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Line Cut Vertex Digraphs of Digraphs

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Abstract: In this paper we define the digraph valued function (digraph operator) namely, the line cut vertex digraph $L_c(D)$ of a digraph D. The problem of reconstructing a digraph from its line cut vertex digraph is presented. Also, outer planarity and maximal outer planarity properties of these digraphs are discussed.

Key Words: Line digraph, cut vertex, complete bipartite subdigraph, line cut vertex digraph, Smarandachely line cut vertex digraph.

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§1. Introduction

Notations and definitions not introduced here can be found in [2,3]. For a simple graph G with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and cut vertex set $C(G) = \{C_1, C_2, \cdots, C_r\}$, detailed by V.R.Kulli et al.[4] gave the following definition. The *line cut vertex graph* of G, written $n(G)$, is the graph whose vertice are the edges $E(G)$ and cut vertices $C(G)$ of G, with two vertices of $n(G)$ adjacent whenever the corresponding edges of G are adjacent or the corresponding members of G are incident, and where the edges and cut vertices of G are called its *members*. Generally, a *Smarandachely line cut vertex digraph* $n_S(G)$ for $S \subset E(G) \bigcup C(G)$ is such a graph with vertices $E(G) \bigcup C(G)$ and members are adjacent if and only if they are adjacent or incident in $\langle S \rangle_G$. Clearly, if $S = E(G) \bigcup C(G)$, then the Smarandachely line cut vertex digraph $n_S(G)$ of G is nothing else but $n(G)$.

Recently, there has been an extension of this topic to trees [5]. In this paper, we extend the definition of the line cut vertex graph of a graph to a directed graph. M.Aigner [1] defines the *line digraph* of a digraph as follows. Let D be a digraph with n vertices $v_1, v_2, \cdots, v_n$ and m arcs and $L(D)$ its associated *line digraph* with $n^{'}$ vertices and $m^{'}$ arcs. We immediately have $n^{'} = m$ and $m^{'} = \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i)$. Furthermore, the in-degree, respectively out-degree of a vertex $v^{'} = (v_i, v_j)$ in $L(D)$ are $d^-(v^{'}) = d^-(v_i)$ and $d^+(v^{'}) = d^+(v_j)$. Also, a digraph D is

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said to be a *line digraph* if it is isomorphic to the line digraph of a certain digraph H [6].

We need some concepts and notations on directed graphs. A *directed graph*(or just *digraph*) D consists of a finite non-empty set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pair of distinct vertices called *arcs*. Here $V(D)$ is the *vertex set* and $A(D)$ is the *arc set* of D. For an arc (u,v) or uv of D, the first vertex u is its *tail*, and the second vertex v is its *head*. A digraph without any arcs is called *totally disconnected*. An out-star (an in-star) in D is a star in the underlying undirected graph of D such that all arcs are directed out of (into) the center. The out-star and in-star of order k is denoted by S_k^+ and S_k^-, respectively. The *out-degree* of a vertex v, written $d^+(v)$, is the number of arcs going out from v and the *in-degree* of a vertex v, written $d^-(v)$, is the number of arcs coming into v. The *total degree* of a vertex v, written $td(v)$, is the number of arcs incident with v. We immediately have $td(v) = d^-(v) + d^+(v)$. A vertex v for which $d^+(v) = d^-(v) = 0$ is called an *isolate*. A vertex v is called a *transmitter* or a *receiver* according as $d^+(v) > 0, d^-(v) = 0$ or $d^+(v) = 0, d^-(v) > 0$.

A *cut set* of D is defined as a minimal set of vertices whose removal increases the number of connected components of D. A cut set of size one is called a *cut vertex*. A *tournament* is a nontrivial complete asymmetric digraph. A tournament of order n is denoted by T_n. A *semi-star digraph*, denoted by $D_{1,n}, n \geq 2$, is a directed graph with $(n+1)$ vertices and n arcs; n vertices have total degree exactly one, and one has n.

Since most of the results and definitions for undirected planar graphs are valid for planar digraphs also, the following definitions hold good for planar digraphs. A *planar drawing* of a digraph D is a drawing of D in which no two distinct arcs intersect. A digraph is said to be *planar* if it admits a planar drawing. If D is a planar digraph, then the *inner vertex number* $i(D)$ of D is the minimum number of vertices not belonging to the boundary of the exterior region in any embedding of D in the plane. A digraph D is *outerplanar* if $i(D) = 0$. An outerplanar digraph is said to be *maximal outerplanar* if no arc can be added to it without losing its outer planarity.

§2. Line Cut Vertex Digraph of a Digraph

For a connected digraph D, the *line cut vertex digraph* $Q = L_c(D)$ has vertex set $V(Q) = A(D) \cup C(D)$, where $C(D)$ is the cut vertex set of D. The arc set

$$A(Q) = \begin{cases} ab : a, b \in A(D), \text{the head of } a \text{ coincides with the tail of } b, \\ Cd : C \in C(D), d \in A(D), \text{the tail of } d \text{ is } C, \\ dC : C \in C(D), d \in A(D), \text{the head of } d \text{ is } C. \end{cases}$$

§3. Decomposition and Reconstruction

A digraph is a *complete bipartite digraph* if its vertex set can be partitioned into two sets A, B in such a way that every arc has its initial vertex in A and its terminal vertex in B and any two vertices $a \in A$ and $b \in B$ are joined by an arc. An arc (u,v) of D is said to be an *end arc* if u is the transmitter and v is the receiver.

Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \cdots, v_n\}$ and cut vertex set $C(D) = \{C_1, C_2, \cdots, C_r\}$. We consider the following three cases.

Case 1. Let v be a vertex of D with $d^-(v) = \alpha$ and $d^+(v) = \beta$. Then α arcs coming into v and the β arcs going out of v give rise to a complete bipartite subdigraph with α tails and β heads and $\alpha \cdot \beta$ arcs joining each tail with each head. This is the decomposition of $L(D)$ into mutually arc disjoint complete bipartite subdigraphs.

Case 2. Let C_j be a cut vertex of D with $d^-(C_j) = \alpha^{'}$. Then $\alpha^{'}$ arcs coming into C_j give rise to a complete bipartite subdigraph with $\alpha^{'}$ tails and a single head(i.e., C_j) and $\alpha^{'}$ arcs joining each tail with C_j.

Case 3. Let C_j be a cut vertex of D with $d^+(C_j) = \beta^{'}$. Then $\beta^{'}$ arcs going out from C_j give rise to a complete bipartite subdigraph with a single tail (i.e., C_j) and $\beta^{'}$ heads and $\beta^{'}$ arcs joining C_j with each head.

Hence by all the above cases, $Q = L_c(D)$ is decomposed into mutually arc-disjoint complete bipartite subdigraphs with $V(Q) = A(D) \cup C(D)$ and arc sets (i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are the sets of in-coming and out-going arcs at v_i of D, respectively; (ii) $\cup_{j=1}^{r} \cup_{k=1}^{r} Z_j^{'} \times C_k$ such that $Z_j^{'} \times C_k = \phi$ for $j \neq k$; (iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} C_k \times Z_j$ such that $C_k \times Z_j = \phi$ for $k \neq j$, where $Z_j^{'}$ and Z_j are the sets of in-coming and out-going arcs at C_j of D, respectively.

Conversely, let H' be a digraph of the type described above. Let us denote each of the complete bipartite subdigraphs obtained by Case 1 by $T_1, T_2, \ldots, T_l$. The vertex set of H is $V(H) = \{t_0, t_1, \ldots, t_l, t_{l+1}\}$. The arcs of H are obtained by the following procedure (see [3]).

For each vertex $v \in L(D)$, we draw an arc a_v to H as follows.

Step 1: If $d^+_{L(D)}(v) > 0$, and $d^-_{L(D)}(v) = 0$, then $a_v = (t_0, t_i)$ is an arc, where i is the base(or index) of T_i such that $v \in X_i$;

Step 2: If $d^+_{L(D)}(v) = 0$, and $d^-_{L(D)}(v) > 0$, then $a_v = (t_j, t_{l+1})$ is an arc, where j is the base of T_j such that $v \in Y_j$;

Step 3: If $d^+_{L(D)}(v) > 0$, and $d^-_{L(D)}(v) > 0$, then $a_v = (t_i, t_j)$ is an arc, where i and j are the indices of T_i and T_j such that $v \in X_j \cap Y_i$. Finally, if $L(D)$ has an isolated vertex, then $a_v = (t_0, t_{l+1})$. Note that this method always constructs H with only one vertex of in-degree zero and one vertex of out-degree zero.

We now mark the cut vertices of H as follows. From Case 2 and Case 3, we observe that for every cut vertex C, there exists at most two complete bipartite subdigraphs, one containing C as the tail, and other as head. Let it be $C_j^{'}$ and $C_j^{''}$, $1 \leq j \leq r$ such that $C_j^{'}$ contains C as the head and $C_j^{''}$ contains C as the tail. Now, if the tails of $C_j^{'}$ and the heads of $C_j^{''}$ are the heads and tails of a single T_i, $1 \leq i \leq l$, then the vertex t_i is a cut vertex in H, where i is the index of T_i. Furthermore, a vertex of an end arc in H whose total degree at least two is a cut vertex. The digraph H thus constructed apparently has H' as line cut vertex digraph. Therefore we have,

Theorem 3.1 *A digraph Q is the line cut vertex digraph of a certain digraph D if and only if $V(Q) = A(D) \cup C(D)$ such that the arc set $A(Q)$ equals: (i) $\cup_{i=1}^{n} X_i \times Y_i$, where X_i and Y_i are* *the sets of in-coming and out-going arcs at v_i of D, respectively, (ii) $\cup_{j=1}^{r} \cup_{k=1}^{r} Z_j^{'} \times C_k$ such that $Z_j^{'} \times C_k = \phi$ for $j \neq k$, (iii) $\cup_{k=1}^{r} \cup_{j=1}^{r} C_k \times Z_j$ such that $C_k \times Z_j = \phi$ for $k \neq j$, where $Z_j^{'}$ and Z_j are the sets of in-coming and out-going arcs at C_j of D, respectively.*

Proposition 3.2 *Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$ and cut vertex set $C(D) = \{C_1, C_2, \ldots, C_r\}$. Then the order and size of $L_c(D)$ are*

$$m + \sum_{j=1}^{r} C_j \quad \text{and} \quad \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + \sum_{j=1}^{r} \{d^-(C_j) + d^+(C_j)\},$$

respectively, where m is the size of D.

Proof Let D be a digraph with $V(D) = \{v_1, v_2, \ldots, v_n\}$ and $C(D) = \{C_1, C_2, \ldots, C_r\}$. Then the order of $L_c(D)$ equals the sum of size and cut vertices of D. Thus, $V(L_c(D)) = m + \sum_{j=1}^{r} C_j$. Now, the size of $L_c(D)$ equals the sum of size of $L(D)$ and the total degree of cut vertices of D. Hence

$$A(L_c(D)) = \sum_{i=1}^{n} d^-(v_i) \cdot d^+(v_i) + \sum_{j=1}^{r} \{d^-(C_j) + d^+(C_j)\}. \qquad \square$$

Proposition 3.3 *Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$. Then $L(D) \simeq L_c(D)$ if and only if D is a block.*

Proof Let D be a digraph with vertex set $V(D) = \{v_1, v_2, \ldots, v_n\}$. Suppose that D is a block. Clearly, D does not have any cut vertex. Thus, the order of $L_c(D)$ is exactly m, where m is the size of D. But, $V(L(D)) = m$. Hence $L(D) \simeq L_c(D)$.

Conversely, suppose that $L(D) \simeq L_c(D)$. Assume that D is not a block. Then there exists at least one cut vertex in D. The order of $L_c(D)$ equals the sum of size and cut vertices of D. But, $V(L(D)) = m$. Thus, the order of $L(D)$ is less than the order of $L_c(D)$. Clearly, $L(D) \neq L_c(D)$, a contradiction. This completes the proof. $\square$

Theorem 3.4 *The line cut vertex digraph $L_c(D)$ of a digraph D is outerplanar if,*

(a) D is a directed path $\vec{P_n}$ on $n \geq 3$ vertices;
(b) D is an in-star(or an out-star) of order $k, k \geq 3$;
(c) D is the semi-star digraph $D_{1,3}$.

Proof We consider the cases following:

Case 1. Suppose that D is a directed path $\vec{P_n}$ on $n \geq 3$ vertices. Then $L_c(D)$ is a connected digraph in which every block is T_3. Clearly, $L_c(D)$ is outerplanar.

Case 2. Suppose that D is an in-star(or an out-star) of order $k, k \geq 3$. Then $L(D)$ is totally disconnected of order $(k-1)$. The number of cut vertex of D is exactly one. Then $L_c(D)$ is an in-star(or an out-star) of order k. Thus, $L_c(D)$ is outerplanar.

Case 3. Suppose that D is the semi-star digraph $D_{1,3}$. Clearly, D contains exactly one cut vertex C. By definition, $L(D)$ is either $D_{1,2}$ or totally disconnected of order three. We consider the following two subcases of Case 3.

Subcase 1 If $L(D)$ is $D_{1,2}$, then $L_c(D)$ is $T_4 - e$, which is outerplanar.

Subcase 2 If $L(D)$ is totally disconnected, then $L_c(D)$ is $D_{1,3}$, which is also outerplanar. This completes the proof. □

Theorem 3.5 *The line cut vertex digraph $L_c(D)$ of a digraph D is maximal outerplanar if and only if D is the directed path $\vec{P_3}$.*

Proof We prove this by the method of contradiction. Suppose $L_c(D)$ is maximal outerplanar. Assume that D is the directed path $\vec{P_4}$. By Proposition 3.2, the order and size of $L_c(D)$ are $n = 5$ and $m = 6$, respectively. But, $m = 6 < 7 = 2n - 3$. Since every maximal outerplanar digraph with n vertices contains $2n-3$ arcs, $L_c(D)$ is not maximal outerplanar, a contradiction.

Conversely, suppose that D is $\vec{P_3}$. Then $L_c(D)$ is T_3, which is maximal outerplanar. This completes the proof. □

§4. Open Problems

We present the following open problems:

(1) Characterize the digraphs whose line cut vertex digraphs are planar, minimally non-outerplanar, and have crossing number one.

(2) One can naturally extend this concept to directed trees. ([See [5]).

References

[1] M.Aigner, On the line graph of a directed graph, *Math.Zeitschr*, 102 (1967) 56-61.

[2] F.Harary, *Graph Theory*, Addison-Wesley, Reading, Mass (1969).

[3] Jorgen Bang-Jensen, Gregory Gutin, *Digraphs Theory, Algorithms and applications*, Springer-Verlag London Limited (2009).

[4] V.R.Kulli, M.H.Muddebihal, On lict and litact graph of a graph, *J.Analy and Computation*, Vol 3 (2006) pp-33-43.

[5] M.H.Muddebihal, R.Chandrasekhar, On pathos lict graph of a tree, *J.Nat.Acad.Sci. letters*, Vol.20 (2003) pp-263-270.

[6] V.V.Malyshko, S.S.Yenchenko, Algorithm for the optimal reconstruction of a digraph, *Cybernetics and Systems Analysis*, Springer (1981).