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Projective Dimension and Betti Number of Some Graphs

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Abstract: Let G be a graph. Then $(G)_i$ denotes a graph such that to every vertex addes i pendant edges. In this paper, we study the projective dimension and Betti number of some graph such as $(S_n)_i$, $(K_{m,n})_i$, $\cdots$.

Key Words: Projective dimension, Betti number, graph.

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§1. Introduction

A *simple graph* is a pair $G = (V, E)$, where $V = V(G)$ and $E = E(G)$ are the sets of vertices and edges of G, respectively. A *path* is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A path with length n denotes by P_n. In a graph G, the *distance* between two distinct vertices x and y, denoted by $d(x, y)$, is the length of the shortest path connecting x and y, if such a path exists: otherwise, we set $d(x, y) = \infty$. The *diameter* of a graph G is $\mathrm{diam(G) = sup\{d(x, y)\ :\ xand\ y\ are\ distinct\ vertices\ of\ G\}}$. A *walk* is an alternating sequence of vertices and connecting edges. Also, a *cycle* is a path that begins and ends on the same vertex. A cycle with length n denotes by C_n. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. For a positive integer r, a *complete r-partite* graph is one in which each vertex is joined to every vertex that is not in the some subset. The *complete bipartite* graph with part sizes m and n is denoted by $K_{m,n}$. The graph $K_{1,n-1}$ is called a *star graph* in which the vertex with degree $n-1$ is called the center of the graph. For any graph G, we denote

$$N[x] = \{y \in V(G) :\ (x, y) \text{ is an edge of } G\} \cup \{x\}.$$

Recall that the *projective dimension* of an R-module M, denoted by $pd(M)$, is the length of the minimal free resolution of M, that is,

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$$pd(M) = \max\{i \mid \beta_{i,j}(M) \neq 0 \text{ for some } j\}.$$

There is a strong connection between the topology of the simplicial complex $\triangle$ and the structure of the free resolution of $\mathbb{K}[\triangle]$. Let $\beta_{i,j}(\triangle)$ denotes the $\mathbb{N}$-graded Betti numbers of the Stanley-Reisner ring $\mathbb{K}[\triangle]$. To any finite simple graph G with the vertex set $V(G) = \{x_1, \cdots, x_n\}$ and the edge set $E(G)$, one can attach an ideal in the Polynomial rings $R = \mathbb{K}[x_1, \cdots, x_n]$ over the field $\mathbb{K}$, whose generators are square-free quadratic monomials $x_i y_j$ such that (x_i, y_j) is an edge of G. This ideal is called the *edge ideal* of G and will be denoted by $I(G)$. Also the edge ring of G, denoted by $\mathbb{K}(G)$ is defined to be the quotient ring $\mathbb{K}(G) = R/I(G)$. Edge ideals and edge rings were first introduced by Villarreal [11] and then they have been studied by many authors in order to examine their algebraic properties according to the combinatorial data of graphs. The most important Algebraic objects among these are Betti numbers and positive dimension. The aim of this paper is to investigate the above mentioned algebraic properties of $(G)_i$, where $(G)_i$ is a graph such that to every vertex adds i pendent edges. In this paper, we denote S_n for a star graph with $n+1$ vertices.

§2. The Projective Dimension of Some Graphs

In this section, we study the projective dimension of some graphs. We begin this section with the following results.

Proposition 2.1([6], Proposition 2.2.8) *If G is the disjoint union of the two graphs G_1 and G_2, then* $\mathrm{pd(G) = pd(G_1) + pd(G_2)}$.

Corollary 2.2([6, Corollary 2.2.9]) *Let components are $G_1, \cdots, G_m$. Then the projective dimension of G is the sum of the projective dimensions of $G_1, \cdots, G_m$, i.e* $\mathrm{pd(G)} = \Sigma_{\mathrm{i=1}}^{\mathrm{m}} \mathrm{pd(G_i)}$.

Throughout this section, v will denote a vertex of T which has all but at most one of its neighbours of degree 1 (and if it has exactly one neighbour then that neighbour also has degree 1). The neighbours of v will be denoted $v_1, \cdots, v_n$ such that $v_1, \cdots, v_{n-1}$ all have degree 1. Also the neighbours of v_n other than v will be denoted by $w_1, \cdots, w_m$.

Let T denoted a forest and let $T^{'}$ denote the subgraph of T which is obtained by deleting the vertex v_1 and let $T^{''}$ denote the subgraph 0f T which is obtained by deleting the vertices $v, v_1, \cdots, v_n$. That is, $T^{'} = T \setminus T\{v_1\}$ and $T^{''} = T \setminus \{v, v_1, \cdots, v_n\}$. Note that $T^{'}$ and $T^{''}$ must both be forests.

Theorem 2.3([6, Theorem 9.4.17]) *Let* $p = \mathrm{pd(T)}$, $p^{'} = \mathrm{pd(T^{'})}$ *and* $p^{''} = \mathrm{pd(T^{''})}$. *Then projective dimension of the forest T is equal to* $p = \max\{\mathrm{p^{'}, p^{''} + n}\}$.

Theorem 2.4([6, Theorem 4.2.6]) *If G is a graph such that G^c is disconnected, then* $\mathrm{pd(G) = |V(G)| - 1}$.

Lemma 2.5([3, Lemma 3.2]) *Let x be a vertex of a graph G. Then* $\mathrm{pd(G) \leq \max\{pd(G - N[x]) + deg(x), pd(G - \{x\}) + 1\}}$.

Lemma 2.6([3, Observation 4.5]) *The maximum size of a minimal vertex cover of G equals BigHeight($I(G)$).*

In the following proposition, we investigate the projective dimension of graph G such that G is the graph obtained from S_n by adding i pendant edges to each vertex.

Proposition 2.7 *If G is the graph obtained from S_n by adding i pendant edges to each vertex, then $pd(G) = ni + 1$.*

Proof Let the set $\{u_0, u_1, \cdots, u_n\}$ be vertex set of S_n and the set $\{u_{j_1}, u_{j_2}, \cdots, u_{j_i}\}$ be the leaves the adjacent with vertex u_i for $0 \leq j \leq n$. Then, by Theorem **??**, we have

$$pd(G) = \max\{pd(G - \{u_1\}), pd(G - \{u_{1_1}, u_{1_2}, \cdots, u_{1_i}, u_1, u_0\}) + i + 1\}.$$

Also, Theorem 2.4 and Corollary 2.2,

$$pd(G - \{u_{1_1}, u_{1_2}, \cdots, u_{1_i}, u_1, u_0\}) + i + 1\} = (n-1)i.$$

By reusing of Theorem 2.3,

$$pd(G - \{u_{1_1}\} = \max\{pd(G - \{u_{1_1}, u_{1_2}\}), ni\}.$$

So we have,

$$pd(G) = \max\{pd(G - \{u_{1_1}, u_{1_2}\}), ni + 1\}.$$

Continuing this process we have,

$$pd(G) = \max\{pd(G - \{u_{1_1}, u_{1_2}, \cdots, u_{1_i}\}), ni + 1\}.$$

Now, let $G_1 = G - \{u_{1_1}, u_{1_2}, \cdots, u_{1_i}\}$. Then with the use of Lemma 2.5, we obtain,

$$pd(G_1) \leq \max\{pd(G_1 - N[u_0]) + \deg(u_0), pd(G_1 - \{u_0\}) + 1\}.$$

Since $pd(G_1 - N[u_0]) = 0, \quad \deg(u_0) = n + i$, we have,

$$pd(G_1) \leq \max\{mi + n, (n-1)i + 1\}.$$

Hence $pd(G) = ni + 1$. This completes the proof. □

In the next proposition, we study the projective dimension of graph G such that G is the graph obtained from $K_{m,n}$ by adding i pendant edges to each vertex.

Proposition 2.8 *If G is the graph obtained from $K_{m,n}$ by adding i pendant edges to each vertex, then $pd(G) = \max\{mi + n, ni + m\}$.*

Proof We do proof by induction on n. Suppose that $n = 1$ and $m \geq 1$. Then by Proposition 2.7, we have, $pd(G) = \max\{mi + 1, i + m\} = mi + 1$. Now, we may assume that $n > 1$ and $m > 1$. Also, let the result is true for each $K_{m,k}$ and $k < n$. Since the sets

$$\{x_1, x_2, \cdots, x_n, y_{1_1}, y_{1_2}, \cdots, y_{1_i}, \cdots, y_{m_1}, y_{m_2}, \cdots, y_{m_i}\},$$

and

$$\{y_1, y_2, \cdots, y_m, x_{1_1}, x_{1_2}, \cdots, x_{1_i}, \cdots, x_{n_1}, x_{n_2}, \cdots, x_{n_i}\},$$

are the two minimal vertex cover of maximal size. By the proof Lemma 2.6, we have

$$pd(G) \geq \text{Bight}(I(G)) = \max\{mi + n, ni + m\}.$$

On the other hand, by Lemma 2.5, we obtain

$$pd(G) \leq \max\{pd(G - N[x_1]) + m + i, pd(G - \{x_1\}) + 1\}.$$

Now, by Corollary 2.2, $pd(G - N[x_1]) = (n - i)$, and so by induction hypothesis,

$$pd(G - \{x_1\}) = \max\{mi + (n - 1), (n - 1)i + m\}.$$

Therefore

$$\begin{aligned} pd(G) &= \max\{ni + m, \max\{mi + (n - 1), (n - 1)i + m\}\} \\ &= \max\{mi + n, ni + m\}. \end{aligned}$$

Hence the result holds. □

Corollary 2.9 *If G is the graph obtained from $S_n \otimes S_m$ by adding i pendent edges to each vertex, then*

$$pd(G) = \max\{(mn + m)i + n + 1, (mn + n)i + m + 1\}$$

for $m, n \geq 1$. In particular, $pd(S_n \otimes S_m) = mn + m + n - 1$.

Proof Since $S_n \otimes S_m = S_{mn} \cup K_{m,n}$, we have for $i \geq 1$, $(S_n \otimes S_m)_i = (S_{mn})_i \cup (K_{m,n})_i$. So by Corollary 2.2, Propositions 2.7 and 2.8, the result holds. □

Lemma 2.10([4, Lemma 5.1]) *Let I be a squar-free monomial ideal and let Λ be any subset of the variables. We relabel the variables so that $\Lambda = \{x_1, \cdots, x_n\}$. Then either there exists a j with $1 \leq j \leq i$ such that* $\mathrm{pd(S/I) = pd(S/(I, x_1, \cdots, x_{j-1}) : x_j)}$ *or* $\mathrm{pd(S/I) = pd(S/(I, x_1, \cdots, x_i)}$.

Lemma 2.11 *Let x be a vertex of a G. Then we have*

(1) $pd(G) = pd(G - \{x\}) + 1$ or $pd(G - N[x]) + \deg(x)$;
(2) *If $pd(G - N[x]) + \deg(x) \geq pd(G - \{x\}) + 1$, then $pd(G) = pd(G - N[x]) + \deg(x)$.*

Proof (1) By the proof of Lemma 2.5, we have

$$pd\left(\frac{R}{(I(G):x)}\right) = pd(G - N[x]) + \deg(x),$$

and

$$pd\left(\frac{R}{(I(G),x)}\right) = pd(G - \{x\}) + 1.$$

Also, by Lemma 2.10, we have

$$pd(G) = pd\left(\frac{R}{(I(G):x)}\right) \quad \text{or} \quad pd(G) = pd\left(\frac{R}{(I(G),x)}\right).$$

Hence the result part (1) holds.

(2) If $pd(G - N[x]) + \deg(x) \geq pd(G - \{x\}) + 1$, then by Lemma 2.5, we have, $pd(G) \leq pd(G - N[x]) + \deg(x)$. Now, we consider the following short exact sequence

$$0 \longrightarrow \frac{R}{(I(G):x)} \longrightarrow \frac{R}{I(G)} \longrightarrow \frac{R}{(I(G),x)} \longrightarrow 0$$

Therefore, $pd(G) = pd\left(\frac{R}{I(G)}\right) \geq pd\left(\frac{R}{(I(G):x)}\right) = pd(G - N[x]) + \deg(x)$. Hence the result holds. □

In the following proposition, we investigate the projective dimension of graphs G and H such that G and H are graphs obtained from P_n and C_n by adding i pendant edges to each vertex, respectively.

Proposition 2.12 *If G and H are graphs obtained from P_n and C_n by adding i pendant edges to each vertex, then*

(1) $pd(G) = \left\lceil \frac{n}{2} \right\rceil i + \left\lfloor \frac{n}{2} \right\rfloor$;

(2) $pd(H) = \begin{cases} \frac{n-1}{2} i + \frac{n+1}{2} & if\ n\ is\ odd, \\ \frac{n}{2} i + \frac{n}{2} & if\ n\ is\ even. \end{cases}$

Proof (1) we do proof by induction on n. If $n = 2$, then G is the double star graph $(s_1)_i$. By Example 2.1.17 in [6], we have $pd(G) = i + 1$. For $n = 3$, let G be the graph shown in Figure 1. Then $pd(G - \{x\}) = pd(P_2)_i = pd(s_1)_i = i + 1$.

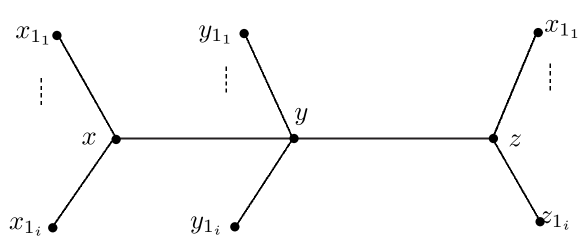

Figure 1

Also we have, $pd(G-N[x]) = pd(s_i) = i$. Hence $pd(G-N[x]) + \deg(x) \geq pd(G-\{x\})+1$, and so by Lemma 2.11, $pd(G) = pd(G-N[x]) + \deg(x) = 2i+1$. Now, let $n \geq 4$ and suppose that for each P_n of order less that n the result is true. Let G be the graph shown in Figure 2.

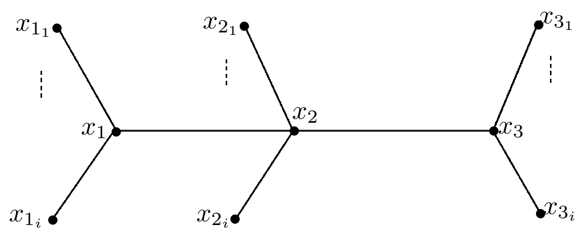

Figure 2

By the inductive hypothesis, we obtain

$$pd(G-\{x_1\}) = pd(P_{n-1})_i = \left\lceil \frac{n-1}{2} \right\rceil i + \left\lfloor \frac{n-1}{2} \right\rfloor,$$

and

$$pd(G-N[x_1]) = pd(P_{n-2})_i = \left\lceil \frac{n-2}{2} \right\rceil i + \left\lfloor \frac{n-2}{2} \right\rfloor.$$

Hence by Lemma 2.11, the proof is complete.

(2) First, Assume that n is a odd number. Then $H-\{x_1\} = (P_{n-1})_i$, and so $H-N[x_1] = (P_{n-3})_i$. If follows from part (1) and Lemma 2.11,

$$\begin{aligned} pd(H) &= pd(H-\{x_1\})+1 = pd(P_{n-1})i+1 \\ &= \left\lceil \frac{n-1}{2} \right\rceil i + \left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \frac{n}{2}i + \frac{n}{2} \end{aligned}$$

or

$$\begin{aligned} pd(H) &= pd(G-N[x_1]) + \deg(x_1) = pd(P_{n-3})_i + i + 2 \\ &= \left\lceil \frac{n-3}{2} \right\rceil i + \left\lfloor \frac{n-3}{2} \right\rfloor + i + 2 = \frac{n}{2}i + \frac{n}{2}. \end{aligned}$$

Hence the result hold. □

§3. The Betti Number of Some Graphs

In this section, we study the Betti number of two special graphs. We begin this section with the basic facts and the following results.

A *simplicial complex* $\triangle$ over a set of vertices $V = \{x_1, \cdots, x_n\}$ is a subset of the powerset of V with that property that, whenever $F \in \triangle$ and $G \subseteq F$, then $G \in \triangle$. The elements of $\triangle$ are called *faces* and the dimension of a face is $\dim(F) = |F| - 1$, where $|F|$ is the cardinality of F. Faces with dimension 0 are called vertices and those with dimension 1 are edges. A maximal face of $\triangle$ with respect to inclusion is called a facet of $\triangle$ and the dimension of $\triangle$, $dim(\triangle)$, is the maximum dimension of its faces. If $\triangle$ has an only facet, then it is called a *simplex*. Let $\triangle$ and $\triangle'$ be two simplicial complexes with vertex sets V and V', respectively. The union $\triangle \cup \triangle'$ defines as the simplicial complex with the vertex set $V \cup V'$ and F is a face of $\triangle \cup \triangle'$ if and only if F is a face of $\triangle$ or $\triangle'$. If $V \cap V' = \emptyset$, then the join $\triangle * \triangle'$ is the simplicial complex on the vertex set $V \cup V'$ with faces $F \cup F'$, where $F \in \triangle$ and $F' \in \triangle'$. The *cone* of $\triangle$, denoted by cone($\triangle$), is the join of a point $\{w\}$ with $\triangle$, that is, $\text{cone}(\triangle) = \triangle * \{w\}$. If $F \in \triangle$, then we define $x_F = \Pi_{x_i \in F} x_i \in R = \mathbb{K}[x_1, \cdots, x_n]$ for some field $\mathbb{K}$. The Stanley-Reisner ideal of $\triangle$, denoted by $I_\triangle$ is $I_\triangle = \langle x_F \mid F \notin \triangle \rangle$ and the Stanley-Reisner ring of $\triangle$ is $\mathbb{K}[\triangle] = \dfrac{R}{I_\triangle}$. Let $\beta_{i,j}(\triangle)$ denotes the $\mathbb{N}$-graded Betti numbers of the Stanley-Reisner ring $\mathbb{K}[\triangle]$. one of the most well-known results is the Hochster's formula.

Theorem 3.1([9, Hochster's formula]) *For $i > 0$, the $\mathbb{N}$-graded Betti number $\beta_{i,j}$ of a simplicial complex $\triangle$ are given by*

$$\beta_{i,j}(\triangle) = \sum_{W \subseteq V(\triangle), |w| = j} \dim_{\mathbb{K}} \widetilde{H}_{j-i-1}(\triangle|_w, \mathbb{K}).$$

Lemma 3.2([9]) *Let $\triangle_1$ and $\triangle_2$ be two simplicial complexes with disjoint vertex sets having m and n vertices, respectively. Also, let $\triangle = \triangle_1 \cup \triangle_2$. Then the $\mathbb{N}$-graded Betti numbers $\beta_{i,d}(\triangle)$ can be expressed as*

$$\begin{cases} \sum\limits_{j=0}^{d-2} \{\beta_{i-j,d-j}(\triangle_1) + \beta_{i-j,d-j}(\triangle_2)\} & if \quad d \neq i+1, \\ \sum\limits_{j=0}^{d-2} \{\beta_{i-j,d-j}(\triangle_1) + \beta_{i-j,d-j}(\triangle_2)\} + \sum\limits_{j=1}^{d-1} & if \quad d = i+1. \end{cases}$$

Lemma 3.3([9]) *Let G and H be two simple graphs whose vertex sets are disjoint. Then $\triangle_{G*H} = \triangle_G \cup \triangle_H$ is the disjoint union of two simplicial complexes.*

Lemma 3.4([6]) *If H is the induced subgraph of G on a subset of the vertices of G, then $\beta_{i,d}(H) \leq \beta_{i,d}(G)$ for all i.*

Proposition 3.5([11, Proposition 5.2.5]) *If $\triangle$ is a simplicial complex and $\text{cn}(\triangle) = w * \triangle$ its cone, then*

$$\widetilde{H}_p(\text{cn}(\triangle), \mathbb{K}) = 0,$$

for all p.

In the following theorem, we find a lower bound for the Betti number of graph $(K_{m,n})_i$.

Theorem 3.6 *Let $G = (K_{m,n})_i$. Then*

$$\beta_l(G) \geq \max\{ \sum_{j+k=l+1} \begin{pmatrix} mi+n \\ j \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix} \quad , \quad \sum_{j+k=l+1} \begin{pmatrix} ni+m \\ j \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix} \}.$$

Proof Suppose that $X = \{x_1, \cdots, x_m\}$ and $Y = \{y_1, \cdots, y_n\}$ be two parts of graph $K_{m,n}$. Also, let $X_r = \{x_{r_1}, \cdots, x_{r_i}\}$ and $Y_s = \{y_{s_1}, \cdots, y_{s_i}\}$ be the leaves, which are adjacent to x_r and y_s, respectively for $1 \leq r \leq m$ and $1 \leq s \leq n$. Now, let $G_1 = (K_{m,n})_i - \cup Y_s$. Then it is easy to see that $\triangle G_1 = \triangle_1 \cup \triangle_2$ such that $\triangle_1 = \langle \{x_1, \cdots, x_m\} \rangle$, and $\triangle_2 = \langle \{y_1, \cdots, y_n, x_{1_2}, \cdots, x_{1_i}, \cdots, x_{m_1}, \cdots, x_{m_i}\} \rangle$. Since $\triangle_1$ and $\triangle_2$ are simplexes, we have by Proposition **??**, $\widetilde{H}_i(\triangle_1, \mathbb{K}) = \widetilde{H}_i(\triangle_2, \mathbb{K}) = 0$ for all field $\mathbb{K}$. Now, let $W \neq \emptyset$. If $W \subseteq V(\triangle_1)$ or $W \subseteq V(\triangle_2)$, then $\triangle_W$ is a simplex. So for all i, $\widetilde{H}_i(\triangle_W, \mathbb{K}) = 0$. Therefore, Suppose that $W \cap V(\triangle_1) \neq \emptyset$ and $W \cap V(\triangle_2) \neq \emptyset$, and so $\triangle_W$ is a simplicial complex with two connected. Thus for all j, we have,

$$\widetilde{H}_j(\triangle_W, \mathbb{K}) = \begin{cases} 0 & j \neq 0, \\ \mathbb{K} & j = 0. \end{cases}$$

If $d = l+1$, the by Hochster's formula, we have

$$\begin{aligned} \beta_{l,d}(G_1) &= \sum_{W \subseteq V(\triangle), |W|=d} \dim \widetilde{H}(\triangle_W, \mathbb{K}) = \sum_{W \subseteq V(\triangle), |W|=d} 1 \\ &= \begin{pmatrix} mi+n \\ 1 \end{pmatrix} \begin{pmatrix} m \\ l \end{pmatrix} + \begin{pmatrix} mi+n \\ 2 \end{pmatrix} \begin{pmatrix} m \\ l-1 \end{pmatrix} \\ &\quad + \cdots + \begin{pmatrix} mi+n \\ l \end{pmatrix} \begin{pmatrix} m \\ 1 \end{pmatrix} \\ &= \sum_{j+k=l+1} \begin{pmatrix} mi+n \\ j \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix}. \end{aligned}$$

Therefore

$$\beta_l(G_1) = \sum_{d=1}^{|V(G_1)|} \beta_{l,d}(G_1) = \sum_{j+k=l+1} \begin{pmatrix} mi+n \\ j \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix}.$$

It follows by Lemma 3.4, $\beta(G) \geq \sum_{j+k=l+1} \begin{pmatrix} mi+n \\ j \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix}$ with using an argument similar, we can see that $\beta(G) \geq \sum_{j+k=l+1} \begin{pmatrix} mi+n \\ j \end{pmatrix} \begin{pmatrix} m \\ k \end{pmatrix}$. This completes the proof. □

As an immediate consequence of the preceding result, we obtain.

Corollary 3.7 *Let* $G = (S_n)_i$. *Then*

$$\beta_l(G) \geq \max\{\sum_{j+k=l+1} \begin{pmatrix} ni+1 \\ j \end{pmatrix} \begin{pmatrix} n \\ k \end{pmatrix}, \begin{pmatrix} n+i \\ l \end{pmatrix}\}.$$

Proof With assume that $m = 1$, the result follows from Theorem 3.5. □

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