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Nonsplit Roman Domination in Graphs

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Abstract: A roman dominating function on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $v \in V(G)$ for which $f(v) = 0$, is adjacent to at least one vertex u with $f(u) = 2$. The weight of a roman dominating function f is the value $w(f) = \sum_{v \in V} f(v)$. The minimum weight of a roman dominating function is called the roman domination number of G and is denoted by $\gamma_R(G)$. A roman dominating function f is called a nonsplit roman dominating function if the subgraph induced by the set $\{v : f(v) = 0\}$ is connected. The minimum weight of a nonsplit roman dominating function is called the nonsplit roman domination number and is denoted by $\gamma_{nsr}(G)$. In this paper, we initiate a study of this parameter.

Key Words: Domination number, roman domination number and nonsplit roman domination number.

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§1. Introduction

The graph $G = (V, E)$ we mean a finite, undirected, connected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m respectively. The degree of a vertex u in G is the number of edges incident with u and is denoted by $d_G(u)$, simply $d(u)$. The minimum and maximum degree of a graph G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1] and Haynes et.al [3, 4].

Let $v \in V$. The open neighborhood and closed neighborhood of v are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$. If $S \subseteq V$ then $N(S) = \bigcup_{v \in S} N(v)$ for all $v \in S$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$ then the private neighbor set of u with respect to S is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$. For any set $S \subseteq V$, the subgraph induced by S is the maximal subgraph of G with vertex set S and is denoted by $\langle S \rangle$.The vertex has degree one is called a pendant vertex. A support is a vertex which is adjacent to a pendant vertex. A weak support is a vertex which is adjacent to exactly one pendant vertex. A strong support is a vertex which is adjacent to at least two pendant vertices. An unicyclic graph is a graph with exactly one cycle. A graph without cycle is called acyclic graph and a connected acyclic graph is called a tree. $H(m_1, m_2, \cdots , m_n)$ denotes the graph obtained from the graph H by attaching m_i

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pendant edges to the vertex $v_i \in V(H), 1 \leq i \leq n$. The graph $K_2(m_1, m_2)$ is called bistar and it is also denoted by $B(m_1, m_2)$. $H(P_{m_1}, P_{m_2}, \cdots, P_{m_n})$ is the graph obtained from the graph H by attaching an end vertex of P_{m_i} to the vertex v_i in $H, 1 \leq i \leq n$. The clique number $\omega(G)$ is the maximum order of the complete subgraph of the graph G.

A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is called the domination number of G and is denoted by $\gamma(G)$. V.R.Kulli and B.Janakiram [5] introduced the concept of nonsplit domination in graphs.Also T.Tamizh Chelvam and B.Jayaparsad [6] studied the same concept in the name of the complementary connected domination in graphs. A dominating set S is called a nonsplit dominating set of a graph G if the induced subgraph $\langle V - S\rangle$ is connected. The minimum cardinality of a nonsplit dominating set of G is called the nonsplit domination number of G and is denoted by $\gamma_{ns}(G)$. A dominating set(nonsplit dominating set) of minimum cardinality is called $\gamma-$set ($\gamma_{ns}-$set) of G. E.J.Cockayne et.al [2] studied the concept of roman domination first. A roman dominating function on a graph G is a function $f : V(G) \longrightarrow \{0, 1, 2\}$ satisfying the condition that every vertex $v \in V$ for which $f(v) = 0$ is adjacent to at least one vertex $u \in V$ with $f(v) = 2$. The weight of a roman dominating function is the value $w(f) = \sum\limits_{v\in V} f(v)$. The minimum weight of a roman dominating function is called the roman dominating number of G and is denoted by $\gamma_R(G)$. P.Roushini Leely Pushpam and S.Padmapriea [6] introduced the concept of restrained roman domination in graphs. A roman dominating function f is called a restrained roman dominating function if the subgraph induced by the set $\{v : f(v) = 0\}$ contains no isolated vertex. The minimum weight of a restrained roman dominating function is called the restrained roman domination number of G and is denoted by $\gamma_{rR}(G)$. In this paper we introduce the concept of nonspilt roman domination and initiate a study of the corresponding parameter.

Theorem 1.1 ([7]) *Let G be a graph. Then $\gamma_{ns}(G) = n - 1$ if and only if G is a star.*

§2. Nonsplit Roman Domination Number

Definition 2.1 *A roman dominating function f is called a nonsplit roman dominating function if the subgraph induced by the set $\{v : f(v) = 0\}$ is connected. The minimum weight of a nonsplit roman dominating function is called the nonsplit roman domination number of G and is denoted by $\gamma_{nsr}(G)$.*

Remark 2.2 For a graph G, let $f : V \longrightarrow \{0, 1, 2\}$ and let (V_0, V_1, V_2) be the ordered partion of V induced by f, where $V_i = \{v \in V : f(v) = i\}$. Note that there exists an one to one correspondence between the function $f : V \longrightarrow \{0, 1, 2\}$ and the ordered partition (V_0, V_1, V_2) of V. Thus we will write $f = (V_0, V_1, V_2)$.

A function $f = (V_0, V_1, V_2)$ is a nonsplit roman dominating function if $V_0 \subseteq N(V_2)$ and the induced subgraph $\langle V_0\rangle$ is connected.The minimum weight of a nonsplit roman dominating function of G is called the nonsplit roman domination number of G and is denoted by $\gamma_{nsr}(G)$. We say that a function $f = (V_0, V_1, V_2)$ is a $\gamma_{nsr}-$function if it is an nonsplit roman dominating function and $w(f)=\gamma_{nsr}(G)$. Also $w(f)=|V_1|+2|V_2|$.

A few nonsplit roman domination number of some standard graphs are listed in the following.

(1) Any nontrivial path P_n, $\gamma_{nsr}(P_n)=n$;
(2) If $n\geq 4$ then $\gamma_{nsr}(C_n)=n$;
(3) If $n\geq 2$ then $\gamma_{nsr}(K_n)=2$;
(4) $\gamma_{nsr}(W_n)=2$;
(5) $\gamma_{nsr}(K_{1,n-1})=n$;
(6) $\gamma_{nsr}(K_{r,s})=4$ where $r,s\geq 2$.

Theorem 2.3 *For a graph G, $\gamma_{ns}(G)\leq\gamma_{nsr}(G)\leq 2\gamma_{ns}(G)$.*

Proof Let $f=(V_0,V_1,V_2)$ be a $\gamma_{nsr}-$function. Then $V_1\cup V_2$ is a nonsplit dominating set of G. Hence $\gamma_{ns}\leq|V_1\cup V_2|=|V_1|+|V_2|\leq|V_1|+2|V_2|=\gamma_{nsr}$. Also, let S be any $\gamma_{ns}-$set of G. Then $f=(V-S,\phi,S)$ is a nonsplit roman dominating function of G. Hence $\gamma_{nsr}(G)\leq 2|S|=2\gamma_{ns}(G)$. □

Observation 2.4 For a nontrivial graph G,

(i) $\gamma(G)\leq\gamma_{ns}(G)\leq\gamma_{nsr}(G)$;
(ii) $2\leq\gamma_{nsr}(G)\leq n$.

Remark 2.5 (i) For any connected graph G, $\gamma_{nsr}(G)=2$ if and only if there exists a non cut vertex v such that $d_G(v)=n-1$. Thus $\gamma_{nsr}(G)=2$ if and only if $G=H+K_1$ for some connected graph H.

(ii) For any connected spanning subgraph H of G, $\gamma_{nsr}(G)\leq\gamma_{nsr}(H)$.

Theorem 2.6 *If G contains a triangle then $\gamma_{nsr}(G)\leq n-1$.*

Proof Let v_1,v_2,v_3 form a triangle in G. Then $f=(\{v_1,v_2\},V-\{v_1,v_2,v_3\},\{v_3\})$ is a nonsplit roman dominating function of G and hence $\gamma_{nsr}(G)\leq n-1$. □

Theorem 2.7 *Let $v\in V(G)$ such that $d_G(v)=\Delta$ and $\langle N(v)\rangle$ be connected. Then $\gamma_{nsr}(G)\leq n-\Delta+1$.*

Proof Let us take $f=(N(v),V-N[v],\{v\})$. Then it is clear that f is a nonsplit roman dominating function. Hence $\gamma_{nsr}(G)\leq|V-N[v]|+2=n-(\Delta+1)+2=n-\Delta+1$. □

Definition 2.8 *Let $f=(V_0,V_1,V_2)$ be a nonsplit roman dominating function and let $u\in V_i, 0\leq i\leq 2$. The function f_u is defined as follows:*

Let V_j and V_k be the two sets in the ordered partition (V_0,V_1,V_2) other than V_i.

$$V_l'=\begin{cases}V_i-\{u\}, & \text{if } l=i\\ V_j\cup\{u\}, & \text{if } l=j\\ V_k, & \text{if } l=k, 0\leq l\leq 2.\end{cases},$$

Then the function $f_u = (V_0', V_1', V_2')$.

It is clear that for every $u \in V_i$ there are two functions f_u.

Definition 2.9 *A nonsplit roman dominating function* $f = (V_0, V_1, V_2)$ *is said to be a minimal nonsplit roman dominating function if for every* $u \in V_i, 0 \leq i \leq 2$ *either* $w(f_u) > w(f)$ *or* f_u *is not a nonsplit roman dominating function.*

We now proceed to obtain a characterization of minimal nonsplit roman dominating function.

Theorem 2.10 *A nonsplit roman dominating function* $f = (V_0, V_1, V_2)$ *is minimal if and only if for each* $u \in V_1$ *and* $v \in V_2$ *the following conditions are true.*

(*i*) $N(u) \cap V_0 = \phi$ *or* $N(u) \cap V_2 = \phi$;

(*ii*) *There exists a vertex* $w \in V_0$ *such that* $N(w) \cap V_2 = \{v\}$.

Proof Let $f = (V_0, V_1, V_2)$ be a minimal nonsplit roman dominating function and let $u \in V_1, v \in V_2$. Suppose $N(u) \cap V_0 \neq \phi$ and $N(u) \cap V_2 \neq \phi$. Then $f_u = (V_0 \cup \{u\}, V_1 - \{u\}, V_2)$ is a nonsplit roman dominating function with $w(f_u) = |V_1| - 1 + 2|V_2| \leq w(f)$ which is a contradiction. Hence either $N(u) \cap V_0 = \phi$ or $N(u) \cap V_2 = \phi$.

Suppose there is no vertex $w \in V_0$ such that $N(w) \cap V_2 = \{v\}$. Then $f_v = (V_0, V_1 \cup \{v\}, V_2 - \{v\})$ is a nonsplit roman dominating function with $w(f_v) = |V_1| + 1 + 2(|V_2| - 1) = |V_1| + 2|V_2| - 1 \leq w(f)$ which is a contradiction. Hence for every $v \in V_2$ there exists a vertex $w \in V_0$ such that $N(w) \cap V_2 = \{v\}$. The converse is straightforward. □

Theorem 2.11 *For a nontrivial graph* G, $\gamma_{nsr}(G) + \omega(G) \leq n + 2$ *where* $\omega(G)$ *is the clique number of* G.

Proof Let S be a set of vertices of G such that $\langle S \rangle$ is complete with $|S| = \omega(G)$. Then $f = (S - \{u\}, V - S, \{u\})$ is a nonsplit roman dominating function of G. Hence $\gamma_{nsr}(G) \leq |V - S| + 2 = n - \omega(G) + 2$. Thus $\gamma_{nsr}(G) + \omega(G) \leq n + 2$. □

Theorem 2.12 *For a graph* G, $\gamma_{nsr}(G) \geq 2n - m - 1$.

Proof Let $f = (V_0, V_1, V_2)$ be a $\gamma_{nsr}-$function. Since $\langle V_0 \rangle$ is connected and every vertex in V_0 is adjacent to at least one vertex in V_2, $\langle V_0 \cup V_2 \rangle$ contains at least $2|V_0| - 1$ edges.

Case 1. $\langle V_1 \rangle$ is connected.

Then $\langle V_1 \rangle$ contains at least $|V_1| - 1$ edges. Since G is connected there should be an edge between a vertex of V_1 and a vertex of $V_0 \cup V_2$. Hence there are at least $|V_1|$ edges other than the edges in $\langle V_0 \cup V_2 \rangle$.

Case 2. $\langle V_1 \rangle$ is disconnected.

Let $G_1, G_2, \cdots, G_k$ be the components of $\langle V_1 \rangle$. Since each G_i contains at least $|V(G_i)| - 1$ edges and since G is connected there exists an edge between a vertex of G_i and a vertex of

$V_0 \cup V_2$. Hence there are at least $\Sigma(|V(G_i)|-1)+k\ (=|V_1|)$ edges.

Hence $m \geq 2|V_0|-1+|V_1| = 2(n-|V_2|-|V_1|)-1+|V_1| = 2n-(2|V_2|+|V_1|)-1 = 2n-\gamma_{nsr}(G)-1$. Hence $\gamma_{nsr}(G) \geq 2n-m-1$. □

Corollary 2.13 *For a tree T, $\gamma_{nsr}(T)=n$.*

Proof $n \geq \gamma_{nsr}(T) \geq 2n-(n-1)-1=n$. Hence $\gamma_{nsr}(T)=n$. □

Corollary 2.14 *For an unicyclic graph G, $n-1 \leq \gamma_{nsr}(G) \leq n$.*

Theorem 2.15 *Let G be an unicyclic graph with cycle $C=(v_1,v_2,\cdots,v_k,v_1)$. Then $\gamma_{nsr}(G)=n-1$ if and only if one of the following is true.*

(i) $C=C_3$;

(ii) $d_G(v) \geq 3$ *for all* $v \in V(H)$ *where* H *is a connected subgraph of* C *of order at least* $k-3$.

Proof Let G be an unicyclic graph with cycle $C=(v_1,v_2,\cdots,v_k,v_1)$. Let $C=C_3$. Then G contains a triangle and hence by Theorem 2.6, $\gamma_{nsr}(G) \leq n-1$ which gives $\gamma_{nsr}(G)=n-1$.

Suppose C contains a connected subgraph H such that $|V(H)| \geq k-3$ and $d_G(v) \geq 3$ for all $v \in V(H)$. It is clear that H is either C or a path. Let P be a path in H of order $k-3$. Let $P=(v_1,v_2,\cdots,v_{k-3})$ and let $u_i \in N(v_i)-V(C), v_i \in V(P)$. Let $X=\{u_1,u_2,\cdots,u_{k-3}\}, V_0=V(P)\cup\{v_k,v_{k-2}\}, V_1=V(G)-(V(C)\cup X), V_2=X\cup\{v_{k-1}\}$. Then $f=(V_0,V_1,V_2)$ is a nonsplit roman dominating function of G. Thus $\gamma_{nsr}(G) \leq n-(k+k-3)+2(k-3+1)=n-1$ and hence $\gamma_{nsr}(G)=n-1$.

Conversely, let us assume $\gamma_{nsr}(G)=n-1$. Let $f=(V_0,V_1,V_2)$ be a γ_{nsr}−function of G. Suppose conditions (i) and (ii) given in the statement of the theorem are not true.

Let $P=(v_1,v_2,\cdots,v_{k-3})$ be a path in C such that $d_G(v_i)=2$ for some $i, 1 \leq i \leq k-3$ and $d_G(v_j)=2, k-2 \leq j \leq k$.

Case 1. $i \neq 1$ and $i \neq k-3$

Then at least one vertex v in the subpath (v_{i-1},v_i,v_{i+1}) with $f(v) \neq 0$ and at least two vertices u and w in the subpath $(v_{k-3},v_{k-2},v_{k-1},v_k,v_1)$ with $f(u) \neq 0$ and $f(w) \neq 0$ and hence either $\langle V_0 \rangle$ is the union of two distinct paths or $V_0=\phi$. Thus either $\langle V_0 \rangle$ is disconnected or $|V_0|=|V_2|=0$. Hence f is not a nonsplit roman dominating function or $\gamma_{nsr}=n$ which is a contradiction.

Case 2. $i=1$ or $i=k-3$

Let $d_G(v_i) \geq 3, 1 \leq i \leq k-2$ and $d_G(v_j)=2, k-3 \leq j \leq k$. Then at least two vertices x and y in $\{v_{k-3},v_{k-2},v_{k-1},v_k\}, d_G(x) \neq 0$ and $d_G(y) \neq 0$. Hence for every vertex v with $f(v)=2$ there exists exactly one vertex u with $f(u)=0$. Thus $\gamma_{nsr}(G)=n$ which is a contradiction. This proves the result. □

Now we characterize the lower bound in Theorem 2.3.

Theorem 2.16 *Let G be a connected graph. Then $\gamma_{ns}=\gamma_{nsr}(G)$ if and only if G is a trivial* *graph.*

Proof Let $f = (V_0, V_1, V_2)$ be a $\gamma_{nsr}-$function of G. Then $\gamma_{ns}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \gamma_{nsr}(G)$ which gives $|V_2| = 0$. Then $V_0 = \phi$ and hence $V_1 = V$. Then $\gamma_{ns}(G) = \gamma_{nsr}(G) = n$ which gives G is a trivial graph. □

Theorem 2.17 *Let G be a nontrivial graph of order n. Then $\gamma_{nsr}(G) = \gamma_{ns}(G)+1$ if and only if there exists a vertex $v \in V(G)$ such that $\langle N(v)\rangle$ has a component of order $n - \gamma_{ns}(G)$.*

Proof Let $v \in V(G)$ such that $\langle N(v)\rangle$ has a component of order $n - \gamma_{ns}(G)$. Let G_1 be the component of $\langle N(v)\rangle$ with $|V(G_1)| = n - \gamma_{ns}(G)$. Let $V_2 = \{v\}, V_1 = V - (V(G_1) \cup \{v\})$ and $V_0 = V - V_1 - V_2$. Then $V_1 \cup V_2$ is a $\gamma_{ns}-$set of G and $f = (V_0, V_1, V_2)$ is a nonsplit roman dominating function and hence $\gamma_{nsr}(G) \leq |V_1| + 2|V_2| = n - (n - \gamma_{ns}(G) + 1) + 2 = \gamma_{ns}(G) + 1$. Since G is nontrivial $\gamma_{ns}(G) + 1 \leq \gamma_{nsr}(G)$ and hence $\gamma_{nsr}(G) = \gamma_{ns}(G) + 1$.

Conversely, let us assume $\gamma_{nsr}(G) = \gamma_{ns}(G)+1$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{nsr}-$function of G. Then $\gamma_{nsr}(G) = |V_1| + 2|V_2|$ which gives $\gamma_{ns}(G) + 1 = |V_1| + 2|V_2|$. Then $|V_1| = \gamma_{ns}(G) + 1 - 2|V_2|$.

Suppose $|V_2| \geq 2$. Since $V_1 \cup V_2$ is a nonsplit dominating set, $\gamma_{ns}(G) \leq |V_1| + |V_2| = \gamma_{ns}(G) + 1 - 2|V_2| + |V_2| = \gamma_{ns}(G) + 1 - |V_2| \leq \gamma_{ns}(G) - 1$ which is a contradiction. Hence $|V_2| \leq 1$.

If $|V_2| = 0$ then $|V_0| = 0$ and hence $|V_1| = V$. Thus $\gamma_{nsr}(G) = n$ and $\gamma_{ns}(G) = n-1$. Then by theorem 1.1 G is a star. Let v be a pendant vertex of G and hence $\langle N(v)\rangle$ is a center vertex of star G. Thus $|N(v)| = 1 = n - (n-1) = n - \gamma_{ns}(G)$.

Suppose $|V_2| = 1$. Let $V_2 = \{v\}$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{nsr}-$function of G. Thus $\gamma_{nsr} = |V_1| + 2$. Then $\gamma_{ns}(G) + 1 - 2 = |V_1|$ which gives $|V_1| = \gamma_{ns}(G) - 1$. Hence $|V_0| = n - |V_1| - |V_2| = n - (\gamma_{ns}(G) - 1) - 1 = n - \gamma_{ns}(G)$ then the result follows. □

Corollary 2.18 *For any graph G, if $\gamma_{nsr}(G) = \gamma_{ns}(G)+1$ then $diam(G) \leq 4$ and $rad(G) \leq 2$.*

Proof Let $\gamma_{nsr}(G) = \gamma_{ns}(G) + 1$. Then there is a vertex $v \in V(G)$ such that $\langle N(v)\rangle$ has a component of order $n - \gamma_{ns}(G)$. Hence every vertex in $V - N[v]$ is adjacent to a vertex in $N(v)$.Thus $diam(G) \leq 4$ and $rad(G) \leq 2$. □

Corollary 2.19 *If T is a tree then $\gamma_{nsr}(T) = \gamma_{ns}(T) + 1$ if and only if T is a star.*

Theorem 2.20 *Let G be an unicyclic graph with the cycle $C = (v_1, v_2, \cdots, v_k, v_1)$. Then $\gamma_{nsr}(G) = \gamma_{ns}(G) + 1$ if and only if G is isomorphic to $C_3(n_1, n_2, 0)$.*

Proof Let us assume $\gamma_{nsr}(G) = \gamma_{ns}(G) + 1$. Then there is a vertex $v \in V(G)$ such that $\langle N(v)\rangle$ has a component of order $n - \gamma_{ns}(G)$. Let G_1 be a component of $\langle N(v)\rangle$ such that $|V(G_1)| = n - \gamma_{ns}(G)$. If $|V(G_1)| \geq 3$ then there is a path $P(u_1, u_2, u_3)$ in G_1. Then the induced subgraph of the sets $\{v, u_1, u_2\}$ and $\{v, u_2, u_3\}$ are cycles which is a contradiction. Hence $|V(G_1)| = 2$ and hence $C = C_3$ so that $C = (v_1, v_2, v_3, v_1)$. If $d_G(v_i) \geq 3$ for all i then $V - \{v_1, v_2, v_3\}$ is a nonsplit dominating set of G and hence $\gamma_{ns}(G) \leq n-3$ then $\gamma_{nsr}(G) \leq n-2$ which is a contradiction. Hence $d_G(v_i) = 2$ for some i. Let $d_G(v_3) = 2$.

Suppose there is a vertex $x \in V(G) - V(C)$ such that $d_G(x) \geq 2$. Let $v_1 \in V(C)$ such that $d(C,x) = d(v_1,x)$. Let $(v_1, x_1, x_2, \cdots, x_r, x), r \geq 1$ be the shortest $v_1 - x$ path. Then $V(G) - \{v_1, v_2, x_1\}$ is a nonsplit dominating set of G and hence $\gamma_{ns}(G) \leq n-3$ which is a contradiction. Hence every vertex in $V - V(C)$ is a pendant vertex which follows the result. □

Theorem 2.21 *Let G be a nontrivial graph of order n. Then $\gamma_{nsr}(G) = \gamma_{ns}(G) + 2$ if and only if*

(i) every vertex $v \in V(G)$ such that $\langle N(v)\rangle$ has no component of order $n - \gamma_{ns}(G)$;

(ii) G has a vertex v such that $\langle N(v)\rangle$ has a component of order $n - \gamma_{ns}(G) - 1$ or G has two vertices u and v such that $\langle N(u) \cup N(v)\rangle$ has a component of order $n - \gamma_{ns}$.

Proof Let the graph G be satisfy the conditions (i) and (ii) in the statement of the theorem. By condition (i) and Theorem 2.17, $\gamma_{nsr}(G) \geq \gamma_{ns}(G) + 2$. Suppose $v \in V(G)$ such that $\langle N(v)\rangle$ has a component G_1 of order $n - \gamma_{ns}(G) - 1$. Then $(V(G_1), V - (V(G_1) \cup \{v\}, \{v\})$ is a nonsplit roman dominating function of G and hence $\gamma_{nsr}(G) \leq n - (n - \gamma_{ns}(G) - 1 + 1) + 2 = \gamma_{ns}(G) + 2$. Hence $\gamma_{nsr}(G) = \gamma_{ns} + 2$. Suppose G has two vertices u and v such that $\langle N(u) \cup N(v)\rangle$ has a component of order $n - \gamma_{ns}(G)$. Let G_2 be the component of $\langle N(u) \cup N(v)\rangle$ with $|V(G_2)| = n - \gamma_{ns}(G)$. Let $V_2 = \{u,v\}, V_1 = V - (V(G_2) \cup \{u,v\})$ and $V_0 = V - V_1 - V_2. = V(G_2)$. Then $V_1 \cup V_2$ is a $\gamma_{ns}-$set of G and $f = (V_0, V_1, V_2)$ is a nonsplit roman dominating function and hence $\gamma_{nsr}(G) \leq |V_1| + 2|V_2| = n - (n - \gamma_{ns}(G) + 2) + 4 = \gamma_{ns}(G) + 2$ and hence $\gamma_{nsr}(G) = \gamma_{ns}(G) + 2$.

Conversely, let us assume $\gamma_{nsr}(G) = \gamma_{ns}(G) + 2$ and let $f = (V_0, V_1, V_2)$ be a $\gamma_{nsr}-$function of G. Then $\gamma_{nsr}(G) = |V_1| + 2|V_2|$ which gives $\gamma_{ns}(G) + 2 = |V_1| + 2|V_2|$. Then $|V_1| = \gamma_{ns}(G) + 2 - 2|V_2|$.

Suppose $|V_2| \geq 3$. Since $V_1 \cup V_2$ is a nonsplit dominating set, $\gamma_{ns}(G) \leq |V_1| + |V_2| = \gamma_{ns}(G) + 2 - 2|V_2| + |V_2| = \gamma_{ns}(G) + 2 - |V_2| \leq \gamma_{ns}(G) - 1$ which is a contradiction. Hence $|V_2| \leq 2$.

If $|V_2| = 0$ then $|V_0| = 0$ and hence $|V_1| = V$. Thus $\gamma_{nsr}(G) = n$ and $\gamma_{ns}(G) = n-2$. Let S be a $\gamma_{ns}-$set of G. Then $\langle V - S\rangle = K_2 = xy$. Suppose $|S| = 1$. Then $G = C_3$ and hence $\gamma_{nsr}(G) = 2$ and $\gamma_{ns}(G) = 1$ which is a contradiction. Thus $|S| \geq 2$. Then S contains two vertices u and v which dominates x and y. Thus G contains two vertices u and v such that $\langle N(u) \cup N(v)\rangle$ contains a component of order $n - \gamma_{ns}(G)$.

Suppose $|V_2| = 1$. Let $V_2 = \{v\}$. Then $\gamma_{nsr} = |V_1| + 2$. Thus $\gamma_{ns}(G) + 2 - 2 = |V_1|$ which gives $|V_1| = \gamma_{ns}(G)$. Then V_0 contains $n - \gamma_{ns}(G) - 1$ vertices. Thus $\langle N(v)\rangle$ has a component a component of order $n - \gamma_{ns} - 1$.

Suppose $|V_2| = 2$. Let $V_2 = \{u,v\}$. Then $\gamma_{nsr} = |V_1| + 4$. Thus $\gamma_{ns}(G) + 2 - 4 = |V_1|$ which gives $|V_1| = \gamma_{ns}(G) - 2$. Hence $|V_0| = n - |V_1| - |V_2| = n - (\gamma_{ns}(G) - 2) - 2 = n - \gamma_{ns}(G)$ then the result follows. □

Corollary 2.22 *If T is a nontrivial tree then $\gamma_{nsr}(T) = \gamma_{ns}(T) + 2$ if and only if T has exactly two support vertices.*

Proof Let T be a tree with $\gamma_{nsr}(T) = \gamma_{ns}(T) + 2$. Then $\gamma_{ns}(T) = n-2$. Let u and v be the support vertices such that $d(u,v)$ is maximum. Let $P(u = u_1, u_2, \cdots, u_k = v)$ be the $u - v$

path. Let u_i be the vertex lie in both $u-w$ and $u-v$ paths such that i is maximum. Then $V-\{u_{i-1},u_i,u_{i+1}\}$ is a nonsplit dominating set which is a contradiction. Hence T contains exactly two support vertices. The converse is obvious. □

Now we characterize the upper bound in Theorem 2.3.

Theorem 2.23 *Let G be a graph. Then $\gamma_{nsr}(G)=2\gamma_{ns}(G)$ if and only if G has a $\gamma_{nsr}-$function $f=(V_0,V_1,V_2)$ with $|V_1|=0$.*

Proof Let $f=(V_0,V_1,V_2)$ be a $\gamma_{nsr}-$function and $|V_1|=0$. Then V_2 is a nonsplit dominating set of G. Suppose, there exists a nonsplit dominating set S of G such that $|S|<|V_2|$. Then $g=(V-S,\phi,S)$ is a nonsplit roman dominating function of G and hence $\gamma_{nsr}(G)\leq 2|S|<2|V_2|$ which is a contradiction. Hence V_2 is a $\gamma_{ns}-$set of G. Hence $\gamma_{nsr}(G)=2|V_2|=2\gamma_{ns}(G)$.

Conversely we assume that $\gamma_{nsr}(G)=2\gamma_{ns}(G)$. Let S be $\gamma_{ns}-$set of G. Take $V_0=V-S, V_1=\phi, V_2=S$.Then $f=(V_0,V_1,V_2)$ is a nonsplit roman donating function of G with $w(f)=2|V_2|=2|S|=2\gamma_{ns}(G)$. Hence f is a $\gamma_{nsr}-$function of G with $|V_1|=0$. □

Theorem 2.24 *Let T be a nontrivial tree. Then $\gamma_{nsr}(T)=2\gamma_{ns}(T)$ if and only if T is isomorphic to $H\circ K_1$ for some tree H.*

Proof Let T be a tree with $\gamma_{nsr}(T)=2\gamma_{ns}(T)$. Then $\gamma_{ns}(T)=\dfrac{n}{2}$. Let S be a $\gamma_{ns}-$set of T. Then $|S|=\dfrac{n}{2}$, $\langle V-S\rangle$ is connected and $|V-S|=\dfrac{n}{2}$. It is clear that any vertex in S cannot adjacent to two or more vertices in $V-S$. If any two distinct vertices of S are adjacent to a vertex in $V-S$ then at least a vertex in $V-S$ is not dominated by S. Hence T is isomorphic to $H\circ K_1$ for some tree H. The converse is obvious. □

Since the graphs P_4 and C_5 are self complementary, the following result is obvious. Hence we omit its proof.

Theorem 2.25 *Let G be a graph such that both G and $\overline{G}$ are connected. Then $\gamma_{nsr}(G)+\gamma_{nsr}(\overline{G})\leq 2n$ and the bound is sharp.*

Theorem 2.26 *Let G be a graph such that G and $\overline{G}$ are connected and $diam(G)\geq 5$. Then $\gamma_{nsr}(G)+\gamma_{nsr}(\overline{G})\leq n+4$.*

Proof Let $S=\{u,v\}$, where $d(u,v)=diam(G)$. Then $f=(V-S,\phi,S)$ is a nonsplit roman dominating function of $\overline{G}$ so that $\gamma_{nsr}(\overline{G})\leq 4$ and hence the result follows. □

Remark 2.27 The bound given in Theorem 2.28 is sharp. The graph $G=P_6$ has diameter 5, $\gamma_{nsr}(G)=6$ and $\gamma_{nsr}(\overline{G})=4$. Thus $\gamma_{nsr}(G)+\gamma_{nsr}(\overline{G})=10=n+4$.

Problem 2.28 *Characterize graphs which attain the bounds given in Theorems* 2.25 *and* 2.26.

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