International J.Math. Combin. Vol.4(2016), 161-167

A Note on Neighborhood Prime Labeling

Mathew Varkey T.K and Rajesh Kumar T.J

(Department of Mathematics, T.K.M College of Engineering, Kollam, Kerala, India)

E-mail: mathewvarkeytk@gmail.com, vptjrk@gmail.com

Abstract: A labeling or numbering of a graph G is an assignment of labels to the vertices of G that induces for each edge uv a labeling depending on the vertex labels $f(u)$ and $f(v)$. In this paper, we investigate neighborhood prime labeling of graph obtained by identifying any two vertices of path P_n. We also discuss neighborhood prime labeling in some graph operations on the cycle C_n.

Key Words: Neighborhood prime labeling,fusion,vertex switching, path union.

AMS(2010): 05C78

§1. Introduction

All graphs in this paper are finite, simple, undirected and having no isolated vertices. For all terminology and notations in graph theory, we follow [2] and for all terminology regarding graceful labeling,we follow [3]. The field of graph theory plays vital role in various fields.Graph labeling is one of the important area in graph theory. Graph labelings where the vertices are assigned values subject to certain conditions have been motivated by practical problems.Labeled graphs serves as useful mathematical models for a broad range of applications such as communication network addressing system,data base management, circuit designs, coding theory, X-ray crystallography, the design of good radar type codes, synch-set codes,missile guidance codes and radio astronomy problems etc. The detailed description of the applications of graph labelings can be seen in [1].

Definition 1.1 *Let $G = (V(G),E(G))$ be a graph with p vertices. A bijection f: $V(G) \rightarrow \{1, 2, 3, \cdots, p\}$ is called prime labeling if for each edge $e = uv$, $gcd(f(u), f(v)) = 1$. A graph which admits prime labeling is called a prime graph.*

The notion of prime labeling was introduced by Roger Entringer and was discussed in a paper by [4]. In [5] the author proved that the path P_n on n vertices is a prime graph. In [6] the author proved that the graph obtained by identifying any two vertices of path P_n is a prime graph. The prime labeling of some cycle related graphs were discussed in [7]. In [9] it is shown that $C_nXP_2; C_nUK_{1,m}, C_nUP_m, K_{1,n}UP_n$, Olive trees, $P_n \circledcirc K_1, n \geq 2, P_1 \bigcup P_2 \bigcup \cdots \bigcup P_n$ have a prime labeling.

[1]Received January 14, 2016, Accepted November 28, 2016.

§2. Neighborhood Prime Labeling

Definition 2.1 ([8]) *Let $G = (V(G), E(G))$ be a graph with p vertices. A bijection $f : V(G) \to \{1, 2, 3, \cdots, p\}$ is called neighborhood prime labeling if for each vertex $v \in V(G)$ with $deg(v) > 1$, $gcd\{f(u) : u \in N(v)\} = 1$. A graph which admits neighborhood prime labeling is called a neighborhood prime graph.*

For a vertex $v \in V(G)$, the neighborhood of v is the set of all vertices in G which are adjacent to v and is denoted by $N(v)$. If in a graph G, every vertex is of degree at most 1, then such a graph is neighborhood prime. S.K.Pater,N.P.Shrimali [8] proved that the path P_n is neighborhood prime graph for every n. They also proved that the cycle C_n is neighborhood prime if $n \not\equiv 2(mod4)$.We consider some results on neighborhood prime labeling of path P_n and cycle C_n.

Definition 2.2 *Let u and v be two distinct vertices of a graph G. A new graph $G_{u,v}$ is constructed by identifying (fusing) two vertices u and v by a single new vertex x such that every edge which was incident with either u or v in G is now incident with x in $G_{u,v}$.*

Definition 2.3 *A vertex switching G_v of a graph G is obtained by taking a vertex v of G,removing all the edges incident with v and adding edges joining v to every vertex which are not adjacent to G.*

Definition 2.4 *Let $G_1, G_2, \cdots, G_n, n \geq 2$ be n copies of a fixed graph G. The graph obtained by adding an edge between G_i and G_{i+1} for $i = 1, 2, \cdots, n-1$ is called the path union of G.*

Theorem 2.1 *The graph obtained by identifying any two vertices of P_n is a neighborhood prime graph if $n \not\equiv 3(mod4)$.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of P_n. Let u be the new vertex of the graph G obtained by identifying two distinct vertices v_1 and v_2 of P_n. Then G is a loop with a path in n-1 vertices.Since the path P_n is neighborhood prime for every n, G is neighborhood prime. Let u be the new vertex of G obtained by identifying two distinct vertices v_a and v_b of P_n. Then G is a cycle (possibly loop) with at most two paths attached at u. The graph G is the disjoint union of cycle C'_n and the path P'_n.Consider the consecutive vertices of C'_n are $u = u_1, u_2, \cdots, u_r$ and the consecutive vertices of P'_n are $v_0 = u, v_1, v_2, \cdots, v_s$. Define a function $f : V(G) \to \{1, 2, 3, \cdots, n-1\}$ as follows.

Case 1. If r and s are both even, define

$$f(u_{2i-1}) = \frac{n-1}{2} + i, 1 \leq i \leq \frac{r}{2}, \quad f(u_{2i}) = i, 1 \leq i \leq \frac{r}{2},$$
$$f(v_{2j-1}) = \frac{n+r-1}{2} + j, 1 \leq j \leq \frac{s}{2}, \quad f(v_{2j}) = \frac{r}{2} + j, 1 \leq j \leq \frac{s}{2}$$

Case 2. If r is even but s is odd, define

$$f(u_{2i-1}) = \frac{n-2}{2} + i, 1 \leq i \leq \frac{r}{2}, \quad f(u_{2i}) = i, 1 \leq i \leq \frac{r}{2},$$
$$f(v_{2j-1}) = \frac{n+r}{2} + j, 1 \leq j \leq \frac{s+1}{2}, \quad f(v_{2j}) = \frac{r}{2} + j, 1 \leq j \leq \frac{s-1}{2}.$$

Case 3. If r is odd but s is even, define

$$f(u_{2i-1}) = \frac{n-2}{2} + i, 1 \leq i \leq \frac{r+1}{2}, \quad f(u_{2i}) = i, 1 \leq i \leq \frac{r-1}{2},$$
$$f(v_{2j-1}) = \frac{n+r-1}{2} + j, 1 \leq j \leq \frac{s}{2}, \quad f(v_{2j}) = \frac{r-1}{2} + j, 1 \leq j \leq \frac{s}{2}.$$

Case 4. If r and s are both odd, define

$$f(u_{2i-1}) = \frac{n-1}{2} + i, 1 \leq i \leq \frac{r+1}{2}, \quad f(u_{2i}) = i, 1 \leq i \leq \frac{r-1}{2},$$
$$f(v_{2j-1}) = \frac{r-1}{2} + j, 1 \leq j \leq \frac{s+1}{2}, \quad f(v_{2j}) = \frac{n+r}{2} + j, 1 \leq j \leq \frac{s-1}{2}.$$

Clearly, f is an injective map. We claim f is neighborhood prime labeling due to the following:

(1) If v_j is a vertex of $P_n^{'}$ and $1 \leq j \leq s-1$, the proof is divided into cases following:

Case 1. If r and s are both even, the neighborhood vertices of each vertex v_j are either $(\frac{n-1+r}{2} + j, \frac{n-1+r}{2} + j + 1)$ or $(\frac{r}{2} + j, \frac{r}{2} + j + 1)$. These are consecutive integers. So the gcd of the neighborhood vertices of v_j is 1.

Case 2. If r is even but s is odd, the neighborhood vertices of each vertex v_j are either $(\frac{n+r}{2} + j, \frac{n+r}{2} + j + 1)$ or $(\frac{r}{2} + j, \frac{r}{2} + j + 1)$. These are consecutive integers. So the gcd of the neighborhood vertices of v_j is 1.

Case 3. If r is odd but s is even, the neighborhood vertices of each vertex v_j are either $(\frac{n-1+r}{2} + j, \frac{n-1+r}{2} + j + 1)$ or $(\frac{r-1}{2} + j, \frac{r-1}{2} + j + 1)$. These are consecutive integers. So the gcd of the neighborhood vertices of v_j is 1.

Case 4. If r and s are both odd the neighborhood vertices of each vertex v_j are either $(\frac{n+r}{2} + j, \frac{n+r}{2} + j + 1)$ or $(\frac{r-1}{2} + j, \frac{r-1}{2} + j + 1)$. These are consecutive integers. So the gcd of the neighborhood vertices of v_j is 1.

(2) If u_i is a vertex of $C_n^{'}$, $2 \leq i \leq r$, the proof is divided into cases following:

Case 1. If r and s are both even, the neighborhood vertices of each vertex u_i are either $(\frac{n-1}{2}+i, \frac{n-1}{2}+i+1)$ or $(i, i+1)$. These are consecutive integers. So the gcd of the neighborhood vertices of u_i is 1.

Case 2. If r is even but s is odd, the neighborhood vertices of each vertex u_i are either $(\frac{n-2}{2}+i, \frac{n-2}{2}+i+1)$ or $(i, i+1)$. These are consecutive integers. So the gcd of the neighborhood vertices of u_i is 1.

Case 3. If r is odd but s is even, the neighborhood vertices of each vertex u_i are either $(\frac{n-2}{2}+i, \frac{n-2}{2}+i+1)$ or $(i, i+1)$. These are consecutive integers. So the gcd of the neighborhood vertices of u_i is 1.

Case 4. If r and s are both odd, the neighborhood vertices of each vertex u_i are either $(\frac{n-1}{2}+i, \frac{n-1}{2}+i+1)$ or $(i, i+1)$. These are consecutive integers. So the gcd of the neighborhood vertices of u_i is 1.

(3) For the vertex $u = u_1$ in $C_n^{'}$, the labeling of one of the neighborhood vertex is one. So the gcd is one.

Finally if we identifying the vertices v_1 and v_n of the path P_n, then the graph G is a cycle with $n-1$ vertices. The cycle C_n is neighborhood prime for $n \ncong 2(mod4)$, G is neighborhood prime if $n \ncong 3(mod4)$. □

§3. Neighborhood Prime Labeling on Cycle Related Graphs

In this section we consider neighborhood prime labeling on cycle with chords,cycle with switching a vertex,path union of cycles and join of two cycles with a path. In [10] Mathew Varkey T.K and Sunoj B.S proved that, every cycle C_n with a chord is prime for $n \geqslant 4$ and every cycle C_n with $[\frac{n-1}{2}] - 1$ chords from a vertex is prime for $n \geqslant 5$. We have the following theorems.

Theorem 3.1 *Every cycle C_n with a chord is neighborhood prime for all $n \geq 4$.*

Proof Let G be a graph such that $G = C_n$ with a chord joining two non-adjacent vertices of C_n,for all $n \geq 4$. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of G. Let the number of vertices of G be n and number of edges of G be $n+1$.

(1) If $n \ncong 2(mod4)$, define a function $f : V(G) \to \{1, 2, \cdots, n\}$ as follows:

Case 1. If n is odd, let

$$f(v_{2j-1}) = \frac{n-1}{2} + j, 1 \leq j \leq \frac{n+1}{2} \text{ and } f(v_{2j}) = j, 1 \leq j \leq \frac{n-1}{2}.$$

Case 2. If n is even, let

$$f(v_{2j-1}) = \frac{n}{2} + j, 1 \leq j \leq \frac{n}{2} \text{ and } f(v_{2j}) = j, 1 \leq j \leq \frac{n}{2}.$$

The neighborhood vertices of each vertex v_i except v_n is $\{v_{i-1}, v_{i+1}\}$ and they are consecutive integers, so it is neighborhood prime. The neighborhood vertices of v_n is $\{v_{n-1}, v_1\}$ and the corresponding labels are consecutive integers $\frac{n-1}{2}$ and $\frac{n+1}{2}$ if n is odd, n and $\frac{n}{2}+1$ if n is even.Now select the vertex v_i and join this to any vertex of G which is not adjacent to v_i. Then it is clear that the gcd of labeling of the neighborhood vertices of each vertex is one and G is neighborhood prime graph.

(2) If $n \cong 2(mod4)$, the labeling of the same function shows that there exists at least one vertex whose neighborhood set is not prime.Let v_i be the vertex whose neighborhood set is not

prime. We choose the vertex v_j which is not adjacent and relatively prime to v_i in G and join with a chord. Then G is a neighborhood prime graph. □

Theorem 3.2 *Every cycle C_n with $n-3$ chords from a vertex is neighborhood prime for $n \geq 5$.*

Proof Let G be a graph such that $G = C_n, n \geq 5$. Let $\{v_1, v_2, \cdots, v_n\}$ be the vertex set of G.Choose an arbitrary vertex v_i and joining v_i to all the vertices which are not adjacent to v_i. Then there are n-3 chords to v_i and from the above theorem G admits neighborhood prime labeling. □

Theorem 3.3 *The graph obtained by switching of any vertex in a cycle C_n is neighborhood prime graph.*

Proof Let $G = C_n$ and $v_1, v_2, \cdots, v_n$ be the successive vertices of C_n.Let G_{v_k} denotes the vertex switching of G with respect to the vertex v_k. Here $|V(G_{v_k})| = n$ and $|E(G_{v_k})| = 2n-5$. Define a labeling $f : V(G) \to \{1, 2, 3, \cdots, n\}$ as follows:

$f(v_k) = 1,$
$f(v_i) = i+1, 1 \leq i \leq k-1,$
$f(v_{k+i}) = f(v_{k-1}) + i, 1 \leq i \leq n-k.$

Then for any vertex v_i other than v_k, the neighborhood vertices containing v_k and so the gcd of the label of vertices in $N(v_i)$ is 1. G_{v_k} is a neighborhood prime graph. □

Theorem 3.4 *Let G be the graph obtained by the path union of finite number of copies of cycle C_n. G is a neighborhood prime graph if $n \not\cong 2(mod 4)$.*

Proof Let G be the path union of cycle C_n and $G_1, G_2, \cdots, G_k$ be k copies of cycle C_n. The vertices of G is nk and edges of G is $(n+1)k$. Let us denote the vertices of G be $v_{ij}, 1 \leq i \leq n, 1 \leq, j \leq k$ and the successive vertices of the graph G_r by $v_{1r}, v_{2r}, \cdots, v_{nr}$. Let $e = v_{1r}v_{1(r+1)}$ be the edge joining G_r and $G_{(r+1)}$ for $r = 1, 2, \cdots, k-1$.

Define the labeling $f : V(G) \to \{1, 2, \cdots, nk\}$ as follows:

Case 1. If n is odd and $1 \leq j \leq k$, define

$$f(v_{(2i-1)j}) = nj + i - \frac{n+1}{2}, 1 \leq i \leq \frac{n+1}{2} \text{ and } f(v_{(2i)j}) = n(j-1) + i, 1 \leq i \leq \frac{n-1}{2}.$$

Case 2. If n is even and $1 \leq j \leq k$, define

$$f(v_{(2i-1)j}) = nj + i - \frac{n}{2}, 1 \leq i \leq \frac{n}{2} \text{ and } f(v_{(2i)j}) = n(j-1) + i, 1 \leq i \leq \frac{n}{2}.$$

We claim that f is a neighborhood prime labeling. If v_{ir} is any vertex of G in the r^th copy of the cycle C_n different from v_{1r}, then $N(v_{ir}) = \{v_{(i-1)r}; v_{(i+1)r}\}$. Since $f(v_{(i-1)r})$ and $f(v_{(i+1)r})$ are consecutive integers, gcd of the labels of the vertices in $N(v_{ir})$ is 1.

Notice that $N(v_{11}) = \{v_{n1}; v_{21}\}$ and $f(v_{21}) = 1$, the gcd of the labels of vertices in $N(v_{11})$ is 1. Now we consider vertices $v_{1r}, 1 \leq r \leq k$.

Case 1. If n is odd, the labels of vertices in $N(v_{1r})$ are $n(r-\frac{3}{2})+\frac{1}{2}$, $n(r+\frac{1}{2})+\frac{1}{2}$, $n(r-1)+1$ and nr. They are relatively prime.

Case 2. If n is even, the labels of vertices in $N(v_{1r})$ are $n(r-\frac{3}{2})+\frac{1}{2}$, $n(r+\frac{1}{2})+\frac{1}{2}$, $n(r-1)+2$ and $n(r-\frac{1}{2})$. They are relatively prime.

Finally we consider v_{1k}.

Case 1. If n is odd, the labels of vertices in $N(v_{1k})$ are $n(k-\frac{3}{2})+\frac{1}{2}$, $n(k-1)+1$ and nk. They are relatively prime.

Case 2. If n is odd, the labels of vertices in $N(v_{1k})$ are $n(k-\frac{3}{2})+\frac{1}{2}$, $n(k-1)+1$ and $n(k-\frac{1}{2})$. They are relatively prime.

The cycle C_n is not neighborhood prime if $n \cong 2(mod4)$. Thus G is not neighborhood prime if $n \cong 2(mod4)$. Hence G is neighborhood prime if $n \ncong 2(mod4)$. □

Theorem 3.5 *The graph obtained by by joining two copies of cycle C_n by a path P_k is a neighborhood prime graph if $n \ncong 2(mod4)$.*

Proof Let G be the graph obtained by joining two copies of cycle C_n by a path P_k. The vertices of G are $2n+k-2$ and edges of G are $2n+k-1$. Let $v_1, v_2, \cdots, v_n$ be the vertices of the first copy of cycle C_n and $w_1, w_2, \cdots, w_n$ be the vertices of the second copy of cycle C_n. Let $u_1, u_2, \cdots, u_k$ be the vertices of path P_k with $v_1 = u_1$ and $w_1 = u_k$.

Define the labeling $f: V(G) \to \{1, 2, 3, \cdots, 2n+k-2\}$ as follows:

Case 1. If n is odd, let the labeling on C_n be

$$f(v_{2i-1}) = \frac{n-1}{2} + i, 1 \le i \le \frac{n+1}{2}, \quad f(v_{2i}) = i, 1 \le i \le \frac{n-1}{2}$$

and

$$f(w_{2i-1}) = \frac{3n-1}{2} + i, 1 \le i \le \frac{n+1}{2}, f(w_{2i}) = n+i, 1 \le i \le \frac{n-1}{2}.$$

Case 2. If n is even, let the labeling on C_n be

$$f(v_{2i-1}) = \frac{n}{2} + i, 1 \le i \le \frac{n}{2}, \quad f(v_{2i}) = i, 1 \le i \le \frac{n}{2}$$

and

$$f(w_{2i-1}) = \frac{3n}{2} + i, 1 \le i \le \frac{n}{2}, \quad f(w_{2i}) = n+i, 1 \le i \le \frac{n}{2}.$$

The labeling on P_k is defined by

Case 1. If k is odd, let

$$f(u_{2i}) = 2n + \frac{k-3}{2} + i, 1 \le i \le \frac{k-1}{2} \text{ and } f(u_{2i+1}) = 2n + i, 1 \le i \le \frac{k-3}{2}.$$

Case 2. If k is even, let

$$f(u_{2i}) = 2n + \frac{k-2}{2} + i, 1 \le i \le \frac{k-2}{2} \text{ and } f(u_{2i+1}) = 2n + i, 1 \le i \le \frac{k-2}{2}.$$

We claim that f is a neighborhood prime labeling. If v_i is any vertex of G in the first copy of the cycle C_n different from v_1, then $N(v_i) = [v_{i-1}, v_{i+1}]$. Since $f(v_{i-1})$ and $f(v_{i+1})$ are consecutive integers, the gcd of the label of the vertices is 1. Also $N(v_1)$ contains the vertex v_2 and $f(v_2) = 1$, the gcd of the label of vertices in $N(v_1)$ is 1.

If w_i is any vertex of G in the second copy of the cycle C_n different from w_1, then $N(w_i) = [w_{i-1}, w_{i+1}]$. Since $f(w_{i-1})$ and $f(w_{i+1})$ are consecutive integers, the gcd of the label of the vertices is 1.

Now, consider w_1.

Case 1. If n is odd, $N(w_1)$ are $\{w_2, w_n, u_{k-1}\}$. They are relatively prime for $n \geq 1$ since $f(w_2) = n+1, f(w_n) = 2n$.

Case 2. If n is even, $N(w_1)$ are $\{w_2, w_n, u_{k-1}\}$. They are relatively prime for $n \geq 2$ since $f(v_2) = n+1, f(v_n) = \frac{3n}{2}$.

Finally, if u_i is any vertex of G in the path P_k different from u_i and u_k,then $N(u_i) = \{u_{i-1}, u_{i+1}\}$. Since $f(u_{i-1})$ and $f(u_{i+1})$ are consecutive integers, the gcd of the label of vertices of $N(u_i)$ is 1. Thus, G is a neighborhood prime labeling graph if $n \not\equiv 2(mod4)$. □

References

[1] G.S.Bloom, S.W.Golomb, *Pro.of the IEEE*, 165(4), (1977), 562-70.

[2] F.Harary, *Graph Theory*, Addision Wesley, Reading, M.A, 1969.

[3] Joseph A.Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 17(2014).

[4] Tout A.,Dabboucy A.N, Howalla K, Prime labeling of graphs, *Nat.Acad.Sci.Letters*, 11(1982),365-368.

[5] Fu H.L, Huang K.C, On Prime labeling, *Discrete Math.*, 127(1994), 181-186.

[6] Samir K.Vaidya, Udayan M.Prajapati, Some new results on prime graphs, *Open journal of Discrete Mathematics*, 2(2012), 99-104.

[7] S.K.Vaidya, K.K.Kanani,Prime labeling for some cycle related graphs, *Journal of Mathematics Research*, Vol.2, No.2, May 2010.

[8] S.K.Patel, N.P.Shrimali,Neighborhood prime labeling, *International journal of Mathematics and Soft Computing*, Vol.5, No.2(2015),135-143.

[9] T.K.Mathew Varkey, *Some Graph Theoretic Operations Associated with Graph Labeling*, PhD thesis, University of Kerala, 2000.

[10] T.K.Mathew Varkey, B.S.sunoj, A note on prime and sequential labelings of finite graphs, *International J.Maths.Combin.*, Vol.1(2015), 80-85.