International J.Math. Combin. Vol.4(2016), 61-69

On the Riemann and Ricci Curvature of Finsler Spaces with Special (α, β)- Metric

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Abstract: In this paper, we study the curvature properties of the special (α, β)- metric $F = \alpha + \epsilon\beta + k\dfrac{\beta^2}{\alpha}$ (where $\epsilon, k \neq 0$ are constants). We find the expressions for Riemann curvature and Ricci curvature of the special (α, β)-metric, when β the 1- form is a killing form of constant length. We give a characterization of the projective flatness for the special (α, β)- metric.

Key Words: Finsler space with (α, β)-metric, Ricci curvature, Riemmann curvature, projectively flat.

AMS(2010): 53B40, 53C60

§1. Introduction

A Finsler metric F(x, y) on an n-dimensional manifold M^n is called an (α, β)-metric ([4]) F(x,y), if F is positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M^n. The (α, β)-metrics form an important class of Finsler metrics appearing iteratively in formulating physics, mechanics, Seismology, Biology, control theory, etc ([1], [6]). There are several interesting curvatures in Finsler geometry ([2], [5]), among them two important curvatures are Riemann curvature and Ricci curvature.

Riemannian metrics on a manifold are quadratic metrics, while Finsler metrics are those without restriction on the quadratic property. The Riemannian curvature in Riemannian geometry can be extended to Finsler metrics as a family of linear transformations on the tangent spaces. The Ricci curvature plays an important role in the geometry of Finsler manifolds and is defined as the trace of the Riemannian curvature on each tangent space.

Consider the Finsler space $F^n = (M^n, F)$ that is equipped with the special (α, β)-metric $F = \alpha + \epsilon\beta + k\dfrac{\beta^2}{\alpha}$ ($\epsilon \neq 0$, $k \neq 0$ are constants), where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric

[1]Received April 27, 2016, Accepted November 10, 2016.

and $\beta = b_i(x)y^i$ is a 1-form on an n-dimensional manifold M^n. Then the space $R^n = (M^n, \alpha)$ is called the associated Riemannian space with $F^n = (M^n, F)$. The covariant differentiation with respect to the Levi Civita connection $\gamma^i_{jk}(x)$ of R^n is denoted by (:). We put $a^{ij} = (a_{ij})^{-1}$

The main purpose of the current paper is to investigate the curvature properties of the special (α, β)-metric $\alpha + \epsilon\beta + k\dfrac{\beta^2}{\alpha}$ $(\epsilon, k \neq 0)$. The paper is organized as follows: Starting with literature survey in section one, we find the Riemann curvature and Ricci curvature of the Finsler space with special (α, β)- metric $\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ in section two (see Theorem 2.1). In section three, we obtain the necessary and sufficient conditions for a Finsler space with (α, β)-metric to be locally projectively flat (see Theorem 3.1).

§2. Riemann curvature and Ricci curvature of special (α, β)-metric $\alpha + \epsilon\beta + k\dfrac{\beta^2}{\alpha}$

Let F be a Finsler metric on an n-dimensional manifold M and G^i be the geodesic coefficient of F, which is defined by

$$G^i = \frac{1}{4}g^{il}\{[F^2]_{x^m y^l}y^m - [F^2]_{x^l}\}. \tag{1}$$

For any $x \in M$ and $y \in T_xM \diagdown \{0\}$, the Riemann curvature $\mathbf{R}_y = R^i_m \frac{\partial}{\partial x^i} \otimes dx^m : T_xM^n \to T_xM^n$ is defined by

$$R^i_m = 2\frac{\partial G^i}{\partial x^m} - \frac{\partial^2 G^i}{\partial x^m \partial y^m}y^m + 2G^m\frac{\partial^2 G^i}{\partial y^m \partial y^m} - \frac{\partial G^i}{\partial y^m}\frac{\partial G^m}{\partial y^m}. \tag{2}$$

The Ricci curvature is the trace of the Riemann curvature, and the Ricci scalar is defined by

$$Ric = R^i_i, \quad R = \frac{1}{n-1}Ric. \tag{3}$$

By definition, an (α, β)-metric on M is expressed in the form $F = \alpha\phi(s), s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}(x)y^i y^m}$ is a positive definite Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form. It is known that (α, β)-metric with $\|\beta_x\|_\alpha < b_0$ is a Finsler metric if and only if $\phi = \phi(s)$ is a positive smooth function on an open interval $(-b_0, b_0)$ satisfying the following conditions:

$$\phi(s) - s\phi' + (b^2 - s^2)\phi''(s) > 0, \quad \forall\ |s| \leq b < b_0. \tag{4}$$

For a special (α, β)-metric $\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$, we have

$$\phi(s) = (1 + \epsilon s + ks^2); \ s = \frac{\beta}{\alpha}. \tag{5}$$

Let $G^i(x, y)$ and $G^i_\alpha(x, y)$ denote the spray coefficients of F and α respectively. To express formula for the spray coefficients G^i of F in terms of α and β, we need to introduce some notations. Let $b_{i:j}$ be a covariant derivative of b_i with respect to y^j. Denote

$$r_{ij}=\frac{1}{2}(b_{i|j}+b_{j|i}),\qquad s_{ij}=\frac{1}{2}(b_{i|j}-b_{j|i}),$$
$$s^i_j=a^{ih}s_{hj},\quad s_j=b_is^i_j=s_{ij}b^i,\quad r_j=r_{ij}b^i,$$
$$r_0=r_jy^j,\quad s_0=s_jy^j,\quad r_{00}=r_{ij}y^iy^j.$$

Lemma 2.1([3]) *For an (α,β)-metric $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$, the geodesic coefficients G^i are given by*

$$G^i=G^i_\alpha+\alpha Qs^i_0+\Theta(-2\alpha Qs_0+r_{00})\frac{y^i}{\alpha}+\psi(-2\alpha Qs_0+r_{00})\big(b^i-\frac{y^i}{\alpha}\big), \tag{6}$$

where

$$Q = \frac{\phi}{\phi-s\phi'},$$
$$\Theta = \frac{(\phi-s\phi')\phi'}{2\phi((\phi-s\phi')+(b^2-s^2)\phi''},$$
$$\Psi = \frac{\phi''}{2((\phi-s\phi')+(b^2-s^2))\phi''}.$$

Here $b^i=a^{ij}b_j$, and $b^2=a^{ij}b_ib_j=b_jb^j$.

Lemma 2.2 *For a special (α,β)-metric $F=\alpha+\epsilon\beta+k\frac{\beta^2}{\alpha}$, the geodesic coefficients G^i are given by*

$$\begin{aligned}G^i &= G^i_\alpha+\alpha\frac{(\epsilon+2ks)}{1-ks^2}s^i_0\\ &+\frac{(\epsilon+2ks-\epsilon ks^2-2k^2s^3)}{2(1+2kb^2-3ks^2)(1+\epsilon s+ks^2)}\Big[-2\alpha\frac{(\epsilon+2ks)}{1-ks^2}s_0\\ &+r_{00}\Big]\frac{y^i}{\alpha}+\frac{k}{1+2kb^2-3ks^2}\Big[-2\alpha\frac{(\epsilon+2ks)}{1-ks^2}s_0+r_{00}\Big]\big(b^i-\frac{y^i}{\alpha}\big)\end{aligned} \tag{7}$$

Proof By a direct computation, we get (7) from (6) □

Theorem 2.1 *For a Finsler space with special (α,β)- metric $F=\alpha+\epsilon\beta+k\frac{\beta^2}{\alpha}$, the Ricci curvature of F is given by*

$$Ric=\overline{Ric}+T, \tag{8}$$

where $\overline{Ric}(={}^{\alpha}Ric)$ denotes the Ricci curvature of α, and

$$\begin{aligned}T &= \frac{4kF\alpha^3}{(\alpha^2-k\beta^2)^2}s_{0j}s^m_0+2\frac{\alpha^2(\epsilon\alpha+2k\beta)}{\alpha^2-k\beta^2}s^m_{0:j}\\ &+\frac{2(\epsilon\alpha^2+2k\alpha\beta)\{\epsilon\alpha^4-\epsilon k\alpha^2\beta^2+2k\alpha^3\beta-2k^2\alpha\beta^3+2sk\alpha^3F\}}{(\alpha^2-k\beta^2)^3}s_{0m}s^m_0\\ &-\alpha^2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s^m_js^j_m\end{aligned}$$

Proof Consider the Finsler space with special (α, β)- metric $F = \alpha + \epsilon\beta + k\frac{\beta^2}{\alpha}$ on an n-dimensional manifold M^n. From Lemma 2.2, the geodesic coefficients G^i of F are related to the coefficients G^i_α of α by

$$G^i = G^i_\alpha + Py^i + T^i, \tag{9}$$

where

$$\begin{aligned} P &= \frac{[\epsilon - 2k + 2k(1-\epsilon)s - (\epsilon + 2k)ks^2 - 2k^2s^3]}{2\alpha(1 + 2kb^2 - 3ks^2)(1 + \epsilon s + ks^2)}\Big[-2\alpha\frac{(\epsilon + 2ks)}{1 - ks^2}s_0 + r_{00}\Big], \\ T^i &= \frac{\alpha(\epsilon + 2ks)}{1 - ks^2}s^i_0 + \frac{k}{1 + 2kb^2 - 3\delta s^2}\Big[\frac{-2\alpha(\epsilon + 2ks)s_0}{1 - ks^2} + r_{00}\Big]b^i. \end{aligned} \tag{10}$$

In this section, we assume that β is a killing form of constant length i. e., β satisfies

$$r_{ij} = 0, \quad and \quad b^j b_{j:m} = 0. \tag{11}$$

Equation (11) implies that

$$s_{ij} = b_{i:j}, \quad s_j = b^i s_{ij} = 0, \quad b^i s^j_i = b^i s_{ri} a^{jr} = -b^i s_{ir} a^{jr} = 0. \tag{12}$$

Thus $P = 0$ and (9) reduces to

$$G^i = G^i_\alpha + T^i, \tag{13}$$

where

$$T^i = \frac{\alpha(\epsilon + 2ks)}{1 - ks^2}s^i_0, \tag{14}$$

Now from (2) and (13), we obtain ([7])

$$R^i_m = {}^\alpha R^i_m + 2T^i_{:m} - y^j T^i_{:j.m} - T^i_{.j} T^j_{.m} + 2T^j T^i_{j.m}, \tag{15}$$

where $T^i_{.j} = \frac{\partial T^i}{\partial y^j}$. Thus the Ricci curvature of F is related to the Ricci curvature of α by

$$Ric = \overline{Ric} + 2T^m_{:m} - y^j T^m_{:j.m} - T^m_{.j} T^j_{.m} + 2T^j T^m_{j.m}, \tag{16}$$

where " : "and " . "denotes the horizontal covariant derivative and vertical covariant derivative with respect to the Berwald connection determined by $\bar{G}^i$ respectively.

Note that

$$\alpha_{:m} = 0, \quad y_{:m} = 0, \quad \beta_{:m} = r_{0m} + s_{om}, \quad b^2_{:m} = 2(r_m + s_m), \quad b^i_{:m} = r^i_m + s^i_m.$$

$$s_{.i} = \frac{b_i}{\alpha} - \frac{sy_i}{\alpha^2}, \quad s_{:i} = \frac{s_{0i}}{\alpha}, \quad s_{.j.i} = \frac{(b_j y_i + b_i y_j)}{\alpha^3} + 3s\frac{y_i y_j}{\alpha^4} - s\frac{a_{ji}}{\alpha^2}$$

We have $F_{:m} = (\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha})_{:m} = \Big(\epsilon + \frac{2k\beta}{\alpha}\Big)b_{0:m}$ and $F_{y^m} = (\alpha + \epsilon\beta + k\frac{\beta^2}{\alpha})_{y^m} = \frac{y_m}{\alpha} +$

$\epsilon b_m + k\dfrac{\beta^2 y_m}{\alpha^3}$. Thus from (14), we have

$$\begin{aligned} T^m_{:j} &= \frac{2kFs_{0j}s^m_0}{(1-ks^2)^2\alpha} + \frac{\alpha(\epsilon+2ks)}{1-ks^2}s^m_{0:j} \\ &= \frac{2kF\alpha^3}{(\alpha^2-k\beta^2)^2}s_{0j}s^m_0 + \frac{\alpha^2(\epsilon\alpha+2k\beta)}{\alpha^2-k\beta^2}s^m_{0:j}. \end{aligned} \tag{17}$$

Using $b_i s^i_0 = 0$, $b_i s^i_j = 0$, $y_i s^i_0 = 0$ & $y_i s^i_{0:j} = 0$, we obtain $T^m_{j.m} = 0$ and $T^m_{j.m}y^j = 0$. Consequently, we obtain the following

$$\begin{aligned} T^j T^k_{.j.k} &= \frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s_{k0}s^k_0 - s\frac{2kF\alpha^3}{(\alpha^2-k\beta^2)^2}s_{j0}s^j_0 \\ &\quad + \frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s^j_0 s^0_j - s\frac{2k\alpha^3F(\epsilon\alpha^2+2k\alpha\beta)}{(\alpha^2-k\beta^2)^3}s^j_0 s^0_j. \end{aligned}$$

$$T^m_{.j}T^j_{.m} = 2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s_{0m}s^m_0 - 2s\frac{2k\alpha^3F(\epsilon\alpha^2+2k\alpha\beta)}{(\alpha^2-k\beta^2)^3}s_{0m}s^m_0 + \alpha^2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s^m_j s^j_m.$$

Plugging these values into (16), we get

$$\begin{aligned} Ric &= \overline{Ric} + 2T^m_{:m} - y^j T^m_{:j.m} - T^m_{.j}T^j_{.m} + 2T^j T^m_{.j.m} \\ &= \overline{Ric} + \frac{4kF\alpha^3}{(\alpha^2-k\beta^2)^2}s_{0j}s^m_0 + 2\frac{\alpha^2(\epsilon\alpha+2k\beta)}{\alpha^2-k\beta^2}s^m_{0:j} + 2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s_{0m}s^m_0 + \\ &\quad \frac{4sk\alpha^3F(\epsilon\alpha^2+2k\alpha\beta)}{(\alpha^2-k\beta^2)^3}s_{0m}s^m_0 - \alpha^2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s^m_j s^j_m + 2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s_{m0}s^m_0 \\ &\quad -2s\frac{2kF\alpha^3}{(\alpha^2-k\beta^2)^2}s_{j0}s^j_0 + 2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s^j_0 s^0_j - 4s\frac{k\alpha^3F(\epsilon\alpha^2+2k\alpha\beta)}{(\alpha^2-k\beta^2)^3}s^j_0 s^0_j. \end{aligned} \tag{18}$$

Since $s_{m0} = -s_{0m}$ and $s^0_j = -s_{j0}$, equation (18) becomes

$$\begin{aligned} Ric &= \overline{Ric} + \frac{4kF\alpha^3}{(\alpha^2-k\beta^2)^2}s_{0j}s^m_0 + 2\frac{\alpha^2(\epsilon\alpha+2k\beta)}{\alpha^2-k\beta^2}s^m_{0:j} + 2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s_{0m}s^m_0 + \\ &\quad \frac{4sk\alpha^3F(\epsilon\alpha^2+2k\alpha\beta)}{(\alpha^2-k\beta^2)^3}s_{0m}s^m_0 - \alpha^2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s^m_j s^j_m \\ &= \overline{Ric} + \frac{4kF\alpha^3}{(\alpha^2-k\beta^2)^2}s_{0j}s^m_0 + 2\frac{\alpha^2(\epsilon\alpha+2k\beta)}{\alpha^2-k\beta^2}s^m_{0:j} + \\ &\quad \frac{2(\epsilon\alpha^2+2k\alpha\beta)\{\epsilon\alpha^4-\epsilon k\alpha^2\beta^2+2k\alpha^3\beta-2k^2\alpha\beta^3+2sk\alpha^3F\}}{(\alpha^2-k\beta^2)^3}s_{0m}s^m_0 - \\ &\quad \alpha^2\frac{(\epsilon\alpha^2+2k\alpha\beta)^2}{(\alpha^2-k\beta^2)^2}s^m_j s^j_m. \end{aligned} \tag{19}$$

This completes the proof. □

§3. Projectively Flat (α, β)-metric

A Finsler metric F = F(x,y) on an open subset $U \subset R^n$ is projectively flat [3] if and only if

$$F_{x^m y^l} y^m - F_{x^l} = 0. \tag{20}$$

By (20), we have the following lemma ([8]).

Lemma 3.1 *An (α, β)- metric $F = \alpha\phi(s)$, where $s = \frac{\beta}{\alpha}$, is projectively flat on an open subset $U \subset R^n$ if and only if*

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + \psi\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0. \tag{21}$$

In this section, we consider the Finsler space with special (α, β)- metric $F = \alpha + \epsilon\beta + k\dfrac{\beta^2}{\alpha}$, where $\epsilon, k \neq 0$ are constants. We have

$$F = \alpha\phi(s), \quad \phi(s) = (1 + \epsilon s + k s^2). \tag{22}$$

Let $b_0 > 0$ be the largest number such that

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad (|s| \leq b < b_0). \tag{23}$$

That is,

$$1 + 2kb^2 - 4ks^2 > 0, \quad (|s| \leq b < b_0). \tag{24}$$

Lemma 3.2 $F = \alpha + \epsilon\beta + k\dfrac{\beta^2}{\alpha}$ *is a Finsler metric iff* $\|\beta\|_\alpha < 1$.

Proof If $F = \alpha + \epsilon\beta + k\dfrac{\beta^2}{\alpha}$ is a Finsler metric, then

$$1 + 2kb^2 - 4ks^2 > 0, \quad (|s| \leq b < b_0). \tag{25}$$

Let s = b, then we get $b < \dfrac{1}{\sqrt{2k}}$, $\forall\ b < b_0$. Let $b \to b_0$, then $b_0 < \dfrac{1}{\sqrt{2k}}$. So $\|\beta\|_\alpha$ <1. Now, if

$$|s| \leq b < \frac{1}{\sqrt{2k}} \tag{26}$$

then

$$1 + 2kb^2 - 4ks^2 > 0, \quad (|s| \leq b < b_0). \tag{27}$$

Thus $F = \alpha + \epsilon\beta + k\dfrac{\beta^2}{\alpha}$ is a Finsler metric. □

By Lemma 2.2, the spray coefficients are given by

$$\begin{aligned} Q &= \frac{\epsilon\alpha^2+2k\alpha\beta}{\alpha^2-k\beta^2}, \\ \Theta &= \frac{\epsilon\alpha^3-2k\alpha^2\beta-\epsilon k\alpha\beta^2-2k^2\beta^3}{2\{(1+2kb^2)\alpha^2-3k\beta^2\}\{\alpha^2+\epsilon\alpha\beta+k\beta^2}, \\ \psi &= \frac{k\alpha^2}{(1+2kb^2)\alpha^2-3k\beta^2}. \end{aligned}$$

Equation (21) is reduced to the following form:

$$\begin{aligned} (a_{ml}\alpha^2-y_my_l)G^m_\alpha+\alpha^3\Big(\frac{\epsilon\alpha^2+2k\alpha\beta}{\alpha^2-k\beta^2}\Big)s_{l0}+\alpha\Big(\frac{k\alpha^2}{(1+2kb^2)\alpha^2-3k\beta^2}\Big) \\ \Big[-2\alpha\Big(\frac{\epsilon\alpha^2+2k\alpha\beta}{\alpha^2-k\beta^2}\Big)s_0+r_{00}\Big]\Big(b_l\alpha-\frac{\beta}{\alpha}y_l\Big)=0. \end{aligned} \tag{28}$$

Lemma 3.3 *If $(a_{ml}\alpha^2-y_my_l)G^m_\alpha=0$, then α is projectively flat.*

Proof If $(a_{ml}\alpha^2-y_my_l)G^m_\alpha=0$, then

$$a_{ml}\alpha^2=y_my_lG^m_\alpha,$$

then there is a $\eta=\eta(x,y)$ such that $y_mG^m_\alpha=\alpha^2\eta$, we get

$$a_{ml}G^m_\alpha=\eta y_l.$$

Contracting with a^{il} yields $G^i_\alpha=\eta y^i$, and thus α is projectively flat. □

Theorem 3.1 *A Finsler space with special (α,β)-metric $F=\alpha+\epsilon\beta+k\frac{\beta^2}{\alpha}$ (where ϵ, $k\neq 0$ are constants) is locally projectively flat iff*

(1) *β is parallel with respect to α;*

(2) *α is locally projectively flat, i. e., of constant curvature.*

Proof Suppose that F is locally projectively flat. First, we rewrite (28) as a polynomial in y^i and α. This gives,

$$\begin{aligned} &(a_{ml}\alpha^2-y_my_l)G^m_\alpha\Big[\{(1+2kb^2)\alpha^2-3k\beta^2\}\{\alpha^2-k\beta^2\}\Big]+2k\alpha^4\beta\{(1+ \\ &2kb^2)\alpha^2-3k\beta^2\}s_{l0}+kr_{00}\alpha^2(\alpha^2-k\beta^2)(b_l\alpha^2-\beta y_l)-4k^2\alpha^4\beta s_0(b_l\alpha^2- \\ &\beta y_l)+\alpha\Big\{\epsilon\alpha^4\{(1+2kb^2)\alpha^2-3k\beta^2\}s_{l0}-2\epsilon k\alpha^4s_0(b_l\alpha^2-\beta y_l)\Big\}=0. \end{aligned} \tag{29}$$

or

$$U+\alpha V=0, \tag{30}$$

where

$$U = (a_{ml}\alpha^2 - y_m y_l)G^m_\alpha\Big[\{(1+2kb^2)\alpha^2 - 3k\beta^2\}\{\alpha^2 - k\beta^2\}\Big] + 2k\alpha^4\beta\{(1+ 2kb^2)\alpha^2 - 3k\beta^2\}s_{l0} + kr_{00}\alpha^2(\alpha^2 - k\beta^2)(b_l\alpha^2 - \beta y_l) - 4k^2\alpha^4\beta s_0 (b_l\alpha^2 - \beta y_l),$$

and

$$V = \epsilon\alpha^4\{(1+2kb^2)\alpha^2 - 3k\beta^2\}s_{l0} - 2\epsilon k\alpha^4 s_0(b_l\alpha^2 - \beta y_l).$$

Now, (30) is a polynomial in (y^i), such that U and V are rational in y^i and α is irrational. Therefore, we must have

$$U = 0 \; and \; V = 0, \tag{31}$$

which implies that

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha\Big[\{(1+2kb^2)\alpha^2 - 3k\beta^2\}\{\alpha^2 - k\beta^2\}\Big] + 2k\alpha^4\beta\{(1 +2kb^2)\alpha^2 - 3k\beta^2\}s_{l0} + kr_{00}\alpha^2(\alpha^2 - k\beta^2)(b_l\alpha^2 - \beta y_l) - 4k^2\alpha^4\beta s_0 (b_l\alpha^2 - \beta y_l) = 0 \qquad (32)$$

and

$$\epsilon\alpha^4\{(1+2kb^2)\alpha^2 - 3k\beta^2\}s_{l0} - 2\epsilon k\alpha^4 s_0(b_l\alpha^2 - \beta y_l) = 0. \tag{33}$$

From (30), considering only terms which do not contain β. There exists a homogenous polynomial V_7 of degree seven in y^i such that

$$\Big\{(1+2kb^2)\epsilon s_{l0} - 2k\epsilon b_l s_0\Big\}\alpha^7 = \beta V_7. \tag{34}$$

Since $\alpha^2 \not\cong o(mod\beta)$, we must have a function $u^l = u^l(x)$ satisfying

$$(1+2kb^2)\epsilon s_{l0} - 2k\epsilon b_l s_0 = u^l\beta. \tag{35}$$

Transvecting (35) by b_l, we have

$$(1+2kb^2)\epsilon s_0 - 2k\epsilon b^2 s_0 = u^l\beta b^l. \tag{36}$$

That is,

$$\epsilon s_j = u^l b_l b_j. \tag{37}$$

Further transvecting by b^j, we have $u^i b_i b^2 = 0$, which implies $u^l b_l = 0$. Substituting this equation into (36), we get $s_0 = 0$. Now, from (32), by contracting with b^l, we get

$$(b_m\alpha^2 - y_m\beta)G^m_\alpha\Big[\{(1+2kb^2)\alpha^2 - 3k\beta^2\}\{\alpha^2 - k\beta^2\}\Big] + 2k\alpha^4\beta\{(1+ 2kb^2)\alpha^2 - 3k\beta^2\}s_0 + kr_{00}\alpha^2(\alpha^2 - k\beta^2)(b^2\alpha^2 - \beta^2) - 4k^2\alpha^4\beta s_0(b_l\alpha^2 - \beta y_l) = 0. \tag{38}$$

Since $s_0 = 0$, we get

$$(b_m\alpha^2 - y_m\beta)G^m_\alpha\Big[\{(1+2kb^2)\alpha^2 - 3k\beta^2\}\{\alpha^2 - k\beta^2\}\Big] + kr_{00}\alpha^2(\alpha^2 - k\beta^2)(b^2\alpha^2 - \beta^2) = 0. \quad (39)$$

Contracting (39) by y^m, we get

$$r_{00} = 0. \quad (40)$$

From (33), we get

$$s_{l0} = 0. \quad (41)$$

Then by (40) and Lemma 3.3, α is projectively flat. From (40) and (41), $b_{i;j} = 0$, i. e., β is parallel to α.

Conversely, if β is parallel with respect to α and α is locally projectively flat, then by Lemma 3.3, we can easily see that F is locally projectively flat. □

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