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Spacelike Curves of Constant Breadth According to Bishop Frame in E_1^3

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Abstract: In this paper, we study a special case of Smarandache breadth curves, and give some characterizations of the space-like curves of constant breadth according to Bishop frame in Minkowski 3-space.

Key Words: Minkowski 3-space, Smarandache breadth curves, curves of constant breadth, Bishop frame.

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§1. Introduction

Curves of constant breadth were introduced by Euler in [3]. Fujivara presented a problem to determine whether there exist space curves of constant breadth or not, and defined the concept of breadth for space curves and also obtained these curves on a surface of constant breadth in [5]. Some geometric properties of curves of constant breadth were given in a plane by [8]. The similar properties were obtained in Euclidean 3-space E^3 in [9]. These kind curves were studied in four dimensional Euclidean space E^4 in [1].

In this paper, we study a special case of Smarandache breadth curves in Minkowski 3-space E_1^3. A Smarandache curve is a regular curve with 2 breadths or more than 2 breadths in Minkowski 3-space E_1^3. Also we investigate position vectors of simple closed space-like curves and give some characterizations of curves of constant breadth according to Bishop frame of type-1 in E_1^3. Thus, we extend this classical topic to the space E_1^3, which is related to the time-like curves of constant breadth in E_1^3, see [10] for details. We also use a method which is similar to one in [9].

§2. Preliminaries

The Minkowski 3-space E_1^3 is an Euclidean 3-space E^3 provided with the standard flat metric

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given by

$$<\ ,\ >= -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. Since $<\ ,\ >$ is an indefinite metric recall that a vector $v \in E_1^3$ can be one of three Lorentzian characters; it can be space-like if $< v, v >> 0$ or $v = 0$, time-like if $< v, v >< 0$ and null if $< v, v >= 0$ and $v \neq 0$. Similarly, an arbitrary curve $\varphi = \varphi(s)$ in E_1^3 can locally be space-like, time-like or null (light-like) if all of its velocity vector φ' is respectively space-like, time-like or null (light-like) for every $s \in J \in \mathbb{R}$. The pseudo-norm of an arbitrary vector $a \in E_1^3$ is given by $\|a\| = \sqrt{|< a, a >|}$. The curve φ is called a unit speed curve if its velocity vector φ' satisfies $\|\varphi'\| = \mp 1$. For any vectors $u, w \in E_1^3$, they are said to be orthogonal if and only if $< u, w >= 0$.

Denote by $\{T, N, B\}$ the moving Frenet frame along curve φ in the space E_1^3. Let φ be a space-like curve with a space-like binormal in the space E_1^3, as similar to in [11], the Frenet formulae are given as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.1}$$

where κ and τ are the first and second curvatures with

$$\begin{aligned} &< T, T >=< B, B >= 1, < N, N >= -1, \\ &< T, N >=< T, B >=< N, B >= 0. \end{aligned}$$

The construction of the Bishop frame is due to L.R.Bishop in [4]. This frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even the space-like curve with a space-like binormal has vanishing second derivative [2]. He used tangent vector and any convenient arbitrary basis for the remainder of the frame. Then, as similar to in [2], the Bishop frame is expressed as

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & -k_2 \\ k_1 & 0 & 0 \\ k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix} \tag{2.2}$$

and

$$\kappa(s) = \sqrt{|k_1^2 - k_2^2|}, \quad \tau(s) = \frac{d\theta}{ds}, \quad \theta(s) = \tanh^{-1}\frac{k_2}{k_1} \tag{2.3}$$

where $k_1$and k_2 are Bishop curvatures.

§3. Spacelike Curves of Constant Breadth According to Bishop Frame in E_1^3

Let $\overrightarrow{\varphi} = \overrightarrow{\varphi}(s)$ and $\overrightarrow{\varphi}^* = \overrightarrow{\varphi}^*(s)$ be simple closed curves of constant breadth in Minkowski 3-space. These curves will be denoted by C and C^*. The normal plane at every point P on the curve meets the curve in the class Γ having parallel tangents $\overrightarrow{\mathbf{T}}$ and $\overrightarrow{\mathbf{T}}^*$ in opposite directions at the opposite points φ and φ^* of the curve as in [5]. A simple closed curve of constant breadth having parallel tangents in opposite directions at opposite points can be represented with respect to Bishop frame by the equation

$$\varphi^*(s) = \varphi(s) + m_1 T + m_2 N_1 + m_3 N_2 \tag{3.1}$$

where $m_i(s)$, $1 \leq i \leq 3$ are arbitrary functions, also φ and φ^* are opposite points. Differentiating (3.1) and considering Bishop equations, we have

$$\begin{aligned} \frac{d\varphi^*}{ds} = \overrightarrow{T}^* \frac{ds^*}{ds} = & \left(\frac{dm_1}{ds} + m_2 k_1 + m_3 k_2 + 1\right) T \\ & + \left(\frac{dm_2}{ds} + m_1 k_1\right) N_1 + \left(\frac{dm_3}{ds} - m_1 k_2\right) N_2. \end{aligned} \tag{3.2}$$

Since $T^* = -T$, rewriting (3.2), we obtain the following equations

$$\begin{cases} \dfrac{dm_1}{ds} = -m_2 k_1 - m_3 k_2 - 1 - \dfrac{ds^*}{ds} \\ \dfrac{dm_2}{ds} = -m_1 k_1 \\ \dfrac{dm_3}{ds} = m_1 k_2. \end{cases} \tag{3.3}$$

If we call θ as the angle between the tangent of the curve (C) at point $\varphi(s)$ with a given direction and consider $\dfrac{d\theta}{ds} = \tau$, we can rewrite (3.3) as follow;

$$\begin{cases} \dfrac{dm_1}{d\theta} = -m_2 \dfrac{k_1}{\tau} - m_3 \dfrac{k_2}{\tau} - f(\theta) \\ \dfrac{dm_2}{d\theta} = -m_1 \dfrac{k_1}{\tau} \\ \dfrac{dm_3}{d\theta} = m_1 \dfrac{k_2}{\tau} \end{cases} \tag{3.4}$$

where

$$f(\theta) = \delta + \delta^* \quad \delta = \frac{1}{\tau}, \delta^* = \frac{1}{\tau^*} \tag{3.5}$$

denote the radius of curvature at the points φ and φ^*, respectively. And using the system (3.4), we have the following differential equation with respect to m_1 as

$$\begin{aligned} & \frac{d^3 m_1}{d\theta^3} - \left(\frac{\kappa}{\tau}\right)^2 \frac{dm_1}{d\theta} + \left[\frac{k_2}{\tau}\frac{d}{d\theta}\left(\frac{k_2}{\tau}\right) - \frac{d}{d\theta}\left(\frac{\kappa}{\tau}\right)^2 - \frac{\kappa}{\tau}\frac{d}{d\theta}\left(\frac{k_2}{\tau}\right)\right] m_1 \\ & + \left(\int_0^\theta m_1 \frac{k_2}{\tau} d\theta\right) \frac{d^2}{d\theta^2}\left(\frac{k_2}{\tau}\right) - \left(\int_0^\theta m_1 \frac{k_1}{\tau} d\theta\right) \frac{d^2}{d\theta^2}\left(\frac{k_1}{\tau}\right) + \frac{d^2 f}{d\theta^2} = 0. \end{aligned} \tag{3.6}$$

The equation (3.6) is characterization of the point φ^*. If the distance between opposite points of (C) and (C^*) is constant, then we write that

$$\|\varphi^* - \varphi\| = -m_1^2 + m_2^2 + m_3^2 = l^2 \text{ is constant.} \tag{3.7}$$

Hence, from (3.7) we obtain

$$-m_1 \frac{dm_1}{d\theta} + m_2 \frac{dm_2}{d\theta} + m_3 \frac{dm_3}{d\theta} = 0 \tag{3.8}$$

Considering system (3.4), we get

$$m_1 \left[2m_3 \frac{k_2}{\tau} + f(\theta) \right] = 0 \tag{3.9}$$

From (3.9), we study the following cases which are depended on the conditions $2m_3 \frac{k_2}{\tau} + f(\theta) = 0$ or $m_1 = 0$.

Case 1. If $2m_3 \frac{k_2}{\tau} + f(\theta) = 0$ then by using (3.4), we obtain

$$\frac{dm_1}{d\theta} - \left(\int_0^\theta m_1 \frac{k_1}{\tau} d\theta\right) \frac{k_1}{\tau} + \frac{f(\theta)}{\tau} = 0. \tag{3.10}$$

Now let us to investigate solution of the equation (3.6) and suppose that m_2, m_3 and $f(\theta)$ are constants, $m_1 \neq 0$, then using (3.4) in (3.6), we have the following differential equation

$$\frac{d^3 m_1}{d\theta^3} - \left(\frac{\kappa}{\tau}\right)^2 \frac{dm_1}{d\theta} - \frac{d}{d\theta}\left(\frac{\kappa}{\tau}\right)^2 m_1 = 0. \tag{3.11}$$

The general solution of (3.11) depends on the character of the ratio $\frac{\kappa}{\tau}$. Suppose that φ is not constant breadth. For this reason, we distinguish the following sub-cases.

Subcase 1.1 Suppose that φ is an inclined curves then the solution of the differential equation (3.11) is

$$m_1 = c_1 + c_2 e^{-\frac{\kappa}{\tau}\theta} + c_3 e^{\frac{\kappa}{\tau}\theta}. \tag{3.12}$$

Therefore, we have m_2 and m_3, respectively,

$$\begin{aligned} m_2 &= -\int_0^\theta \left(c_1 + c_2 e^{-\frac{\kappa}{\tau}\theta} + c_3 e^{-\frac{\kappa}{\tau}\theta}\right) \frac{k_1}{\tau} d\theta \\ m_3 &= \int_0^\theta \left(c_1 + c_2 e^{-\frac{\kappa}{\tau}\theta} + c_3 e^{\frac{\kappa}{\tau}\theta}\right) \frac{k_2}{\tau} d\theta \end{aligned} \tag{3.13}$$

where c_1, c_2 and c_3 are real numbers.

Subcase 1.2 Suppose that φ is a line. The solution is in the following form

$$m_1 = A_1 \frac{\theta^2}{2} + A_2\theta + A_3. \tag{3.14}$$

Hence, we have m_2 and m_3 as follows

$$\begin{aligned} m_2 &= -\int_0^\theta (A_1 \frac{\theta^2}{2} + A_2\theta + A_3)\frac{k_1}{\tau}d\theta \\ m_3 &= \int_0^\theta (A_1 \frac{\theta^2}{2} + A_2\theta + A_3)\frac{k_2}{\tau}d\theta \end{aligned} \tag{3.15}$$

where A_1, A_2 and A_3 are real numbers.

Case 2. If $m_1 = 0$, then $m_2 = M_2$ and $m_3 = M_3$ are constants. Let us suppose that $m_2 = m_3 = c$ (constant). Thus, the equation (3.4) is obtained as

$$f(\theta) = \frac{-c(k_1 + k_2)}{\tau}.$$

This means that the curve is a circle. Moreover, the equation (3.6) has the form

$$\frac{d^2 f}{d\theta^2} = 0. \tag{3.16}$$

The solution of (3.16) is

$$f(\theta) = l_1\theta + l_2 \tag{3.17}$$

where l_1 and l_2 are real numbers. Therefore, we write the position vector φ^* as follows

$$\varphi^* = \varphi + M_2 N_1 + M_3 N_2 \tag{3.18}$$

where M_2 and M_3 are real numbers.

Finally, the distance between the opposite points of the curves (C) and (C^*) is

$$\|\varphi^* - \varphi\| = M_2^2 + M_3^2 = \text{ constant}. \tag{3.19}$$

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