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A Note on (j,k)-Symmetric Harmonic Functions Defined by Sälägean Derivatives

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Abstract: In this paper, using the concept of (j,k)-symmetric functions and Sälägean operator we introduce the class $\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$ of functions $f = h + \overline{g}$ which are harmonic in $\mathcal{U}$. Coefficients bound for functions to be in this class. We also shown that this coefficient bound is also necessary for the class of functions of the form $f = h_\lambda + \overline{g_\lambda}$, belonging to the class $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. Distortion bounds, extreme points and neighborhood for the class $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ are also obtained.

Key Words: (j,k)-Symmetric functions, harmonic functions, Sälägean operator, distortion bounds, neighborhood.

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§1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simply connected domain $\mathcal{D} \in \mathbb{C}$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$, (see [14]).

Denote by $\mathcal{SH}$ the class of functions $f(z) = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \quad and \quad |z| < 1\}$, for which $f(0) = f_z(0) - 1 = 0$. Then for $f(z) = h + \overline{g} \in \mathcal{SH}$, we may express the analytic functions f and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad |b_1| < 1. \tag{1}$$

Note that $\mathcal{SH}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the

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co-analytic part of its members is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{2}$$

A function $f(z) = h + \overline{g}$ with h and g given by (1) is said to be harmonic starlike of order β for $(0 \leq \beta < 1$, for $|z| = r < 1$ if

$$\frac{\partial}{\partial\theta}(\arg f(re^{i\theta})) = \Im\left\{\frac{\frac{\partial}{\partial\theta} f(re^{i\theta})}{f(re^{i\theta})}\right\} = \Re\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \geq \beta.$$

The class of all harmonic starlike functions of order β is denoted by $\mathcal{S}^*_{\mathcal{H}}(\beta)$ and extensively studied by Jahangiri ([1]). The cases $\beta = 0$ and $b_1 = 1$ were studied by Silverman and Silvia ([2]) and Silverman ([6]).

Definition 1.1 *Let k be a positive integer. A domain $\mathcal{D}$ is said to be k-fold symmetric if a rotation of $\mathcal{D}$ about the origin through an angle $\frac{2\pi}{k}$ carries $\mathcal{D}$ onto itself. A function f is said to be k-fold symmetric in $\mathcal{U}$ if for every z in $\mathcal{U}$*

$$f(e^{\frac{2\pi i}{k}} z) = e^{\frac{2\pi i}{k}} f(z).$$

The family of all k-fold symmetric functions is denoted by $\mathcal{S}^k$ and for $k = 2$ we get class of the odd univalent functions.

The notion of (j,k)-symmetrical functions $(k = 2, 3, \cdots\ ;\ j = 0, 1, 2, \cdots, k-1)$ is a generalization of the notion of even, odd, k-symmetrical functions and also generalize the well-known result that each function defined on a symmetrical subset can be uniquely expressed as the sum of an even function and an odd function.

The theory of (j,k) symmetrical functions has many interesting applications, for instance in the investigation of the set of fixed points of mappings, for the estimation of the absolute value of some integrals, and for obtaining some results of the type of Cartan uniqueness theorem for holomorphic mappings ([12]).

Definition 1.2 *Let $\varepsilon = (e^{\frac{2\pi i}{k}})$ and $j = 0, 1, 2, \cdots, k-1$ where $k \geq 2$ is a natural number. A function $f : \mathcal{U} \mapsto \mathbb{C}$ is called (j,k)-symmetrical if*

$$f(\varepsilon z) = \varepsilon^j f(z), \qquad z \in \mathcal{U}.$$

We note that the family of all (j,k)-symmetric functions is denoted be $\mathcal{S}^{(j,k)}$. Also, $\mathcal{S}^{(0,2)}$, $\mathcal{S}^{(1,2)}$ and $\mathcal{S}^{(1,k)}$ are called even, odd and k-symmetric functions respectively. We have the following decomposition theorem.

Theorem 1.3([12]) *For every mapping $f : \mathcal{D} \mapsto \mathbb{C}$, and $\mathcal{D}$ is a k-fold symmetric set, there* *exists exactly the sequence of (j,k)- symmetrical functions $f_{j,k}$,*

$$f(z)=\sum_{j=0}^{k-1} f_{j,k}(z)$$

where

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f(\varepsilon^{v}z), \tag{3}$$

$$(f\in\mathcal{A};\ \ k=1,2,\cdots;j=0,1,2,\cdots,k-1)$$

From (3) we can get

$$f_{j,k}(z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}f(\varepsilon^{v}z)=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{-vj}\left(\sum_{n=1}^{\infty}a_n(\varepsilon^{v}z)^n\right),$$

then

$$f_{j,k}(z)=\sum_{n=1}^{\infty}\delta_{n,j}a_nz^n,\quad a_1=1 \qquad ,\delta_{n,j}=\frac{1}{k}\sum_{v=0}^{k-1}\varepsilon^{(n-j)v}=\begin{cases}1, & n=lk+j;\\ 0, & n\neq lk+j;\end{cases}, \tag{4}$$

Ahuja and Jahangiri ([3]) discussed the class $\mathcal{SH}(\beta)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1) and satisfying

$$\Im\left\{\frac{2\frac{\partial}{\partial\theta}f(re^{i\theta})}{f(re^{i\theta})-f(-re^{i\theta})}\right\}>\beta,\qquad 0\le\beta<1.$$

The authors ([13]) introduced and discussed the class $\mathcal{SH}_s^(j,k)(\beta)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1) and satisfying

$$\Im\left\{\frac{\frac{\partial}{\partial\theta}f(re^{i\theta})}{f_{j,k}(re^{i\theta})}\right\}>\beta,\qquad 0\le\beta<1.$$

For $f=h+\overline{g}$ where h and g are given by (1), Jahangiri ([15]) defined the modified Sälägean operator of f as

$$D^{\lambda}f(z)=D^{\lambda}h(z)+(-1)^{\lambda}\overline{D^{\lambda}g(z)},\quad \lambda=0,1,2,\cdots, \tag{5}$$

where

$$D^{\lambda}h(z)=z+\sum_{n=2}^{\infty}n^{\lambda}a_nz^n,\qquad D^{\lambda}g(z)=\sum_{n=1}^{\infty}n^{\lambda}b_nz^n. \tag{6}$$

Now using Sälägean operator D^{λ} and the concepts of $(j,k)-$symmetric points we define the following.

Definition 1.4 *For $0\le\beta<1$ and $k=1,2,3,\cdots,\ j=0,1,\cdots,k-1,\lambda\in\mathbb{N}_0,\ b\neq 0,$*

let $\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$ *denote the class of harmonic functions* f *of the form (1) which satisfy the condition*

$$\Re\left\{1-\frac{2}{b}+\frac{2D^{\lambda+1}f(z)}{bD^{\lambda}f_{j,k}(z)}\right\}>\beta, \tag{7}$$

where

$$D^{\lambda}f_{j,k}(z)=\delta_{1,j}z+\sum_{n=2}^{\infty}n^{\lambda}\delta_{n,j}a_nz^n+(-1)^{\lambda}\overline{\sum_{n=1}^{\infty}n^{\lambda}\delta_{n,j}b_nz^n} \tag{8}$$

and $\delta_{n,j}$ *is defined by (4).*

Let $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ denote the subclass of $\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$ consist of harmonic functions of $f_{\lambda}=h_{\lambda}+\overline{g_{\lambda}}$ such that h_{λ} and $\overline{g_{\lambda}}$ are of the form

$$h_{\lambda}(z)=z-\sum_{n=2}^{\infty}|a_n|z^n,\qquad g_{\lambda}(z)=(-1)^{\lambda}\sum_{n=1}^{\infty}|b_n|z^n. \tag{9}$$

Since

$$rrh_{j,k}=\delta_{1,j}z+\sum_{n=2}^{\infty}\delta_{n,j}a_nz^n,\qquad g_{j,k}(z)=\overline{\sum_{n=1}^{\infty}\delta_{n,j}b_nz^n}, \tag{10}$$

also, let $f_{\lambda,j,k}h_{\lambda,j,k}+\overline{g_{\lambda,j,k}}$ such that $h_{\lambda,j,k}$ and $\overline{g_{\lambda,j,k}}$ are of the form

$$h_{\lambda,j,k}(z)=\delta_{-1,j}z-\sum_{n=2}^{\infty}\delta n,j|a_n|z^n,\qquad g_{\lambda,j,k}(z)=(-1)^{\lambda}\sum_{n=1}^{\infty}\delta n,j|b_n|z^n, \tag{11}$$

where $\delta n,j$ is given by (4).

The following special cases are of interest

(1) $\mathcal{SH}_s^{1,k}(\beta,0,2)=\mathcal{SH}_s^{k}(\beta)$, the class introduced by AL-Shaqsi and Darus in [15];

(2) $\mathcal{SH}_s^{1,2}(\beta,0,2)=\mathcal{SH}(\beta)$, the class introduced byAhuja and Jahangiri in [3];

(3) $\mathcal{SH}_s^{1,1}(\beta,0,2)=\mathcal{S_H}^{*}(\beta)$ the class introduced by Jahangiri in [1];

(4) $\mathcal{SH}_s^{1,1}(0,0,2)=\mathcal{S_H}^{*}$ the class introduced by Silverman and Silvia in [2];

(5) $\mathcal{SH}_s^{1,1}(\beta,\lambda,2)=\mathcal{SH}(\beta,\lambda)$ the class introduced by Jahangiri in [5];

(6) $\mathcal{SH}_s^{j,k}(\beta,0,2)=\mathcal{SH}_s^{j,k}(\beta)$ the class introduced Fuad Alsarari and S.Latha ([13]).

§2. Coefficient Bounds

Theorem 2.1 *If* $f=h+\overline{g}$ *with* h *and* g *given by* (**??**) *and* $D^{\lambda}f_{j,k}$ *is defined by (8)*

$$\sum_{n=1}^{\infty}n^{\lambda}\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|+\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|\right\}\leq 2, \tag{12}$$

where $b\neq 0, a_1=1,\quad 0\leq\beta<1,\quad \lambda\in\mathbb{N}_0$ *and* $\delta_{n,j}$ *is defined by (4), then* f *is sense-preserving, harmonic univalent in* $\mathcal{U}$*, and* $f\in\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$.

Proof If $z_1 \neq z_2$, then

$$\begin{aligned}
\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|,\\
&= 1-\left|\frac{\sum_{n=1}^{\infty} b_n(z_1^n-z_2^n)}{(z_1-z_2)+\sum_{n=2}^{\infty} a_n(z_1^n-z_2^n)}\right|,\\
&> 1-\frac{\sum_{n=1}^{\infty}|b_n|}{1-\sum_{n=2}^{\infty}|a_n|},\\
&> 1-\frac{\sum_{n=1}^{\infty}\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|b_n|}{1-\sum_{n=2}^{\infty}\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|a_n|}\geq 0,
\end{aligned}$$

which proves univalence. Note that f is sense-preserving in $\mathcal{U}$. This is because

$$\begin{aligned}
|h'(z)| &\geq 1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1} > 1-\sum_{n=2}^{\infty}\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|a_n|\\
&\geq \sum_{n=1}^{\infty}\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|b_n| > \sum_{n=1}^{\infty}\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|b_n||z|^{n-1}\\
&\geq \sum_{n=1}^{\infty} n|b_n||z|^{n-1} \geq |g'(z)|.
\end{aligned}$$

Using the fact $\Re\{w\} > \beta$ if and only if $|1-\beta+w| \geq |1+\beta-w|$ it suffices that. Let

$$\Re\left\{\frac{A(z)}{B(z)}\right\} = \Re\left\{\frac{(b-2)D^{\lambda}f_{j,k}(z)+2D^{\lambda+1}f(z)}{bD^{\lambda}f_{j,k}(z)}\right\} > \beta, \tag{13}$$

it suffices to show that

$$|A(z)+(1-\beta)B(z)| - |A(z)-(1+\beta)B(z)| \geq 0.$$

Now we have

$$\begin{aligned}
|(1-\beta)B(z)+A(z)| &= \left|(1-\beta)bD^{\lambda}f_{j,k}(z)+(b-2)D^{\lambda}f_{j,k}(z)+2D^{\lambda+1}f(z)\right|\\
&= \left|[b(2-\beta)-2]D^{\lambda}f_{j,k}(z)+2D^{\lambda+1}f(z)\right|\\
&= \Bigg|(b(2-\beta)-2)[\delta_{1,j}z+\sum_{n=2}^{\infty} n^{\lambda}\delta_{n,j}a_n z^n+(-1)^{\lambda}\sum_{n=1}^{\infty} n^{\lambda}\delta_{n,j}b_n z^n]\\
&\quad + 2[z+\sum_{n=2}^{\infty} n^{\lambda+1}a_n z^n-(-1)^{\lambda}\sum_{n=1}^{\infty} n^{\lambda+1}b_n z^n]\Bigg|.
\end{aligned}$$

So

$$\begin{aligned}
|(1-\beta)B(z)+A(z)| \geq [(b(2-\beta)-2)\delta_{1,j}+2]|z| - \sum_{n=2}^{\infty}[2n+(b(2-\beta)\\
-2)\delta_{n,j}]n^{\lambda}|a_n||z|^n - \sum_{n=1}^{\infty}[2n-(b(2-\beta)-2)\delta_{n,j}]n^{\lambda}|b_n||z|^n.
\end{aligned} \tag{14}$$

Also

$$
\begin{aligned}
|(1+\beta)B(z)-A(z)| &= \left|(1+\beta)bD^{\lambda}f_{j,k}(z)-(b-2)D^{\lambda}f_{j,k}(z)-2D^{\lambda+1}f(z)\right| \\
&= \Bigg|(2+b\beta)[\delta_{1,j}z+\sum_{n=2}^{\infty}n^{\lambda}\delta_{n,j}a_nz^n+(-1)^{\lambda}\sum_{n=1}^{\infty}n^{\lambda}\delta_{n,j}b_nz^n] \\
&\quad -2[z+\sum_{n=2}^{\infty}n^{\lambda+1}a_nz^n-(-1)^{\lambda}\sum_{n=1}^{\infty}n^{\lambda+1}b_nz^n]\Bigg|.
\end{aligned}
$$

So

$$
\begin{aligned}
|(1+\beta)B(z)-A(z)| &\leq [(2+b\beta)\delta_{1,j}-2]|z|+\sum_{n=2}^{\infty}[2n-(2+b\beta)\delta_{n,j}]n^{\lambda}|a_n||z|^n+\sum_{n=1}^{\infty} \\
&\times [2n+(2+b\beta)\delta_{n,j}]n^{\lambda}|b_n||z|^n. \qquad (15)
\end{aligned}
$$

By (14) and (15), we have

$$
\begin{aligned}
&|(1-\beta)B(z)+A(z)|-|(1+\beta)B(z)-A(z)| \\
&\geq 2\{[b(1-\beta)-2]\delta_{1,j}+2\}|z|-2\sum_{n=2}^{\infty}\{[b(1-\beta)-2]\delta_{n,j}+2n\}n^{\lambda}|a_n||z|^n \\
&\quad -2\sum_{n=1}^{\infty}\{[2-b(1-\beta)]\delta_{n,j}+2n\}n^{\lambda}|b_n||z|^n \\
&\geq 2\{[b(1-\beta)-2]\delta_{1,j}+2\} \\
&\quad \times\left\{1-\sum_{n=2}^{\infty}\frac{\{[b(1-\beta)-2]\delta_{n,j}+2n\}}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|a_n|-\sum_{n=1}^{\infty}\frac{\{[2-b(1-\beta)]\delta_{n,j}+2n\}}{[b(1-\beta)-2]\delta_{1,j}+2}n^{\lambda}|b_n|\right\}.
\end{aligned}
$$

This last expression is nonnegative by (12), and so the proof is complete. □

The harmonic function

$$
f(z)=z+\sum_{n=2}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{n,j}+2n\}n^{\lambda}}x_nz^n+\sum_{n=1}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[2-b(1-\beta)]\delta_{n,j}+2n\}n^{\lambda}}\overline{y_nz^n}, \quad (16)
$$

where $b\neq 0,\ \ 0\leq\beta<1,\ \ \lambda\in\mathbb{N}_0$ and $\delta_{n,j}$ is defined by (4), and $\sum_{n=2}^{\infty}|x_n|+\sum_{n=1}^{\infty}|y_n|=1$ shows that the coefficient given by (12) is sharp, and the functions of form (160 are in $\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$.

We next show that condition (12) is also necessary for functions in $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$.

Theorem 2.2 *Let $f_\lambda=h_\lambda+\overline{g_\lambda}$ be given by (9). Then $f_\lambda\in\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ if and only if*

$$
\sum_{n=1}^{\infty}n^{\lambda}\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|+\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|\right\}\leq 2, \qquad (17)
$$

where $b>0, a_1=1,\ \ 0\leq\beta<1,\ \ \lambda\in\mathbb{N}_0$ and $\delta_{n,j}$ is defined by (4).

Proof Since $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)\subset\mathcal{SH}_s^{j,k}(\beta,\lambda,b)$, we only need to prove the necessity part of the theorem. To this end, for functions f_λ and $f_{\lambda,j,k}$ of the form (9) and (11) respectively, we notice that the condition (7) is equivalent to

$$\Re\left\{\frac{[b(1-\beta)-2]D^\lambda f_{\lambda,j,k}(z)+2D^{\lambda+1}(z)}{bD^\lambda f_{\lambda,j,k}(z)}\right\}\geq 0$$

$$\Re\left\{\frac{[(b(1-\beta)-2)\delta_{1,j}+2]z-\sum_{n=2}^\infty[2n+(b(1-\beta)-2)\delta_{n,j}]n^\lambda|a_n|z^n-\sum_{n=1}^\infty[2n-(b(1-\beta)-2)\delta_{n,j}]n^\lambda|b_n|\overline{z^n}}{b[\delta_{n,1}z-\sum_{n=2}^\infty n^\lambda\delta_{n,j}|a_n|z^n+\sum_{n=1}^\infty n^\lambda\delta_{n,j}]|b_n|\overline{z^n}}\right\}\geq 0.$$

The above required in the above inequality must hold for all values of z in $\mathcal{U}$. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we have

$$\frac{[(b(1-\beta)-2)\delta_{1,j}+2]-\sum_{n=2}^\infty[2n+(b(1-\beta)-2)\delta_{n,j}]n^\lambda|a_n|r^{n-1}-\sum_{n=1}^\infty[2n-(b(1-\beta)-2)\delta_{n,j}]n^\lambda|b_n|r^{n-1}}{b[\delta_{n,1}z-\sum_{n=2}^\infty n^\lambda\delta_{n,j}|a_n|r^{n-1}+\sum_{n=1}^\infty n^\lambda\delta_{n,j}|b_n|r^{n-1}]}\geq 0.$$

If the condition (7) does not hold, then the numerator in the above inequality is negative for r sufficiently close to 1. Hence there exist $z_0=r_0\in(0,1)$ for which the quotient in the above inequality is negative. This contradicts the required condition for $f_\lambda\in\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ so the proof is complete. □

§3. Distortion Bounds

Theorem 3.1 *Let $f_\lambda\in\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. Then for $|z|=r<1$, one has*

$$|f_\lambda(z)|\geq(1-|b_1|)r-r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right),$$

$$|f_\lambda(z)|\leq(1-|b_1|)r+r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right).$$

Proof Let $f_\lambda\in\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. Taking the obsolete vale of f_λ, we obtain

$$\begin{aligned}
&|f_\lambda(z)|=\left|z-\sum_{n=2}^\infty a_nz^n+(-1)^\lambda\sum_{n=1}^\infty b_nz^n\right|\\
&\geq(1-|b_1|)r-\sum_{n=2}^\infty(|a_n|+|b_n|)r^n\geq(1-|b_1|)r-r^2\sum_{n=2}^\infty(|a_n|+|b_n|)\\
&\geq(1-|b_1|)r-r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(\sum_{n=2}^\infty\frac{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}{b(1-\beta)-2]\delta_{1,j}+2}(|a_n|+|b_n|)\right)\\
&\geq(1-|b_1|)r-r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\sum_{n=2}^\infty n^\lambda\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|\right\}\\
&\quad-r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\sum_{n=2}^\infty n^\lambda\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|\right\}\\
&\geq(1-|b_1|)r-r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right).
\end{aligned}$$

Also

$$
\begin{aligned}
|f_\lambda(z)| &= \left|z-\sum_{n=2}^{\infty}a_nz^n+(-1)^\lambda\sum_{n=1}^{\infty}b_nz^n\right|\\
&\le (1-|b_1|)r+\sum_{n=2}^{\infty}(|a_n|+|b_n|)r^n \le (1-|b_1|)r+r^2\sum_{n=2}^{\infty}(|a_n|+|b_n|)\\
&\le (1-|b_1|)r+r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(\sum_{n=2}^{\infty}\frac{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}{b(1-\beta)-2]\delta_{1,j}+2}(|a_n|+|b_n|)\right)\\
&\le (1-|b_1|)r+r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\sum_{n=2}^{\infty}n^\lambda\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|\right\}\\
&\quad +r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\sum_{n=2}^{\infty}n^\lambda\left\{\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|\right\}\\
&\le (1-|b_1|)r+r^2\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^\lambda}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right). \qquad \square
\end{aligned}
$$

The following covering result follows form left-hand inequality in Theorem 3.1.

Corollary 3.2 *Let* $f_\lambda = h_\lambda + \overline{g_\lambda}$ *be given by (9) are in* $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$*, then*

$$
\left\{w:|w|<\frac{2^{\lambda+1}[4+\{b(1-\beta)-2\}\delta_{2,j}]-[b(1-\beta)-2]\delta_{1,j}+2}{2^\lambda[4+\{b(1-\beta)-2\}\delta_{2,j}]}\right.
\left.-\frac{2^{\lambda+1}[4+\{b(1-\beta)-2\}\delta_{2,j}-[2+\{2-b(1-\beta)\}\delta_{1,j}]]}{2^\lambda[4+\{b(1-\beta)-2\}\delta_{2,j}]}|b_1|\right\}\subset f_\lambda(\mathcal{U}).
$$

Next we determine the extreme points of closed convex hulls of $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ denoted by $clco\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. $[2n+\{b(1-\beta)-2\}\delta_{n,j}], [b(1-\beta)-2]\delta_{1,j}+2, [2n+\{2-b(1-\beta)\}\delta_{n,j}]$.

Theorem 3.3 *Let* $f_\lambda = h_\lambda + \overline{g_\lambda}$ *be given by (9). Then* $f_\lambda \in \overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ *if and only if*

$$
f_\lambda(z)=\sum_{n=1}^{\infty}(X_nh_{\lambda_n}(z)+Y_ng_{\lambda_n}(z)),
$$

where $h_{\lambda_1}=z$,

$$
\begin{aligned}
h_{\lambda_n}(z) &= z-([b(1-\beta)-2]\delta_{1,j}+2)/n^\lambda[2n+\{b(1-\beta)-2\}\delta_{n,j}]z^n,\ n=2,3,4,\cdots,\\
g_{\lambda_n}(z) &= z+(-1)^\lambda([b(1-\beta)-2]\delta_{1,j}+2)/n^\lambda[2n+\{2-b(1-\beta)\}\delta_{n,j}]\overline{z}^n,\ n=1,2,3,\cdots
\end{aligned}
$$

and

$$
\sum_{n=1}^{\infty}(X_n+Y_n)=1,\ X_n\ge 0,\ Y_n\ge 0.
$$

In particular, the extreme point of $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ *are* $\{h_{\lambda_n}\}$ *and* $\{g_{\lambda_n}\}$.

Proof Since

$$\begin{aligned} f_\lambda(z) &= \sum_{n=1}^{\infty}(X_n h_{\lambda_n}(z)+Y_n g_{\lambda_n}(z)) \\ &= \sum_{n=1}^{\infty}(X_n+Y_n)z-\sum_{n=2}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{n^\lambda[2n+\{b(1-\beta)-2\}\delta_{n,j}]}X_n z^n \\ &\quad +(-1)^\lambda\sum_{n=1}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{n^\lambda[2n+\{2-b(1-\beta)\}\delta_{n,j}]}Y_n\overline{z}^n. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=2}^{\infty}n^\lambda\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n| &+ \sum_{n=1}^{\infty}n^\lambda\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n| \\ &= \sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n=1-X_1\leq 1, \end{aligned}$$

and so $f_\lambda \in clco\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$.

Conversely suppose that $f_\lambda \in clco\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. Setting

$$X_n=\frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|a_n|, \qquad 0\leq X_n\leq 1, \quad n=2,3,4,\cdots,$$

$$Y_n=\frac{[2n+\{2-b(1-\beta)\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_n|, \qquad 0\leq Y_n\leq 1, \quad n=1,2,3,\cdots,$$

and

$$X_1=1-\sum_{n=2}^{\infty}X_n-\sum_{n=2}^{\infty}Y_n.$$

Therefore f_λ can be written as

$$\begin{aligned} f_\lambda(z) &= z-\sum_{n=2}^{\infty}|a_n|z^n+(-1)^\lambda-\sum_{n=1}^{\infty}|b_n|\overline{z}^n \\ &= z-\sum_{n=2}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{n^\lambda[2n+\{b(1-\beta)-2\}\delta_{n,j}]}X_n z^n \\ &\quad +(-1)^\lambda-\sum_{n=1}^{\infty}\frac{[b(1-\beta)-2]\delta_{1,j}+2}{n^\lambda[2n+\{2-b(1-\beta)\}\delta_{n,j}]}Y_n\overline{z}^n \\ &= z-\sum_{n=2}^{\infty}(h_{\lambda_n}(z)-z)X_n+(-1)^\lambda\sum_{n=1}^{\infty}(g_{\lambda_n}(z)-z)Y_n \\ &= \sum_{n=2}^{\infty}h_{\lambda_n}(z)X_n+\sum_{n=1}^{\infty}g_{\lambda_n}(z)Y_n+z\left(1-\sum_{n=2}^{\infty}X_n-\lambda\sum_{n=1}^{\infty}Y_n\right) \\ &= \sum_{n=1}^{\infty}\left(h_{\lambda_n}(z)X_n+g_{\lambda_n}(z)Y_n\right). \end{aligned}$$

□

§4. Neighborhood Result

In this section, we will prove that the functions in neighborhood of $\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$ are starlike harmonic functions.

Using (4), we define the ϱ-neighborhood of function $f \in \tau H$

$$\mathcal{N}_\rho(f) = \left\{ F(z) = z - \sum_{n=2}^{\infty} A_n z^n - \sum_{n=1}^{\infty} B_n \overline{z}^n,\ \sum_{n=2}^{\infty} n[|a_n - A_n| + |b_n - B_n| + |b_1 - B_1| \le \rho \right\},$$

where $\rho > 0$.

Theorem 4.1 *Let*

$$\rho = \frac{2^\lambda[\{2+\{\frac{b}{2}(1-\beta)-1\}\delta_{2,j}] - [b(1-\beta)-2]\delta_{1,j} + 2 - \{2^\lambda[2+\{\frac{b}{2}(1-\beta)-1\}\delta_{2,j} - [2+\{2-b(1-\beta)\}\delta_{1,j}]]\}}{2^\lambda[\{2+\{\frac{b}{2}(1-\beta)-1\}\delta_{2,j}]}.$$

Then $N_\rho(\overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)) \subset \tau H$.

Proof Suppose $f \in \overline{\mathcal{SH}}_s^{j,k}(\beta,\lambda,b)$. Let $F_\lambda = H_\lambda + \overline{G_\lambda} \in N_\rho(f_\lambda)$, where $H_\lambda = z - \sum_{n=2}^{\infty} A_n z^n$ and $G_\lambda = (-1)^\lambda \sum_{n=1}^{\infty} B_n \overline{z}^n$, we need to show that $F_\lambda \in \tau H$. In other words, it suffices to show that F_λ satisfies the condition $\tau(F) = \sum_{n=2}^{\infty} n[|A_n| + |B_n|] + |B_1| \le 1$. We observe that

$$\begin{aligned}
\tau(F) &= \sum_{n=2}^{\infty} n[|A_n| + |B_n|] + |B_1| \\
&= \sum_{n=2}^{\infty} n[|A_n - a_n + a_n| + |B_n b_n + b_n|] + |B_1 - b_1 + b_1| \\
&= \sum_{n=2}^{\infty} n[|A_n - a_n| + |B_n - b_n|] + \sum_{n=2}^{\infty} n[|a_n| + |b_n|] + |B_1 - b_1| + |b_1| \\
&= \rho + |b_1| + \sum_{n=2}^{\infty} n[|a_n| + |b_n|] \\
&= \rho + |b_1| + \frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^{\lambda-1}} \left(\sum_{n=2}^{\infty} \frac{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^{\lambda-1}}{b(1-\beta)-2]\delta_{1,j}+2} (|a_n| + |b_n|) \right) \\
&\le \rho + |b_1| + \frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^{\lambda-1}} \sum_{n=2}^{\infty} n^\lambda \left\{ \frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2} |a_n| \right\} \\
&\quad + \frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^{\lambda-1}} \sum_{n=2}^{\infty} n^\lambda \left\{ \frac{[2n+\{b(1-\beta)-2\}\delta_{n,j}]}{[b(1-\beta)-2]\delta_{1,j}+2} |b_n| \right\} \\
&\le \rho + |b_1| + \frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^{\lambda-1}} \left(1 - \frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2} |b_1| \right).
\end{aligned}$$

Now this last expression is never greater than one if

$$\begin{aligned}\rho &\le 1-|b_1|-\frac{[b(1-\beta)-2]\delta_{1,j}+2}{\{[b(1-\beta)-2]\delta_{2,j}+4\}2^{\lambda-1}}\left(1-\frac{[2+\{2-b(1-\beta)\}\delta_{1,j}]}{[b(1-\beta)-2]\delta_{1,j}+2}|b_1|\right)\\ &=\frac{2^{\lambda}[\{2+\{\frac{b}{2}(1-\beta)-1\}\delta_{2,j}]-[b(1-\beta)-2]\delta_{1,j}+2-\{2^{\lambda}[2+\{\frac{b}{2}(1-\beta)-1\}\delta_{2,j}-[2+\{2-b(1-\beta)\}\delta_{1,j}]]\}}{2^{\lambda}[\{2+\{\frac{b}{2}(1-\beta)-1\}\delta_{2,j}]}.\end{aligned}$$

□

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