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Directed Pathos Block Line Cut-Vertex Digraph of an Arborescence

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Abstract: For an arborescence T, a *directed pathos block line cut-vertex digraph* $Q = DPBL_c(T)$ has vertex set $V(Q) = A(T) \cup C(T) \cup B(T) \cup P(T)$, where $C(T)$ is the cut-vertex set, $B(T)$ is the block set, and $P(T)$ is a directed pathos set of T. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(T)$ and the head of a coincides with the tail of b; Cd such that $C \in C(T)$ and $d \in A(T)$ and the tail of d is C; dC such that $C \in C(T)$ and $d \in A(T)$ and the head of d is C; Bc such that $B \in B(T)$ and $c \in A(T)$ and the arc c lies on the block B; Pa such that $a \in A(T)$ and $P \in P(T)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(T)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j. The problem of reconstructing an arborescence from its $DPBL_c(T)$ is discussed. We present the characterization of digraphs whose $DPBL_c(T)$ are planar and outer planar. In addition, a necessary and sufficient condition for $DPBL_c(T)$ to have crossing number one is presented. Further we show that for any arborescence T, $DPBL_c(T)$ never be maximal outer planar and minimally nonouterplanar.

Key Words: Crossing number, inner vertex number, complete bipartite digraph.

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§1. Introduction

We shall assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [1,4]. The concept of *pathos* of a graph G was introduced by Harary [2] as a collection of minimum number of edge disjoint open paths whose union is G.

The *path number* of a graph G is the number of paths in any pathos. The path number of

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a tree T equals k, where $2k$ is the number of odd degree vertices of T. Harary [3] and Stanton [8] calculated the path number of certain classes of graphs like trees and complete graphs.

H. M. Nagesh and R. Chandrasekhar [7] introduced the concept of a pathos block line cut-vertex graph of a tree.

A *pathos block line cut-vertex graph* of a tree T, written $PBL_c(T)$, is a graph whose vertices are the edges, paths of a pathos, cut-vertices, and blocks of T, with two vertices of $PBL_c(T)$ adjacent whenever the corresponding edges of T are adjacent or the edge lies on the corresponding path of the pathos or the edge incident with the cut-vertex or the edge lies on the corresponding block; two distinct pathos vertices P_m and P_n of $PBL_c(T)$ are adjacent whenever the corresponding paths of the pathos $P_m(v_i, v_j)$ and $P_n(v_k, v_l)$ have a common vertex.

The characterization of graphs whose $PBL_c(T)$ are planar, outer planar, maximal outer planar, and minimally nonouterplanar were presented.

In this paper, we extend the definition of a pathos block line cut-vertex graph of a tree to an arborescence. Furthermore, some of its characterizations such as the planarity, outer planarity, etc., are discussed.

We need some concepts and notations on directed graphs. A *directed graph* (or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pair of distinct vertices called *arcs*. Here $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. If (u, v) or uv is an arc in D, then we say that u is a *neighbor* of v. A digraph D is *semicomplete* if for each pair of distinct vertices u and v, at least one of the arcs (u, v) and (v, u) exists in D. A semicomplete digraph of order n is denoted by D_n.

For a connected digraph D, a vertex z is called a *cut-vertex* if $D - \{z\}$ has more than one connected component. A *block* B of a digraph D is a maximal weak subdigraph of D, which has no vertex v such that $B - v$ is disconnected. An entire digraph is a block if it has only one block. There are exactly three categories of blocks: strong, strictly unilateral, and strictly weak". The *out-degree* of a vertex v, written $d^+(v)$, is the number of arcs going out from v and the *in-degree* of a vertex v, written $d^-(v)$, is the number of arcs coming into v. The *total degree* of a vertex v, written $td(v)$, is the number of arcs incident with v. We immediately have $td(v) = d^-(v) + d^+(v)$.

A vertex with an in-degree (out-degree) zero is called a *source (sink)*. The directed path on $n \geq 2$ vertices is the digraph $\vec{P}_n = \{V(\vec{P}_n), E(\vec{P}_n), \eta\}$, where $V(\vec{P}_n) = \{u_1, u_2, \cdots, u_n\}$, $E(\vec{P}_n) = \{e_1, e_2, \cdots, e_{n-1}\}$, where η is given by $\eta(e_i) = (u_i, u_{i+1})$, for all $i \in \{1, 2, \cdots, (n-1)\}$.

An *arborescence* is a directed graph in which, for a vertex u called the *root* (a vertex of in-degree zero) and any other vertex v, there is exactly one directed path from u to v. We shall use T to denote an arborescence. A *root arc* of T is an arc which is directed out from the root of T, i.e., an arc whose tail is the root of T.

Since most of the results and definitions for undirected planar graphs are valid for planar digraphs also, the following definitions hold good for planar digraphs.

If D is a planar digraph, then the *inner vertex number* $i(D)$ of D is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of D in the plane. A digraph D is *outerplanar* if $i(D) = 0$ and *minimally nonouterplanar* if $i(D) = 1$ [5]. The *crossing number* of a digraph D, denoted by $\mathrm{cr}(D)$, is the minimum number of crossings of its arcs when the digraph D is drawn in the plane.

§2. Definitions

Definition 2.1 *The line digraph $L(D)$ of a digraph D has the arcs of D as vertices. There is an arc from $D-$ arc pq towards $D-$ arc uv if and only if $q = u$.*

Definition 2.2 *If a directed path $\vec{P}_n$ starts at one vertex and ends at a different vertex, then $\vec{P}_n$ is called an open directed path.*

Definition 2.3 *The directed pathos of an arborescence T is defined as a collection of minimum number of arc disjoint open directed paths whose union is T.*

Definition 2.4 *The directed path number $k^{'}$ of T is the number of open directed paths in any directed pathos of T, and is equal to the number of sinks in T.*

Definition 2.5 *For an arborescence T, a directed pathos line cut-vertex digraph $Q = DPL_c(T)$ has vertex set $V(Q) = A(T) \cup C(T) \cup P(T)$, where $C(T)$ is the cut-vertex set and $P(T)$ is a directed pathos set of T. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(T)$ and the head of a coincides with the tail of b; Cd such that $C \in C(T)$ and $d \in A(T)$ and the tail of d is C; dC such that $C \in C(T)$ and $d \in A(T)$ and the head of d is C; Pa such that $a \in A(T)$ and $P \in P(T)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(T)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j.*

Definition 2.6 *For an arborescence T, a directed pathos block line cut-vertex digraph $Q = DPBL_c(T)$ has vertex set $V(Q) = A(T) \cup C(T) \cup B(T) \cup P(T)$, where $C(T)$ is the cut-vertex set, $B(T)$ is the block set, and $P(T)$ is a directed pathos set of T. The arc set $A(Q)$ consists of the following arcs: ab such that $a, b \in A(T)$ and the head of a coincides with the tail of b; Cd such that $C \in C(T)$ and $d \in A(T)$ and the tail of d is C; dC such that $C \in C(T)$ and $d \in A(T)$ and the head of d is C; Bc such that $B \in B(T)$ and $c \in A(T)$ and the arc c lies on the block B; Pa such that $a \in A(T)$ and $P \in P(T)$ and the arc a lies on the directed path P; P_iP_j such that $P_i, P_j \in P(T)$ and it is possible to reach the head of P_j from the tail of P_i through a common vertex, but it is possible to reach the head of P_i from the tail of P_j.*

Note that the directed path number k' of an arborescence T is minimum only when the out-degree of the root of T is one. Therefore, unless otherwise specified, the out- degree of the root of every arborescence is exactly one. Finally, we assume that the direction of the directed pathos is along the direction of the arcs in T. Since the pattern of directed pathos for an arborescence is not unique, the corresponding directed pathos block line cut-vertex digraph is also not unique.

§3. Basic Properties of $DPBL_c(T)$

Remark 3.1 Since every arc of T is a block (strictly unilateral), the arcs directed out of block vertices reaches the vertices of $L(T)$ does not affect the crossing number of $DPBLc(T)$.

Observation 3.2 If T is an arborescence of order n ($n \geq 3$), then $L(T) \subseteq L_c(T) \subseteq DPL_c(T) \subseteq DPBL_c(T)$.

Remark 3.3 The number of arcs whose tail and head are the directed pathos vertices in $DPBL_c(T)$ is $k^{'} - 1$.

Proposition 3.4 *Let T be an arborescence with vertex set $V(T) = \{v_1, v_2, \cdots, v_n\}$, cut-vertex set $C(T) = \{C_1, C_2, \cdots, C_r\}$, and block set $B(T) = \{B_1, B_2, \cdots, B_s\}$. Then the order and size of $DPBL_c(T)$ are*

$$2(n-1) + k^{'} + \sum_{j=1}^{r} C_j \quad \textit{and} \quad \sum_{i=1}^{n} d^{-}(v_i) \cdot d^{+}(v_i) + \sum_{j=1}^{r} \{d^{-}(C_j) + d^{+}(C_j)\} + k^{'} + 2n - 3,$$

respectively.

Proof Let T be an arborescence with vertex set $V(T) = \{v_1, v_2, \cdots, v_n\}$, cut-vertex set $C(T) = \{C_1, C_2, \cdots, C_r\}$, and block set $B(T) = \{B_1, B_2, \cdots, B_s\}$. Then the order of $DPBL_c(T)$ equals the sum of size, cut-vertices, blocks, and the directed path number $k^{'}$ of T. Since every arc of an arborescence is a block, the order of $DPBL_c(T)$ is

$$n - 1 + \sum_{j=1}^{r} C_j + n - 1 + k^{'},$$
$$\Rightarrow 2(n-1) + k^{'} + \sum_{j=1}^{r} C_j.$$

The size of $DPBL_c(T)$ equals the sum of size of T and $L(T)$; total degree of cut-vertices; and the number of arcs whose tail and head are the directed pathos vertices. By Remark 3.3, the size of $DPBL_c(T)$ is,

$$\sum_{i=1}^{n} d^{-}(v_i) \cdot d^{+}(v_i) + \sum_{j=1}^{r} \{d^{-}(C_j) + d^{+}(C_j)\} + 2(n-1) + k^{'} - 1,$$
$$\Rightarrow \sum_{i=1}^{n} d^{-}(v_i) \cdot d^{+}(v_i) + \sum_{j=1}^{r} \{d^{-}(C_j) + d^{+}(C_j)\} + k^{'} + 2n - 3. \qquad \square$$

§4. A Criterion for Directed Pathos Block Line Cut-Vertex Digraphs

The main objective is to determine a necessary and sufficient condition that a digraph be a directed pathos block line cut-vertex digraph.

A *complete bipartite digraph* is a directed graph D whose vertices can be partitioned into non-empty disjoint sets A and B such that each vertex of A has exactly one arc directed towards each vertex of B and such that D contains no other arc.

Theorem 4.1 *A digraph $T^{'}$ is a directed pathos block line cut-vertex digraph of an arborescence T if and only if $V(T^{'}) = A(T) \cup C(T) \cup B(T) \cup P(T)$ and arc sets*

(1) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of T, respectively;

(2) $\cup_{j=1}^{r} \cup_{k=1}^{r} Z_j^{'} \times C_k$ such that $Z_j^{'} \times C_k = \phi$ for $j \neq k$;

(3) $\cup_{k=1}^{r} \cup_{j=1}^{r} C_k \times Z_j$ such that $C_k \times Z_j = \phi$ for $k \neq j$, where $Z_j^{'}$ and Z_j are the sets of in-coming and out-going arcs at C_k of T, respectively.

(iv) $\cup_{k=1}^{t} \cup_{j=1}^{t} P_k \times Y_j$ such that $P_k \times Y_j = \phi$ for $k \neq j$;

(4) $\cup_{k=1}^{t} \cup_{j=1}^{t} P_k \times Y_j^{'}$ such that $P_k \times Y_j^{'} = \phi$ for $k \neq j$, where Y_j is the set of arcs on which P_k lies and $Y_j^{'}$ is the set of directed paths whose heads are reachable from the tail of P_k through a common vertex in T;

(5) $\cup_{l=1}^{s} \cup_{l'=1}^{s} B_l \times N_{l'}$ such that $B_l \times N_{l'} = \phi$ for $l \neq l^{'}$, where $N_{l'}$ is the set of arcs lies on B_l in T.

Proof Let T be an arborescence with vertex set $V(T) = \{v_1, v_2, \cdots, v_n\}$, cut-vertex set $C(T) = \{C_1, C_2, \cdots, C_r\}$, block set $B(T) = \{B_1, B_2, \cdots, B_s\}$, and a directed pathos set $P(T) = \{P_1, P_2, \cdots, P_t\}$. We consider the following cases.

Case 1. Let v be a vertex of T with $d^-(v) = \alpha$ and $d^+(v) = \beta$. Then α arcs coming into v and the β arcs going out of v give rise to a complete bipartite subdigraph with α tails and β heads and $\alpha \cdot \beta$ arcs joining each tail with each head. This is the decomposition of $L(T)$ into mutually arc disjoint complete bipartite subdigraphs.

Case 2. Let C_i be a cut-vertex of T with $d^-(C_i) = \alpha^{'}$. Then $\alpha^{'}$ arcs coming into C_i give rise to a complete bipartite subdigraph with $\alpha^{'}$ tails and a single head (i.e., C_i) and $\alpha^{'}$ arcs joining each tail with C_i.

Case 3. Let C_i be a cut-vertex of T with $d^+(C_i) = \beta^{'}$. Then $\beta^{'}$ arcs going out of C_i give rise to a complete bipartite subdigraph with a single tail (i.e., C_i) and $\beta^{'}$ heads and $\beta^{'}$ arcs joining C_i with each head.

Case 4. Let P_j be a directed path which lies on $\alpha^{''}$ arcs in T. Then $\alpha^{''}$ arcs give rise to a complete bipartite subdigraph with a single tail (i.e., P_j) and $\alpha^{''}$ heads and $\alpha^{''}$ arcs joining P_j with each head.

Case 5. Let P_j be a directed path, and let $\beta^{''}$ be the number of directed paths whose heads are reachable from the tail of P_j through the common vertex in T. Then $\beta^{''}$ arcs give rise to a complete bipartite subdigraph with a single tail (i.e., P_j) and $\beta^{''}$ heads and $\beta^{''}$ arcs joining P_j with each head.

Case 6. Let B_p be a block of T. Then the arcs, say γ lies on B_p give rise to a complete bipartite subdigraph with a single tail (i.e., B_p) and γ heads and γ arcs joining B_p with each head.

Hence by all the above cases, $Q = DPBL_c(T)$ is decomposed into mutually arc-disjoint complete bipartite subdigraphs with $V(Q) = A(T) \cup C(T) \cup B(T) \cup P(T)$ and arc sets (i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of T, respectively.

(2) $\cup_{j=1}^{r}\cup_{k=1}^{r} Z_j^{'} \times C_k$ such that $Z_j^{'} \times C_k = \phi$ for $j \neq k$.

(3) $\cup_{k=1}^{r}\cup_{j=1}^{r} C_k \times Z_j$ such that $C_k \times Z_j = \phi$ for $k \neq j$, where $Z_j^{'}$ and Z_j are the sets of in-coming and out-going arcs at C_k of T, respectively.

(4) $\cup_{k=1}^{t}\cup_{j=1}^{t} P_k \times Y_j$ such that $P_k \times Y_j = \phi$ for $k \neq j$.

(5) $\cup_{k=1}^{t}\cup_{j=1}^{t} P_k \times Y_j^{'}$ such that $P_k \times Y_j^{'} = \phi$ for $k \neq j$, where Y_j is the set of arcs on which P_k lies and $Y_j^{'}$ is the set of directed paths whose heads are reachable from the tail of P_k through a common vertex in T.

(6) $\cup_{l=1}^{s}\cup_{l^{'}=1}^{s} B_l \times N_{l^{'}}$ such that $B_l \times N_{l^{'}} = \phi$ for $l \neq l^{'}$, where $N_{l^{'}}$ is the set of arcs lies on B_l in T.

Conversely, let $T^{'}$ be a digraph of the type described above. Let $t_1, t_2, \cdots, t_l$ be the vertices corresponding to complete bipartite subdigraphs $T_1, T_2, \cdots, T_l$ of Case 1, respectively; and let $w^1, w^2, \cdots, w^t$ be the vertices corresponding to complete bipartite subdigraphs $P_1^{'}, P_2^{'}, \cdots, P_t^{'}$ of Case 4, respectively. Finally, let t_0 be a vertex chosen arbitrarily.

For each vertex v of the complete bipartite subdigraphs $T_1, T_2, \cdots, T_l$, we draw an arc a_v as follows.

(a) If $d^+(v) > 0$, $d^-(v) = 0$, then $a_v := (t_0, t_i)$, where i is the base (or index) of T_i such that $v \in Y_i$.

(b) If $d^+(v) > 0$, $d^-(v) > 0$, then $a_v := (t_i, t_j)$, where i and j are the indices of T_i and T_j such that $v \in X_j \cap Y_i$.

(c) If $d^+(v) = 0$, $d^-(v) = 1$, then $a_v := (t_j, w^n)$ for $1 \leq n \leq t$, where j is the base of T_j such that $v \in X_j$.

Note that, in (t_j, w^n) no matter what the value of j is, n varies from 1 to t such that the number of arcs of the form (t_j, w^n) is exactly t.

We mark the cut-vertices as follows. From Case 2 and Case 3, we observe that for every cut vertex C, there exists exactly two complete bipartite subdigraphs, one containing C as the tail, and other as head. Let it be $C_j^{'}$ and $C_j^{''}$ for $1 \leq j \leq r$ such that $C_j^{'}$ contains C as the tail and $C_j^{''}$ as head. If the heads of $C_j^{'}$ and tails of $C_j^{''}$ are the heads and tails of a single T_i for $1 \leq i \leq l$, then the vertex t_i is a cut-vertex, where i is the index of T_i.

We now mark the directed pathos as follows. It is easy to observe that the directed path number $k^{'}$ equals the number of subdigraphs of Case 4. Let $\psi_1, \psi_2, \cdots, \psi_t$ be the number of heads of subdigraphs $P_1^{'}, P_2^{'}, \cdots, P_t^{'}$, respectively. Suppose we mark the directed path P_1. For this we choose any ψ_1 number of arcs and mark P_1 on ψ_1 arcs. Similarly, we choose ψ_2 number of arcs and mark P_2 on ψ_2 arcs. This process is repeated until all directed pathos are marked. The digraph T with directed pathos and cut-vertices thus constructed apparently has $T^{'}$ as directed pathos block line cut- vertex digraph. □

Given a directed pathos block line cut-vertex digraph Q, the proof of the sufficiency of above theorem shows how to find an arborescence T such that $DPBL_c(T) = Q$. This obviously raises the question of whether Q determines T uniquely. Although the answer to this in general is no, the extent to which T is determined is given as follows.

One can easily check that using reconstruction procedure of the sufficiency of above theorem, any arborescence (without directed pathos) is uniquely reconstructed from its directed pathos block line cut-vertex digraph. Since the pattern of directed pathos for an arborescence is not unique, there is freedom in marking directed pathos for an arborescence in different ways. This clearly shows that if the directed path number is one, any arborescence with directed pathos is uniquely reconstructed from its directed pathos block line cut-vertex digraph. It is known that a directed path is a special case of an arborescence. Since the directed path number $k^{'}$ of a directed path $\vec{P_n}$ of order n ($n \geq 3$) is exactly one, a directed path with a directed pathos is uniquely reconstructed from its directed pathos block line cut-vertex digraph.

§5. Characterization of $DPBL_c(T)$

Theorem 5.1 *A directed pathos block line cut-vertex digraph $DPBL_c(T)$ of an arborescence T is planar if and only if the total degree of each vertex of T is at most three.*

Proof Suppose $DPBL_c(T)$ is planar. Assume that $td(v) \geq 4$, for every vertex $v \in T$. Suppose there exists a vertex v of total degree four in T, that is, T is an arborescence whose underlying graph is $K_{1,4}$. Let $V(T) = \{a, b, c, d, e\}$ and $A(T) = \{(a, c), (c, b), (c, d), (c, e)\}$ such that a and (a, c) are the root and root arc of T, respectively. By definition, $A(L(T)) = \{(ac, cb), (ac, cd), (ac, ce)\}$. Since c is the cut-vertex of T, it is the tail of arcs $(c, b), (c, d), (c, e)$; and the head of an arc (a, c). Then c is a neighbor of vertices cb, cd, ce; and ac is a neighbor of c. This shows that $\mathrm{cr}(L_c(T)) = 0$. Let $P(T) = \{P_1, P_2, P_3\}$ be a directed pathos set of T such that P_1 lies on the arcs $(a, c), (c, b)$; P_2 lies on (c, d); and P_3 lies on (c, e). Then P_1 is a neighbor of ac, cb, P_2, P_3; P_2 is a neighbor of cd; and P_3 is a neighbor of ce. Clearly $\mathrm{cr}(DPL_c(T)) = 1$. By Remark 3.1, $\mathrm{cr}(DPBL_c(T)) = 1$, a contradiction.

Conversely, suppose that the total degree of each vertex of T is at most three. Let $V(T) = \{v_1, v_2, \cdots, v_n\}$ and $A(T) = \{e_1, e_2, \cdots, e_{n-1}\}$ such that v_1 and $e_1 = (v_1, v_2)$ are the root and root arc of T, respectively. By definition, $L(T)$ is an out-tree of order $n - 1$. The number of cut-vertices of T equals the number of vertices whose total degree is at least two. Then $L_c(T)$ is a connected digraph in which every block is either D_3 or $D_4 - e$. Furthermore, the directed path number $k^{'}$ is the number of sinks in T. Then the arcs joining vertices of $L(T)$ and directed pathos vertices; and arcs joining directed pathos vertices gives $DPL_c(T)$ such that $\mathrm{cr}(DPL_c(T)) = 0$. By Remark 3.1, $\mathrm{cr}(DPBL_c(T)) = 0$. This completes the proof. □

Theorem 5.2 *A directed pathos block line cut-vertex digraph $DPBL_c(T)$ of an arborescence T is outer planar if and only if T is a directed path $\vec{P_n}$ of order n ($n \geq 3$).*

Proof Suppose $DPBL_c(T)$ is outer planar. Assume that T is an arborescence whose underlying graph is $K_{1,3}$. Let $V(T) = \{a, b, c, d\}$ and $A(T) = \{(a, b), (b, c), (b, d)\}$ such that a and (a, b) are the root and root arc of T, respectively. Then $A(L(T)) = \{(ab, bc), (ab, bd)\}$. Since b is the cut-vertex of T, it is the tail of arcs $(b, c), (b, d)$; and the head of an arc (a, b). By definition, $L_c(T) = D_4 - e$. Clearly $i(L_c(T)) = 0$. Let $P(T) = \{P_1, P_2\}$ be a directed pathos set of T such that P_1 lies on the arcs $(a, b), (b, c)$; and P_2 lies on (b, d). Then P_1 is a neighbor of ab, bc, P_2; and P_2 is a neighbor of bd. This shows that $i(DPL_c(A_r)) = 1$. Since every arc of T is a block, let B_1, B_2, B_3 be blocks corresponding to arcs $(a, b), (b, c), (b, d)$, respectively.

Then the arcs joining B_1 and ab; B_2 and bc; and B_3 and bd increases the inner vertex number of $DPL_c(T)$ by one. Thus $i(DPBL_c(T)) = 2$, a contradiction.

Conversely, suppose that T is a directed path of order n $(n \geq 3)$. Let $V(T) = \{v_1, v_2, \cdots, v_n\}$ and $A(T) = \{e_1, e_2, \cdots, e_{n-1}\}$. Clearly, the directed path number of T is one. Then the underlying graph of $DPL(T)$ is the fan graph $F_{1,n-1}$. Let $C(T) = \{C_1, C_2, \cdots, C_{n-2}\}$ be the cut-vertex set of T such that the arcs e_i are directed into the cut-vertices C_i, and e_{i+1} are directed out of C_i for $1 \leq i \leq n-2$. Then the vertices e_i are the neighbors of C_i, and C_i are the neighbors of e_{i+1}. This shows that $i(DPL_c(T)) = 0$. Since every arc of T is a block, by Remark 3.1, $i(DPBL_c(T)) = 0$. □

Theorem 5.3(F. Harary, [1]) *Every maximal outer planar graph G with n vertices has $2n-3$ edges.*

Theorem 5.4 *For any arborescence T, $DPBL_c(T)$ is not maximal outerplanar.*

Proof We use contradiction. Suppose that $DPBL_c(T)$ is maximal outer planar. We consider the following three cases.

Case 1. Suppose that $td(v) \geq 4$, for every vertex $v \in T$. By Theorem 5.1, $DPBL_c(T)$ is nonplanar, a contradiction.

Case 2. Suppose there exits a vertex of total degree three in T. By necessity of Theorem 5.2, $DPBL_c(T)$ nonouterplanar, a contradiction.

Case 3. Suppose that T is a directed path $\vec{P}_n$ of order n $(n \geq 3)$. By Proposition 3.4, the order and size of $DPBL_c(T)$ are $3\alpha+3$ and $5\alpha+2$, respectively, where $\alpha = (n-2), n \geq 3$. But $5\alpha + 2 < 6\alpha + 3 = 2(3\alpha + 3) - 3$. By Theorem 5.3, $DPBL_c(T)$ is not maximal outerplanar, again a contradiction. Hence by all the above cases, $DPBL_c(T)$ is not maximal outerplanar. □

Theorem 5.5 *For any arborescence T, $DPBL_c(T)$ is not minimally nonouter planar.*

Proof We use contradiction. Suppose that $DPBL_c(T)$ is minimally nonouter planar. We consider the following three cases.

Case 1. Suppose that $td(v) \geq 4$, for every vertex $v \in T$. By Theorem 5.1, $DPBL_c(T)$ is nonplanar, a contradiction.

Case 2. Suppose there exits a vertex of total degree three in T. By necessity of Theorem 5.2, $i(DPBL_c(T)) = 2$, a contradiction.

Case 3. Suppose that T is a directed path $\vec{P}_n$ of order n $(n \geq 3)$. By Theorem 5.2, $DPBL_c(T)$ is outer planar, again a contradiction. Hence by all the above cases, $DPBL_c(T)$ is not minimally nonouterplanar. □

Theorem 5.6 *A directed pathos block line cut-vertex digraph $DPBL_c(T)$ of an arborescence T has crossing number one if and only if the underlying graph of T is $K_{1,4}$.*

Proof Suppose $DPBL_c(T)$ has crossing number one. Assume that T is an arborescence whose underlying graph is a star graph $K_{1,n}$ on $n \geq 5$ vertices. Suppose $T = K_{1,5}$. Let $V(T) = \{a, b, c, d, e, f\}$ and $A(T) = \{(a, c), (c, b), (c, d), (c, e), (c, f)\}$ such that a and (a, c) are the root and root arc of T, respectively. Then $A(L(T)) = \{(ac, cb), (ac, cd), (ac, ce), (ac, cf)\}$. Since c is the cut-vertex of T, it is the tail of arcs $(c, b), (c, d), (c, e), (c, f)$; and the head of an arc (a, c). Then c is a neighbor of vertices cb, cd, ce, cf; and ac is a neighbor of c. This shows that $\mathrm{cr}(L_c(T)) = 0$. Let $P(T) = \{P_1, P_2, P_3, P_4\}$ be a directed pathos set of T such that P_1 lies on the arcs $(a, c), (c, b)$; P_2 lies on (c, d); P_3 lies on (c, e); and P_4 lies on (c, f). Then P_1 is a neighbor of ac, cb, P_2, P_3, P_4; P_2 is a neighbor of cd; P_3 is a neighbor of ce; and P_4 is a neighbor of cf. This shows that $\mathrm{cr}(DPL_c(T)) = 2$. By Remark 3.1, $\mathrm{cr}(DPBL_c(T)) = 2$, a contradiction.

Conversely, suppose that T is an arborescence whose underlying graph is $K_{1,4}$. By necessity of Theorem 5.1, $\mathrm{cr}(DPBL_c(T)) = 1$. This completes the proof. □

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