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On Linear Operators Preserving Orthogonality of Matrices over Fuzzy Semirings

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Abstract: In this paper, we investigate the linear operators preserving orthogonality of matrices over fuzzy semirings. We firstly characterize invertible linear operators preserving orthogonality of fuzzy matrices. And then, based on the obtained results, we study the invertible linear operators preserving orthogonality of matrices over the direct product of fuzzy semirings, and give some complete characterizations.

Key Words: Linear operator; orthogonality; fuzzy semirings.

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§1. Introduction

Let $\mathbb{F}=[0,1]$ be a set of reals between 0 and 1 with addition $(+)$, and multiplication $(\cdot)$ and the ordinary order $\leq$ such that

$$x+y=max\{x,y\}\,\text{and}\,x\cdot y=min\{x,y\}$$

for all $x,y\in\mathbb{F}$. We call $\mathbb{F}$ a fuzzy semiring. For any $x,y\in\mathbb{F}$, we omit the dot of $x\cdot y$ and simply write xy.

Let $M_n(\mathbb{F})$ denote the set of all $n\times n$ matrices over $\mathbb{F}$. Define $+$ and $\cdot$ on $M_n(\mathbb{F})$as follows:

$$(\forall A,B\in M_n(\mathbb{F}))\ A+B=[a_{ij}+b_{ij}]_{n\times n},\ A\cdot B=[\sum_{k=1}^{n}a_{ik}b_{kj}]_{n\times n}.$$

It is easy to verify that $(M_n(\mathbb{F}),+,\cdot)$ is a semiring with the operations defined above. And the matrices in $(M_n(\mathbb{F}),+,\cdot)$ are called fuzzy matrices.

Let $\mathbb{F}$ be a fuzzy semiring and $A\in M_n(\mathbb{F})$. We denote the transpose of A by A^t and the entry of A in the ith row and jth column by a_{ij}.

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For any $A \in M_n(\mathbb{F})$ and any $\lambda \in \mathbb{F}$, we define

$$\lambda A = [\lambda a_{ij}]_{n\times n}.$$

A mapping $T : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ is called a *linear operator* if

$$T(aA + bB) = aT(A) + bT(B)$$

for all $a, b \in \mathbb{F}$ and $A, B \in M_n(\mathbb{F})$. Notice that if T is a linear operator on $M_n(\mathbb{F})$, then $T(O) = O$.

A, B in $M_n(\mathbb{F})$ are said to be orthogonal (see [**?**]) if $AB = BA = O$. Let T be an operator on $M_n(\mathbb{F})$. We say that T preserves orthogonality if $T(A)$ and $T(B)$ are orthogonal whenever A and B are orthogonal.

During the past 100 years, one of the most active and fertile subjects in matrix theory is the linear preserver problem (LPP for short), which concerns the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. The first paper can be traced down to Frobenius's work in 1897. Since then, a number of works in the area have been published. Among these works, although the linear operators concerned are mostly linear operators on matrix spaces over some fields or rings, the same problem has been extended to matrices over various semirings.

Many authors have studied the linear operators that preserve invariants of matrices over semirings. For example, idempotent preservers were investigated by Song, Kang and Beasley ([16]), Dolžan and Oblak ([6]), Orel ([14])et al. Nilpotent preservers were discussed by Song, Kang and Jun ([19]), Li and Tan ([12]) et al. Regularity preservers were studied by Song, Kang, Jun, Beasley and Sze in [10] and [21] et al. Pshenitsyna ([15]) considered invertibility preservers. Besides, Beasley, Guterman, Jun and Song ([1])investigated the linear preservers of extremes of rank inequalities over semirings, Beasley and Lee([2])studied the linear operators that strongly preserve r-potent matrices over semirings, Song and Kang ([20]) discussed commuting pairs of matrices preservers and so on.

The linear preserver problems about orthogonality of matrices are more and more caused people's attention. In [17] and [18], Šemrl studied maps on idempotents matrices that preserve orthogonality over a division ring. Burgos et al. ([3]) studied orthogonality preserving operators between C*-algebras, JB*-algebras and JB*-triples. Cui, Hou and Park ([5]) described the additive maps preserving the indefinite orthogonality of operators acting on indefinite inner product spaces. Also, there are some literature on maps that approximately preserve orthogonality (see [4],[9] et al).

Note that the researches about linear operators preserving orthogonality of matrices over semiring are not much, and fuzzy semirings are the ones which have bright background. In this paper our purpose is to obtain characterizations of invertible linear operators that preserve orthogonality matrices over fuzzy semirings. In Section 2 we characterize invertible linear operators preserving orthogonality of fuzzy matrices. Based on the obtained results, we study the invertible linear operators preserving orthogonality of matrices over the direct product of fuzzy semirings in Sections 3, and obtain some complete characterizations.

For notations and terminologies occurred but not mentioned in this paper, the readers are referred to [8].

§2. Linear Operators Preserving Orthogonality of Fuzzy Matrices

In this section, we will study the complete characterizations of linear operators that preserve orthogonality of fuzzy matrices.

Let $\mathbb{S}$ be a semiring. A matrix $P \in M_n(\mathbb{S})$ is called a *permutation matrix* (see [21]) if it has exactly one entry 1 in each row and each column and 0's elsewhere. Observe that if $P \in M_n(\mathbb{S})$ is a permutation matrix, then $PP^t = P^tP = I$.

For each $x \in \mathbb{F}$, define

$$x^* = \begin{cases} 0, & \text{if } x = 0, \\ 1, & \text{if } x \neq 0. \end{cases}$$

Then the mapping

$$\varphi : \mathbb{F} \to \mathbb{B}_1, x \mapsto x^*$$

is a homomorphism. Its entrywise extension to a mapping

$$\psi : M_n(\mathbb{F}) \to M_n(\mathbb{B}_1), A \mapsto A^*$$

preserves sums, products and multiplication by scalars.

It is well known the only invertible matrices in $M_n(\mathbb{B}_1)$ are permutation matrices (see [20]).

In fact, we can also obtain the following theorem.

Theorem 2.1 *The permutation matrices are the only invertible matrices in* $M_n(\mathbb{F})$.

Proof Let $A \in M_n(\mathbb{F})$ be an invertible matrix. Then there exists a matrix $B \in M_n(\mathbb{F})$ such that $AB = BA = I_n$. This implies $A^*B^* = B^*A^* = I_n$, and thus A^* and B^* are permutation matrices with $B^* = (A^*)^t$. Notice that any product of two elements in $\mathbb{F}$ is their minimum, the nonzero entries in A are 1's. Thus, A is a permutation matrix. □

Let $E_{i,j} \in M_n(\mathbb{F})$ is the matrix with 1 as its (i,j)-entry and 0 elsewhere. We call such $E_{i,j}$ a *cell* (see [19]) and denote $\mathbb{E}_n = \{E_{i,j} | i,j \in \underline{n}\}$, where $\underline{n} = \{1, 2, \cdots, n\}$. By virtue of definition, for any $E_{i,j}, E_{k,l} \in \mathbb{E}_n$, we can easily have that

$$E_{i,j}E_{k,l} = \begin{cases} E_{i,l}, & if\ j = k, \\ O, & otherwise. \end{cases}$$

From [21], a semiring $\mathbb{S}$ with 0 and 1 is said to be *commutative* if $(S, \cdot, 1)$ is commutative; a semiring $\mathbb{S}$ is called an *antiring* if $a + b = 0$ implies $a = b = 0$ for any $a, b \in S$, i.e., 0 is the unique invertible element in $(S, +, 0)$; a semiring $\mathbb{S}$ is said to be *entire* if $a \neq 0, b \neq 0$ imply $ab \neq 0$ for any $a, b \in S$. It is obvious that fuzzy semiring $\mathbb{F}$ is a commutative entire antiring.

Lemma 2.2([16]) *Let* $\mathbb{S}$ *be a commutative antiring and* T *a linear operator on* $M_n(\mathbb{S})$*. Then*

T is invertible if and only if there exist a permutation α on the set $\{(i,j)|i,j \in \underline{n}\}$ and unit elements $b_{ij} \in \mathbb{S}, i,j \in \underline{n}$ such that $T(E_{i,j}) = b_{ij}E_{\alpha(i,j)}$.

Lemma 2.2 shows that if T is a linear operator on $M_n(\mathbb{S})$ in which $\mathbb{S}$ is a commutative antiring, then T permutes $\mathbb{E}_n$ with unit scalar multiplication.

Theorem 2.3 *Let $\mathbb{F}$ be a fuzzy semiring. If T is a linear operator on $M_n(\mathbb{F})$ with $n = 1$, then T preserves orthogonality of fuzzy matrices.*

Proof Let $\mathbb{F}$ be a fuzzy semiring and T a linear operator on $M_n(\mathbb{F})$ with $n = 1$. Suppose that $A, B \in M_n(\mathbb{F})$ such that A and B are orthogonal. Then, we must have that $A = O$ or $B = O$. It follows from the linearity of T that $T(O) = O$. Furthermore, $T(A)T(B) = T(B)T(A) = O$. Hence, $T(A)$ and $T(B)$ are orthogonal. So T preserves orthogonality of fuzzy matrices. □

Theorem 2.4 *Let $\mathbb{F}$ be a fuzzy semiring and $T : M_n(\mathbb{F}) \longrightarrow M_n(\mathbb{F})$ a linear operator with $n \geq 2$. Then T is an invertible linear operator that preserves orthogonality of fuzzy matrices if and only if there exists a permutation matrix $P \in M_n(\mathbb{B}_1)$ such that either $T(X) = PXP^t$ for all $X \in M_n(\mathbb{F})$, or $T(X) = PX^tP^t$ for all $X \in M_n(\mathbb{F})$.*

Proof ($\Longrightarrow$) Let T be an invertible linear operator on $M_n(\mathbb{F})$ which preserves orthogonality of fuzzy matrices. Note that fuzzy semiring $\mathbb{F}$ is a commutative entire antiring, by the virtue of Lemma 2.2, there exists a permutation α on the set $\{(i,j)|i,j \in \underline{n}\}$ such that $T(E_{i,j}) = E_{\alpha(i,j)}$. For any $i \neq j$, denote $T(E_{i,j}) = E_{p,q}$. If $p = q$ then it follows from $E_{i,j}E_{i,j} = O$ that

$$(T(E_{i,j}))^2 = (E_{p,p})^2 = E_{p,p} = O,$$

it is a contradiction. Thus, $p \neq q$. Note that α is a permutation, then there is a permutation σ of $\{1, 2, \cdots, n\}$ such that $T(E_{i,i}) = E_{\sigma(i),\sigma(i)}$ for each $i = 1, 2, \cdots, n$.

Define an operator L on $M_n(\mathbb{F})$ by

$$L(X) = P^tT(X)P$$

for all $X \in M_n(\mathbb{F})$, where P is a permutation matrix corresponding to σ such that $L(E_{i,i}) = E_{\sigma(i),\sigma(i)}$ for each $i = 1, 2, \cdots, n$.

It is easy to see that L is an invertible linear operator on $M_n(\mathbb{F})$ that preserves orthogonality of matrices. By Lemma 2.2, L permutes $\mathbb{E}_n$. Therefor, for any cell $E_{r,s}$ in $\mathbb{E}_n$, there exists exactly one cell $E_{p,q}$ in $\mathbb{E}_n$ such that $L(E_{r,s}) = E_{p,q}$.

Suppose that $r \neq s$. Since L is injective, we have $p \neq q$ because $L(E_{i,i}) = E_{\sigma(i),\sigma(i)}$ for each $i = 1, 2, \cdots, n$. Assume that $p \neq r$ and $p \neq s$. Since $E_{r,s}E_{p,p} = E_{p,p}E_{r,s} = O$, we have

$$L(E_{p,p})L(E_{r,s}) = E_{p,p}E_{p,q} = E_{p,q} = O,$$

it is a contradiction. Hence, $p = r$ or $p = s$. Similarly, $q = r$ or $q = s$. Therefore, for each $E_{r,s}$

in $\mathbb{E}_n$,

$$L(E_{r,s}) = E_{r,s}, \text{ or } L(E_{r,s}) = E_{s,r}.$$

Suppose that $L(E_{r,s}) = E_{r,s}$ for some $E_{r,s} \in \mathbb{E}_n$ with $r \neq s$ and $L(E_{r,t}) = E_{t,r}$ for some $t \in \underline{n}$ with $t \neq r, s$. It follows form $E_{r,s}E_{r,t} = E_{r,t}E_{r,s} = O$ that

$$L(E_{r,t})L(E_{r,s}) = E_{t,r}E_{r,s} = E_{t,s} = O,$$

it is a contradiction. It follows that if $L(E_{i,j}) = E_{i,j}$ for some $E_{i,j} \in \mathbb{E}_n$ with $i \neq j$, then we have $L(E_{r,s}) = E_{r,s}$ for all $E_{r,s} \in \mathbb{E}_n$.

Consequently, we have established that $L(X) = X$ or $L(X) = X^t$ for all $X \in M_n(\mathbb{F})$.

If $L(X) = X$ for all $X \in \mathbb{M}_n(\mathbb{F})$. By the definition of L, we have

$$P^tT(X)P = X,$$

or equivalently

$$T(X) = PXP^t$$

for all $X \in M_n(\mathbb{F})$.

Similarly, if $L(X) = X^t$ for all $X \in M_n(\mathbb{F})$, we can get

$$T(X) = PX^tP^t.$$

($\Longleftarrow$) Suppose that $T(X) = PXP^t$ for all $X \in M_n(\mathbb{F})$. It's a routine matter to verify that T is invertible. For any $X, Y \in M_n(\mathbb{F})$, if X and Y are orthogonal, then $XY = YX = O$. It follows that

$$T(X)T(Y) = T(Y)T(X) = O.$$

That is to say, $T(X)$ and $T(Y)$ are orthogonal. Thus, T preserves orthogonality of fuzzy matrices.

Similarly, if $T(X) = PX^tP^t$ for all $X \in M_n(\mathbb{F})$, then T is also an invertible linear operator preserving orthogonality of fuzzy matrices. $\square$

Example 2.5 Let

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

ba a matrix in $M_4(\mathbb{F})$. Define an operator T on $M_4(\mathbb{F})$ by

$$T(X) = PX^tP^t$$

for all $X \in M_4(\mathbb{F})$. By Theorem 2.4, T is an invertible linear operator preserving orthogonality of fuzzy matrices.

§3. Linear Operators Preserving Orthogonality of Matrices Over the Direct Product of Fuzzy Semirings

In this section we will study the invertible linear operators that preserve orthogonality of matrices over the direct product of fuzzy semirings.

Hereafter, let $\mathbb{S} = \prod_{\lambda \in \wedge} \mathbb{S}_\lambda$, where $\mathbb{S}_\lambda = \mathbb{F}$ is a fuzzy semiring for any $\lambda \in \wedge$. For any $\lambda \in \wedge$ and any $s \in \mathbb{S}$, we denote $s(\lambda)$ by s_λ. Define

$$(a+b)_\lambda = a_\lambda + b_\lambda, (ab)_\lambda = a_\lambda b_\lambda \ (a, b \in \mathbb{S}, \ \lambda \in \wedge).$$

It is easy to verify that $(\mathbb{S}, +, \cdot)$ is a semiring with 0 and 1 under the operations defined above. For any $A = [a_{ij}] \in M_n(\mathbb{S})$ and any $\lambda \in \wedge, A_\lambda := [(a_{ij})_\lambda] \in M_n(\mathbb{S}_\lambda)$. It is obvious that

$$(A+B)_\lambda = A_\lambda + B_\lambda, (AB)_\lambda = A_\lambda B_\lambda \text{ and } (sA)_\lambda = s_\lambda A_\lambda$$

for all $A, B \in M_n(\mathbb{S})$ and all $s \in \mathbb{S}$.

By the above definition, it is not hard to obtain the following result.

Lemma 3.1 *Let $A, B \in M_n(\mathbb{S})$. Then the following statements hold:*

(i) $A = B$ if and only if $A_\lambda = B_\lambda$ for any $\lambda \in \wedge$;
(ii) A and B are orthogonal if and only if A_λ and B_λ are orthogonal for any $\lambda \in \wedge$.

The following lemma is due to Orel [14].

Lemma 3.2 *If $T : M_n(\mathbb{S}) \to M_n(\mathbb{S})$ is a linear operator, then for any $\lambda \in \wedge$, there exists a unique linear operator $T_\lambda : M_n(\mathbb{S}_\lambda) \to M_n(\mathbb{S}_\lambda)$ such that $(T(A))_\lambda = T_\lambda(A_\lambda)$ for any $A \in M_n(\mathbb{S})$.*

Theorem 3.3 *Let $\mathbb{S} = \prod_{\lambda \in \wedge} \mathbb{S}_\lambda$, where $\mathbb{S}_\lambda = \mathbb{F}$ is a fuzzy semiring for any $\lambda \in \wedge$. If T is a linear operator on $M_n(\mathbb{S})$ with $n = 1$, then T preserves orthogonality of matrices.*

Proof Assume that $A, B \in \mathbb{M}_n(\mathbb{S})$, and A and B are orthogonal. By Lemma 3.1 (ii), we have A_λ and B_λ are orthogonal for any $\lambda \in \wedge$. It follows from Theorem 2.3 that $(T(A))_\lambda$ and $(T(B))_\lambda$ are orthogonal. Again by Lemma 3.1 (ii), we obtain that $T(A)$ and $T(B)$ are orthogonal. Hence T preserves orthogonality of matrices. □

Proposition 3.4 *Let T be a linear operator on $M_n(\mathbb{S})$. Then T is invertible if and only if T_λ is invertible for any $\lambda \in \wedge$.*

Proof ($\Longrightarrow$) Let T be a linear operator on $M_n(\mathbb{S})$. Suppose that T is invertible. For any $\lambda \in \wedge$ and $A, B \in M_n(\mathbb{S}_\lambda)$, there exist $X, Y \in M_n(\mathbb{S})$ such that $X_\lambda = A, Y_\lambda = B$, and $X_\mu = Y_\mu = O$ for any $\mu \neq \lambda$. If $T_\lambda(A) = T_\lambda(B)$ then

$$(T(X))_\lambda = T_\lambda(A) = T_\lambda(B) = (T(Y))_\lambda.$$

Also,

$$(T(X))_\mu = (T(Y))_\mu = T_\mu(O) = O$$

for any $\mu \neq \lambda$. This shows that $T(X) = T(Y)$. Since T is injective, we have $X = Y$. Further,

$$A = X_\lambda = Y_\lambda = B.$$

Thus T_λ is injective.

On the other hand, since T is surjective, there exists $Q \in M_n(\mathbb{S})$ such that $T(Q) = Y$. We can deduce that

$$B = Y_\lambda = T(Q)_\lambda = T_\lambda(Q_\lambda).$$

That is to say, T_λ is surjective. Hence T_λ is invertible.

($\Longleftarrow$) Assume that T_λ is invertible for any $\lambda \in \wedge$. For any $A, B \in \mathbb{M}_n(\mathbb{S})$, if $T(A) = T(B)$ then

$$T_\lambda(A_\lambda) = (T(A))_\lambda = (T(B))_\lambda = T_\lambda(B_\lambda).$$

Since T_λ is injective, we have $A_\lambda = B_\lambda$. By Lemma 3.1 (i) it follows that $A = B$. So T is injective. Since T_λ is surjective, there exists X_λ such that $T_\lambda(X_\lambda) = B_\lambda$. Take $A \in M_n(\mathbb{S})$ with $A_\lambda = X_\lambda$ for any $\lambda \in \wedge$. It is clear that $T(A) = B$, and so T is surjective. Thus T is invertible. □

Proposition 3.5 *Let T be a linear operator on $M_n(\mathbb{S})$. Then T preserves orthogonality of matrices if and only if T_λ preserves orthogonality of fuzzy matrices for any $\lambda \in \wedge$.*

Proof ($\Longrightarrow$) For any $\lambda \in \wedge$ and any $A, B \in M_n(\mathbb{F})$, there exist $X, Y \in M_n(\mathbb{S})$ such that $X_\lambda = A, Y_\lambda = B$ and $X_\mu = Y_\mu = O$ for any $\mu \neq \lambda$. If A and B are orthogonal, then $XY = YX = O$. Since T preserves orthogonality of matrices, we have $T(X)T(Y) = T(Y)T(X) = O$. Further,

$$T_\lambda(A)T_\lambda(B) = (T(X))_\lambda(T(Y))_\lambda = ((T(X)T(Y))_\lambda = O.$$

Similarly, $T_\lambda(B)T_\lambda(A) = O$. This shows that $T_\lambda(A)$ and $T_\lambda(B)$ are orthogonal. So T_λ preserves orthogonality of fuzzy matrices as required.

($\Longleftarrow$) For any $X, Y \in M_n(\mathbb{S})$, if X and Y are orthogonal, then X_λ and Y_λ are orthogonal for any $\lambda \in \wedge$ by Lemma 3.1 (i). Since T_λ preserves orthogonality of fuzzy matrices, we have

$$(T(X))_\lambda(T(Y))_\lambda = T_\lambda(X_\lambda)T_\lambda(Y_\lambda) = O.$$

Similarly, $(T(Y))_\lambda(T(X))_\lambda = O$. So $T(X)_\lambda$ and $T(Y)_\lambda$ are orthogonal. Again by Lemma 3.1 (ii), we can show that $T(X)$ and $T(Y)$ are orthogonal. Therefore, T preserves orthogonality of matrices. □

In the following, we will give the main theorem of this section.

Theorem 3.6 *Let $\mathbb{S} = \prod_{\lambda \in \wedge} \mathbb{S}_\lambda$, where $\mathbb{S}_\lambda = \mathbb{F}$ is a fuzzy semiring for any $\lambda \in \wedge$. Let $T : M_n(\mathbb{S}) \to M_n(\mathbb{S}$ be a linear operator with $n \geq 2$. Then T is an invertible linear operator preserving orthogonality of matrices if and only if there exist $P \in M_n(\mathbb{S})$ and $s_1, s_2 \in \mathbb{S}$ such that*

$$T(X) = P(s_1X + s_2X^t)P^t$$

for all $X \in M_n(\mathbb{S})$*, where* $(s_1)_\lambda, (s_2)_\lambda \in \{0,1\}$*,* $(s_1)_\lambda \neq (s_2)_\lambda$ *and* $P_\lambda \in M_n(\mathbb{F})$ *is a permutation matrix for any* $\lambda \in \wedge$*.*

Proof ($\Longrightarrow$) It follows from Propositions 3.4 and 3.5 that T_λ is an invertible linear operator preserving orthogonality of matrices. For any $X \in M_n(\mathbb{S})$, $X_\lambda \in M_n(\mathbb{F})$. By virtue of Theorem 2.4, there exists permutation matrix $P_\lambda \in M_n(\mathbb{F})$ such that either

$$T_\lambda(X_\lambda) = P_\lambda X_\lambda P_\lambda^t \tag{1}$$

for all $X_\lambda \in M_n(\mathbb{S}_\lambda)$, or

$$T_\lambda(X_\lambda) = P_\lambda X_\lambda^t P_\lambda^t \tag{2}$$

for all $X_\lambda \in M_n(\mathbb{S}_\lambda)$. Let $\wedge_1 := \{\lambda \in \wedge | T_\lambda \text{ is the form of (1)}\}$ and $\wedge_2 := \{\lambda \in \wedge | T_\lambda \text{ is the form of (2)}\}$. It is clear that $\wedge_1 \bigcap \wedge_2 = \emptyset, \wedge_1 \bigcup \wedge_2 = \wedge$. For $i = 1, 2$, let $s_i \in \mathbb{S}$, where $(s_i)_\lambda = 1$ if $\lambda \in \wedge_i$ and 0 otherwise. Thus, for any $X \in M_n(\mathbb{S})$, there exist $P \in M_n(\mathbb{S})$ and $s_1, s_2 \in \mathbb{S}$ such that

$$T(X) = P(s_1X + s_2X^t)P^t,$$

where $(s_1)_\lambda, (s_2)_\lambda \in \{0,1\}$, $(s_1)_\lambda \neq (s_2)_\lambda$ and $P_\lambda \in M_n(\mathbb{F})$ is a permutation matrix for any $\lambda \in \wedge$.

($\Longleftarrow$) For any $\lambda \in \wedge$ and any $A \in M_n(\mathbb{S}_\lambda)$, there exists $X \in M_n(\mathbb{S})$ such that $A = X_\lambda$. We have

$$T_\lambda(A) = T_\lambda(X_\lambda) = (T(X))_\lambda = (P(s_1X + s_2X^t)P^t)_\lambda.$$

If $(s_1)_\lambda = 1$, $(s_2)_\lambda = 0$, then $T_\lambda(A) = P_\lambda A P_\lambda^t$ for any $A \in M_n(\mathbb{S}_\lambda)$. Otherwise, $T_\lambda(A) = P_\lambda A^t P_\lambda^t$ for any $A \in M_n(\mathbb{S}_\lambda)$. It follows from Theorem 2.4 that T_λ is an invertible linear operator preserving orthogonality. Hence T is an invertible linear operator preserving orthogonality of matrices by Propositions 3.4 and 3.5. □

Thus we have obtained complete characterizations of invertible linear operators preserving orthogonality of matrices over the direct product of fuzzy semirings by Theorems 3.3 and 3.6.

Example 3.7 Let $\mathbb{S} = \mathbb{F} \times \mathbb{F} \times \mathbb{F}$. Take

$$P = \begin{bmatrix} (0,1,1) & (1,0,0) & (0,0,0) \\ (1,0,0) & (0,1,0) & (0,0,1) \\ (0,0,0) & (0,0,1) & (1,1,0) \end{bmatrix} \in M_3(\mathbb{S})$$

and $s_1 = (0,1,0), s_2 = (1,0,1)$ in $\mathbb{S}$. Define an operator on $M_3(\mathbb{S})$ by

$$T(X) = P(s_1X + s_2X^t)P^t$$

for all $X \in M_3(\mathbb{S})$.

It is obvious that $P_\lambda (\lambda = 1, 2, 3)$ are all permutation matrices. Thus, by Theorem 3.6, T is an invertible linear operator that preserves orthogonality of matrices over $\mathbb{S}$.

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