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On Wiener and Weighted Wiener Indices of Neighborhood Corona of Two Graphs

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Abstract: In this paper, we compute the Wiener index, degree distance index and Gutman index of neighborhood corona of two graphs.

Key Words: Neighborhood corona, Wiener index, degree distance index, Gutman index.

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§1. Introduction

Let $G = (V, E)$ be a connected simple graph. The distance between two vertices u and v in G, denoted by $d_G(u, v)$ is the length of a shortest path between u and v in G. The degree of a vertex u in G, denoted by $d_G(u)$ is the number of vertices that are adjacent to u in G. The Wiener index $W(G)$ of a graph G is a distance based graph invariant introduced by H. Wiener [18] in order to determine the boiling point of paraffin. It is defined as the sum of distance between all pairs of vertices in G. i.e., $W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v)$. The degree distance index $DD(G)$ and Gutman index $Gut(G)$ of a graph are weighted versions of Wiener index, which are defined as follows:

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} (d_G(u) + d_G(v))d_G(u, v)$$

and

$$Gut(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u)\ d_G(v)\ d_G(u, v).$$

The degree distance index which is a degree distance based graph invariant, was introduced independently by A. A. Dobrynin, A. A. Kochetova [6] and I. Gutman [10]. The Gutman index, earlier known as Schultz index of the second kind was introduced in 1994 by Gutman [10]. It may be noted that if G is a tree on n vertices, then the Wiener index, degree distance index and Gutman index are closely related by the identities $DD(G) = 4W(G) - n(n - 1)$ and

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$Gut(G) = 4W(G) - (2n-1)(n-1)$. More details about Wiener index and its variants can be found in [2, 3, 4, 5, 6, 7, 12, 14, 17] and the references cited therein.

The corona [9] of two graphs G_1 and G_2 is the graph obtained by taking one copy of G_1, $|V(G_1)|$ copies of G_2 and joining each i-th vertex of G_1 to every vertex in the i-th copy of G_2. The neighborhood corona [13] of two graphs G_1 and G_2 denoted by $G_1 * G_2$, is a variant of corona of two graphs and is defined as the graph obtained by taking one copy of G_1 and $|V(G_1)|$ copies of G_2, and joining every neighbour of the i-th vertex of G_1 to every vertex in the i-th copy of G_2. Recently, various graph invariants of corona product of two graphs have been studied, for example, see [1, 15, 19].

Example 1.1 The neighborhood corona $P_3 * P_2$.

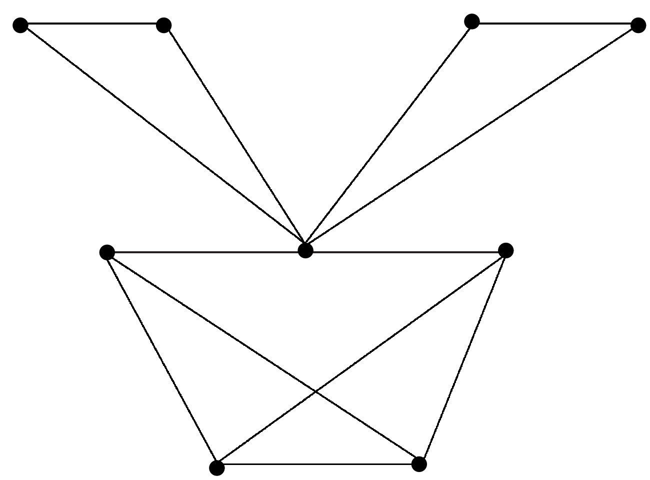

Fig. 1. $P_3 * P_2$.

In this paper, we compute Wiener index, degree distance index and Gutman index of $G_1 * G_2$.

§2. Main Results

Let G_1 be a graph with vertex set $V(G_1) = \{v_1, v_2, \cdots, v_{n_1}\}$, edge $E(G_1) = \{e_1, e_2, \cdots, e_{m_1}\}$ and let G_2 be a graph with vertex set $V(G_2) = \{u_1, u_2, \cdots, u_{n_2}\}$ and edge set $E(G_2) = \{e'_1, e'_2, \cdots, e'_{m_2}\}$. We denote the vertex set of the i-th copy of G_2 by $V_i(G_2) = \{u_{i1}, u_{i2}, \cdots, u_{in_2}\}$.

To prove our main results we need the following definitions and two lemmas whose proofs follows directly by the definition of neighborhood corona.

Definition 2.1 *For a graph G, we define*

$$E_\Delta(G) := \{e \in E(G) : e \text{ is contained in a triangle of } G\},$$

$$T_1(G) := \sum_{uv \in E_\Delta(G)} d(u) + d(v) \text{ and } T_2(G) := \sum_{uv \in E_\Delta(G)} d(u)d(v).$$

Clearly, if G has a vertex v of degree of $|V(G)| - 1$ and $G - v$ is connected graph with at least two vertices, then $E_\Delta(G) = E(G)$, $T_1(G) = M_1(G)$ and $T_2(G) = M_2(G)$.

Lemma 2.2 *Let $G = G_1 * G_2$. Then*

$$d_G(x) = \begin{cases} (n_2 + 1)d_{G_1}(x), & \text{if } x \in V(G_1), \\ d_{G_2}(x) + d_{G_1}(v_i), & \text{if } x \in V_i(G_2). \end{cases}$$

Lemma 2.3 *If $G = G_1 * G_2$, then*

(1) $d_G(v_i,\ v_j) = d_{G_1}(v_i,\ v_j),\ \forall\ v_i,\ v_j \in V(G_1)$;

(2) $d_G(u_{ij}, u_{ik}) = \begin{cases} 1, & \text{if } u_j u_k \in E(G_2), \\ 2, & \text{if } u_j u_k \notin E(G_2). \end{cases}$;

(3) *for* $i \neq k$, $d_G(u_{ij}, u_{km}) = \begin{cases} 3, & \text{if } v_i v_k \in E(G_1) \text{ and } v_i v_k \notin E_\Delta(G_1), \\ 2, & \text{if } v_i v_k \in E_\Delta(G_1), \\ d_{G_1}(v_i, v_k), & \text{if } v_i v_k \notin E(G_1). \end{cases}$;

(4) $d_G(u_{ij}, v_k) = \begin{cases} d_{G_1}(v_i, v_k), & \text{if } v_i \neq v_k, \\ 2, & \text{if } v_i = v_k. \end{cases}$.

Theorem 2.4 *The Wiener index of $G = G_1 * G_2$ is given by*

$$W(G) = (n_2 + 1)^2 W(G_1) + n_1(n_2(n_2 - 1) - m_2) + n_2^2(2m_1 - |E_\Delta(G_1)|) + 2n_1 n_2.$$

Proof We know that

$$W(G) = \sum_{\{x,\ y\} \subseteq V(G)} d_G(x,\ y) = A_1 + A_2 + A_3 + A_4, \tag{2.1}$$

where

$$A_1 = \sum_{\{v_i, v_j\} \subseteq V(G_1)} d_G(v_i,\ v_j),$$

$$A_2 = \sum_{v_i \in V(G_1)} \sum_{\{u_{ij}, u_{ik}\} \subseteq V_i(G_2)} d_G(u_{ij},\ u_{ik}),$$

$$A_3 = \sum_{\{v_i, v_j\} \subseteq V(G_1)} \sum_{\substack{u_{ik} \in V_i(G_2) \\ u_{jm} \in V_j(G_2)}} d_G(u_{ik},\ u_{jm})$$

and

$$A_4 = \sum_{v_i \in V(G_1)} \sum_{\substack{u_{ij} \in V_i(G_2) \\ v_k \in V(G_1)}} d_G(u_{ij},\ v_k).$$

By Lemma 2.3, we have

$$A_1 = \sum_{\{v_i,\ v_j\} \subseteq V(G_1)} d_G(v_i,\ v_j) = \sum_{\{v_i, v_j\} \subseteq V(G_1)} d_{G_1}(v_i,\ v_j) = W(G_1), \tag{2.2}$$

$$\begin{aligned} A_2 &= \sum_{v_i \in V(G_1)} \sum_{\{u_{ij}, u_{ik}\} \subseteq V_i(G_2)} d_G(u_{ij},\ u_{ik}) \\ &= \sum_{v_i \in V(G_1)} \left\{ \sum_{\{u_{ij}, u_{ik}\} \subseteq V_i(G_2)} 2 - \sum_{u_j u_k \in E(G_2)} 1 \right\} \\ &= \sum_{v_i \in V(G_1)} (n_2(n_2 - 1) - m_2) = n_1(n_2(n_2 - 1) - m_2), \end{aligned} \tag{2.3}$$

$$\begin{aligned} A_3 &= \sum_{\{v_i,\ v_j\} \subseteq V(G_1)} \sum_{\substack{u_{ik} \in V_i(G_2) \\ u_{jm} \in V_j(G_2)}} d_G(u_{ik}, u_{jm}) \\ &= n_2^2 \left\{ \sum_{\{v_i,\ v_j\} \subseteq V(G_1)} d_{G_1}(v_i,\ v_j) + \sum_{v_i v_j \in E(G_1)} 2 - \sum_{v_i v_j \in E_\Delta(G_1)} 1 \right\} \\ &= n_2^2(W(G_1) + 2m_1 - |E_\Delta(G_1)|) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} A_4 &= \sum_{v_i \in V(G_i)} \left\{ \sum_{\substack{u_{ij} \in V_i(G_2) \\ v_k \in V(G_1) \\ v_k \neq v_i}} d_G(v_k,\ u_{ij}) + \sum_{u_{ij} \in V_i(G_2)} 2 \right\} \\ &= n_2 \sum_{v_i \in V(G_1)} \sum_{v_k \in V(G_1)} d_{G_1}(v_i,\ v_k) + 2n_1 n_2 \\ &= 2n_2 W(G_1) + 2n_1 n_2. \end{aligned} \tag{2.5}$$

Applying (2.2), (2.3), (2.4) and (2.5) in (2.1), we obtain the desired result. □

Corollary 2.5 *If G_1 is a triangle free graph, then the Wiener index of $G = G_1 * G_2$ is given by*

$$W(G) = (n_2+1)^2 W(G_1) + n_1(n_2(n_2-1) - m_2) + 2n_2^2 m_1 + 2n_1 n_2.$$

Corollary 2.6 *If $G_1 = H_1 \vee H_2$ (join of two connected graphs H_1 and H_2 with $|V(H_1)| \geqslant 2$), then the Wiener index of $G = G_1 * G_2$ is given by*

$$W(G) = (n_2+1)^2 W(G_1) + n_1(n_2(n_2-1) - m_2) + n_2^2 m_1 + 2n_1 n_2.$$

Lemma 2.7([4]) *Let P_n and C_n denote the path and cycle on n vertices, respectively. Then*

$$W(P_n) = \frac{n(n^2-1)}{6}$$

and

$$W(C_n) = \begin{cases} n^3/8, & \text{if } n \text{ is even,} \\ n(n^2-1)/8, & \text{if } n \text{ is odd.} \end{cases}$$

Applying the above lemma in Theorem 2.4, we obtain the following corollary.

Corollary 2.8 (1) $W(P_n * P_m) = \frac{1}{6}((m+1)^2 n^3 + (17m^2 - 2m + 5)n) - 2m^2$;

(2) $W(C_{2n} * C_m) = ((m+1)^2 n^2 + 6m^2)n$;

(3) For $n \neq 1$, $W(C_{2n+1} * C_m) = (2n+1)(m^2n^2 + m^2n + 2mn^2 + 6m^2 + 2mn + n^2 + n)/2$;

(4) *For* $n \neq 1$, $W(C_{2n+1} * P_m) = (2n+1)(m^2n^2 + m^2n + 2mn^2 + 6m^2 + 2mn + n^2 + n + 2)/2$;

(5) $W(C_{2n} * P_m) = m^2n^3 + 2mn^3 + 6m^2n + n^3 + 2n$;

(6) $W(P_n * C_m) = \frac{1}{6}((m+1)^2 n^3 + (17m^2 - 2m - 1)n) - 2m^2$.

The first and second Zagreb indices of a graph denoted by $M_1(G)$ and $M_2(G)$, respectively, are degree based topological indices introduced by Gutman and N. Trinajstić ([11]). These two indices are defined as

$$M_1(G) = \sum_{e_i = v_l v_m \in E(G)} d_G(v_l) + d_G(v_m) = \sum_{v_i \in G} d_G^2(v_i)$$

and

$$M_2(G) = \sum_{e_i = v_l v_m \in E(G)} d_G(v_l) d_G(v_m).$$

Now, we derive a formula for $DD(G_1 * G_2)$ in terms of degree distance of G_1, Wiener index of G_1 and first Zagreb index of G_1 and G_2.

Theorem 2.9 *The degree distance index of $G = G_1 * G_2$ is given by*

$$\begin{aligned} DD(G) = &(2n_2^2 + 3n_2 + 1)DD(G_1) + 4m_2(n_2 + 1)W(G_1) + n_2^2(2M_1(G_1) - T_1(G_1)) \\ &- n_1M_1(G_2) + (8n_2^2 + (8m_2 + 4)n_2 - 4m_2)m_1 + 4m_2n_2(n_1 - |E_\Delta(G_1)|). \end{aligned}$$

Proof We know that

$$DD(G) = \sum_{\{x,y\}\subseteq V(G)} (d_G(x) + d_G(y))\ d_G(x, y) = A_1 + A_2 + A_3 + A_4, \tag{2.6}$$

where

$$\begin{aligned} A_1 &= \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} (d_G(v_i) + d_G(v_j))\ d_G(v_i,\ v_j), \\ A_2 &= \sum_{v_i\in V(G_1)} \sum_{\{u_{ij},u_{ik}\}\subseteq V_i(G_2)} [d_G(u_{ij}) + d_G(u_{ik})]\ d_G(u_{ij},\ u_{ik}), \\ A_3 &= \sum_{\{v_i,v_j\}\subseteq V(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2) \\ u_{jm}\in V_j(G_2)}} [d_G(u_{ik}) + d_G(u_{jm})]\ d_G(u_{ik},\ u_{jm}) \end{aligned}$$

and $A_4 = \sum_{v_i\in V(G_1)} \sum_{\substack{u_{ij}\in V_i(G_2) \\ v_k\in V(G_1)}} [d_G(u_{ij}) + d_G(v_k)]d_G(u_{ij}, v_k).$

Applying Lemmas 2.2 and 2.3, we compute A_1, A_2, A_3 and A_4 as follows:

$$\begin{aligned} A_1 &= \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} (d_G(v_i) + d_G(v_j))\ d_G(v_i,\ v_j) \\ &= (n_2 + 1) \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} (d_{G_1}(v_i) + d_{G_1}(v_j))\ d_{G_1}(v_i,\ v_j) \\ &= (n_2 + 1)DD(G_1). \end{aligned} \tag{2.7}$$

$$\begin{aligned} A_2 = &\sum_{v_i\in V(G_1)} \sum_{\{u_{ij},u_{ik}\}\subseteq V_i(G_2)} [d_G(u_{ij}) + d_G(u_{ik})]\ d_G(u_{ij},\ u_{ik}) \\ &= \sum_{v_i\in V(G_1)} \left\{ 2 \sum_{\{u_{ij},u_{ik}\}\subseteq V_i(G_2)} [2d_{G_1}(v_i) + \ d_{G_2}(u_j) + \ d_{G_2}(u_k)] \right. \\ &\left. - \sum_{u_ju_k\in E(G_2)} [2d_G(v_i) + d_{G_2}(u_j) + d_{G_2}(u_k)] \right\} \\ &= \sum_{v_i\in V(G_1)} \{2(n_2(n_2 - 1)d_{G_1}(v_i) + 2(n_2 - 1)m_2) - 2m_2d_G(v_i) - M_1(G_2)\} \\ &= 4(n_2(n_2 - 1) - m_2)m_1 + 4n_1m_2(n_2 - 1) - n_1M_1(G_2). \end{aligned} \tag{2.8}$$

$$
\begin{aligned}
A_3 &= \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} [d_G(u_{ik}) + d_G(u_{jm})]\ d_G(u_{ik},\ u_{jm}) \\
&= \sum_{\{v_i,v_j\}\subseteq V(G_1)} d_{G_1}(v_i,\ v_j) \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} [d_{G_1}(v_i) + d_{G_1}(v_j) + d_{G_2}(u_{ik}) + d_{G_2}(u_{jm})] \\
&+ 2\sum_{v_iv_j\in E(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} [d_{G_1}(v_i) + d_{G_1}(v_j) + d_{G_2}(u_{ik}) + d_{G_2}(u_{jm})] \\
&- \sum_{v_iv_j\in E_\Delta(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} [d_{G_1}(v_i) + d_{G_1}(v_j) + d_{G_2}(u_{ik}) + d_{G_2}(u_{jm})] \\
&= \sum_{\{v_i,v_j\}\subseteq V(G_1)} d_{G_1}(v_i,\ v_j)\ [n_2^2(d_{G_1}(v_i) + d_{G_1}(v_j)) + 4n_2m_2] \\
&+ 2\sum_{v_iv_j\in E(G_1)} [n_2^2(d_{G_1}(v_i) + d_{G_1}(v_j)) + 4n_2m_2] \\
&- \sum_{v_iv_j\in E_\Delta(G_1)} [n_2^2(d_{G_1}(v_i) + d_{G_1}(v_j)) + 4n_2m_2] \\
&= n_2^2DD(G_1) + 4n_2m_2W(G_1) + n_2^2(2M_1(G_1) - T_1(G_1)) + 4n_2m_2(2m_1 - |E_\Delta(G_1)|). \qquad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
A_4 &= \sum_{v_i\in V(G_1)} \sum_{\substack{u_{ij}\in V_i(G_2)\\ v_k\in V(G_1)}} [d_G(u_{ij}) + d_G(v_k)] d_G(u_{ij},\ v_k) \\
&= \sum_{v_i\in V(G_1)} \left\{ \sum_{\substack{u_{ij}\in V_i(G_2)\\ v_k\in V(G_1)\\ v_i\neq v_k}} [d_G(u_{ij}) + d_G(v_k)] d_G(u_{ij},\ v_k) + 2\sum_{u_{ij}\in V_i(G_2)} (d_G(u_{ij}) + d_G(v_i)) \right\} \\
&= \sum_{v_i\in V(G_1)} \left\{ \sum_{v_k\in V(G_1)} (n_2d_{G_1}(v_i) + n_2(n_2+1)d_{G_1}(v_k) + 2m_2)d_{G_1}(v_i,\ v_k) + 2[2m_2 + n_2(n_2+2)d_G(v_i)] \right\} \\
&= (n_2^2 + 2n_2)DD(G_1) + 4m_2W(G_1) + 4n_1m_2 + 4m_1(n_2+2)n_2. \qquad (2.10)
\end{aligned}
$$

Applying (2.7), (2.8), (2.9) and (2.10) in (2.6), we obtain the desired result. □

Corollary 2.10 *If G_1 is a triangle free graph, then the degree distance index of $G = G_1 * G_2$ is given by*

$$
\begin{aligned}
DD(G) &= (2n_2^2 + 3n_2 + 1)DD(G_1) + 4m_2(n_2+1)W(G_1) + 2n_2^2M_1(G_1) \\
&- n_1M_1(G_2) + (8n_2^2 + (8m_2+4)n_2 - 4m_2)m_1 + 4m_2n_2n_1.
\end{aligned}
$$

Corollary 2.11 *If $G_1 = H_1 \vee H_2$ (join of two connected graphs H_1 and H_2 with $|V(H_1)| \geqslant 2$),* *then the degree distance index of $G = G_1 * G_2$ is given by*

$$DD(G) = (2n_2^2 + 3n_2 + 1)DD(G_1) + 4m_2(n_2 + 1)W(G_1) + n_2^2 M_1(G_1) \\ - n_1 M_1(G_2) + (2n_2^2 + (m_2 + 1)n_2 - m_2)4m_1 + 4m_2 n_2 n_1.$$

Lemma 2.12([7,16]) *Let P_n and C_n denote the path and cycle on n vertices, respectively. Then*

$$DD(P_n) = \frac{n(n-1)(2n-1)}{3}$$

and

$$DD(C_n) = \begin{cases} n^3/2, & \text{if n is even,} \\ \\ n(n^2-1)/2, & \text{if n is odd.} \end{cases}$$

Using Lemmas 2.7, 2.12 and also the facts that $M_1(P_n) = 4n - 6$ $(n \geqslant 2), M_2(P_n) = 4n - 8$ $(n \geqslant 3)$, $M_1(C_n) = M_2(C_n) = 4n$ in Theorem 2.9, we obtain the following corollary.

Corollary 2.13 (1) *$DD(P_n * P_m) = (2n^3 - 2n^2 + 28n - 28)m^2 + (2n^3 - 3n^2 - 15n + 8)m - n^2 + 11n - 4$;*

(2) *$DD(C_{2n} * C_m) = 4n(3m^2n^2 + 4mn^2 + 14m^2 + n^2 - 2m)$;*

(3) *for $n \neq 1$, $DD(C_{2n+1} * C_m) = 2(2n+1)(3m^2n^2 + 3m^2n + 4mn^2 + 14m^2 + 4mn + n^2 - 2m + n)$;*

(4) *for $n \neq 1$, $DD(C_{2n+1} * P_m) = 2(2n+1)(3m^2n^2 + 3m^2n + 3mn^2 + 14m^2 + 3mn - 8m + 5)$;*

(5) *$DD(C_{2n} * P_m) = 4n(3m^2n^2 + 3mn^2 + 14m^2 - 8m + 5)$;*

(6) *$DD(P_n * C_m) = \dfrac{1}{3}((6m^2 + 8m + 2)n^3 - (6m^2 + 9m + 3)n^2 + (84m^2 - 11m + 1)n) - 28m^2$.*

Now, we derive a formula for $Gut(G_1 * G_2)$ in terms of degree distance of G_1, Gutman index of G_1, Wiener index of G_1 and Zagreb indices of G_1 and G_2.

Theorem 2.14 *The Gutman index of $G = G_1 * G_2$ is given by*

$$Gut(G) = (2n_2 + 1)^2 Gut(G_1) + 2m_2(2n_2 + 1)DD(G_1) + 4m_2^2 W(G_1) - (n_1 + 2m_1)M_1(G_2) \\ - n_1 M_2(G_2) + (n_2(3n_2 + 4m_2 + 1) - m_2)M_1(G_1) + n_2^2(2M_2(G_1) - T_2(G_1)) \\ - 2n_2 m_2 T_1(G_1) - 4m_2^2 |E_\Delta(G_1)| + 8m_1 m_2[2n_2 + m_2] + 4n_1 m_2^2.$$

Proof Notice that

$$Gut(G) = \sum_{\{x,y\} \subseteq V(G)} (d_G(x)\ d_G(y))\ d_G(x,\ y) = A_1 + A_2 + A_3 + A_4, \tag{2.11}$$

where

$$A_1 = \sum_{\{v_i,v_j\}\subseteq V(G_1)} d_G(v_i)\ d_G(v_j))\ d_G(v_i,\ v_j),$$
$$A_2 = \sum_{v_i\in V(G_1)} \sum_{\{u_{ij},u_{ik}\}\subseteq V_i(G_2)} d_G(u_{ij})\ d_G(u_{ik})\ d_G(u_{ij},\ u_{ik}),$$
$$A_3 = \sum_{\{v_i,v_j\}\subseteq V(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} d_G(u_{ik})\ d_G(u_{jm})\ d_G(u_{ik},\ u_{jm})$$

and $$A_4 = \textstyle\sum_{v_i\in V(G_1)} \frac{dv}{dx} \sum_{\substack{u_{ij}\in V_i(G_2)\\ v_k\in V(G_1)}} d_G(u_{ij})\ d_G(v_k)\ d_G(u_{ij},\ v_k).$$

Applying Lemmas 2.2 and 2.3, $A_i (i = 1, 2, 3, 4)$ can be computed as follows:

$$\begin{aligned} A_1 &= \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} d_G(v_i)\ d_G(v_j)\ d_G(v_i,\ v_j) \\ &= (n_2+1)^2 \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} [d_{G_1}(v_i)\ d_{G_1}(v_j)]\ d_{G_1}(v_i,\ v_j) = (n_2+1)^2 Gut(G_1), \end{aligned} \tag{2.12}$$

$$\begin{aligned} A_2 &= \sum_{v_i\in V(G_1)} \sum_{\{u_{ij},\ u_{ik}\}\subseteq V_i(G_2)} d_G(u_{ij})\ d_G(u_{ik})\ d_G(u_{ij},\ u_{ik}) \\ &= \sum_{v_i\in V(G_1)} \left\{ 2 \sum_{\{u_{ij},\ u_{ik}\}\subseteq V_i(G_2)} d_G(u_{ij})\ d_G(u_{ik}) - \sum_{u_j u_k\in E(G_2)} d_G(u_{ij})\ d_G(u_{ik}) \right\} \\ &= \sum_{v_i\in V(G_1)} \left\{ 2 \sum_{\{u_{ij},\ u_{ik}\}\subseteq V_i(G_2)} [d_{G_2}(u_{ij})\ d_{G_2}(u_{ik}) + d_{G_1}(v_i)\ (d_{G_2}(u_{ij}) + d_{G_2}(u_{ik})) + d_G^2(v_i)] \right. \\ &\quad \left. - \sum_{u_j u_k\in E(G_2)} (d_{G_2}(u_{ij})\ d_{G_2}(u_{ik}) + d_{G_1}(v_i)\ [d_{G_2}(u_{ij}) + d_{G_2}(u_{ik})] + d_G^2(v_i)) \right\} \\ &= \sum_{v_i\in V(G_1)} \Big\{ 4m_2^2 - M_1(G_2) + 4(n_2-1)m_2 d_{G_1}(v_i) + n_2(n_2-1)d_G^2(v_i) \\ &\quad - M_2(G_2) - d_G(v_i)M_1(G_2) - m_2 d_G^2(v_i) \Big\} \\ &= n_1(4m_2^2 - M_2(G_2)) - (n_1+2m_1)M_1(G_2) + 8m_1 m_2(n_2-1) + (n_2(n_2-1) - m_2)M_1(G_1), \end{aligned} \tag{2.13}$$

$$\begin{aligned} A_3 &= \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} d_G(u_{ik})\ d_G(u_{jm})\ d_G(u_{ik},\ u_{jm}) \\ &= \sum_{\{v_i,\ v_j\}\subseteq V(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} d_G(u_{ik})\ d_G(u_{jm})\ d_{G_1}(v_i,\ v_j) \\ &\quad + 2 \sum_{v_i v_j\in E(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} d_G(u_{ik})\ d_G(u_{jm}) - \sum_{v_i v_j\in E_\Delta(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} d_G(u_{ik})\ d_G(u_{jm}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\{v_i,v_j\}\subseteq V(G_1)} d_{G_1}(v_i,\ v_k) \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} [d_{G_1}(v_i)+d_{G_2}(u_k)]\ [d_{G_1}(v_j)+d_{G_2}(u_m)]\\
&\quad + 2\sum_{v_iv_j\in E(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} [d_{G_1}(v_i)+d_{G_2}(u_k)]\ [d_{G_1}(v_j)+d_{G_2}(u_m)]\\
&\quad - \sum_{v_iv_j\in E_\Delta(G_1)} \sum_{\substack{u_{ik}\in V_i(G_2)\\ u_{jm}\in V_j(G_2)}} [d_{G_1}(v_i)+d_{G_2}(u_k)]\ [d_{G_1}(v_j)+d_{G_2}(u_m)]\\
&= \sum_{\{v_i,v_j\}\subseteq V(G_1)} d_{G_1}(v_i,\ v_k) \left\{n_2^2 d_{G_1}(v_i)d_{G_1}(v_j)+2n_2m_2(d_{G_1}(v_i)+d_{G_1}(v_j))+4m_2^2\right\}\\
&+2\sum_{v_iv_j\in E(G_1)} \left\{n_2^2 d_{G_1}(v_i)d_{G_1}(v_j)+2n_2m_2\ (d_{G_1}(v_i)+d_{G_1}(v_j))+4m_2^2\right\}\\
&\quad - \sum_{v_iv_j\in E_\Delta(G_1)} \left\{n_2^2 d_{G_1}(v_i)d_{G_1}(v_j)+2n_2m_2\ (d_{G_1}(v_i)+d_{G_1}(v_j))+4m_2^2\right\}\\
&= n_2^2 Gut(G_1)+2n_2m_2DD(G_1)+4m_2^2W(G_1)+n_2^2(2M_2(G_1)-T_2(G_1))\\
&\quad +2n_2m_2(2M_1(G_1)-T_1(G_1))+4m_2^2(2m_1-|E_\Delta(G_1)|).
\end{aligned}
\tag{2.14}
$$

$$
\begin{aligned}
A_4 &= \sum_{v_i\in V(G_1)} \left\{ \sum_{\substack{u_{ij}\in V_i(G_2)\\ v_k\in V(G_1)\\ v_i\neq v_k}} d_G(u_{ij})\ d_G(v_k)\ d_{G_1}(v_i,\ v_k) + 2\sum_{u_{ij}\in V_i(G_2)} d_G(u_{ij})\ d_G(v_i) \right\}\\
&= (n_2+1)\sum_{v_i\in V(G_1)} \left\{ \sum_{v_k\in V(G_1)} d_{G_1}(v_i,\ v_k)\ d_{G_1}(v_k)\ (2m_2+n_2\ d_{G_1}(v_i)) \right.\\
&\quad \left. +2\ (n_2 d_{G_1}^2(v_i)+\ 2m_2\ d_{G_1}(v_i)) \right\}\\
&= 2(n_2+1)(m_2DD(G_1)+n_2Gut(G_1)+n_2M_1(G_1)+4m_1m_2).
\end{aligned}
\tag{2.15}
$$

Using (2.12), (2.13), (2.14) and (2.15) in (2.11), we obtain the required result. □

Corollary 2.15 *If G_1 is a triangle free graph, then the Gutman index of $G = G_1 * G_2$ is given by*

$$
\begin{aligned}
Gut(G) &= (2n_2+1)^2 Gut(G_1)+2m_2(2n_2+1)DD(G_1)+4m_2^2W(G_1)\\
&\quad -(n_1+2m_1)M_1(G_2)-n_1M_2(G_2)+(n_2(3n_2+4m_2+1)-m_2)M_1(G_1)\\
&\quad +2n_2^2M_2(G_1)+8m_1m_2[2n_2+m_2]+4n_1m_2^2.
\end{aligned}
$$

Corollary 2.16 *If $G_1 = H_1 \vee H_2$ (join of two connected graphs H_1 and H_2 with $|V(H_1)| \geqslant 2$), then the Gutman index of $G = G_1 * G_2$ is given by*

$$\begin{aligned}Gut(G) &= (2n_2+1)^2 Gut(G_1) + 2m_2(2n_2+1)DD(G_1) + 4m_2^2 W(G_1)\\ &\quad - (n_1+2m_1)M_1(G_2) - n_1 M_2(G_2) + (n_2(3n_2+2m_2+1) - m_2)M_1(G_1)\\ &\quad + n_2^2 M_2(G_1) + 4m_1 m_2[4n_2+m_2] + 4n_1 m_2^2.\end{aligned}$$

Lemma 2.17([8]) *Let P_n and C_n denote the path and the cycle on n vertices, respectively. Then*

$$Gut(P_n) = (n-1)(2n^2-4n+3)/3$$

and

$$Gut(C_n) = \begin{cases} n^3/2, \text{ if n is even,} \\ \\ n(n^2-1)/2, \text{ if n is odd.} \end{cases}$$

Applying Lemmas 2.7, 2.12 and 2.17 in Theorem 2.14, we obtain the following corollary.

Corollary 2.18 (1) *For $n, m \geqslant 3$, $Gut(P_n * P_m) = (6n^3 - 12n^2 + 74n - 86)m^2 - (6n^2 + 62n - 60)m + 43n - 27$;*

(2) *$Gut(C_{2n} * C_m) = 4n(9m^2n^2 + 6mn^2 + 32m^2 + n^2 - 8m)$;*

(3) *For $n \neq 1$, $Gut(C_{2n+1} * C_m) = 2(2n+1)(9m^2n^2 + 9m^2n + 6mn^2 + 32m^2 + 6mn + n^2 - 8m + n)$;*

(4) *For $n \neq 1$, $Gut(C_{2n+1} * P_m) = 2(2n+1)(9m^2n^2 + 9m^2n + 32m^2 - 36m + 21)$;*

(5) *$Gut(C_{2n} * P_m) = 4n(9m^2n^2 + 32m^2 - 36m + 21)$;*

(6) *$Gut(P_n * C_m) = 6(m+1/3)^2n^3 - \frac{1}{3}((36m^2 + 30m + 6)n^2 + (222m^2 - 18m + 7)n) - 86m^2 + 4m - 1$.*

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