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Further Results on Super Geometric Mean Graphs

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Abstract: Let G be a graph and $f : V(G) \rightarrow \{1, 2, 3, \cdots, p + q\}$ be an injection. For each edge uv, the induced edge labeling f^* is defined as $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$. Then f is called a super geometric mean labeling if $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1, 2, 3, \cdots, p + q\}$. A graph that admits a super geometric mean labeling is called a super geometric mean graph. In this paper, we have discussed the super geometric meanness of the graphs $P_n \cup C_m, T_n \cup C_m, mC_n$, the complete graph $K_n, [P_n; S_m]$, subdivision of $P_n \odot K_1, TW(P_n)$, middle graph of a path, triangular ladder, $C_n \odot K_1$, duplication of a vertex of the cycle, duplication of an edge of the cycle, triangular grid graph and edge identification of two cycles.

Key Words: Labeling, super geometric mean labeling, super geometric mean graph.

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§1. Introduction

Throughout this paper, by a graph we mean a finite, undirected and simple graph. Let $G(V, E)$ be a graph with p vertices and q edges. For notations and terminology, we follow [5]. For a detailed survey on graph labeling, we refer [4].

A path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n. The union of m copies of a graph G is denoted by mG. A complete graph K_n is a graph on n vertices in which every pair of distinct vertices are joined by an edge. A star graph S_n is a complete bipartite graph $K_{1,n}$. Let $v_1^{(i)}, v_2^{(i)}, v_3^{(i)}, \cdots, v_{m+1}^{(i)}$ and $u_1, u_2, u_3, \ldots, u_n$ be the vertices of the i^{th} copy of the star graph $S_m, 1 \leq i \leq n$ and the path P_n respectively. The graph $[P_n; S_m]$ is obtained from n copies of S_m and the path P_n by joining u_i with the central vertex $v_1^{(i)}$ of the i^{th} copy of S_m by means of an edge, for $1 \leq i \leq n$. For a graph G, the graph $S(G)$ is obtained by subdividing each edge of G by a vertex. A twig $TW(P_n), n \geq 3$ is a graph obtained from a path by attaching exactly two pendant vertices to each internal vertices to each internal vertices of the path.

The middle graph $M(G)$ of a graph G is the graph whose vertex set is $\{v : v \in V(G)\} \cup \{e : e \in E(G)\}$ and the edge set is $\{e_1e_2 : e_1, e_2 \in E(G)$ and e_1 and e_2 are adjacent edges of $G\} \cup \{ve : v \in V(G), e \in E(G)$ and e is incident with $v\}$.

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A ladder L_n is a graph $P_2 \times P_n$ with $V(L_n) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(L_n) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\}$. A triangular ladder $TL_n, n \geq 2$ is a graph obtained by completing the ladder $L_n \cong P_2 \times P_n$ by adding the edges u_iv_{i+1} for $1 \leq i \leq n-1$. $G \odot K_1$ is the graph obtained from G by attaching a new pendant vertex at each vertex of G.

Duplication of a vertex v_k of a graph G produces a new graph G' by adding a vertex v'_k with $N(v_k) = N(v'_k)$. Duplication of an edge $e = uv$ of a graph G by adding an edge $e' = u'v'$ such that $N(u') = N(u) \cup \{v'\} - \{v\}$ and $N(v') = N(v) \cup \{u'\} - \{u\}$.

In [6], S.K. Vaidya et al. discussed the harmonic mean labeling of duplication of a vertex and edge of a cycle. In [7], R. Vasuki et al. discussed the super mean labeling of some standard graphs. A. Durai Baskar et al. [1,2] discussed the geometric mean labeling some standard graphs. Motivated by these works, the concept of super geometric mean labeling was introduced and studied in [3].

A vertex labeling of G is an assignment $f : V(G) \rightarrow \{1, 2, 3, \ldots, p+q\}$ be an injection. For a vertex labeling f, the induced edge labeling f^* is defined as $f^*(uv) = \left\lceil \sqrt{f(u)f(v)} \right\rceil$. Then f is called a super geometric mean labeling if $f(V(G)) \cup \{f^*(uv) : uv \in E(G)\} = \{1, 2, 3, \ldots, p+q\}$. A graph that admits a super geometric mean labeling is called a super geometric mean graph.

The graph shown in Figure 1 is a super geometric mean graph.

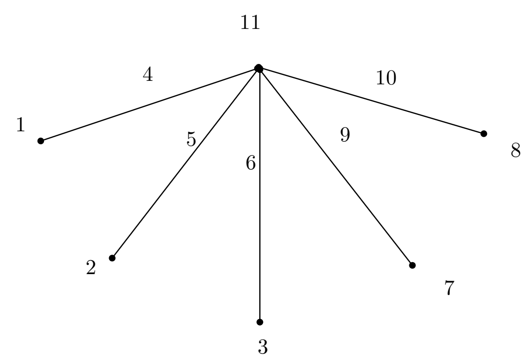

Figure 1

In this paper, we have established the super geometric meanness of the graphs $P_n \cup C_m$ for $n \geq 1$ and $m \geq 3, T_n \cup C_m$ for $n \geq 4$ and $m \geq 3, mC_n$, the complete graph $K_n, [P_n; S_m]$ for $n \geq 1$ and $m \leq 2$, subdivision of $P_n \odot K_1, TW(P_n)$ for $n \geq 3$, middle graph of a path, triangular ladder, $C_n \odot K_1$ for $n \geq 3$, duplication of a vertex of the cycle, duplication of an edge of the cycle, triangular grid graph and edge identification of two cycles.

§2. Main Results

Theorem 2.1 *$P_n \cup C_m$ is a super geometric mean graph, for $n \geq 1$ and $m \geq 3$.*

Proof Let $u_1, u_2, \cdots, u_m$ and $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_m and the path P_n respectively.

Case 1. $m \geq 4$.

We define $f : V(P_n \cup C_m) \cup E(P_n \cup C_m) \to \{1, 2, 3, \ldots, 2m + 2n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 1 & i = 1 \\ 4i - 4 & 2 \le i \le \left\lfloor \frac{m}{2} \right\rfloor \\ 2m - 3 & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is odd} \\ 2m & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \text{ and } m \text{ is even} \\ 2m & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is odd} \\ 2m - 3 & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is even} \\ 4m + 5 - 4i & \left\lfloor \frac{m}{2} \right\rfloor + 3 \le i \le m \end{cases}$$

and $f(v_i) = 2m + 2i - 1$ for $1 \le i \le n$. The induced edge labeling is as follows

$$f^*(u_i u_{i+1}) = \begin{cases} 4i - 2 & 1 \le i \le \left\lfloor \frac{m}{2} \right\rfloor \\ 2m - 1 & i = \left\lfloor \frac{m}{2} \right\rfloor + 1 \\ 2m - 2 & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is odd} \\ 2m - 5 & i = \left\lfloor \frac{m}{2} \right\rfloor + 2 \text{ and } m \text{ is even} \\ 4m + 3 - 4i & \left\lfloor \frac{m}{2} \right\rfloor + 3 \le i \le m - 1, \end{cases}$$

$f^*(u_1 u_m) = 3$ and $f^*(v_i v_{i+1}) = 2m + 2i$ for $1 \le i \le n - 1$.

Case 2. $m = 3$.

We define $f : V(P_n \cup C_3) \cup E(P_n \cup C_3) \to \{1, 2, 3, \cdots, 2n + 5\}$ as follows $f(u_1) = 1, f(u_2) = 4, f(u_3) = 6$ and $f(v_i) = 5 + 2i$ for $1 \le i \le n$. The induced edge labeling is as follows:

$f^*(u_1 u_2) = 2$, $f^*(u_2 u_3) = 5$, $f^*(u_3 u_1) = 3$ and $f^*(v_i v_{i+1}) = 6 + 2i$ for $1 \le i \le n - 1$.

Hence, f is a super geometric mean labeling of $P_n \cup C_m$. Thus the graph $P_n \cup C_m$ is a super geometric mean graph for $n \ge 1$ and $m \ge 3$. □

The super geometric mean labeling of $P_5 \cup C_6$ and $P_4 \cup C_3$ are shown in Figure 2.

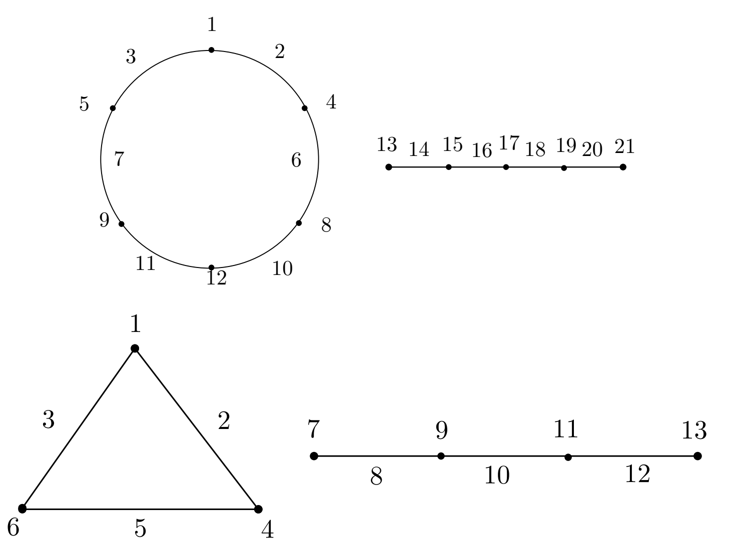

Figure 2

Theorem 2.2 *For a T-graph $T_n, T_n \cup C_m$ is a super geometric mean graph, for $n \geq 4$ and $m \geq 3$.*

Proof Let $u_1, u_2, \cdots, u_m$ be the vertices of the cycle C_m and $v_1, v_2, \cdots, v_{n-1}$ be the vertices of the path P_{n-1} and let v_n be the pendant vertex identified with v_{n-2} in T_n.

Case 1. $m \geq 4$.

We define $f : V(T_n \cup C_m) \cup E(T_n \cup C_m) \to \{1, 2, 3, \cdots, 2m + 2n - 1\}$ as follows:

$$f(u_i) = \begin{cases} 1 & i = 1 \\ 4i - 4 & 2 \leq i \leq \lfloor \frac{m}{2} \rfloor \\ 2m - 3 & i = \lfloor \frac{m}{2} \rfloor + 1 \text{ and } m \text{ is odd} \\ 2m & i = \lfloor \frac{m}{2} \rfloor + 1 \text{ and } m \text{ is even} \\ 2m & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is odd} \\ 2m - 3 & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is even} \\ 4m + 5 - 4i & \lfloor \frac{m}{2} \rfloor + 3 \leq i \leq m, \end{cases}$$

$f(v_i) = 2m + 2i - 1$ for $1 \leq i \leq n - 3$, $f(v_{n-2}) = 2m + 2n - 3$, $f(v_{n-1}) = 2m + 2n - 6$ and $f(v_n) = 2m + 2n - 1$.

The induced edge labeling is as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 4i - 2 & 1 \leq i \leq \lfloor \frac{m}{2} \rfloor \\ 2m - 1 & i = \lfloor \frac{m}{2} \rfloor + 1 \\ 2m - 2 & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is odd} \\ 2m - 5 & i = \lfloor \frac{m}{2} \rfloor + 2 \text{ and } m \text{ is even} \\ 4m + 3 - 4i & \lfloor \frac{m}{2} \rfloor + 3 \leq i \leq m - 1, \end{cases}$$

$$f^*(u_1 u_m) = 3, f^*(v_i v_{i+1}) = 2m + 2i \text{ for } 1 \leq i \leq n - 4,$$
$$f^*(v_{n-3} v_{n-2}) = 2m + 2n - 5, f^*(v_{n-2} v_{n-1}) = 2m + 2n - 4 \text{ and}$$
$$f^*(v_{n-2} v_n) = 2m + 2n - 2.$$

Case 2. $m = 3$.

We define $f : V(T_n \cup C_3) \cup E(T_n \cup C_3) \to \{1, 2, 3, \cdots, 2n + 5\}$ as follows:

$f(u_1) = 1, f(u_2) = 4, f(u_3) = 6, f(v_i) = 5 + 2i$ for $1 \leq i \leq n - 3$, $f(v_{n-2}) = 2n + 3$, $f(v_{n-1}) = 2n$ and $f(v_n) = 2n + 5$.

The induced edge labeling as follows:

$$f^*(u_1u_2) = 2, f^*(u_2u_3) = 5, f^*(u_3u_1) = 3,$$
$$f^*(v_iv_{i+1}) = 6 + 2i \text{ for } 1 \le i \le n-4,$$
$$f^*(v_{n-3}v_{n-2}) = 2n+1, f^*(v_{n-2}v_{n-1}) = 2n+2 \text{ and}$$
$$f^*(v_{n-2}v_n) = 2n+4.$$

Hence, f is a super geometric mean labeling of $T_n \cup C_m$. Thus the graph $T_n \cup C_m$ is a super geometric mean graph for $n \ge 4$ and $m \ge 3$. □

The super geometric mean labeling of $T_6 \cup C_7$ and $T_7 \cup C_3$ are shown in Figure 3.

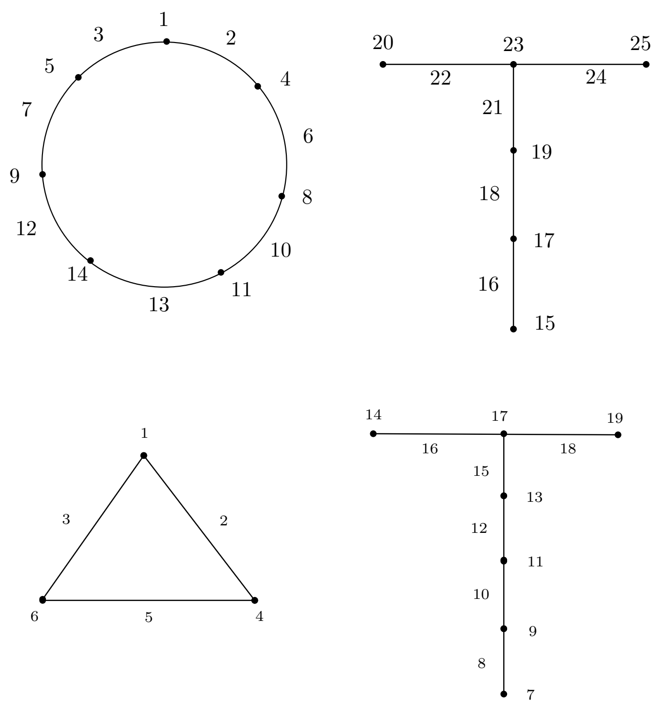

Figure 3

Theorem 2.3 *mC_n is a super geometric mean graph, for any m and n.*

Proof Let $\left\{v_j^{(i)} : 1 \le j \le n\right\}$ be the vertices of the i^{th} copy of the cycle C_n, $1 \le i \le m$.

Case 1. $n \ge 5$.

We define $f : V(mC_n) \cup E(mC_n) \to \{1, 2, 3, \cdots, 2mn\}$ as follows:

$$f\left(v_j^{(1)}\right) = \begin{cases} 1 & j = 1 \\ 4j-4 & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2n-3 & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 2n & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 2n & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 2n-3 & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 4n+5-4i & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le i \le n, \end{cases}$$

$$f\left(v_j^{(2)}\right) = \begin{cases} 2n+1 & j = 1 \\ 2n+4j-5 & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \\ 4n-3 & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 4n & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 4n & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 4n-3 & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 6n+6-4j & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n \end{cases}$$

and $f\left(v_j^{(i)}\right) = 2n + f\left(v_j^{(i-1)}\right)$ for $3 \le i \le m$ and $1 \le j \le n$.

The induced edge labeling is as follows:

$$f^*\left(v_j^{(1)}v_{j+1}^{(1)}\right) = \begin{cases} 4j-2 & 1 \le j \le \left\lfloor \frac{n}{2} \right\rfloor \\ 2n-1 & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 2n-2 & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 2n-5 & j = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 4n+3-4j & \left\lfloor \frac{n}{2} \right\rfloor + 3 \le j \le n-1, \end{cases}$$

$$f^*\left(v_1^{(1)}v_n^{(1)}\right) = 3,$$

$$f^*\left(v_j^{(2)}v_{j+1}^{(2)}\right) = \begin{cases} 2n+2 & j = 1 \\ 2n+4j-3 & 2 \le j \le \left\lfloor \frac{n}{2} \right\rfloor\text{-1} \\ 4n-5 & j = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is odd} \\ 4n-2 & j = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is even} \\ 4n-1 & j = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 6n+4-4j & \left\lfloor \frac{n}{2} \right\rfloor + 2 \le j \le n-1, \end{cases}$$

$$f^*\left(v_1^{(2)}v_n^{(2)}\right) = 2n+4,$$

$$f^*\left(v_j^{(i)}v_{j+1}^{(i)}\right) = 2n + f^*\left(v_j^{(i-1)}v_{j+1}^{(i-1)}\right) \text{ for } 3 \le i \le m \text{ and } 1 \le j \le n-1$$

and $$f^*\left(v_1^{(i)}v_n^{(i)}\right) = 2n + f^*\left(v_1^{(i-1)}v_n^{(i-1)}\right) \text{ for } 3 \le i \le m.$$

Case 2. $n = 4$.

We define $f : V(mC_4) \cup E(mC_4) \to \{1, 2, 3, \cdots, 8m\}$ as follows:

$$f\left(v_j^{(1)}\right) = \begin{cases} 1 & j = 1 \\ 5j - 6 & 2 \le j \le 3 \\ 5 & j = 4 \end{cases} \text{ and } f\left(v_j^{(2)}\right) = \begin{cases} 8 & j = 1 \\ 6j & 2 \le j \le 3 \\ 13 & j = 4. \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_j^{(1)} v_{j+1}^{(1)}\right) = \begin{cases} 4j - 2 & 1 \le j \le 2 \\ 7 & j = 3 \end{cases}, f^*\left(v_1^{(1)} v_4^{(1)}\right) = 3$$

$$f^*\left(v_j^{(2)} v_{j+1}^{(2)}\right) = \begin{cases} 5j + 5 & 1 \le j \le 2 \\ 16 & j = 3 \end{cases} \text{ and } f^*\left(v_1^{(2)} v_4^{(2)}\right) = 11.$$

Subcase 2.1 m is odd and $m \ge 3$.

$$f\left(v_j^{(3)}\right) = \begin{cases} 14 & j = 1 \\ 2j + 16 & 2 \le j \le 4, \end{cases}$$

$$f\left(v_j^{(4)}\right) = \begin{cases} 2j + 23 & 1 \le j \le 3 \\ 34 & j = 4, \end{cases}$$

$$f\left(v_j^{(5)}\right) = \begin{cases} 31 & j = 1 \\ 3j + 29 & 2 \le j \le 3 \\ 40 & j = 4 \end{cases} \text{ and}$$

$$f\left(v_j^{(i)}\right) = f\left(v_j^{(i-2)}\right) + 16 \text{ for } 6 \le i \le m \text{ and } 1 \le j \le 4.$$

The induced edge labeling is as follows

$$f^*\left(v_j^{(3)} v_{j+1}^{(3)}\right) = \begin{cases} 17 & j = 1 \\ 2j + 17 & 2 \le j \le 3 \end{cases}, f^*\left(v_i^{(3)} v_4^{(3)}\right) = 19,$$

$$f^*\left(v_j^{(4)} v_{j+1}^{(4)}\right) = \begin{cases} 26 & j = 1 \\ 4j + 20 & 2 \le j \le 3 \end{cases}, f^*\left(v_1^{(4)} v_4^{(4)}\right) = 30,$$

$$f^*\left(v_j^{(5)} v_{j+1}^{(5)}\right) = \begin{cases} 33 & j = 1 \\ 2j + 33 & 2 \le j \le 3 \end{cases}, f^*\left(v_1^{(5)} v_4^{(5)}\right) = 36,$$

$$f^*\left(v_j^{(i)} v_{j+1}^{(i)}\right) = f^*\left(v_j^{(i-2)} v_{j+1}^{(i-2)}\right) + 16 \text{ for } 6 \le i \le m \text{ and } 1 \le j \le 3$$

and $$f^*\left(v_1^{(i)} v_4^{(i)}\right) = f^*\left(v_1^{(i-2)} v_4^{(i-2)}\right) + 16 \text{ for } 6 \le i \le m.$$

Subcase 2.2 m is even and $m \geq 4$.

$$f\left(v_j^{(3)}\right) = \begin{cases} 14 & j=1 \\ 3j+13 & 2 \leq j \leq 3 \\ 26 & j=4 \end{cases}, f\left(v_j^{(4)}\right) = \begin{cases} 23 & j=1 \\ 3j+21 & 2 \leq j \leq 3 \\ 32 & j=4, \end{cases}$$

$$f\left(v_j^{(5)}\right) = \begin{cases} 2j+31 & 1 \leq j \leq 3 \\ 42 & j=4 \end{cases} \text{ and}$$

$$f\left(v_j^{(i)}\right) = f\left(v_j^{(i-2)}\right) + 16 \text{ for } 6 \leq i \leq m \text{ and } 1 \leq j \leq 4.$$

The induced edge labeling is as follows

$$f^*\left(v_j^{(3)}v_{j+1}^{(3)}\right) = \begin{cases} 17 & j=1 \\ 3j+15 & 2 \leq j \leq 3 \end{cases}, f^*\left(v_1^{(3)}v_4^{(3)}\right) = 20,$$

$$f^*\left(v_j^{(4)}v_{j+1}^{(4)}\right) = \begin{cases} 25 & j=1 \\ 2j+25 & 2 \leq j \leq 3 \end{cases}, f^*\left(v_1^{(4)}v_4^{(4)}\right) = 28,$$

$$f^*\left(v_j^{(5)}v_{j+1}^{(5)}\right) = \begin{cases} 34 & j=1 \\ 4j+28 & 2 \leq j \leq 3 \end{cases}, f^*\left(v_1^{(5)}v_4^{(5)}\right) = 38,$$

$$f^*\left(v_j^{(i)}v_{j+1}^{(i)}\right) = f^*\left(v_j^{(i-2)}v_{j+1}^{(i-2)}\right) + 16 \text{ for } 6 \leq i \leq m \text{ and } 1 \leq j \leq 3$$

$$\text{and } f^*\left(v_1^{(i)}v_4^{(i)}\right) = f^*\left(v_1^{(i-2)}v_4^{(i-2)}\right) + 16 \text{ for } 6 \leq i \leq m.$$

Case 3. $n = 3$.

We define $f : V(mC_3) \cup E(mC_3) \to \{1, 2, 3, \cdots, 6m\}$ as follows:

$$f\left(v_1^{(i)}\right) = 6i-5 \text{ for } 1 \leq i \leq m, \ f\left(v_2^{(i)}\right) = \begin{cases} 4 & i=1 \\ 6i-3 & 2 \leq i \leq m \end{cases}$$

and $f\left(v_3^{(i)}\right) = 6i$ for $1 \leq i \leq m$. The induced edge labeling is as follows:

$$f^*\left(v_1^{(i)}v_2^{(i)}\right) = 6i-4 \text{ for } 1 \leq i \leq m, \ f^*\left(v_2^{(i)}v_3^{(i)}\right) = 6i-1 \text{ for } 1 \leq i \leq m \text{ and}$$

$$f^*\left(v_3^{(i)}v_1^{(i)}\right) = \begin{cases} 3 & i=1 \\ 6i-2 & 2 \leq i \leq m \end{cases}.$$

Hence, f is super geometric mean labeling of mC_n. Thus the graph mC_n is a super geometric mean graph for any m and n. □

The super geometric mean labeling of $4C_6, 7C_4$ and $5C_3$ are shown in Figure 4.

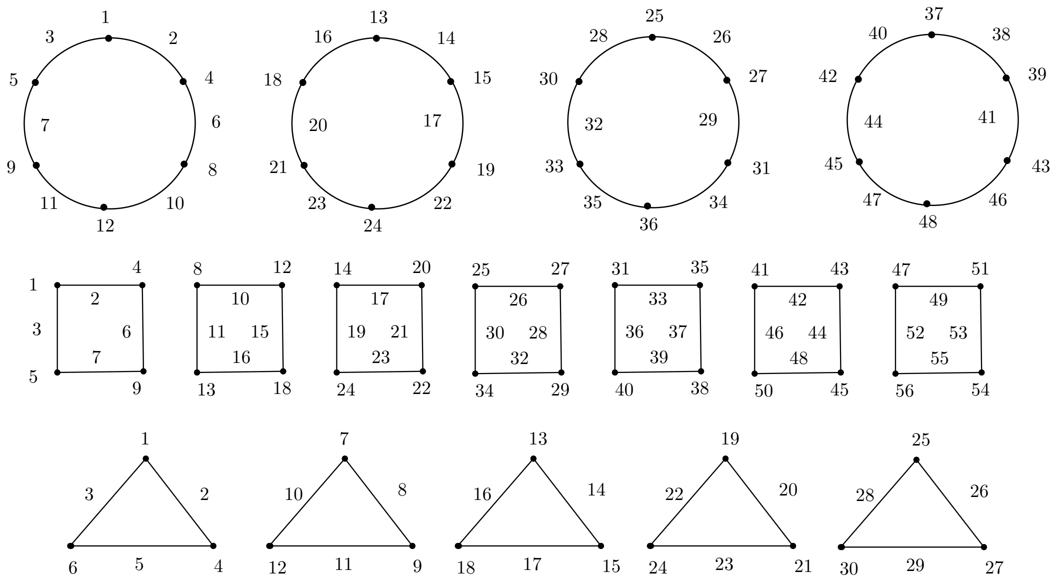

Figure 4

Corollary 2.4 *$mC_n \cup P_k$ is a super geometric mean graph for any m, n and k.*

Proof By the above Theorems 2.1 and 2.3 the results follows. □

Theorem 2.5 *K_n is a super geometric mean graph if and only if $n \leq 3$.*

Proof Based on the definition of super geometric mean labeling, 1 and $p+q$ should be the vertex labels.

For all $p \geq 5$, the edge having the end vertices whose labels are 1 and $p+q$ is less than or equal to $p-1$. So we cannot have distinct edge labels for the edges incident with a vertex whose vertex label is 1.

When $p = 4, 1, p+q = 10$ and $p+q-2 = 8$ are to be the vertex labels whose induced edge labels are $3, 4$ and 9. So we cannot label for the 4^{th} vertex in which the edge label is 2. Also 2 cannot be the vertex label. □

The super geometric mean labeling of K_1, K_2 and K_3 are shown in Figure 5.

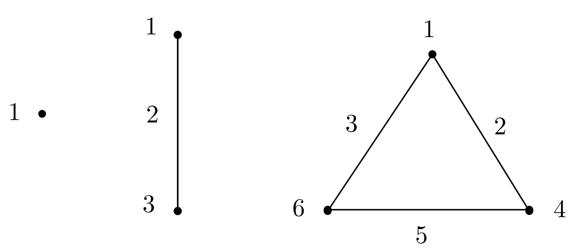

Figure 5

Theorem 2.6 *$[P_n; S_m]$ is a super geometric mean graph, for $n \geq 1$ and $m \leq 2$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the path P_n and $v_1^{(i)}, v_2^{(i)}, \cdots, v_m^{(i)}$ be the pendant vertices at each vertex u_i of the path P_n, for $1 \leq i \leq n$.

Case 1. $m = 1$.

We define $f : V([P_n; S_1]) \cup E([P_n; S_1]) \to \{1, 2, 3, \cdots, 6n-1\}$ as follows:

$$f(u_i) = \begin{cases} 5 & i = 1 \\ 6i-5 & 2 \leq i \leq n \end{cases}, f\left(v_1^{(i)}\right) = 6i-3 \text{ for } 1 \leq i \leq n,$$

$$f\left(v_2^{(i)}\right) = \begin{cases} 1 & i = 1 \\ 6i & 2 \leq i \leq n-1 \end{cases} \text{ and } f\left(v_2^{(n)}\right) = 6n-1.$$

The induced edge labeling is as follows:

$$f^*(u_i u_{i+1}) = \begin{cases} 6 & i = 1 \\ 6i-2 & 2 \leq i \leq n-1, \end{cases}$$

$$f^*\left(u_i v_1^{(i)}\right) = \begin{cases} 4 & i = 1 \\ 6i-4 & 2 \leq i \leq n, \end{cases}$$

$$f^*\left(v_1^{(i)} v_2^{(i)}\right) = \begin{cases} 2 & i = 1 \\ 6i-1 & 2 \leq i \leq n-1 \end{cases} \text{ and } f^*\left(v_1^{(n)} v_2^{(n)}\right) = 6n-2.$$

Case 2. $m = 2$.

We define $f : V([P_n; S_2]) \cup E([P_n; S_2]) \to \{1, 2, 3, \cdots, 8n-1\}$ as follows:

$$f(u_i) = \begin{cases} 2i+5 & 1 \leq i \leq 2 \\ 8i-8 & 3 \leq i \leq n, \end{cases}$$

$$f\left(v_1^{(i)}\right) = \begin{cases} 5 & i = 1 \\ 8i-5 & 2 \leq i \leq n-1, \end{cases}$$

$$f\left(v_1^{(n)}\right) = 8n-3,$$

$$f\left(v_2^{(i)}\right) = \begin{cases} 1 & i = 1 \\ 8i-1 & 2 \leq i \leq n-1, \end{cases} f\left(v_2^{(n)}\right) = 8n-6,$$

$$f\left(v_3^{(i)}\right) = \begin{cases} 2 & i = 1 \\ 8i+1 & 2 \leq i \leq n-1 \end{cases} \text{ and } f\left(v_3^{(n)}\right) = 8n-1.$$

The induced edge labeling is as follows

$$f^*(u_iu_{i+1}) = \begin{cases} 8 & i=1 \\ 8i-4 & 2 \le i \le n-1, \end{cases}$$

$$f^*\left(u_iv_1^{(i)}\right) = \begin{cases} 6 & i=1 \\ 8i-6 & 2 \le i \le n-1, \end{cases}$$

$$f^*\left(u_nv_1^{(n)}\right) = 8n-5, f^*\left(v_1^{(i)}v_2^{(i)}\right) = \begin{cases} 3 & i=1 \\ 8i-3 & 2 \le i \le n-1, \end{cases}$$

$$f^*\left(v_1^{(n)}v_2^{(n)}\right) = 8n-4 \text{ and } f^*\left(v_1^{(i)}v_3^{(i)}\right) = \begin{cases} 4 & i=1 \\ 8i-2 & 2 \le i \le n. \end{cases}$$

Hence, f is a super geometric mean labeling of $[P_n; S_m]$. Thus the graph $[P_n; S_m]$ is a super geometric mean graph, for $n \ge 1$ and $m \le 2$. □

The super geometric mean labeling of $[P_6; S_1]$ and $[P_5; S_2]$ are shown in Figure 6.

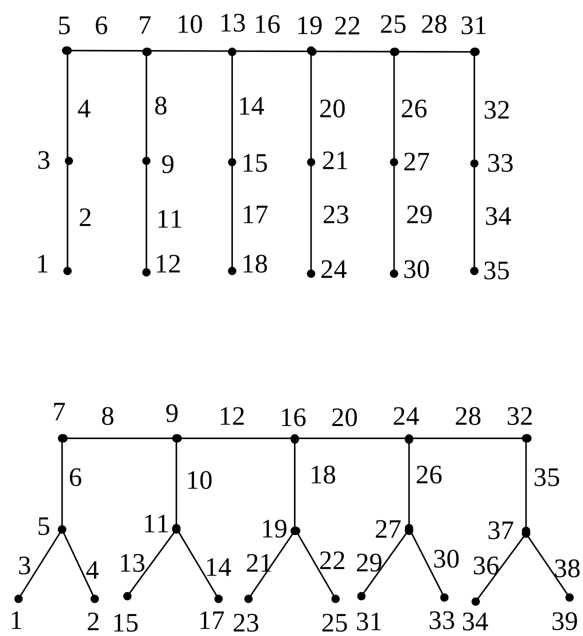

Figure 6

Theorem 2.7 *$S(P_n \odot K_1)$ is a super geometric mean graph, for $n \ge 1$.*

Proof Let $V(P_n \odot K_1) = \{u_i, v_i : 1 \le i \le n\}$. Let x_i be the vertex which divides the edge

u_iv_i, for $1 \leq i \leq n$ and y_i be the vertex which divides the edge u_iv_{i+1}, for $1 \leq i \leq n-1$. Then

$$V(S(P_n \odot K_1)) = \{u_i, v_i, x_i, y_j : 1 \leq i \leq n, 1 \leq j \leq n-1\}$$
$$E(S(P_n \odot K_1)) = \{u_ix_i, v_ix_i : 1 \leq i \leq n\} \cup \{u_iy_i, y_iu_{i+1} : 1 \leq j \leq n-1\}$$

We define $f : V(S(P_n \odot K_1)) \bigcup E(S(P_n \odot K_1)) \to \{1, 2, 3, \cdots, 8n-3\}$ as follows:

$$f(u_i) = \begin{cases} 5 & i = 1 \\ 8i-7 & 2 \leq i \leq n \end{cases}, f(y_i) = 8i-1 \text{ for } 1 \leq i \leq n-1,$$

$$f(x_i) = 8i-5 \text{ for } 1 \leq i \leq n, f(v_i) = \begin{cases} 1 & i = 1 \\ 8i-2 & 2 \leq i \leq n-1 \end{cases} \text{ and}$$

$$f(v_n) = 8n-3.$$

The induced edge labeling is as follows

$$f^*(u_iy_i) = \begin{cases} 6 & i = 1 \\ 8i-4 & 2 \leq i \leq n-1 \end{cases}, f^*(y_iu_{i+1}) = 8i \text{ for } 1 \leq i \leq n-1,$$

$$f^*(u_ix_i) = \begin{cases} 4 & i = 1 \\ 8i-6 & 2 \leq i \leq n \end{cases}, f^*(x_iv_i) = \begin{cases} 2 & i = 1 \\ 8i-3 & 2 \leq i \leq n-1 \end{cases},$$

and $f^*(x_nv_n) = 8n-4$.

Hence, f is a super geometric mean labeling of $S(P_n \odot K_1)$. Thus the graph $S(P_n \odot K_1)$ is a super geometric mean graph, for $n \geq 1$. □

A super geometric mean labeling of $S(P_4 \odot K_1)$ is shown in Figure 7.

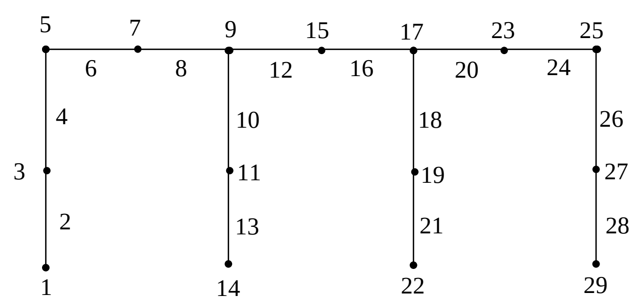

Figure 7

Theorem 2.8 *$TW(P_n)$ is a super geometric mean graph, for $n \geq 3$.*

Proof Let $u_1, u_2, \cdots, u_n$ be the vertices of the path P_n and $v_1^{(i)}, v_2^{(i)}$ be the pendant vertices at each vertex u_i of the path P_n, for $2 \leq i \leq n-1$. Then

$$V\left(TW(P_n)\right) = V(P_n) \cup \left\{v_1^{(i)}, v_2^{(i)} : 2 \leq i \leq n-1\right\} \text{ and}$$
$$E\left(TW(P_n)\right) = E(P_n) \cup \left\{u_i v_1^{(i)}, u_i v_2^{(i)} : 2 \leq i \leq n-1\right\}.$$

We define $f : V(TW(P_n)) \cup E(TW(P_n)) \to \{1, 2, 3, \ldots, 6n-9\}$ as follows

$$f(u_i) = \begin{cases} 1 & i = 1 \\ 6i - 7 & 2 \leq i \leq n-2, \end{cases}$$
$$f(u_{n-1}) = 6n - 11, f(u_n) = 6n - 9,$$
$$f\left(v_1^{(i)}\right) = \begin{cases} 2 & i = 2 \\ 6i - 9 & 3 \leq i \leq n-2 \end{cases}, f\left(v_1^{(n-1)}\right) = 6n - 16,$$
$$f\left(v_2^{(i)}\right) = 6i - 5 \text{ for } 2 \leq i \leq n-2 \textit{ and } f\left(v_2^{(n-1)}\right) = 6n - 14.$$

The induced edge labeling is as follows

$$f^*(u_i u_{i+1}) = \begin{cases} 3 & i = 1 \\ 6i - 4 & 2 \leq i \leq n-3 \end{cases}, f^*(u_{n-2} u_{n-1}) = 6n - 15,$$
$$f^*(u_{n-1} u_n) = 6n - 10,$$
$$f^*\left(u_i v_1^{(i)}\right) = 6i - 8 \text{ for } 2 \leq i \leq n-2, f^*\left(u_{n-1} v_1^{(n-1)}\right) = 6n - 13 \text{ and}$$
$$f^*\left(u_i v_2^{(i)}\right) = 6i - 6 \text{ for } 2 \leq i \leq n-1.$$

Hence, f is a super geometric mean labeling of $TW(P_n)$. Thus the graph $TW(P_n)$ is a super geometric mean graph, for $n \geq 3$. □

A super geometric mean labeling of $TW(P_8)$ is shown in Figure 8.

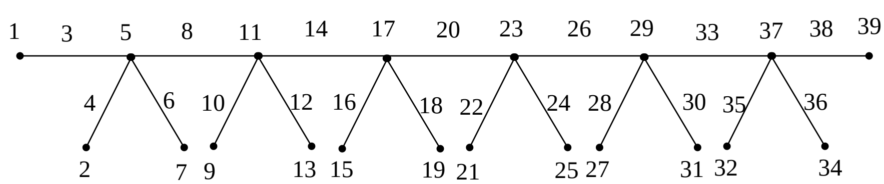

Figure 8

Theorem 2.9 *$M(P_n)$ is a super geometric mean graph, for $n \geq 4$.*

Proof Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(P_n) = \{e_i = v_i v_{i+1} : 1 \leq i \leq n-1\}$ be the vertex set and edge set of the path P_n. Then

$$V(M(P_n)) = \{v_1, v_2, \ldots, v_n, e_1, e_2, \ldots, e_{n-1}\} \text{ and}$$
$$E(M(P_n)) = \{v_ie_i, e_iv_{i+1} : 1 \le i \le n-1\} \cup \{e_ie_{i+1} : 1 \le i \le n-2\}.$$

We define $f : V(M(P_n)) \cup E(M(P_n)) \to \{1, 2, 3, \cdots, 5n-5\}$ as follows:

$$f(v_i) = \begin{cases} 1 & i = 1 \\ 2i+1 & 2 \le i \le 3 \\ 5i-5 & 4 \le i \le n \end{cases} \text{ and } f(e_i) = \begin{cases} 8i-5 & 1 \le i \le 2 \\ 5i-2 & 3 \le i \le n-1. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(e_ie_{i+1}) = \begin{cases} 6i & 1 \le i \le 2 \\ 5i+1 & 3 \le i \le n-2 \end{cases}, f^*(e_iv_i) = \begin{cases} 2 & i = 1 \\ 2i+4 & 2 \le i \le 3 \\ 5i-3 & 4 \le i \le n-1 \end{cases}$$

and $f^*(e_iv_{i+1}) = 5i-1$ for $1 \le i \le n-1$.

Hence, f is a super geometric mean labeling of $M(P_n)$. Thus the graph $M(P_n)$ is a super geometric mean graph for $n \ge 4$. □

A super geometric mean labeling of $M(P_8)$ is shown in Figure 9.

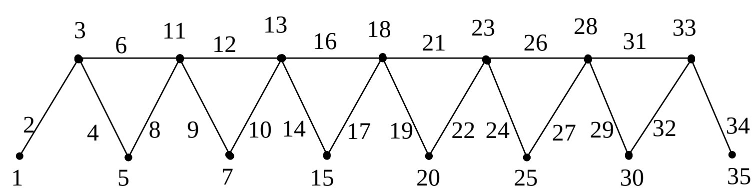

Figure 9

Theorem 2.10 *TL_n is a super geometric mean graph, for $n \ge 3$.*

Proof Let the vertex set of TL_n be $\{v_1, v_2, \cdots, v_n, u_1, u_2, \cdots, u_n\}$ and the edge set of TL_n be $\{u_iu_{i+1}, u_iv_{i+1}, v_iv_{i+1} : 1 \le i \le n-1\} \cup \{u_iv_i : 1 \le i \le n\}$. Then TL_n has $2n$ vertices and $4n-3$ edges. We define $f : V(TL_n) \cup E(TL_n) \to \{1, 2, 3, \cdots, 6n-3\}$ as follows:

$$f(v_i) = \begin{cases} 1 & i = 1 \\ 6i-6 & 2 \le i \le n \end{cases}, f(u_i) = 6i-2 \text{ for } 1 \le i \le n-1$$

and $f(u_n) = 6n-3$. The induced edge labeling is as follows:

$$f^*(v_iv_{i+1}) = 6i-3 \text{ for } 1 \le i \le n-1, f^*(u_iu_{i+1}) = 6i+1 \text{ for } 1 \le i \le n-1,$$
$$f^*(u_iv_i) = 6i-4 \text{ for } 1 \le i \le n \text{ and } f^*(u_iv_{i+1}) = 6i-1 \text{ for } 1 \le i \le n-1.$$

Hence, f is a super geometric mean labeling of TL_n. Thus the graph TL_n is a super geometric mean graph for $n \ge 3$. □

A super geometric mean labeling of TL_7 are shown in Figure 10.

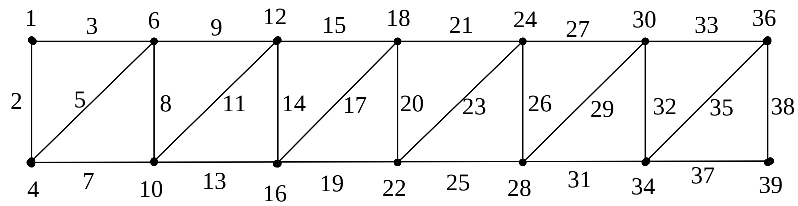

Figure 10

Theorem 2.11 *$C_n \odot K_1$ is a super geometric mean graph.*

Proof Let $v_1, v_2, \cdots, v_n$ be the vertices of the cycle C_n and $u_1, u_2, \cdots, u_n$ be the pendant vertices of the cycle C_n.

Case 1. $n \geq 7$.

We define $f : V(C_n \odot K_1) \cup E(C_n \odot K_1) \to \{1, 2, 3, \cdots, 4n\}$ as follows:

$$f(v_i) = \begin{cases} 3 & i = 1 \\ 8i - 11 & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 4n - 7 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 4n - 2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 8n + 12 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n \end{cases}$$

$$f(u_i) = \begin{cases} 7i - 6 & 1 \leq i \leq 3 \\ 8i - 9 & 4 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 4n - 5 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 4n & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 4n - 2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 4n - 7 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 8n + 10 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(v_i v_{i+1}) = \begin{cases} 4 & i = 1 \\ 8i - 7 & 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \\ 4n - 11 & i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is odd} \\ 4n - 6 & i = \left\lfloor \frac{n}{2} \right\rfloor \text{ and } n \text{ is even} \\ 4n - 3 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 8n + 8 - 8i & \left\lfloor \frac{n}{2} \right\rfloor + 2 \leq i \leq n - 1, \end{cases}$$

$f^*(v_1 v_n) = 6$ and

$$f^*(u_iv_i) = \begin{cases} 5i-3 & 1 \leq i \leq 2 \\ 8i-10 & 3 \leq i \leq \lfloor \frac{n}{2} \rfloor \\ 4n-6 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is odd} \\ 4n-1 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is even} \\ 8n+11-8i & \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n \end{cases}$$

Case 2. $n = 3, 4, 5, 6$.

In this case, the super geometric mean labelings are given in Figure 11.

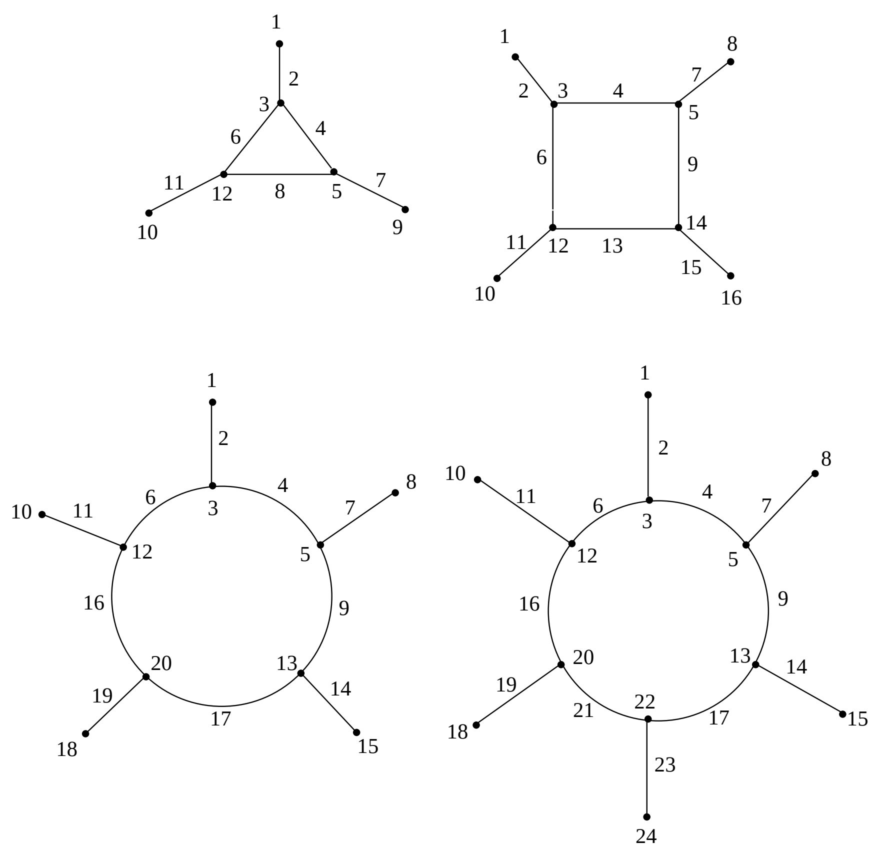

Figure 11

Hence, f is a super geometric mean labeling of $C_n \odot K_1$. Thus the graph $C_n \odot K_1$ is a super geometric mean graph. □

A super geometric mean labeling of $C_9 \odot K_1$ is shown in Figure 12.

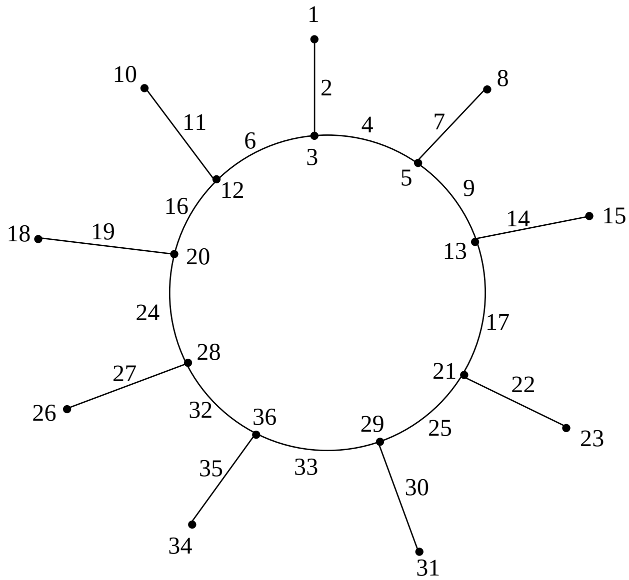

Figure 12

Theorem 2.12 *The graph obtained by duplication of an arbitrary vertex in cycle C_n is a super geometric mean graph, for $n \geq 4$.*

Proof Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of the cycle C_n, for $n \geq 4$. Without loss of generality we duplicate the vertex $v = v_1$ and its duplicated vertex is v_1'. Then the resultant graph G will have $n+1$ vertices and $n+2$ edges.

We define $f : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, 2n+3\}$ as follows:

$$f(v_1') = 1,$$

$$f(v_i) = \begin{cases} 8-2i & 1 \leq i \leq 2 \\ 4i & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 2n+3 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 2n & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 2n-1 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 4n+7-4i & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n-1 \end{cases}$$

and $$f(v_n) = 9.$$

The induced edge labeling is as follows

$$f^*(v_1'v_2) = 2, \quad f^*(v_1'v_n) = 3,$$

$$f^*(v_iv_{i+1}) = \begin{cases} 2i+3 & 1 \leq i \leq 2 \\ 4i+2 & 3 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1 \\ 2n+1 & i = \lfloor \frac{n}{2} \rfloor \text{ and } n \text{ is odd} \\ 2n+2 & i = \lfloor \frac{n}{2} \rfloor \text{ and } n \text{ is even} \\ 2n+2 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is odd} \\ 2n+1 & i = \lfloor \frac{n}{2} \rfloor + 1 \text{ and } n \text{ is even} \\ 4n+5-4i & \lfloor \frac{n}{2} \rfloor + 2 \leq i \leq n-2, \end{cases}$$

$$f^*(v_{n-1}v_n) = 10 \text{ and } f^*(v_1v_n) = 8.$$

Hence, f is a super geometric mean labeling of G. Thus the graph obtained by duplication of an arbitrary vertex in the cycle C_n is a super geometric mean graph, for $n \geq 4$. □

The graph obtained by duplication of a vertex in C_9 and its super geometric mean labeling is shown in Figure 13.

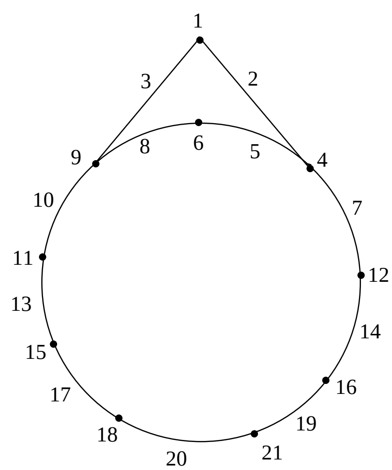

Figure 13

Theorem 2.13 *The graph obtained by duplication of an arbitrary edge in cycle C_n is a super geometric mean graph, for $n \geq 3$.*

Proof Let $v_1, v_2, v_3, \cdots, v_n$ be the vertices of the cycle C_n. Without loss of generality we duplication an edge $e = v_1v_2$ and its duplicated edge is $e' = v_1'v_2'$. Then the resultant graph G will have $n+2$ vertices and $n+3$ edges.

Case 1. $n \geq 6$.

We define $f : V(G) \cup E(G) \to \{1, 2, 3, \cdots, 2n+5\}$ as follows:

$f(v_1') = 1, f(v_2') = 3$ and

$$f(v_i') = \begin{cases} 9 & i = 1 \\ 5i-5 & 2 \leq i \leq 3 \\ 4i-2 & 4 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 \\ 2n+5 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \\ 2n+2 & i = \left\lfloor \frac{n}{2} \right\rfloor + 3 \text{ and } n \text{ is odd} \\ 2n+1 & i = \left\lfloor \frac{n}{2} \right\rfloor + 3 \text{ and } n \text{ is even} \\ 4n+13-4i & \left\lfloor \frac{n}{2} \right\rfloor + 4 \leq i \leq n. \end{cases}$$

The induced edge labeling is as follows:

$$f^*(v_1'v_2') = 2, f^*(v_1'v_n') = 4, f^*(v_2'v_3') = 6,$$

$$f^*(v_iv_{i+1}) = \begin{cases} i+6 & 1 \leq i \leq 2 \\ 4i & 3 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\ 2n+3 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is odd} \\ 2n+4 & i = \left\lfloor \frac{n}{2} \right\rfloor + 1 \text{ and } n \text{ is even} \\ 2n+4 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is odd} \\ 2n+3 & i = \left\lfloor \frac{n}{2} \right\rfloor + 2 \text{ and } n \text{ is even} \\ 4n+11-4i & \left\lfloor \frac{n}{2} \right\rfloor + 3 \leq i \leq n-1 \end{cases}$$

and $$f^*(v_1v_n) = 11.$$

Case 2. $n = 3, 4, 5$.

In this case, the super geometric mean labelings are given in Figure 14.

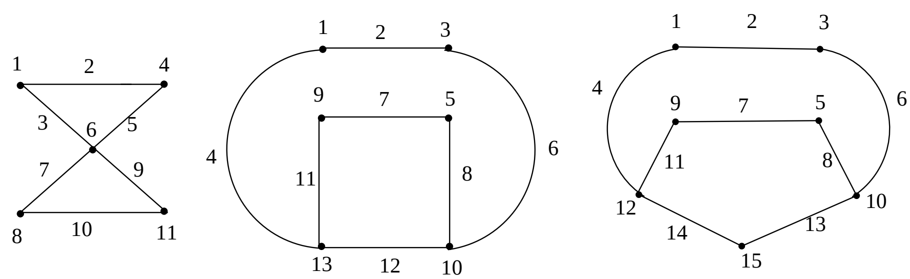

Figure 14

Hence, f is a super geometric mean labeling of G. Thus the graph obtained by duplication of an arbitrary edge in cycle C_n is a super geometric mean graph, for $n \geq 3$. □

The graph obtained by duplication of an edge in C_8 and its super geometric mean labeling is shown in Figure 15.

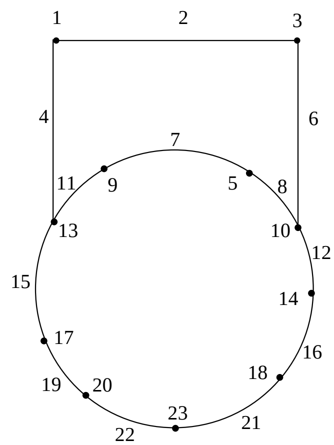

Figure 15

A triangular grid $T_n(G)$ with n vertices in each side are constructed as follows: The vertices of $T_n(G)$ are $\left\{v_i^{(j)} : 1 \leq j \leq n,\ \ 1 \leq i \leq n+1-j\right\}$ and the edges are $\left\{v_i^{(j)} v_{i+1}^{(j)} : 1 \leq j \leq n-1,\right.$ $\left. 1 \leq i \leq n-j\right\} \bigcup \left\{v_i^{(j)} v_i^{(j+1)} : 1 \leq j \leq n-1, 1 \leq i \leq n-j\right\} \bigcup \left\{v_{i+1}^{(j)} v_i^{(j+1)} : 1 \leq j \leq n-1, 1 \leq\right.$ $\left. i \leq n-j\right\}$. The triangular grid graph $T_6(G)$ is shown in Figure 16.

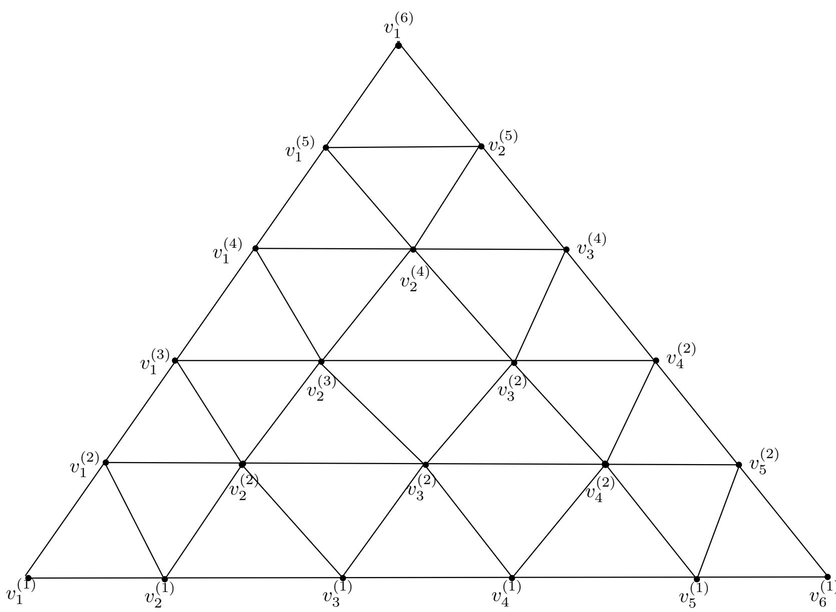

Figure 16

Theorem 2.14 *The triangular grid graph $T_n(G)$ is a super geometric mean graph.*

Proof Let $\left\{v_i^{(j)} : 1 \leq j \leq n, 1 \leq i \leq n+1-j\right\}$ be the vertex set of $T_n(G)$. Then the edge set of $T_n(G)$ are $\left\{v_i^{(j)} v_{i+1}^{(j)} : 1 \leq j \leq n-1, 1 \leq i \leq n-j\right\} \bigcup \left\{v_i^{(j)} v_i^{(j+1)} : 1 \leq j \leq n-1, 1 \leq i \leq\right.$ $\left. n-j\right\} \bigcup \left\{v_{i+1}^{(j)} v_i^{(j+1)} : 1 \leq j \leq n-1,\right.$
$\left. 1 \leq i \leq n-j\right\}$.

We define $f : V(T_n(G)) \cup E(T_n(G)) \to \{1, 2, 3, \cdots, n(2n-1)\}$ as follows

$$f\left(v_i^{(1)}\right) = i(2i-1) \text{ for } 1 \leq i \leq n \text{ and}$$
$$f\left(v_i^{(j)}\right) = f\left(v_{i+1}^{(j-1)}\right) - 2 \text{ for } 2 \leq j \leq n \text{ and } 1 \leq i \leq n+1-j.$$

The induced edge labeling is as follows

$$f^*\left(v_i^{(1)}v_{i+1}^{(1)}\right) = i(2i+1) \text{ for } 1 \leq i \leq n-1,$$
$$f^*\left(v_i^{(j)}v_{i+1}^{(j)}\right) = f^*\left(v_{i+1}^{(j-1)}v_{i+2}^{(j-1)}\right) - 2 \text{ for } 2 \leq j \leq n-1 \text{ and } 1 \leq i \leq n-j,$$
$$f^*\left(v_i^{(1)}v_i^{(2)}\right) = (i+1)(2i-1) \text{ for } 1 \leq i \leq n-1,$$
$$f^*\left(v_i^{(j)}v_i^{(j+1)}\right) = f^*\left(v_{i+1}^{(j-1)}v_{i+1}^{(j)}\right) - 2 \text{ for } 2 \leq j \leq n-1 \text{ and } 1 \leq i \leq n-j,$$
$$f^*\left(v_{i+1}^{(1)}v_i^{(2)}\right) = i(2i-1) + 4i \text{ for } 1 \leq i \leq n-1 \text{ and}$$
$$f^*\left(v_{i+1}^{(j)}v_i^{(j+1)}\right) = f^*\left(v_{i+2}^{(j-1)}v_{i+1}^{(j)}\right) - 2 \text{ for } 2 \leq j \leq n-1 \text{ and } 1 \leq i \leq n-j.$$

Hence, f is a super geometric mean labeling of $T_n(G)$. Thus the graph $T_n(G)$ is a super geometric mean graph. □

A super geometric mean labeling of $T_7(G)$ is shown in Figure 17.

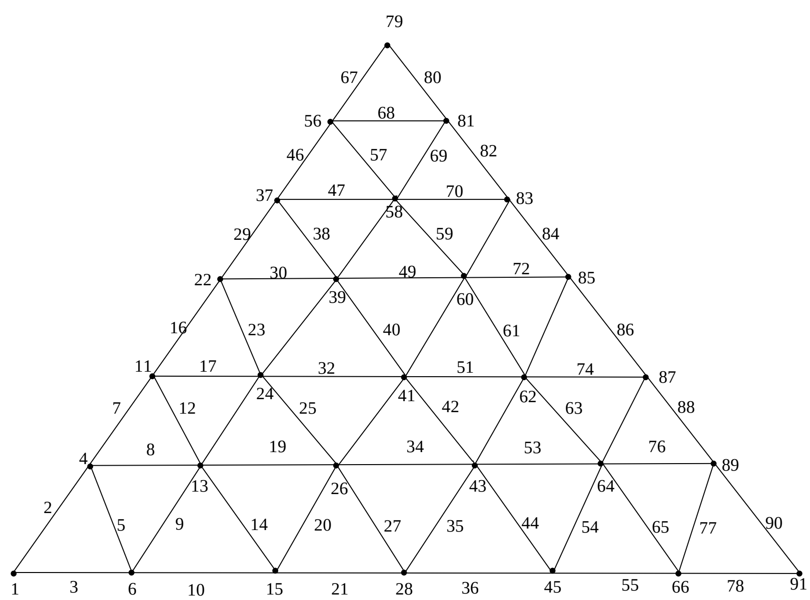

Figure 17

The graph $G'(p_1, p_2, \cdots, p_n)$ is obtained from n cycles of length $p_1, p_2, \ldots, p_n$ by identifying the j^{th} cycle and $(j+1)^{th}$ cycle by the edges $v_{\frac{p_j+1}{2}}^{(j)} v_{\frac{p_j+3}{2}}^{(j)}$ and $v_1^{(j+1)} v_{p_{j+1}}^{(j+1)}$, for $1 \leq j \leq n-1$.

Theorem 2.15 *The graph $G'(p_1, p_2, \ldots, p_n)$ is a super geometric mean graph all p_j's are odd or all p_j's are even with $p_j \neq 4$ for $1 \leq j \leq n$.*

Proof Let $\left\{v_i^{(j)} : 1 \le j \le n \textit{ and } 1 \le i \le p_j\right\}$ be the vertices of the n number of cycles with $p_j \ne 4$. For $1 \le j \le n-1$, the j^{th} cycle and $(j+1)^{th}$ cycle by the edges $v^{(j)}_{\frac{p_j+1}{2}} v^{(j)}_{\frac{p_j+3}{2}}$ and $v_1^{(j+1)} v_{p_{j+1}}^{(j+1)}$. We define $f : V(G') \cup E(G') \to \left\{1, 2, 3, \cdots, \sum\limits_{i=1}^{n} 2p_i - 3n + 3\right\}$ as follows.

Case 1. p_j is odd.

When $p_1 = 5$, define

$f\left(v_1^{(1)}\right) = 3$, $f\left(v_2^{(1)}\right) = 1$, $f\left(v_3^{(1)}\right) = 10$, $f\left(v_4^{(1)}\right) = 8$ and $f\left(v_5^{(1)}\right) = 6$.

The induced edge labeling is as follows:

$f^*\left(v_1^{(1)} v_2^{(1)}\right) = 2$, $f^*\left(v_2^{(1)} v_3^{(1)}\right) = 4$, $f^*\left(v_3^{(1)} v_4^{(1)}\right) = 9$,

$f^*\left(v_4^{(1)} v_5^{(1)}\right) = 7$ and $f^*\left(v_1^{(1)} v_5^{(1)}\right) = 5$.

When $p_1 \ge 7$, define

$$f\left(v_i^{(1)}\right) = \begin{cases} 4i-3 & 1 \le i \le \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4i-2 & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4i-9 & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \\ 4p_1 + 4 - 4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 3 \le i \le p_1 \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_i^{(1)} v_{i+1}^{(1)}\right) = \begin{cases} 4i-1 & 1 \le i \le \left\lfloor \frac{p_1}{2} \right\rfloor - 1 \\ 4i & i = \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4i-3 & i = \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \\ 4p_1 + 2 - 4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 2 \le i \le p_1 - 1 \end{cases}$$

and $$f^*\left(v_1^{(1)} v_{p_1}^{(1)}\right) = 2.$$

For $2 \le j \le n$, define

$$f\left(v_i^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 5 & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \le i \le p_j - 1 \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 2 & i = 1 \\ \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 3 & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i - 2 & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \le i \le p_j - 2 \end{cases}$$

and $$f^*\left(v_{p_j-1}^{(j)} v_{p_j}^{(j)}\right) = \sum_{k=1}^{j-1} 2p_k - 3(j-2) + 1.$$

Case 2. p_j is even.

When $p_1 = 6$, define

$f\left(v_1^{(1)}\right) = 6, f\left(v_2^{(1)}\right) = 8, f\left(v_3^{(1)}\right) = 10, f\left(v_4^{(1)}\right) = 12, f\left(v_5^{(1)}\right) = 1$ and $f\left(v_6^{(1)}\right) = 3$.

The induced edge labeling is as follows:

$f^*\left(v_1^{(1)}v_2^{(1)}\right) = 7, f^*\left(v_2^{(1)}v_3^{(1)}\right) = 9, f^*\left(v_3^{(1)}v_4^{(1)}\right) = 11,$

$f^*\left(v_4^{(1)}v_5^{(1)}\right) = 4, f^*\left(v_5^{(1)}v_6^{(1)}\right) = 2$ and $f^*\left(v_1^{(1)}v_6^{(1)}\right) = 5$.

When $p_1 \geq 8$, define

$$f\left(v_i^{(1)}\right) = \begin{cases} 4i-3 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4p_1+4-4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq i \leq p_1. \end{cases}$$

The induced edge labeling is as follows:

$$f^*\left(v_i^{(1)}v_{i+1}^{(1)}\right) = \begin{cases} 4i-1 & 1 \leq i \leq \left\lfloor \frac{p_1}{2} \right\rfloor \\ 4p_1+2-4i & \left\lfloor \frac{p_1}{2} \right\rfloor + 1 \leq i \leq p_1-1 \end{cases}$$

and $f^*\left(v_1^{(1)}v_{p_1}^{(1)}\right) = 2$.

Subcase 2.1 $2 \leq j \leq n$ and j is odd.

$$\text{Let } f\left(v_i^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 5 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 6 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 3 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \leq i \leq p_j - 1. \end{cases}$$

The induced edge labeling is as follows

$$f^*\left(v_i^{(j)}v_{i+1}^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 2 & i = 1 \\ \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 3 & 2 \leq i \leq \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 4 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 5 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i - 2 & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \leq i \leq p_j - 2 \end{cases}$$

and
$$f^*\left(v_{p_j-1}^{(j)}v_{p_j}^{(j)}\right) = \sum_{k=1}^{j-1} 2p_k - 3(j-2) + 1.$$

Subcase 2.2 $2 \leq j \leq n$ and j is even.

Let $f\left(v_i^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 4 & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 3 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 6 & i = \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i - 1 & \left\lfloor \frac{p_j}{2} \right\rfloor + 2 \le i \le p_j - 1. \end{cases}$

The induced edge labeling is as follows:

$$f^*\left(v_i^{(j)} v_{i+1}^{(j)}\right) = \begin{cases} \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 1 & i = 1 \\ \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 4i - 2 & 2 \le i \le \left\lfloor \frac{p_j}{2} \right\rfloor - 2 \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 5 & i = \left\lfloor \frac{p_j}{2} \right\rfloor - 1 \\ \sum\limits_{k=1}^{j-1} 2p_k + 2p_j - 3(j-2) - 4 & i = \left\lfloor \frac{p_j}{2} \right\rfloor \\ \sum\limits_{k=1}^{j-1} 2p_k + 4p_j - 3(j-2) - 4i - 3 & \left\lfloor \frac{p_j}{2} \right\rfloor + 1 \le i \le p_j - 2 \end{cases}$$

and $f^*\left(v_{p_j-1}^{(j)} v_{p_j}^{(j)}\right) = \sum\limits_{k=1}^{j-1} 2p_k - 3(j-2) + 2.$

Hence, f is a super geometric mean labeling of $G'(p_1, p_2, \ldots, p_n)$. Thus it is a super geometric mean graph with $p_j \ne 4$ for $1 \le j \le n$. □

A super geometric mean labeling of $G'(7, 13, 11, 5)$ and $G'(8, 10, 12, 8)$ are shown in Figure 18.

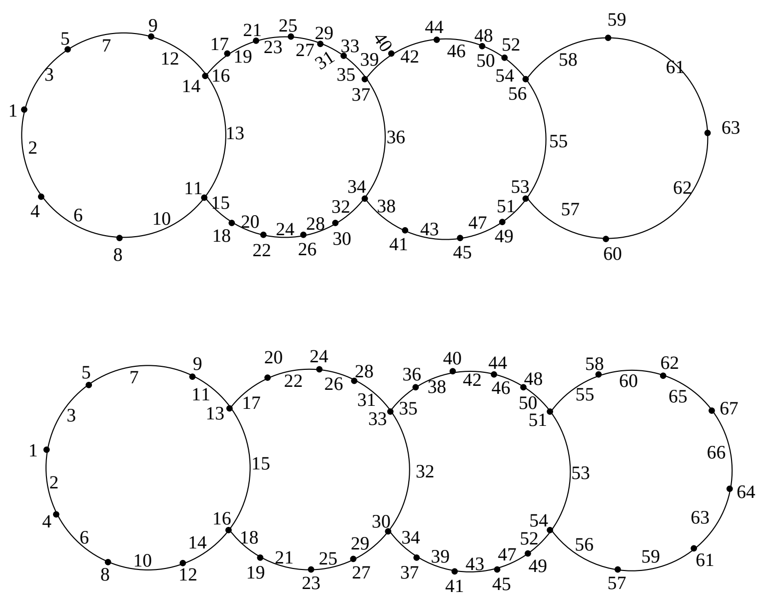

Figure 18

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