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k-Metric Dimension of a Graph

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Abstract: In a connected graph $G(V, E)$, a set $S \subseteq V$ is said to be a *k-resolving set* of G, if for every pair of distinct vertices $u, v \in V - S$, there exists a vertex $w \in S$ such that $|d(u, w) - d(v, w)| \geq k$ for some $k \in Z^+$. Among all k-resolving sets of G, a set having minimum cardinality is called a *k-metric basis* of G and its cardinality is called the *k-metric dimension* of G, denoted by $\beta_k(G)$. In this paper, we characterize graphs with prescribed k-metric dimension. We also extend some of the earlier known results on metric dimension.

Key Words: Metric dimension, k-metric dimension, landmarks.

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§1. Introduction

All graphs considered in this paper are simple, finite, undirected and connected. A vertex $w \in V(G)$ is said to resolve a pair of vertices $u, v \in V(G)$ if $d(u, w) \neq d(v, w)$. A set $S \subseteq V(G)$ resolves G if every pair of distinct vertices of G is resolved by some vertex in S. Further, the set S is called a *resolving set* of G. In other words, a resolving set of G is a set $S = \{w_1, w_2, \ldots, w_t\}$ of vertices in G such that for each $u \in V(G)$, the vector $r(u|S) = (d(u, w_1), d(u, w_2), \cdots, d(u, w_t))$ uniquely identifies u. The k-vector $r(u|S)$ is called the *metric code*, *S-location* or S-code of $u \in V(G)$. A resolving set of minimum cardinality in a graph is called a *minimum resolving set* or *metric basis*, the elements of which, are called *landmarks*. The *metric dimension* of G, denoted by $\beta(G)$, is the cardinality of a minimum resolving set in G.

The concept of resolving sets for a connected graph was introduced in the year 1975 by Slater [15] using the term *locating set*. He called the minimum resolving set a *reference set* and the cardinality of a reference set the *locating number* of the graph. In fact, resolving sets were studied much earlier in the context of the coin-weighing problem [3, 4, 8]. In the year 1976, Harary and Melter [11] independently introduced these concepts, however, under different terminologies. They used the term *metric dimension* instead of locating number. Since then,

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a significant amount of work has been carried out on resolving sets [2, 18, 23, 21, 17, 19, 7, 12, 25]. Also, there have been many instances where the concept of resolving sets has arisen, some of which include navigation of robots, solution of the Mastermind game and network discovery & verification.

The following are some of the results on metric dimension obtained by various authors and are used for immediate reference in the subsequent sections of this paper.

Theorem 1.1([Khuller, Raghavachari and Rosenfeld, [13]) *For a simple connected graph G, $\beta(G) = 1$ if and only if $G \cong P_n$.*

Theorem 1.2([Harary and Melter [11]) *For any positive integer n, $\beta(G) = n - 1$ if and only if $G \cong K_n$.*

Theorem 1.3(Chartrand, Erwin, Harary and Zhang [6]) *If G is a connected graph of order n, then $\beta(G) \leqslant n - diam(G)$.*

Lemma 1.4 *For any connected graph G on n vertices which is not a path,*

$$2 \leq \beta(G) \leq n - diam(G).$$

In this paper, we establish certain bounds on k-metric dimension $\beta_k(G)$, introduced by Sooryanarayana [22], as a generalization of metric dimension. Further, we obtain a bound on the degree of a vertex and order of a graph in terms of its k-metric dimension. We also characterize graphs G with $\beta_k(G) = k$.

§2. k-Metric Dimension

The k-metric dimension $\beta_k(G)$ was introduced by Sooryanarayana in [22] as a generalization to metric dimension. In particular, some work was carried out by Geetha and Sooryanarayana [24] for $k = 2$.

Definition 2.1 *Let $G(V,E)$ be a connected graph and $l, k \in Z^+$ with $k \leqslant l$. A subset S of V is said to be a (l,k)-resolving set of G, if for every $u, v \in V - S$ and $u \neq v$, there exists a vertex $w \in S$ with the property that $k \leq |d(u,w) - d(v,w)| \leqslant l$. Further if $l \geqslant diam(G)$, then every (l,k)-resolving set is simply called a k-resolving set.*

Definition 2.2 *A k-resolving set S is said to be a minimal k-resolving set if none of its proper subsets is a k-resolving set. Further a minimal k-resolving set of minimum cardinality is called a lower k-metric basis or simply a k-metric basis of G and is denoted by S_k and its cardinality is called the k-metric dimension of G and is denoted by $\beta_k(G)$.*

Some of the results that follow directly from the above definition are stated below.

Remark 2.3 For any graph G on n vertices, $1 \leq \beta_k(G) \leq n - 1$ for all $k \in Z^+$. Further, if

$k \geq d$, the diameter of G, then $\beta_k(G) = n - 1$.

Remark 2.4 For $k = 1$, the k-metric dimension is same as metric dimension of a graph and for $k \geq 2$, it follows that $\beta_k(G) \geq \beta(G)$. Further, as $1 \leq \beta(G) \leqslant \beta_k(G) \leqslant |V(G)|$, it follows for any integer $k \geq 1$ that $\beta_k(K_n) = n - 1$ whenever $n \geq 2$.

Lemma 2.5 *For any integer $k \geq 1$, if S is a k-resolving set of a connected graph G and $v \in S$, then $V - S$ has at most one pendant vertex adjacent to v.*

Proof If two or more pendant vertices are adjacent to v, then for each vertex $w \in S$ the distance from these vertices is identical. Hence S will not resolve these vertices. □

Lemma 2.6 *For any connected non-trivial graph G and an integer $k \geq 2$, if S is a k-resolving set of G, then $d(x, y) \geq k$ for any two distinct vertices $x, y \in V - S$.*

Proof Suppose, to the contrary, that $d(x, y) \leq k - 1$ for some $x, y \in V - S$. Let $w \in S$ be arbitrary. Without loss of generality, we assume that $d(x, w) \geq d(y, w)$. Then by triangular inequality, we have $d(x, w) \leq d(x, y) + d(y, w) \Rightarrow d(x, w) - d(y, w) \leq d(x, y) \leq k - 1$, a contradiction since w is arbitrary. □

If S_k is a k-metric basis for a graph G with $|V - S_k| > 1$, then, by Lemmas 2.5 and 2.6, it follows that

1. $V - S_k$ is an independent set and S_k is a dominating set.
2. At least $k - 1$ vertices in any shortest path between two distinct vertices of $V - S_k$ are in S_k.
3. The cardinality of S_k is at least $k - 1$, i.e., $\beta_k(G) \geq k - 1$.
4. $n - i(G) \leqslant \beta_k(G) \leqslant n - 1$, where $i(G)$ denotes the independence number of the graph G
5. $\gamma(G) \leqslant \beta_k(G)$, where $\gamma(G)$ is the lower domination number of G.

Combining the above results, we have

Lemma 2.7 *For any $k \in Z^+$ and a connected non-trivial graph G on n vertices,*

$$k - 1 \leqslant \beta_k(G) \leqslant n - 1.$$

The following result shows the cases where the lower bound in Lemma 2.7 is attained.

Theorem 2.8 *For any connected non-trivial graph G of order n and an integer $k \in Z^+$, $\beta_k(G) = k - 1$ if and only if $n = k$.*

Proof Let S_k be a metric basis with $|S_k| = \beta_k(G) = k - 1$. Then, as $\beta_k(G) \leqslant n - 1$, it follows that $n \geqslant k$. If $n > k$, then there exist at least two vertices $x, y \in V - S_k$. But then, the second condition stated above implies that $< S_k > \cong P_{k-1}$ and x is adjacent to one of the end vertices of $< S_k >$ and y is adjacent to the other. Hence $|V - S_k| = 2$ and $G \cong P_{k+1}$. This shows that $diam(G) = k \Rightarrow |d(x,w) - d(y,w)| < k$ for any $w \in S_k$, a contradiction. Thus, $n = k$. The converse follows immediately from Remark 2.3 by noting the fact that $k = n > n - 1 \geqslant diam(G)$. □

Corollary 2.9 *For any connected graph G and any integer $k \geq 2$, $\beta_k(G) = 1$ if and only if $G \cong K_2$.*

The following result is an extension of Theorem 1.2 and shows the cases where the upper bound in Lemma 2.7 is attained.

Theorem 2.10 *For any connected non-trivial graph G on n vertices and any integer $k \geq 1$, $\beta_k(G) = n - 1$ if and only if $diam(G) \leq k$.*

Proof For $k = 1$, the result follows by Theorem 1.2. Suppose that $k \geq 2$ and let G be a connected non-trivial graph on n vertices with $\beta_k(G) = n - 1$. Assume, to the contrary, that $diam(G) \geq k+1$. Then there exists a pair of vertices $u, v \in V$ such that $d(u,v) = k+1$. Let $P : u{-}x_1{-}x_2{-}\cdots{-}x_k{-}v$ be a shortest path from u to v. Let $S = V - \{x_1, v\}$. Then $V - S = \{x_1, v\}$ and for these $x_1, v \in V - S$, the vertex $u \in S$ is such that $d(u,v) - d(u,x) = (k+1) - 1 = k$. So, S is a k-resolving set of G and hence $\beta_k(G) \leq |S| = n-2$, a contradiction. The converse follows from the fact that for any three distinct vertices x, y and u in G, $|d(x,u) - d(y,u)| \leq k-1$ since $diam(G) \leqslant k$. □

Remark 2.11 From Theorem 2.10, it follows for any $k \geq 2$ that the k-metric dimension of the graphs on n vertices such as, Petersen graph, complete p-partite graphs for any $p, 2 \leqslant p \leq n$, $H + K_1$ for any graph H on $n - 1$ vertices, etc., is $n - 1$.

§3. Bounds on Order of a Graph and Degree of a Vertex in Terms of k-Metric Dimension

In this section, we present some bounds on the order of a graph and degree of a vertex in a graph in terms of its k-metric dimension.

Theorem 3.1 *For any connected non-trivial graph G of order n and an integer $k \geq 3$, if $\beta_k(G) = m$, then $m + 1 \leq n \leq m\left(\dfrac{k+1}{k-1}\right) + 1$ for odd k and $m + 1 \leq n \leq m\left(\dfrac{k+2}{k}\right) + 1$ for even k.*

Proof The lower bound follows from Lemma 2.7. To establish the upper bound, consider a k-metric basis S_k for G with $|S_k| = m$. Then, $V - S_k$ is totally disconnected and $|V - S_k| = n - m$. Since G is connected, by Lemma 2.6, the length of a shortest path between any two vertices $u, v \in V - S_k$ should include at least $k - 1$ vertices of S_k such that none of them is adjacent to any other vertex in $V - S_k$. Thus, for $n - m$ vertices in $V - S_k$, we must have at least $(n - m - 1)\left\lfloor \frac{k-1}{2} \right\rfloor$ distinct vertices in S. □

Remark 3.2 The above theorem need not be true for the case $k = 2$. For instance, for the graph shown in Figure 1, the set $S_2 = \{w_1, w_2, w_3\}$ is a metric basis with $m = 3$ and $n = 8$.

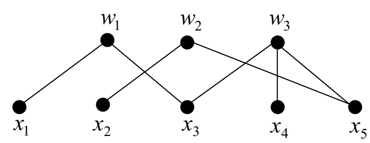

Figure 1 A graph G on 8 vertices with $\beta_2(G) = 3$

Theorem 3.3 *For any connected non-trivial graph G of order n, if $\beta_2(G) = m$, then* $m + 1 \leq n \leq \dfrac{m(m+3)}{2}$.

Proof The lower bound follows from Lemma 2.7. For the upper bound, let S_k be a k-metric basis for G with $|S_k| = m$ with $w_1, w_2, \cdots, w_m$ being the vertices in S_k. Let $N_{\bar{S}_k}(w_j)$ denote the set of vertices in $V - S_k$ adjacent to the vertex w_j, for $1 \leqslant j \leqslant m$. Then for each pair of vertices $x, y \in N_{\bar{S}_k}(w_1)$, S_k should contain at least one vertex w_i which is adjacent to exactly one of these vertices (clearly $w_i \neq w_1$). Hence for the vertex $w_1 \in S_k$, the set S_k should contain at least $N_{\bar{S}_k}(w_1) - 1$ new vertices other than w_1. This is possible only if $N_{\bar{S}_k}(w_1) \leqslant m$. We now define $N(w_j)$ recursively as (i) $N(w_1) = N_{\bar{S}_k}(w_1)$ and (ii) for $j \geqslant 2$, $N(w_j) = N_{\bar{S}_k}(w_j) - N_{\bar{S}_k}(w_{j-1})$. Then, for each pair of vertices in $x, y \in N(w_2)$, we require at least $N(w_2) - 1$ vertices in $S_k - \{w_1, w_2\}$ adjacent to exactly one of these vertices (since $N(w_1) \cap N(w_2) = \emptyset$). This is possible only if $N(w_2) \leqslant m - 1$. Continuing the same argument, we get, for each $1 \leqslant j \leqslant m$, that $N(w_j) \leqslant m - j + 1$. Further, since the graph G is connected and the set $V - S_k$ is independent, the way $N(w_j)$ is constructed implies that

$$|V - S_k| = \sum_{j=1}^{m} N(w_j) = \sum_{j=1}^{m} (m - j + 1) = \frac{m(m+3)}{2}. \qquad \square$$

Lemma 3.4 *For any integer $k \geqslant 2$ and a k-resolving set S of a graph G of order n with $|S| \leqslant n - 2$, if $v \in S$ is a vertex that lies in a shortest path between two vertices x and y in $V - S$, then* $\deg(v) \leqslant |S| - k + 2$.

Proof We prove the result in two cases based on whether v is adjacent to any vertex in $V - S$ or not.

Case 1. x (similarly y) is a vertex adjacent to v.

In this case any shortest xy-path P should contain at least $k-1$ vertices of S for any other vertex $y \in V - S$. Such a vertex y exists as $|V - S| \geqslant 2$. Further, we note that exactly two vertices in P are adjacent to v.

Subcase 1 P contains exactly $k - 1$ vertices of S.

In such a case, v is adjacent to at most $|S| - (k-1)$ vertices of $S - P$. Further if v is adjacent to exactly $|S| - k + 1$ vertices of $S - P$ then no vertex $w \in S$ will resolve x and y since in this case $d(x, y) = k$ and $d(x, w) = 2$, $d(y, w) \leqslant k$. Hence v is adjacent to at most $|S| - k$

vertices in $S-P$. Now if v is adjacent to any other vertex in $z \in V-S$, then $k=2$ since $d(x,z)=2$. Thus, in order to resolve x and z, we require a vertex $w \in S$ non adjacent to v. This shows that v is adjacent to a vertex in $V-S$ only by being non-adjacent to a vertex in S. Thus $deg(v) \leqslant |S|-k+2$.

Subcase 2. P contains more than $k-1$ vertices of S.

In this case v is adjacent to at most $|S|-k$ vertices of S not in P and the vertex adjacent to y in P will resolve x and y. Hence $deg(v) \leqslant |S|-k+2$.

Case 2. v is not adjacent to any vertex in $V-S$.

In this case v is adjacent to exactly two vertices in P and at most $|S|-(l(P)-1)$ vertices in $S-P$. However, as discussed earlier, if v is adjacent to exactly $|S|-(l(P)-1)$ vertices in $S-P$, then no vertex in S will resolve x and y unless $l(P)>k$, which implies that $\deg(v) \leqslant |S|-k+2 = |S|-k+2$. □

Lemma 3.5 *For any integer $k \geqslant 2$ and a k-resolving set S of a graph G of order n with $|S| \leqslant n-2$, if $v \in S$ is a vertex not in any shortest path between any two vertices x and y in $V-S$, then* $\deg(v) \leqslant |S|-k+1$.

Proof The vertex v is adjacent to at most two adjacent vertices in a shortest path P between two vertices x and y in $V-S$. Otherwise, v lies in a shortest xy-path or P will not remain a shortest path.

Case 1. v is adjacent to two adjacent vertices u_1 and u_2 in P.

In this case no vertex $z \in V-S$ is adjacent to v. Otherwise, it is easy to observe that v lies in a shortest path between x and z which is not possible. Also, neither v nor any vertex v_1 adjacent to v will resolve x and y whenever $l(P) \leqslant k$. Without loss of generality, let $d(x,v) \geqslant d(y,v)$ and u_1 be nearer to x than u_2. Then $d(x,v) \geqslant d(x,u_1)$. If not, extending xv-path to u_2 and then from u_2 to y along P yields an xy-path containing v that has length at most that of P, a contradiction. Also $d(x,v) \leq d(x,u_1)+1$ as v is adjacent to u_1 which implies that $d(x,u_1) \leqslant d(x,v) \leqslant d(x,u_1)+1$. Similarly $d(y,u_2) \leqslant d(y,v) \leqslant d(y,u_2)+1$. Hence $|d(x,v)-d(y,v)| \leqslant |d(x,u_1)+1-d(y,u_2)| = |d(x,u_2)-d(y,u_2)| = |d(x,u_2)+d(y,u_2)-2d(y,u_2)| = |l(P)-2d(x,u_2)| = |l(P)-2| = k-2 < k$, a contradiction. Similarly we can show that v_1 will not resolve x and y. Thus, $l(P) \geqslant k+1$ so that v is adjacent to two vertices in P and at most $|S|-k-1$ vertices of $S-V(P)$. Hence $\deg(v) \leqslant |S|-k+1$.

Case 2. v is adjacent to at most one vertex u_1 in P.

In this case v can be adjacent to at most $|S|-k$ elements of $S-V(P)$. Further, if v is adjacent to any vertex in $V-S$, then $k \leqslant 4$. When $k=3$ or 4 and v is adjacent to exactly one vertex $z \in V-S$, we require at least one vertex in $S-V(P)$ not adjacent to v to resolve each pair of vertices in $\{x,y,z\}$. When $k=2$ and v is adjacent to $z_1, z_2, z_3, \cdots, z_i$ in $V-S$, we require i vertices in S not adjacent to v to resolve each pair in $\{x,y,z_1,z_2,\cdots,z_i\}$. Hence $\deg(v) \leqslant |S|-k$.

Thus, in each of the cases, we see that $\deg(v) \leqslant |S|-k+1$. □

The following lemma is based on the fact that the set $V-S$ is an independent set and for each $x, y \in V-S$, $d(x,y) \geqslant k$, the vertex x cannot be adjacent to at least $k-1$ vertices in S.

Lemma 3.6 *For any integer $k \geqslant 2$ and a k-resolving set S of a graph G of order n with $|S| \leqslant n-2$, if $v \in V-S$, then* $\deg(v) \leqslant |S|-k+1$.

Summarizing the above results, we have the following theorem.

Theorem 3.7 *For any integer $k \geqslant 2$ and a graph G of order $n \geqslant k$*

$$\Delta(G) \leqslant \beta_k(G) - k + 2.$$

In the following theorem, we establish a bound on the order of a graph in terms of its k-metric dimension and diameter.

Theorem 3.8 *Suppose G is a graph on n vertices with diameter $d \geq 2$ and metric dimension $\beta_k(G) = m$. Then*

$$n \leqslant m + 1 + \binom{m}{k} \sum_{i=1}^{d-1} (d-i-k+1)^{m-k}.$$

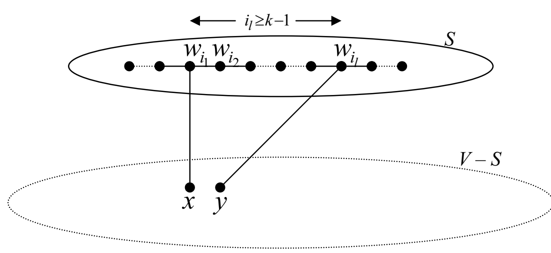

Figure 2 A k-resolving set for the proof of Theorem 3.8.

Proof Let S_k be a k-resolving set with $|S-k| = m$ and $x, y \in V - S_k$. Then, as $d(x,y) \geqslant k$, there are vertices $w_{i_1}, w_{i_2}, \cdots, w_{i_l}$ of S_k in a shortest xy-path, where $i_l \geqslant k-1$. The coordinates of the vertex x corresponding to these i_l vertices are respectively $1, 2, \ldots l$ and that of the vertex y are $l, l-1, \cdots 1$. Hence these coordinates are fixed. Now, for any other $w_j \in S_k$, if the coordinate of x corresponding to w_j is l_j, then, as $d(x,y) \geqslant k$, the difference between l_j and coordinate of y corresponding to w_j should be at least k. Without loss of generality we assume $d(x, w_j) \leqslant d(y, w_j)$. Then, there are at most $(d - l_j - k + 1)$ possibilities for the coordinate of y corresponding to the vertex w_j, where $1 \leqslant l_j \leqslant d$. Thus, there are at most

$$\sum_{i=1}^{d-1} (d-i-k+1)^{m-k}$$

possible vectors that can be assigned for the vertex y. Therefore

$$|V - S_k| \leqslant \left(\begin{array}{c} m \\ k \end{array} \right) \sum_{i=1}^{d-1} (d-i-k+1)^{m-k}. \qquad \square$$

§4. Characterization of Graphs with $\beta_k(G) = k$

S. Khuller *et al.* [13] in the year 1996, proved that $\beta(G) = 1$ if and only if G is a path. In a similar manner, we characterize classes of graphs for which $\beta_2(G) = 2$ in this section. Further, we establish a characterization of graphs with $\beta_k(G) = k$.

Theorem 4.1 *For a connected graph G, $\beta_2(G) = 2$ if and only if $G \cong P_3$ or P_4 or P_5 or C_3.*

Proof Let G be a connected graph such that $\beta_2(G) = 2$ and $S = \{w_1, w_2\}$ be a 2-metric basis of G. Then, by Corollary 2.9, $|V| \geq 3$.

We first claim that $|V(G)| \leq 5$. By Lemma 2.6, the set $V - S$ is an independent set. So, as the graph G is connected, every vertex in $V - S$ is adjacent to a vertex in S. If two or more vertices in $V - S$ are adjacent to both the vertices in S, then by Definition 2.2, we see that S is not a 2-metric basis. Hence, at most one vertex can be adjacent to both the vertices in S. Similarly, at most one vertex $x \in V - S$ can be adjacent to one of the vertices w_1 or w_2 (since if $x, y \in V - S$ are adjacent to w_1, then, as S is a 2-metric basis, $|d(x, w_2) - d(y, w_2)| \geq 2$ which is not possible because S is independent). Hence $|V - S| \leq 3$ and $|V| \leq 5$.

Suppose $|V| = 3$, then G is one of P_3 or C_3 as G is connected. Similarly, if $|V| = 4$, then by Theorem 2.10, $diam(G) > 2$ and hence G must be P_4. In the case of $|V| = 5$, we have $|V - S| = 3$ and by the same argument, we see that at most one vertex can be adjacent to either w_1 or w_2 and at most one vertex can be adjacent to both w_1 and w_2. If w_1 and w_2 are non-adjacent, then G is a path P_5. Else, as seen in Figure 3, for any vertex $v \in V - S$, we have $1 \leq d(v, w_i) \leq 2$, for each $i = 1, 2$ and hence S is not 2-metric basis. Thus if $|V| = 5$, then G must be a path.

Conversely, it is easy to verify that each the graphs P_3, P_4, P_5 and C_3 has its 2-metric dimension 2. This completes the proof. $\square$

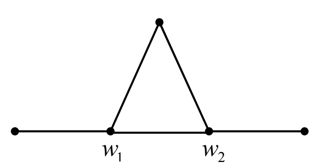

Figure 3 Graph with $\beta_2(G) = 2$

Theorem 4.2 *For any integer $k \geq 3$, $\beta_k(G) = k$ if and only if G is a connected graph on $k+1$ vertices or $G \cong P_{k+2}$.*

Proof Let S_k be a k-metric basis for G with $|S_k| = k$. Then, by Theorem 3.1, we get

$$k+1 \le n \le k\left(\frac{k}{k-1}\right)+1 = \frac{(k+2)(k-1)+1}{k-1} = (k+2)+\frac{1}{k-1} \Rightarrow k+1 \le n \le k+2$$

(since $k \ge 3$) and hence $|V - S_k| \le 2$.

Case 1. $|V - S_k| = 2$.

Let $V - S_k = \{x, y\}$. $w \in S_k$ resolves x and y, and $x_1 \in S_k$ is adjacent to x. Then the only two possibilities are that (i) $x_1 = w$ and $d(y, w) = k+1$ or (ii) $x_1 \neq w$ and y is adjacent to w (since $|S_k| = k$ and $d(x, y) \ge k$). So, the graph in this case is P_{k+2}.

Case 2. $|V - S_k| = 1$

In this case, $|V(G)| = k+1$ and hence $diam(G) \le k$. Thus, by Theorem 2.10, it follows that G is any connected graph on $k+1$ vertices for $\beta_k(G) = k$.

Conversely for a connected graph on $k+1$ vertices we have $diam(G) \le k$ and hence by Theorem 2.10, $\beta_k(G) = (k+1) - 1 = k$. Further, for any path on P_{k+2} vertices and any k-resolving set S_k of P_{k+2}, the distance between any two vertices in $V - S_k$ is at least k, which implies that, $|V - S_k| \le 2$. Hence $|S_k| \ge k$.

Let $\{v_1, v_2, \cdots, v_k, v_{k+1}, v_{k+2}\}$ be the vertices of the path P_{k+2} such that v_i is adjacent to only v_{i+1} for each $i, 1 \le i \le k+1$. Consider the set $S_k = \{v_2, v_3, \cdots, v_k, v_{k+2}\}$. v_1, v_{k+1} are the only vertices in $V - S_k$ that are k-resolved by the vertex v_{k+2} in S_k. Hence S_k is a k-resolving set with $|S_k| = k$. Therefore $\beta_k(P_{k+2}) = k$. □

Problem 4.3 *Solve for G, the equation $\beta_k(G) = k+1$ for all $k \leqslant diam(G)$.*

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