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The Geodesic Irredundant Sets in Graphs

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Abstract: For two vertices u and v of a connected graph G, the set $I[u,v]$ consists of all those vertices lying on $u-v$ geodesics in G. Given a set S of vertices of G, the union of all sets $I[u,v]$ for $u,v \in S$ is denoted by $I[S]$. A convex set S satisfies $I[S]=S$. The convex hull $[S]$ of S is the smallest convex set containing S. The hull number $h(G)$ is the minimum cardinality among the subsets S of V with $[S]=V$. In this paper, we introduce and study the geodesic irredundant number of a graph. A set S of vertices of G is a geodesic irredundant set if $u \notin I[S-\{u\}]$ for all $u \in S$ and the maximum cardinality of a geodesic irredudant set is its irredundant number $gir(G)$ of G. We determine the irredundant number of certain standard classes of graphs. Certain general properties of these concepts are studied. We characterize the classes of graphs of order n for which $gir(G)=2$ or $gir(G)=n$ or $gir(G)=n-1$, respectvely. We prove that for any integers a and b with $2 \leq a \leq b$, there exists a connected graph G such that $h(g)=a$ and $gir(G)=b$. A graph H is called a maximum irredundant subgraph if there exists a graph G containing H as induced subgraph such that $V(H)$ is a maximum irredundant set in G. We characterize the class of maximum irredundant subgraphs.

Key Words: Interior vertex, extreme vertex, hull number, geodesic irredundant sets, irredundant number.

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§1. Introduction

By a graph $G=(V,E)$ we mean a finite undirected connected graph without loops or multiple edges. The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u - v path in G. An u - v path of length $d(u,v)$ is called an u - v *geodesic*. It is known that the distance is a metric on the vertex set V. The set $I[u,v]$ consists of all vertices lying on some u - v geodesic of G, while for $S \subseteq V$, $I[S]=\bigcup_{u,v\in S} I[u,v]$. The set S is *convex* if $I[S]=S$. The *convex hull* $[S]$ is the smallest convex containing S. The convex hull $[S]$

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can also be formed from the sequence $\{I^k[S]\}$, $k \geq 0$, where $I^0[S] = S$, $I^1[S] = I[S]$ and $I^k[S] = I[I^{k-1}[S]]$ for $k \geq 2$. From some term on, this sequence must be constant. Let p be the smallest number such that $I^p[S] = I^{p+1}[S]$. Then $I^p[S]$ is the convex hull $[S]$. A set S of vertices of G is a *hull set* of G if $[S] = V$, and a hull set of minimum cardinality is a *minimum hull set* or *h-set* of G. The cardinality of a minimum hull set of G is the *hull number* $h(G)$ of G. To illustrate these concepts, consider the graph G in Figure 1.1 and the set $S = \{s, t, y\}$. Since $I[S] = \{s, t, u, v, w, x, y\}$ and $I^2[S] = V$, it follows that S is a hull set of G. In fact, S is a minimum hull set and so $h(G) = 3$.

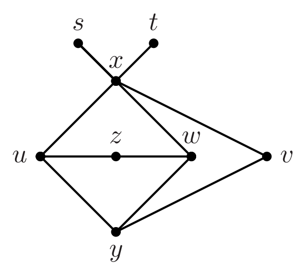

Figure 1.1

A vertex x is an *extreme vertex* of G if the induced subgraph of the neighbors of x is complete or equivalently, $V - \{x\}$ is convex in G. The hull number is an important graph parameter. The hull number of a graph was introduced by Everett and Seidman [7] and further studied in [2, 3, 4, 5, 8].

These concepts have many applications in location theory and convexity theory. There are interesting applications of these concepts to the problem of designing the route for a shuttle and communication network design. For basic graph theoretic terminology, we refer to [6]. We also refer to [1] for results on distance in graphs.

If S is hull set of a connected graph G and $u, v \in S$, then each vertex of every $u-v$ geodesic of G belongs to $I[S]$. This gives the following observation.

Observation 1.1([3]) *Let S be a h-set of a connected graph G and let $u, v \in S$. If $w \neq (u, v)$ lies on a $u - v$ geodesic in G, then $w \notin S$.*

The above observation motivate us to study a new type of sets, called geodesic irredundant sets, which generalizes minimum hull sets in a graph. In the next section, we introduce and study geodesic irredundant sets and the irredundant number of a graph. The irredundant number of certain standard classes of graphs are determined. Various characterization results are proved.

Theorem 1.2([3]) *For integers $m, n \geq 2$, $h(K_{m,n}) = 2$.*

Theorem 1.3([3]) *Each extreme vertex of a connected graph G belongs to every hull set of G. In particular, if the set S of all extreme vertices is a hull set of G, then S is the unique h-set of G.*

§2. Geodesic Irredundant Sets in Graphs

Let S be a set of vertices in a connected graph G. A vertex v in S is called an *interior vertex* of S, if $v \in I[S-\{v\}]$. The set of all interior vertices of S is denoted by S^0. It can be observe that if $S^0 = \emptyset$, then $T^0 = \emptyset$ for any subset T of S. A set S of vertices is called a *geodesic irredundant set* or simply *irredundant set* if $S^0 = \emptyset$. An irredundant set of maximum cardinality is called a *maximum irredundant set* or a $gir-set$ of G. The cardinality of a $gir-set$ is the *irredundant number* $gir(G)$ of G. It follows from Observation 1.1 that every minimum hull set of a connected graph G is an irredundant set in G and so we have that $2 \leq h(G) \leq gir(G) \leq n$, where n is the order of G. To illustrate these concepts, consider the graph G in Figure 2.1. Let $S = \{v_2, v_3, v_5\}$. Then it is clear that $S^0 = \emptyset$ and so S is an irredundant set. It can be easily verified that any set with four or more vertices is not an irredundant set of G and so $gir(G) = |S| = 3$. On the other hand, let $S' = \{v_1, v_4\}$. Then $I[S'] = V$ and so we have that $h(G) = 2$. Since the irredundant number of a disconnected graph is the sum of the irredundant numbers of its components, we are only concerned with connected graphs. One can note that for each integer n, there is only one connected graph of order n having the largest possible irredundant number, namely n, and this is the complete graph K_n.

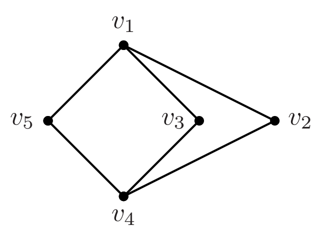

Figure 2.1

Theorem 2.1 *For a connected graph G of order n, $gir(G) = n$ if and only if $G = K_n$.*

We determine the geodesic irredundant number of certain standard classes of graphs.

Proposition 2.2 *For integers $m \geq n \geq 2$, $gir(K_{m,n}) = m$.*

Proof It is clear that $gir(K_{2,2}) = 2$ and so we can assume that $m \geq 3$. Let V_1 and V_2 be the partite sets of $K_{m,n}$ with $|V_1| = m$ and $|V_2| = n$. Then it is obvious that both V_1 and V_2 are irredundant sets of $K_{m,n}$. Now, let S be any set of cardinality greater than m. Then $S \cap V_1 \neq \emptyset$ and $S \cap V_2 \neq \emptyset$. Since $|S| \geq 3$, it follows that either $|S \cap V_1| \geq 2$ or $|S \cap V_2| \geq 2$. This shows that $S^0 \neq \emptyset$ and hence $gir(K_{m,n}) = |V_1| = m$. □

Proposition 2.3 *For any cycle C_n $(n \geq 5)$, $gir(C_n) = 3$.*

Proof Let $S = \{x_1, x_2, \cdots, x_k\}$ be any set of vertices in C_n of cardinality $k \geq 4$. We prove that $S^0 \neq \emptyset$. Assume the contrary that $S^0 = \emptyset$. Then we consider the following two cases.

Case 1. n is even. Now, let v be the antipodal vertex of x_1. If $v \in S$ and since $|S| \geq 4$, it follows that $S^0 \neq \emptyset$. So we can assume that $v \notin S$. Let $P_1 : x_1 = u_1, u_2, \cdots, u_{\frac{n}{2}+1} = v$ and $P_2 : x_1 = v_1, v_2, \cdots, v_{\frac{n}{2}+1} = v$ be the two $x_1 - v$ geodesics in C_n. Since $S^0 = \emptyset$, without loss of generality we can assume that $x_2 = u_r \in P_1$; and $x_3 = v_s \in P_2$ and $x_4 = u_t \in P_1$. If $t < r$, then $x_4 \in S^0$; and it $t > r$, then $x_2 \in S^0$. This is a contradiction. Thus S is not an irredundant set and hence $gir(C_n) \leq 3$. Now, since $T = \{u_1, u_{\frac{n}{2}}, v_{\frac{n}{2}}\}$ is an irredundant set of cardinality 3, we have that $gir(C_n) = 3$.

Case 2. n is odd. Let $x_1 \in S$ and let v, v' be the two antipodal vertices of x_1. Let $P_1 : x_1 = u_1, u_2, \cdots, u_{\frac{n+1}{2}} = v'$ and $P_2 : x_1 = v_1, v_2, \cdots, v_{\frac{n+1}{2}} = v$ be the $x_1 - v'$ and $x_1 - v$ geodesics in C_n, respectively. Since S is an irredundant set containing at least four vertices, it follows that either $v \notin S$ or $v' \notin S$.

Subcase 2.1 $v \notin S$ and $v' = x_2 \in S$. Then it is clear that $x_3, x_4 \notin P_1$ and so $x_3, x_4 \in P_2$. This implies that either $x_3 \in S^0$ or $x_4 \in S^0$. This leads to a contradiction to the fact that $S^0 = \emptyset$.

Subcase 2.2 $v \notin S$ and $v' \notin S$. Now, since $S^0 = \emptyset$, we have that P_1 contains at most one of x_2 and x_3. Also, P_2 contains at most one of x_2 and x_3. Hence without loss of generality, we may assume that $x_2 \in P_1$ and $x_3 \in P_2$. Now, since $|S| \geq 4$, as in Case 1, it follows that $S^0 \neq \emptyset$. This is impossible and hence $gir(C_n) \leq 3$. Now, since $T = \{x_1, v, v'\}$ is an irredundant set of C_n, we have that $gir(C_n) = 3$. □

The irredundant number of a graph has certain properties that are also possessed by the hull number of a graph. In [6], it was shown that if G is a connected graph of order $n \geq 2$ and diameter d, then $h(G) \leq n - d + 1$. The same result is also true for the irredundant number of a graph.

Theorem 2.4 *Let G be a connected graph of order n and diameter d. Then $gir(G) \leq n - d + 1$.*

Proof Let S be any set of cardinality greater than $n - d + 1$. Let $P : u_0, u_1, \cdots, u_d = v$ be a diameteral path in G. Since $|S| > n - d + 1$, it follows that S contains at least three vertices from the diameteral path P, say, u_i, u_j and u_k with $0 \leq i < j < k \leq d$. This implies that $u_j \in I[u_i, u_k]$ and so $S^0 \neq \emptyset$. Thus $gir(G) \leq n - d + 1$. □

We determine $gir(T)$ for T a tree.

Theorem 2.5 *For any tree T with k end vertices, $gir(T) = k$.*

Proof Let S be a gir-set of T. Suppose that the set S contains a cut vertex, say, v of T. Let $C_1, C_2, \ldots, C_l (l \geq 2)$ be the components of $T - v$. It is clear that each component C_i of $T - v$ contains at least one end vertex, say, u_i of T . Since S is an irredundant set of T containing the cut vertex v, without loss of generality, we may assume that $C_1 \cap S \neq \phi$ and $C_i \cap S = \emptyset$ for all $i = 2, 3, \cdots, l$. First, we prove that $l = 2$. Otherwise, if $l \geq 3$, then the set $S' = (S - \{v\}) \cup \{u_2, u_3\}$ is an irredundant set in T with $|S'| = gir(G) + 1$. This is a contradiction. Hence $l = 2$. Now, let $S_1 = (S - \{v\}) \cup \{u_2\}$. Then S_1 is an irredundant set of cardinality $gir(G)$. Moreover, S_1 excludes the cut vertex v and includes a new end vertex u_2. We can continue this process until the resultant gir-set has no cut vertices. This is possible only when S has k vertices or less. Now, since the set of all end vertices of T is an irredundant set, the result follows. □

A *caterpillar* is a tree of order 3 or more, the removal of whose end-vertices produces a path.

Theorem 2.6 *For any non trivial tree T of order n and diameter d, $gir(T) = n - d + 1$ if and only if T is a caterpillar.*

Proof Let T be any non trivial tree. Let u, v be two vertices in T such that $d(u, v) = d$; and let $P : u = v_0, v_1, \ldots, v_{d-1}, v_d = v$ be a diameteral path. Let k be the number of end vertices of T and l the number of internal vertices of T other than $v_1, v_2, \ldots, v_{d-1}$. Then $d - 1 + l + k = n$. By Theorem 2.5, $gir(T) = k = n - d - l + 1$. Hence $gir(T) = n - d + 1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the diameteral path P, if and only if T is a caterpillar. □

Remark 2.7 Every minimum hull set of a connected graph G contains its extreme vertices. This is, in fact, true for non-minimum hull sets and follows directly from the fact that an extreme vertex v is either an initial or terminal vertex of any geodesic containing v. One might be led to believe that every maximum irredundant set of a graph G must contains its extreme vertices, but this is not so, as the graph G in Figure 2.2, the set $S = \{u_1, u_2, u_3, u_4\}$ is the unique *gir*-set of G. Moreover, any irredundant set of G containing the extreme vertex u_5 is of cardinality less than or equal to 3.

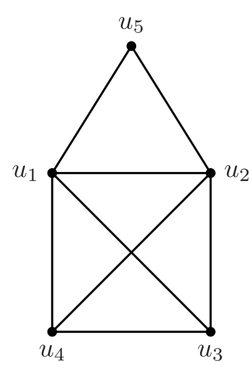

Figure 2.2

Remark 2.8 In a connected graph G, cut vertices do not belong to any h-set of G. But cut vertices may belong to *gir*-sets of a graph. For the graph G in Figure 2.3, the set $S = \{u_1, u_2, u_3\}$ is an *gir*-set containing the cut vertex u_1.

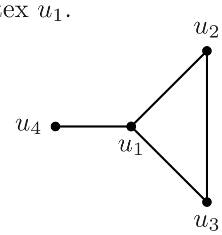

Figure 2.3

Theorem 2.9 *In a connected graph G, a cut vertex v belongs to an gir-set in G if and only if $G-v$ has exactly two components and at least one of them is K_1.*

Proof First, let S be an *gir*-set of G containing the cut vertex v. Suppose that $G-v$ has three components, say C_1, C_2 and C_3. Since S is an irredundant set containing the cut vertex v, it follows that S intersect with at most one of C_1, C_2 and C_3. Assume without loss of generality that $S \cap V(C_2) = \emptyset$ and $S \cap V(C_3) = \emptyset$. Choose vertices x and y in G such that $x \in V(C_2)$ and $y \in V(C_3)$. Then it is obvious that the set $T = (S - \{v\}) \cup \{x, y\}$ is an irredundant set in G. This is a contradiction to the maximality of S. Hence $G-v$ has exactly two components, say C_1 and C_2. Now, suppose that $C_1 \neq K_1$ and $C_2 \neq K_1$. Then as above, we have that $S \cap V(C_1) = \emptyset$ or $S \cap V(C_2) = \emptyset$. Since $|S| \geq 2$, we can assume that $S \cap V(C_1) \neq \emptyset$ and $S \cap V(C_2) = \emptyset$. Let x and y be any two distinct vertices in C_2. Then the set $T = (S - \{v\}) \cup \{x, y\}$ is an irredundant set in G, which is impossible. Hence either $C_1 = K_1$ or $C_2 = K_1$. Conversely, suppose that $G-v$ has exactly two components, say, C_1 and C_2 such that $V(C_1) = \{u\}$. Let S be any *gir*-set of G. Suppose that $v \notin S$. Since S is a maximum irredundant set and $V(C_2)$ is convex in G, it follows that the vertex u belongs to S. This implies that the set $T = (S - \{u\}) \cup \{v\}$ is an irredundant set of cardinality $gir(G)$ containing the cut vertex v. Hence the result follows. □

Next theorem is a characterization of classes of graphs G for which $gir(G) = 2$. The length of a shortest cycle in a connected graph G is the girth of G, denoted by $girth(G)$.

Theorem 2.10 *For a connected graph G, $gir(G) = 2$ if and only if $G = P_n$ or $G = C_4$.*

Proof If $G = P_n$ or $G = C_4$, then it follows from Theorem 2.5 and Proposition 2.2 that $gir(G) = 2$. Conversely, assume that $gir(G) = 2$. If G is acyclic, then it follows from Theorem 2.5 that $G = P_n$. So, assume that G contains cycles. First, we prove that $girth(G) = 4$. Suppose that $girth(G) = r \geq 5$. Let $C : u_1, u_2, \cdots, u_r, u_1$ be a shortest cycle in G. If $r = 2n$, then it clear that $d(u_1, u_n) = n-1; d(u_1, u_{n+2}) = n-1$ and $d(u_n, u_{n+2}) = 2$. Hence it follows that the set $S = \{u_1, u_n, u_{n+2}\}$ is an irredundant set in G, which is a contradiction to the fact that $gir(G) = 2$. Similarly, if $r = 2n+1$, then we have that $d(u_1, u_{n+1}) = n; d(u_1, u_{n+2}) = n$ and $d(u_{n+1}, u_{n+2}) = 1$. Hence it follows that the set $S = \{u_1, u_{n+1}, u_{n+2}\}$ is an irredundant set, which is also impossible. This implies that $girth(G) \leq 4$. Now, if $girth(G) = 3$, then there exist three mutually adjacent vertices in G, say, u, v and w and so G has an irredundant set of cardinality 3. Therefore, we have that $girth(G) = 4$. Let $C : u, v, w, x, u$ be a shortest cycle in G. If $G \neq C$, then without loss of generality, we can assume that there exists a vertex y in G such that $y \notin V(C)$ and y is adjacent to u in G. Since $girth(G) = 4$, it follows that y is not adjacent to both x and v. This shows that the set $T = \{x, y, v\}$ is an irredundant set in G, which leads to a contradiction. Hence we have that $G = C_4$. □

For any connected graph G, we have that $2 \leq h(G) \leq gir(G)$. The following theorem is a realization of this result.

Theorem 2.11 *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $h(G) = a$ and $gir(G) = b$.*

Proof If $a = 2$, then it follows from Theorem 1.2 and Proposition 2.2 that $h(K_{2,b}) = 2$ and $gir(K_{2,b}) = b$. So, assume that $a > 2$. Let G be the graph obtained from the complete graph K_b with vertex set $V(K_b) = \{x_1, x_2, \cdots, x_b\}$ by adding new vertices u and v; and the edges $ux_i (1 \leq i \leq b - a + 2)$ and $vx_i (1 \leq i \leq b - a + 2)$. We first show that $h(G) = a$. Since the set $S = \{u, v, x_{b-a+3}, x_{b-a+4}, \cdots, x_b\}$ of all extreme vertices of G is a hull set of G, it follows directly from Theorem 1.3 that $h(G) = |S| = a$. Also, it is clear that the set $T = \{x_1, x_2, \cdots, x_b\}$ is an irredundant set and so $gir(G) \geq |T| = b$. Now, it follows from Theorems 2.1 and 3.1 that $gir(G) = b$. □

§3. Maximum Irredundant Subgraphs

In this section, we present a characterization of graphs of order n having the irredundant number $n-1$. By Theorem 2.5, the star $K_{1,n-1}$ of order $n \geq 3$, which can also be expressed as $K_1 + \overline{K_{n-1}}$, has irredundant number $n - 1$. Our characterization of graphs of order n having the irredundant number $n - 1$ shows that the class of stars can be generalized to produce all graphs having the irredundant number $n - 1$.

Theorem 3.1 *Let G be a connected graph of order n. Then $gir(G) = n - 1$ if and only if $G = K_1 + \bigcup_j m_j K_j$ with $\sum m_j \geq 2$ or $G = K_n - \{e_1, e_2, \cdots, e_k\}$ with $1 \leq k \leq n - 3$, where e_i's all are edges in K_n which are incident to a common vertex v.*

Proof Suppose that $G = K_1 + \bigcup_j m_j K_j$ and let v be the cut vertex of G. Then it is clear that $V - \{v\}$ is an irredundant set in G. Also, if $G = K_n - \{vx_1, vx_2, \cdots, vx_k\}$, then $V - \{v\}$ is an irredundant set in G. Hence it follows from Theorem 2.1 that $gir(G) = n-1$. Conversely, assume that $gir(G) = n - 1$, then it follows from Theorems 2.1 and 2.4that $diam(G) = 2$ and so G contains interior vertices. We consider the following two cases.

Case 1. G has a unique interior vertex, say v. Choose vertices u and w both are different from v such that $v \in I[u, w]$. In this case, we prove that $G = K_1 + \bigcup_j m_j K_j$. For, if G has no cut vertices, then the vertices u and w lie on a common cycle C; and so there exist vertices x, y and z on the cycle C such that $P : x, y, z$ is a geodesic of length 2 with $y \neq v$. This leads to a contradiction and hence G has cut vertices. Now, since every cut vertex of G is also an interior vertex, it follows that v is the only cut vertex in G. Since $diam(G) = 2$ and v is the unique interior vertex in G, we have that the vertex v must be adjacent to every other vertices in G. Now, let $C_1, C_2, \ldots C_k$ $(k \geq 2)$ be the components of $G - v$. We claim that each C_i is complete. Suppose there exists j with $1 \leq j \leq k$ such that $diam(C_j) \geq 2$. Then there exists a geodesic $Q : u_1, u_2, u_3$ in G with $u_2 \neq v$. This is a contradiction to the fact that v is the unique interior vertex in G. Hence each component of $G - v$ is complete and so $G = K_1 + \bigcup_j m_j K_j$.

Case 2. G has at least two interior vertices. Let S be an irredundant set of cardinality $n - 1$ and let $V - S = \{v\}$. We first claim that $< S >$ is complete. If not, assume that there exist vertices x and y in S which are not adjacent in G. Then $d(x, y) = 2$. Also, since S is an irredundant set of cardinality $n - 1$, we have that v is the only vertex adjacent to both x and y in G. Moreover, one can observe that if u_1 and u_2 are non-adjacent vertices in S, then the vertex v is only vertex adjacent to both u_1 and u_2 in G. Now, Since G contains at least two interior vertices, it follows that there exist vertices u and z in G such that $u \neq z$ and $z \in I[u, v]$. It follows from the above observation that the vertex u is adjacent to both x and y. Hence $u \in S^0$. This is a contradiction. Thus $< S >$ is complete. Now, since G is connected. By Theorem 2.1, we have that $G = K_n - \{vx_1, vx_2, \cdots, vx_k\}$. □

We now introduce a concept that will turn out to be closely connected to the result already stated in this section. A graph H is called a *maximum irredundant subgraph*, if there exists a graph G containing H as an induced subgraph such that $V(H)$ is a maximum irredundant set of G. For example, consider the graphs H and G in Figure 3.1. It follows from Theorems 2.1 and 3.1 that the irredundant set $S = \{u, v, w\}$ is maximum in G, and H is an induced subgraph of G. Hence H is a maximum irredundant subgraph of the graph G. Also, by Theorem 3.1, for positive integers $n_1, n_2, \cdots, n_r$ with $r \geq 1$, the graph $K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}$ is a maximum irredundant subgraph. The analog concepts of minimum hull subgraph was studied in [3]. A graph H is a minimum hull subgraph if there exists a graph G containing H as an induced subgraph such that $V(H)$ is a minimum hull set of G. Next, we characterize the class of all maximum irredundant subgraphs.

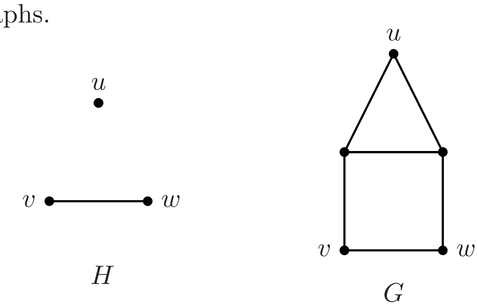

Figure 3.1 *G&H*

Theorem 3.2 *A non trivial graph H is a maximum irredundant subgraph of some connected graph if and only if every component of H is complete.*

Proof First, let H be a maximum irredundant subgraph of a connected graph G. Assume to the contrary, that H contains a component that is not complete. Then there exist $u, v \in V(H)$ such that $d_H(u, v) = 2$ and so H has at least one vertex, say, w different from both u and v such that w lies on some $u - v$ geodesic in H . This is a contradiction to the fact that $V(H)$ is an irredundant set in G. We now verify the converse. Let H be a graph such that every component of H is complete. If H is connected, then H is the maximum irredundant subgraph of H itself. Otherwise, $H = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r}$ for positive integers $n_1, n_2, \cdots, n_r$, where $r \geq 2$. Let $G = K_1 + H$. Then by Theorem 3.1, $V(H)$ is a maximum irredundant set in G. This completes the proof. □

We leave the following problem as open.

Problem 3.3 *Characterize the classes of graphs G for which $gir(G) = h(G)$.*

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