International J.Math. Combin. Vol.4(2014), 120-126

Smarandache Lattice and Pseudo Complement

N.Kannappa

(Mathematics Department, TBML College, Porayar, Tamil Nadu, India)

K.Suresh

(Department of Mathematics, Mailam Engineering College, Vilupuram,Tamil Nadu, India)

E-mail: Porayar.sivaguru91@yahoo.com, Mailam.suya24su02@gmail.com

Abstract: In this paper, we introduce Smarandache - 2-algebraic structure of lattice S, namely *Smarandache lattices*. A Smarandache 2-algebraic structure on a set N means a weak algebraic structure A_0 on N such that there exists a proper subset M of N which is embedded with a stronger algebraic structure A_1, where a stronger algebraic structure means such a structure which satisfies more axioms, by proper subset one can understands a subset different from the empty set, by the unit element if any, and from the whole set. We obtain some of its characterization through pseudo complemented.

Key Words: Lattice, Boolean algebra, Smarandache lattice and Pseudo complemented lattice.

AMS(2010): 53C78

§1. Introduction

New notions are introduced in algebra to study more about the congruence in number theory by Florentin Smarandache [1]. By <proper subset> of a set A, we consider a set P included in A, different from A, also different from the empty set and from the unit element in A - if any they rank the algebraic structures using an order relationship.

The algebraic structures $S_1 \ll S_2$ if both of them are defined on the same set; all S_1 laws are also S_2 laws; all axioms of S_1 law are accomplished by the corresponding S_2 law; S_2 law strictly accomplishes more axioms than S_1 laws, or in other words S_2 laws has more laws than S_1. For example, a semi-group $\ll$ monoid $\ll$ group $\ll$ ring $\ll$ field, or a semi group $\ll$ commutative semi group, ring $\ll$ unitary ring, $\cdots$ etc. They define a general special structure to be a structure SM on a set A, different from a structure SN, such that a proper subset of A is an SN structure, where SM $\ll$ SN.

§2. Preliminaries

Definition 2.1 *Let P be a lattice with* 0 *and $x \in P$. We say $x*$ is a pseudo complemented of*

[1]Received September 1, 2014, Accepted December 10, 2014.

x iff $x* \in P$ and $x \wedge x* = 0$, and for every $y \in P$, if $x \wedge y = 0$ then $y \leq x*$.

Definition 2.2 *Let P be a pseudo complemented lattice. $N_P = \{\ x* : x \in P\}$ is the set of complements in P. $N_P = \{\ N_P, \leq N, \neg N, 0_N, 1_N, \wedge N, \vee N\}$, where*

(1) $\leq_N$ *is defined by: for every $x, y \in N_P$, $x \leq_N y$ iff $x \leq_P b$;*
(2) $\neg N$ *is defined by: for every $x \in N_P$, $\neg N(x) = x*$;*
(3) $\wedge_N$ *is defined by: for every $x, y \in N_P$, $x \wedge_N y = x \wedge_P y$;*
(4) $\vee_N$ *is defined by: for every $x, y \in N_P$, $x \vee_N y = (x * \wedge_P y*)*$;*
(5) $1_N = 0_P*$, $0_N = 0_P$.

Definition 2.3 *Let P be a lattice with 0. Define I_P to be the set of all ideals in P, i.e., $I_P =< I_P, \leq_I, \wedge_I, \vee_I, 0_I, 1_I >$, where*

$$\leq = \subseteq, i \wedge_I j = I \cap J, i \vee_I j = (I \cup J], 0_I = 0_A, 1_I = A.$$

Definition 2.4 *If P is a distributive lattice with 0, I_P is a complete pseudo complemented lattice, let P be a lattice with 0 and NI_P, the set of normal ideals in P, is given by $NI_P = \{\ I* \in I_P : I \in I_P\}$. Alternatively, $NI_P = \{\ I \in I_P : I = I**\}$. Thus $NI_P = \{\ NI_P, \subseteq, \cap, \cup, \wedge_{NI}, \vee_{NI}\}$, which is the set of pseudo complements in I_P.*

Definition 2.5 *A Pseudo complemented distributive lattice P is called a stone lattice if, for all $a \in P$, it satisfies the property $a * \vee a ** = 1$.*

Definition 2.6 *Let P be a pseudo complemented distributive lattice. Then for any filter F of P, define the set $\delta(F)$ by $\delta(F) = \{\ a* \in P/a* \in F\}$.*

Definition 2.7 *Let P be a pseudo complemented distributive lattice. An ideal I of P is called a δ-ideal if $I = \delta(F)$ for some filter F of P.*

Now we have introduced a definition by [4]:

Definition 2.8 *A lattice S is said to be a Smarandache lattice if there exist a proper subset L of S, which is a Boolean algebra with respect to the same induced operations of S.*

§3. Characterizations

Theorem 3.1 *Let S be a lattice. If there exist a proper subset N_P of S defined in Definition 2.2, then S is a Smarandache lattice.*

Proof By hypothesis, let S be a lattice and whose proper subset $N_P = \{\ x^* : x \in P\}$ is the set of all pseudo complements in P. By definition, if P is a pseudo complement lattice, then $N_P = \{\ x^* : x \in P\}$ is the set of complements in P, i.e., $N_P = \{\ N_P, \leq_N, \neg_N, 0_N, 1_N, \wedge_N, \vee_N\}$, where

(1) $\leq_N$ is defined for every $x, y \in N_P$, $x \leq_N y$ iff $x \leq_P b$;

(2) $\neg N$ is defined for every $x \in N_P$, $\neg N(x) = x*$;

(3) $\wedge_N$ is defined for every $x, y \in N_P$, $x \wedge_N y = x \wedge_P y$;

(4) $\vee_N$ is defined for every $x, y \in N_P, x \vee_N y = (x * \wedge_P y*)*$;

(5) $1_N = 0_P*$, $0_N = 0_P$.

It is enough to prove that N_P is a Boolean algebra.

(1) For every $x, y \in N_P$, $x \wedge_N y \in N_P$ and $\wedge_N$ is meet under $\leq_N$. If $x, y \in N_P$, then $x = x**$ and $y = y**$.

Since $x \wedge_P y \leq_P x$, by result if $x \leq_N y$ then $y* \leq_N x*$, $x* \leq_P (x \wedge_P y)*$, and with by result if $x \leq_N y$ then $y* \leq_N x*$, $(x \wedge_P y)** \leq_P x$. Similarly, $(x \wedge_P y)** \leq_P y$. Hence $(x \wedge_P y)** \leq_P (x \wedge_P y)**$.

By result, $x \leq_N x**$, $(x \wedge_P y) \leq_P (x \wedge_P y)**$. Hence $(x \wedge_P y) \in N_P$, $(x \wedge_N y) \in N_P$. If $a \in N_P$ and $a \leq_N x$ and $a \leq_N y$, then $a \leq_P x$ and $a \leq_P y$, $a \leq_P (x \wedge_P y)$. Hence $a \leq_N (x \wedge_N y)$. So, indeed $\wedge_N$ is meet in $\leq_N$.

(2) For every $x, y \in N_P$, $x \vee_N y \in N_P$ and $\vee_N$ is join under $\leq_N$. Let $x, y \in N_P$. Then $x*, y* \in N_P$. By (1), $(x * \wedge_P y*) \in N_P$. Hence $(x * \wedge_P y*)* \in N_P$, and hence $(x \vee_N y) \in N_P$, $(x * \wedge_P y*) \leq_P x*$. By result $x \leq_N x**$, $x** \leq_P (x * \wedge_P y*)*$ and $N_P = \{ x \in P; x = x**\}$, $x \leq_P (x * \wedge_P y*)*$.

Similarly, $y \leq_P (x * \wedge_P y*)*$. If $a \in N_P$ and $x \leq_N a$ and $y \leq_N a$, then $x \leq_P a$ and $y \leq_P a$, then by result if $x \leq_N y$, then $y* \leq_N x*$, $a* \leq_P x*$ and $a* \leq_P y*$. Hence $a* \leq_P (x* \wedge_P y*)$. By result if $x \leq_N y$ then $y* \leq_N x*$, $(x* \wedge_P y*)* \leq_P a**$. Thus, by result $N_P = \{ x \in P : x = x**\}$, $(x * \wedge_P y*)* \leq_P x$ and $x \vee_N y \leq_N a$. So, indeed $\vee_N$ is join in $\leq_N$.

(3) 0_N, $1_N \in N_P$ and 0_N, 1_N are the bounds of N_P. Obviously $1_N \in N_P$ since $1_N = 0_P*$ and for every $a \in N_P$, $a \wedge_P 0_P = 0_P$. For every $a \in N_P$, $a \leq_P 0_P*$. Hence $a \leq_N 1_N$, $0*_P$, $0_P** \in N_P$ and $0_P * \wedge_P 0_P ** \in N_P$. But of course, $0 *_P \wedge_P$, $0_P ** = 0_P$. Thus, $0_P \in N_P$, $0_N \in N_P$. Obviously, for every $a \in N_P$, $0_P \leq_P a$. Hence for every $a \in N_P$, $0_N \leq_N a$. So N_P is bounded lattice.

(4) For every $a \in N_P$, $\neg N(a) \in N_P$ and for every $a \in N_P$, $a \wedge_N \neg N(a) = 0_N$, and for every $a \in N_P$, $a \vee_N \neg N(a) = 1_N$.

Let $a \in N_P$. Obviously, $\neg_N(a) \in N_P$, $a \vee_N \neg N(a) = a \vee_N a* = ((a * \wedge_P b**))* = (a* \wedge_P a)* = 0_P* = 1_N$, $a \wedge_N \neg N(a) = a \wedge_P a* = 0_P = 0_N$. So N_P is a bounded complemented lattice.

(5) Since $x \leq_N (x \vee_N (y \wedge_N z))$, $(x \wedge_N z) \leq_N x \vee_N (y \wedge_N z)$. Also $(y \wedge_N z) \leq_N x \vee_N (y \wedge_N z)$. Obviously, if $a \leq_N b$, then $a \wedge_N b* = 0_N$. Since $b \wedge b* = 0_N$, so $(x \wedge_N z) \wedge_N (x \vee_N (y \wedge_N z))* = 0_N$ and $(y \wedge_N z) \wedge_N (x \vee_N (y \wedge_N z))* = 0_N$, $x \wedge_N (z \wedge_N (x \vee_N (y \wedge_N z))*) = 0_N$, $y \wedge_N (z \wedge_N (x \vee_N (y \wedge_N z))*) = 0_N$.

By definition of pseudo complement: $z \wedge_N (x \vee_N (y \wedge_N z))* \leq_N x*$, $z \wedge_N (x \vee_N (y \wedge_N z))* \leq_N y*$, Hence, $z \wedge_N (x \vee_N (y \wedge_N z))* \leq_N x * \wedge_N y*$. Once again, If $a \leq_N b$, then $a \wedge_N b* = 0_N$. Thus, $z \wedge_N (x \vee_N (y \wedge_N z)) * \wedge (x * \wedge_N y*)* = 0_N$, $z \wedge_N (x * \wedge_N y*)* \leq_N (x \vee_N (y \wedge_N z))**$. Now, by definition of $\wedge_N$: $z \wedge_N (x * \vee_N y*)* = z \wedge_N (x \vee_N y)$ and by $N_P = \{ x \in P : x = x**\} :$ $(x \vee_N (y \wedge_N z))** = x \vee_N (y \wedge_N z)$. Hence, $z \wedge_N (x \vee_N y) \leq_N x \vee_N (y \wedge_N z)$.

Hence, indeed N_P is a Boolean Algebra. Therefore by definition, S is a Smarandache lattice. □

For example, a distributive lattice D_3 is shown in Fig.1,

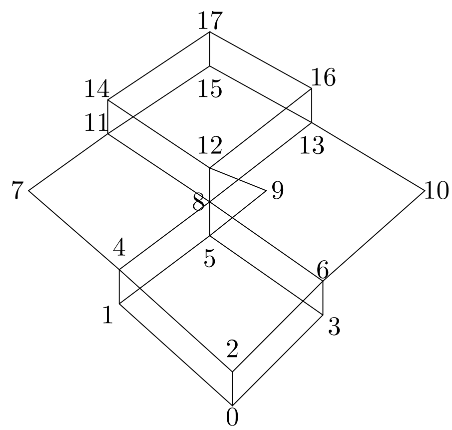

Fig.1

where D_3 is pseudo coplemented because

$0* = 17,$

$8* = 11* = 12* = 13* = 14* = 15* = 16* = 17* = 0,$

$1* = 10, 6* = 10* = 1,$

$2* = 9, 5* = 9* = 2,$

$3* = 7, 4* = 7* = 3$

and its correspondent Smarandache lattice is shown in Fig.2.

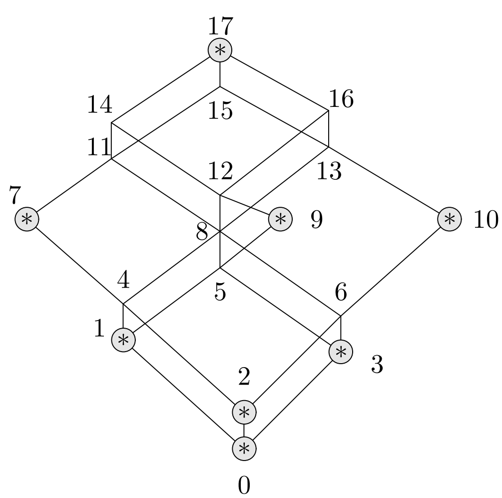

Fig.2

Theorem 3.2 *Let S be a distributive lattice with 0. If there exist a proper subset NI_P of S, defined Definition 2.4. Then S is a Smarandache lattice.*

Proof By hypothesis, let S be a distributive lattice with 0 and whose proper subset $NI_P = \{ I* \in I_P, I \in I_P\}$ is the set of normal ideals in P. We claim that NI_P is Boolean algebra since $NI_P = \{ I* \in I_P : I \in I_P\}$ is the set of normal ideals in P.

Alternatively, $NI_P = \{ I \in I_P : I = I**\}$. Let $I \in I_P$. Take $I* = \{y \in P : \text{forevery} i \in I : y \wedge i = 0\}$, $I* \in I_P$. Namely, if $a \in I*$ then for every $i \in I : a \wedge i = 0$. Let $b \leq a$. Then, obviously, for every $i \in I, b \wedge i = 0$. Thus $b \in I*$. If $a, b \in I*$, then for every $i \in I, a \wedge i = 0$, and for every $i \in I, b \wedge i = 0$.

Hence for every $i \in I, (a \wedge i) \vee (b \wedge i) = 0$. By distributive, for every $i \in I, i \wedge (a \vee b) = 0$, i.e., $a \vee b \in I*$. Thus $I* \in I_P$, $I \cap I* = I \cap \{ y \in P, \text{forevery } i \in I, y \wedge i = 0\} = \{ 0\}$.

Let $I \cap J = \{ 0\}$ and $j \in$ J. Suppose that for some $i \in I, i \wedge j \neq 0$. Then $i \wedge j \in I \cap J$. Because I and j are ideals, so $I \cap J \neq \{0\}$. Hence, for every $i \in I, j \wedge i = 0$, and $j \subseteq I*$.

Consequently, $I*$ is a pseudo complement of I and I_P is a pseudo complemented. Therefore I_P is a Boolean algebra. Thus NI_P is the set of all pseudo complements lattice in I_P.

Notice that we have proved that pseudo complemented form a Boolean algebra in Theorem 3.1. Whence, $NI_P$is a Boolean algebra. By definition, S is a Smarandache lattice. □

Theorem 3.3 *Let S be a lattice. If there exist a pseudo complemented distributive lattice P, $X*(P)$ is a sub-lattice of the lattice $I^\delta(P)$ of all δ-ideals of P, which is the proper subset of S. Then S is a Smarandache lattice.*

Proof By hypothesis, let S be a lattice and there exist a pseudo complemented distributive lattice P, $X*(P)$ is a sub-lattice of the lattice $I^\delta(P)$ of all δ-ideals of P, which is the proper subset of S.

Let $(a*], (b*] \in X*(P)$ for some $a, b \in P$. Clearly, $(a*] \cap (b*] \in X*(P)$. Again, $(a*] \cup (b*] = \delta([a)) \cup \delta([b)) = \delta[(a) \cup ([b)) = \delta([a \cap b)) = ((a \cap b)*] \in X*(P)$. Hence $X*(P)$ is a sub-lattice of $I^\delta(P)$ and it is a distributive lattice. Clearly $(0**]$ and $(0*]$ are the least and greatest elements of $X*(P)$.

Now for any $a \in$ P, $(a*] \cap (a**] = (0]$ and $(a*] \cup (b**] = \delta([a)) \cup \delta([a*)) = \delta([a)) \cup ([a*)) = \delta([a \cap a*)) = \delta([0)) = \delta(P) = P$. Hence (a**] is the complement of (a*] in X*(P).
Therefore { X*(P),∪ ,∩} is a bounded distributive lattice in which every element is complemented.
Thus $X*(P)$ is also a Boolean algebra, which implies that S is a Smarandache lattice. □

Theorem 3.4 *Let S be a lattice and P is a pseudo complemented distributive lattice. If S is a Smarandache lattice. Then the following conditions are equivalent:*

(1) *P is a Boolean algebra;*
(2) *every element of P is closed;*
(3) *every principal ideal is a δ-ideal;*
(4) *for any ideal I, $a \in I$ implies $a** \in I$;*

(5) *for any proper ideal* I, $I \cap D(P) = \phi$;

(6) *for any prime ideal* A, $A \cap D(P) = \phi$;

(7) *every prime ideal is a minimal prime ideal;*

(8) *every prime ideal is a* δ*-ideal;*

(9) *for any* $a, b \in P$, $a* = b*$ *implies* $a = b$;

(10) $D(P)$ *is a singleton set.*

Proof Since S is a Smarandache lattice. By definition and previous theorem, we observe that there exists a proper subset P of S such that which is a Boolean algebra. Therefore, P is a Boolean algebra.

(1) $\Longrightarrow$ (2) Assume that P is a Boolean algebra. Then clearly, P has a unique dense element, precisely the greatest element. Let $a \in P$. Then $a * \wedge a = 0 = a * \wedge a * *$. Also $a * \vee a$, $a * \vee a * * \in D(P)$. Hence $a * \vee a = a * \vee a * *$. By the cancellation property of P, we get $a = a * *$. Therefore every element of P is closed.

(2) $\Longrightarrow$ (3) Let I be a principal ideal of P. Then $I = (a]$ for some $a \in P$. By condition (2), $a = a * *$. Now, $(a] = (a * *] = \delta([a*))$. So $(a)]$ is a δ-ideal.

(3) $\Longrightarrow$ (4) Notice that I be a proper ideal of P. Let $a \in I$. Then there must be $(a] = \delta(F)$ for some filter F of P. Hence, we get that $a * ** = a* \in F$. Therefore $a * * \in \delta(F) = (a] \subseteq I$.

(4) $\Longrightarrow$ (5) Let I be a proper ideal of P. Suppose $a \in I \cap D(P)$. Then $a * * \in P$ and $a* = 0$. Therefore $1 = 0* = a * * \in P$, a contradiction.

(5) $\Longrightarrow$ (6) Let I be a proper ideal of P, $I \cap D(P) = \phi$. Then P is a prime ideal of P, $A \cap D(P) = \phi$.

(6) $\Longrightarrow$ (7) Let A be a prime ideal of P such that $A \cap D(P) = \phi$ and $a \in A$. Clearly $a \wedge a* = 0$ and $a \vee a* \in D(P)$. So $a \vee a* \notin A$, i.e., $a* \notin A$. Therefore A is a minimal prime ideal of P.

(7) $\Longrightarrow$ (8) Let A be a minimal prime ideal of P. It is clear that $P \setminus A$ is a filter of P. Let $a \in$ A. Since A is minimal, there exists $b \notin A$ such that $a \wedge b = 0$. Hence $a * \wedge b = b$ and $a* \notin A$. Whence, $a* \in (P \setminus A)$, which yields $a \in \delta(P \setminus A)$. Conversely, let $a \in \delta(P \setminus A)$. Then we get $a* \notin A$. Thus, we have $a \in A$ and $P = \delta(P \setminus A)$. Therefore A is δ-ideal of P.

(8) $\Longrightarrow$ (9) Assume that every prime ideal of P is a δ-ideal. Let $a, b \in P$ be chosen that $a* = b*$. Suppose $a \neq b$. Then there exists a prime ideal A of P such that $a \in A$ and $b \notin A$. By hypothesis, A is a δ- ideal of P. Hence $A = \delta(F)$ for some filter F of P. Consequently, $a \in A = \delta(F)$, We get $b* = a* \in F$. Thus, $b \in \delta(F) = A$, a contradiction. Therefore $a = b$.

(9) $\Longrightarrow$ (10) Suppose x, y be two elements of $D(P)$. Then $x* = 0 = y*$, which implies that $x = y$. Therefore $D(P)$ is a singleton set.

(10) $\Longrightarrow$ (1) Assume that $D(P) = \{d\}$ is singleton set. Let $a \in P$. We always have $a \vee a* \in D(P)$. Whence, $a \wedge a* = 0$ and $a \vee a* = d$. This true for all $a \in P$. Also $0 \leq a \leq a \vee a* = d$.

Therefore P is a bounded distributive lattice, in which every element is complemented, Hence the above conditions are equivalent. □

References

[1] Florentin Smarandache, *Special Algebraic Structures*, University of New Mexico,1991, SC: 06A99.

[2] Gratzer G., *General Lattice Theory*, Academic Press, New York, San Francisco, 1978.

[3] Monk, J.Donald, Bonnet and Robert eds(1989), *Hand book of Boolean Algebras*, Amsterdam, North Holland Publishing co.

[4] Padilla, Raul, Smarandache algebraic structure, *International Conference on Semi-Groups*, Universidad do minho, Bra go, Portugal, 18-23 June 1999.

[5] Padilla, Raul, Smarandache algebraic structure, *Smarandache Notions journal*, Vol.1, 36-38.

[6] Padilla, Raul, Smarandache algebraic structures, *Bulletin of Pure and Applied Science*, Vol.17.E, NO.1(1998),119-121.

[7] Samba Siva Rao, Ideals in pseudo complemented distributive Lattices, *Archivum Mathematical(Brno)* , No.48(2012), 97-105.

[8] Sabine Koppel Berg, *General Theory Boolean Algebra*, North Holland, Amsterdam, 1989.

[9] T.Ramaraj and N.Kannappa, On finite Smarandache near rings, *Scienctia Magna*, Vol.1, 2(2005), 49-51.

[10] N.Kannappa and K.Suresh, On some characterization Smarandache Lattice, *Proceedings of the International Business Management*, Organized by Stella Marie College, Vol.II, 2012, 154-155.