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The Neighborhood Pseudochromatic Number of a Graph

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Abstract: A *pseudocoloring* of G is a coloring of G in which adjacent vertices can receive the same color. The *neighborhood pseudochromatic number* of a non-trivial connected graph G, denoted $\psi_{nhd}(G)$, is the maximum number of colors used in a pseudocoloring of G such that every vertex has at least two vertices in its closed neighborhood receiving the same color. In this paper, we obtain $\psi_{nhd}(G)$ of some standard graphs and characterize all graphs for which $\psi_{nhd}(G)$ is $1, 2, n-1$ or n.

Key Words: Coloring, pseudocoloring, neighborhood, domination.

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§1. Introduction

Historically, the coloring terminology comes from the map-coloring problem which involves coloring of the countries in a map in such a way that no two adjacent countries are colored with the same color. The committee scheduling problem is another problem which can be rephrased as a vertex coloring problem. As such, the concept of graph coloring motivates varieties of graph labelings with an addition of various conditions and has a wide range of applications - channel assignment in wireless communications, traffic phasing, fleet maintenance and task assignment to name a few. More applications of graph coloring can be found in [2,17]. A detailed discussion on graph coloring and some of its variations can be seen in [1,5-8,16,18].

Throughout this paper, we consider a graph G which is simple, finite and undirected. A vertex k-coloring of G is a surjection $c : V(G) \to \{1, 2, \cdots, k\}$. A vertex k-coloring c of a graph G is said to be a *proper k-coloring* if vertices of G receive different colors whenever they are adjacent in G. Thus for a proper k-coloring, we have $c(u) \neq c(v)$ whenever $uv \in E(G)$. The minimum k for which there is a proper k-coloring of G is called the chromatic number of G, denoted $\chi(G)$. It can be seen that a proper k-coloring of G is simply a vertex partition of $V(G)$ into k independent subsets called color classes. For any vertex $v \in V(G)$, $N[v] = N(v) \bigcup \{v\}$

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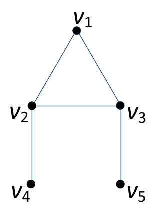
where $N(v)$ is the set of all the vertices in $V(G)$ which are adjacent to v. As discussed in [14], a dominating set S of a graph $G(V, E)$ is a subset of V such that every vertex in V is either an element of S or is adjacent to an element of S. The minimum cardinality of a dominating set of a graph G is called its dominating number, denoted $\gamma(G)$. Further, a dominating set of G with minimum cardinality is called a γ-set of G.

As introduced in 1967 by Harary et al. [10, 11], a *complete k-coloring* of a graph G is a proper k-coloring of G such that, for any pair of colors, there is at least one edge of G whose end vertices are colored with this pair of colors. The greatest k for which G admits a complete k-coloring is the *achromatic number* $\alpha(G)$. In 1969, while working on the famous Nordaus - Gaddum inequality [16], R. P. Gupta [9] introduced a new coloring parameter, called the *pseudoachromatic number*, which generalizes the achromatic number. A pseudo k-coloring of G is a k-coloring in which adjacent vertices may receive same color. A *pseudocomplete k-coloring* of a graph G is a pseudo k-coloring such that, for any pair of distinct colors, there is at least one edge whose end vertices are colored with this pair of colors. The *pseudoachromatic number* $\psi(G)$ is the greatest k for which G admits a pseudocomplete k-coloring. This parameter was later studied by V. N. Bhave [3], E. Sampath Kumar [19] and V. Yegnanarayanan [20]. Motivated by the above studies, we introduce here a new graph invariant and study some of its properties in this paper. We use standard notations, the terms not defined here may be found in [4, 12, 14, 15].



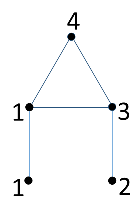
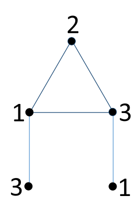


Figure 1. The graph G **Figure 2. A pseudo 4-coloring of G**



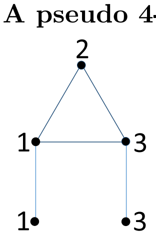


Figure 3. A complete 3-coloring of G **Figure 4. A pseudocomplete 3-coloring of G**

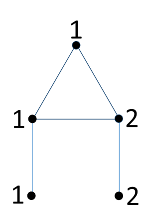


Figure 5. A neighborhood pseudo 2-coloring of G

Definition 1.1 *A neighborhood pseudo k-coloring of a connected graph $G(V, E)$ is a pseudo k-coloring $c : V(G) \to \{1, 2, \cdots, k\}$ of G such that for every $v \in V$, $c|_{N[v]}$ is not an injection.*

In other words, a connected graph $G = (V, E)$ is said to have a neighborhood pseudo k-coloring if there exists a pseudo k-coloring c of G such that $\forall v \in V(G)$, $\exists u, w \in N[v]$ with $c(u) = c(w)$.

Definition 1.2 *The maximum k for which G admits a neighborhood pseudo k-coloring is called the neighborhood pseudochromatic number of G, denoted $\psi_{nhd}(G)$. Further, a coloring c for which k is maximum is called a maximal neighborhood pseudocoloring of G.*

The Figures 1-5 show a graph G and its various colorings. The above Definition 1.1 can be extended to disconnected graphs as follows.

Definition 1.3 *If G is a disconnected graph with k components $H_1, H_2, \ldots, H_k$, then*

$$\psi_{nhd}(G) = \sum_{i=1}^{k} \psi_{nhd}(H_i).$$

Observation 1.4 For any graph G of order n, $1 \leq \psi_{nhd}(G) \leq n$. In particular, if G is connected, then $1 \leq \psi_{nhd}(G) \leq n - 1$.

Observation 1.5 If H is any connected subgraph of a graph G, then $\psi_{nhd}(H) \leq \psi_{nhd}(G)$.

§2. Preliminary Results

In this section, we study the neighborhood pseudochromatic number of standard graphs. We also obtain certain bounds on the neighborhood pseudochromatic number of a graph. We end the section with a few characterizations. We first state the following theorem whose proof is immediate.

Theorem 2.1 *If n is an integer and $n_i \in Z^+$ for each $i = 1, 2, \cdots$,*

(1) $\psi_{nhd}(\overline{K}_n) = n$ *for* $n \geq 1$;

(2) $\psi_{nhd}(K_n) = n - 1$ *for* $n \geq 2$;

(3) $\psi_{nhd}(P_n) = \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 2$;

(4) $\psi_{nhd}(C_n) = \begin{cases} 2 & \text{for } n = 3 \\ \lfloor \frac{n}{2} \rfloor & \text{for } n > 3 \end{cases}$

(5) $\psi_{nhd}(K_{1,n}) = 1$ *for* $n \geq 1$;

(6) $\psi_{nhd}(K_{n_1, n_2, \ldots, n_k}) = \sum\limits_{i=1}^{k} n_i - 2$ *where each* $n_i \geq 2$.

Corollary 2.2 *For any graph G having k components, $\psi_{nhd}(G) \geq k$.*

Corollary 2.3 *For a connected graph G with diameter d, $\psi_{nhd}(G) \geq \lfloor \frac{d}{2} \rfloor$.*

Corollary 2.4 *If G is a graph with k non-trivial components $H_1, H_2, \ldots, H_k$ and $\omega(H_i)$ is the clique number of H_i, then*

$$\psi_{nhd}(G) \geq \sum_{i=1}^{k} \omega(H_i) - k.$$

Corollary 2.5 *$\psi_{nhd}(G) \leq n-k$ for a connected graph G of order $n \geq 3$ with k pendant vertices.*

Lemma 2.6 *For a connected graph G, $\psi_{nhd}(G) \geq 2$ if and only if G has a subgraph isomorphic to C_3 or P_4.*

Proof If G contains C_3 or P_4, from Observation 1.5 and Theorem 2.1, $\psi_{nhd}(G) \geq 2$. Conversely, let G be a connected graph with $\psi_{nhd}(G) \geq 2$. If possible, suppose that G has neither a C_3 or nor a P_4 as its subgraph, then G is isomorphic to $K_{1,n}$. But then, $\psi_{nhd}(G) = 1$, a contradiction by Theorem 2.1. □

As a consequence of Lemma 2.6, we have a consequence following.

Corollary 2.7 *A non-trivial graph G is a star if and only if $\psi_{nhd}(G) = 1$.*

Theorem 2.8 *A graph G of order n is totally disconnected if and only if $\psi_{nhd}(G) = n$.*

Proof If a graph G of order n is totally disconnected, then by Theorem 2.1, $\psi_{nhd}(G) = n$. Conversely, if G is not totally disconnected, then G has an edge, say, e. Now for an end vertex of e, at least one color should repeat in G, so $\psi_{nhd}(G) < |V| = n$. Hence the theorem. □

§3. Characterization of a Graph G with $\psi_{nhd}(G) = n - 1$

Theorem 3.1 *For a connected graph G of order n, $\psi_{nhd}(G) = n - 1$ if and only if $G \cong G_1 + P_2$ for some graph G_1 of order $n - 2$.*

Proof Let G_1 be any graph on $n - 2$ vertices and $G = G_1 + P_2$. By Observation 1.4, $\psi_{nhd}(G) \leq n - 1$. Now to prove the reverse inequality, let $V(G) = \{v_1, v_2, \cdots, v_{n-2}, v_{n-1}, v_n\}$ with v_1, v_2 being the vertices of P_2. Define a coloring $c : V(G) \to \{1, 2, \cdots, n - 1\}$ as follows:

$$c(v_i) = \begin{cases} 1 & \text{if} \quad i = 1, 2 \\ i - 1 & \text{otherwise} \end{cases}$$

It can be easily seen that c is a neighborhood pseudo k-coloring of G with $k = n - 1$ implies that $\psi_{nhd}(G) \geqslant n - 1$. Hence $\psi_{nhd}(G) = n - 1$.

Conversely, let $G = (V, E)$ be a connected graph of order n with $\psi_{nhd}(G) = n - 1$. Thus there exists a neighborhood pseudo k-coloring, say c with $k = n - 1$ colors. This implies that all vertices but two in V receive different colors under c. Without loss of generality, let the only two vertices receiving the same color be v_1 and v_2 and other $n-2$ vertices of G be $v_3, v_4, \cdots, v_n$.

Now, for each i, $3 \leqslant i \leqslant n$, we have known that $c|_{N[v_i]}$ is an injection unless both v_1 and v_2 are in $N[v_i]$. Thus each v_i is adjacent to both v_1 and v_2 in G. Further, if v_1 is not adjacent to v_2, then, as c assigns $n-1$ colors to the graph $G - \{v_2\}$, we get that $c|_{N[v_1]}$ is an injection from $V(G)$ onto $\{1, 2, \cdots, n-1\}$, which is a contradiction to the fact that c is a neighborhood pseudo $n-1$ coloring of G. Thus, $G \cong P_2 + G_1$ for some graph G_1 on $n-2$ vertices. □

§4. A Bound in Terms of the Domination Number

In this section, we establish a bound on the neighborhood pseudochromatic number of a graph in terms of its domination number. Using this result, we give a characterization of graphs G with $\psi_{nhd}(G) = 2$.

Lemma 4.1 *Every connected graph $G(V, E)$ has a γ-set S satisfying the property that for every $v \in S$, there exists a vertex $u \in V - S$ such that $N(u) \bigcap S = \{v\}$.*

Proof Consider any γ-set S of a connected graph G. We construct a γ-set with the required property as follows. Firstly, we obtain a γ-set of G with the property that $deg_G(v) \geq 2$ whenever $v \in S$. Let S_1 be the set of all pendant vertices of G in S. If $S_1 = \emptyset$, then S itself is the required set. Otherwise, consider the set $S_2 = (S - S_1) \bigcup_{v \in S_1} N(v)$. It is easily seen that S_2 is a dominating set. Also, $|S| = |S_2|$ since each vertex of degree 1 in S_1 is replaced by a unique vertex in $V - S$. Otherwise, at least two vertices in S_1, say u and v, are replaced by a unique vertex in $V - S$, say w, in which case $S' = (S - \{u, v\}) \bigcup \{w\}$ is a dominating set of G with $|S'| < |S|$, a contradiction to the fact that S is a γ-set. Further, $deg(v) \geq 2$ for all $v \in S_2$ failing which G will not remain connected. In this case, S_2 is the required set. Now, we replace S by S_2 and proceed further.

If for all $v \in S$, there exists $u \in V - S$ such that $N(u) \bigcap S = \{v\}$, then we are done with the proof. If not, let $D = \{v \in S : N(u) \bigcap S - \{v\} \neq \emptyset,\ u \in N(v)\}$. Then, for each vertex v in D, every vertex $u \in N(v)$ is dominated by some vertex $w \in S - \{v\}$. We now claim that w is adjacent to another vertex $x \neq u \in V - S$. Otherwise, $(S - \{v, w\}) \bigcup \{u\}$ is a dominating set having lesser elements than in S, again a contradiction. Now, replace S by $(S - \{v\}) \bigcup \{u\}$. Repeating this procedure for every vertex in D will provide a γ-set S of G with the property that for all $v \in S$, there exists $u \in V - S$ such that $N(u) \bigcap S = \{v\}$. □

Theorem 4.2 *For any graph G, $\psi_{nhd}(G) \geq \gamma(G)$.*

Proof Let $G = (V, E)$ be a connected graph with $V = \{v_1, v_2, \cdots, v_n\}$ with $\gamma(G) = k$. By Lemma 4.1, G has a γ-set, say S, satisfying the property that for all $v \in S$, there exists $u \in V - S$ such that $N(u) \bigcap S = \{v\}$. Without loss in generality, we take $S = \{v_1, v_2, \cdots, v_k\}$, S_1 as the set of all those vertices in $V - S$ which are adjacent to exactly one vertex in S and S_2 as the set of all the remaining vertices in $V - S$ so that $V = S \cup S_1 \cup S_2$.

We define a coloring $c : V(G) \to \{1, 2, \cdots, k\}$ as follows:

$$c(v_i) = \begin{cases} i & \text{if } v_i \in S \\ j & \text{if } v_i \in S_1 \text{ where } j \text{ is the index of the vertex in } S \text{ adjacent to } v_i \\ k & \text{otherwise where } k \text{ is the index of any vertex in } S \text{ adjacent to } v_i \end{cases}$$

Then for every vertex $v_i \in S$, there exists a vertex, say v_j in $V - S$ with $c(v_i) = c(v_j)$ and vice-versa. This ensures that c is a neighborhood pseudocoloring of G. Hence $\psi_{nhd}(G) \geq k = \gamma(G)$. The result obtained for connected graphs can be easily extended to disconnected graphs. □

§5. Characterization of a Graph G with $\psi_{nhd}(G) = 2$

Using the results in Section 4, we give a characterization of a graph G with pseudochromatic number 2 through the following observation in this section.

Observation 5.1 The following are the six forbidden subgraphs in any non-trivial connected graph G with $\psi_{nhd}(G) \leq 2$, i. e., a non-trivial connected graph G has $\psi_{nhd}(G) \geq 3$ if G has a subgraph isomorphic to one of the six graphs in Figure 6.

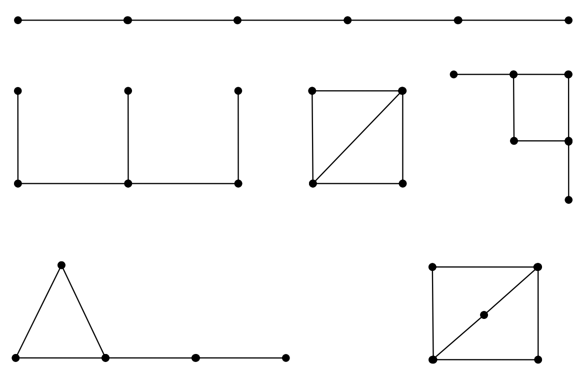


Figure 6. Forbidden subgraphs in a non-trivial connected graph with $\psi_{nhd}(G) \leq 2$

Proof The result follows directly from Observation 1.5 and the fact that the neighborhood pseudochromatic number of each of the graphs in Figure 6 is 3. □

Theorem 5.2 *For a non-trivial connected graph G, $\psi_{nhd}(G) = 2$ if and only if G is isomorphic to one of the three graphs G_1, G_2 or G_3 or is a member of one of the graph families G_4, G_5, G_6, G_7 or G_8 in Figure 7.*

Proof Let G be a non-trivial connected graph. Suppose G is isomorphic to one of the three graphs G_1, G_2 or G_3 or is a member of one of the graph families G_4, G_5, G_6, G_7 or G_8 in Figure 7. Then it is easy to observe that $\psi_{nhd}(G) = 2$.

Conversely, suppose $\psi_{nhd}(G) = 2$. By Theorem 4.2, $\gamma(G) \leq \psi_{nhd}(G) = 2$ implies that either $\gamma(G) = 1$ or $\gamma(G) = 2$.

If $\gamma(G) = 1$, then G is a star $K_{1,n}, n \geq 1$ or is isomorphic to G_1 or a member of the family G_4 in Figure 7 or has a subgraph isomorphic to H_3 in Figure 6. Similarly, if $\gamma(G) = 2$, then G is one of the graphs G_2 or G_3 or is a member of the family G_5, G_6, G_7 or G_8 in Figure 7 or has a subgraph isomorphic to one of the graphs in Figure 6.

However, since $\psi_{nhd}(G) = 2$, by Observation 5.1, G cannot have a subgraph isomorphic to any of the graphs in Figure 6. Thus, the only possibility is that G is isomorphic to one of the three graphs G_1, G_2 or G_3 or is a member of one of the graph families G_4, G_5, G_6, G_7 or G_8 in Figure 7. □

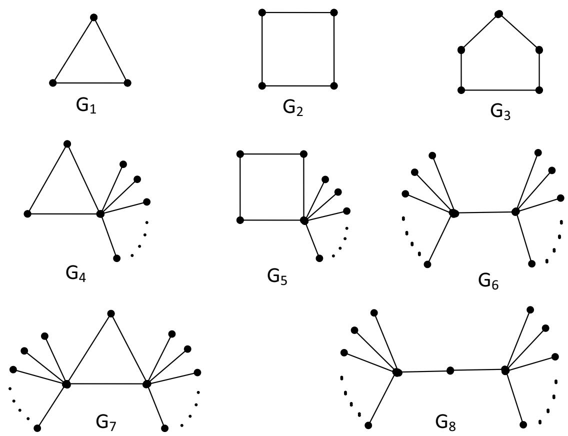


Figure 7. Graphs or graph families with $\psi_{nhd}(G) = 2$

§6. Conclusion

In this paper, we have obtained the neighborhood pseudochromatic number of some standard graphs. We have established some trivial lower bounds on this number. Improving on these lower bounds remains an interesting open problem.

We have also characterized graphs G for which $\psi_{nhd}(G) = 1, 2, n-1$ or n. However, the problem of characterizing graphs for which $\psi_{nhd}(G) = 3$ still remains open.

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