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Transformation Graph G^{xy} with $xy = +-$

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Abstract: For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The transformation graph G^{+-} of G is the graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two adjacent edges of G or to a vertex and an edge not incident with it in G. In this paper, we obtain structural properties and eccentricity properties of G^{+-}. We establish characterization of graphs whose G^{+-} are Eulerian. In addition, we obtain middle graphs, total graphs and quasi-total graphs of G, which are isomorphic to G^{+-}.

Key Words: Eccentricity, transformation graph, Smarandachely transformation graph, middle graph, total graph, quasi-total graph.

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§1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph without loops or multiple edges. For any graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. The vertices and edges of a graph are called its elements. Two elements of a graph are neighbors if they are either incident or adjacent. For graph theoretic terminology, we refer to [3].

The eccentricity of a vertex $u \in V(G)$ is defined as $e_G(u) = \max\{d_G(u, v) : v \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. If there is no confusion, then we simply denote the eccentricity of a vertex v in G as $e(v)$ and use $d(u, v)$ to denote the distance between two vertices u, v in G. The minimum and maximum eccentricities are the radius $r(G)$ and diameter $diam(G)$ of G, respectively.

A set of vertices which covers all the edges of a graph G is called a vertex cover for G, while a set of edges which covers all the vertices is an edge cover. The smallest number of vertices in any vertex cover for G is called its vertex covering number and is denoted by $\alpha_0(G)$ or α_0. Similarly, $\alpha_1(G)$ or α_1 is the smallest number of edges in any edge cover of G and is called its edge covering number. A set of vertices in G is independent if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence number of G and is denoted by $\beta_0(G)$ or β_0. Analogously, an independent set of edges of G has no two of its edges adjacent and the maximum cardinality of such a set is the edge independence number

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$\beta_1(G)$ or β_1.

When a new concept is developed in graph theory, it is often first applied to particular classes of graphs. Afterwards more general graphs are studied. The adjacency relation between two vertices or two edges and incidence relationship between vertices and edges define new structure from the given graph.

A connected graph G is said to be geodetic, if a unique shortest path joins any two of its vertices. The shortest path between two vertices u and v is called geodetic path between u and v. A subset S of V is called a dominating set if $N[S] = V$. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$ ([4]).

The line graph of G, denoted by $L(G)$, is the graph whose vertex set is $E(G)$ with two vertices adjacent in $L(G)$ whenever the corresponding edges of G are adjacent ([3]). The middle graph $M(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent in $M(G)$ whenever either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it ([1]). Clearly, $E(M(G)) = E(T(G)) \setminus E(G)$. Sampathkumar and Chikkodimath also studied it independently and they called it the semi-total graph $T_1(G)$ of a graph G ([5]). The total graph $T(G)$ of G has vertex set $V(G) \cup E(G)$ and vertices of $T(G)$ are adjacent whenever they are neighbors in G ([2]). The quasi-total graph $P(G)$ of G is a graph with vertex set $V(G) \cup E(G)$ and two vertices are adjacent if and only if they correspond to two nonadjacent vertices of G or to two adjacent edges of G or to a vertex and an edge incident with it in G ([6]).

Let $G = (V, E)$ be a graph and α, β be elements of $V(G) \cup E(G)$. We say that the associativity of α and β is $+$ if they are adjacent or incident in G, otherwise $-$. Let xy be a 2-permutation of the set $\{+, -\}$. We say that α and β correspond to the first term x of xy if both α and β are in $E(G)$. We say that α and β correspond to the second term y of xy if one of α and β is in $V(G)$ and the other in $E(G)$. The transformation graph G^{xy} of G is defined on the vertex set $V(G) \cup E(G)$. Two vertices α and β of G^{xy} are joined by an edge if and only if their associativity in G is consistent with the corresponding term of xy. Since there are four distinct 2-permutation of $\{+, -\}$, we obtain four graph transformations of G namely G^{++}, G^{+-}, G^{-+}, G^{--}. It is interesting to see that G^{++} is exactly the middle graph $M(G)$ of G.

We define the transformation graph G^{+-} as follows: G^{+-} of G is the graph with vertex set $V(G) \bigcup E(G)$ and two vertices are adjacent if and only if they correspond to adjacent edges of G or to a vertex and an edge not incident with it in G. The vertex v_i (e'_i) of G^{+-} corresponding to a vertex v_i (*edge* e_i) of G and is referred to as point (line) vertex. Generally, a *Smarandachely transformation graph* G_S^{+-} for a subset $S \subset (V(G) \cup E(G))$ is defined to be a graph with vertex set $V(G) \bigcup E(G)$ and two vertices are adjacent if and only if they correspond to adjacent edges of G or to a vertex and an edge not incident with it in $\langle S \rangle_G$. Thus, if $S = V(G) \bigcup E(G)$, then $G_S^{+-} = G^{+-}$.

§2. Main Results

Observations Let G be a (p, q) graph. Then

1. The line graph $L(G)$ of G is an induced subgraph of G^{+-};

2. The number of vertices of G^{+-} is $p + q$;
3. The degree of a point vertex v in G^{+-} is $q - deg_G(v)$;
4. For every edge $e = uv$ of G the degree of a line vertex e' in G^{+-} is $p + deg_G(u) + deg_G(v) - 4$.

Theorem 2.1 *Let G be a (p, q) graph. Then the number of edges $q^{'}$ of G^{+-} is $q^{'} = \frac{1}{2}\sum d_i^2 + q(p-3)$, where $d_i = deg_G(v_i)$.*

Proof By the definition of G^{+-}, each line vertex is adjacent to $p - 2$ point vertices. Since the number of edges in line graph is given by $-q + \frac{1}{2}\sum d_i^2$, it follows that

$$q^{'} = \frac{1}{2}\sum d_i^2 + q(p-3). \qquad \square$$

In the following we characterize disconnected G^{+-}.

Theorem 2.2 *G^{+-} is disconnected if and only if G is one of the following graphs: $2K_2$ and $K_{1,n} \cup lK_1$, $n, l \geq 0$.*

Proof Let G be a (p, q) graph. Assume that G^{+-} is disconnected. We consider the following cases.

Case 1. $q = 0$ and $p \geq 1$. Then $G \cong lK_1$, $l \geq 1$. It follows that G is totally disconnected, so that G^{+-} is totally disconnected.

Case 2. $q = 1$ and $p \geq 2$. Then $G \cong K_2 \cup lK_1$, $l \geq 0$.

Case 3. $q = 2$ and $p \geq 3$. Then $G \cong 2K_2 \cup lK_1$ or $G \cong K_{1,2} \cup lK_1$. But for $G \cong 2K_2 \cup lK_1$, $l \geq 1$, G^{+-} is connected.

Case 4. $q \geq 3$ and $p \geq 3$, we consider the following subcases.

Subcase 4.1 $q = 3$ and $p = 3$. Then $G = C_3$, for which G^{+-} is connected.

Subcase 4.2 $q \geq 3$ and $p \geq 4$. Then G can have at most one vertex of degree greater than or equal three, and all other vertices are either isolated vertices or pendant vertices, otherwise G^{+-} is connected. Hence, $G \cong K_{1,n} \cup lK_1$, $n \geq 3$, $l \geq 0$.

From all the above cases, it follows that, if G^{+-} is disconnected, then $G \cong 2K_2$ or $G \cong K_{1,n} \cup lK_1$, $n, l \geq 0$.

The converse is obvious. $\square$

Corollary 2.3 *G^{+-} is connected if and only if G is none of the following graphs: $2K_2$ and $K_{1,n} \cup lK_1$, $n, l \geq 0$.*

Proof Proof follows from above theorem. $\square$

Theorem 2.4 *Let G be a connected graph, and v be a cut vertex of G incident with $q-1$ edges. Then G^{+-} has a cut vertex such that v is pendant vertex.*

Proof Assume that G is a connected graph having a cut vertex v. Then $G - v$ is disconnected. Since v is incident with $q-1$ edges in G, there exists an edge e of G which is not incident with v in G. By the definition of G^{+-}, the point vertex v is adjacent with only one vertex e' in G^{+-}. Hence the line vertex e' is a cut vertex of G^{+-}, and hence $G^{+-} - e'$ is disconnected graph and v is pendant vertex. □

Corollary 2.5 *Let G be any graph, such that G^{+-} is disconnected. Then G^{+-} contains a cut vertex if and only if G is any graph of $2K_2$, $K_{1,2} \cup lK_1$, $l \geq 0$ and $K_2 \cup lK_l$, $l \geq 2$.*

In the following, we find the girth of G^{+-}.

Theorem 2.6 *For any connected (p, q) graph G with $p \geq 4$, the girth of G^{+-} is 3.*

Proof If G contains a triangle or $K_{1,3}$, then the line graph $L(G)$ of G contains triangle. Since $L(G)$ is a subgraph of G^{+-}, it follows that girth of G^{+-} is 3. Assume that G is triangle-free and $K_{1,3}$ - free. Then $\beta_0 \geq 2$. Since $p \geq 4$, let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ be edges in G such that (u_1, v_1, u_2, v_2) is a path of length 3 in G. Let $e_3 = (v_1, u_2)$. Then the subgraph induced by the vertices e_1', e_3', v_2 and e_1' is a triangle in G^{+-}. Thus, the girth of G^{+-} is 3. □

In the following theorem, we find the graph G for which G^{+-} is geodetic.

Theorem 2.7 *Let G be a (p, q) graph such that G^{+-} is connected. Then G^{+-} is geodetic if and only if G is one of the following graphs: P_4, $K_3 \cup K_1$, $nK_2 \cup K_1$, $n \geq 2$.*

Proof Since G^{+-} is connected, it follows from Theorem 2.2 that, G cannot be any of the following graphs $2K_2$ or $K_{1,n} \cup lK_1$, $n, l \geq 0$. Assume that G^{+-} is geodetic. We consider the following cases.

Case 1. $p \leq 3$. In this case, by Theorem 2.2, G^{+-} is disconnected, a contradiction.

Case 2. $p \geq 5$. Let $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$. If e_1 is adjacent to e_2, then $v_1 = u_2$ and (u_1, v_1, v_2) is P_3. Since $p \geq 5$, there exist at least two more vertices, say v_3 and v_4. Both v_3 and v_4 cannot be adjacent to v_1, and both cannot be isolated vertices in G, by Theorem 2.2. Now, we consider the following subcases.

Subcase 2.1 $e_3 = (v_2, v_3)$. If v_4 is isolated or $e_4 = (v_2, v_4)$ or $e_4 = (v_3, v_4)$, then $\langle e_2', u_1, e_3', v_4 \rangle$ is $K_4 - e$, where e is an edge of K_4, so that G^{+-} is not geodetic.

Subcase 2.2 $e_3 = (v_3, v_4)$. Then $\langle e_1', v_3, e_2', v_4 \rangle$ is $K_4 - e$, and G^{+-} is not geodetic.

In all the above cases, we have a contradiction to the fact that G^{+-} is geodetic. Hence, no two edges are adjacent in G. Thus, for $p \geq 5$, $G \cong nK_2 \cup K_1$, $n \geq 2$.

Case 3: $p = 4$. Then G is connected or $G \cong K_3 \cup K_1$, since otherwise G^{+-} is disconnected by Theorem 2.2. Also, $G \ncong K_{1,3}$, again by Theorem 2.2. So, G is one of the following graphs: P_4, $K_3 \cup K_1$, C_4, $K_3 \bullet K_2$, $K_4 - e$ or K_4.

If $G \cong C_4$, $K_3 \bullet K_2$, $K_4 - e$ or K_4, then G^{+-} contains C_4 or $K_4 - e$ as induced subgraph and hence not geodetic. So, $G \cong P_4$ or $G \cong K_3 \cup K_1$, for which G^{+-} is geodetic.

The converse is obvious. □

Theorem 2.8 *G^{+-} contains $K_{1,3}$ as an induced subgraph if and only if G contains $K_2 \cup 3K_1$ or $3K_2 \cup K_1$ as an induced subgraph.*

Proof Assume G^{+-} contains $K_{1,3}$ as an induced subgraph.

(1) If the center vertex of $K_{1,3}$ in G^{+-} is a line vertex then G contains $K_2 \cup 3K_1$ as an induced subgraph.

(2) If the center vertex of $K_{1,3}$ in G^{+-} is a point vertex then G contains $3K_2 \cup K_1$ as an induced subgraph.

Hence G contains $K_2 \cup 3K_1$ or $3K_2 \cup K_1$ as an induced subgraph.

The converse is obvious. □

Corollary 2.9 *If G contains $K_2 \cup 3K_1$ or $3K_2 \cup K_1$ as an induced subgraph then G^{+-} cannot be the line graph of any graph.*

Proof Suppose G contains $K_2 \cup 3K_1$ or $3K_2 \cup K_1$ as an induced subgraph, then by above Theorem, G^{+-} contains $K_{1,3}$ as an induced subgraph. Since $K_{1,3}$ is forbidden induced subgraph for line graphs, the result follows. □

In the following, we find the domination number of G^{+-}.

Theorem 2.10 *For any (p, q) graph G, which is not totally disconnected,*
$\gamma(G^{+-}) \leq 3$.

Proof Clearly G has at least one edge, let e' be a line vertex of G^{+-} corresponding to an edge $e = (u, v)$ of G. Since each line vertex of G^{+-} is adjacent to $p - 2$ point vertices, e' is adjacent to all point vertices except u and v. By the definition of G^{+-}, any line vertex in G^{+-} other than e' is adjacent to u or v or both and hence $\{u, v, e'\}$ forms a dominating set of G^{+-}. But for $G \cong 2K_2$, we have $\gamma(G^{+-}) = 2$. Thus, $\gamma(G^{+-}) \leq 3$. □

Now, we establish a criterion for G^{+-} to be Eulerian.

Theorem 2.11 *Let G be a (p, q) graph such that G^{+-} is connected, then G^{+-} is Eulerian if and only if one of the following holds:*

(1) p is even, q is odd and $deg_G(u)$ is odd for all $u \in V(G)$;

(2) p is even, q is even and $deg_G(u)$ is even for all $u \in V(G)$.

Proof Suppose that G^{+-} is Eulerian. Then the degree of each vertex in G^{+-} is even. By Proposition 3, $deg_{G^{+-}}(u) = q - deg_G(u)$, for every point vertex u of G^{+-}. So, q and $deg_G(u)$ are both even or both odd. Also by Proposition 4, $deg_{G^{+-}}(e') = deg_G(u) + deg_G(v) + p - 4$ for every line vertex e' corresponding to $e = (u, v)$ of G. Since $u, v \in V(G^{+-})$, $deg_G(u)$ and $deg_G(v)$ are both even or both odd. So, $deg_G(u) + deg_G(v) + p - 4$ is even, and hence p is even.

Conversely, suppose that (1) holds. Since $deg_{G^{+-}}(u) = q - deg_G(u)$, it follows that for all $u \in V(G)$, $deg_{G^{+-}}(u)$ is even. Also, for every line vertex e' in G^{+-} corresponding to an edge $e = (x, y)$ of G, we have $deg_{G^{+-}}(e') = deg_G(x) + deg_G(y) + p - 4$. Since $deg_G(x)$ and $deg_G(y)$

are odd, and p is even, it follows that $deg_{G^{+-}}(e')$ is even. So, every vertex of G^{+-} has even degree. Thus, G^{+-} is Eulerian. The proof is similar if (2) holds. □

Theorem 2.12 *For any connected graph G with at least three vertices, such that G^{+-} is connected, diam(G^{+-}) is at most* 4.

Proof Let G be a connected graph with at least three vertices such that G^{+-} is connected. We consider the following cases.

Case 1. Let e_1' and e_2' be line vertices of G^{+-}. If e_1 and e_2 are adjacent in G, then $d_{G^{+-}}(e_1', e_2') = 1$. If e_1 and e_2 are not adjacent in G, then there exists an edge e in G adjacent to both e_1 and e_2 in G or there exists a vertex in G not incident with both e_1 and e_2, since otherwise G^{+-} would be disconnected. In both cases, $d_{G^{+-}}(e_1', e_2') = 2$, so that, the distance between any two line vertices in G^{+-} is at most 2.

Case 2. Let u and v be point vertices of G^{+-}. We consider the following subcases.

Subcase 2.1 u and v are not adjacent in G and e is an edge in G not incident with both u and v. Then, (u, e', v) is geodetic path in G^{+-}, and hence $d_{G^{+-}}(u, v) = 2$.

Subcase 2.2 u and v are not adjacent in G and e is an edge in G incident with u but not incident with v. Since G is connected, u and v are connected by path. Let $(u, e, v_1, e_1, v_2, e_2, v_3 \ldots, e_k, v)$ be the $u - v$ geodetic path in G. If $k = 1$, then (u, e_1', e', v) is geodetic path in G^{+-} and $d_{G^{+-}}(u, v) = 3$. If $k \geq 2$, then (u, e_k', e_{k-1}', v) is geodetic path in G^{+-} and $d_{G^{+-}}(u, v) = 3$.

Subcase 2.3 u and v are adjacent in G. Let $e = (u, v)$. Since $p \geq 3$ and G is connected, there exists a vertex w in G such that (u, v, w) is a path in G. Since G^{+-} is also connected, we consider the following subcases.

Subcase 2.3.1 There exists a vertex x in G such that x is adjacent to w, let $e_1 = (w, x)$. Then (u, e_1', v) is a path in G^{+-} and $d_{G^{+-}}(u, v) = 2$.

Subcase 2.3.2 There exists a vertex x in G such that x is adjacent to u, let $e_2 = (x, u)$. Then (u, e_1', e', e_2', v) is a geodetic path in G^{+-} and $d_{G^{+-}}(u, v) = 4$.

Case 3. Let u and e' be point vertex and line vertex, respectively of G^{+-}. If u is not incident with e in G, then $d_{G^{+-}}(u, e') = 1$. If u is incident with e in G, then let $e = (u, v)$. Since $p \geq 3$, and G is connected, it follows that there exists an edge $e_1 = (v, w)$, say. Then $d_{G^{+-}}(u, e') = 2$.

Hence, from all the above cases, $diam(G^{+-}) \leq 4$. □

In the following theorem, we obtain graphs whose middle graph and G^{+-} are isomorphic.

Theorem 2.13 *For any $G(p, q)$ graph, G^{+-} and middle graph $M(G)$ are isomorphic if and only if G is one of the following graphs $\overline{K}_p$, $2K_2$, P_4, C_4, $P_3 \cup K_1$, $K_4 - e$ or K_4.*

Proof Assume that $G^{+-} \cong M(G)$. If $q = 0$, then $G \cong G^{+-} \cong M(G) \cong \overline{K}_p$. If $q \neq 0$, then $|E(G^{+-})| = |E(M(G))|$. That is $|E(L(G))| + q(p-2) = |E(L(G))| + 2q$, so that, $q(p-2) = 2q$. Then $p = 4$ and G is one of the graphs $K_2 \cup 2K_1$, $2K_2$, P_4, $K_3 \cup K_1$, C_4, $P_3 \cup K_1$, $K_{1,3}$, $K_3 \bullet K_2$,

$K_4 - e$ or K_4. Among these graphs, if G is $K_2 \cup 2K_1$ or $K_{1,3}$ or $K_3 \cup K_1$ or $K_3 \bullet K_2$, G^{+-} and $M(G)$ are not isomorphic. Hence, G is one of the graphs: $2K_2$, P_4, C_4, $P_3 \cup K_1$, $K_4 - e$ or K_4.

Conversely, if G is one of the following graphs $\overline{K}_p$, $2K_2$, P_4, C_4, $P_3 \cup K_1$, $K_4 - e$ or K_4, then clearly G^{+-} and middle graph $M(G)$ are isomorphic. □

In the following theorem, we obtain graphs whose total graph and G^{+-} are isomorphic.

Theorem 2.14 *For any graph $G(p,q)$, G^{+-} and total graph $T(G)$ are isomorphic if and only if $G \cong \overline{K_p}$.*

Proof Assume that $G^{+-} \cong T(G)$. If $q = 0$, then $G \cong G^{+-} \cong T(G) \cong \overline{K}_p$. If $q \neq 0$, then $|E(G^{+-})| = |E(T(G))|$. That is, $|E(L(G)| + q(p-2) = |E(G)| + |E(L(G))| + 2q$, so that $q(p-2) = |E(G)| + 2q$. Hence, $p = 5$. If G has at least one edge then G^{+-} and $T(G)$ are non isomorphic, a contradiction.

Conversely, clearly if $G \cong \overline{K_p}$, then G^{+-} and total graph $T(G)$ are isomorphic. □

In the following theorem, we obtain graphs whose quasi-total graph and G^{+-} are isomorphic.

Theorem 2.15 *Let G be any connected (p,q) graph such that $p \geq 5$. Then G^{+-} and quasi-total graph $P(G)$ are non isomorphic.*

Proof Assume that G^{+-} and $P(G)$ are isomorphic. Then

$$\begin{aligned} |E(L(G))| + q(p-2) &= |E(\overline{G})| + |E(L(G))| + 2q \\ &\Longrightarrow q(p-2) = \frac{p(p-1)}{2} - q + 2q \\ &\Longrightarrow \frac{p(p-1)}{2} = q(p-3) \Longrightarrow p(p-1) = 2q(p-3). \end{aligned} \quad (1)$$

We consider the following cases.

Case 1. $p = q$. Then G is unicyclic. From Equation (1), $p = q = 5$. But $G^{+-} \ncong P(G)$.

Case 2. $q < p$. Since G is connected, $q = p - 1$. Then G is a tree. From Equation (1), $p = 6$ and $q = 5$. It can be verified that $G^{+-} \ncong P(G)$.

Case 3. $q > p$. Then from Equation (1), $p \leq 4$. But $p \geq 5$. So, $G^{+-} \ncong P(G)$.

Hence, from all the above cases, G^{+-} and $P(G)$ are non isomorphic. □

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