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A Study on Cayley Graphs of Non-Abelian Groups

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Abstract: All connected Cayley graphs over Abelian groups are Hamiltonian. However, for Cayley graphs over non-Abelian groups, Chen and Quimpo prove in [2] that Cayley graphs over Hamiltonian groups (i.e, non- Abelian groups in which every subgroup is normal) are Hamiltonian. In this paper we discuss a few of the ideas which have been developed to establish the existence of Hamiltonian cycles and paths in the vertex induced subgraphs of Cayley graphs over non-Abelian groups.

Key Words: Cayley graphs, Hamiltonian cycles and paths, complete graph, orbit and centralizer of an element in a group, centre of a group.

AMS(2010): 05C25

§1. Introduction

Let G be a finite group and S be a non-empty subset of G. The graph $Cay(G, S)$ is defined as the graph whose vertex set is G and whose edges are the pairs (x, y) such that $sx = y$ for some $s \in S$ and $x \neq y$. Such a graph is called the Cayley graph of G relative to S. The definition of Cayley graphs of groups was introduced by Arthur Cayley in 1878 and the Cayley graphs of groups have received serious attention since then.Finding Hamiltonian cycles in graphs is a difficult problem,of interest in combinatorics, computer science and applications. In this paper,we present a short survey of various results in that direction and make some observations.

§2. Preliminaries

In this section deals with the basic definitions of graph theory and group theory which are needed in sequel. A graph (V, E) is said to be connected if there is a path between any two vertices of (V, E). Every pair of arbitrary vertices in (V, E) can be joined by an edge,then it is complete. A subgraph (U, F) of a graph (V, E) is said to be vertex induced subgraph if F consists of all the edges of (V, E) joining pairs of vertices of U. A Hamiltonian path is a path in (V, E) which goes through all the vertices in (V, E) exactly once. A hamiltonian cycle is a closed Hamiltonian path.A graph is said to be hamiltonian if it contains a hamiltonian cycle.

Let G be a group. The orbit of an element x under G is usually denoted as $\bar{x}$ and is defined

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as $\bar{x} = \{gx/g \in G\}$. Let x be a fixed element of G. The centralizer of an element x in G, $C_G(x)$, is the set of all elements in G that commute with x. In symbols, $C_G(x) = \{g \in G/gx = xg\}$. The centre of a group is denoted as $Z(G)$ and is defined as $Z(G) = \{g \in G/gx = xg \forall x \in G\}$. A group G acts on G by conjugation means $gx = gxg^{-1}$ for all $x \in G$. An element $x \in G$ is called an involution if $x^2 = e$, where e is the identity.

Theorem 2.1 *Let G be a finite non-Abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian, provided there exist an element $a \in \bar{x}$, which generates $C_G(x)$.*

Proof Since $a \in \bar{x}$ which generates $C_G(x)$, we have $C_G(x) = \{a, a^2, a^3, \cdots, a^n = e\}$ and $a \neq e$, where e is the identity. Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose an element $s \in \bar{x}$ such that $s = (ua)a(ua)^{-1}$.

Now $su = (ua)a(ua)^{-1}u = (ua)a(a^{-1}u^{-1})u = (ua)aa^{-1}(u^{-1}u) = (ua)aa^{-1}e = (ua)aa^{-1} = (ua)e = ua$, then there is an edge from u to ua. Again,

$$s(ua) = (ua)a(ua)^{-1}(ua) = (ua)a(a^{-1}u^{-1})(ua) = (ua)(aa^{-1})(u^{-1}u)a = (ua)(ea) = ua^2,$$

then there is an edge from ua to ua^2, so there exist a path from u to ua^2. Continuing in this way, we get a path $u \to ua \to ua^2 \to ua^3 \to \cdots \to ua^n = ue = u$ in the induced subgraph with vertex set $C_G(x)$ of $Cay(G, \bar{x})$, which is hamiltonian. □

Example 1 Let $G = S_5$ and let $x = (123)(45)$.From the composition table we have $C_G(x) = \{(), (45), (123), (132), (123)(45), (132)(45)\}$ and $\bar{x} = \{(123)(45), (124)(35), (125)(34), (132)(45), (134)(25), (135)(24), (142)(35), (143)(25), (145)(23), (152)(34), (153)(24), (154)(23), (15)(234), (14)(235), (15)(243), (13)(245), (14)(253), (13)(254), (12)(345), (12)(354)\}$. We observe that either $(123)(45)$ or $(132)(45)$ in $\bar{x}$ generates $C_G(x)$. Then, Theorem 2.1 implies that the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian and is given in Figure 1.

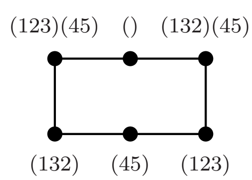

Figure 1

Theorem 2.2 *Let G be a finite non-Abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is Hamiltonian, provided $\bar{x}$ contains two involutions a and b which generates $C_G(x)$ and they commute.*

Proof Since $\bar{x}$ has two involutions a and b which generates $C_G(x)$, we have $C_G(x) = \{a, b, ab, e\}$. Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. Since $\bar{x}$ is the orbit of $x \in G$

and G act on G by conjugation, we can choose two involutions s_1 and s_2 in $\bar{x}$ such that $s_1 = (ua)a(ua)^{-1}$ and $s_2 = (ub)b(ub)^{-1}$. Now $s_1u = (ua)a(ua)^{-1}u = (ua)a(a^{-1}u^{-1})u = (ua)(aa^{-1})(u^{-1}u) = ((ua)e)e = ua$, so there is an edge from u to ua. Again $s_2(ua) = (ub)b(ub)^{-1}ua = (ub)b(b^{-1}u^{-1})ua = (ub)(bb^{-1})(u^{-1}u)a = ((ub)e)ea = uba = uab$, then there is an edge from ua to uab, so there exist a path from u to uab. Again $s_1(uab) = (ua)a(ua)^{-1}(uab) = (ua)a(a^{-1}u^{-1})(uab) = (ua)(aa^{-1})(u^{-1}u)ab = ((ua)e)eab = (ua)ab = u(aa)b = (ue)b = ub$, so there is an edge from uab to ub. Again $s_2(ub) = (ub)b(ub)^{-1}(ub) = (ub)be = ub^2 = ue = u$. Thus we get a Hamiltonian cycle $u \to ua \to uab \to ub \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$. □

Example 2 Let $G = S_4$ and let $x = (13)$. From the composition table we have $C_G(x) = \{(), (13), (24), (13)(24)\}$ and $\bar{x} = \{(12), (13), (14), (23), (24), (34))\}$. We can observe that $\bar{x}$ has two involutions (13) and (24) which generates $C_G(x)$ and(13)(24)=(24)(13). Then, Theorem 2.2 implies that the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is Hamiltonian and is given in Figure 2.

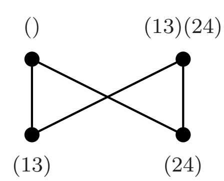

Figure 2

Theorem 2.3 *Let G be a finite non-Abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has disjoint Hamiltonian cycles, provided $\bar{x}$ has three elements a, b, c which do not generate $C_G(x)$ and they together with identity is isomorphic to V_4, the Klein-4 group.*

Proof We have $\{e, a, b, c\} \cong V_4$, so $ab = ba = c, bc = cb = a, ac = ca = b$ and a, b, c are involutions. Since $\bar{x}$ has three elements a, b, c which do not generate $C_G(x)$, we see that $x \neq e$. To prove that the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has disjoint hamiltonian cycles, it is enough to show that there exist at least two closed disjoint hamiltonian paths in it. Let $u \in \{e, a, b, c\}$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose two elements $s_1, s_2 \in \bar{x}$ such that $s_1 = (ua)a(ua)^{-1}$ and $s_2 = (ub)b(ub)^{-1}$. Now $s_1u = (ua)a(ua)^{-1}u = (ua)a(a^{-1}u^{-1})u = (ua)(aa^{-1})(u^{-1}u) = ((ua)e)e = ua$ then there is an edge from u to ua. Again $s_2(ua) = (ub)b(ub)^{-1}ua = (ub)b(b^{-1}u^{-1})ua = (ub)(bb^{-1})(u^{-1}u)a = ((ub)e)ea = uba = uc$ then there is an edge from ua to uc,consequently there exist a path from u to uc. Again $s_1(uc) = (ua)a(ua)^{-1}(uc) = (ua)a(a^{-1}u^{-1})(uc) = (ua)(aa^{-1})(u^{-1}u)c = ((ua)e)c = uac = ub$, so there is an edge from uc to ub and hence there exist a path from u to ub. Again $s_2(ub) = (ub)b(ub)^{-1}(ub) = (ub)b(b^{-1}u^{-1})(ub) = (ub)(bb^{-1})(u^{-1}u)b = ((ub)e)b = ubb = ue = u$. Thus we get a hamiltonian cycle $C_1 : u \to ua \to$

$uc \to ub \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$. In particular, for $u = a$, we get a hamiltonian cycle $a \to e \to b \to c \to a$.

Since a, b, c do not generate $C_G(x)$, clearly $C_G(x)$ contains at least one element $u_1 \notin V_4$. Now $s_1u_1 = (ua)a(ua)^{-1}u_1 = (ua)a(a^{-1}u^{-1})u_1 = (ua)(aa^{-1})(u^{-1}u_1) = (ua)e(u^{-1}u_1) = (ua)(u^{-1}u_1)$. Since $u \in V_4$, we have $ua = au$, then $(ua)(u^{-1}u_1) = (au)(u^{-1}u_1) = a(uu^{-1})u_1 = (ae)u_1 = au_1$. Clearly $au_1 \notin V_4$. For if $au_1 \in V_4$, then $au_1 = u_2 \in V_4$, which implies $u_1 = a^{-1}u_2 \in V_4$, it is a contradiction to our assumption that $u_1 \notin V_4$. So there exist an edge from u_1 to au_1. Again $s_2(au_1) = (ub)b(ub)^{-1}(au_1) = (ub)b(b^{-1}u^{-1})(au_1) = (ub)(bb^{-1})u^{-1}(au_1) = (ub)eu^{-1}(au_1) = (ub)u^{-1}(au_1) = (bu)u^{-1}(au_1) = b(uu^{-1})(au_1) = be(au_1) = bau_1 = cu_1$, as above we can show that $cu_1 \notin V_4$. Thus there exist an edge from au_1 to cu_1 and consequently a path from u_1 to cu_1. Also $s_1(cu_1) = (ua)a(ua)^{-1}(cu_1) = (ua)a(a^{-1}u^{-1})(cu_1) = (ua)(aa^{-1})u^{-1}(cu_1) = (ua)eu^{-1}(cu_1) = (ua)u^{-1}(cu_1) = (au)u^{-1}(cu_1) = a(uu^{-1})(cu_1) = ae(cu_1) = acu_1 = bu_1$. Here also $bu_1 \notin V_4$, so there is a path from u_1 to bu_1. Again $s_2(bu_1) = (ub)b(ub)^{-1}(bu_1) =$
$(ub)b(b^{-1}u^{-1})(bu_1) = (ub)(bb^{-1})u^{-1}(bu_1) = (ub)eu^{-1}(bu_1) = (ub)u^{-1}(bu_1) = (bu)u^{-1}(bu_1) = b(uu^{-1})(bu_1) = be(bu_1) = (bb)u_1 = eu_1 = u_1$. Thus we get another hamiltonian cycle $C_2 : u_1 \to au_1 \to cu_1 \to bu_1 \to u_1$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$, which is disjoint from C_1. □

Example 3 Let $G = S_4$ and let $x = (12)(34)$. From the composition table we have $C_G(x) = \{(), (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$ and $\bar{x} = \{(12)(34), (13)(24), (14)(23)\}$. We can observe that $\bar{x}$ has three elements which do not generate $C_G(x)$ and they together with identity is V_4 in S_4. Then, Theorem 2.3 implies that the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has disjoint hamiltonian cycles and are given in Figure 3.

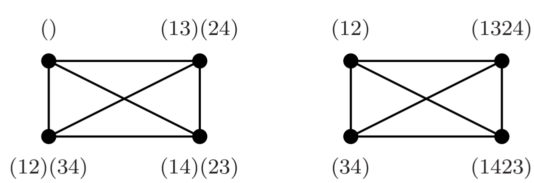

Figure 3

Theorem 2.4 *Let G be a finite non-Abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has two complete hamiltonian cycles, one with vertex set P_1 and other with vertex set P_2, provided $C_G(x)$ has a partition (P_1, P_2), where $\bar{x}$ generates $P_1 \cong V_4$ and P_2 is the generating set of P_1.*

Proof Since $P_1 \cong V_4$ and $\bar{x}$ generates P_1, we have $P_1 = \{e, u_1, u_2, u_3\}$. Then by Theorem 2.3, for every $u \in P_1$, we get a hamiltonian cycle $C_1 : u \to uu_1 \to uu_3 \to uu_2 \to u$ in the induced subgraph with vertex set P_1 of the Cayley graph $Cay(G, \bar{x})$. To prove that it is complete, it is enough to show that every pair of vertices in P_1 has an edge. Let u_1 and u_2 are two arbitrary vertices in P_1. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose an element $s \in \bar{x}$ such that $s = (u_1u_2)(u_1^{-1}u_2)(u_1u_2)^{-1}$. Now $su_1 = (u_1u_2)(u_1^{-1}u_2)(u_1u_2)^{-1}u_1 = (u_1u_2)(u_1^{-1}u_2)(u_2^{-1}u_1^{-1})u_1 = (u_1u_2)u_1^{-1}(u_2u_2^{-1})(u_1^{-1}u_1) = (u_1u_2)u_1^{-1}e = (u_1u_2)u_1^{-1}$ $= (u_2u_1)u_1^{-1} = u_2(u_1u_1^{-1}) = u_2e = u_2$. Thus there exist an edge from u_1 to u_2 and hence it is complete.

Since P_2 is the generating set of P_1, we have $P_2P_2 = P_1, P_2P_1 = P_2, P_1P_2 = P_2, P_1P_1 = P_1$. Let $u_4 \in P_2$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose two elements $s_1, s_2 \in \bar{x}$ such that $s_1 = (uu_1)u_1(uu_1)^{-1}$ and $s_2 = (uu_2)u_2(uu_2)^{-1}$ for $u \in P_1$.

Now $s_1u_4 = (uu_1)u_1(uu_1)^{-1}u_4 = (uu_1)u_1(u_1^{-1}u^{-1})u_4 = (uu_1)(u_1u_1^{-1})u^{-1}u_4 = (uu_1)eu^{-1}u_4$ $= (uu_1)u^{-1}u_4 = (u_1u)u^{-1}u_4 = u_1(uu^{-1})u_4 = u_1eu_4 = u_1u_4$. Clearly $u_1u_4 \notin P_1$, since $P_1P_2 = P_2$. So there is an edge from u_4 to u_1u_4.

Again

$$\begin{aligned} s_2(u_1u_4) &= (uu_2)u_2(uu_2)^{-1}(u_1u_4) = (uu_2)u_2(u_2^{-1}u^{-1})(u_1u_4) \\ &= (uu_2)(u_2u_2^{-1})u^{-1}(u_1u_4) = (uu_2)eu^{-1}(u_1u_4) \\ &= (u_2u)u^{-1}(u_1u_4) = u_2(uu^{-1})u_1u_4 \\ &= u_2e(u_1u_4) = (u_2u_1)u_4 = u_3u_4, \end{aligned}$$

as above we can show that $u_3u_4 \notin P_1$. Thus there is an edge from u_1u_4 to u_3u_4 and consequently a path from u_4 to u_3u_4.

Also

$$\begin{aligned} s_1(u_3u_4) &= (uu_1)u_1(uu_1)^{-1}(u_3u_4) = (uu_1)u_1(u_1^{-1}u^{-1})(u_3u_4) = (uu_1)(u_1u_1^{-1})u^{-1}(u_3u_4) \\ &= (uu_1)eu^{-1}(u_3u_4) = (uu_1)u^{-1}(u_3u_4) = (u_1u)u^{-1}(u_3u_4) \\ &= u_1(uu^{-1})(u_3u_4) = u_1e(u_3u_4) = (u_1u_3)u_4 = u_2u_4. \end{aligned}$$

Here also $u_2u_4 \notin P_1$, so there exist a path from u_4 to u_2u_4.

Again

$$\begin{aligned} s_2(u_2u_4) &= (uu_2)u_2(uu_2)^{-1}(u_2u_4) = (uu_2)u_2(u_2^{-1}u^{-1})(u_2u_4) \\ &= (uu_2)(u_2u_2^{-1})u^{-1}(u_2u_4) = (uu_2)eu^{-1}(u_2u_4) = (u_2u)u^{-1}(u_2u_4) \\ &= u_2(uu^{-1})(u_2u_4) = u_2e(u_2u_4) = (u_2u_2)u_4 = eu_4 = u_4. \end{aligned}$$

Thus we get another hamiltonian cycle $C_2 : u_4 \to u_1u_4 \to u_3u_4 \to u_2u_4 \to u_4$ in the induced subgraph with vertex set P_2 of the Cayley graph $Cay(G, \bar{x})$, which is disjoint from C_1. Let $u_4, u_5 \in P_2$. We can choose an element $s \in \bar{x}$ such that $s = (u_4u_5^{-1})(u_5u_4^{-1})(u_4u_5^{-1})^{-1}$. Then $su_4 = (u_4u_5^{-1})(u_5u_4^{-1})(u_4u_5^{-1})^{-1}u_4 = u_4(u_5^{-1}u_5)u_4^{-1}u_5(u_4^{-1}u_4) = (u_4e)u_4^{-1}u_5e = (u_4u_4^{-1})u_5 = eu_5 = u_5$. Thus for any two arbitrary elements $u_4, u_5 \in P_2$ is connected by an edge, so the induced subgraph with vertex set P_2 of the Cayley graph $Cay(G, \bar{x})$ is complete. □

Example 4 Let $G = S_5$ and let $x = (12)(34)$. From the composition table we have $C_G(x) =$

$\{(), (12), (34), (12)(34), (13)(24), (14)(23), (1324), (1423)\}$ and $\bar{x} = \{(12)(34), (12)(35), (12)(45),$ $(14)(23), (13)(24), (13)(25), (13)(45), (14)(25), (14)(35), (15)(23), (15)(24), (15)(34), (23)(45),$ $(25)(34), (24)(35)\}$. We can observe that $C_G(x)$ has a partition (P_1, P_2) where $P_1 = \{(), (12)(34),$ $(13)(24), (14)(23)\}$, which is V_4 in S_5 and P_2 is the generating set of P_1. Then, Theorem 2.4 implies that the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ has two complete Hamiltonian cycles and are given in Figure 4.

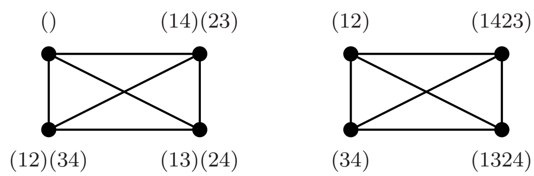

Figure 4

Theorem 2.5 *Let G be a finite non-Abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x} \cup V_4)$ is complete, provided there exist an element $a \in \bar{x}$, which generates $C_G(x)$ and $|C_G(x)| \leqslant 4$.*

Proof Since $a \in \bar{x}$ which generates $C_G(x)$ with $|C_G(x)| \leqslant 4$, by Theorem 2.1, for $u \in C_G(x)$ we get a hamiltonian path $u \to ua \to ua^2 \to ua^3 \to ua^4 = ue = u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$. Then clearly the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x} \cup V_4)$ is hamiltonian. Since the graph is hamiltonian, we know that there exist an edge from ua^i to ua^{i+1} . To prove that this graph is complete, it is enough to show that there exist an edge from ua^i to ua^{i+2} for $i = 0, 1$. We can choose an element $s \in V_4$ such that $s = ua^2u^{-1}$.Now $s(ua^i) = ua^2u^{-1}(ua^i) = ua^{i+2}$. So there exist an edge from ua^i to ua^{i+2}. Thus the graph is complete. □

Example 5 Let $G = S_5$ and let $x = (1423)$. From the composition table we have $C_G(x) =$ $\{(), (12)(34), (1423), (1324)\}$ and $\bar{x} = \{(1234), (1235), (1245), (1423), (1523), (2345), (1534),$ $(2534), (1342), (1352), (1452), (1432), (1532), (2453), (1543), (2543), (1354), (1324), (1325), (1345),$ $(1425), (1435), (1524), (2435), (1254), (1243), (1253), (2354), (1542), (1453)\}$. We can observe that either (1423) or (1324) in $\bar{x}$ generates $C_G(x)$ with $|C_G(x)| \leqslant 4$. Then, Theorem 2.5 implies that the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x} \cup V_4)$ is complete and is given in Figure 5.

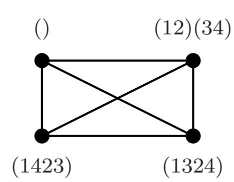

Figure 5

Theorem 2.6 *Let G be a finite non-Abelian group and G act on G by conjugation. Then for $x \in G$, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x} \cup V_4)$ is complete, provided there exist two involutions $a, b \in \bar{x}$ satisfy the conditions $ab = ba$ and $(ab)^2 = e$, which generates $C_G(x)$.*

Proof Since $\bar{x}$ contains two involutions a and b which generates $C_G(x)$ and $ab = ba$, by Theorem 2.2 we get a hamiltonian path $u \to ua \to uab \to ub \to u$ in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$. Then clearly the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x} \cup V_4)$ is hamiltonian. To prove that it is complete, it is enough to show that there exist edges from u to uab and ua to ub . Since V_4 is the klein-4 group, we can choose an element $s \in V_4$ such that $s = u(ab)u^{-1}$. Now $su = u(ab)u^{-1}u = (uab)(u^{-1}u) = uabe = uab$, so there is an edge from u to uab.

Similarly $s(ua) = u(ab)u^{-1}(ua) = uab(u^{-1}u)a = uab(ea) = u(ab)a = u(ba)a = uba^2 = ube = ub$, so there is an edge from ua to ub. Thus the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x} \cup V_4)$ is complete. □

Example 6 Let $G = S_4$ and let $x = (13)$. By Example 2, we get a hamiltonian cycle in the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$. If we add $(13)(24) \in V_4$ in $\bar{x}$, then it makes the graph complete and is given in Figure 6.

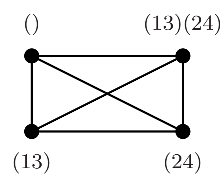

Figure 6

Theorem 2.7 *Let G be a finite non-Abelian group and G act on G by conjugation. Then for $x \in G$, where x is not an involution, the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian provided $|C_G(x)| \leqslant 5$.*

Proof Since x is not an involution, we see that $x \neq e$, where e is the identity. Let $u \in C_G(x)$. Then $ux = xu$ for $x \in G$. Since $\bar{x}$ is the orbit of $x \in G$ and G act on G by conjugation, we can choose an element $s = (ux)x(ux)^{-1} \in \bar{x}$ such that $s \in \bar{x} \cap C_G(x)$. Now $su = (ux)x(ux)^{-1}u = (ux)x(x^{-1}u^{-1})u = (ux)xx^{-1}(u^{-1}u) = (ux)xx^{-1}e = (ux)xx^{-1} = (ux)e = ux$. Then there is an edge from u to ux. Again $s(ux) = (ux)x(ux)^{-1}(ux) = (ux)x(e) = (ux)x = ux^2$, then there is an edge from ux to ux^2 so there exist a path from u to ux^2. Continuing in this way, we get a path from u to ux^i for $i \in N$. Since G is finite and $x \in G$, we have $ux^i = ux^j$ for some i and j. Now $(ux^j)x^{-i} = (ux^i)x^{-i} = ue = u$. Thus the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian. □

Example 7 Let $G = S_5$ and let $x = (13245)$. From the composition table we have $C_G(x) = \{(), (15423), (13245), (12534), (14352)\}$ and $\bar{x} = \{(12345), (14532), (12435), (15423), (13245),$

$(15324), (15243), (12453), (14325), (15432), (13452), (14523), (15342), (12534), (13425), (14235), (13542), (15234), (14352), (13254), (12354), (14253), (12543), (13524)\}$. We observe that $x^2 \neq e$ with $|C_G(x)| \leqslant 5$. Then, Theorem 2.7 implies that the induced subgraph with vertex set $C_G(x)$ of the Cayley graph $Cay(G, \bar{x})$ is hamiltonian and is given in Figure 7.

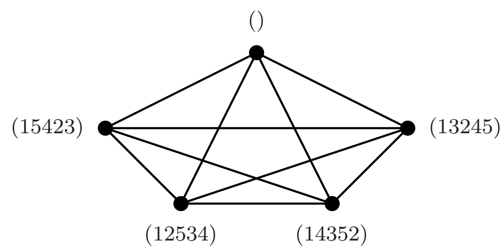

Figure 7

Theorem 2.8 *Let G be a finite non-Abelian group and N be a non-trivial normal subgroup of G. Then $Cay(\frac{G}{N}, Z(\frac{G}{N}))$ is complete, provided $Z(\frac{G}{N}) \neq e$.*

Proof Let $u, v \in \frac{G}{N}$ with $u \neq v$. Then $u = g_1h$ and $v = g_2h$ for $g_1, g_2 \in G$ and $h \in N$. Since $s \in Z(\frac{G}{N})$ and $Z(\frac{G}{N}) \neq e$, we have an element $s = (g_1^{-1}g_2)h \in Z(\frac{G}{N})$ such that $sx = xs$ for every $x \in Z(\frac{G}{N})$. Now $su = (g_1^{-1}g_2)h(g_1h) = (g_1h)(g_1^{-1}g_2)h = ((g_1g_1^{-1})g_2)h = (eg_2)h = g_2h = v$. So for any two arbitrary vertices u, v in $\frac{G}{N}$ has an edge. Thus the Cayley graph $Cay(\frac{G}{N}, Z(\frac{G}{N}))$ is complete. $\square$

Example 8 Let $G = S_4$. We observe that $Cay(\frac{G}{A_4}, Z(\frac{G}{A_4}))$ is complete where as $Cay(\frac{G}{V_4}, Z(\frac{G}{V_4}))$ is not, since $Z(\frac{G}{V_4}) = e$.

Suppose $G = D_4$.We have $N = ((), (13)(24))$ is a normal subgroup of G with $Z(\frac{D_4}{N}) \neq e$. Then, Theorem 2.8 implies that $Cay(\frac{G}{N}, Z(\frac{G}{N}))$ is complete and is shown in Figure 8.

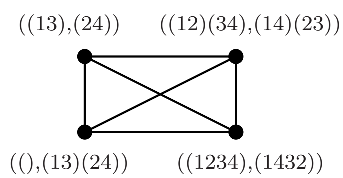

Figure 8

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