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On $(m,(c_1,c_2))$-Regular Bipolar Fuzzy Graphs

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Abstract: In this paper d_m- degree and total d_m-degree of a vertex in bipolar fuzzy graphs are defined. Also (m,(c_1,c_2))-regularity and totally $(m,(c_1,c_2))$-regularity of bipolar fuzzy graphs are defined. A relation between $(m,(c_1,c_2))$-regularity and totally $(m,(c_1,c_2))$-regularity on bipolar fuzzy graph is studied. $(m,(c_1,c_2))$-regularity on some bipolar fuzzy graphs whose underlying crisp graphs are path on $2m$ vertices, a cycle C_n are studied with some specific membership functions.

Key Words: Degree of a vertex in fuzzy graph, regular fuzzy graph, bipolar fuzzy graph, total degree, totally regular fuzzy graph, d_2- degree of a vertex in bipolar fuzzy graph, total d_2- degree, $(2,(c_1,c_2))$-regular fuzzy graphs, totally $(2,(c_1,c_2))$-regular.

AMS(2010): 05C12, 03E72, 05C72.

§1. Introduction

In 1965, Lofti A.Zadeh [16] introduced the concept of fuzzy subset of a set as method of representing the phenomena of uncertainty in real life situation. Azriel Rosenfeld introduced fuzzy graphs in 1975 [12]. It has been growing fast and has numerous application in various fields. Nagoor Gani and Radha [10]introduced regular fuzzy graphs, total degree, totally regular fuzzy graphs. Alison Northup introduced semiregular graphs that we call it as $(2,k)$- regular graphs and discussed some properties of (2,k)-regular graphs. In 1994 W.R.Zhang [15] initiated the concept of bipolar fuzzy sets as generalization of fuzzy sets. Bipolar fuzzy sets are extension of fuzzy sets whose membership value in $[-1,1]$. N.R.Santhi Maheswari and C.Sekar introduced d_2- degree of vertex in graphs and discussed some properties of d_2- degree of a vertex in graphs [13]. S.Ravi Narayanan and N.R.Santhi Maheswari introduced d_2-degree of a vertex in bipolar fuzzy graphs, total d_2-degree of a vertex in bipolar fuzzy graph and discussed some properties of d_2-degree of the vertex in bipolar fuzzy graph [11].

This paper motivates us to introduce d_m- degree in bipolar fuzzy graph. Throughout this paper, the vertices take the membership values (m_1^+,m_1^-) and edges take the membership values (m_2^+,m_2^-) where m_1^+,$m_2^+ \in [0,1]$ and m_1^-,$m_2^- \in [-1,0]$.

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§2. Preliminaries

We present some known definitions related to fuzzy graphs and bipolar fuzzy graphs for ready reference to go through the work presented in this paper.

Definition 2.1([9]) *A fuzzy graph $G : (\sigma, \mu)$ is a pair of functions (σ, μ), where $\sigma : V \rightarrow [0,1]$ is a fuzzy subset of a non empty set V and $\mu : VXV \rightarrow [0, 1]$ is a symmetric fuzzy relation on σ such that for all u, v in V, the relation $\mu(u, v) \le \sigma(u) \wedge \sigma(v)$ is satisfied. A fuzzy graph G is called complete fuzzy graph if the relation $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ is satisfied.*

Definition 2.2([2]) *A bipolar fuzzy graph with an underlying set V is defined to be the pair $G = (A, B)$ where $A = (m_1^+, m_1^-)$ is a bipolar fuzzy set on V and $B = (m_2^+, m_2^-)$ is a bipolar fuzzy set on E such that $m_2^+(x, y) \le \min\{m_1^+(x), m_1^+(y)\}$ and $m_2^-(x, y) \ge \max\{m_1^-(x), m_1^-(y)\}$ for all $(x, y) \in E$. Here A is called a bipolar fuzzy vertex set of V and B is called a bipolar fuzzy edge set of E.*

Definition 2.3([9]) *The strength of connectedness between two vertices u and v is defined as $\mu^{\infty}(u, v) = sup\{\mu^k(u, v) : k = 1, 2, \cdots\}$ where $\mu^k(u, v) = sup\ \{\mu(u, u_1) \wedge \mu(u_1, u_2) \wedge \cdots \wedge \mu(u_{k-1}, v)$: $u, u_1, u_2, \cdots, u_{k-1}, v$ is path connecting u and v of length $k\}$.*

Definition 2.4([14]) *The positive degree of a vertex $u \in G$ is $d^+(u) = \sum m_2^+(u, v)$. The negative degree of a vertex $u \in G$ is $d^-(u) = \sum m_2^-(u, v)$. The degree of the vertex u is defined as $d(u) = (d^+(u), d^-(u))$.*

Definition 2.5([14]) *Let $G = (A, B)$ be a bipolar fuzzy graph where $A = (m_1^+, m_1^-)$ and $B = (m_2^+, m_2^-)$ be two bipolar fuzzy sets on a non-empty finite set V. Then G is said to be regular bipolar fuzzy graph if all the vertices of G has same degree (c_1, c_2).*

Definition 2.6([14]) *Let $G = (A, B)$ be a bipolar fuzzy graph. The total degree of a vertex $u \in V$ is denoted by $td(u)$ and defined as $td(u) = (td^+(u), td^-(u))$ where $td^+(u) = \sum m_2^+(u, v) + m_1^+(u)$ and $td^-(u) = \sum m_2^-(u, v) + m_1^-(u)$.*

Definition 2.7([11]) *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^*(V, E)$. The positive d_2 - degree of a vertex $u \in G$ is defined as $d_2^+(u) = \sum m_2^{(2,+)}(u, v)$, where $m_2^{(2,+)}(u, v) = \sup\{m_2^+(u, u_1) \wedge m_2^+(u_1, v) : u, u_1, v$ is the shortest path connecting u and v of length 2. The negative d_2- degree of a vertex $u \in G$ is defined as $d_2^-(u) = \sum m_2^{(2,-)}(u, v)$ where $m_2^{(2,-)}(u, v) = \inf\{m_2^-(u, u_1) \vee m_2^-(u_1, v) : u, u_1, v$ is the shortest path connecting u and v of length 2. The d_2 - degree of a vertex u is defined as $d_2(u) = (d_2^+(u), d_2^-(u))$. The minimum d_2 - degree of G is $\delta_2(G) = \wedge\{d_2(v) : v \in V\}$. The maximum d_2 - degree of G is $\Delta_2(G) = \vee\{d_2(v) : v \in V\}$*

Definition 2.8([11]) *Let $G : (\sigma, \mu)$ be a bipolar fuzzy graph on $G^* : (V, E)$. If $d_2(v) = (c_1, c_2)$, for all $v \in V$, then G is said to be $(2, (c_1, c_2))$-regular bipolar fuzzy graph.*

Definition 2.9([11]) *Let $G = (A, B)$ be a bipolar fuzzy graph. Then the total d_2- degree of a vertex $u \in V$ is defined as $td_2(u) = (td_2^+(u), td_2^-(u))$ where $td_2^+(u) = d_2^+(u) + m_1^+(u)$ and $td_2^-(u) = d_2^-(u) + m_1^-(u)$. Also it can be defined as $td_2(u) = d_2(u) + A(u)$ where $A(u) =$*

$(m_1^+(u), m_1^-(u))$.

Definition 2.10([11]) *Let $G=(A,B)$ be a bipolar fuzzy graph. If all the vertices of G have the same total d_2-degree (c_1,c_2) then G is said to be totally $(2,(c_1,c_2))$- regular bipolar fuzzy graph.*

§3. d_m- Degree of Vertex in Bipolar Fuzzy Graphs

Definition 3.1 *Let $G=(A,B)$ be a bipolar fuzzy graph on $G^*(V,E)$. The positive d_m - degree of a vertex $u \in G$ is defined as $d_m^+(u) = \sum m_2^{(m,+)}(u,v)$, where $m_2^{(m,+)}(u,v) = sup\{m_2^+(u,u_1) \wedge m_2^+(u_1,u_2) \wedge \cdots \wedge m_2^+(u_{m-1},v) : u,u_1,u_2,\cdots,u_{m-1},v$ is the shortest path connecting u and v of length $m)\}$. The negative d_m- degree of a vertex $u \in G$ is defined as $d_2^-(u) = \sum m_2^{(m,-)}(u,v)$ where $m_2^{(m,-)}(u,v) = inf\,\{m_2^-(u,u_1) \vee m_2^-(u_1,u_2) \vee \cdots \vee m_2^-(u_{m-1},v) : u,u_1,u_2,\ldots,u_{m-1},v$ is the shortest path connecting u and v of length $m)\}$.The d_m - degree of a vertex u is defined as $d_m(u) = (d_m^+(u), d_m^-(u))$.*

The minimum d_m - degree of G is $\delta_m(G) = \wedge\{d_m(v) : v \in V\}$ and the maximum d_m - degree of G is $\Delta_m(G) = \vee\{d_m(v) : v \in V\}$.

Example 3.2 Consider a bipolar fuzzy graph on $G^*(V,E)$

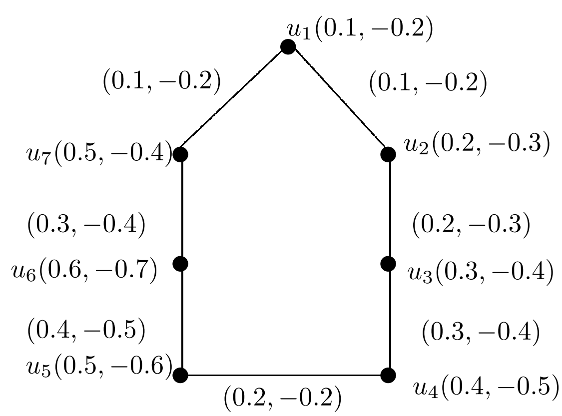

Figure 1

$$d_3^+(u_1) = (0.1 \wedge 0.2 \wedge 0.3) + (0.1 \wedge 0.3 \wedge 0.4) = 0.1 + 0.1 = 0.2$$
$$d_3^-(u_1) = (-0.2 \vee -0.3 \vee -0.2) + (-0.2 \vee -0.4 \vee -0.5)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_1) = (0.2, -0.4)$$

$$d_3^+(u_2) = (0.2 \wedge 0.2 \wedge 0.3) + (0.1 \wedge 0.1 \wedge 0.3) = 0.2 + 0.1 = 0.3$$
$$d_3^-(u_2) = (-0.3 \vee -0.2 \vee -0.4) + (-0.2 \vee -0.2 \vee -0.4)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_2) = (0.3, -0.4)$$

$$d_3^+(u_3) = (0.2 \wedge 0.1 \wedge 0.1) + (0.2 \wedge 0.3 \wedge 0.4) = 0.1 + 0.2 = 0.3$$

$$d_3^-(u_3) = (-0.3 \vee -0.2 \vee -0.2) + (-0.2 \vee -0.5 \vee -0.4)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_3) = (0.3, -0.4)$$

$$d_3^+(u_4) = (0.2 \wedge 0.2 \wedge 0.1) + (0.3 \wedge 0.4 \wedge 0.3) = 0.1 + 0.3 = 0.4$$
$$d_3^-(u_4) = (-0.2 \vee -0.3 \vee -0.2) + (-0.4 \vee -0.5 \vee -0.4)$$
$$= (-0.2) + (-0.4) = -0.6$$
$$d_3(u_4) = (0.4, -0.6)$$

$$d_3^+(u_5) = (0.3 \wedge 0.2 \wedge 0.2) + (0.4 \wedge 0.3 \wedge 0.1) = 0.2 + 0.1 = 0.3$$
$$d_3^-(u_5) = (-0.4 \vee -0.2 \vee -0.3) + (-0.5 \vee -0.4 \vee -0.2)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_5) = (0.3, -0.4)$$

$$d_3^+(u_6) = (0.4 \wedge 0.3 \wedge 0.2) + (0.3 \wedge 0.1 \wedge 0.1) = 0.2 + 0.1 = 0.3$$
$$d_3^-(u_6) = (-0.5 \vee -0.4 \vee -0.2) + (-0.4 \vee -0.2 \vee -0.2)$$
$$= (-0.2) + (-0.2) = -0.4$$
$$d_3(u_6) = (0.3, -0.4)$$

$$d_3^+(u_7) = (0.3 \wedge 0.4 \wedge 0.3) + (0.1 \wedge 0.1 \wedge 0.2) = 0.3 + 0.1 = 0.4$$
$$d_3^-(u_7) = (-0.4 \vee -0.5 \vee -0.4) + (-0.2 \vee -0.2 \vee -0.3)$$
$$= (-0.4) + (-0.2) = -0.6$$
$$d_3(u_7) = (0.4, -0.6)$$

§4. $(m, (c_1, c_2))$-Regular Bipolar Fuzzy Graphs

Definition 4.1 *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^*(V, E)$. If $d_m(v) = (c_1, c_2)$, for all $v \in V$ then G is said to be $(m, (c_1, c_2))$-regular bipolar fuzzy graph.*

Example 4.2 Consider a bipolar fuzzy graph on $G^*(V, E)$

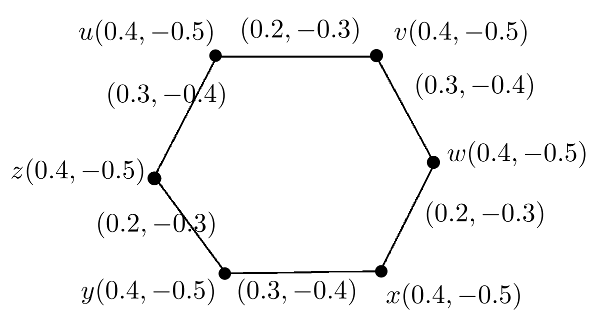

Figure 2

$$d_3^+(u) = sup\{(0.2 \wedge 0.3 \wedge 0.2), (0.3 \wedge 0.2 \wedge 0.3)\}$$

$= sup\{0.2, 0.2\} = 0.2$
$d_3^-(u) = inf\{(-0.3 \vee -0.4 \vee -0.3), (-0.4 \vee -0.3 \vee -0.4)\}$
$= inf\{-0.3, -0.3\} = -0.3$
$d_3(u) = (0.2, -0.3)$

$d_3^+(v) = sup\{(0.2 \wedge 0.3 \wedge 0.2), (0.3 \wedge 0.2 \wedge 0.3)\}$
$= sup\{0.2, 0.2\} = 0.2$
$d_3^-(v) = inf\{(-0.3 \vee -0.4 \vee -0.3), (-0.4 \vee -0.3 \vee -0.4)\}$
$= inf\{-0.3, -0.3\} = -0.3$
$d_3(v) = (0.2, -0.3)$

$d_3^+(w) = sup\{(0.3 \wedge 0.2 \wedge 0.3), (0.2 \wedge 0.3 \wedge 0.2)\}$
$= sup\{0.2, 0.2\} = 0.2$
$d_3^-(w) = inf\{(-0.4 \vee -0.3 \vee -0.4), (-0.3 \vee -0.4 \vee -0.3)\}$
$= inf\{-0.3, -0.3\} = -0.3$
$d_3(w) = (0.2, -0.3)$

$d_3^+(x) = sup\{(0.2 \wedge 0.3 \wedge 0.2), (0.3 \wedge 0.2 \wedge 0.3)\}$
$= sup\{0.2, 0.2\} = 0.2$
$d_3^-(x) = inf\{(-0.3 \vee -0.4 \vee -0.3), (-0.4 \vee -0.3 \vee -0.4)\}$
$= inf\{-0.3, -0.3\} = -0.3$
$d_3(x) = (0.2, -0.3)$

$d_3^+(y) = sup\{(0.3 \wedge 0.2 \wedge 0.3), (0.2 \wedge 0.3 \wedge 0.2)\}$
$= sup\{0.2, 0.2\} = 0.2$
$d_3^-(y) = inf\{(-0.4 \vee -0.3 \vee -0.4), (-0.3 \vee -0.4 \vee -0.3)\}$
$= inf\{-0.3, -0.3\} = -0.3$
$d_3(y) = (0.2, -0.3)$

$d_3^+(z) = sup\{(0.3 \wedge 0.2 \wedge 0.3), (0.2 \wedge 0.3 \wedge 0.2)\}$
$= sup\{0.2, 0.2\} = 0.2$
$d_3^-(z) = inf\{(-0.4 \vee -0.3 \vee -0.4), (-0.3 \vee -0.4 \vee -0.3)\}$
$= inf\{-0.3, -0.3\} = -0.3$
$d_3(z) = (0.2, -0.3)$
Here, G is $(3,(0.2,-0.3))$-regular bipolar fuzzy graph.

§5. Totally $(m,(c_1,c_2))$-Regular Bipolar Fuzzy Graphs

Definition 5.1 *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^*(V, E)$. The total d_m-degree of a vertex $u \in V$ is defined as $td_m(u) = d_m(u) + A(u)$.*

The minimum td_m-degree is $t\delta_m(G) = \wedge\{td_m(v)\colon v \in G\}$ and the maximum td_m-degree is $t\Delta_m(G) = \vee\{td_m(v)\colon v \in G\}$

Definition 5.2 *Let $G = (A, B)$ be a bipolar fuzzy graph on $G^*(V, E)$. If each vertex of G has same total d_m - degree, then G is said to be totally $(m,(c_1,c_2))$ - regular bipolar fuzzy graph.*

Example 5.3 In Figure.2,

$td_3(u) = d_3(u) + A(u) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(v) = d_3(v) + A(v) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(w) = d_3(w) + A(w) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(x) = d_3(x) + A(x) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(y) = d_3(y) + A(y) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$
$td_3(z) = d_3(z) + A(z) = (0.2, -0.3) + (0.4, -0.5) = (0.6, -0.8)$

Since all the vertices have the same total td_3- degree$(0.6, -0.8)$ This graph is totally $(3, (0.6, -0.8))$-regular bipolar fuzzy graph.

Remark 5.4 *A $(m, (c_1, c_2))$-regular bipolar fuzzy graph need not be totally $(m, (c_1, c_2))$-regular bipolar fuzzy graph*

Example 5.5 Consider a bipolar fuzzy graph on $G^*(V, E)$.

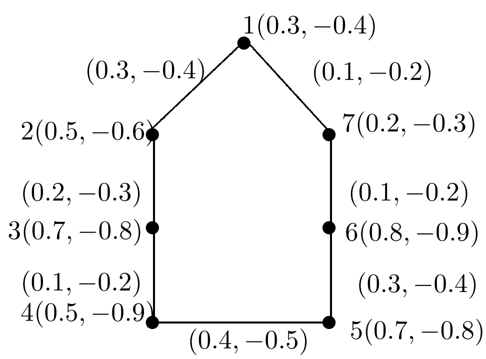

Figure 3

Here, $d_3(v) = (0.2, -0.4)$ for all $v \in V$.Hence G is a $(3, (0.2, -0.4))$-regular bipolar fuzzy graph. But $td_3(1) \neq td_3(7)$.Hence G is not totally $(3, (c_1, c_2))$-regular bipolar fuzzy graph.

Remark 5.6 *A totally $(m, (c_1, c_2))$-regular bipolar fuzzy graph need not be $(m, (c_1, c_2))$-regular bipolar fuzzy graph*

Example 5.7 Consider a bipolar fuzzy graph on $G^*(V, E)$.

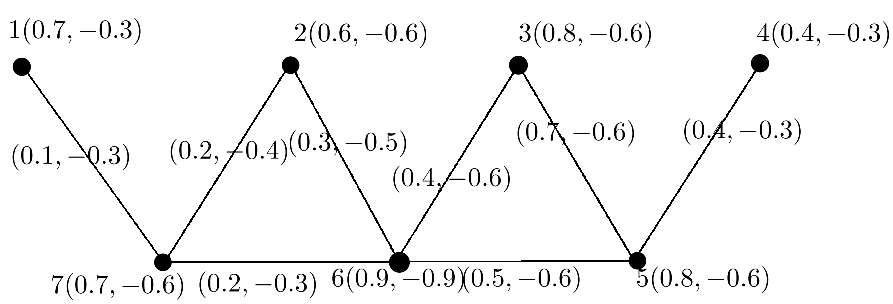

Figure 4

Here, $td_3(v) = (0.9, -0.9)$, for all $v \in V$. Hence G is a totally $(3,(0.9,-0.9))$-regular bipolar fuzzy graph. But $d_3(1) \neq d_3(2)$. Hence G is not $(3,(c_1,c_2))$-regular bipolar fuzzy graph.

Remark 5.8 *A $(m,(c_1,c_2))$-regular bipolar fuzzy graph which is a totally $(m,(c_1,c_2))$-regular bipolar fuzzy graph.*

Example 5.9 Consider a bipolar fuzzy graph on $G^*(V,E)$.

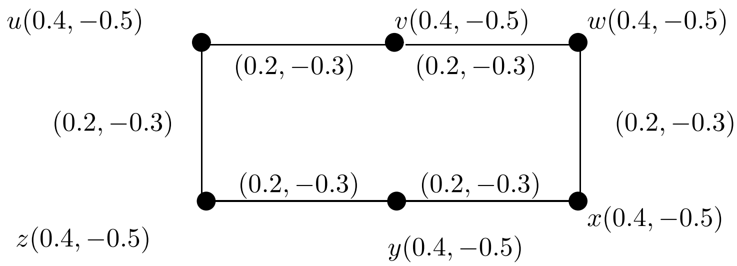

Figure 5

Here, $d_3(v) = (0.2, -0.3)$ and $td_3(v) = (0.6, -0.8)$, for all $v \in V$. Hence G is $(3,(0.2,-0.3))$-regular bipolar fuzzy graph and totally $(3,(0.6,-0.8))$-regular bipolar fuzzy graph.

Theorem 5.10 *Let $G = (A,B)$ be a bipolar fuzzy graph on $G^*(V,E)$. Then $A(u) = (k_1,k_2)$ for all $u \in V$ if and only if the following conditions are equivalent.*

(1) *$G = (A,B)$ is $(m,(c_1,c_2))$ - regular bipolar fuzzy graph;*

(2) *$G = (A,B)$ is totally $(m,(c_1+k_1,c_2+k_2))$ - regular bipolar fuzzy graph.*

Proof Suppose $A(u) = (k_1,k_2)$ for all $u \in V$. Assume that G is a $(m,(c_1,c_2))$ - regular bipolar fuzzy graph. Then $d_m(u) = (c_1,c_2)$, for all $u \in V$. So $td_m(u) = d_m(u) + A(u) = (c_1,c_2)+(k_1,k_2) = (c_1+k_1,c_2+k_2)$. Hence G is a totally $(m,(c_1+k_1,c_2+k_2))$ - regular bipolar fuzzy graph.. Thus (i) $\Rightarrow$ (ii) is proved.

Now suppose G is totally $(m,(c_1+k_1,c_2+k_2))$ - regular bipolar fuzzy graph.

$\Rightarrow td_m(u) = (c_1+k_1,c_2+k_2)$, for all $u \in V$

$\Rightarrow d_m(u) + A(u) = (c_1+k_1,c_2+k_2)$, for all $u \in V$

$\Rightarrow d_m(u) + (k_1,k_2) = (c_1,c_2) + (k_1,k_2)$, for all $u \in V$

$\Rightarrow d_m(u) = (c_1,c_2)$, for all $u \in V$.

Hence G is $(m,(c_1,c_2))$ - regular bipolar fuzzy graph. Thus (i) and (ii) are equivalent.

Conversely assume (i) and (ii) are equivalent. Let G be a $(m,(c_1,c_2))$ - regular bipolar fuzzy graph and totally $(m,(c_1+k_1,c_2+k_2))$ - regular bipolar fuzzy graph.

$\Rightarrow td_m(u) = (c_1+k_1,c_2+k_2)$ and $d_m(u) = (c_1,c_2)$, for all $u \in V$

$\Rightarrow d_m(u) + A(u) = (c_1+k_1,c_2+k_2)$ and $d_m(u) = (c_1,c_2)$, for all $u \in V$

$\Rightarrow d_m(u) + A(u) = (c_1,c_2) + (k_1,k_2)$ and $d_m(u) = (c_1,c_2)$, for all $u \in V$

$\Rightarrow A(u) = (k_1,k_2)$, for all $u \in V$.

Hence $A(u) = (k_1,k_2)$. □

§6. $(m,(c_1,c_2))$- Regularity on Path of $2m$ Vertices with Specific Membership Function

Theorem 6.1 *Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is path on $2m$ vertices. If B is constant function then G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph.*

Proof Suppose that B is constant function, say $B(uv)=(c_1,c_2)$, for all $uv \in E$. Then $d_m(u)=(c_1,c_2)$. Hence G is $(m,(c_1,c_2))$ - regular bipolar fuzzy graph. □

Remark 6.2 *The converse of Theorem* 6.1 *need not be true.*

Example 6.3 For example consider $G=(A,B)$ be bipolar fuzzy graph such that $G^*(V,E)$ is path on 6 vertices.

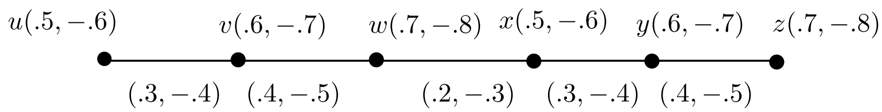

Figure 6

Note that $d_3(u)=(0.2,-0.3)$, for all $u \in V$. So, G is a $(3,(0.2,-0.3))$ regular bipolar fuzzy graph. But B is not constant function.

Theorem 6.4 *Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is path on $2m$ vertices. If the alternate edges have same membership values then G is a $(m,(c_1,c_2))$- regular bipolar fuzzy graph where $c_1=min\{m_2^{(m,+)}\}$ and $c_2=max\{m_2^{(m,-)}\}$.*

Proof Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is path on $2m$ vertices. Let $e_1,e_2,\ldots,e_{2m-1}$ be the edges of path G^* in that order. If alternate edges have same membership values then

$$m_2^+(e_i)=\begin{cases} k_1 & \text{if } i \text{ is odd} \\ k_2 & \text{if } i \text{ is even} \end{cases} \quad \text{and} \quad m_2^-(e_i)=\begin{cases} r_1 & \text{if } i \text{ is odd} \\ r_2 & \text{if } i \text{ is even} \end{cases}.$$

For $i=1,2,\cdots,m$,

$d_m^+(v_i)=\{B^+(e_i)\wedge B^+(e_{i+1})\wedge\cdots\wedge B^+(e_{m-2+i})\wedge B^+(e_{m-1+i})\}=min\{k_1,k_2\}$,

$d_m^+(v_i)=c_1$ where $c_1=min\{k_1,k_2\}$.

For $i=1,2,\cdots,m$,

$$d_m^+(v_{m+i})=\{B^+(e_i)\wedge B^+(e_{i+1})\wedge\cdots\wedge B^+(e_{m-2+i})\wedge B^+(e_{m-1+i})\}$$
$$=min\{k_1,k_2\},$$

$d_m^+(v_{m+i})=c_1$ where $c_1=min\{k_1,k_2\}$.

For $i=1,2,\cdots,m$,

$d_m^-(v_i)=\{B^-(e_i)\vee B^-(e_{i+1})\vee\cdots\vee B^-(e_{m-2+i})\vee B^-(e_{m-1+i})\}=max\{r_1,r_2\}$,

$d_m^-(v_i) = c_2$ where $c_2 = max\{r_1, r_2\}$.

For $i = 1, 2, \cdots, m$,

$$d_m^-(v_{m+i}) = \{B^-(e_i) \vee B^-(e_{i+1}) \vee \cdots \vee B^-(e_{m-2+i}) \vee B^+(e_{m-1+i})\}$$
$$= max\{r_1, r_2\}.$$

$d_m^-(v_{m+i}) = c_2$, where $c_2 = max\{r_1, r_2\}$.

Since $d_m(v) = (d_m^+(v), d_m^-(v)) = (c_1, c_2)$, for all $v \in V$, we know that G is an $(m,(c_1,c_2))$-regular bipolar fuzzy graph. □

Theorem 6.5 *Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is path on $2m$ vertices. If the middle edge have positive membership value less than positive membership value of remaining edges and negative membership value greater than negative membership value of remaining edges, then G is a $(m,(c_1,c_2))$ - regular bipolar fuzzy graph where c_1 and c_2 are membership values of the middle edge.*

Proof Let $G = (A, B)$ be a bipolar fuzzy graph such that $G^*(V, E)$ is path on $2m$ vertices. Let $e_1, e_2, \ldots, e_{2m-1}$ be the edges of path G^* in that order. Let the positive membership values of the edges $e_1, e_2, \cdots, e_{2m-1}$ be $k_1, k_2, \cdots, k_{m-1}, k_m, k_{m+1}, \cdots, k_{2m-1}$ such that $k_m = c_1 \leq k_1, k_2, \cdots, k_{2m-1}$ and the negative values of the edges $e_1, e_2, \cdots, e_{2m-1}$ be then $r_1, r_2, \cdots, r_{m-1}, r_m, r_{m+1}, \ldots, r_{2m-1}$ such that $r_m = c_2 \geq r_1, r_2, \cdots, r_{2m-1}$.

For $i = 1, 2, \cdots, m$,

$$d_m^+(v_i) = \{B^+(e_i) \wedge B^+(e_{i+1}) \wedge \cdots \wedge B^+(e_{m-2+i}) \wedge B^+(e_{m-1+i})\}$$
$$= min\{k_i, k_{i+1}, k_{m-2+i}, k_{m-1+i}\} = k_m,$$

$d_m^+(v_i) = c_1$, where $c_1 = k_m$.

For $i = 1, 2, \cdots, m$,

$$d_m^+(v_{m+i}) = \{B^+(e_i) \wedge B^+(e_{i+1}) \wedge \cdots \wedge B^+(e_{m-2+i}) \wedge B^+(e_{m-1+i})\}$$
$$= min\{k_i, k_{i+1}, k_{m-2+i}, k_{m-1+i}\} = k_m,$$

$d_m^+(v_{m+i}) = c_1$ where $c_1 = k_m$.

For $i = 1, 2, \cdots, m$,

$$d_m^-(v_i) = \{B^-(e_i) \vee B^-(e_{i+1}) \vee \cdots \vee B^-(e_{m-2+i}) \vee B^-(e_{m-1+i})\}$$
$$= \max\{r_i, r_{i+1}, \cdots, r_{m-2+i}, r_{m-1+i}\} = r_m,$$

$d_m^-(v_i) = c_2$ where $c_2 = r_m$.

For $i = 1, 2, \cdots, m$,

$$d_m^-(v_{m+i}) = \{B^-(e_i) \vee B^-(e_{i+1}) \vee \cdots \vee B^-(e_{m-2+i}) \vee B^+(e_{m-1+i})\}$$
$$= \max\{r_i, r_{i+1}, \ldots, r_{m-2+i}, r_{m-1+i}\} = r_m,$$

$d_m^-(v_{m+i}) = c_2$, where $c_2 = r_m$.

Since $d_m(v) = (d_m^+(v), d_m^-(v)) = (c_1, c_2)$, for all $v \in V$ we know that G is an $(m,(c_1,c_2))$-regular bipolar fuzzy graph. □

Remark 6.6 *If A is constant function, then Theorems 6.1, 6.4 and 6.5 hold good for totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph.*

§7. $(m,(c_1,c_2))$ - Regularity on a Cycle with Some Specific Membership Functions

Theorem 7.1 *Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is the cycle of length $\geq 2m+1$. If m_2^+ and m_2^- are constant functions, then G is a $(m,(c_1,c_2))$ - regular bipolar fuzzy graph. where $(c_1,c_2)=2(m_2^+,m_2^-)$.*

Remark 7.2 *The converse of the Theorem 7.1 need not be true. For example consider $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is an odd cycle of length five.*

Example 7.3 See the graph in Figure 7.

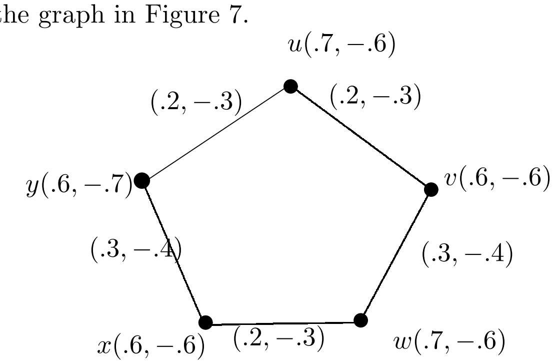

Figure 7

Note that $d_2(u)=(0.4,-0.6)$ for all $u\in V$. So G is a $(2,(0.4,-0.6))$- regular bipolar fuzzy graph. But m_2^+ and m_2^- are not constant functions.

Theorem 7.4 *Let $G=(A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is an even cycle of length $\geq 2m+2$. If the alternate edges have same positive and negative membership values then G is a $(m,(c_1,c_2))$- regular bipolar fuzzy graph.*

Proof If alternate edges have same positive and negative membership values then

$$m_2^+(e_i)=\begin{cases} k_1 & \text{if } i \text{ is odd} \\ k_2 & \text{if } i \text{ is even}\end{cases} \quad \text{and} \quad m_2^-(e_i)=\begin{cases} k_3 & \text{if } i \text{ is odd} \\ k_4 & \text{if } i \text{ iseven}\end{cases}.$$

Here we have 4 possible cases:

(1) $k_1>k_2$ and $k_3>k_4$,

$d_m^+(v)=min\{k_1,k_2\}+min\{k_1,k_2\}=k_2+k_2=2k_2=c_1$,

$d_m^-(v)=max\{k_3,k_4\}+min\{k_3,k_4\}=k_3+k_3=2k_3=c_2$.

(2) $k_1>k_2$ and $k_3<k_4$,

$d_m^+(v)=min\{k_1,k_2\}+min\{k_1,k_2\}=k_2+k_2=2k_2=c_1$,

$d^-_m(v) = max\{k_3,k_4\} + min\{k_3,k_4\} = k_4 + k_4 = 2k_4 = c_2$.

(3) $k_1 < k_2$ and $k_3 > k_4$,

$d^+_m(v) = min\{k_1,k_2\} + min\{k_1,k_2\} = k_1 + k_1 = 2k_1 = c_1$,

$d^-_m(v) = max\{k_3,k_4\} + min\{k_3,k_4\} = k_3 + k_3 = 2k_3 = c_2$.

(4) $k_1 < k_2$ and $k_3 < k_4$,

$d^+_m(v) = min\{k_1,k_2\} + min\{k_1,k_2\} = k_1 + k_1 = 2k_1 = c_1$,

$d^-_m(v) = max\{k_3,k_4\} + min\{k_3,k_4\} = k_4 + k_4 = 2k_4 = c_2$.

Hence G is a $(m,(c_1,c_2))$- regular bipolar fuzzy graph where $d_m(v) = (c_1,c_2)$. □

Remark 7.5 *Let $G = (A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is an even cycle of length $> 2m+2$. Even if the alternate edges have same positive and same negative membership values, then G need not be a $(m,(c_1,c_2))$- regular bipolar fuzzy graph. Since if $A = (m^+_1, m^-_1)$ is not a constant function, G is not totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph.*

Theorem 7.6 *Let $G = (A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is any cycle of length $> 2m+1$. Let $k_2 \geq k_1$ and $k_4 \geq k_3$ and*

$$m^+_2(e_i) = \begin{cases} k_1 & if\ i\ is\ odd \\ k_2 & if\ i\ is\ even \end{cases} \quad and \quad m^-_2(e_i) = \begin{cases} k_3 & if\ i\ is\ odd \\ k_4 & if\ i\ is\ even \end{cases}.$$

Then G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph.

Proof The proof is divided into two cases.

Case 1. Let $G = (A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is an even cycle of length $\leq 2m+2$. Then by Theorem 7.4, G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph.

Case 2. Let $G = (A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is an odd cycle of length $\leq 2m+1$.For any $m > 1$, $d_m = (c_1,c_2)$, for all $v \in V$. Hence G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph. □

Remark 7.7 *Let $G = (A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is any cycle of length $> 2m+1$. Even if $k_2 \geq k_1$ and $k_4 \geq k_3$,*

$$m^+_2(e_i) = \begin{cases} k_1 & if\ i\ is\ odd \\ k_2 & if\ i\ is\ even \end{cases} \quad and \quad m^-_2(e_i) = \begin{cases} k_3 & if\ i\ is\ odd \\ k_4 & if\ i\ is\ even \end{cases}.$$

Then G need not be totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph, since if $A = (m^+_1, m^-_1)$ is not a constant function G is not totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph

Theorem 7.8 *Let $G = (A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is odd cycle of length*

$> 2m+1$. Let

$$m_2^+(e_i) = \begin{cases} k_1 & if\ i\ is\ odd \\ membership\ value\ x \geq k_1 & if\ i\ is\ even \end{cases},$$

$$m_2^-(e_i) = \begin{cases} k_2 & if\ i\ is\ odd \\ membership\ value\ y \geq k_2 & if\ i\ is\ even \end{cases},$$

where x is not constant. Then G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph.

Proof Let

$$m_2^+(e_i) = \begin{cases} k_1 & \text{if } i \text{ is odd} \\ \text{membership value } x \geq k_1 & \text{if } i \text{ is even} \end{cases},$$

and

$$m_2^-(e_i) = \begin{cases} k_2 & \text{if } i \text{ is odd} \\ \text{membership value } y \geq k_2 & \text{if } i \text{ is even} \end{cases},$$

where x is not constant. Then

$$\begin{aligned} d_m(v_i) &= (min\{k_1,x\} + max\{k_2,y\}) + (min\{k_1,x\} + max\{k_2,y\}) \\ &= (k_1,y) + (k_1,y) = (2k_1,2y), \end{aligned}$$

$d_m(v_i) = (c_1,c_2)$, where $c_1 = 2k_1$,$c_2 = 2y$, for all $v \in V$. Hence G is $(m,(c_1,c_2))$- regular bipolar fuzzy graph. □

Remark 7.9 *Let $G = (A,B)$ be a bipolar fuzzy graph such that $G^*(V,E)$ is odd cycle of length $> 2m+1$. Even if*

$$m_2^+(e_i) = \begin{cases} k_1 & if\ i\ is\ odd \\ membership\ value\ x \geq k_1 & if\ i\ is\ even \end{cases}$$

and

$$m_2^-(e_i) = \begin{cases} k_2 & if\ i\ is\ odd \\ membership\ value\ y \geq k_2 & if\ i\ is\ even \end{cases},$$

where x is not a constant function. Then G need not be totally $(m,(c_1,c_2))$- regular bipolar fuzzy graph, since if $A = (m_1^+, m_1^-)$ is not a constant function G is not totally $(m,(c_1,c_2))$-regular bipolar fuzzy graph.

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