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Radial Signed Graphs

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Abstract: In this paper we introduced a new notion radial signed graph of a signed graph and its properties are obtained. Also, we obtained the structural characterization of radial signed graphs. Further, we presented some switching equivalent characterizations.

Key Words: Signed graphs, balance, switching, radial signed graph, negation of a signed graph.

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§1. Introduction

For standard terminology and notation in graph theory we refer Harary [4] and Zaslavsky [40] for signed graphs. Throughout the text, we consider finite, undirected graph with no loops or multiple edges.

Within the rapid growth of the Internet and the Web, and in the ease with which global communication now takes place, connectedness took an important place in modern society. Global phenomena, involving social networks, incencitives and the behavior of people based on the links that connect us appear in a regular manner. Motivated by these developments, there is a growing multidisciplinary interest to understand how highly connected systems operate [3].

In social sciences we often deal with relations of opposite content, e.g., "love"- "hatred", "likes"-"dislikes", "tells truth to"-"lies to" etc. In common use opposite relations are termed positive and negative relations. A signed graph is one in which relations between entities may be of various types in contrast to an unsigned graph where all relations are of the same type. In signed graphs edge-coloring provides an elegant and uniform representation of the various types of relations where every type of relation is represented by a distinct color.

In the case where precisely one relation and its opposite are under consideration, then instead of two colors, the signs + and - are assigned to the edges of the corresponding graph in order to distinguish a relation from its opposite. In the case where precisely one relation and its opposite are under consideration, then instead of two colors, the signs + and $-$ are assigned to the edges of the corresponding graph in order to distinguish a relation from its opposite. Formally, a signed graph $\Sigma = (\Gamma, \sigma) = (V, E, \sigma)$ is a graph Γ together with a function that assigns a sign $\sigma(e) \in \{+, -\}$, to each edge in Γ. σ is called the signature or sign function. In such a signed graph, a subset A of $E(\Gamma)$ is said to be positive if it contains an even number

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of negative edges, otherwise is said to be negative. Balance or imbalance is the fundamental property of a signed graph. A signed graph Σ is balanced if each cycle of Σ is positive. Otherwise it is unbalanced.

Signed graphs Σ_1 and Σ_2 are isomorphic, written $\Sigma_1 \cong \Sigma_2$, if there is an isomorphism between their underlying graphs that preserves the signs of edges.

The theory of balance goes back to Heider [7] who asserted that a social system is balanced if there is no tension and that unbalanced social structures exhibit a tension resulting in a tendency to change in the direction of balance. Since this first work of Heider, the notion of balance has been extensively studied by many mathematicians and psychologists. In 1956, Cartwright and Harary [2] provided a mathematical model for balance through graphs.

A *marking* of Σ is a function $\zeta : V(\Gamma) \to \{+,-\}$. Given a signed graph Σ one can easily define a marking ζ of Σ as follows: For any vertex $v \in V(\Sigma)$,

$$\zeta(v) = \prod_{uv \in E(\Sigma)} \sigma(uv),$$

the marking ζ of Σ is called *canonical marking* of Σ.

The following are the fundamental results about balance, the second being a more advanced form of the first. Note that in a bipartition of a set, $V = V_1 \cup V_2$, the disjoint subsets may be empty.

Theorem 1.1 *A signed graph Σ is balanced if and only if either of the following equivalent conditions is satisfied:*

(1)(Harary [5]) *Its vertex set has a bipartition $V = V_1 \cup V_2$ such that every positive edge joins vertices in V_1 or in V_2, and every negative edge joins a vertex in V_1 and a vertex in V_2;*

(2)(Sampathkumar [13]) *There exists a marking μ of its vertices such that each edge uv in Γ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$.*

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. *Complement* of Σ is a signed graph $\overline{\Sigma} = (\overline{\Gamma}, \sigma')$, where for any edge $e = uv \in \overline{\Gamma}$, $\sigma'(uv) = \zeta(u)\zeta(v)$. Clearly, $\overline{\Sigma}$ as defined here is a balanced signed graph due to Theorem 1.1. For more new notions on signed graphs refer the papers (see [10-37]).

A switching function for Σ is a function $\zeta : V \to \{+,-\}$. The switched signature is $\sigma^\zeta(e) := \zeta(v)\sigma(e)\zeta(w)$, where e has end points v, w. The switched signed graph is $\Sigma^\zeta := (\Sigma|\sigma^\zeta)$. We say that Σ switched by ζ. Note that $\Sigma^\zeta = \Sigma^{-\zeta}$ (see [1]).

If $X \subseteq V$, switching Σ by X (or simply switching X) means reversing the sign of every edge in the cut set $E(X, X^c)$. The switched signed graph is Σ^X. This is the same as Σ^ζ where $\zeta(v) := -$ if and only if $v \in X$. Switching by ζ or X is the same operation with different notation. Note that $\Sigma^X = \Sigma^{X^c}$.

Signed graphs Σ_1 and Σ_2 are switching equivalent, written $\Sigma_1 \sim \Sigma_2$ if they have the same underlying graph and there exists a switching function ζ such that $\Sigma_1^\zeta \cong \Sigma_2$. The equivalence class of Σ,

$$[\Sigma] := \{\Sigma' : \Sigma' \sim \Sigma\},$$

is called the its switching class.

Similarly, Σ_1 and Σ_2 are switching isomorphic, written $\Sigma_1 \cong \Sigma_2$, if Σ_1 is isomorphic to a switching of Σ_2. The equivalence class of Σ is called its switching isomorphism class.

Two signed graphs $\Sigma_1 = (\Gamma_1, \sigma_1)$ and $\Sigma_2 = (\Gamma_2, \sigma_2)$ are said to be *weakly isomorphic* (see [**?**]) or *cycle isomorphic* (see [**?**]) if there exists an isomorphism $\phi : \Gamma_1 \to \Gamma_2$ such that the sign of every cycle Z in Σ_1 equals to the sign of $\phi(Z)$ in Σ_2. The following result is well known.

Theorem 1.2(T. Zaslavsky [39]) *Two signed graphs Σ_1 and Σ_2 with the same underlying graph are switching equivalent if and only if they are cycle isomorphic.*

In [16], the authors introduced the switching and cycle isomorphism for signed digraphs.

In this paper, we initiate a study of the *radial signed graph* of a given signed graph and solve some important signed graph equations and equivalences involving it. Further, we obtained the structural characterization of radial signed graphs.

§2. Radial Signed Graph of a Signed Graph

In a graph Γ, the distance $d(u,v)$ between a pair of vertices u and v is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex u is the distance to a vertex farthest from u. The radius $r(\Gamma)$ of Γ is defined by $r(\Gamma) = \min\{e(u) : u \in \Gamma\}$ and the diameter $d(\Gamma)$ of Γ is defined by $d(\Gamma) = \max\{e(u) : u \in \Gamma\}$. A graph for which $r(\Gamma) = d(\Gamma)$ is called a *self-centered graph* of radius $r(\Gamma)$. A vertex v is called an eccentric vertex of a vertex u if $d(u,v) = e(u)$. A vertex v of Γ is called an eccentric vertex of Γ if it is an eccentric vertex of some vertex of Γ. Let S_i denote the subset of vertices of Γ whose eccentricity is equal to i.

Kathiresan and Marimuthu [8] introduced a new type of graph called radial graph. Two vertices of a graph Γ are said to be radial to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph Γ, denoted by $R(\Gamma)$, has the vertex set as in Γ and two vertices are adjacent in $R(\Gamma)$ if, and only if, they are radial in Γ. If Γ is disconnected, then two vertices are adjacent in $R(\Gamma)$ if they belong to different components of Γ. A graph Γ is called a *radial graph* if $R(\Gamma') = \Gamma$ for some graph Γ'.

Motivated by the existing definition of complement of a signed graph, we now extend the notion of radial graphs to signed graphs as follows: The *radial signed graph* $R(\Sigma)$ of a signed graph $\Sigma = (\Gamma, \sigma)$ is a signed graph whose underlying graph is $R(\Gamma)$ and sign of any edge uv is $R(\Sigma)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of Σ. Further, a signed graph $\Sigma = (\Gamma, \sigma)$ is called radial signed graph, if $\Sigma \cong R(\Sigma')$ for some signed graph Σ'. following result restricts the class of radial graphs.

Theorem 2.1 *For any signed graph $\Sigma = (\Gamma, \sigma)$, its radial signed graph $R(\Sigma)$ is balanced.*

Proof Since sign of any edge $e = uv$ in $R(\Sigma)$ is $\zeta(u)\zeta(v)$, where ζ is the canonical marking of Σ, by Theorem 1.1, $R(\Sigma)$ is balanced. □

For any positive integer k, the k^{th} iterated radial signed graph, $R^k(\Sigma)$ of Σ is defined as follows:

$$R^0(\Sigma) = \Sigma, \qquad R^k(\Sigma) = R(R^{k-1}(\Sigma)).$$

Corollary 2.2 *For any signed graph $\Sigma = (\Gamma, \sigma)$ and for any positive integer k, $R^k(\Sigma)$ is balanced.*

The following result characterize signed graphs which are radial signed graphs.

Theorem 2.3 *A signed graph $\Sigma = (\Gamma, \sigma)$ is a radial signed graph if, and only if, Σ is balanced signed graph and its underlying graph Γ is a radial graph.*

Proof Suppose that Σ is balanced and Γ is a radial graph. Then there exists a graph Γ' such that $R(\Gamma') \cong \Gamma$. Since Σ is balanced, by Theorem 1, there exists a marking ζ of Γ such that each edge uv in Σ satisfies $\sigma(uv) = \zeta(u)\zeta(v)$. Now consider the signed graph $\Sigma' = (\Gamma', \sigma')$, where for any edge e in Γ', $\sigma'(e)$ is the marking of the corresponding vertex in Γ. Then clearly, $R(\Sigma') \cong \Sigma$. Hence Σ is a radial signed graph.

Conversely, suppose that $\Sigma = (\Gamma, \sigma)$ is a radial signed graph. Then there exists a signed graph $\Sigma' = (\Gamma', \sigma')$ such that $R(\Sigma') \cong \Sigma$. Hence, Γ is the radial graph of Γ' and by Theorem 3, Σ is balanced. □

The following result characterizes the signed graphs which are isomorphic to radial signed graphs. In case of graphs the following result is due to Kathiresan and Marimuthu [9].

Theorem 2.4 *Let Γ be a graph of order n. Then $R(\Gamma) \cong \Gamma$ if, and only if, Γ is a connected graph with $r(\Gamma) = d(\Gamma) = 1$ or $r(\Gamma) = 1$ and $d(\Gamma) = 2$.*

Theorem 2.5 *For any connected signed graph $\Sigma = (\Gamma, \sigma)$, $\Sigma \sim R(\Sigma)$ if, and only if, Σ is balanced and the underlying graph Γ with $r(\Gamma) = d(\Gamma) = 1$ or $r(\Gamma) = 1$ and $d(\Gamma) = 2$.*

Proof Suppose $R(\Sigma) \sim \Sigma$. This implies, $R(\Gamma) \cong \Gamma$ and hence by Theorem 2.4, we see that the graph Γ satisfies the conditions in Theorem 2.4. Now, if Σ is any signed graph with underlying graph being $r(\Gamma) = d(\Gamma) = 1$ or $r(\Gamma) = 1$ and $d(\Gamma) = 2$, Theorem 2.1 implies that $R(\Sigma)$ is balanced and hence if Σ is unbalanced and its radial signed graph $R(\Sigma)$ being balanced can not be switching equivalent to Σ in accordance with Theorem 1.2. Therefore, Σ must be balanced.

Conversely, suppose that Σ balanced signed graph with the underlying graph Γ with $r(\Gamma) = d(\Gamma) = 1$ or $r(\Gamma) = 1$ and $d(\Gamma) = 2$. Then, since $R(\Sigma)$ is balanced as per Theorem 3 and since $R(\Gamma) \cong \Gamma$ by Theorem 2.4, the result follows from Theorem 1.2 again. □

In [9], the authors characterize the graphs for which $R(\Gamma) \cong \overline{\Gamma}$.

Theorem 2.6 *Let Γ be a graph of order n. Then $R(\Gamma) \cong \overline{\Gamma}$ if, and only if, either $S_2(\Gamma) = V(\Gamma)$ or Γ is disconnected in which each component is complete.*

In view of the above result, we have the following result that characterizes the family of signed graphs satisfies $R(\Sigma) \sim \overline{\Sigma}$.

Theorem 2.7 *For any signed graph $\Sigma = (\Gamma, \sigma)$, $R(\Sigma) \sim \overline{\Sigma}$ if, and only if, either $S_2(\Gamma) = V(\Gamma)$ or Γ is disconnected in which each component is complete.*

Proof Suppose that $R(\Sigma) \sim \overline{\Sigma}$. Then clearly, $R(\Gamma) \cong \overline{\Gamma}$. Hence by Theorem 2.6, Γ is either $S_2(\Gamma) = V(\Gamma)$ or disconnected in which each component is complete.

Conversely, suppose that Σ is a signed graph whose underlying graph is either $S_2(\Gamma) = V(\Gamma)$ or Γ is disconnected in which each component is complete. Then by Theorem 2.6, $R(\Gamma) \cong \overline{\Gamma}$. Since for any signed graph Σ, both $R(\Sigma)$ and $\overline{\Sigma}$ are balanced, the result follows by Theorem 1.2. □

The following result due to Kathiresan and Marimuthu [9]gives a characterization of graphs for which $R(\Gamma) \sim R(\overline{\Gamma})$.

Theorem 2.8 *Let* Γ *be a graph. Then* $R(\Gamma) \sim R(\overline{\Gamma})$ *if, and only if,* Γ *satisfies any one the following conditions:*

(1) Γ *or* $\overline{\Gamma}$ *is complete;*

(2) Γ *or* $\overline{\Gamma}$ *is disconnected with each component complete out of which one is an isolated vertex.*

We now give a characterization of signed graphs whose radial signed graphs are switching equivalent to their radial signed graph of complementary signed graphs.

Theorem 2.9 *For any signed graph* $\Sigma = (\Gamma, \sigma)$, $R(\Sigma) \sim R(\overline{\Sigma})$ *if, and only if,* Γ *satisfies the conditions of Theorem* 2.8.

The notion of *negation* $\eta(\Sigma)$ of a given signed graph Σ defined in [6] as follows:

$\eta(\Sigma)$ has the same underlying graph as that of Σ with the sign of each edge opposite to that given to it in Σ. However, this definition does not say anything about what to do with nonadjacent pairs of vertices in Σ while applying the unary operator $\eta(.)$ of taking the negation of Σ.

For a signed graph $\Sigma = (\Gamma, \sigma)$, the $E_k(\Sigma)$ is balanced (Theorem 2.1). We now examine, the conditions under which negation $\eta(\Sigma)$ of $E_k(\Sigma)$ is balanced.

Theorem 2.10 *Let* $\Sigma = (\Gamma, \sigma)$ *be a signed graph. If* $R(\Gamma)$ *is bipartite then* $\eta(R(\Sigma))$ *is balanced.*

Proof Since, by Theorem 2.1, $R(\Sigma)$ is balanced, if each cycle C in $R(\Sigma)$ contains even number of negative edges. Also, since $R(\Gamma)$ is bipartite, all cycles have even length; thus, the number of positive edges on any cycle C in $R(\Sigma)$ is also even. Hence $\eta(R(\Sigma))$ is balanced. □

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