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Spherical Chains Inside a Spherical Segment

Buddhadev Pal

Institute of Science, Department of Mathematics

Banaras Hindu University, Varanasi, Uttar Pradesh 221005, India

Santu Dey and Arindam Bhattacharyya

Faculty of Science, Department of Mathematics, Jadavpur University, Kolkata 700032, India

E-mail: buddha.pal@rediffmail.com, santu.mathju@gmail.com, bhattachar1968@yahoo.co.in

Abstract: The present paper deals with a spherical chain whose centers lie on a horizontal plane which can be drawn inside a spherical fragment and we display some geometric properties related to the chain itself. Here, we also grant recursive and non recursive formulas for calculating the coordinates of the centers and the radii of the spheres.

Key Words: Spherical chain, horizontal plane, continued fraction.

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§1. Introduction

Let us consider a sphere "$ABEFA$" with diameter AE and center L. If we cut this sphere by a plane, parallel to the coordinate planes then we get a circle and this intersection plane that contains the circle is nothing but the $Y_1 = 0$ plane. Because we construct the coordinate system at the point I (see Figure 1), it is the intersection between the diameter(AE) of the sphere"$ABEFA$" and the plane $Y_1 = 0$. It is possible to construct an infinite chain of spheres inside a spherical fragment where the centers of all sphere of the chain lie on a horizontal plane, parallel to the X_1Y_1 plane or may be X_1Y_1 plane and each sphere tangent to the plane $Y_1 = 0$ and spherical fragment, that contains FEB and to its two immediate neighbors.

Let $2(a_1 + b_1)$ be the diameter of the sphere and $2b_1$ be the length of the segment IE. Here we have set up a Cartesian coordinate system with origin at I and let us consider sphere with center (x_i^1, y_i^1, k_1) which lie on a horizontal plane, parallel to the x^1y^1 plane or may be x^1y^1 plane, it depends upon the value of k_1 and radius r_0^1 tangent to the plane $Y_1 = 0$ and the spherical fragment, that containing FEB. Now, we construct a infinite chain of tangent spheres, with centers (x_i^1, y_i^1, k_1) which lie on a horizontal plane, parallel to the X_1Y_1 plane or may be X_1Y_1 plane, it depends upon the value of k_1 and radii r_i^1 for integer value of i, positive and negative and k_1 is fixed but the values of k_1 may be positive, negative or zero. That means for particular values of k_1, we get a sequence of horizontal planes parallel to X_1Y_1 plane. Therefore

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if we consider any horizontal parallel plane corresponding to the X_1Y_1 plane then there exist a spherical chain for which the center of the spheres of the spherical chains lies on that plane.

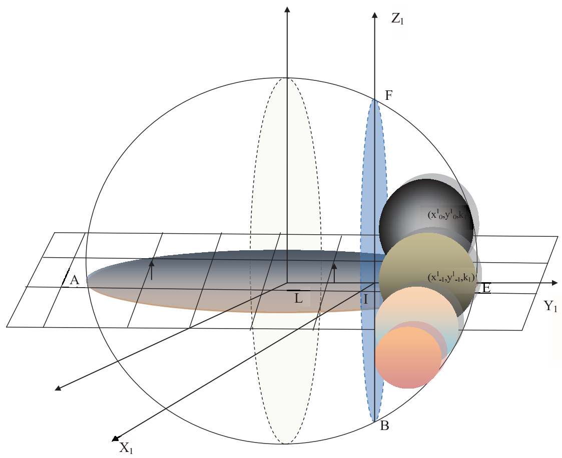

Figure 1. Spherical Chains inside a spherical segment

In this paper, we have learnt that the locus of the centers of the spherical chain mentioned above is a certain type of curve. Here, we have exhibited that locus of the point of centers of the spheres of the chain lie on a sphere. We have also inferred recursive and non recursive formulae to find coordinates of centers and radii of the spheres of a spherical chain.

§2. Some Geometric Properties of the Spherical Chain

Here we have speculated some basic properties of the infinite chain of spheres as mentioned above.

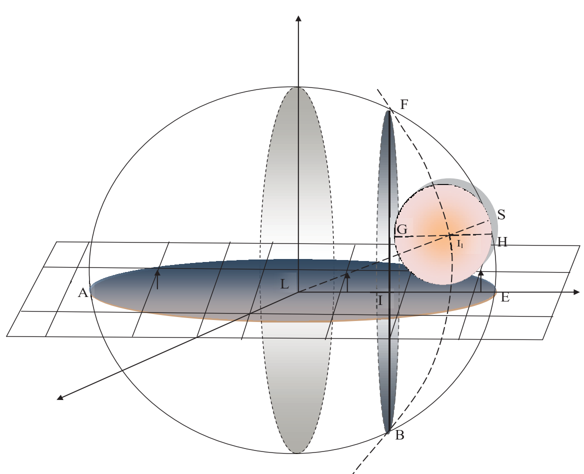

Figure 2.1 Centers of the spheres in chain on a parabola

Proposition 2.1 *The centers of the spheres of the spherical chain on a horizontal plane, lie on a parabola with axis parallel to Y_1-axis, focus is at a height k_1 from L and the vertex is at* $(0, b_1 - \frac{k^2}{4a_1}, k_1)$. If we draw Figure 2.1 explicitly, then

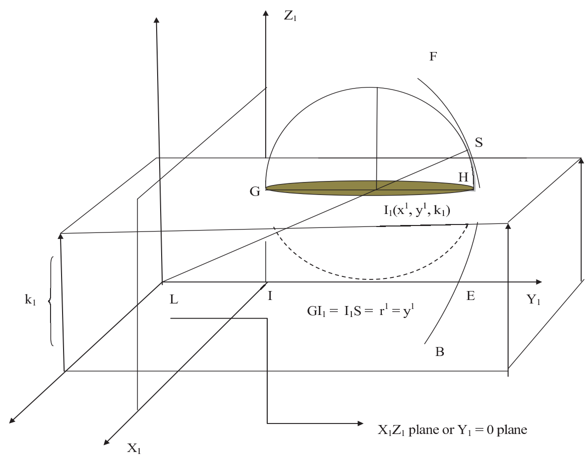

Figure 2.2 Centers of the spheres in chain on a parabola

Proof Let us consider a sphere of the chain with center $I_1(x^1, y^1, k_1)$, lie on a horizontal plane which is parallel to the coordinate plane X_1Y_1, diameter GH, radius r^1, tangent to the spherical arc FEB at S. Since LS contains I_1 (see Figure 2.2), we have by taking into account that L, where L is the center of the sphere which contains the spherical chains, has coordinate $(0, b_1 - a_1, 0)$ and

$$LS = a_1 + b_1,$$

$$LI_1 = \sqrt{(x^1)^2 + (y^1 - b_1 + a_1)^2 + (k_1)^2},$$

$$I_1S = GI_1 = r^1 = y^1.$$

Now, it is clear that

$$LI_1 = LS - I_1S.$$

From these, we have

$$\sqrt{(x^1)^2 + (y^1 - b_1 + a_1)^2 + (k_1)^2} = a_1 + b_1 - y^1,$$

which simplifies into

$$(x^1)^2 = -4a_1\left\{y^1 - (b_1 - \frac{k_1^2}{4a_1})\right\}. \tag{1}$$

This clearly represents a parabola which is symmetric with respect to the axis parallel to Y_1-axis with vertex $\left(0, b^1 - \frac{(k_1)^2}{4a_1}, k_1\right)$ and focus $\left(0, b^1 - a_1 - \frac{(k_1)^2}{4a_1}, k_1\right)$, where L is the center of the sphere. □

Proposition 2.2 *The points of tangency between consecutive spheres of the chain lie on a sphere.*

Proof Consider two neighboring spheres with centers (x_i^1, y_i^1, k_1), $(x_{i+1}^1, y_{i+1}^1, k_1)$, radii r_i^1, r_{i+1}^1 respectively, tangent to each other at U_i, see Figure 3.

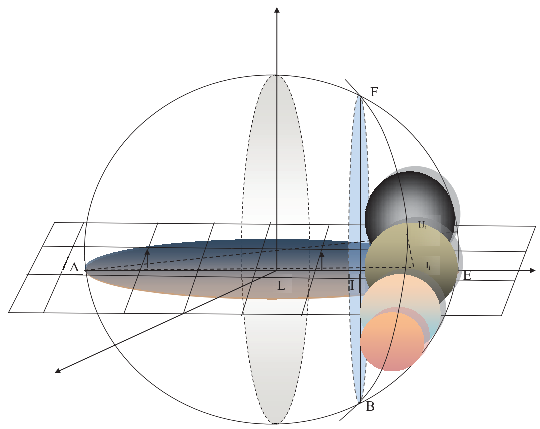

Figure 3. Points of tangency on a spherical arc

By using Proposition 2.1 and noting that A has coordinate $(0, -2a_1, 0)$, we have

$$AI_i^2 = (x_i^1)^2 + (y_i^1 + 2a_1)^2 + (k_1)^2 = (x_i^1)^2 + \left\{ -\frac{(x^1)^2}{4a_1} + b_1 - \frac{(k_1)^2}{4a_1} \right\}^2 + (k_1)^2,$$

$$(r_i^1)^2 = (y_i^1)^2 = \left\{ -\frac{(x^1)^2}{4a_1} + b_1 - \frac{k_1^2}{4a_1} \right\}^2.$$

Applying the Pythagorean theorem to the right triangle AI_iU_i, we have

$$AU_i^2 = AI_i^2 - (r_i^1)^2 = 4a_1(a_1 + b_1) = AI.AE = AF^2.$$

Thus it follows that U_i lie on the sphere with center at A and radius AF. □

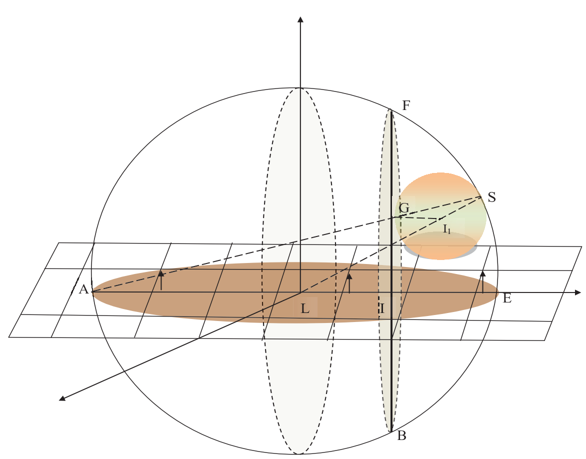

Figure 4. Line joining points of tangency

Proposition 2.3 *If a sphere of the chain touches the plane $Y_1 = 0$ at G and touches the spherical fragment FEB at S, then the points A* (end point of the diameter opposite to plane $Y_1 = 0$), *G, S are collinear.*

Proof Suppose a sphere has center I_1 of a spherical chain which touches the plane $Y_1 = 0$ at G and the spherical fragment FEB at S, see Figure 4. Note that triangles LAS and I_1GS are isosceles triangles where $\angle LSA = \angle LAS = \angle I_1SG = \angle I_1GS$. Thus A, G, S must be collinear as the triangles LAS and I_1GS are similar.

§3. Recursive and Non-Recursive Formulae to Find Coordinates of Centers and Radii of the Spheres of a Spherical Chain

From Figure 5, the triangle $I_iI_{i-1}A_i$ is right angle triangle (as I_iA_i is perpendicular drawn on r^1_{i-1}) with the centers I_{i-1} and I_i of two neighboring spheres of the chain.

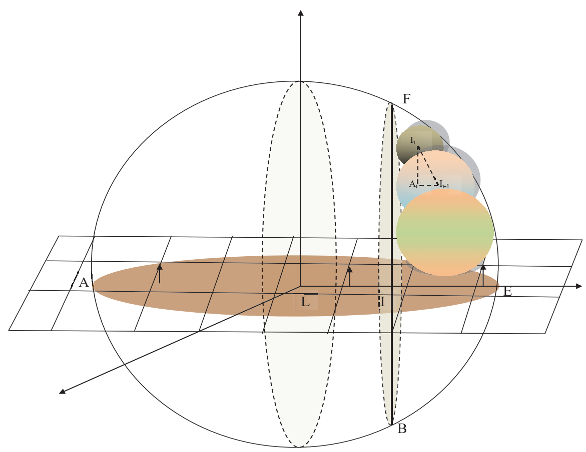

Figure 5. Construction for determination of recursive formula

Since these spheres have radii $r^1_{i-1} = y^1_{i-1}$ and $r^1_i = y^1_i$ respectively, we have

$$(x^1_i - x^1_{i-1})^2 + (y^1_i - y^1_{i-1})^2 + (k_1 - k_1)^2 = (r^1_i + r^1_{i-1})^2 = (y^1_i + y^1_{i-1})^2,$$

$$(x^1_i - x^1_{i-1})^2 = 4y^1_i y^1_{i-1}.$$

Using (1), we can write

$$(x^1_i - x^1_{i-1})^2 = 4\left\{b_1 - \frac{k_1^2}{4a_1} - \frac{(x^1_i)^2}{4a_1}\right\}\left\{b_1 - \frac{k_1^2}{4a_1} - \frac{(x^1_{i-1})^2}{4a_1}\right\},$$

or

$$\frac{4a_1\left\{a_1 + b_1 - k_1^2/4a_1\right\} - (x^1_{i-1})^2}{4a_1^2}.(x^1_i)^2 - 2x^1_{i-1}x^1_i + \frac{\left\{a_1 + b_1 - k_1^2/4a_1\right\}x^2_{i-1} - 4a_1\left\{b_1 - k_1^2/4a_1\right\}^2}{a_1} = 0. \quad (2)$$

If we index the spheres in the chain in such a way that the coordinate x^1_i increases with the index i, then from (2) we have

$$x_i^1 = \frac{2x_{i-1}^1 - \Big\{(x_{i-1}^1)^2/a_1 - 4(b_1 - k_1^2/4a_1)\Big\}\sqrt{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1}}}{2\Big\{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1} - \frac{(x_{i-1}^1)^2}{4a_1^2}\Big\}}. \tag{3}$$

This is a recursive formula that can be applied provided that x_0^1 of the first circle is known. Note that x_0^1 must be chosen in the interval $\Big\{-2\sqrt{a_1(b_1 - k_1^2/4a_1)}, 2\sqrt{a_1(b_1 - k_1^2/4a_1)}\Big\}$. Now the z^1 coordinate is k_1 and y_i^1 are radii derived from (1), by

$$y_i^1 = r_i^1 = b_1 - \frac{k_1^2}{4a_1} - \frac{(x^1)_i^2}{4a_1}. \tag{4}$$

Now, it is possible to transform the recursion formula into a continued fraction and after some calculations, we get

$$x_i^1 = 2a_1\left\{\sqrt{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1}} - \frac{1}{\frac{x_{i-1}^1}{2a_1} + \sqrt{1 + \frac{(b_1 - k^2/4a_1)}{a_1}}}\right\}. \tag{5}$$

Let

$$\wp = 2\sqrt{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1}}, \quad \textit{and} \quad \xi_i = \frac{x_i^1}{2a_1} - \sqrt{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1}}, \quad i = 1, 2, ..., \tag{6}$$

then, we have

$$\xi_i = -\frac{1}{\wp + \xi_{i-1}}.$$

Thus, for positive integral values of i,

$$\xi_i = -\frac{1}{\wp - \frac{1}{\wp - \frac{1}{\frac{\cdots}{-\frac{1}{\wp + \xi_{0+}}}}}}.$$

Here we have used ξ_{0+} in place of ξ_0 and

$$\xi_{0+} = \frac{x_0^1}{2a_1} - \sqrt{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1}}.$$

Now, if we solve equation (2) for x_{i-1}^1 then we get

$$x_{i-1}^1 = \frac{2x_i^1 + \Big\{(x_i^1)^2/a_1 - 4(b_1 - k_1^2/4a_1)\Big\}\sqrt{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1}}}{2\Big\{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1} - \frac{(x_i^1)^2}{4a_1^2}\Big\}}. \tag{7}$$

Thus, for negative integral values of i, with

$$\xi_{-i} = \frac{x^1_{-i}}{2a_1} + \sqrt{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1}},$$

we have

$$\xi_{-i} = -\frac{1}{-\wp - \frac{1}{-\wp - \frac{1}{\ddots - \frac{1}{-\wp + \xi_{0+}}}}},$$

where

$$\xi_{0-} = \frac{x^1_0}{2a_1} + \sqrt{1 + \frac{(b_1 - k_1^2/4a_1)}{a_1}}.$$

Therefore it is possible to give nonrecursive formulae for calculating x^1_i and x^1_{-i}. In the following, here we shall consider only x^1_i for positive integer indices because, as far as x^1_{-i} is concerned, it is enough to change, in all the formulae involved, $\wp$ into $-\wp$, x^1_i into x^1_{i-1}. Starting from (5), and by considering its particular structure, one can write, for $i = 1, 2, 3, ...$

$$\xi_i = -\frac{\mu_{i-1}(\wp)}{\mu_i(\wp)},$$

where $\mu_i(\wp)$ are polynomials with integer coefficients. Here are the first five of them.

$\mu_0(\wp)$	1
$\mu_1(\wp)$	$\wp + \xi_{0+}$
$\mu_2(\wp)$	$(\wp^2 - 1) + \wp\xi_{0+}$
$\mu_3(\wp)$	$(\wp^3 - 2\wp) + (\wp^2 - 1)\xi_{0+}$
$\mu_4(\wp)$	$(\wp^4 - 3\wp^2 + 1) + (\wp^3 - 2\wp)\xi_{0+}$
$\mu_5(\wp)$	$(\wp^5 - 4\wp^3 + 3\wp) + (\wp^4 - 3\wp^2 + 1)\xi_{0+}$

According to a fundamental property of continued fraction [1], these polynomials satisfy the second order linear recurrence

$$\mu_i(\wp) = \wp\mu_{i-1}(\wp) - \mu_{i-2}(\wp). \tag{8}$$

We can further write

$$\mu_i(\wp) = \varphi_i(\wp) + \varphi_{i-1}(\wp)\xi_{0+}, \tag{9}$$

for a sequence of simpler polynomials $\varphi_i(\wp)$, each either odd or even. In fact, from (8) and (9), we have

$$\varphi_{i+2}(\wp) = \wp\varphi_{i+1}(\wp) - \varphi_i(\wp).$$

Explicitly,

$$\varphi_i(\wp) = \left\{ \begin{array}{ll} 1, & i = 0 \\ \sum_{n=0}^{\frac{1}{2}} (-1)^{\frac{1}{2}+n} \begin{pmatrix} \frac{1}{2}+n \\ 2n \end{pmatrix} \wp^{2n}, & i = 2,4,6,... \\ \sum_{n=1}^{\frac{i+1}{2}} (-1)^{\frac{i+1}{2}+n} \begin{pmatrix} \frac{i-1}{2}+n \\ 2n-1 \end{pmatrix} \wp^{2n-1}, & i = 1,3,5,... \end{array} \right\}$$

From (6), we have

$$x_i^1 = a_1(\wp - 2\frac{\mu_{i-1}(\wp)}{\mu_i(\wp)}), \tag{10}$$

for $i = 1, 2, ...$

Note 3.1 One can also consider the planes parallel to Y^1Z^1 plane and Z^1X^1 plane.

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