynomial of thorn star S_P defined in Definition 2.8 is*

$$TH(S_P, \lambda) = a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \ldots + a_{2k}\lambda^{2k} + a_{2k+2}\lambda^{2k+2},$$

where

$$\begin{aligned} a_1 &= 0 \\ a_{2k} &= \sum_{i=1}^{n} \binom{p_i}{2} \\ a_{2k+2} &= \sum_{1 \leq i < j \leq n} p_i p_j. \end{aligned}$$

Proof There are no pair of pendant vertices which are at odd distance. Therefore, $a_{2k+1} = 0$ and the further proof follows from Theorem 2.7. $\square$

Definition 2.10 *Let C_n be the n-vertex cycle labeled consecutively as $u_1, u_2, \cdots, u_n$, $n \geq 3$. and let $\mathbb{C}_P = C(p_1, p_2, \cdots, p_n : k)$ be the thorn ring obtained from C_n by attaching p_i paths of length k to the vertex u_i, $i = 1, 2, \cdots, n$.*

Theorem 2.11 *The terminal Hosoya polynomial of thorn ring $\mathbb{C}_P$ defined in Definition* 2.10 *is*

$$TH(\mathbb{C}, \lambda) = a_1\lambda + a_2\lambda^2 + \ldots + a_{2k}\lambda^{2k} + a_{2k+1}\lambda^{2k+1},$$

where

$$\begin{aligned} a_1 &= 0 \\ a_{2k} &= \sum_{i=1}^{n} \binom{p_i}{2} \\ a_{2k+1} &= \sum_{i=1}^{n} (2k + d(v_i, v_j|G))p_i p_j. \end{aligned}$$

Proof The proof is analogous to that of Theorem 2.7. □

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On the Distance Eccentricity Zagreb Indeices of Graphs

Akram Alqesmah, Anwar Alwardi and R. Rangarajan

(Department of Studies in Mathematics, University of Mysore, Mysore 570 006, India)

E-mail: aalqesmah@gmail.com

Abstract: Let $G = (V, E)$ be a connected graph. The distance eccentricity neighborhood of $u \in V(G)$ denoted by $N_{De}(u)$ is defined as $N_{De}(u) = \{v \in V(G) : d(u, v) = e(u)\}$, where $e(u)$ is the eccentricity of u. The cardinality of $N_{De}(u)$ is called the distance eccentricity degree of the vertex u in G and denoted by $deg^{De}(u)$. In this paper, we introduce the first and second distance eccentricity Zagreb indices of a connected graph G as the sum of the squares of the distance eccentricity degrees of the vertices, and the sum of the products of the distance eccentricity degrees of pairs of adjacent vertices, respectively. Exact values for some families of graphs and graph operations are obtained.

Key Words: First distance eccentricity Zagreb index, Second distance eccentricity Zagreb index, Smarandachely distance eccentricity.

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§1. Introduction

In this research work, we concerned about connected, simple graphs which are finite, undirected with no loops and multiple edges. Throughout this paper, for a graph $G = (V, E)$, we denote $p = |V(G)|$ and $q = |E(G)|$. The complement of G, denoted by $\overline{G}$, is a simple graph on the same set of vertices $V(G)$ in which two vertices u and v are adjacent if and only if they are not adjacent in G. The open neighborhood and the closed neighborhood of u are denoted by $N(u) = \{v \in V : uv \in E\}$ and $N[u] = N(u) \cup \{u\}$, respectively. The degree of a vertex u in G, is denoted by $deg(u)$, and is defined to be the number of edges incident with u, shortly $deg(u) = |N(u)|$. The maximum and minimum degrees of G are defined by $\Delta(G) = max\{deg(u) : u \in V(G)\}$ and $\delta(G) = min\{deg(u) : u \in V(G)\}$, respectively. If $\delta = \Delta = k$ for any graph G, we say G is a regular graph of degree k. The distance between any two vertices u and v in G denoted by $d(u, v)$ is the number of edges of the shortest path joining u and v. The eccentricity $e(u)$ of a vertex u in G is the maximum distance between u and any other vertex v in G, that is $e(u) = \max\{d(u, v), v \in V(G)\}$.

The path, wheel, cycle, star and complete graphs with p vertices are denoted by P_p, W_p, C_p, S_p and K_p, respectively, and $K_{r,m}$ is the complete bipartite graph on $r + m$ vertices. All the definitions and terminologies about graph in this paper available in [6].

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The Zagreb indices have been introduced by Gutman and Trinajestic [5].

$$M_1(G)=\sum_{u\in V(G)}\big[deg(u)\big]^2=\sum_{u\in V(G)}\sum_{v\in N(u)}deg(v)=\sum_{uv\in E(G)}\big[deg(u)+deg(v)\big].$$

$$M_2(G)=\sum_{uv\in E(G)}deg(u)deg(v)=\frac{1}{2}\sum_{u\in V(G)}deg(u)\sum_{v\in N(u)}deg(v).$$

Here, $M_1(G)$ and $M_2(G)$ denote the first and the second Zagreb indices, respectively. For more details about Zagreb indices, we refer to [2, 4, 9, 13, 11, 12, 7, 10, 8].

Let $u\in V(G)$. The distance eccentricity neighborhood of u denoted by $N_{De}(u)$ is defined as $N_{De}(u)=\{v\in V(G):\ d(u,v)=e(u)\}$. The cardinality of $N_{De}(u)$ is called the distance eccentricity degree of the vertex u in G and denoted by $deg^{De}(u)$, and $N_{De}[u]=N_{De}(u)\cup\{u\}$, note that if u has a full degree in G, then $deg(u)=deg^{De}(u)$. And generally, a Smarandachely distance eccentricity neighborhood $N^S_{De}(u)$ of u on subset $S\subset V(G)$ is defined to be $N^S_{De}(u)=\{v\in V(G)\backslash S:\ d_{G\backslash S}(u,v)=e(u)\}$ with Smarandachely distance eccentricity $\left|N^S_{De}(u)\right|$. Clearly, $\left|N^{\emptyset}_{De}(u)\right|=deg^{De}(u)$. The maximum and minimum distance eccentricity degree of a vertex in G are denoted respectively by $\Delta^{De}(G)$ and $\delta^{De}(G)$, that is $\Delta^{De}(G)=\max_{u\in V}|N_{De}(u)|$, $\delta^{De}(G)=\min_{u\in V}|N_{De}(u)|$. Also, we denote to the set of vertices of G which have eccentricity equal to α by $V_e^{\alpha}(G)\subseteq V(G)$, where $\alpha=1,2,\cdots,diam(G)$. In this paper, we introduce the distance eccentricity Zagreb indices of graphs. Exact values for some families of graphs and some graph operations are obtained.

§2. Distance Eccentricity Zagreb Indices of Graphs

In this section, we define the first and second distance eccentricity Zagreb indices of connected graphs and study some standard graphs.

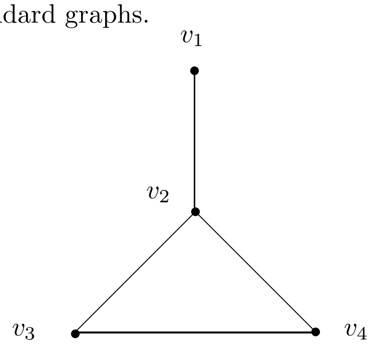

Fig.1

Definition 2.1 *Let $G=(V,E)$ be a connected graph. Then the first and second distance eccentricity Zagreb indices of G are defined by*

$$\begin{aligned}M_1^{De}(G)&=\sum_{u\in V(G)}\big[deg^{De}(u)\big]^2,\\ M_2^{De}(G)&=\sum_{uv\in E(G)}deg^{De}(u)deg^{De}(v).\end{aligned}$$

Example 2.2 Let G be a graph as in Fig.1. Then

$$(i)\quad M_1^{De}(G)=\sum_{u\in V(G)}\big[deg^{De}(u)\big]^2=\sum_{i=1}^{4}\big(deg^{De}(v_i)\big)^2$$
$$=\big(deg^{De}(v_1)\big)^2+\big(deg^{De}(v_2)\big)^2+\big(deg^{De}(v_3)\big)^2+\big(deg^{De}(v_4)\big)^2$$
$$=\big(2\big)^2+\big(3\big)^2+\big(1\big)^2+\big(1\big)^2=15.$$

$$(ii)\ M_2^{De}(G)=\sum_{uv\in E(G)}deg^{De}(u)deg^{De}(v)$$
$$=deg^{De}(v_1)deg^{De}(v_2)+deg^{De}(v_2)deg^{De}(v_3)+deg^{De}(v_2)deg^{De}(v_4)$$
$$+\,deg^{De}(v_3)deg^{De}(v_4)=13.$$

Calculation immediately shows results following.

Proposition 2.3 (i) *For any path P_p with $p\geq 2$,* $M_1^{De}(P_p)=\begin{cases} p+3, & p \text{ is odd,}\\ p, & p \text{ is even;}\end{cases}$

(ii) *For $p\geq 3$,* $M_1^{De}(C_p)=\begin{cases} 4p, & p \text{ is odd,}\\ p, & p \text{ is even;}\end{cases}$

(iii) $M_1^{De}(K_p)=M_1(K_p)=p\big(p-1\big)^2$*;*

(iv) *For $r,m\geq 2$,* $M_1^{De}(K_{r,m})=r\big(r-1\big)^2+m\big(m-1\big)^2$*;*

(v) *For $p\geq 3$,* $M_1^{De}(S_p)=(p-1)\big(p-2\big)^2+\big(p-1\big)^2$*;*

(vi) *For $p\geq 5$,* $M_1^{De}(W_p)=(p-1)\big(p-4\big)^2+\big(p-1\big)^2$.

Proposition 2.4 (i) *For $p\geq 2$,* $M_2^{De}(P_p)=\begin{cases} p+1, & p \text{ is odd,}\\ p-1, & p \text{ is even;}\end{cases}$

(ii) *For $p\geq 3$,* $M_2^{De}(C_p)=\begin{cases} 4p, & p \text{ is odd,}\\ p, & p \text{ is even;}\end{cases}$

(iii) $M_2^{De}(K_p)=M_2(K_p)=\frac{p(p-1)}{2}\big(p-1\big)^2$*;*

(iv) *For $r,m\geq 2$,* $M_2^{De}(K_{r,m})=rm\big(r-1\big)\big(m-1\big)$*;*

(v) *For $p\geq 3$,* $M_2^{De}(S_p)=\big(p-1\big)^2(p-2)$*;*

(vi) *For $p\geq 5$,* $M_2^{De}(W_p)=(p-1)(p-4)(2p-5)$.

Proposition 2.5 *For any graph G with $e(v)=2$, $\forall\, v\in V(G)$,*

(i) $M_1^{De}(G)=M_1(\overline{G})$*;*

(ii) $M_2^{De}(G)=q(p-1)^2-(p-1)M_1(G)+M_2(G)$.

Proof Since $e(v)=2$, $\forall v\in V(G)$, then $deg_G^{De}(v)=deg_{\overline{G}}(v)$. Hence the result. □

Corollary 2.6 *For any k-regular (p,q)-graph G with diameter two,*

(i) $M_1^{De}(G)=p(p-k-1)^2$*;*

(ii) $M_2^{De}(G)=\frac{1}{2}pk(p-k-1)^2$.

§3. Distance Eccentricity Zagreb Indices for Some Graph Operations

In this section, we compute the first and second distance eccentricity Zagreb indices for some graph operations.

Cartesian Product. The Cartesian product of two graphs G_1 and G_2, where $|V(G_1)| = p_1$, $|V(G_2)| = p_2$ and $|E(G_1)| = q_1$, $|E(G_2)| = q_2$ is denoted by $G_1\Box G_2$ has the vertex set $V(G_1)\times V(G_2)$ and two vertices (u,u') and (v,v') are connected by an edge if and only if either $([u = v$ and $u'v' \in E(G_2)])$ or $([u' = v'$ and $uv \in E(G_1)])$. By other words, $|E(G_1\Box G_2)| = q_1p_2 + q_2p_1$. The degree of a vertex (u,u') of $G_1\Box G_2$ is as follows:

$$deg_{G_1\Box G_2}(u,u') = deg_{G_1}(u) + deg_{G_2}(u').$$

The Cartesian product of more than two graphs is denoted by $\prod_{i=1}^n G_i$ $\big(\prod_{i=1}^n G_i = G_1\Box G_2\Box\ldots\Box G_n = (G_1\Box G_2\Box\ldots\Box G_{n-1})\Box G_n\big)$, in which any two vertices $u = (u_1,u_2,\ldots,u_n)$ and $v = (v_1,v_2,\ldots,v_n)$ are adjacent in $\prod_{i=1}^n G_i$ if and only if $u_i = v_i, \forall i \neq j$ and $u_jv_j \in E(G_j)$, where $i,j = 1,2,\ldots,n$. If $G_1 = G_2 = \cdots = G_n = G$, we have the n-th Cartesian power of G, which is denoted by G^n.

Lemma 3.1([8]) *Let $G = \prod_{i=1}^n G_i$ and let $u = (u_1,u_2,\cdots,u_n)$ be a vertex in $V(G)$. Then*

$$e(u) = \sum_{i=1}^n e(u_i).$$

Lemma 3.2 *Let $G = \prod_{i=1}^n G_i$ and let $u = (u_1,u_2,\ldots,u_n)$ be a vertex in G. Then*

$$deg_G^{De}(u) = \prod_{i=1}^n deg_{G_i}^{De}(u_i).$$

Proof Since $e(u) = \sum_{i=1}^n e(u_i)$ (Lemma 3.1), then each distance eccentricity neighbor of u_1 in G_1 corresponds $deg_{G_2}^{De}(u_2)$ vertices in G_2 and each distance eccentricity neighbor of u_2 in G_2 corresponds $deg_{G_3}^{De}(u_3)$ vertices in G_3 and so on. Thus by using the Principle of Account

$$deg_G^{De}(u) = deg_{G_1}^{De}(u_1)deg_{G_2}^{De}(u_2)\cdots deg_{G_n}^{De}(u_n). \qquad \Box$$

Theorem 3.3 *Let $G = \prod_{i=1}^n G_i$. Then*

(i) $M_1^{De}(G) = \prod_{i=1}^n M_1^{De}(G_i)$;

(ii) $M_2^{De}(G) = \sum_{j=1}^n \prod_{\substack{i=1\\ i\neq j}}^n M_1^{De}(G_i)M_2^{De}(G_j)$.

Proof Let $u = (u_1,u_2,\cdots,u_n)$ and $v = (v_1,v_2,\cdots,v_n)$ be any two vertices in $V(G)$. Then

$$
\begin{aligned}
(i)\ \ M_1^{De}(G) &= \sum_{u\in V(G)} \left(deg_G^{De}(u)\right)^2 = \sum_{u\in V(G)} \left(deg_{G_1}^{De}(u_1)deg_{G_2}^{De}(u_2)\dots deg_{G_n}^{De}(u_n)\right)^2 \\
&= \sum_{u_1\in V(G_1)} \sum_{u_2\in V(G_2)} \dots \sum_{u_n\in V(G_n)} \left(deg_{G_1}^{De}(u_1)\right)^2 \left(deg_{G_2}^{De}(u_2)\right)^2 \dots \left(deg_{G_n}^{De}(u_n)\right)^2 \\
&= \prod_{i=1}^{n} M_1^{De}(G_i).
\end{aligned}
$$

(ii) To prove the second distance eccentricity Zagreb index we will use the mathematical induction. First, if $n = 2$, then

$$
\begin{aligned}
M_2^{De}(G_1\Box G_2) &= \sum_{(u_1,u_2)(v_1,v_2)\in E(G_1\Box G_2)} deg_{G_1}^{De}(u_1)deg_{G_1}^{De}(v_1)deg_{G_2}^{De}(u_2)deg_{G_2}^{De}(v_2) \\
&= \sum_{u_1\in V(G_1)} \sum_{(u_1,u_2)(u_1,v_2)\in E(G_1\Box G_2)} \left(deg_{G_1}^{De}(u_1)\right)^2 deg_{G_2}^{De}(u_2)deg_{G_2}^{De}(v_2) \\
&\quad + \sum_{u_2\in V(G_2)} \sum_{(u_1,u_2)(v_1,u_2)\in E(G_1\Box G_2)} \left(deg_{G_2}^{De}(u_2)\right)^2 deg_{G_1}^{De}(u_1)deg_{G_1}^{De}(v_1) \\
&= M_1^{De}(G_1)M_2^{De}(G_2) + M_1^{De}(G_2)M_2^{De}(G_1) \\
&= \sum_{j=1}^{2} \prod_{\substack{i=1 \\ i\neq j}}^{2} M_1^{De}(G_i)M_2^{De}(G_j).
\end{aligned}
$$

Now, suppose the claim is true for $n-1$. Then

$$
\begin{aligned}
M_2^{De}\left(\Box_{i=1}^{n-1} G_i \Box G_n\right) &= M_1^{De}\left(\Box_{i=1}^{n-1} G_i\right)M_2^{De}(G_n) + M_1^{De}(G_n)M_2^{De}\left(\Box_{i=1}^{n-1} G_i\right) \\
&= \prod_{i=1}^{n-1} M_1^{De}(G_i)M_2^{De}(G_n) + M_1^{De}(G_n)\sum_{j=1}^{n-1} \prod_{\substack{i=1 \\ i\neq j}}^{n-1} M_1^{De}(G_i)M_2^{De}(G_j) \\
&= \sum_{j=1}^{n} \prod_{\substack{i=1 \\ i\neq j}}^{n} M_1^{De}(G_i)M_2^{De}(G_j). \qquad \Box
\end{aligned}
$$

Composition. The composition $G = G_1[G_2]$ of two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and edge sets $E(G_1)$ and $E(G_2)$, where $|V(G_1)| = p_1$, $|E(G_1)| = q_1$ and $|V(G_2)| = p_2$, $|E(G_2)| = q_2$ is the graph with vertex set $V(G_1) \times V(G_2)$ and any two vertices (u, u') and (v, v') are adjacent whenever u is adjacent to v in G_1 or $u = v$ and u' is adjacent to v' in G_2. Thus, $|E(G_1[G_2])| = q_1p_2^2 + q_2p_1$. The degree of a vertex (u, u') of $G_1[G_2]$ is as follows:

$$deg_{G_1[G_2]}(u, u') = p_2deg_{G_1}(u) + deg_{G_2}(u').$$

Lemma 3.4([8]) *Let $G = G_1[G_2]$ and $e(v) \neq 1$, $\forall\, v \in V(G_1)$. Then $e_G((u, u')) = e_{G_1}(u)$.*

Lemma 3.5 *Let $G = G_1[G_2]$ and $e(v) \neq 1$, $\forall\, v \in V(G_1)$. Then*

$$
deg_G^{De}(u, u') = \begin{cases} p_2deg_{G_1}^{De}(u) + deg_{\overline{G_2}}(u'), & \text{if } u \in V_e^2(G_1); \\ p_2deg_{G_1}^{De}(u), & \text{otherwise.} \end{cases}
$$

Proof From Lemma 3.4, we have $e_G(u,u') = e_{G_1}(u)$. Therefore, $N_G^{De}(u,u') = \{(x,x') \in V(G) : d\big((u,u'),(x,x')\big) = e_{G_1}(u)\}$. Now, if $u \notin V_e^2(G_1)$, then $N_G^{De}(u,u') = \{(x,x') \in V(G) : x \in N_{G_1}^{De}(u)\}$ and hence, $deg_G^{De}(u,u') = p_2 deg_{G_1}^{De}(u)$ and if $u \in V_e^2(G_1)$, then $deg_G^{De}(u,u') = p_2 deg_{G_1}^{De}(u) + deg_{\overline{G_2}}(u')$ (note that all the vertices of the copy of G_2 with the projection $u \in V(G_1)$ which are not adjacent to (u,u') have distance two from (u,u')). $\square$

Theorem 3.6 *Let $G = G_1[G_2]$ and $e(v) \neq 1$, $\forall\, v \in V(G_1)$. Then*

$$M_1^{De}(G) = p_2^3 M_1^{De}(G_1) + |V_e^2(G_1)| M_1(\overline{G_2}) + 4p_2\overline{q_2} \sum_{u \in V_e^2(G_1)} deg_{G_1}^{De}(u).$$

Proof By definition, we know that

$$\begin{aligned}
M_1^{De}(G) &= \sum_{(u,u') \in V(G)} \big(deg_G^{De}(u,u')\big)^2 = \sum_{u \in V(G_1)} \sum_{u' \in V(G_2)} \big(deg_G^{De}(u,u')\big)^2 \\
&= \sum_{u \in V_e^2(G_1)} \sum_{u' \in V(G_2)} \big(p_2 deg_{G_1}^{De}(u) + deg_{\overline{G_2}}(u')\big)^2 \\
&\quad + \sum_{u \in V(G_1) - V_e^2(G_1)} \sum_{u' \in V(G_2)} \big(p_2 deg_{G_1}^{De}(u)\big)^2 \\
&= \sum_{u \in V(G_1)} \sum_{u' \in V(G_2)} \big(p_2 deg_{G_1}^{De}(u)\big)^2 + \sum_{u \in V_e^2(G_1)} M_1(\overline{G_2}) \\
&\quad + \sum_{u \in V_e^2(G_1)} \sum_{u' \in V(G_2)} 2p_2 deg_{\overline{G_2}}(u') deg_{G_1}^{De}(u) \\
&= p_2^3 M_1^{De}(G_1) + |V_e^2(G_1)| M_1(\overline{G_2}) + 4p_2\overline{q_2} \sum_{u \in V_e^2(G_1)} deg_{G_1}^{De}(u).
\end{aligned}$$
$\square$

Theorem 3.7 *Let $G = G_1[G_2]$ and $e(v) \neq 1$ or 2, $\forall\, v \in V(G_1)$. Then*

$$M_2^{De}(G) = p_2^4 M_2^{De}(G_1) + p_2^2 q_2 M_1^{De}(G_1).$$

Proof By deifnition, we know that

$$\begin{aligned}
M_2^{De}(G) &= \frac{1}{2} \sum_{(u,u') \in V(G)} deg_G^{De}(u,u') \sum_{(v,v') \in N_G(u,u')} deg_G^{De}(v,v') \\
&= \frac{1}{2} \sum_{u \in V(G_1)} \sum_{u' \in V(G_2)} deg_G^{De}(u,u') \Big[\sum_{v \in N_{G_1}(u)} \sum_{v' \in V(G_2)} deg_G^{De}(v,v') + \sum_{v' \in N_{G_2}(u')} deg_G^{De}(u,v') \Big] \\
&= \frac{1}{2} \sum_{u \in V(G_1)} \sum_{u' \in V(G_2)} p_2 deg_{G_1}^{De}(u) \Big[\sum_{v \in N_{G_1}(u)} \sum_{v' \in V(G_2)} p_2 deg_{G_1}^{De}(v) + \sum_{v' \in N_{G_2}(u')} p_2 deg_{G_1}^{De}(u) \Big] \\
&= p_2^4 M_2^{De}(G_1) + p_2^2 q_2 M_1^{De}(G_1).
\end{aligned}$$

This completes the proof. $\square$

Disjunction and Symmetric Difference. The disjunction $G_1 \vee G_2$ of two graphs G_1 and G_2 with $|V(G_1)| = p_1$, $|E(G_1)| = q_1$ and $|V(G_2)| = p_2$, $|E(G_2)| = q_2$ is the graph with vertex set $V(G_1)\times V(G_2)$ in which (u,u') is adjacent to (v,v') whenever u is adjacent to v in G_1 or u' is adjacent to v' in G_2. So, $|E(G_1\vee G_2)|=q_1p_2^2+q_2p_1^2-2q_1q_2$. The degree of a vertex (u,u') of $G_1\vee G_2$ is as follows:

$$deg_{G_1\vee G_2}(u,u')=p_2deg_{G_1}(u)+p_1deg_{G_2}(u')-deg_{G_1}(u)deg_{G_2}(u').$$

Also, the symmetric difference $G_1\oplus G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1)\times V(G_2)$ in which (u,u') is adjacent to (v,v') whenever u is adjacent to v in G_1 or u' is adjacent to v' in G_2, but not both. From definition one can see that, $|E(G_1\oplus G_2)|=q_1p_2^2+q_2p_1^2-4q_1q_2$. The degree of a vertex (u,u') of $G_1\oplus G_2$ is as follows:

$$deg_{G_1\oplus G_2}(u,u')=p_2deg_{G_1}(u)+p_1deg_{G_2}(u')-2deg_{G_1}(u)deg_{G_2}(u').$$

The distance between any two vertices of a disjunction or a symmetric difference cannot exceed two. Thus, if $e(v)\neq 1$, $\forall\, v\in V(G_1)\cup V(G_2)$, the eccentricity of all vertices is constant and equal to two. We know the following lemma.

Lemma 3.8 *Let G_1 and G_2 be two graphs with $e(v)\neq 1$, $\forall\, v\in V(G_1)\cup V(G_2)$. Then*

(i) $deg^{De}_{G_1\vee G_2}(u,u')=deg_{\overline{G_1\vee G_2}}(u,u')$;
(ii) $deg^{De}_{G_1\oplus G_2}(u,u')=deg_{\overline{G_1\oplus G_2}}(u,u')$.

Theorem 3.9 *Let G_1 and G_2 be two graphs with $e(v)\neq 1$, $\forall\, v\in V(G_1)\cup V(G_2)$. Then*

(i) $M_1^{De}(G_1\vee G_2)=M_1(\overline{G_1\vee G_2})$;
(ii) $M_2^{De}(G_1\vee G_2)=q_{G_1\vee G_2}(p_1p_2-1)^2-(p_1p_2-1)M_1(G_1\vee G_2)+M_2(G_1\vee G_2)$.

Proof The proof is straightforward by Proposition 2.5. □

Theorem 3.10 *Let G_1 and G_2 be any two graphs with $e(v)\neq 1$, $\forall\, v\in V(G_1)\cup V(G_2)$. Then*

(i) $M_1^{De}(G_1\oplus G_2)=M_1(\overline{G_1\oplus G_2})$;
(ii) $M_2^{De}(G_1\oplus G_2)=q_{G_1\oplus G_2}(p_1p_2-1)^2-(p_1p_2-1)M_1(G_1\oplus G_2)+M_2(G_1\oplus G_2)$.

Proof The proof is straightforward by Proposition 2.5. □

Join. The join G_1+G_2 of two graphs G_1 and G_2 with disjoint vertex sets $|V(G_1)|=p_1$, $|V(G_2)|=p_2$ and edge sets $|E(G_1)|=q_1$, $|E(G_2)|=q_2$ is the graph on the vertex set $V(G_1)\cup V(G_2)$ and the edge set $E(G_1)\cup E(G_2)\cup\{u_1u_2: u_1\in V(G_1); u_2\in V(G_2)\}$. Hence, the join of two graphs is obtained by connecting each vertex of one graph to each vertex of the other graph, while keeping all edges of both graphs. The degree of any vertex $u\in G_1+G_2$ is given by

$$deg_{G_1+G_2}(u)=\begin{cases} deg_{G_1}(u)+p_2, & \text{if } u\in V(G_1);\\ deg_{G_2}(u)+p_1, & \text{if } u\in V(G_2).\end{cases}$$

By using the definition of the join graph $G=\sum\limits_{i=1}^{n}G_i$, we get the following lemma.

Lemma 3.11 *Let* $G = \sum_{i=1}^{n} G_i$ *and* $u \in V(G)$. *Then*

$$deg_G^{De}(u) = \begin{cases} |V(G)| - 1, & u \in V_e^1(G_i); \\ p_i - 1 - deg_{G_i}(u), & u \in V(G_i) - V_e^1(G_i), \text{ for } i = 1, 2, \ldots, n. \end{cases}$$

Theorem 3.12 *Let* $G = \sum_{i=1}^{n} G_i$. *Then*

$$M_1^{De}(G) = \big(|V(G)| - 1\big)^2 \sum_{i=1}^{n} |V_e^1(G_i)| + \sum_{i=1}^{n} \Big[M_1(G_i) + p_i\big(p_i - 1\big)^2 - 4q_i(p_i - 1)\Big].$$

Proof By definition,

$$\begin{aligned} M_1^{De}(G) &= \sum_{u \in V(G)} \big[deg_G^{De}(u)\big]^2 = \sum_{i=1}^{n} \sum_{u \in V(G_i)} \big[deg_G^{De}(u)\big]^2 \\ &= \sum_{i=1}^{n} \sum_{u \in V_e^1(G_i)} \big[deg_G^{De}(u)\big]^2 + \sum_{i=1}^{n} \sum_{u \in V(G_i) - V_e^1(G_i)} \big[p_i - 1 - deg_{G_i}(u)\big]^2 \\ &= \big(|V(G)| - 1\big)^2 \sum_{i=1}^{n} |V_e^1(G_i)| + \sum_{i=1}^{n} M_1(\overline{G_i}). \end{aligned}$$

This completes the proof. □

Theorem 3.13 *Let* $G = \sum_{i=1}^{n} G_i$. *Then*

$$\begin{aligned} M_2^{De}(G) = \frac{1}{2}\big(|V(G)| - 1\big) \sum_{i=1}^{n} |V_e^1(G_i)| \Big[\big(|V(G)| - 1\big)\big(-1 + \sum_{j=1}^{n} |V_e^1(G_j)|\big) \\ + 2\sum_{j=1}^{n} \big(p_j^2 - p_j - 2q_j\big)\Big] + \sum_{i=1}^{n} \big[q_i(p_i - 1)^2 - (p_i - 1)M_1(G_i) + M_2(G_i)\big] \\ + \sum_{i=1}^{n-1} \big(p_i^2 - p_i - 2q_i\big) \sum_{j=i+1}^{n} \big(p_j^2 - p_j - 2q_j\big). \end{aligned}$$

Proof By definition, we get that

$$M_2^{De}(G) = \sum_{uv \in E(G)} deg_G^{De}(u) deg_G^{De}(v) = \frac{1}{2} \sum_{u \in V(G)} deg_G^{De}(u) \sum_{v \in N_G(u)} deg_G^{De}(v)$$

$$
\begin{aligned}
=&\frac{1}{2}\sum_{i=1}^{n}\sum_{u\in V(G_i)} deg_G^{De}(u)\Big[\sum_{v\in N_{G_i}(u)} deg_G^{De}(v)+\sum_{\substack{j=1\\ j\neq i}}^{n}\sum_{v\in V(G_j)} deg_G^{De}(v)\Big]\\
=&\frac{1}{2}\sum_{i=1}^{n}\sum_{u\in V_e^1(G_i)}\big(|V(G)|-1\big)\Big[\big(|V(G)|-1\big)\big(|V_e^1(G_i)|-1\big)+\sum_{v\in V(G_i)-V_e^1(G_i)} deg_{\overline{G_i}}(v)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\big[\big(|V(G)|-1\big)|V_e^1(G_j)|+\sum_{v\in V(G_j)-V_e^1(G_j)} deg_{\overline{G_j}}(v)\big]\Big]\\
&+\frac{1}{2}\sum_{i=1}^{n}\sum_{u\in V(G_i)-V_e^1(G_i)} deg_{\overline{G_i}}(u)\Big[\big(|V(G)|-1\big)|V_e^1(G_i)|+\sum_{v\in N_{G_i}(u)-V_e^1(G_i)} deg_{\overline{G_i}}(v)\\
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\big[\big(|V(G)|-1\big)|V_e^1(G_j)|+\sum_{v\in V(G_j)-V_e^1(G_j)} deg_{\overline{G_j}}(v)\big]\Big]\\
=&\frac{1}{2}\big(|V(G)|-1\big)\sum_{i=1}^{n}|V_e^1(G_i)|\Big[\big(|V(G)|-1\big)\big(-1+\sum_{j=1}^{n}|V_e^1(G_j)|\big)\\
&+\sum_{j=1}^{n}\big(p_j^2-p_j-2q_j\big)\Big]+\frac{1}{2}\sum_{i=1}^{n}\big(p_i^2-p_i-2q_i\big)\Big[\big(|V(G)|-1\big)\sum_{j=1}^{n}|V_e^1(G_j)|\\
&+\sum_{\substack{j=1\\ j\neq i}}^{n}\big(p_j^2-p_j-2q_j\big)\Big]+\sum_{i=1}^{n}\big[q_i(p_i-1)^2-(p_i-1)M_1(G_i)+M_2(G_i)\big]\\
=&\frac{1}{2}\big(|V(G)|-1\big)\sum_{i=1}^{n}|V_e^1(G_i)|\Big[\big(|V(G)|-1\big)\big(-1+\sum_{j=1}^{n}|V_e^1(G_j)|\big)\\
&+2\sum_{j=1}^{n}\big(p_j^2-p_j-2q_j\big)\Big]+\sum_{i=1}^{n}\big[q_i(p_i-1)^2-(p_i-1)M_1(G_i)+M_2(G_i)\big]\\
&+\sum_{i=1}^{n-1}\big(p_i^2-p_i-2q_i\big)\sum_{j=i+1}^{n}\big(p_j^2-p_j-2q_j\big).
\end{aligned}
$$

Note that, the equality

$$\frac{1}{2}\sum_{i=1}^{n}\big(p_i^2-p_i-2q_i\big)\sum_{\substack{j=1\\ j\neq i}}^{n}\big(p_j^2-p_j-2q_j\big)=\sum_{i=1}^{n-1}\big(p_i^2-p_i-2q_i\big)\sum_{j=i+1}^{n}\big(p_j^2-p_j-2q_j\big),$$

is applied in the previous calculation. □

Corollary 3.14 *If G_i $(i=1,2,\cdots,n)$ has no vertices of full degree ($V_e^1(G_i)=\phi$), then*

(i) $M_1^{De}\big(\sum_{i=1}^{n}G_i\big)=\sum_{i=1}^{n}M_1(\overline{G_i})$;

(ii)
$$M_2^{De}\big(\sum_{i=1}^{n}G_i\big)=\sum_{i=1}^{n}\big[q_i(p_i-1)^2-(p_i-1)M_1(G_i)+M_2(G_i)\big]+\sum_{i=1}^{n-1}\big(p_i^2-p_i-2q_i\big)\sum_{j=i+1}^{n}\big(p_j^2-p_j-2q_j\big).$$

Corona Product. The corona product $G_1 \circ G_2$ of two graphs G_1 and G_2, where $|V(G_1)| = p_1$, $|V(G_2)| = p_2$ and $|E(G_1)| = q_1$, $|E(G_2)| = q_2$ is the graph obtained by taking $|V(G_1)|$ copies of G_2 and joining each vertex of the i-th copy with vertex $u \in V(G_1)$. Obviously, $|V(G_1 \circ G_2)| = p_1(p_2+1)$ and $|E(G_1 \circ G_2)| = q_1 + p_1(q_2+p_2)$. It follows from the definition of the corona product $G_1 \circ G_2$, the degree of each vertex $u \in G_1 \circ G_2$ is given by

$$deg_{G_1 \circ G_2}(u) = \begin{cases} deg_{G_1}(u) + p_2, & \text{if } u \in V(G_1); \\ deg_{G_2}(u) + 1, & \text{if } u \in V(G_2). \end{cases}$$

We therefore know the next lemma.

Lemma 3.15 *Let $G = G_1 \circ G_2$ be a connected graph and let $u \in V(G)$. Then*

$$deg_G^{De}(u) = \begin{cases} p_2 deg_{G_1}^{De}(u), & u \in V(G_1); \\ p_2 deg_{G_1}^{De}(v), & u \in V(G) - V(G_1), \text{ where } v \in V(G_1) \text{ is adjacent to } u. \end{cases}$$

Theorem 3.16 *Let $G = G_1 \circ G_2$ be a connected graph. Then*

(*i*) $M_1^{De}(G) = p_2^2(p_2+1)M_1^{De}(G_1)$;
(*ii*) $M_2^{De}(G) = p_2^2 M_2^{De}(G_1) + p_2^2(q_2+p_2)M_1^{De}(G_1)$.

Proof By definition, calculation shows that

$$\begin{aligned}
(i)\ M_1^{De}(G) &= \sum_{u \in V(G)} \left[deg_G^{De}(u)\right]^2 \\
&= \sum_{u \in V(G_1)} \left[deg_G^{De}(u)\right]^2 + \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} \left[deg_G^{De}(u)\right]^2 \\
&= \sum_{u \in V(G_1)} \left[p_2 deg_{G_1}^{De}(u)\right]^2 + \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} \left[p_2 deg_{G_1}^{De}(v)\right]^2 \\
&= p_2^2 M_1^{De}(G_1) + p_2^3 M_1^{De}(G_1).
\end{aligned}$$

$$\begin{aligned}
(ii)\ M_2^{De}(G) &= \frac{1}{2} \sum_{u \in V(G)} deg^{De}(u) \sum_{v \in N(u)} deg^{De}(v) \\
&= \frac{1}{2} \sum_{u \in V(G_1)} deg_G^{De}(u) \left[\sum_{v \in N_{G_1}(u)} deg_G^{De}(v) + \sum_{v \in V(G_2)} deg_G^{De}(v)\right] \\
&\quad + \frac{1}{2} \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} deg_G^{De}(u) \left[\sum_{w \in N_{G_2}(u)} deg_G^{De}(w) + deg_G^{De}(v)\right] \\
&= \frac{1}{2} \sum_{u \in V(G_1)} p_2 deg_{G_1}^{De}(u) \left[\sum_{v \in N_{G_1}(u)} p_2 deg_{G_1}^{De}(v) + p_2^2 deg_{G_1}^{De}(u)\right] \\
&\quad + \frac{1}{2} \sum_{v \in V(G_1)} \sum_{u \in V(G_2)} p_2 deg_{G_1}^{De}(v) \left[p_2 deg_{G_1}^{De}(v) deg_{G_2}(u) + p_2 deg_{G_1}^{De}(v)\right] \\
&= p_2^2 M_2^{De}(G_1) + p_2^2(q_2+p_2)M_1^{De}(G_1).
\end{aligned}$$

This completes the proof. □

Example 3.17 For any cycle C_{p_1} and any path P_{p_2},

$$(i)\ \ M_1^{De}(C_{p_1} \circ P_{p_2}) = \begin{cases} 4p_1p_2^2(p_2+1), & p_1 \text{ is odd;} \\ p_1p_2^2(p_2+1), & p_1 \text{ is even.} \end{cases}$$

$$(ii)\ M_2^{De}(C_{p_1} \circ P_{p_2}) = \begin{cases} 8p_1p_2^3, & p_1 \text{ is odd;} \\ 2p_1p_2^3, & p_1 \text{ is even.} \end{cases}$$

Example 3.18 For any two cycles C_{p_1} and C_{p_2},

$$(i)\ M_1^{De}(C_{p_1} \circ C_{p_2}) = \begin{cases} 4p_1p_2^2(p_2+1), & p_1 \text{ is odd;} \\ p_1p_2^2(p_2+1), & p_1 \text{ is even.} \end{cases}$$

$$(ii)\ M_2^{De}(C_{p_1} \circ C_{p_2}) = \begin{cases} 4p_1p_2^2(2p_2+1), & p_1 \text{ is odd;} \\ p_1p_2^2(2p_2+1), & p_1 \text{ is even.} \end{cases}$$

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Clique-to-Clique Monophonic Distance in Graphs

I. Keerthi Asir and S. Athisayanathan

(Department of Mathematics, St. Xavier's College (Autonomous), Palayamkottai - 627002, Tamil Nadu, India.)

E-mail: asirsxc@gmail.com, athisxc@gmail.com

Abstract: In this paper we introduce the clique-to-clique $C - C'$ monophonic path, the clique-to-clique monophonic distance $d_m(C, C')$, the clique-to-clique $C - C'$ monophonic, the clique-to-clique monophonic eccentricity $e_{m_3}(C)$, the clique-to-clique monophonic radius R_{m_3}, and the clique-to-clique monophonic diameter D_{m_3} of a connected graph G, where C and C' are any two cliques in G. These parameters are determined for some standard graphs. It is shown that $R_{m_3} \leq D_{m_3}$ for every connected graph G and that every two positive integers a and b with $2 \leq a \leq b$ are realizable as the clique-to-clique monophonic radius and the clique-to-clique monophonic diameter, respectively, of some connected graph. Further it is shown that for any three positive integers a, b, c with $3 \leq a \leq b \leq c$ are realizable as the clique-to-clique radius, the clique-to-clique monophonic radius, and the clique-to-clique detour radius, respectively, of some connected graph and also it is shown that for any three positive integers a, b, c with $4 \leq a \leq b \leq c$ are realizable as the clique-to-clique diameter, the clique-to-clique monophonic diameter, and the clique-to-clique detour diameter, respectively, of some connected graph. The clique-to-clique monophonic center $C_{m_3}(G)$ and the clique-to-clique monophonic periphery $P_{m_3}(G)$ are introduced. It is shown that the clique-to-clique monophonic center a connected graph does not lie in a single block of G.

Key Words: Clique-to-clique distance, clique-to-clique detour distance, clique-to-clique monophonic distance.

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§1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected simple graph. For basic graph theoretic terminologies, we refer to Chartrand and Zhang [2]. If $X \subseteq V$, then $\langle X \rangle$ is the subgraph induced by X. A clique C of a graph G is a maximal complete subgraph and we denote it by its vertices. A $u - v$ path P beginning with u and ending with v in G is a sequence of distinct vertices such that consecutive vertices in the sequence are adjacent in G. A chord of a path $u_1, u_2, ..., u_n$ in G is an edge u_iu_j with $j \geq i + 2$. For a graph G, the length of a path is the number of edges on the path. In 1964, Hakimi [3] considered the facility location problems as vertex-to-vertex distance in graphs. For any two vertices u and v in a connected graph G, the

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distance $d(u,v)$ is the length of a shortest $u-v$ path in G. For a vertex v in G, the eccentricity of v is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is its radius and the maximum eccentricity is its diameter, denoted by $rad(G)$ and $diam(G)$ respectively. A vertex v in G is a central vertex if $e(v) = rad(G)$ and the subgraph induced by the central vertices of G is the center $Cen(G)$ of G. A vertex v in G is a peripheral vertex if $e(v) = diam(G)$ and the subgraph induced by the peripheral vertices of G is the periphery $Per(G)$ of G. If every vertex of a graph is central vertex then G is called self-centered.

In 2005, Chartrand et. al. [1] introduced and studied the concepts of detour distance in graphs. For any two vertices u and v in a connected graph G, the detour distance $D(u,v)$ is the length of a longest $u-v$ path in G. For a vertex v in G, the detour eccentricity of v is the detour distance between v and a vertex farthest from v in G. The minimum detour eccentricity among the vertices of G is its detour radius and the maximum detour eccentricity is its detour diameter, denoted by $rad_D(G)$ and $diam_D(G)$ respectively. Detour center, detour self-centered and detour periphery of a graph are defined similarly to the center, self-centered and periphery of a graph respectively.

In 2011, Santhakumaran and Titus [7] introduced and studied the concepts of monophonic distance in graphs. For any two vertices u and v in G, a $u-v$ path P is a $u-v$ monophonic path if P contains no chords. The monophonic distance $d_m(u,v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. For a vertex v in G, the monophonic eccentricity of v is the monophonic distance between v and a vertex farthest from v in G. The minimum monophonic eccentricity among the vertices of G is its monophonic radius and the maximum monophonic eccentricity is its monophonic diameter, denoted by $rad_m(G)$ and $diam_m(G)$ respectively. Monophonic center, monophonic self-centered and monophonic periphery of a graph are defined similar to the center and periphery respectively of a graph.

In 2002, Santhakumaran and Arumugam [6] introduced the facility locational problem as clique-to-clique distance $d(C,C')$ in graphs as follows. Let ζ be the set of all cliques in a connected graph G the clique-to-clique distance is defined by $d(C,C') = \min\{d(u,v) : u \in C, v \in C'\}$. For our convenience a $C-C'$ path of length $d(C,C')$ is called a clique-to-clique $C-C'$ geodesic or simply $C-C'$ geodesic. The clique-to-clique eccentricity $e_3(C)$ of a clique C in G is the maximum clique-to-clique distance from C to a clique $C' \in \zeta$ in G. The minimum clique-to-clique eccentricity among the cliques of G is its clique-to-clique radius and the maximum clique-to-clique eccentricity is its clique-to-clique diameter, denoted by r_3 and d_3 respectively. A clique C in G is called a clique-to-clique central clique if $e_3(C) = r_3$ and the subgraph induced by the clique-to-clique central cliques of G are clique-to-clique center of G. A clique C in G is called a clique-to-clique peripheral clique if $e_3(C) = d_3$ and the subgraph induced by the clique-to-clique peripheral cliques of G are clique-to-clique periphery of G. If every clique of G is clique-to-clique central clique then G is called clique-to-clique self-centered.

In 2015, Keerthi Asir and Athisayanathan [4] introduced and studied the concepts of clique-to-clique detour distance $D(C,C')$ in graphs as follows. Let ζ be the set of all cliques in a connected graph G and $C, C' \in \zeta$ in G. A clique-to-clique $C-C'$ path P is a $u-v$ path, where $u \in C$ and $v \in C'$, in which P contains no vertices of C and C' other than u and v and the clique-to-clique detour distance, $D(C, C')$ is the length of a longest $C - C'$ path in G. A $C - C'$ path of length $D(C, C')$ is called a $C - C'$ detour. The clique-to-clique detour eccentricity of a clique C in G is the maximum clique-to-clique detour distance from C to a clique $C' \in \zeta$ in G. The minimum clique-to-clique detour eccentricity among the cliques of G is its clique-to-clique detour radius and the maximum clique-to-clique detour eccentricity is its clique-to-clique detour diameter, denoted by R_3 and D_3 respectively. The clique-to-clique detour center $C_{D3}(G)$, the clique-to-clique detour self-centered, the clique-to-clique detour periphery $P_{D3}(G)$ are defined similar to the clique-to-clique center. the clique-to-clique self-centered and the clique-to-clique periphery of a graph respectively.

These motivated us to introduce the concepts of clique-to-clique monophonic distance in graphs and investigate certain results related to clique-to-clique monophonic distance and other distances in graphs. These ideas have intresting applications in channel assignment problem in radio technologies and capture different aspects of certain molecular problems in theoretical chemistry. Also there are useful applications of these concepts to security based communication network design. In a social network a clique represents a group of individuals having a common interest. Thus the clique-to-clique monophonic centrality have intresting application in social networks. Throughout this paper, G denotes a connected graph with at least two vertices.

§2. Clique-to-Clique Monophonic Distance

Definition 2.1 *Let ζ be the set of all cliques in a connected graph G and $C, C' \in \zeta$. A clique-to-clique $C - C'$ path P is said to be a clique-to-clique $C - C'$ monophonic path if P contains no chords in G. The clique-to-clique monophonic distance $d_m(C, C')$ is the length of a longest $C - C'$ monophonic path in G. A $C - C'$ monophonic path of length $d_m(C, C')$ is called a clique-to-clique $C - C'$ monophonic or simply $C - C'$ monophonic.*

Example 2.2 Consider the graph G given in Fig 2.1. For the cliques $C = \{u, w\}$ and $C' = \{v, z\}$ in G, the $C - C'$ paths are $P_1 : u, v$, $P_2 : w, x, z$ and $P_3 : w, x, y, z$. Now P_1 and P_2 are $C - C'$ monophonic paths, while P_3 is not so. Also the clique-to-clique distance $d(C, C') = 1$, the clique-to-clique monophonic distance $d_m(C, C') = 2$, and the clique-to-clique detour distance $D(C, C') = 3$. Thus the clique-to-clique monophonic distance is different from both the clique-to-clique distance and the clique-to-clique detour distance. Now it is clear that P_1 is a $C - C'$ geodesic, P_2 is a $C - C'$ monophonic, and P_3 is a $C - C'$ detour.

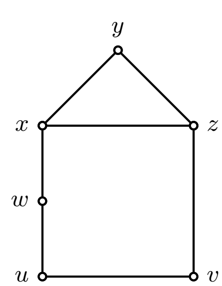

Fig.2.1

Keerthi Asir and Athisayanathan [4] showed that for any two cliques C and C' in a non-trivial connected graph G of order n, $0 \leq d(C,C') \leq D(C,C') \leq n-2$. Now we have the following theorem.

Theorem 2.3 *For any two cliques C and C' in a non-trivial connected graph G of order n, $0 \leq d(C,C') \leq d_m(C,C') \leq D(C,C') \leq n-2$.*

Proof By definition $d(C,C') \leq d_m(C,C')$. If P is a unique $C-C'$ path in G, then $d(C,C') = d_m(C,C') = D(C,C')$. Suppose that G contains more than one $C-C'$ path. Let Q be a longest $C-C'$ path in G.

Case 1. If Q does not contain a chord, then $d_m(C,C') = D(C,C')$.

Case 2. If Q contains a chord, then $d_m(C,C') < D(C,C')$. □

Remark 2.4 The bounds in Theorem 2.3 are sharp. If $G = K_2$, then $0 = d(C,C) = d_m(C,C) = D(C,C) = n-2$. Also if G is a tree, then $d(C,C') = d_m(C,C') = D(C,C')$ for every cliques C and C' in G and the graph G given in Fig. 2.1, $0 < d(C,C') < d_m(C,C') < D(C,C') < n-2$.

Theorem 2.5 *Let C and C' be any two adjacent cliques ($C \neq C'$) in a connected graph G. Then $d_m(C,C') = n-2$ if and only if G is a cycle $C_n (n > 3)$.*

Proof Assume that G is cycle $C_n : u_1, u_2, \cdots, u_{n-1}, u_n, u_1 (n \geq 4)$. Since any edge in G is a clique, without loss of generality we assume that $C = \{u_1, u_2\}$, $C' = \{u_n, u_1\}$ be any two adjacent cliques. Then there exists two distinct $C-C'$ paths, say P_1 and P_2 such that $P_1 : u_1$ is a trivial $C-C'$ path of length 0 and $P_2 : u_2, u_3, \cdots, u_{n-1}, u_n$ is $C-C'$ monophonic path of length $n-2$. It is clear that $d_m(C,C') = n-2$. Conversely assume that for any two distinct adjacent cliques C and C' in a connected graph G, $d_m(C,C') = n-2$. We prove that G is a cycle. Suppose that G is not a cycle. Then G must be either a tree or a cyclic graph.

Case 1. If G is a tree, then $C-C'$ path is trivial. So that $d_m(C,C') = 0 < n-2$, which is a contradiction.

Case 2. If G is a cyclic graph, then G must contain a cycle $C_d : x_1, x_2, \cdots, x_d, x_1$ of length $d < n$. If $C = \{x_1, x_2\}$ and $C' = \{x_n, x_1\}$ then $d_m(C,C') < n-2$, which is a contradiction. □

Since the length of a clique-to-clique monophonic path between any two cliques in $K_{n,m}$ is 2, we have the following theorem.

Theorem 2.6 *Let $K_{n,m} (n \leq m)$ be a complete bipartite graph with the partition V_1, V_2 of $V(K_{n,m})$ such that $|V_1| = n$ and $|V_2| = m$. Let C and C' be any two cliques in $K_{n,m}$, then $d_m(C,C') = 2$.*

Since every tree has unique clique-to-clique monophonic path, we have the following theorem.

Theorem 2.7 *If G is a tree, then $d(C,C') = d_m(C,C') = D(C,C')$ for every cliques C and C' in G.*

The converse of the Theorem 2.7 is not true. For the graph G obtained from a complete bipartite graph $K_{2,n}(n \geq 2)$ by joining the vertices of degree n by an edge. In such a graph every clique C is isomorphic to K_3 and so for any two cliques C and C', $d(C, C') = d_m(C, C') = D(C, C') = 0$, but G is not tree.

§3. Clique-to-Clique Monophonic Center

Definition 3.1 *Let G be a connected graph and let ζ be the set of all cliques in G. The clique-to-clique monophonic eccentricity $e_{m_3}(C)$ of a clique C in G is defined by $e_{m_3}(C) = \max\{d_m(C, C') : C' \in \zeta\}$. A clique C' for which $e_{m_3}(C) = d_m(C, C')$ is called a clique-to-clique monophonic eccentric clique of C. The clique-to-clique monophonic radius of G is defined as, $R_{m_3} = rad_{m_3}(G) = \min\{e_{m_3}(C) : C \in \zeta\}$ and the clique-to-clique monophonic diameter of G is defined as, $D_{m_3} = diam_{m_3}(G) = \max\{e_{m_3}(C) : C \in \zeta\}$. A clique C in G is called a clique-to-clique monophonic central clique if $e_{m_3}(C) = R_{m_3}$ and the clique-to-clique monophonic center of G is defined as, $C_{m_3}(G) = Cen_{m_3}(G) = \langle\{C \in \zeta : e_{m_3}(C) = R_{m_3}\}\rangle$. A clique C in G is called a clique-to-clique monophonic peripheral clique if $e_{m_3}(C) = D_{m_3}$ and the clique-to-clique monophonic periphery of G is defined as, $P_{m_3}(G) = Per_{m_3}(G) = \langle\{C \in \zeta : e_{m_3}(C) = D_{m_3}\}\rangle$. If every clique of G is a clique-to-clique monophonic central clique, then G is called a clique-to-clique monophonic self centered graph.*

Example 3.2 For the graph G given in Fig.3.1, the set of all cliques are given by, $\zeta = \{C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9\}$ where $C_1 = \{v_1, v_2, v_3\}$, $C_2 = \{v_3, v_4\}$, $C_3 = \{v_4, v_5, v_6\}$, $C_4 = \{v_6, v_7\}$, $C_5 = \{v_7, v_8\}$, $C_6 = \{v_8, v_{10}\}$, $C_7 = \{v_9, v_{10}\}$, $C_8 = \{v_4, v_9\}$, $C_9 = \{v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$.

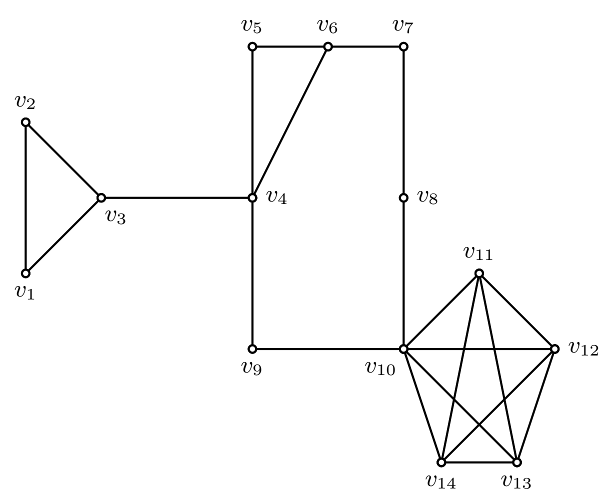

Fig.3.1

The clique-to-clique eccentricity $e_3(C)$, the clique-to-clique detour eccentricity $e_{D3}(C)$, the clique-to-clique monophonic eccentricity $e_{m_3}(C)$ of all the cliques of G are given in Table 1.

Cliques C	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
$e_3(C)$	3	2	2	2	3	3	2	2	3
$e_{m_3}(C)$	5	4	4	5	4	4	5	4	5
$e_{D3}(C)$	6	5	4	5	5	5	6	5	6

Table 1

The clique-to-clique monophonic eccentric clique of all the cliques of G are given in Table 2.

Cliques C	Clique-to-Clique Monophonic Eccentric Cliques
C_1	C_4, C_5, C_6, C_7, C_9
C_7	C_1, C_2, C_3, C_8
C_9	C_1, C_2, C_3, C_8

Table 2

The clique-to-clique radius $r_3 = 2$, the clique-to-clique diameter $d_3 = 3$, the clique-to-clique detour radius $R_3 = 4$, the clique-to-clique detour diameter $D_3 = 6$, the clique-to-clique monophonic radius $R_{m_3} = 4$ and the clique-to-clique monophonic diameter $D_{m_3} = 5$. Also it is clear that the clique-to-clique center $C_3(G) = \langle\{C_2, C_3, C_4, C_7, C_8\}\rangle$, the clique-to-clique periphery $P_3(G) = \langle\{C_1, C_5, C_6, C_9\}\rangle$, the clique-to-clique detour center $C_{D3}(G) = \langle\{C_3\}\rangle$, the clique-to-clique detour periphery $P_{D3}(G) = \langle\{C_1, C_7, C_9\}\rangle$, the clique-to-clique monophonic center $C_{m_3}(G) = \langle\{C_2, C_3, C_5, C_6, C_8\}\rangle$, the clique-to-clique monophonic periphery $P_{m_3}(G) = \langle\{C_1, C_4, C_7, C_9\}\rangle$.

The clique-to-clique monophonic radius R_{m_3} and the clique-to-clique monophonic diameter D_{m_3} of some standard graphs are given in Table 3.

Graph G	K_n	$P_n(n \geq 3)$	$C_n(n \geq 4)$	$W_n(n \geq 5)$	$K_{n,m}(m \geq n)$
R_{m_3}	0	$\lfloor \frac{n-3}{2} \rfloor$	$n-2$	$n-3$	2
D_{m_3}	0	$n-3$	$n-2$	$n-3$	2

Table 3

Remark 3.3 The complete graph K_n, the cycle C_n, the wheel W_n and the complete bipartite graph $K_{n,m}$ are the clique-to-clique monophonic self centered graphs.

Remark 3.4 In a connected graph G, $C_3(G)$, $C_{D3}(G)$, $C_{m_3}(G)$ and $P_3(G)$, $P_{D3}(G)$, $P_{m_3}(G)$ need not be same. For the graph G given in Fig 3.1, it is shown that $C_3(G)$, $C_{D3}(G)$, $C_{m_3}(G)$ and $P_3(G)$, $P_{D3}(G)$, $P_{m_3}(G)$ are distinct.

Theorem 3.5 *Let G be a connected graph of order n. Then*

(i) $0 \leq e_3(C) \leq e_{m_3}(C) \leq e_{D3}(C) \leq n-2$ *for every clique C in G;*
(ii) $0 \leq r_3 \leq R_{m_3} \leq R_3 \leq n-2$;
(iii) $0 \leq d_3 \leq D_{m_3} \leq D_3 \leq n-2$.

Proof This follows from Theorem 2.3. □

Remark 3.6 The bounds in Theorem 3.5(i) are sharp. If $G = K_2$, then $0 = e_3(C) = e_{m_3}(C) = e_{D3}(C) = n-2$. Also if G is a tree, then $e_3(C) = e_{m_3}(C) = e_{D3}(C)$ for every clique C in G and the graph G given in Fig. 2.1, $e_3(C) < e_{m_3}(C) < e_{D3}(C)$, where $C = \{u, w\}$.

In [1, 2] it is shown that in a connected graph G, the radius and diameter are related by $rad(G) \leq diam(G) \leq 2rad(G)$, the detour radius and detour diameter are related by $rad_D(G) \leq diam_D(G) \leq 2rad_D(G)$, and Santhakumaran et. al. [7]showed that the monophonic radius and monophonic diameter are related by $rad_m(G) \leq diam_m(G)$. Also Santhakumaran et. al. [6] showed that the clique-to-clique radius and clique-to-clique diameter are related by $r_3 \leq d_3 \leq 2r_3+1$ and Keerthi Asir et. al. [4] showed that the upper inequality does not hold for the clique-to-clique detour distance. The following example shows that the similar inequality does not hold for the clique-to-clique monophonic distance.

Remark 3.7 For the graph G of order $n \geq 7$ obtained by identifying the central vertex of the wheel $W_{n-1} = K_1 + C_{n-2}$ and an end vertex of the path P_2. It is easy to verify that $D_{m_3} > 2R_{m_3}$ and $D_{m_3} > 2R_{m_3} + 1$.

Ostrand [5] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the radius and diameter respectively of some connected graph, Chartrand et. al. [1] showed that every two positive integers a and b with $a \leq b \leq 2a$ are realizable as the detour radius and detour diameter respectively of some connected graph, and Santhakumaran et. al. [7] showed that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter respectively of some connected graph. Also Santhakumaran et. al. [6] showed that every two positive integers a and b with $a \leq b \leq 2a+1$ are realizable as the clique-to-clique radius and clique-to-clique diameter respectively of some connected graph. Keerthi Asir et. al. [4] showed that every two positive integers a and b with $2 \leq a \leq b$ are realizable as the clique-to-clique detour radius and clique-to-clique detour diameter respectively of some connected graph. Now we have a realization theorem for the clique-to-clique monophonic radius and the clique-to-clique monophonic diameter for some connected graph.

Theorem 3.8 *For each pair a, b of positive integers with $2 \leq a \leq b$, there exists a connected graph G with $R_{m_3} = a$ and $D_{m_3} = b$.*

Proof Our proof is divided into cases following.

Case 1. $a = b$.

Let $G = C_{a+2} : u_1, u_2, \cdots, u_{a+2}, u_1$ be a cycle of order $a+2$. Then $e_{m_3}(u_iu_{i+1}) = a$ for $1 \leq i \leq a+2$. It is easy to verify that every clique S in G with $e_{m_3}(S) = a$. Thus $R_{m_3} = a$

and $D_{m_3} = b$ as $a = b$.

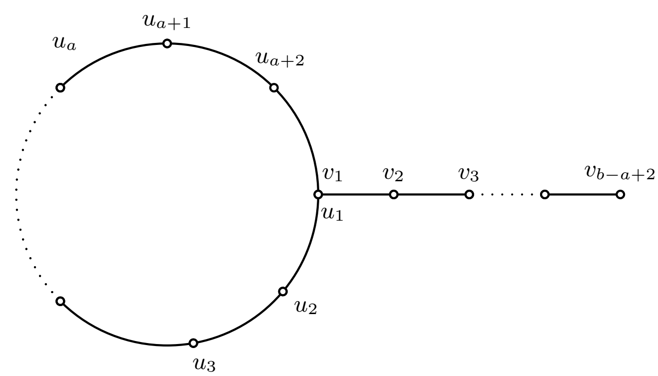

Fig. 3.2

Case 2. $2 \leq a < b \leq 2a$.

Let $C_{a+2} : u_1, u_2, \cdots, u_{a+2}, u_1$ be a cycle of order $a+2$ and $P_{b-a+2} : v_1, v_2, \cdots, v_{b-a+2}$ be a path of order $b-a+2$. We construct the graph G of order $b+3$ by identifying the vertex u_1 of C_{a+2} and v_1 of P_{b-a+2} as shown in Fig. 3.2. It is easy to verify that

$$e_{m_3}(u_i u_{i+1}) = \begin{cases} b-i+2, & \text{if } \ 2 \leq i \leq \left\lceil \frac{a+2}{2} \right\rceil \\ b-a+i-1, & \text{if } \ \left\lceil \frac{a+2}{2} \right\rceil < i \leq a+1, \end{cases}$$

and $e_{m_3}(v_i v_{i+1}) = a+i-1$ if $2 \leq i \leq b-a+1$, $e_{m_3}(u_2u_3) = e_{m_3}(u_{a+1}u_{a+2}) = e_{m_3}(u_{b-a}u_{b-a+1}) = b$, $e_{m_3}(u_1u_2) = e_{m_3}(u_1u_{a+2}) = e_{m_3}(v_1v_2) = a$. It is easy to verify that there is no clique S in G with $e_{m_3}(S) < a$ and there is no clique S' in G with $e_{m_3}(S') > b$. Thus $R_{m_3} = a$ and $D_{m_3} = b$ as $a < b$.

Case 3. $a < b > 2a$.

Let G be a graph of order $b + 2a + 4$ obtained by identifying the central vertex of the wheel $W_{b+3} = K_1 + C_{b+2}$ and an end vertex of the path P_{2a+2}, where $K_1 : v_1$, $C_{b+1} : u_1, u_2, \cdots, u_{b+2}, u_1$ and $P_{2a+2} : v_1, v_2, \cdots, v_{2a+2}$. The resulting graph G is shown in Fig.3.3.

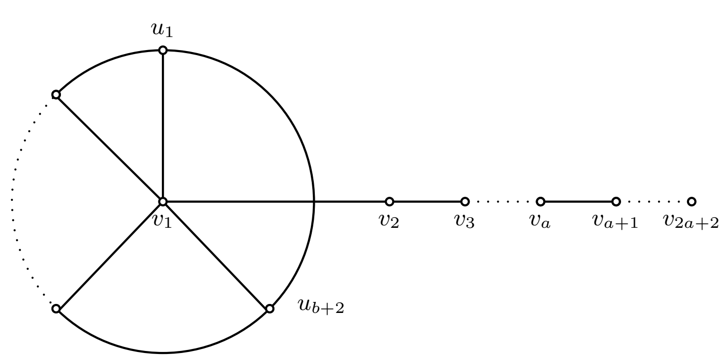

Fig. 3.3

It is easy to verify that $e_{m_3}(v_1u_iu_{i+1}) = b$ if $1 \le i \le b+2$ and

$$e_{m_3}(v_iv_{i+1}) = \begin{cases} 2a - i, & \text{if } 1 \le i \le a, \\ i - 1, & \text{if } a < i < 2a + 2. \end{cases}$$

It is also easy to verify that there is no clique S in G with $e_{m_3}(S) < a$ and there is no clique S' in G with $e_{m_3}(S') > b$. Thus $R_{m_3} = a$ and $D_{m_3} = b$ as $b > 2a$. □

Santhakumaran et. al. [7] showed that every three positive integers a, b and c with $3 \le a \le b \le c$ are realizable as the radius, monophonic radius and detour radius respectively of some connected graph. Now we have a realization theorem for the clique-to-clique radius, clique-to-clique monophonic radius and clique-to-clique detour radius respectively of some connected graph.

Theorem 3.9 *For any three positive integers a, b, c with $3 \le a \le b \le c$, there exists a connected graph G such that $r_3 = a$, $R_{m_3} = b$, $R_3 = c$.*

Proof The proof is divided into cases following.

Case 1. $a = b = c$.

Let $P_1 : u_1, u_2, \cdots, u_{a+2}$ and $P_2 : v_1, v_2, \ldots, v_{a+2}$ be two paths of order $a+2$. We construct the graph G of order $2a+4$ by joining u_1 in P_1 and v_1 in P_2 by an edge. It is easy to verify that $e_3(u_1v_1) = e_{m_3}(u_1v_1) = e_{D3}(u_1v_1) = a$, $e_3(u_iu_{i+1}) = e_{m_3}(u_iu_{i+1}) = e_{D3}(u_iu_{i+1}) = a+i$ if $1 \le i \le a+1$.

It is also easy to verify that there is no clique S in G with $e_3(S) < a, e_{m_3}(S) < b$ and $e_{D3}(S) < c$. Thus $r_3 = a$, $R_{m_3} = b$ and $R_3 = c$ as $a = b = c$.

Case 2. $3 \le a \le b < c$.

Let $P_1 : u_1, u_2, \cdots, u_{a+2}$ and $P_2 : v_1, v_2, \cdots, v_{a+2}$ be two paths of order $a + 2$. Let $Q_1 : w_1, w_2, \ldots, w_{b-a+3}$ and $Q_2 : z_1, z_2, \cdots, z_{b-a+3}$ be two paths of order $b - a + 3$. Let $K_1 : x_1, x_2, \cdots, x_{c-b+1}$ and $K_2 : y_1, y_2, \cdots, y_{c-b+1}$ be two complete graphs of order $c - b + 1$. We construct the graph G of order $2c + 4$ as follows: (i) identify the vertices u_1 in P_1 with w_1 in Q_1 and also identify the vertices v_1 in P_2 with z_1 in Q_2; (ii) identify the vertices u_3 in P_1 with w_{b-a+3} in Q_1 and also identify the vertices z_{b-a+3} in Q_2 with v_3 in P_2; (iii) identify the vertices u_{a+1} in P_1 with x_1 in K_1 and also identify the vertices x_{c-b+1} in K_1 with u_a in P_1; (iv) identify the vertices v_{a+1} in P_2 with y_1 in K_2 and also identify the vertices y_{c-b+1} in K_2 with v_a in P_2; (v) join each vertex $w_i (2 \le i \le b - a + 2)$ in Q_1 with u_2 in P_1 and join each vertex $z_i (2 \le i \le b - a + 2)$ in Q_2 with v_2 in P_2 (vi) join u_1 in P_1 with v_1 in P_2. The resulting graph G is shown in Fig.3.4.

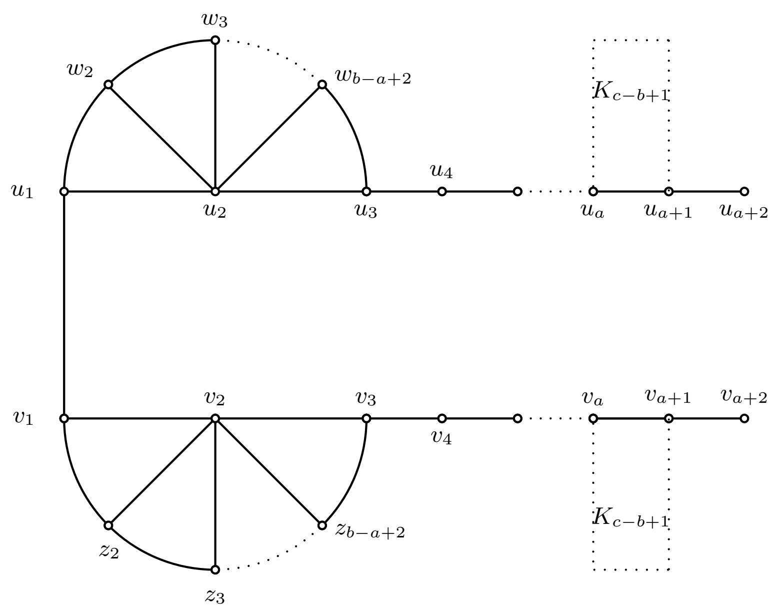

Fig.3.4

It is easy to verify that $e_3(u_1v_1) = a$,

$$e_3(u_2w_iw_{i+1}) = \begin{cases} a+1, & \text{if } i = 1, \\ a+2, & \text{if } 2 \leq i \leq b-a+2, \end{cases}$$

$$e_3(v_2z_iz_{i+1}) = \begin{cases} a+1, & \text{if } i = 1, \\ a+2, & \text{if } 2 \leq i \leq b-a+2, \end{cases}$$

$$e_3(u_iu_{i+1}) = \begin{cases} a+i, & \text{if } 3 \leq i < a, \\ 2a+1, & \text{if } i = a+1, \end{cases}$$

$$e_3(v_iv_{i+1}) = \begin{cases} a+i, & \text{if } 3 \leq i < a, \\ 2a+1, & \text{if } i = a+1, \end{cases}$$

$$e_3(K_1) = 2a, \quad e_3(K_2) = 2a, \quad e_{m_3}(u_1v_1) = b,$$

and $e_{m_3}(u_2w_iw_{i+1}) = b+i$, if $1 \leq i \leq b-a+2$, $e_{m_3}(v_2z_iz_{i+1}) = b+i$, if $1 \leq i \leq b-a+2$,

$$e_{m_3}(u_iu_{i+1}) = \begin{cases} 2b-a+i, & \text{if } 3 \leq i < a, \\ 2b+1, & \text{if } i = a+1, \end{cases}$$

$$e_{m_3}(v_iv_{i+1}) = \begin{cases} 2b-a+i, & \text{if } 3 \leq i < a, \\ 2b+1, & \text{if } i = a+1, \end{cases}$$

$$e_{m_3}(K_1) = 2b, \quad e_{m_3}(K_2) = 2b, \quad e_{D3}(u_1v_1) = c,$$

$e_{D3}(u_2w_iw_{i+1}) = c+i$, if $1 \le i \le b-a+2$, $e_{D3}(v_2z_iz_{i+1}) = c+i$, if $1 \le i \le b-a+2$,

$$e_{D3}(u_iu_{i+1}) = \begin{cases} c+b-a+1+i, & \text{if } 3 \le i < a, \\ c+b+2, & \text{if } i = a+1, \end{cases}$$

$$e_{D3}(v_iv_{i+1}) = \begin{cases} c+b-a+1+i, & \text{if } 3 \le i < a, \\ c+b+2, & \text{if } i = a+1, \end{cases}$$

$$e_{D3}(K_1) = c+b+1, \qquad e_{D3}(K_2) = c+b+1.$$

It is also easy to verify that there is no clique S in G with $e_3(S) < a$, $e_{m_3}(S) < b$ and $e_{D3}(S) < c$. Thus $r_3 = a$, $R_{m_3} = b$ and $R_3 = c$ as $a \le b < c$.

Case 3. $3 \le a < b = c$.

Let $P_1 : u_1, u_2, \cdots, u_a, u_{a+2}$ and $P_2 : v_1, v_2, \cdots, v_a, v_{a+2}$ be two paths of order $a+2$. Let $Q_1 : w_1, w_2, \cdots, w_{b-a+3}$ and $Q_2 : z_1, z_2, \cdots, z_{b-a+3}$ be two paths of order $b-a+3$. Let $E_i : x_i(3 \le i \le b-a+2)$ and $F_i : y_i(3 \le i \le b-a+2)$ be $2(b-a)$ copies of K_1. We construct the graph G of order $4b-2a+6$ as follows: (i) identify the vertices u_1 in P_1 with w_1 in Q_1 and also identify the vertices v_1 in P_2 with z_1 in Q_2; (ii) identify the vertices u_3 in P_1 with w_{b-a+3} in Q_1 and also identify the vertices z_{b-a+3} in Q_2 with v_3 in P_2 (iii) join each vertex $x_i(3 \le i \le b-a+2)$ with $w_i(3 \le i \le b-a+2)$ and u_1 and also join each vertex $y_i(3 \le i \le b-a+2)$ with $z_i(3 \le i \le b-a+2)$ and v_1 (iv) join u_1 in P_1 with v_1 in P_2. The resulting graph G is shown in Fig. 3.5.

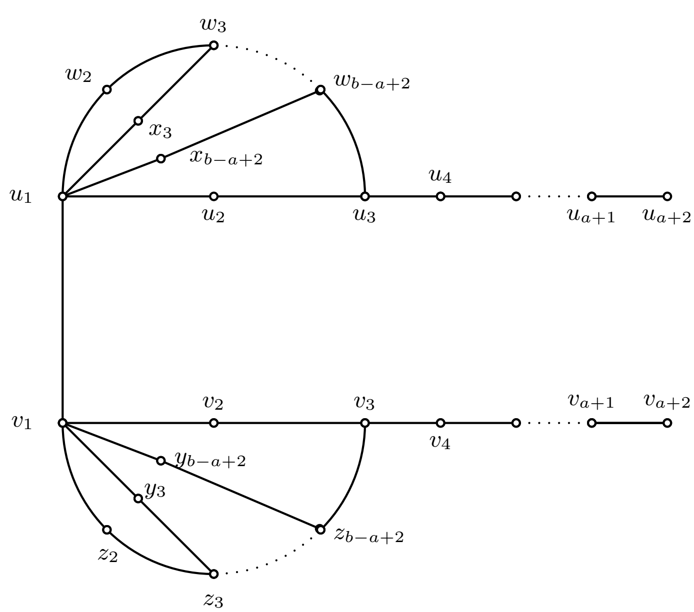

Fig.3.5

It is easy to verify that $e_3(u_1v_1) = a$,

$$e_3(w_iw_{i+1}) = \begin{cases} a+1, & \text{if } i = 1, \\ a+2, & \text{if } i = 2, \\ a+3, & \text{if } 3 \leq i \leq b-a+2, \end{cases}$$

and $e_3(u_iu_{i+1}) = a+i$, if $1 \leq i \leq a+1$, $e_3(u_1x_i) = a+1$, if $3 \leq i \leq b-a+2$, $e_3(w_ix_i) = a+2$, if $3 \leq i \leq b-a+2$, $e_{m_3}(u_1v_1) = b$,

$$e_{m_3}(w_iw_{i+1}) = \begin{cases} b+1, & \text{if } i = 1 \\ 2b-a+5-i, & \text{if } 2 \leq i \leq \left\lfloor \frac{b-a+5}{2} \right\rfloor \\ b+i, & \text{if } \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \leq b-a+2 \end{cases}$$

$$e_{m_3}(u_iu_{i+1}) = \begin{cases} b+1, & \text{if } i = 1, \\ 2b-a+3, & \text{if } i = 2, \\ 2b-a+i, & \text{if } 3 \leq i \leq a+1, \end{cases}$$

and $e_{m_3}(u_1x_i) = b+1$, if $3 \leq i \leq b-a+2$,

$$e_{m_3}(w_ix_i) = \begin{cases} 2b-a+6-i, & \text{if } 3 \leq i \leq \left\lfloor \frac{b-a+5}{2} \right\rfloor, \\ b+i, & \text{if } \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \leq b-a-2, \end{cases}$$

and $e_{D3}(u_1v_1) = c$,

$$e_{D3}(w_iw_{i+1}) = \begin{cases} c+1, & \text{if } i = 1, \\ 2c-a+5-i, & \text{if } 2 \leq i \leq \left\lfloor \frac{b-a+5}{2} \right\rfloor, \\ c+i, & \text{if } \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \leq b-a+2, \end{cases}$$

$$e_{D3}(u_iu_{i+1}) = \begin{cases} c+1, & \text{if } i = 1, \\ 2c-a+3, & \text{if } i = 2, \\ 2c-a+i, & \text{if } 3 \leq i \leq a+1, \end{cases}$$

and $e_{D3}(u_1x_i) = c+1$, if $3 \leq i \leq b-a+2$,

$$e_{D3}(w_ix_i) = \begin{cases} 2c-a+6-i, & \text{if } 3 \leq i \leq \left\lfloor \frac{b-a+5}{2} \right\rfloor, \\ c+i, & \text{if } \left\lfloor \frac{b-a+5}{2} \right\rfloor < i \leq b-a+2. \end{cases}$$

It is easy to verify that there is no clique S in G with $e_3(S) < a$, $e_{m_3}(S) < b$ and $e_{D3}(S) < c$. Thus $r_3 = a$, $R_{m_3} = b$ and $R_3 = c$ as $a < b = c$. □

Santhakumaran et. al. [7] showed that every three positive integers a, b and c with $5 \leq a \leq b \leq c$ are realizable as the diameter, monophonic diameter and detour diameter respectively of some connected graph. Now we have a realization theorem for the clique-to-clique diameter, clique-to-clique monophonic diameter and clique-to-clique detour diameter respectively of some connected graph.

Theorem 3.10 *For any three positive integers a, b, c with $4 \leq a \leq b \leq c$, there exists a connected graph G such that $d_3 = a$, $D_{m_3} = b$ and $D_3 = c$.*

Proof The proof is divided into cases following.

Case 1. $a = b = c$.

Let $G = P_{a+3} : u_1, u_2, \cdots, u_{a+3}$ be a path. Then

$$e_3(u_iu_{i+1}) = e_{m_3}(u_iu_{i+1}) = e_{D3}(u_iu_{i+1}) = \begin{cases} a - i + 1, & \text{if } 1 \leq i \leq \left\lfloor \frac{a+1}{2} \right\rfloor, \\ i - 2, & \text{if } \left\lfloor \frac{a+1}{2} \right\rfloor < i \leq a + 2. \end{cases}$$

It is easy to verify that there is no clique S in G with $e_3(S) > a$, $e_{m_3}(S) > b$ and $e_{D3}(S) > c$. Thus $d_3 = a$, $D_{m_3} = b$ and $D_3 = c$ as $a = b = c$.

Case 2. $4 \leq a \leq b < c$.

Let $P_1 : u_1, u_2, \cdots, u_{a+2}$ be a path of order $a+2$. Let $P_2 : w_1, w_2, \cdots, w_{b-a+3}$ be a path of order $b - a + 3$. Let $P_3 : x_1, x_2$ be a path of order 2. Let $K_1 : y_1, y_2, \cdots, y_{c-b+1}$ be a complete graph of order $c - b + 1$. We construct the graph G of order $c + 3$ as follows: (i) identify the vertices u_1 in P_1, w_1 in P_2 with x_1 in P_3 and identify the vertices u_3 in P_1 with w_{b-a+3} in P_2; (ii) identify the vertices u_{a+1} in P_1 with y_1 in K_1 and identify the vertices u_a in P_1 with y_{c-b+1} in K_1; (iii) join each vertex $w_i (2 \leq i \leq b - a + 2)$ in P_2 with u_2 in P_1. The resulting graph G is shown in Fig.3.6. It is easy to verify

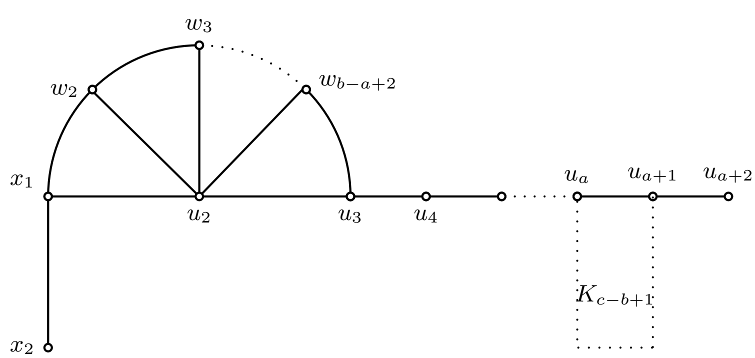

Fig.3.6

that $e_3(x_1x_2) = a$, $e_3(K_1) = a - 1$,

$$e_3(u_iu_{i+1}) = \begin{cases} a - i, & \text{if } 3 \leq i \leq \left\lfloor \frac{a}{2} \right\rfloor, \\ i - 1, & \text{if } \left\lfloor \frac{a}{2} \right\rfloor < i \leq a, \end{cases}$$

$$e_3(u_2w_iw_{i+1}) = \begin{cases} a-1, & \text{if } 1 \le i \le b-a+1, \\ a-2, & \text{if } i = b-a+2, \end{cases}$$

and $e_{m_3}(x_1x_2) = b$, $e_{m_3}(K_1) = b-1$,

$$e_{m_3}(u_iu_{i+1}) = \begin{cases} b-a+i-1, & \text{if } 3 \le i \le a \text{ for } b-a+i \ge a-i, \\ a-i, & \text{if } 3 \le i \le a \text{ for } b-a+i \le a-i, \end{cases}$$

$$e_{m_3}(u_2w_iw_{i+1}) = \begin{cases} b-i, & \text{if } 1 \le i \le \left\lfloor \frac{b}{2} \right\rfloor \text{ for } \left\lfloor \frac{b}{2} \right\rfloor < b-a+3, \\ i-1, & \text{if } \left\lfloor \frac{b}{2} \right\rfloor < i \le b-a+3 \text{ for } \left\lfloor \frac{b}{2} \right\rfloor \le b-a+3, \\ b-i, & \text{if } 1 \le i \le b-a+3 \text{ for } \left\lfloor \frac{b}{2} \right\rfloor \ge b-a+3, \end{cases}$$

and $e_{D3}(x_1x_2) = c$, $e_{D3}(K_1) = b$,

$$e_{D3}(u_iu_{i+1}) = \begin{cases} b-a+i, & \text{if } 3 \le i \le a \text{ for } b-a+i \ge c-b+a-i-1, \\ c-b+a-i-1, & \text{if } 3 \le i \le a \text{ for } b-a+i \le c-b+a-i-1, \end{cases}$$

$$e_{D3}(u_2w_iw_{i+1}) = \begin{cases} c-i-1, & \text{if } 1 \le i \le \left\lfloor \frac{b}{2} \right\rfloor \text{ for } \left\lfloor \frac{b}{2} \right\rfloor < c-b+1, \\ i-1, & \text{if } \left\lfloor \frac{b}{2} \right\rfloor < i \le b-a+3 \text{ for } \left\lfloor \frac{b}{2} \right\rfloor \le c-b+1, \\ c-i-1, & \text{if } 1 \le i \le b-a+3 \text{ for } \left\lfloor \frac{b}{2} \right\rfloor \ge c-b+1 \end{cases}$$

It is easy to verify that there is no clique S in G with $e_3(S) > a$, $e_{m_3}(S) > b$ and $e_{D3}(S) > c$. Thus $d_3 = a$, $D_{m_3} = b$ and $D_3 = c$ as $a \le b < c$.

Case 3. $4 \le a < b = c$.

Let $P_1 : u_1, u_2, \cdots, u_{a+2}$ be a path of order $a+2$. Let $P_2 : w_1, w_2, \ldots, w_{b-a+3}$ be a path of order $b-a+3$. Let $P_3 : x_1, x_2$ be a path of order 2. Let $E_i : x_i(3 \le i \le b-a+2)$ be a $b-a$ copies of K_1. We construct the graph G of order $2b-a+4$ as follows: (i) identify the vertices u_1 in P_1,w_1 in P_2 with x_1 in P_3 and also identify the vertices u_3 in P_1 with w_{b-a+3} in P_2; (ii) join each vertex $x_i(3 \le i \le b-a+2)$ with u_3 in P_1 and w_i in P_2. The resulting graph G is shown in Fig.3.7.

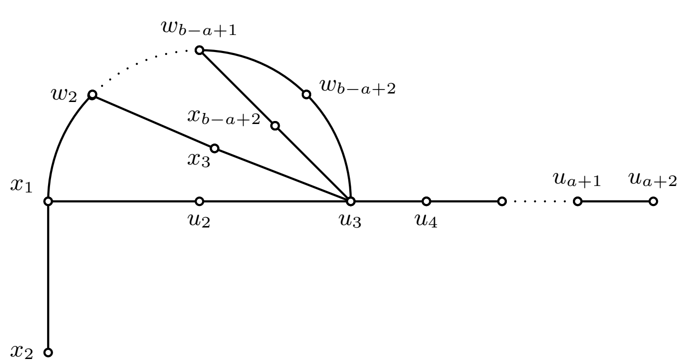

Fig.3.7

It is easy to verify that $e_3(x_1x_2) = a$,

$$e_3(u_iu_{i+1}) = \begin{cases} a-i, & \text{if } 1 \le i \le \left\lfloor \frac{a}{2} \right\rfloor, \\ i-1, & \text{if } \left\lfloor \frac{a}{2} \right\rfloor < i \le a, \end{cases}$$

$e_3(u_3x_i) = a-2$, if $3 \le i \le b-a+2$,

$$e_3(w_iw_{i+1}) = \begin{cases} a, & \text{if } 1 \le i \le b-a, \\ a-1, & \text{if } i = b-a+1, \\ a-2, & \text{if } i = b-a+2, \end{cases}$$

and $e_3(x_iw_{i-1}) = a-1$, if $3 \le i \le b-a+2$, $e_{m_3}(x_1x_2) = b$,

$$e_{m_3}(u_3x_i) = \begin{cases} b-a+2, & \text{if } 3 \le i \le b-a+2 \text{ for } b-a+3 \ge a-2, \\ a-2, & \text{if } 3 \le i \le b-a+2 \text{ for } b-a+3 \le a-2, \end{cases}$$

$$e_{m_3}(u_iu_{i+1}) = \begin{cases} b, & \text{if } i = 1, \\ b-a+2, & \text{if } i = 2 \text{ for } b-a+3 \ge a-2 \quad, \\ a-2, & \text{if } i = 2 \text{ for } b-a+3 \le a-2 \quad, \\ b-a+i-1, & \text{if } 3 \le i \le a \text{ for } b-a+3 \ge a-2 \quad, \\ a-i, & \text{if } 3 \le i \le a \text{ for } b-a+3 \le a-2 \quad, \end{cases}$$

$$e_{m_3}(w_iw_{i+1}) = \begin{cases} b-i, & \text{if } 1 \le i \le \left\lfloor \frac{b-a+1}{2} \right\rfloor, \\ a+i-1, & \text{if } \left\lfloor \frac{b-a+1}{2} \right\rfloor < i \le b-a+1, \\ i-1, & \text{if } i = b-a+2 \text{ for } i \ge a-2, \\ a-2, & \text{if } i = b-a+2 \text{ for } i \le a-2, \end{cases}$$

$$e_{m_3}(x_iw_{i-1}) = \begin{cases} e_{m_3}(w_{i-2}w_{i-1}) & \text{if } 3 \le i \le \left\lfloor \frac{b-a+3}{2} \right\rfloor, \\ e_{m_3}(w_{i-1}w_i) & \text{if } \left\lfloor \frac{b-a+3}{2} \right\rfloor < i \le b-a+2, \end{cases}$$

$e_{D3}(x_1x_2) = c$,

$$e_{D3}(u_iu_{i+1}) = \begin{cases} c, & \text{if } i = 1, \\ c-a+2, & \text{if } i = 2 \text{ for } b-a+3 \ge a-2 \quad, \\ a-2, & \text{if } i = 2 \text{ for } b-a+3 \le a-2 \quad, \\ c-a+i-1, & \text{if } 3 \le i \le a \text{ for } b-a+3 \ge a-2 \quad, \\ a-i, & \text{if } 3 \le i \le a \text{ for } b-a+3 \le a-2 \quad, \end{cases}$$

$$e_{D3}(w_iw_{i+1}) = \begin{cases} c-i, & \text{if } 1 \le i \le \left\lfloor \frac{b-a+1}{2} \right\rfloor, \\ a+i-1, & \text{if } \left\lfloor \frac{b-a+1}{2} \right\rfloor < i \le b-a+1, \\ i-1, & \text{if } i = b-a+2 \text{ for } i \ge a-2, \\ a-2, & \text{if } i = b-a+2 \text{ for } i \le a-2 \end{cases}$$

$$e_{D3}(x_iw_{i-1}) = \begin{cases} e_{D3}(w_{i-2}w_{i-1}) & \text{if } 3 \le i \le \left\lfloor \frac{b-a+3}{2} \right\rfloor, \\ e_{D3}(w_{i-1}w_i) & \text{if } \left\lfloor \frac{b-a+3}{2} \right\rfloor < i \le b-a+2, \end{cases}$$

$$e_{D3}(u_3x_i) = \begin{cases} c-a+2, & \text{if } 3 \le i \le b-a+2 \text{ for } b-a+3 \ge a-2, \\ a-2, & \text{if } 3 \le i \le b-a+2 \text{ for } b-a+3 \le a-2. \end{cases}$$

It is easy to verify that there is no clique S in G with $e_3(S) > a$, $e_{m_3}(S) > b$ and $e_{D3}(S) > c$. Thus $d_3 = a$, $D_{m_3} = b$ and $D_3 = c$ as $a < b = c$. □

In [2], it is shown that the center of every connected graph G lies in a single block of G, Chartrand et. al. [1] showed that the detour center of every connected graph G lies in a single block of G, and Santhakumaran et. al. [7] showed that the monophonic center of every connected graph G lies in a single block of G. But Keerthi Asir et. al. [4] showed that the clique-to-clique detour center of every connected graph G does not lie in a single block of G. However the similar result is not true for the clique-to-clique monophonic center of a graph.

Remark 3.11 The clique-to-clique monophonic center of every connected graph G does not lie in a single block of G. For the Path P_{2n+1}, the clique-to-clique monophonic center is always P_3, which does not lie in a single block.

We leave the following open problems.

Problem 3.12 *Does there exist a connected graph G such that $e_3(C) \ne e_{m_3}(C) \ne e_{D3}(C)$ for every clique C in G?*

Problem 3.13 *Is every graph a clique-to-clique monophonic center of some connected graph?*

Problem 3.14 *Characterize clique-to-clique monophonic self-centered graphs.*

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Some Parameters of Domination on the Neighborhood Graph

M. H. Akhbari

(Institute of Mathematics and Mechanics, Kazan Federal University, 18 Kremlyovskaya Str., Kazan 420008, Russia)

F. Movahedi

(Department of Mathematics, Golestan University, Gorgan, Iran)

S. V. R. Kulli

(Department of Mathematics, Gulbarga University, Gulbarga 585106, India)

E-mail: mhakhbari20@gmail.com, f.movahedi62@gmail.com, vrkulli@gmail.com

Abstract: Let $G = (V, E)$ be a simple graph. The neighborhood graph $N(G)$ of a graph G is the graph with the vertex set $V \cup S$ where S is the set of all open neighborhood sets of G and with vertices $u, v \in V(N(G))$ adjacent if $u \in V$ and v is an open neighborhood set containing u. In this paper, we obtain the domination number, the total domination number and the independent domination number in the neighborhood graph. We also investigate these parameters of domination on the join and the corona of two neighborhood graphs.

Key Words: Neighborhood graph, domination number, Smarandachely dominating k-set, total domination, independent domination, join graph, corona graph.

AMS(2010): 05C69, 05C72.

§1. Introduction

Let $G = (V, E)$ be a simple graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The neighborhood of a vertex u is denoted by $N_G(u)$ and its degree $|N_G(u)|$ by $deg_G(u)$. The minimum and maximum degree of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The open neighborhood of a set $S \subseteq G$ is the set $N(S) = \bigcup_{v\in V(G)} N(v)$, and the closed neighborhood of S is the set $N[S] = N(S) \cup S$. A cut-vertex of a graph G is any vertex $u \in V(G)$ for which induced subgraph $G \setminus \{u\}$ has more components than G. A vertex with degree 1 is called an end-vertex [1].

A dominating set is a set D of vertices of G such that every vertex outside D is dominated by some vertex of D. The domination number of G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G, and generally, a vertex set D^k_S of G is a Smarandachely dominating k-set if each vertex of G is dominated by at least k vertices of S. Clearly, if $k = 1$, such a set D^k_S is nothing else but a dominating set of G. A dominating set D is a total dominating set of G if every vertex of the graph is adjacent to at least one vertex in D. The total domination number of G, denoted by $\gamma_t(G)$ is the minimum size of a total dominating set of G. A dominating

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set D is called an independent dominating set if D is an independent set. The independent domination number of G denoted by $\gamma_i(G)$ is the minimum size of an independent dominating set of G [1].

The join of two graphs G_1 and G_2, denoted by $G_1 + G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv \,|\, u \in V(G_1)\, and\, v \in V(G_2)\}$. The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2. For every $v \in V(G_1)$, G_2^v is the copy of G_2 whose vertices are attached one by one to the vertex v. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added [2].

We use K_n, C_n and P_n to denote a complete graph, a cycle and a path of the order n, respectively. A complete bipartite graph denotes by $K_{m,n}$ and the graph $K_{1,n}$ of order $n+1$ is a star graph with one vertex of degree n and n end-vertices.

The neighborhood graph $N(G)$ of a graph G is the graph with the vertex set $V \cup S$ where S is the set of all open neighborhood sets of G and two vertices u and v in $N(G)$ are adjacent if $u \in V$ and v is an open neighborhood set containing u. In Figure 1, a graph G and its neighborhood graph are shown. The open neighborhood sets in graph G are $N(1) = \{2, 3, 4\}$, $N(2) = \{1, 3\}$, $N(3) = \{1, 2\}$ and $N(4) = \{1\}$ [3].

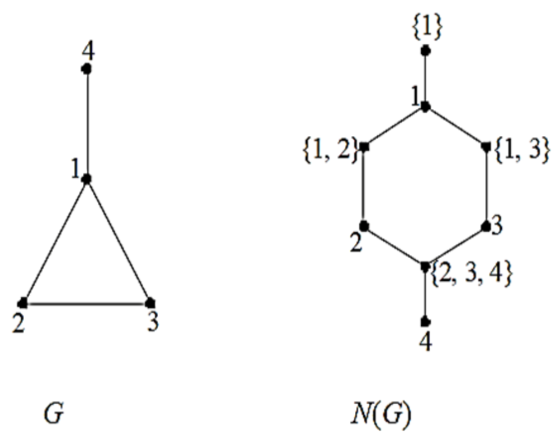

Figure 1 The graph G and the neighborhood graph of G.

In this paper, we determine the domination number, total domination number and independent domination number for the neighborhood graph of a graph G. Also, we consider the join graph and the corona graph of two neighborhood graphs and investigate some parameters of domination of these graphs.

§2. Lemma and Preliminaries

In the text follows we recall some results that establish the domination number, the total domination number and the independent domination number for graphs, that are of interest for our work.

Lemma 2.1([3]) *If G be a graph without isolated vertex of order n and the size of m, then $N(G)$ is a bipartite graph with $2n$ vertices and $2m$ edges.*

Lemma 2.2([3]) *If T be a tree with $n \geq 2$, then $N(T) = 2T$.*

Lemma 2.3 ([3]) *For a cycle C_n with $n \geq 3$ vertices,*

$$N(C_n) = \begin{cases} 2C_n & \text{if } n \text{ is even,} \\ C_{2n} & \text{if } n \text{ is odd.} \end{cases}$$

Lemma 2.4 ([3])

(*i*) *For $1 \leq m \leq n$, $N(K_{m,n}) = 2K_{m,n}$;*
(*ii*) *For $n \geq 1$, $N(\bar{K_n}) = \bar{K_n}$;*
(*iii*) *A graph G is a r-regular if and only if $N(G)$ is a r-regular graph.*

Lemma 2.5 ([1]) *Let $\gamma(G)$ be the domination number of a graph G, then*

(*i*) *For $n \geq 3$, $\gamma(C_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$;*
(*ii*) $\gamma(K_n) = \gamma(K_{1,n}) = 1$;
(*iii*) $\gamma(K_{m,n}) = 2$;
(*iv*) $\gamma(\bar{K_n}) = n$.

Lemma 2.6 ([4]) *If T be a tree of order n and l end-vertices, then*

$$\gamma(T) \geq \frac{n - l + 2}{3}.$$

Lemma 2.7 ([5]) *Let G be a r-regular graph of order n. Then*

$$\gamma(G) \geq \frac{n}{r+1}.$$

Lemma 2.8 ([6]) *Let γ_t be the total domination number of G. Then*
(*i*) $\gamma_t(K_n) = \gamma_t(K_{n,m}) = 2$;
(*ii*) $$\gamma_t(P_n) = \gamma_t(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \ (mod\, 4), \\ \frac{n+2}{2} & \text{if } n \equiv 2 \ (mod\, 4), \\ \frac{n+1}{2} & \text{otherwise.} \end{cases}$$
(*iiii*) *Let T be a nontrivial tree of order n and l end-vertices, then*

$$\gamma_t(T) \geq \frac{n - l + 2}{2};$$

(*iv*) *Let G be a graph , then $\gamma_t(G) \geq 1 + \frac{|C|}{2}$, where C is the set of cut-vertices of G.*

Lemma 2.9 ([7]) *Let γ_i be the independent domination number of G. Then*

(*i*) $\gamma_i(P_n) = \gamma_i(C_n) = \lceil \frac{n}{3} \rceil$;

(*ii*) $\gamma_i(K_{n,m}) = min\{n, m\}$;

(*iii*) *For a graph G with n vertices and the maximum degree Δ,*

$$\left\lceil \frac{n}{1+\Delta} \right\rceil \leq \gamma_i(G) \leq n - \Delta.$$

(*iv*) *If G is a bipartite graph of order n without isolated vertex, then*

$$\gamma_i(G) \leq \frac{n}{2};$$

(*v*) *For any tree T with n vertices and l end-vertices,*

$$\gamma_i(T) \leq \frac{n+l}{3}.$$

Lemma 2.10 ([8]) *For any graph G, $\chi(G) \leq \triangle(G) + 1$ where $\chi(G)$ is the chromatic number of G.*

Lemma 2.11 ([9]) *For any graph G, $\kappa(G) \leq \delta(G)$, where $\kappa(G)$ is the connectivity of G.*

§3. The Domination Number, the Total Domination Number and the Independent Domination Number on $N(G)$

In this section, we propose the obtained results of some parameters of domination on a neighborhood graph.

Theorem 3.1 *Let the neighborhood graph of G be $N(G)$, then*

(*i*) $\gamma(N(P_n)) = 2\lceil \frac{n}{3} \rceil$;

(*ii*) $\gamma(N(C_n)) = \begin{cases} 2\lceil \frac{n}{3} \rceil & \text{if } n \text{ is even,} \\ \lceil \frac{2n}{3} \rceil & \text{if } n \text{ is odd.} \end{cases}$

(*iii*) $\gamma(N(K_{1,n})) = \gamma(N(K_n)) = 2$;

(*iv*) *For* $2 \leq n \leq m$, $\gamma(N(K_{n,m})) = 4$;

(*v*) *For* $n \geq 2$, $\gamma(N(\bar{K_n})) = 2n$.

Proof (*i*) Using Lemma 2.2, for $n \geq 2$, $N(P_n) = 2P_n$. So, it is sufficient to consider a dominating set of P_n. By Lemma 2.5(i), $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. Therefore,

$$\gamma(N(P_n)) = 2\gamma(P_n) = 2\lceil \frac{n}{3} \rceil.$$

(*ii*) If n is even then by Lemma 2.3, $N(C_n) = 2C_n$. So, we consider a cycle C_n of order n

and using Lemma 2.5(i), we have

$$\gamma(N(C_n)) = 2\gamma(C_n) = 2\lceil \frac{n}{3} \rceil.$$

If n is odd, then since $N(C_n)$ is a cycle of order $2n$ so, $\gamma(N(C_n)) = \gamma(C_{2n}) = \lceil \frac{2n}{3} \rceil$.

The segments on (iii), (iv) and (v) can be obtained similarly by applying Lemma 2.1, Lemma 2.4 and Lemma 2.5. □

Theorem 3.2 *Let T be a tree of order n with l end-vertices. Then*

$$\frac{2}{3}(n - l + 2) \leq \gamma(N(T)) \leq n.$$

Proof Using Lemma 2.2, for every tree T, $N(T) = 2T$. So, we consider a tree T to investigate its domination number. Thus, by Lemma 2.6, for every tree T of order n with l end-vertices,

$$\gamma(T) \geq \frac{n - l + 2}{3}.$$

Therefore,

$$\gamma(N(T)) = 2\gamma(T) \geq 2\big(\frac{n - l + 2}{3}\big).$$

Since T is without isolated vertices so, $N(T)$ is a graph without any isolated vertex. Therefore, $V(T) \subseteq V(N(T))$ is a dominating set of $N(T)$. Thus, $\gamma(N(T)) \leq n$. It completes the result. □

Theorem 3.3 *Let G be a r-regular graph. Then,*

$$\gamma(N(G)) \geq \frac{2n}{r + 1}.$$

Proof Using Lemma 2.5(iii), since G is an r-regular graph so, $N(G)$ is a r-regular graph too. According to Lemma 2.1 and Lemma 2.7, we have

$$\gamma(N(G)) \geq \frac{2n}{r + 1}.$$ □

Theorem 3.4 *Let $N(G)$ be a neighborhood graph of G. Then for every vertex $x \in V(G)$, $deg_G(x)$ is equal with $deg_{N(G)}(x)$.*

Proof Assume $x \in V(G)$ and $deg_G(x) = k$. So, the neighborhood set of x is $N(x) = \{y_1, \cdots, y_k\}$ where $y_i \in V(G)$. In graph $N(G)$, x is adjacent to a vertex such as $N(u)$ that consists x. Then, x is adjacent to $N(y_i)$ for every $1 \leq i \leq k$. Thus, degree of x is k in $N(G)$. Therefore, $deg_G(x) = deg_{N(G)}(x)$. □

Theorem 3.5 *Let $\gamma(N(G))$ be the domination number of $N(G)$. For any graph G of order n*

with the maximum degree $\Delta(G)$,

$$\left\lceil \frac{2n}{1+\Delta(G)} \right\rceil \leq \gamma(N(G)) \leq 2n - \Delta(G).$$

Proof Let D be a dominating set of $N(G)$. Each vertex of D can dominate at most itself and $\Delta(N(G))$ other vertices. Since by Theorem 3.4, $\Delta(N(G)) = \Delta(G)$ so,

$$\gamma(N(G)) = |D| \geq \left\lceil \frac{2n}{1+\Delta(G)} \right\rceil.$$

Now, let v be a vertex with the maximum degree $\Delta(N(G))$ and $N[v]$ be a closed neighborhood set of v in $N(G)$. Then v dominates $N[v]$ and the vertices in $V(N(G)) \setminus N[v]$ dominate themselves.

Hence, $V(N(G)) \setminus N[v]$ is the dominating set of cardinality $2n - \Delta(N(G))$. So,

$$\gamma(N(G)) \leq 2n - \Delta(N(G)) = 2n - \Delta(G). \qquad \square$$

We establish a relation between the domination number of $N(G)$ and the chromatic number $\chi(G)$ of the graph G.

Theorem 3.6 *For any graph* G,

$$\gamma(N(G)) + \chi(G) \leq 2n + 1.$$

Proof By Theorem 3.5, $\gamma(N(G)) \leq 2n - \Delta(G)$ and by Lemma 2.10, $\chi(G) \leq \Delta(G) + 1$. Thus,

$$\gamma(N(G)) + \chi(G) \leq 2n + 1. \qquad \square$$

We obtain a relation between the domination number of $N(G)$ and the connectivity $\kappa(G)$ of G following.

Theorem 3.7 *For any graph* G,

$$\gamma(N(G)) + \kappa(G) \leq 2n.$$

Proof By Theorem 3.5, $\gamma(N(G)) \leq 2n - \Delta(G)$ and by Lemma 2.11, $\kappa(G) \leq \delta(G)$. Therefore,

$$\gamma(N(G)) + \kappa(G) \leq 2n - \Delta(G) + \delta(G),$$

since, $\delta(G) \leq \Delta(G)$ so,

$$\gamma(N(G)) + \kappa(G) \leq 2n. \qquad \square$$

The following theorem is an easy consequence of the definition of $N(G)$, Lemmas 2.2–2.4

and Lemma 2.8.

Theorem 3.8 *Let the neighborhood graph of G be $N(G)$ and $\gamma_t(N(G))$ be the total domination number of $N(G)$. Then*

$$(i)\quad \gamma_t(N(P_n)) = \gamma_t(N(C_n)) = \begin{cases} n & \text{if } n \equiv 0 \ (mod\, 4), \\ n+2 & \text{if } n \equiv 2 \ (mod\, 4), \\ n+1 & \text{otherwise,} \end{cases}$$

(ii) *For every* $n, m \geq 1$, $\gamma_t(N(K_{m,n})) = 4$;

(iii) *For* $n \geq 2$, $\gamma_t(N(K_n)) = 4$.

Theorem 3.9 *Let G be a graph of order n without isolated vertices and with the maximum degree Δ. Then,*

$$\gamma_t(N(G)) \geq \frac{2n}{\Delta}.$$

Proof Let D be a total dominating set of $N(G)$. Then, every vertex of $V(N(G))$ is adjacent to some vertices of D. Since, every $v \in D$ can have at most $\Delta(N(G))$ neighborhood, it follows that $\Delta(N(G))\gamma_t(N(G)) \geq |V(N(G))| = 2n$. By Theorem 3.4, $\Delta(N(G)) = \Delta(G) = \Delta$ so, $\Delta\gamma_t(N(G)) \geq 2n$. Therefore,

$$\gamma_t(N(G)) \geq \frac{2n}{\Delta}. \qquad \square$$

Theorem 3.10 *Let T be a nontrivial tree of order n and l end-vertices. Then,*

$$\gamma_t(N(T)) \geq n + 2 - l.$$

Proof Using Lemma 2.2, $N(T) = 2T$ and so, $\gamma_t(N(T)) = 2\gamma_t(T)$. By Lemma 2.8(iv),

$$\gamma_t(T) \geq \frac{n+2-l}{2}.$$

Therefore,

$$\gamma_t(N(T)) = 2\gamma_t(T) \geq 2\Big(\frac{n+2-l}{2}\Big) = n + 2 - l. \qquad \square$$

Theorem 3.11 *Let G be a graph with x cut-vertices. Then,*

$$\gamma_t(N(G)) \geq 1 + x.$$

Proof Let C be the set of cut-vertices of $N(G)$. Since for every cut-vertex u of G, u and $N(u)$ are both cut-vertices in $N(G)$ so, $|C| = 2x$. By Lemma 2.8(iv), $\gamma_t(N(G)) \geq 1 + \frac{|C|}{2}$. Therefore, we have

$$\gamma_t(N(G)) \geq 1 + \frac{|C|}{2} = 1 + \frac{2x}{2} = 1 + x. \qquad \square$$

Theorem 3.12 *Let $\gamma_i(G)$ be the independent domination number of G. Then*

(i) $\gamma_i(N(K_{n,m})) = 2m$;
(ii) $\gamma_i(N(K_{1,n})) = 2$;
(iii) $\gamma_i(N(\bar{K_n})) = 2n$;
(iv) $\gamma_i(N(P_n)) = 2\lceil \frac{n}{3} \rceil$;
(v) $\gamma_i(N(C_n)) = \begin{cases} 2\lceil \frac{n}{3} \rceil & \text{if } n \text{ is even,} \\ \lceil \frac{2n}{3} \rceil & \text{if } n \text{ is odd.} \end{cases}$

Proof The theorem easily proves using Lemma 2.3, Lemma 2.4(i, ii), Lemma 2.5 and Lemma 2.9(i, ii). □

Theorem 3.13 *For a graph G with n vertices and the maximum degree Δ,*

$$\left\lceil \frac{2n}{1+\Delta} \right\rceil \leq \gamma_i(N(G)) \leq 2n - \Delta.$$

Proof It is easy to see that $N(G)$ is a graph of order $2n$ and the maximum degree Δ. So, using Lemma 2.9(iii) we have the result. □

We establish a relation between the independent domination number of $N(G)$ and the chromatic number $\chi(G)$ of G.

Theorem 3.14 *For any graph G,*

$$\gamma_i(N(G)) + \chi(G) \leq 2n + 1.$$

Proof By Theorem 3.13, $\gamma_i(N(G)) \leq 2n - \Delta(G)$ and by Lemma 2.10, $\chi(G) \leq \Delta(G) + 1$. So,

$$\gamma_i(N(G)) + \chi(G) \leq 2n + 1.$$ □

The following theorem is the relation between the independent domination number of $N(G)$ and the connectivity $\kappa(G)$ of G.

Theorem 3.15 *For any graph G,*

$$\gamma_i(N(G)) + \kappa(G) \leq 2n.$$

Proof By Theorem 3.13, $\gamma_i(N(G)) \leq 2n - \Delta(G)$ and by Lemma 2.11, $\kappa(G) \leq \delta(G)$. So,

$$\gamma_i(N(G)) + \kappa(G) \leq 2n - \Delta(G) + \delta(G) \leq 2n.$$ □

Theorem 3.16 *Let G be a simple graph of order n and without any isolated vertex. Then*

$$\gamma_i(N(G)) \leq n.$$

Proof For every graph G with n vertices, $N(G)$ is a bipartite graph of order $2n$. Since G doesn't have any isolated vertex so, $N(G)$ is a graph without isolated vertex. Thus, by Lemma 2.9(iv) we have

$$\gamma_i(N(G)) \leq \frac{2n}{2} = n. \qquad \square$$

Theorem 3.17 *Let T be a tree with n vertices and l end-vertices without isolated vertices. Then*

$$\gamma_i(N(T)) \leq \frac{2}{3}(n+2l).$$

Proof For every tree T, $N(T) = 2T$. Let v be an end-vertex of G. Then, the corresponding vertices of v and $N(v)$ are end-vertices in $N(G)$. Thus, if T has l end-vertices then $2l$ end-vertices are in $N(T)$. So, by Lemma 2.9(v) we have

$$\gamma_i(N(T)) = 2\gamma_i(T) \leq 2\Big(\frac{n+2l}{3}\Big). \qquad \square$$

§4. The Results of the Combination of Neighborhood Graphs

In this section, we consider two graphs G_1 and G_2 and study the join and the corona of their neighborhood graphs in two cases. In Section 4.1, we consider two cases for the join of graphs: i) the neighborhood graph of G_1+G_2 that denotes by $N(G_1+G_2)$, ii) the join of two graphs $N(G_1)$ and $N(G_2)$. So, the domination number, the total domination number and the independent domination number of these graphs are obtained. In Section 4.2, we study the domination number, the total domination number and the independent domination number on two cases of the corona graphs: *i*) $N(G_1 \circ G_2)$ and *ii*) $N(G_1) \circ N(G_2)$.

4.1 The Join of Neighborhood Graphs

Let G_1 be a simple graph of order n_1 with m_1 edges and G_2 be a simple graph with n_2 vertices and m_2 edges. By the definition of the join of two graphs, $G_1 + G_2$ has $n_1 + n_2$ vertices and $m_1 + m_2 + m_1m_2$ edges. So, the neighborhood graph of $G_1 + G_2$ has $2(n_1 + n_2)$ vertices and $2m$ edges where $m = m_1 + m_2 + m_1m_2$. For every $x \in V(G_1 + G_2)$ that $x \in V(G_1)$, we have $deg_{G_1+G_2}(x) = deg_{G_1}(x) + n_2$. Also, if $y \in V(G_1 + G_2)$ and $y \in V(G_2)$ then $deg_{G_1+G_2}(y) = deg_{G_2}(y) + n_1$. On the other hand, using Theorem 3.4 we know that $deg_G(x) = deg_{N(G)}(x)$. So, $deg_{G_1+G_2}(x) = deg_{N(G_1+G_2)}(x)$. Thus, if $x \in V(G_1)$, then $deg_{N(G_1+G_2)}(x) = deg_{G_1}(x) + n_2$ and if $y \in V(G_2)$ then $deg_{N(G_1+G_2)}(y) = deg_{G_2}(y) + n_1$.

Now, let G_1 and G_2 be simple graphs without any isolated vertex. Thus, the join of $N(G_1)$ and $N(G_2)$ denotes $N(G_1) + N(G_2)$ of order $2(n_1 + n_2)$. Also, $N(G_1 + G_2)$ has $2m_1 + 2m_2 + 4m_1m_2$ edges. Therefore, $E(N(G_1) + N(G_2)) = E(N(G_1 + G_2)) + 2m_1m_2$. Also, we can obtain for every $x \in V(N(G_1))$, $deg_{N(G_1)+N(G_2)}(x) = deg_{N(G_1)}(x) + 2n_2$ and for every $y \in V(G_2)$, $deg_{N(G_1)+N(G_2)}(y) = deg_{N(G_2)}(y) + 2n_1$.

Theorem 4.1 *Let G_1 and G_2 be simple graphs without isolated vertex. If order of G_1 is n_1 and $\Delta(G_1) \geq n_1 - 1$, then*

$$\gamma(N(G_1 + G_2)) = \gamma_i(N(G_1 + G_2)) = 2.$$

Proof Let $x \in V(G_1)$ be a vertex with the maximum degree at least n_1-1. So, x dominates $n_1 - 1$ vertices of G_1. Let $D = \{x, N_{G_1+G_2}(x)\}$ and $N_{G_1+G_2}(x)$ be the open neighborhood set of x in $G_1 + G_2$. Since, every vertex of G_1 is adjacent to all of vertices of G_2 in $G_1 + G_2$ so, the degree of x in $G_1 + G_2$ is $n_1 + n_2 - 1$ and x dominates $n_1 + n_2 - 1$ in $N(G_1 + G_2)$. Similarity, $N_{G_1+G_2}(x)$ dominates $n_1 + n_2 - 1$ vertices of $N(G_1 + G_2)$. So, $\gamma(N(G_1 + G_2)) = |D| = 2$.

Since, x and $N_{G_1+G_2}(x)$ are not adjacent in $N(G_1 + G_2)$. Thus, D is an independent dominating set in $N(G_1 + G_2)$. Therefore, $\gamma_i(N(G_1 + G_2)) = 2$. □

Theorem 4.2 *Let G_1 and G_2 be simple graphs without isolated vertices. Then*

$$2 \leq \gamma(N(G_1 + G_2)) \leq 4.$$

Proof It is clearly to obtain $\gamma(N(G_1 + G_2)) \geq 2$. Let $S = \{x, N_{G_1+G_2}(x), y, N_{G_1+G_2}(y)\}$ where $x \in V(G_1)$ and $y \in V(G_2)$. Then, x dominates all of vertices of G_2 in $G_1 + G_2$ and so, all of vertices of $N(G_1 + G_2)$ that are the corresponding set to the neighborhoods of $V(G_2)$. Similarity, $y \in V(G_2)$ dominates n_1 vertices of $N(G_1 + G_2)$. It is shown that S is a dominating set of $N(G_1 + G_2)$. Therefore, the result holds. □

Theorem 4.3 *For graphs G_1 and G_2,*

$$\gamma_t(N(G_1 + G_2)) = 4.$$

Proof Assume $S = \{x, N_{G_1+G_2}(x), y, N_{G_1+G_2}(y)\}$ where $x \in V(G_1)$ and $y \in V(G_2)$. The vertex of $N_{G_1+G_2}(x)$ in $N(G_1 + G_2)$ is the corresponding vertex to the neighborhood of x in G_1. So, x dominates all of the vertices of G_1 and y dominates all of vertices of G_2. It is clearly to see that x is adjacent to $N_{G_1+G_2}(y)$ and y is adjacent to $N_{G_1+G_2}(x)$. Therefore, S is a total dominating set of $N(G_1 + G_2)$ and we have $\gamma_t(N(G_1 + G_2)) \leq |S| = 4$.

Let D be a total dominating set of $N(G_1 + G_2)$ that $|D| \leq 3$. We can assume that $D = \{x, y, z\}$. Thus, we have the following cases.

Case 1. If $x, y, z \in V(G_1 + G_2)$, then since $V(N(G_1 + G_2)) = V(G_1 + G_2) \cup S$ so, all of the vertices S are dominated by D where S is the set of all open neighborhood sets of $G_1 + G_2$. But, each of vertices of $V(G_1 + G_2)$ in $V(N(G_1 + G_2))$ is not dominated by D. Thus, it is a contradiction.

Case 2. Let one of vertices of D be in $V(G_1+G_2)$ and remained vertices be in S of $N(G_1+G_2)$. Without loss of generality suppose that $x \in V(G_1)$. So, $x \in V(G_1 + G_2)$ and $y, z \in S$. since x

doesn't dominate $N_{G_1+G_2}(x)$ and y, z don't dominate the corresponding vertices to y and z in $V(G_1+G_2)$ so, D is not the dominate set in $N(G_1+G_2)$. So, it is a contradiction.

Therefore, $\gamma_t(N(G_1+G_2)) \geq 4$. □

Theorem 4.4 *For graphs G_1 and G_2,*

(i) $\gamma(N(G_1)+N(G_2)) = 2$*;*

(ii) $\gamma_t(N(G_1)+N(G_2)) = 2$.

Proof Using the definition of the total dominating set and the structure of the join of two graphs, the result is hold. □

4.2 **The Corona of Neighborhood Graphs**

In this section, the results of the investigating of the corona on the neighborhood graphs are proposed.

Theorem 4.5 *Let G be a connected graph of order m and H any graph of order n. Then*

$$\gamma(N(G) \circ N(H)) = 2m.$$

Proof According to the definition of the corona G and H, for every $v \in N(G)$, $V(v + N(H)^v) \cap V(N(G)) = \{v\}$ in which $N(H)^v$ is copy of $N(H)$ whose vertices are attached one by one to the vertex v. Thus, $\{v\}$ is a dominating set of $v + N(H)^v$ for $v \in V(N(G))$. Therefore, $V(N(G))$ is a dominating set of $N(G) \circ N(H)$ and $\gamma(N(G) \circ N(H)) \leq 2m$.

Let D be a dominating set of $N(G) \circ N(H)$. We show that $D \cap V(v + N(H)^v)$ is a dominating set of $v + N(H)^v$ for every $v \in V(N(G))$.

If $v \in D$, then $\{v\}$ is a dominating set of $v + N(H)^v$. It follows that $V(v + N(H)^v) \cap D$ is a dominating set of $v + N(H)^v$. If $v \notin D$ and let $x \in V(v + N(H)^v) \setminus D$ with $x \neq v$. Since, D is a dominating set of $N(G) \circ N(H)$, there exists $y \in D$ such that $xy \in E(N(G) \circ N(H))$. Then, $y \in V(N(H)^v) \cap D$ and $xy \in E(v + N(H)^v)$. Therefore, it completes the result.

Since $D \cap V(v + N(H)^v)$ is a dominating set of $v + N(H)^v$ for every $v \in V(N(G))$ so, $\gamma(N(G) \circ N(H)) = |D| \geq 2m$. It completes the proof. □

Theorem 4.6 *Let G be a connected graph of order m and H any graph of order n. Then*

$$\gamma_t(N(G) \circ N(H)) = 2m.$$

Proof It is easily to obtain that $V(N(G))$ is a total dominating set for $N(G) \circ N(H)$. So, $\gamma_t(N(G) \circ N(H)) \leq 2m$.

Let D be a total dominating set of $N(G) \circ N(H)$. Then, for every $v \in V(N(G))$, $|V(v + N(H)^v) \cap D| \geq 1$. So, $\gamma_t((N(G) \circ N(H)) = |D| \geq 2m$. Therefore, $\gamma_t(N(G) \circ N(H)) = 2m$. □

Theorem 4.7 *Let G be a simple graph of order n without isolated vertex. Then*

$$\gamma_i(N(G) \circ K_1) = 2n.$$

Proof It is clearly that there exists $2n$ end-vertices in $N(G) \circ K_1$. Since, the set of these end-vertices is the dominating set and the independent set in $N(G) \circ K_1$ so, the result holds. □

Theorem 4.8 *Let G be a simple graph without isolated vertex. Then*

$$N(G \circ K_1) \cong N(G) \circ K_1.$$

Proof Two graphs are isomorphism, if there exists the function bijection between the vertex sets of these graphs. So, we consider the function $f : V(N(G \circ K_1)) \longrightarrow V(N(G) \circ K_1)$ where for every u and v in $V(N(G \circ K_1))$ if $uv \in E(N(G \circ K_1))$ then $f(u)f(v) \in E(N(G) \circ K_1)$. It means that there exists an one to one correspondence between the vertex sets and the edge sets of $N(G \circ K_1)$ and $N(G) \circ K_1$. We easily obtain the following results:

For $N(G \circ K_1)$, $|V(N(G \circ K_1))| = 2|V(G \circ K_1)| = 2(2n) = 4n$ and $|E(N(G \circ K_1))| = 2|E(G \circ K_1)| = 2(m + n)$. Also, for graph $N(G) \circ K_1$, we have

$$\begin{aligned} |V(N(G) \circ K_1)| &= 2|V(N(G))| = 4n, \\ |E(N(G) \circ K_1)| &= 2n + |E(N(G))| = 2n + 2m = 2(n + m). \end{aligned}$$

For any $x \in V(N(G \circ K_1))$ with $deg_{N(G \circ K_1)}(x) = 1$, then $x \notin V(G)$ and $x \in V(N(G))$. On the other hand, if $y \in V(N(G) \circ K_1)$ and $deg_{N(G) \circ K_1}(y) = 1$ then, $y \notin V(N(G))$. Thus, $x \in N(G \circ K_1)$ is corresponding to y in $N(G) \circ K_1$. Also, using Theorem 3.4, if $x \in V(G)$, then $deg_{N(G \circ K_1)}(x) = deg_{G \circ K_1}(x)$ and $deg_{N(G) \circ K_1}(x) = deg_{G \circ K_1}(x)$. Therefore, if $x \in V(G)$ then, the degree of x in $N(G \circ K_1)$ is equal with the degree of x in $N(G) \circ K_1$. These results are shown that there exists an one to one correspondence between two graphs $N(G) \circ K_1$ and $N(G \circ K_1)$. □

Theorem 4.8 is shown that the obtained results on some parameters of domination of two graphs $N(G \circ K_1)$ and $N(G) \circ K_1$ are equal. So, Theorems 4.5–4.7 hold for $N(G \circ K_1)$ for any graph G.

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Primeness of Supersubdivision of Some Graphs

Ujwala Deshmukh

(Department of Mathematics, Mithibai College, Vile Parle(West), Mumbai, India)

Smita A. Bhatavadekar

(Department of Applied Mathematics, Lokmanya Tilak College of Engineering, University of Mumbai, India)

E-mail: ujwala_deshmukh@rediffmail.com, smitasj1@gmail.com

Abstract: A graph with n vertices is said to admit a prime labeling if it's vertices are labeled with distinct integers $1, 2, \cdots, n$ such that for edge xy , the labels assigned to x and y are relatively prime. The graph that admits a prime labeling is said to be prime. G. Sethuraman has introduced concept of supersubdivision of a graph. In the light of this concept, we have proved that supersubdivision by $K_{2,2}$ of star, cycle and ladder are prime.

Key Words: Star, ladder, cycle, subdivision of graphs, supersubdivision of graphs, prime labeling, Smarandachely prime labeling.

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§1. Introduction

We consider finite undirected graphs without loops, also without multiple edges. G Sethuraman and P. Selvaraju [2] have introduced supersubdivision of graphs and proved that there exists a graceful arbitrary supersubdivision of $C_n, n \geq 3$ with certain conditions. Alka Kanetkar has proved that grids are prime [1]. Some results on prime labeling for some cycle related graphs were established by S.K. Vaidya and K.K.Kanani [6]. It was appealing to study prime labeling of supersubdivisions of some families of graphs.

§2. Definitions

Definition 2.1(Star) *A star S_n is the complete bipartite graph $K_{1,n}$ a tree with one internal node and n leaves, for $n > 1$.*

Definition 2.2(Ladder) *A ladder L_n is defined by $L_n = P_n \times P_2$ here P_n is a path of length n , $\times$ denotes Cartesian product. L_n has $2n$ vertices and $3n - 2$ edges.*

Definition 2.3(Cycle) *A cycle is a graph with an equal number of vertices and edges where vertices can be placed around circle so that two vertices are adjacent if and only if they appear*

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consecutively along the circle. The cycle is denoted by C_n.

Definition 2.4(Subdivision of a Graph) *Let G be a graph with p vertices and q edges. A graph H is said to be a subdivision of G if H is obtained by subdividing every edge of G exactly once. H is denoted by $S(G)$. Thus, $|V| = p + q$ and $|E| = 2q$.*

Definition 2.5(Supersubdivision of a Graph) *Let G be a graph with p vertices and q edges. A graph H is said to be a supersubdivision of G if it is obtained from G by replacing every edge e of G by a complete bipartite graph $K_{2,m}$. H is denoted by $SS(G)$. Thus, $|V| = p + mq$ and $|E| = 2mq$.*

Definition 2.6(Prime Labelling) *A prime labeling of a graph is an injective function $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$ such that for every pair of adjacent vertices u and v, $gcd\left(f\left(u\right), f\left(v\right)\right) = 1$ i.e.labels of any two adjacent vertices are relatively prime. A graph is said to be prime if it has a prime labeling.*

Generally, a labeling is called Smarandachely prime on a graph H by Smarandachely denied axiom ([5], [8]) if there is such a labeling $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$ on G that for every edge uv not in subgraphs of G isomorphic to H, $gcd\left(f\left(u\right), f\left(v\right)\right) = 1$.

For a complete bipartite graph $K_{2,m}$, we call the part consisting of two vertices, the 2 *vertices part* of $K_{(2,m)}$ and the part consisting of m vertices, the *m-vertices part* of $K_{2,m}$ in this paper.

§3. Main Results

Theorem 3.1 *A supersubdivision of S_n, i.e. $SS\left(S_n\right)$ is prime for $m = 2$.*

Proof Let u be the internal node i.e.centre vertex. Let $v_1, v_2, \cdots, v_n$ be endpoints. Let $v_i^1, v_i^2, i = 1, 2, \cdots, n$ be vertices of graph $K_{2,2}$ replacing edge uv_i. Here, $|V| = 3n + 1$.

Let $f : V \to \{1, 2, \ldots, 3n + 1\}$ be defined as follows:

$$
\begin{aligned}
& f\left(u\right) = 1, && \\
& f\left(v_i\right) = 3i, && i = 1, 2, \cdots, n, \\
& f\left(v_i^1\right) = 3i - 1, && i = 1, 2, \cdots, n, \\
& f\left(v_i^2\right) = 3i + 1, && i = 1, 2, \cdots, n.
\end{aligned}
$$

As $f\left(u\right) = 1$, $gcd\left(f\left(u\right), f\left(v_i^1\right)\right) = 1$ and $gcd\left(f\left(u\right), f\left(v_i^2\right)\right) = 1$.

As successive integers are coprime, $gcd\left(f\left(v_i^1\right), f\left(v_i\right)\right) = (3i - 1, 3i) = 1$ and $gcd\left(f\left(v_i^2\right), f\left(v_i\right)\right) = (3i + 1, 3i) = 1$. Thus $SS\left(S_n\right)$ is prime. □

Let C_n be a cycle of length n. Let $c_1, c_2, \cdots, c_n$ be the vertices of cycle. Let $c_{i,i+1}^k$, $k = 1, 2$ be the vertices of the bipartite graph that replaces the edge $c_i c_{i+1}$ for $i = 1, 2, \cdots, n - 1$ Let $c_{n,1}^k$, $k = 1, 2$ be the vertices of the bipartite graph that replaces the edge $c_n c_1$. To illustrate these notations a figure is shown below.

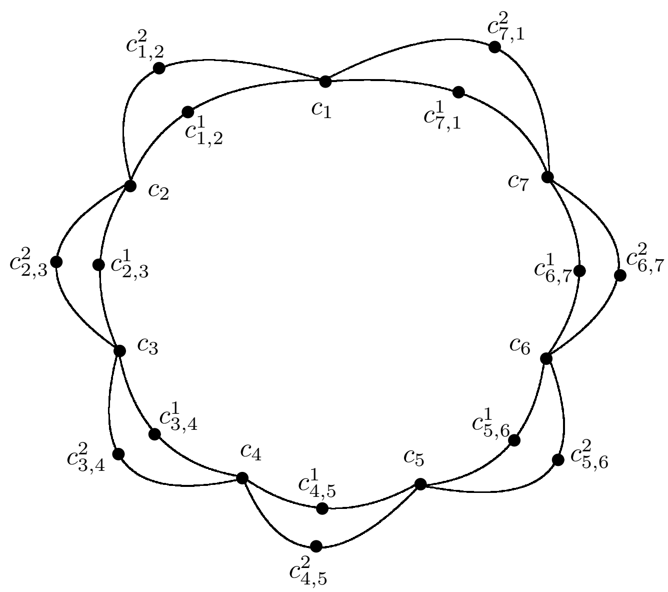

Fig.1 Graph with $n = 7$ with general vertex labels

Theorem 3.2 *A supersubdivision of C_n, i.e. $SS(C_n)$ is prime for $m = 2$.*

Proof Let $p_1, p_2, \cdots, p_k$ be primes such that $3 \leq p_1 < p_2 < p_3 \cdots < p_k < 3n$ such that if p is any prime from 3 to $3n$ then $p = p_i$ for some i between 1 to k.

Define $S_2 = \{S_{2_i}/S_{2_i} = 2^i, i \in \mathbb{N} \ \ such \ \ that \ \ S_{2_i} \leq 3n\}$. Choose greatest i such that $p_i \leq n$ and denote it by l. Let $S_{p_1} = \{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i, \ \ i \in \{2, 3, \cdots, n\} \backslash \{p_l, p_{l-1}, \cdots, p_{l-(n-k-2)}\}$. Define $f : V \to \{1, 2, \ldots, 3n\}$ using following algorithm.

Case 1. $n = 3$ to 8.

In this case, $k = n$.

Step 1. $f(c_r) = p_r \quad for \quad r = 1, 2, \cdots, k$ and $f\left(c_{1,2}^1\right) = 1$.

Step 2. Choose greatest i, such that $2p_i < 3n$ and denote it by r. Define S_{p_j} for $j = 2, 3, \cdots, r$ such that $S_{p_{j_{i-1}}} < S_{p_{j_i}}$ to be $S_{p_j} = \left\{S_{p_{j_i}}/S_{p_{j_i}} = p_j \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{3n}{p_j} \right\rceil\right\}\right\}$.

Step 3. For $i = 2, 3, \cdots, n$, $k = 1, 2$. Label $c_{i,i+1}^k$ using elements of S_{p_j} in increasing order starting from $j = 1, 2, \cdots, r$ and then by elements of S_2 in increasing order.

Step 4. Choose greatest i such that $2^i \leq 3n$. Label $c_{n,1}^k$, $\ k = 1, 2$ as $2^{i-1}, 2^{i-2}$.

Step 5. Label $c_{1,2}^2$ as 2^i.

Case 2. $n = 9$ to 11

In this case, $k + 1 = n$.

Step 1. $f(c_r) = p_r \quad for \quad r = 1, 2, \ldots, k$ and $f(c_n) = 1$.

Step 2. Choose greatest i, such that $2p_i < 3n$ and denote it by r. Define S_{p_j} for $j = 2, 3, \cdots, r$ such that $S_{p_{j_{i-1}}} < S_{p_{j_i}}$ to be $S_{p_j} = \left\{S_{p_{j_i}}/S_{p_{j_i}} = p_j \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{3n}{p_j} \right\rceil\right\}\right\}$.

Step 3. For $i = 2, 3, \cdots, n$ and $k = 1, 2$, label $c^k_{i,i+1}$ using elements of S_{p_j} in increasing order starting from $j = 1, 2, \ldots, r$ and then by elements of S_2 in increasing order.

Step 4. Choose greatest i such that $2^i \leq 3n$. Label $c^k_{n,1}, \quad k = 1, 2$ as $2^{i-2}, 2^{i-3}$.

Step 5. Label $c^k_{1,2}, k = 1, 2$ as 2^i and 2^{i-1}.

Case 3. $n \geq 12$.

Step 1. $f(c_r) = p_r \qquad for \quad r = 1, 2, \cdots, k$.

Step 2. $f(c_{k+1}) = 1$.

For $j = 1, 2, \cdots, n-k-2$, $f(c_{n-j}) = 3p_{l-j}$.

Step 3. Choose greatest i, such that $2p_i < 3n$ and denote it by r. Define S_{p_j} for $j = 2, 3, \cdots, r$ such that $S_{p_{j_{i-1}}} < S_{p_{j_i}}$ to be

$$S_{p_j} = \left\{ S_{p_{j_i}} / S_{p_{j_i}} = p_j \times i, i \in \left\{ 2, 3, \cdots, \left\lceil \frac{3n}{p_j} \right\rceil \right\} \setminus \bigcup_{r=1}^{j-1} \{k \times p_r / k \in \mathbb{N} \right\} \right\}.$$

Step 4. For $i = 2, 3, \cdots, n$ and $k = 1, 2$. Label $c^k_{i,i+1}$ using elements of S_{p_j} in increasing order starting from $j = 1, 2, \ldots, r$ and then by elements of S_2 in increasing order.

Step 5. Choose greatest i such that $2^i \leq 3n$. Label $c^k_{n,1}, \quad k = 1, 2$ as $2^{i-2}, 2^{i-3}$.

Step 6. Label $c^k_{1,2}, k = 1, 2$ as 2^i and 2^{i-1}.

In this case, labels of vertices $c_1, c_2, \cdots, c_k$ are prime . Vertices c_{k+1}, to c_n get labels which are multiples by 3 of $p_l, p_{l-1}, \cdots, p_{l-(n-k-2)}$. Apart from these labels and 3 itself, we have $k-1$ more multiples of 3. Thus $k-1$ vertices of the type $c^j_{i,i+1}, \quad 2 \leq i \leq \left\lceil \frac{k-1}{2} \right\rceil, j = 1, 2$ will get labels as multiples of 3. And hence are relatively prime to labels of corresponding $c'_i s$. Similarly, for multiples of $5, 7$ and so on. Thus, $SS(C_n)$ is prime. □

Theorem 3.3 *A supersubdivision of L_n, i.e. $SS(L_n)$ is prime for $m = 2$.*

Proof Let $u_1, u_2, \cdots .u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of the two paths in L_n. Let u_iu_{i+1}, v_iv_{i+1} for $i = 1, 2, \cdots, n-1$ and u_iv_i for $i = 1, 2, \cdots, n-1, n$ be the edges of L_n. Let $x^k_i, k = 1, 2$ be the vertices of bipartite graph $K_{2,2}$ replacing the edge $u_iu_{i+1}, i = 1, 2, \cdots, n-1$. Let $y^k_i, k = 1, 2, \cdots, m$ be the vertices of the bipartite graph $K_{2,2}$ replacing the edge $v_{n-i}v_{n-i-1}, i = 1, 2, \cdots, n-1$. Let $w^k_i, k = 1, 2$ be the vertices of the bipartite graph $K_{2,2}$ replacing the edge u_iv_i for $i = 1, 2, \cdots, n-1, n$.

Thus, $|V| = 2n + 2n + 2(n-1) + 2(n-1) = 8n - 4$. Let $p_1, p_2, \cdots, p_k$ be primes such that $3 \leq p_1 < p_2 < p_3 \cdots < p_k < 3n$ such that if p is any prime between 3 to $3n$ then $p = p_i$ for some i between 1 to k. Choose greatest i, such that $2p_i < 8n - 4$ and denote it by r.

Define S_{p_j} for $j = 2, 3, \cdots, r$ such that $S_{p_{j_{i-1}}} < S_{p_{j_i}}$ to be

$$S_{p_j} = \left\{ S_{p_{j_i}} / S_{p_{j_i}} = p_j \times i, i \in \left\{ 2, 3, \cdots, \left\lceil \frac{8n-4}{p_j} \right\rceil \right\} \setminus \bigcup_{r=1}^{j-1} \{k \times p_r / k \in \mathbb{N}\} \right\}.$$

Define $S_2 = \left\{ S_{2_i} / S_{2_i} = 2^i, i \in \mathbb{N} \text{ such that } S_{2_i} \leq 3n \right\}$ and a labeling from $V \to \{1, 2, \cdots, 8n - 4\}$ as follows.

Case 1. $n = 2$.

In this case, $k = 2n$. Let $X = \{w_2^1, w_2^2, y_1^1, y_1^2, w_1^1, w_1^2, x_1^2\}$ be an ordered set. Define S_{p_1} such that $S_{p_1} = \left\{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i = 3 \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{8n-4}{p_j} \right\rceil \right\}\right\}$.

Step 1. $f(u_r) = p_r$ for $r = 1, 2$.

Step 2. $f(v_{n-r}) = p_{n+r+1}$ for $r = 0, 1$.

Step 3. $f(x_1^1) = 1$.

Step 4. Label elements of X in order by using elements of S_{p_j} in increasing order starting with $j = 1, 2, \cdots, r$ and then using elements of S_2 in increasing order.

Case 2. $n = 3$ and 6.

In this case, $k = 2n+1$. Let $X = \{x_2^1, x_2^2, x_3^1, \cdots, x_{n-1}^1, x_{n-1}^2, y_1^1, y_1^2, y_2^1, \cdots, y_{n-1}^1, y_{n-1}^2, w_1^1, w_1^2, w_2^1, \cdots, w_n^1, w_n^2\}$ be an ordered set. Define S_{p_1} such that

$$S_{p_1} = \left\{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i = 3 \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{8n-4}{p_j} \right\rceil \right\}\right\}.$$

Step 1. $f(u_r) = p_r$ for $r = 1, 2, \cdots, n$.

Step 2. $f(v_{n-r}) = p_{n+r+1}$ for $r = 0, 1, \cdots, n-1$.

Step 3. $f(x_1^1) = 1$ and $f(x_1^2) = p_k$.

Step 4. Label elements of X in order by using elements of S_{p_j} in increasing order starting with $j = 1, 2, \cdots, r$ and then using elements of S_2 in increasing order.

Case 3. $n = 4, 5$ and 7 to 11.

In this case, $k = 2n$. Let $X = \{x_2^1, x_2^2, x_3^1, \cdots, x_{n-1}^1, x_{n-1}^2, y_1^1, y_1^2, y_2^1, \cdots, y_{n-1}^1, y_{n-1}^2, w_1^1, w_1^2, w_2^1, \cdots, w_n^1, w_n^2, x_1^2\}$ be an ordered set. Define S_{p_1} such that

$$S_{p_1} = \left\{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i = 3 \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{8n-4}{p_j} \right\rceil \right\}\right\}.$$

Step 1. $f(u_r) = p_r$ for $r = 1, 2, \cdots, n$.

Step 2. $f(v_{n-r}) = p_{n+r+1}$ for $r = 0, 1, \ldots, n-1$.

Step 3. $f(x_1^1) = 1$.

Step 4. Label elements of X in order by using elements of S_{p_j} in increasing order starting with $j = 1, 2, \cdots, r$ and then using elements of S_2 in increasing order.

Case 4. $n \geq 12$.

Let $X = \{x_2^1, x_2^2, x_3^1, \cdots, x_{n-1}^1, x_{n-1}^2, y_1^1, y_1^2, y_2^1, \cdots, y_{n-1}^1, y_{n-1}^2, w_n^1, w_n^2, w_{n-1}^1, \cdots, w_1^1, w_1^2\}$ be an ordered set. Choose greatest i, such that $p_i \leq \left\lceil \frac{8n-4}{3} \right\rceil$ and denote it by l.

Step 1. $f(u_r) = p_r$ for $r = 1, 2, \cdots, n$.

Step 2. $f(v_r) = 3p_{l-(r-1)}$ for $r = 1, 2, \cdots, 2n-k$.

Step 3. $f(v_{n-r}) = p_{n+r+1}$ for $r = 0, 1, \cdots, n-(2n-k+1)$.

Step 4. $S_{p_1} = \left\{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i, \quad i \in \left\{2, 3, \cdots, \left\lceil \frac{8n-4}{3} \right\rceil \right\}\right\} \setminus \left\{p_l, p_{l-1}, \cdots, p_{l-(2n-k-1)}\right\}$.

Step 5. Label elements of X in order by using elements of S_{p_j} in increasing order starting with $j = 1, 2, \cdots, r$ and then using elements of S_2 in increasing order.

Step 6. Choose greatest i such that $2^i \leq 3n$. Label x_1^1, x_1^2 as 2^i and 2^{i-1}.

In the above labeling, vertices $u_i's$ and $v_i's$ receive prime labels. Vertices $x_i's$, $y_i's$,$w_i's$ adjacent to $u_i's$,$v_i's$ are labeled with numbers which are multiples of 3 followed by multiples of 5 and so on. Since $m = 2$(small), labels are not multiples of respective primes. Thus $SS(L_n)$ prime. $\square$

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We know nothing of what will happen in future, but by the analogy of past experience.

By Abraham Lincoln, an American president.

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